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# On the motivic cohomology of mixed characteristic schemes

*Sur la cohomologie motivique des schémas en caractéristique mixte*

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## Tess BOUIS


THESE DE DOCTORAT


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## Composition du jury

Membres du jury avec voix délibérative

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**Titre :** Sur la cohomologie motivique des schémas en caractéristique mixte

**Mots clés :** Cohomologie motivique, $K$-théorie algébrique, géométrie $p$-adique

**Résumé :** Dans cette thèse, nous construisons une théorie de cohomologie motivique pour les schémas quasi-compacts quasi-séparés, qui généralise la construction d'Elmanto–Morrow dans le cas des schémas au-dessus d'un corps. Notre construction n'est pas $\mathbb{A}^1$-invariante en général, mais elle utilise la cohomologie motivique classique $\mathbb{A}^1$-invariante des schémas lisses sur $\mathbb{Z}$. La nouveauté principale de notre construction est la définition et l'étude d'une filtration globale sur l'homologie cyclique topologique, dont les parties graduées unifient la cohomologie syntomique de Bhatt–Morrow–Scholze et la cohomologie de de Rham dérivée. Nous établissons un grand nombre des propriétés attendues de la cohomologie motivique, notamment une suite spectrale d'Atiyah–Hirzebruch vers la $K$-théorie algébrique non-connective, la formule des fibrés projectifs et la descente pro cdh. Les résultats du Chapitre 11 sont ceux de [Bou23].

**Title:** On the motivic cohomology of mixed characteristic schemes

**Keywords:** Motivic cohomology, algebraic $K$-theory, $p$-adic geometry

**Abstract:** In this thesis, we construct a theory of motivic cohomology for quasi-compact quasi-separated schemes, which generalises the construction of Elmanto–Morrow in the case of schemes over a field. Our construction is non-$\mathbb{A}^1$-invariant in general, but it uses the classical $\mathbb{A}^1$-invariant motivic cohomology of smooth $\mathbb{Z}$-schemes as an input. The main new input of our construction is a global filtration on topological cyclic homology, whose graded pieces provide a common generalisation of derived de Rham cohomology and Bhatt–Morrow–Scholze's syntomic cohomology. Our theory satisfies various expected properties of motivic cohomology, including a relation to non-connective algebraic $K$-theory via an Atiyah–Hirzebruch spectral sequence, the projective bundle formula, and pro cdh descent. The results of Chapter 11 have appeared as [Bou23].

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# Contents

# Chapter 1

# Introduction (en français)

La cohomologie motivique est un analogue de la cohomologie singulière en géométrie algébrique. Beilinson et Lichtenbaum ont d'abord prédit son existence pour les schémas $X$ de type fini sur $\mathbb{Z}$ [Lic73, Lic84, Beĭ86, Beĭ87, BMS87], afin de mieux comprendre les valeurs spéciales de leurs fonctions $L$. La cohomologie motivique, sous la forme de complexes de groupes abéliens $\mathbb{Z}(i)^{\mathrm{mot}}(X)$ indexés par les entiers $i \geq 0$, devrait être une interpolation entière entre la cohomologie étale et les espaces propres d'Adams sur la $K$-théorie algébrique rationalisée. En d'autres termes, il devrait exister une filtration naturelle $\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{K}(X)$ sur la $K$-théorie algébrique non connective $\mathrm{K}(X)$, qui se scinde rationnellement, et dont les parties graduées décalées

$$\mathbb{Z}(i)^{\mathrm{mot}}(X) \simeq \mathrm{gr}^i_{\mathrm{mot}}\mathrm{K}(X)[-2i]$$

sont données modulo $p$, lorsque $p$ est inversible dans $X$, et en degrés au plus $i$, par la cohomologie étale $R\Gamma_{\text{ét}}(X, \mu_p^{\otimes i})$ :

$$\tau^{\leq i}\,\mathbb{F}_p(i)^{\mathrm{mot}}(X) \simeq \tau^{\leq i} R\Gamma_{\text{ét}}(X, \mu_p^{\otimes i}).$$

Une telle théorie a d'abord été développée dans le cas lisse sur l'initiative de Bloch et Voevodsky [Blo86, VSF00], en utilisant des cycles algébriques et la théorie de l'$\mathbb{A}^1$-homotopie. Dans cette généralité, l'utilisation des techniques $\mathbb{A}^1$-invariantes est rendue possible par le théorème fondamental de Quillen en $K$-théorie algébrique [Qui73], qui affirme que la $K$-théorie algébrique est $\mathbb{A}^1$-invariante sur les schémas réguliers. Sur des schémas plus généraux, la $K$-théorie algébrique n'est pas $\mathbb{A}^1$-invariante, et la cohomologie motivique se doit donc, elle aussi, d'être non $\mathbb{A}^1$-invariante en général. La première théorie de cohomologie motivique non $\mathbb{A}^1$-invariante est due au travail récent d'Elmanto et Morrow [EM23], qui utilisent des avancées récentes en $K$-théorie algébrique et en homologie cyclique topologique. Leur théorie est développée dans la généralité des schémas quasi-compacts et quasi-séparés (qcqs) sur un corps quelconque, et coïncide avec la théorie classique $\mathbb{A}^1$-invariante sur les variétés lisses.

Dans cette thèse, nous étendons le travail d'Elmanto–Morrow en caractéristique mixte, produisant ainsi une théorie de cohomologie motivique dans la généralité initialement prévue par Beilinson et Lichtenbaum. Notre théorie s'appuie notamment sur les progrès récents en théorie de Hodge $p$-adique entière [BMS19, BS22, BL22], et fournit en retour une description complète de la cohomologie motivique modulo $p$, y compris lorsque $p$ n'est pas inversible dans le schéma $X$.

## 1.1 Une théorie de cohomologie motivique non $\mathbb{A}^1$-invariante

Le point de départ de notre construction est le résultat suivant, dû à Kerz–Strunk–Tamme [KST18] (qui prouvent que la $K$-théorie homotopique est la faisceautisation cdh de la $K$-théorie algébrique) et à Land–Tamme [LT19] (qui prouvent que la fibre $\mathrm{K}^{\mathrm{inf}}$ de la trace cyclotomique satisfait la descente cdh).

**Théorème 1.1.1** ([KST18, LT19]). *Soit $X$ un schéma qcqs. Alors le diagramme commutatif naturel*

$$\begin{array}{ccc} \mathrm{K}(X) & \longrightarrow & \mathrm{TC}(X) \\ \downarrow & & \downarrow \\ \mathrm{KH}(X) & \longrightarrow & \big(L_{\mathrm{cdh}}\mathrm{TC}\big)(X) \end{array}$$

*est un carré cartésien de spectres, où $\mathrm{KH}(X)$ est la $K$-théorie homotopique de $X$, $\mathrm{TC}(X)$ est l'homologie cyclique topologique de $X$, $L_{\mathrm{cdh}}$ est le foncteur de cdh faisceautisation, la flèche horizontale du haut est la trace cyclotomique, et la flèche horizontale du bas est la faisceautisation cdh de la trace cyclotomique.*

Le Théorème 1.1.1 affirme que la $K$-théorie algébrique des schémas peut être reconstruite uniquement en termes de la $K$-théorie homotopique (donc d'informations provenant de la théorie de l'$\mathbb{A}^1$-homotopie) et de l'homologie cyclique topologique (donc d'informations provenant des méthodes de trace). La topologie cdh est une topologie de Grothendieck, introduite par Voevodsky [SV00, Voe10] afin d'appliquer des techniques de théorie des topos à l'étude de la résolution des singularités. En particulier, si la résolution des singularités était vraie, tout schéma qcqs serait localement régulier dans la topologie cdh. Alors que la $K$-théorie homotopique et l'homologie cyclique topologique ont, à l'origine, été introduites comme des approximations utiles de la $K$-théorie algébrique déjà existante, nous construisons la cohomologie motivique des schémas en utilisant des raffinements de la $K$-théorie homotopique et de l'homologie cyclique topologique. Plus précisément, notre filtration motivique sur la $K$-théorie algébrique est définie par le recollement de certaines filtrations sur la $K$-théorie homotopique, sur l'homologie cyclique topologique, et sur l'homologie cyclique topologique cdh faisceautisée.

Pour la $K$-théorie homotopique, nous utilisons les travaux récents de Bachmann–Elmanto–Morrow [BEM24], qui construisent une filtration $\mathrm{Fil}^{\star}_{\mathrm{cdh}}\mathrm{KH}(X)$, qui est fonctorielle, multiplicative et indexée par $\mathbb{N}$, sur la $K$-théorie homotopique des schémas qcqs $X$. Les parties graduées décalées de cette filtration, que nous noterons $\mathbb{Z}(i)^{\mathrm{cdh}}(X)$, fournissent une bonne théorie de *cohomologie motivique cdh-locale* pour les schémas qcqs. Leur construction, que nous rappelons en Section 4.2, repose sur la cohomologie motivique classique $\mathbb{A}^1$-invariante des schémas lisses sur $\mathbb{Z}$, et étend la plupart de ses propriétés aux schémas qcqs généraux.

Notre première construction est celle d'une filtration $\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{TC}(X)$ fonctorielle, multiplicative et indexée par $\mathbb{Z}$, pour les schémas qcqs $X$. Cette filtration coïncide avec la filtration HKR sur $\mathrm{HC}^{-}(X/\mathbb{Q})$ en caractéristique zéro [Ant19, MRT22, Rak20], et avec la filtration motivique sur $\mathrm{TC}(X;\mathbb{Z}_p)$ après $p$-complétion [BMS19, Mor21, BL22, HRW22]. La définition qui suit décrit les parties graduées décalées de cette filtration :

$$\mathbb{Z}(i)^{\mathrm{TC}}(X) \simeq \mathrm{gr}^{i}_{\mathrm{mot}}\mathrm{TC}(X)[-2i].$$

**Définition 1.1.2** (Voir Définition 3.3.3). Pour tout schéma qcqs $X$ et tout entier $i \in \mathbb{Z}$, le complexe $\mathbb{Z}(i)^{\mathrm{TC}}(X) \in \mathcal{D}(\mathbb{Z})$ est défini par un carré cartésien naturel

$$\begin{array}{ccc} \mathbb{Z}(i)^{\mathrm{TC}}(X) & \longrightarrow & R\Gamma_{\mathrm{Zar}}\big(X, \widehat{\mathbb{L}\Omega}^{\geq i}_{-/\mathbb{Z}}\big) \\ \downarrow & & \downarrow \\ \prod_{p \in \mathbb{P}} \mathbb{Z}_p(i)^{\mathrm{BMS}}(X) & \longrightarrow & \prod_{p \in \mathbb{P}} R\Gamma_{\mathrm{Zar}}\big(X, (\widehat{\mathbb{L}\Omega}^{\geq i}_{-/\mathbb{Z}})^\wedge_p\big). \end{array}$$

Une conséquence de cette définition est que le préfaisceau $\mathbb{Z}(i)^{\mathrm{TC}}$ s'identifie naturellement avec le complexe syntomique de Bhatt–Morrow–Scholze $\mathbb{Z}_p(i)^{\mathrm{BMS}}$ en caractéristique $p$, et avec le complexe de de Rham dérivé Hodge-complété $R\Gamma_{\mathrm{Zar}}(-, \widehat{\mathbb{L}\Omega}^{\geq i}_{-/\mathbb{Q}})$ en caractéristique zéro. D'après [EM23], le complexe motivique $\mathbb{Z}(i)^{\mathrm{mot}}$ est défini en caractéristique $p$ et zéro respectivement par des carrés cartésiens

$$\begin{array}{ccc} \mathbb{Z}(i)^{\mathrm{mot}}(X) & \longrightarrow & \mathbb{Z}_p(i)^{\mathrm{BMS}}(X) \\ \downarrow & & \downarrow \\ \mathbb{Z}(i)^{\mathrm{cdh}}(X) & \longrightarrow & \big(L_{\mathrm{cdh}}\, \mathbb{Z}_p(i)^{\mathrm{BMS}}\big)(X) \end{array} \qquad \begin{array}{ccc} \mathbb{Z}(i)^{\mathrm{mot}}(X) & \longrightarrow & R\Gamma_{\mathrm{Zar}}\big(X, \widehat{\mathbb{L}\Omega}^{\geq i}_{-/\mathbb{Q}}\big) \\ \downarrow & & \downarrow \\ \mathbb{Z}(i)^{\mathrm{cdh}}(X) & \longrightarrow & R\Gamma_{\mathrm{cdh}}\big(X, \widehat{\mathbb{L}\Omega}^{\geq i}_{-/\mathbb{Q}}\big). \end{array}$$

La définition suivante est donc une généralisation naturelle en caractéristique mixte de la définition d'Elmanto–Morrow au-dessus d'un corps.

**Définition 1.1.3** (Cohomologie motivique ; voir Section 4.3). Pour tout schéma qcqs $X$ et tout entier $i \in \mathbb{Z}$, le *complexe motivique de poids* $i$

$$\mathbb{Z}(i)^{\mathrm{mot}}(X) \in \mathcal{D}(\mathbb{Z})$$

de $X$ est défini par un carré cartésien naturel

$$\begin{array}{ccc} \mathbb{Z}(i)^{\mathrm{mot}}(X) & \longrightarrow & \mathbb{Z}(i)^{\mathrm{TC}}(X) \\ \downarrow & & \downarrow \\ \mathbb{Z}(i)^{\mathrm{cdh}}(X) & \longrightarrow & \big(L_{\mathrm{cdh}}\, \mathbb{Z}(i)^{\mathrm{TC}}\big)(X), \end{array}$$

où la flèche horizontale du bas est induite par un raffinement filtré de la trace cyclotomique cdh faisceautisée.

Établir la relation attendue entre les complexes motiviques $\mathbb{Z}(i)^{\mathrm{mot}}$ et la $K$-théorie algébrique (Théorème C ci-dessous) demande cependant plus d'efforts en caractéristique mixte qu'au-dessus d'un corps. Le principal obstacle est ici de prouver que les préfaisceaux $\mathbb{Z}(i)^{\mathrm{mot}}$ s'annulent en poids $i < 0$. Nous commençons par rappeler comment cet énoncé est prouvé au-dessus d'un corps. Nous utiliserons dans ce qui suit le fait que, par construction, les préfaisceaux $\mathbb{Z}(i)^{\mathrm{cdh}}$ s'annulent en poids $i < 0$ (Section 4.2).

En caractéristique $p$, les préfaisceaux $\mathbb{Z}_p(i)^{\mathrm{BMS}}$ s'annulent en poids $i < 0$, donc il en est de même pour les préfaisceaux $\mathbb{Z}(i)^{\mathrm{mot}}$, et l'étape zéro de la filtration motivique associée $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}$ coïncide avec la $K$-théorie algébrique. Si l'on ignore pour le moment les problèmes de complétude de cette filtration motivique, cela signifie que les préfaisceaux $\mathbb{Z}(i)^{\mathrm{mot}}$ fournissent un raffinement cohomologique naturel de la $K$-théorie algébrique sur les schémas de caractéristique $p$.

En caractéristique zéro, les préfaisceaux $R\Gamma_{\text{Zar}}(-, \widehat{\mathbb{L}\Omega}^{\geq i}_{-/\mathbb{Q}})$ ne s'annulent pas en poids $i < 0$. Au lieu de cela, ils s'identifient au préfaisceau $R\Gamma_{\text{Zar}}(-, \widehat{\mathbb{L}\Omega}_{-/\mathbb{Q}})$, qui s'avère être un faisceau cdh sur les schémas qcqs de caractéristique zéro, d'après les résultats de Cortiñas–Haesemeyer–Schlichting–Weibel [CHSW08], Antieau [Ant19] et Elmanto–Morrow [EM23]. En d'autres termes, la flèche verticale de droite du diagramme précédent est une équivalence en poids $i < 0$, de sorte que les préfaisceaux $\mathbb{Z}(i)^{\text{mot}}$ s'annulent en poids $i < 0$, et que l'étape zéro de la filtration motivique associée $\text{Fil}^{\star}_{\text{mot}}\text{K}$ coïncide avec la $K$-théorie algébrique. Cela signifie que les préfaisceaux $\mathbb{Z}(i)^{\text{mot}}$ fournissent un raffinement cohomologique naturel de la $K$-théorie algébrique sur les schémas de caractéristique zéro.

En caractéristique mixte, nous prouvons également qu'en poids $i < 0$, les préfaisceaux $\mathbb{Z}(i)^{\text{TC}}$ sont des faisceaux cdh sur les schémas qcqs, *i.e.*, que les préfaisceaux $\mathbb{Z}(i)^{\text{mot}}$ s'annulent. Le résultat modulo un nombre premier $p$ est une conséquence, comme en caractéristique $p$, du fait que les préfaisceaux $\mathbb{Z}_p(i)^{\text{BMS}}$ s'annulent en poids $i < 0$. La difficulté est alors de prouver que les préfaisceaux $\mathbb{Q}(i)^{\text{TC}}$ sont des faisceaux cdh en poids $i < 0$.

Le principal résultat de descente cdh utilisé en caractéristique zéro n'est plus vrai en caractéristique mixte. Plus précisément, le préfaisceau $R\Gamma_{\text{Zar}}(-, \widehat{\mathbb{L}\Omega}_{-/\mathbb{Z}})$ (ou sa rationalisation) n'est pas un faisceau cdh sur les schémas qcqs. Nous contournerons cette difficulté en prouvant un analogue rigide-analytique de ce résultat de descente cdh. Pour formuler ce résultat, nous noterons

$$R\Gamma_{\text{Zar}}\big(X, \underline{\widehat{\mathbb{L}\Omega}}^{\geq i}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p}\big)$$

la *cohomologie de Rham dérivée rigide-analytique* d'un schéma qcqs $X$ au-dessus de $\mathbb{Z}_{(p)}$, que nous définissons comme la colimite du diagramme

$$R\Gamma_{\text{Zar}}\big(X, (\widehat{\mathbb{L}\Omega}^{\geq i}_{-/\mathbb{Z}})^{\wedge}_p\big) \longleftarrow R\Gamma_{\text{Zar}}\big(X, \widehat{\mathbb{L}\Omega}^{\geq i}_{-/\mathbb{Z}}\big) \longrightarrow R\Gamma_{\text{Zar}}(X, \widehat{\mathbb{L}\Omega}^{\geq i}_{-_{\mathbb{Q}}/\mathbb{Q}})$$

dans la catégorie dérivée $\mathcal{D}(\mathbb{Z})$. Nous limitons ici notre attention aux schémas qcqs au-dessus de $\mathbb{Z}_{(p)}$ par souci de simplicité, et renvoyons au Chapitre 5 pour les définitions et les énoncés analogues sur $\mathbb{Z}$. La Définition 1.1.2 implique qu'il existe un carré cartésien naturel

$$\begin{array}{ccc} \mathbb{Q}(i)^{\text{TC}}(X) & \longrightarrow & R\Gamma_{\text{Zar}}(X, \widehat{\mathbb{L}\Omega}^{\geq i}_{-_{\mathbb{Q}}/\mathbb{Q}}) \\ \big\downarrow & & \big\downarrow \\ \mathbb{Q}_p(i)^{\text{BMS}}(X) & \longrightarrow & R\Gamma_{\text{Zar}}(X, \underline{\widehat{\mathbb{L}\Omega}}^{\geq i}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p}) \end{array}$$

dans la catégorie dérivée $\mathcal{D}(\mathbb{Q})$. En poids $i < 0$, les préfaisceaux $\mathbb{Q}_p(i)^{\text{BMS}}$ sont nuls. Comme déjà mentionné, le préfaisceau $R\Gamma_{\text{Zar}}(-, \widehat{\mathbb{L}\Omega}_{-_{\mathbb{Q}}/\mathbb{Q}})$ est par ailleurs un faisceau cdh sur les schémas qcqs. Le fait que les préfaisceaux $\mathbb{Q}(i)^{\text{TC}}$ soient des faisceaux cdh en poids $i < 0$ est donc équivalent au résultat suivant, qui peut être vu comme un analogue rigide-analytique de l'énoncé rationnel précédent de descente cdh.

**Théorème A** (Descente cdh pour la cohomologie de de Rham rigide-analytique ; voir Corollaire 5.4.4)**.** *Pour tout nombre premier $p$, le préfaisceau $R\Gamma_{\text{Zar}}\big(-, \underline{\widehat{\mathbb{L}\Omega}}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p}\big)$ satisfait la descente cdh sur les schémas qcqs au-dessus de $\mathbb{Z}_{(p)}$.*

La preuve moderne du résultat rationnel analogue de descente cdh repose sur la théorie des invariants tronquants de Land–Tamme [LT19] et sur un théorème de Goodwillie [Goo85], qui prouvent respectivement que tout invariant tronquant est un faisceau cdh sur les schémas qcqs et que l'homologie cyclique périodique sur $\mathbb{Q}$ est un invariant tronquant. Par définition, un invariant tronquant est un invariant localisant $E$ tel que pour tout $\mathbb{E}_1$-anneau connectif $R$, la flèche naturelle $E(R) \to E(\pi_0(R))$ est une équivalence. Pour prouver le Théorème A, nous utilisons les mathématiques condensées de Clausen–Scholze [CS19], et prouvons qu'une certaine variante rigide-analytique de l'homologie cyclique périodique est un invariant tronquant. En particulier, la preuve du Théorème A utilise un résultat sur les anneaux associatifs (et même sur les $\mathbb{E}_1$-anneaux solides connectifs généraux).

Le Théorème A nous permet d'obtenir la description cohomologique suivante de la cohomologie motivique rationnelle.

**Théorème B** (Cohomologie motivique $p$-adique et rationnelle ; voir Corollaires 4.3.12 et 5.6.6)**.** *Soit $X$ un schéma qcqs et $p$ un nombre premier. Alors pour tous entiers $i \in \mathbb{Z}$ et $k \geq 1$, les diagrammes commutatifs naturels*

$$\begin{array}{ccc} \mathbb{Z}/p^k(i)^{\mathrm{mot}}(X) & \longrightarrow & \mathbb{Z}/p^k(i)^{\mathrm{BMS}}(X) \\ \downarrow & & \downarrow \\ \mathbb{Z}/p^k(i)^{\mathrm{cdh}}(X) & \longrightarrow & \big(L_{\mathrm{cdh}}\, \mathbb{Z}/p^k(i)^{\mathrm{BMS}}\big)(X) \end{array} \qquad \begin{array}{ccc} \mathbb{Q}(i)^{\mathrm{mot}}(X) & \longrightarrow & R\Gamma_{\mathrm{Zar}}\big(X, \widehat{\mathbb{L}\Omega}^{\geq i}_{-_{\mathbb{Q}}/\mathbb{Q}}\big) \\ \downarrow & & \downarrow \\ \mathbb{Q}(i)^{\mathrm{cdh}}(X) & \longrightarrow & R\Gamma_{\mathrm{cdh}}\big(X, \widehat{\mathbb{L}\Omega}^{\geq i}_{-_{\mathbb{Q}}/\mathbb{Q}}\big) \end{array}$$

*sont des carrés cartésiens dans la catégorie dérivée $\mathcal{D}(\mathbb{Z})$.*

Ensemble, ces deux carrés cartésiens retrouvent les carrés cartésiens d'Elmanto–Morrow définissant les complexes motiviques $\mathbb{Z}(i)^{\mathrm{mot}}$ au-dessus d'un corps, et en sont donc des analogues naturels en caractéristique mixte. La partie $p$-adique du Théorème B est une conséquence formelle des Définitions 1.1.2 et 1.1.3. La partie rationnelle du Théorème B implique, quant à elle, que la différence entre $\mathbb{Q}(i)^{\mathrm{mot}}(X)$ et $\mathbb{Q}(i)^{\mathrm{cdh}}(X)$ ne dépend que de la rationalisation $X_{\mathbb{Q}}$ du schéma $X$. Si $X$ est régulier, cette différence devrait s'annuler, et ce résultat est donc plus intéressant en présence de singularités. Plus précisément, le Théorème B peut être utilisé pour extraire des informations intéressantes sur les singularités d'un anneau commutatif général $R$ : les faisceaux cdh sont typiquement insensibles aux singularités, de sorte que l'information singulière dans le complexe motivique $\mathbb{Z}(i)^{\mathrm{mot}}(R)$ est en fait contrôlée par les complexes $\mathbb{Z}/p^k(i)^{\mathrm{BMS}}(R)$ et $\mathbb{L}\Omega^{<i}_{(R\otimes_{\mathbb{Z}}\mathbb{Q})/\mathbb{Q}}$, pour lesquels il existe des techniques de calcul.

Le Théorème B implique également que les préfaisceaux $\mathbb{Z}(i)^{\mathrm{mot}}$ s'annulent en poids $i < 0$, ce qui était la partie manquante essentielle pour établir les propriétés fondamentales suivantes de la cohomologie motivique.

**Théorème C** (Lien avec la $K$-théorie algébrique)**.** *Il existe un faisceau Nisnevich finitaire de spectres filtrés*

$$\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{K}(-) : \mathrm{Sch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

*qui admet les propriétés suivantes :*

*(1)* (Suite spectrale d'Atiyah–Hirzebruch ; voir Section 5.5) *Pour tout schéma qcqs $X$, la filtration $\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{K}(X)$ est une filtration multiplicative et indexée par $\mathbb{N}$ sur la $K$-théorie algébrique non connective $\mathrm{K}(X)$, dont les parties graduées sont naturellement données par*

$$\mathrm{gr}^i_{\mathrm{mot}}\mathrm{K}(X) \simeq \mathbb{Z}(i)^{\mathrm{mot}}(X)[2i], \quad i \geq 0.$$

*En particulier, en écrivant* $\mathrm{H}^j_{\mathrm{mot}}(X, \mathbb{Z}(i)) := \mathrm{H}^j(\mathbb{Z}(i)^{\mathrm{mot}}(X))$ *pour les* groupes de cohomologie motivique *correspondants, il existe une suite spectrale d'Atiyah–Hirzebruch*

$$E_2^{i,j} = \mathrm{H}^{i-j}_{\mathrm{mot}}(X, \mathbb{Z}(-j)) \Longrightarrow \mathrm{K}_{-i-j}(X).$$

*Si $X$ est de dimension valuative*[1] *finie, alors la filtration* $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X)$ *est complète, et la suite spectrale d'Atiyah–Hirzebruch est convergente.*

*(2)* (Décomposition d'Adams ; voir Corollaire 5.5.11) *Pour tout schéma qcqs $X$, la suite spectrale d'Atiyah–Hirzebruch dégénère rationnellement et, pour tout entier $n \in \mathbb{Z}$, il existe un isomorphisme naturel de groupes abéliens*

$$\mathrm{K}_n(X) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \bigoplus_{i \geq 0} \big(\mathrm{H}^{2i-n}_{\mathrm{mot}}(X, \mathbb{Z}(i)) \otimes_{\mathbb{Z}} \mathbb{Q}\big)$$

*induite par les opérateurs d'Adams sur la $K$-théorie algébrique rationalisée.*

L'une des principales motivations initiales pour développer la cohomologie motivique était d'appliquer des techniques cohomologiques à l'étude de la $K$-théorie algébrique [BMS87]. Le théorème suivant résume nos résultats sur les relations entre la cohomologie motivique et un certain nombre d'autres invariants cohomologiques. Lorsque $X$ est lisse sur $\mathbb{Z}$, nous noterons

$$\mathbb{Z}(i)^{\mathrm{cla}}(X) := z^i(X, \bullet)[-2i]$$

pour le *complexe motivique classique de poids $i$*, où $z^i(X, \bullet)$ est le complexe de cycles de Bloch (et où $\bullet$ est l'indice cohomologique).

**Théorème D.** *Soit $X$ un schéma qcqs, et $i \geq 0$ un entier.*

*(1)* (Poids zéro ; voir Exemple 5.6.8) *Il existe une équivalence naturelle*

$$\mathbb{Z}(0)^{\mathrm{mot}}(X) \xrightarrow{\sim} R\Gamma_{\mathrm{cdh}}(X, \mathbb{Z})$$

*dans la catégorie dérivée $\mathcal{D}(\mathbb{Z})$.*

*(2)* (Poids un ; voir Exemple 8.1.15) *Il existe une flèche naturelle*

$$R\Gamma_{\mathrm{Nis}}(X, \mathbb{G}_m)[-1] \longrightarrow \mathbb{Z}(1)^{\mathrm{mot}}(X)$$

*dans la catégorie dérivée $\mathcal{D}(\mathbb{Z})$, qui est un isomorphisme en degrés au plus trois.*

*(3)* (Cohomologie étale ; voir Corollaire 6.1.6) *Pour tout nombre premier $\ell$ qui est inversible dans $X$ et tout entier $k \geq 1$, il existe une flèche naturelle*

$$\mathbb{Z}/\ell^k(i)^{\mathrm{mot}}(X) \longrightarrow R\Gamma_{\mathrm{\acute{e}t}}(X, \mu_{\ell^k}^{\otimes i})$$

*dans la catégorie dérivée $\mathcal{D}(\mathbb{Z}/\ell^k)$, qui est un isomorphisme en degrés au plus $i$.*

[1] La dimension valuative d'un anneau commutatif, définie en termes de rangs de certains anneaux de valuation, a été introduite par Jaffard [Jaf60, Chapitre IV], et été généralisée aux schémas dans [EHIK21, Section 2.3]. La dimension valuative d'un schéma est toujours au moins égale à sa dimension de Krull, et les deux notions coïncident sur les schémas noethériens. Dans ce qui suit, la dimension valuative d'un schéma qcqs $X$ sera utilisée comme une borne supérieure de la dimension cohomologique du topos cdh de $X$ ([EHIK21, Theorem 2.4.15]).

(4) (Cohomologie syntomique ; voir Corollaire 6.2.5) *Pour tout nombre premier $p$ et tout entier $k \geq 1$, il existe une flèche naturelle*

$$\mathbb{Z}/p^k(i)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{syn}}(X)$$

*dans la catégorie dérivée $\mathcal{D}(\mathbb{Z}/p^k)$, qui est un isomorphisme en degrés au plus $i$, et où $\mathbb{Z}/p^k(i)^{\mathrm{syn}}(X)$ désigne la cohomologie syntomique de poids $i$ de $X$ au sens de [BL22].*

(5) ($K$-théorie de Milnor ; voir Théorème 8.2.6) *Si $X = \mathrm{Spec}(A)$ est le spectre d'un anneau local hensélien $A$, alors pour tout entier $n \geq 1$, il existe un isomorphisme naturel*

$$\widehat{\mathrm{K}}_i^{\mathrm{M}}(A)/n \xrightarrow{\cong} \mathrm{H}^i_{\mathrm{mot}}(A, \mathbb{Z}(i))/n$$

*de groupes abéliens, où $\widehat{\mathrm{K}}_i^{\mathrm{M}}(A)$ désigne le $i^{\text{ème}}$ $K$-groupe amélioré de Milnor de $A$ au sens de [Ker10].*

(6) (Cohomologie motivique classique ; voir Chapitre 7) *Si $X$ est lisse sur $\mathbb{Z}$, alors il existe une flèche naturelle*

$$\mathbb{Z}(i)^{\mathrm{cla}}(X) \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(X)$$

*dans la catégorie dérivée $\mathcal{D}(\mathbb{Z})$, qui est un isomorphisme en degrés au plus $i+1$ en général, et un isomorphisme en tous degrés si $X$ est de dimension au plus un sur $\mathbb{Z}$.*

(7) (Cohomologie motivique lisse ; voir Corollary 8.1.12) *Si $X = \mathrm{Spec}(A)$ est le spectre d'un anneau local $A$, alors pour tout entier $i \geq 0$, il existe une équivalence naturelle*

$$\mathbb{Z}(i)^{\mathrm{lisse}}(A) \xrightarrow{\sim} \tau^{\leq i}\,\mathbb{Z}(i)^{\mathrm{mot}}(A)$$

*dans la catégorie dérivée $\mathcal{D}(\mathbb{Z})$, où $\mathbb{Z}(i)^{\mathrm{lisse}}$ désigne la cohomologie motivique lisse de poids $i$ de $A$, définie comme l'extension de Kan à gauche depuis les $\mathbb{Z}$-algèbres lisses du complexe motivique classique $\mathbb{Z}(i)^{\mathrm{cla}}$. En particulier, le foncteur $\tau^{\leq i}\,\mathbb{Z}(i)^{\mathrm{mot}}$ est étendu de Kan à gauche sur les anneaux locaux depuis les algèbres locales essentiellement lisses sur $\mathbb{Z}$.*

(8) (Cohomologie motivique $\mathbb{A}^1$-invariante ; voir Théorème 12.1.5) *Il existe une équivalence naturelle*

$$\big(L_{\mathbb{A}^1}\,\mathbb{Z}(i)^{\mathrm{mot}}\big)(X) \xrightarrow{\sim} \mathbb{Z}(i)^{\mathbb{A}^1}(X)$$

*dans la catégorie dérivée $\mathcal{D}(\mathbb{Z})$, où $\mathbb{Z}(i)^{\mathbb{A}^1}(X)$ désigne la cohomologie motivique $\mathbb{A}^1$-invariante de poids $i$ de $X$ au sens de [BEM24].*

Le résultat suivant est une conséquence du Théorème D (8) et du fait que les complexes motiviques $\mathbb{A}^1$-invariants $\mathbb{Z}(i)^{\mathbb{A}^1}$ coïncident avec les complexes motiviques classiques $\mathbb{Z}(i)^{\mathrm{cla}}$ sur les schémas lisses sur $\mathbb{Z}$ [BEM24]. En particulier, et bien que nous nous attendions à ce que nos complexes motiviques $\mathbb{Z}(i)^{\mathrm{mot}}$ coïncident avec les complexes motiviques classiques $\mathbb{Z}(i)^{\mathrm{cla}}$ sur les schémas lisses sur $\mathbb{Z}$, cela signifie que nous pouvons au moins prouver ce résultat après avoir imposé que les complexes $\mathbb{Z}(i)^{\mathrm{mot}}$ soient $\mathbb{A}^1$-invariants.

**Corollaire E** (Voir Corollaire 12.1.9)**.** *Soit $X$ un schéma lisse sur $\mathbb{Z}$. Alors, pour tout entier $i \geq 0$, il existe une équivalence naturelle*

$$\mathbb{Z}(i)^{\mathrm{cla}}(X) \simeq \big(L_{\mathbb{A}^1}\,\mathbb{Z}(i)^{\mathrm{mot}}\big)(X)$$

*dans la catégorie dérivée $\mathcal{D}(\mathbb{Z})$.*

En supposant l'existence d'une bonne catégorie dérivée de motifs DM, les groupes de cohomologie motivique d'un schéma $X$ devraient être donnés par

$$\mathrm{H}^j_{\mathrm{mot}}(X, \mathbb{Z}(i)) \cong \mathrm{Hom}_{\mathrm{DM}}(\mathrm{M}(X), \mathbb{Z}(i)[j]),$$

où $\mathrm{M}(X) \in \mathrm{DM}$ est le motif associé à $X$, et $\mathbb{Z}(i) \in \mathrm{DM}$ sont les motifs de Tate, qui prennent part, pour chaque entier $r \geq 0$, à une décomposition naturelle dans DM :

$$\mathrm{M}(\mathbb{P}^r_{\mathbb{Z}}) \cong \bigoplus_{j=0}^{r} \mathbb{Z}(j)[2j].$$

Dans le contexte $\mathbb{A}^1$-invariant, Voevodsky a construit une telle catégorie dérivée de motifs, dans laquelle les complexes motiviques classiques $\mathbb{Z}(i)^{\mathrm{cla}}$ peuvent ainsi être interprétés en termes de ces motifs de Tate. Sans supposer l'$\mathbb{A}^1$-invariance, Annala–Iwasa [AI23] et Annala–Hoyois–Iwasa [AHI23, AHI24] ont récemment introduit une catégorie dérivée de motifs plus générale, dans laquelle la décomposition du motif $\mathrm{M}(\mathbb{P}^r_{\mathbb{Z}})$, *i.e.*, la *formule des fibrés projectifs*, est isolée comme propriété définissante de leur construction. Le résultat suivant signifie que les complexes motiviques $\mathbb{Z}(i)^{\mathrm{mot}}$ s'intègrent naturellement dans cette théorie de motifs non $\mathbb{A}^1$-invariants.

**Théorème F** (Formule des fibrés projectifs ; voir Théorème 9.3.2). *Soit $X$ un schéma qcqs, $i \geq 0$ un entier, et $\mathcal{E}$ un fibré vectoriel de rang $r$ sur $X$. Alors, pour tout entier $i \geq 0$, les puissances de la première classe de Chern motivique $c_1^{\mathrm{mot}}(\mathcal{O}(1)) \in \mathrm{H}^2_{\mathrm{mot}}(\mathbb{P}_X(\mathcal{E}), \mathbb{Z}(1))$ induisent une équivalence naturelle*

$$\bigoplus_{i=0}^{r-1} \mathbb{Z}(i-j)^{\mathrm{mot}}(X)[-2j] \xrightarrow{\sim} \mathbb{Z}(i)^{\mathrm{mot}}(\mathbb{P}_X(\mathcal{E}))$$

*dans la catégorie dérivée $\mathcal{D}(\mathbb{Z})$.*

Le Théorème F est prouvé par Elmanto–Morrow dans le cas d'égale caractéristique [EM23], où la preuve repose sur la formule des fibrés projectifs pour les complexes $\mathbb{Z}(i)^{\mathrm{cdh}}$ [BEM24]. En caractéristique mixte, la formule des fibrés projectifs pour les complexes motiviques cdh-locaux $\mathbb{Z}(i)^{\mathrm{cdh}}$ n'est cependant connue que conditionnellement à une certaine propriété des anneaux de valuation, appelée $F$-lissité [BM23, BEM24]. Comme nous l'expliquons maintenant, il est possible de prouver cette propriété en caractéristique mixte, dans le cas des anneaux de valuation au-dessus d'une base perfectoïde. Le cas des anneaux de valuation généraux restant ouvert, notre preuve du Théorème F est différente de celle d'Elmanto–Morrow, et utilise en particulier notre description de la cohomologie motivique à coefficients finis en fonction de la cohomologie syntomique (Théorème 6.2.4).

## 1.2 Cohomologie motivique des anneaux de valuation

Ces dernières années, les anneaux de valuation ont été utilisés comme un moyen pour contourner la résolution des singularités, afin d'adapter des arguments de caractéristique zéro en caractéristique positive [KST21, KM21]. Nous expliquons ici comment adapter certaines de ces idées en caractéristique mixte, à commencer par l'isomorphisme de Cartier. La plupart des résultats qui suivent sur les anneaux de valuation sont ceux de [Bou23].

Soit $p$ un nombre premier, que nous fixons pour le reste de l'introduction. L'isomorphisme de Cartier affirme que pour toute $\mathbb{F}_p$-algèbre lisse $A$, le morphisme de Cartier inverse

$$C^{-1} : \begin{cases} \Omega^n_{A/\mathbb{F}_p} \longrightarrow \mathrm{H}^n(\Omega^\bullet_{A/\mathbb{F}_p}) \\ f dg_1 \wedge \cdots \wedge dg_n \longmapsto f^p g_1^{p-1} \ldots g_n^{p-1} dg_1 \wedge \cdots \wedge dg_n \end{cases}$$

est un isomorphisme de $\mathbb{F}_p$-espaces vectoriels pour chaque $n \geq 0$. En supposant la résolution des singularités en caractéristique $p$, toute $\mathbb{F}_p$-algèbre serait localement lisse dans la topologie cdh. Les anneaux locaux pour la topologie cdh sont des anneaux de valuation, donc les anneaux de valuation de caractéristique $p$ devraient être des colimites filtrées de $\mathbb{F}_p$-algèbres lisses, et en particulier devraient satisfaire l'isomorphisme de Cartier. Sans supposer la résolution des singularités, Gabber a prouvé que les anneaux de valuation de caractéristique $p$ satisfont effectivement cette propriété fondamentale des $\mathbb{F}_p$-algèbres lisses qu'est l'isomorphisme de Cartier. Motivés par ce résultat et par le manque général de compréhension des anneaux de valuation, Kelly et Morrow [KM21] ont alors introduit les *algèbres Cartier lisses* comme des $\mathbb{F}_p$-algèbres satisfaisant l'isomorphisme de Cartier, et ont étudié leur $K$-théorie algébrique.

Nous développons ici un analogue de cette théorie en caractéristique mixte. Le morphisme de Cartier inverse, en tant que généralisation du morphisme de Frobenius, est spécifique à la caractéristique $p$, et nous définissons le fait d'être $p$-Cartier lisse pour les homomorphismes d'anneaux généraux essentiellement en fonction de leur réduction modulo $p$. Cette notion plus générale coïncide avec celle introduite par Kelly–Morrow [KM21] dans le cas particulier où l'anneau de base est $\mathbb{F}_p$. Pour une $\mathbb{F}_p$-algèbre $R$, nous noterons $\phi_R$ son endomorphisme de Frobenius.

**Définition 1.2.1** (Cartier lisse). (1) Un morphisme $R \to S$ de $\mathbb{F}_p$-algèbres est *Cartier lisse*, ou $S$ est une $R$-algèbre *Cartier lisse*, s'il est cotangent lisse, *i.e.*, son complexe cotangent $\mathbb{L}_{S/R}$ est un $S$-module plat en degré zéro donné par $\Omega^1_{S/R}$, et si le morphisme de Cartier inverse

$$C^{-1} : \Omega^n_{S/R} \otimes_{R,\phi_R} R \longrightarrow \mathrm{H}^n(\Omega^\bullet_{S/R})$$

est un isomorphisme de $R$-modules pour chaque $n \geq 0$.

(2) Un morphisme $R \to S$ d'anneaux commutatifs est *$p$-Cartier lisse*, ou $S$ est une $R$-algèbre *$p$-Cartier lisse*, si la flèche naturelle $S \otimes^{\mathbb{L}}_R R/p \longrightarrow S/p[0]$ est une équivalence dans la catégorie dérivée $\mathcal{D}(R/p)$, et si la réduction $R/p \longrightarrow S/p$ modulo $p$ est Cartier lisse.

Autrement dit, un morphisme $R \to S$ d'anneaux commutatifs est $p$-Cartier lisse s'il est $p$-cotangent lisse (Définition 11.1.3), et si sa réduction $R/p \to S/p$ modulo $p$ satisfait l'isomorphisme de Cartier. L'hypothèse de $p$-cotangent lissité est nécessaire pour ne pas perdre le contrôle des invariants que nous allons étudier lors de la spécialisation en caractéristique $p$. Tout morphisme lisse d'anneaux commutatifs est $p$-Cartier lisse (pour tout nombre premier $p$). En ce qui concerne les anneaux de valuation, une extension d'anneaux de valuation (c'est-à-dire un morphisme injectif d'anneaux de valuation) en caractéristique mixte n'est en général pas $p$-Cartier lisse. Par exemple, le morphisme $\mathbb{Z}_p \to \overline{\mathbb{Z}}_p$, où $\overline{\mathbb{Z}}_p$ est l'anneau des entiers d'une clôture algébrique $\overline{\mathbb{Q}}_p$ de $\mathbb{Q}_p$, n'est pas $p$-cotangent lisse, et ne satisfait pas non plus l'isomorphisme de Cartier : son complexe cotangent est en degré zéro, donné par le $\overline{\mathbb{Z}}_p$-module de torsion $\overline{\mathbb{Q}}_p/\overline{\mathbb{Z}}_p$, et $\Omega^0_{(\overline{\mathbb{Z}}_p/p)/\mathbb{F}_p} = \overline{\mathbb{Z}}_p/p$ contient des éléments nilpotents ; le complexe de Rham de $\mathbb{F}_p \to \overline{\mathbb{Z}}_p/p$ est en fait nul en degrés positifs, et ne contient donc pas beaucoup d'informations. En revanche, si l'anneau de valuation de base est suffisamment ramifié, ces obstructions disparaissent et nous pouvons prouver le résultat général qui suit. Ce résultat généralise un théorème de Gabber en caractéristique $p$, qui est valable sur les anneaux de valuation parfaits (Théorème 11.2.4 ci-dessous). Nous renvoyons le lecteur ou la lectrice à [BMS18, Definition 3.5] pour la définition d'un anneau perfectoïde.

**Théorème G** (Voir Théorème 11.2.1). *Soit $V_0$ un anneau de valuation dont la $p$-complétion est un anneau perfectoïde. Soit $V$ un anneau de valuation qui contient $V_0$ comme un sous-anneau. Alors le morphisme d'inclusion $V_0 \to V$ est $p$-Cartier lisse.*

La preuve du Théorème G procède par réduction au cas des anneaux de valuation de caractéristique $p$, et utilise la théorie des déformations. Nous expliquons maintenant comment appliquer le Théorème G à des calculs de la cohomologie motivique des anneaux de valuation.

Rappelons que sur les schémas lisses sur $\mathbb{F}_p$, la conjecture de Beilinson–Lichtenbaum, prouvée par Suslin et Voevodsky comme une conséquence de la conjecture de Bloch–Kato [SV00], calcule la partie $\ell$-adique de la cohomologie motivique en termes de la cohomologie du faisceau étale $\mu_{\ell^k}$ des $\ell^k$-racines de l'unité. Pour décrire la partie $p$-adique de la cohomologie motivique, il faut remplacer $\mu_{\ell^k}^{\otimes i}$ (qui est nul sur les variétés lisses lorsque $\ell = p$ et $i > 0$) par les faisceaux de Rham–Witt logarithmiques $W_k\Omega^i_{-,\log}$ [GL00]. La description correspondante de la $K$-théorie algébrique, en terme de ces faisceaux de Rham–Witt logarithmiques, est généralisée dans [KM21] à toutes les $\mathbb{F}_p$-algèbres Cartier lisses.

Sur les schémas lisses sur un domaine de Dedekind de caractéristique mixte, la partie $p$-adique de la cohomologie motivique classique est décrite de façon similaire en faibles degrés par la cohomologie étale de la fibre générique [Gei04]. Ce résultat est une conséquence de la conjecture de Gersten prouvée par Geisser [Gei04], et n'est pas connu pour les schémas réguliers généraux. Combiné avec le Théorème D (6), le résultat suivant étend cette description de la cohomologie motivique classique au cas régulier. Plus précisément, la notion de schéma $F$-lisse est introduite par Bhatt–Mathew [BM23] comme une généralisation non noethérienne des schémas réguliers, et notre résultat s'applique naturellement à tout schéma $F$-lisse et sans $p$-torsion.

**Théorème H** (Conjecture de Beilinson–Lichtenbaum pour les schémas $F$-lisses ; voir Corollaire 6.2.6). *Soit $X$ un schéma $F$-lisse et sans $p$-torsion (e.g., un schéma régulier, plat sur $\mathbb{Z}$). Alors pour tous entiers $i \geq 0$ et $k \geq 1$, la flèche de comparaison de Beilinson–Lichtenbaum*

$$\mathbb{Z}/p^k(i)^{\mathrm{mot}}(X) \longrightarrow R\Gamma_{\text{ét}}(X[\tfrac{1}{p}], \mu_{p^k}^{\otimes i})$$

*est un isomorphisme en degrés au plus $i-1$, et est injectif en degré $i$.*

Sur un anneau de base perfectoïde, les notions d'algèbres $F$-lisse et $p$-Cartier lisse coïncident d'après le Théorème 11.1.19. Au vu des Théorèmes G et H, nous obtenons ainsi une description complète de la cohomologie motivique $p$-adique des anneaux de valuation au-dessus d'une base perfectoïde.

**Corollaire I** (Cohomologie motivique des anneaux de valuation ; voir Théorème 11.4.6). *Soit $V_0$ un anneau de valuation sans $p$-torsion et dont la $p$-complétion est un anneau perfectoïde, et $V$ un anneau de valuation qui contient $V_0$ comme sous-anneau. Alors, pour tous entiers $i \geq 0$ et $k \geq 1$, le complexe motivique $\mathbb{Z}/p^k(i)^{\mathrm{mot}}(V)$ est en degrés au plus $i$, et la flèche de comparaison de Beilinson–Lichtenbaum*

$$\mathbb{Z}/p^k(i)^{\mathrm{mot}}(V) \longrightarrow R\Gamma_{\text{ét}}(\mathrm{Spec}(V[\tfrac{1}{p}]), \mu_{p^k}^{\otimes i})$$

*est un isomorphisme en degrés au plus $i-1$. En degré $i$, cette flèche est injective, et son image est engendrée par les symboles, via le morphisme de symboles*

$$(V^\times)^{\otimes i} \to \mathrm{H}^i_{\text{ét}}(\mathrm{Spec}(V[\tfrac{1}{p}]), \mu_{p^k}^{\otimes i}).$$

La preuve du Théorème H repose sur un théorème de comparaison syntomique-étale de Bhatt–Mathew [BM23]. Nous reprouvons ce théorème de comparaison syntomique-étale dans le cas des algèbres $p$-Cartier lisses sur une $\mathbb{Z}_p^{\mathrm{cyc}}$-algèbre perfectoïde, en utilisant une propriété des algèbres $p$-Cartier lisses sur la filtration de Nygaard de leur cohomologie prismatique relative (Théorème J ($\mathcal{N}^{\geq}$) ci-dessous). Pour prouver cette propriété, nous étendons un certain nombre de résultats sur la cohomologie prismatique relative des algèbres lisses aux algèbres $p$-Cartier lisses générales. Il est *a priori* surprenant que ces propriétés soient vraies en ne supposant qu'une propriété sur la fibre spéciale (l'isomorphisme de Cartier) ; nous démontrons que certaines de ces propriétés caractérisent le fait d'être $p$-Cartier lisse.

**Théorème J** (Voir Théorème 11.1.16)**.** *Soit $(A, I)$ un prisme borné, et soit $S$ une $A/I$-algèbre $p$-cotangent lisse. Les propositions suivantes sont équivalentes :*

($CSm$) *$S$ est $p$-Cartier lisse sur $A/I$.*

($\mathbb{L}\Omega = \widehat{\mathbb{L}\Omega}$) *La flèche canonique $(\mathbb{L}\Omega_{S/(A/I)})^\wedge_p \to (\widehat{\mathbb{L}\Omega}_{S/(A/I)})^\wedge_p$ est une équivalence dans la catégorie dérivée $\mathcal{D}(A/I)$, où $(\widehat{\mathbb{L}\Omega}_{S/(A/I)})^\wedge_p$ désigne la complétion de Hodge du complexe de Rham dérivé $p$-complété.*

($\mathbb{L}\Omega = \Omega$) *La flèche de counité $(\mathbb{L}\Omega_{S/(A/I)})^\wedge_p \to (\Omega_{S/(A/I)})^\wedge_p$ est une équivalence dans la catégorie dérivée $\mathcal{D}(A/I)$, où $(\mathbb{L}\Omega_{S/(A/I)})^\wedge_p$ désigne le complexe de Rham dérivé $p$-complété.*

($dR$) *La flèche de comparaison de de Rham $\mathbb{\Delta}^{(1)}_{S/A} \otimes^{\mathbb{L}}_A A/I \to (\Omega_{S/(A/I)})^\wedge_p$ est une équivalence dans la catégorie dérivée $\mathcal{D}(A/I)$.*

($\mathbb{\Delta} = \widehat{\mathbb{\Delta}}$) *La flèche canonique* can : $\mathbb{\Delta}^{(1)}_{S/A} \to \widehat{\mathbb{\Delta}}^{(1)}_{S/A}$ *est une équivalence dans la catégorie dérivée $\mathcal{D}(A)$, où $\widehat{\mathbb{\Delta}}^{(1)}_{S/A}$ désigne la complétion de Nygaard du complexe prismatique.*

($L\eta$) *La flèche de Frobenius $\tilde{\phi} : \mathbb{\Delta}^{(1)}_{S/A} \to L\eta_I \mathbb{\Delta}_{S/A}$ est une équivalence dans la catégorie dérivée $\mathcal{D}(A)$.*

($\mathcal{N}^{\geq}$) *La flèche de Frobenius $\tau^{\leq i}\phi : \tau^{\leq i}\mathcal{N}^{\geq i}\mathbb{\Delta}^{(1)}_{S/A} \to \tau^{\leq i} I^i \mathbb{\Delta}_{S/A}$ est une équivalence dans la catégorie dérivée $\mathcal{D}(A)$ pour tout $i \geq 0$.*

## 1.3 Cohomologie motivique des schémas singuliers

Pour le reste de cette introduction, nous nous concentrons sur les propriétés de la cohomologie motivique qui sont spécifiques aux schémas singuliers. L'une des caractéristiques les plus intéressantes, mais aussi les plus mystérieuses, de la $K$-théorie algébrique des schémas singuliers est la présence de $K$-groupes négatifs non nuls. La plupart des propriétés connues sur ces $K$-groupes négatifs reposent sur des résultats concernant la $K$-théorie algébrique des éclatements [CHSW08, KST18]. Il a notamment été prouvé par Thomason [Tho93] que la $K$-théorie algébrique associe une suite exacte longue de $K$-groupes à la donnée d'un éclatement par rapport à une immersion fermée régulière. Le résultat suivant est un raffinement cohomologique du résultat de Thomason.

**Théorème K** (Formule des éclatements réguliers ; voir Théorème 9.3.1)**.** *Pour toute immersion fermée régulière $Y \to X$ de schémas qcqs (i.e., le sous-schéma fermé $Y$ est défini, localement sur $X$ pour la topologie de Zariski, par une suite régulière) et tout entier $i \geq 0$, le diagramme commutatif*

$$\begin{array}{ccc} \mathbb{Z}(i)^{\mathrm{mot}}(X) & \longrightarrow & \mathbb{Z}(i)^{\mathrm{mot}}(Y) \\ \downarrow & & \downarrow \\ \mathbb{Z}(i)^{\mathrm{mot}}(\mathrm{Bl}_Y(X)) & \longrightarrow & \mathbb{Z}(i)^{\mathrm{mot}}(\mathrm{Bl}_Y(X) \times_X Y) \end{array}$$

*est un carré cartésien dans la catégorie dérivée $\mathcal{D}(\mathbb{Z})$.*

La $K$-théorie algébrique n'associe en revanche pas de suite exacte longue à tout éclatement. Motivées par le théorème de Grothendieck sur les fonctions formelles pour la cohomologie quasi-cohérente [Gro61, Theorem 4.1.5], de nombreuses personnes ont espéré un analogue formel de ces suites exactes longues en $K$-théorie algébrique, qui serait valable pour des éclatements généraux. Un tel résultat a finalement été démontré par Kerz–Strunk–Tamme [KST18], sous la forme d'une propriété d'excision pro cdh pour la $K$-théorie algébrique des schémas noethériens quelconques. Notons que la topologie pro cdh a récemment été introduite par Kelly–Saito [KS24], comme un moyen d'encoder cette propriété d'excision pro cdh en une propriété de descente pour cette topologie de Grothendieck. Le résultat suivant s'appuie, en particulier, sur le Théorème B et sur le théorème de Grothendieck sur les fonctions formelles, expliquant ainsi en partie l'analogie entre les techniques quasi-cohérentes et celles utilisées en $K$-théorie algébrique.

**Théorème L** (Descente pro cdh ; voir Théorème 10.2.11)**.** *Pour tout entier $i \geq 0$, le préfaisceau $\mathbb{Z}(i)^{\mathrm{mot}}$ satisfait la descente pro cdh sur les schémas noethériens. En d'autres termes, pour chaque carré d'éclatement abstrait*

$$\begin{array}{ccc} Y' & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

*de schémas noethériens, le diagramme commutatif associé*

$$\begin{array}{ccc} \mathbb{Z}(i)^{\mathrm{mot}}(X) & \longrightarrow & \mathbb{Z}(i)^{\mathrm{mot}}(X') \\ \downarrow & & \downarrow \\ \{\mathbb{Z}(i)^{\mathrm{mot}}(rY)\}_r & \longrightarrow & \{\mathbb{Z}(i)^{\mathrm{mot}}(rY')\}_r \end{array}$$

*est un carré faiblement cartésien de pro objets dans la catégorie dérivée $\mathcal{D}(\mathbb{Z})$.*

Kelly et Saito ont prouvé que la $K$-théorie algébrique non connective s'identifiait en fait à la faisceautisation pro cdh de la $K$-théorie algébrique connective [KS24]. Combinée avec l'observation de Bhatt–Lurie que la $K$-théorie algébrique connective est étendue de Kan à gauche sur les anneaux commutatifs depuis les $\mathbb{Z}$-algèbres lisses [EHK+20], cette propriété a motivé la définition des *complexes de cohomologie motivique pro cdh* $\mathbb{Z}(i)^{\mathrm{procdh}}$, comme la faisceautisation pro cdh de l'extension de Kan à gauche des complexes motiviques classiques $\mathbb{Z}(i)^{\mathrm{cla}}$. Le résultat suivant repose sur la comparaison avec la cohomologie motivique lisse (Théorème D (7)) et sur la descente pro cdh (Théorème L).

**Corollaire M** (Comparaison avec la cohomologie motivique pro cdh ; voir Théorème 10.4.2). *Soit $X$ un schéma noethérien. Alors, pour tout entier $i \geq 0$, il existe une équivalence naturelle*

$$\mathbb{Z}(i)^{\mathrm{procdh}}(X) \xrightarrow{\sim} \mathbb{Z}(i)^{\mathrm{mot}}(X)$$

*dans la catégorie dérivée $\mathcal{D}(\mathbb{Z})$.*

Notons que les complexes motiviques pro cdh $\mathbb{Z}(i)^{\mathrm{procdh}}$ ne sont pas finitaires, et ne peuvent donc pas coïncider avec les complexes motiviques $\mathbb{Z}(i)^{\mathrm{mot}}$ sur les schémas qcqs généraux.

Une conjecture importante de Weibel [Wei80] énonce que pour tout schéma noethérien $X$ de dimension au plus $d$, les $K$-groupes négatifs $\mathrm{K}_{-n}(X)$ s'annulent pour les entiers $n > d$. Cette conjecture a été établie par Kerz–Strunk–Tamme [KST18], comme une conséquence de la descente pro cdh pour la $K$-théorie algébrique. La preuve du résultat suivant utilise les techniques de Kerz–Strunk–Tamme [KST18] telles que reformulées par Elmanto–Morrow [EM23], qui prouvent le même résultat au-dessus d'un corps. En particulier, le Théorème N repose sur la descente pro cdh pour la cohomologie motivique (Théorème L).

**Théorème N** (Annulation de Weibel motivique ; voir Théorème 10.3.3). *Soit $X$ un schéma noethérien de dimension finie $d$, et $i \geq 0$ un entier. Alors pour tout entier $j > i + d$, le groupe de cohomologie motivique $\mathrm{H}^j_{\mathrm{mot}}(X, \mathbb{Z}(i))$ est nul.*

Via la suite spectrale d'Atiyah–Hirzebruch du Théorème C, le Théorème N est un raffinement motivique de la conjecture d'annulation de Weibel en $K$-théorie.

# Chapter 2

# Introduction (in english)

Motivic cohomology is an analogue in algebraic geometry of singular cohomology. It was first envisioned to exist for schemes $X$ of finite type over $\mathbb{Z}$ by Beilinson and Lichtenbaum [Lic73, Lic84, Beĭ86, Beĭ87, BMS87], as a way to better understand the special values of their $L$-functions. Motivic cohomology, in the form of complexes of abelian groups $\mathbb{Z}(i)^{\mathrm{mot}}(X)$ indexed by integers $i \geq 0$, should be an integral interpolation between étale cohomology, and Adams eigenspaces on rationalised algebraic $K$-theory. That is, there should be a natural filtration $\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{K}(X)$ on the non-connective algebraic $K$-theory $\mathrm{K}(X)$, which splits rationally, and whose shifted graded pieces

$$\mathbb{Z}(i)^{\mathrm{mot}}(X) \simeq \mathrm{gr}^i_{\mathrm{mot}}\mathrm{K}(X)[-2i]$$

are given mod $p$, when $p$ is invertible in $X$, and in degrees at most $i$, by the étale cohomology $R\Gamma_{\text{ét}}(X, \mu_p^{\otimes i})$:

$$\tau^{\leq i}\,\mathbb{F}_p(i)^{\mathrm{mot}}(X) \simeq \tau^{\leq i} R\Gamma_{\text{ét}}(X, \mu_p^{\otimes i}).$$

Such a theory was first developed in the smooth case at the initiative of Bloch and Voevodsky [Blo86, VSF00], using algebraic cycles and $\mathbb{A}^1$-homotopy theory. In this generality, the use of $\mathbb{A}^1$-invariant techniques is permitted by Quillen's fundamental theorem of algebraic $K$-theory [Qui73], stating that algebraic $K$-theory is $\mathbb{A}^1$-invariant on regular schemes. On more general schemes, algebraic $K$-theory fails to be $\mathbb{A}^1$-invariant, so motivic cohomology itself needs to be non-$\mathbb{A}^1$-invariant in general. The first non-$\mathbb{A}^1$-invariant theory of motivic cohomology was recently introduced by Elmanto and Morrow [EM23], using recent advances in algebraic $K$-theory and topological cyclic homology. Their theory is developed in the generality of quasi-compact quasi-separated (qcqs) schemes over an arbitrary field, and recovers on smooth varieties the classical $\mathbb{A}^1$-invariant theory.

In this thesis, we extend the work of Elmanto–Morrow to mixed characteristic, thus producing a theory of motivic cohomology in the originally expected generality of Beilinson and Lichtenbaum. Our theory relies on recent progress in integral $p$-adic Hodge theory [BMS19, BS22, BL22], and offers in return a complete description of mod $p$ motivic cohomology, even when $p$ is not invertible in the qcqs scheme $X$.

## 2.1 A non-$\mathbb{A}^1$-invariant theory of motivic cohomology

The starting point of our construction is the following result, due to Kerz–Strunk–Tamme [KST18] (who prove that homotopy $K$-theory is the cdh sheafification of algebraic $K$-theory) and Land–Tamme [LT19] (who prove that the fibre $\mathrm{K}^{\mathrm{inf}}$ of the cyclotomic trace map satisfies cdh descent).

**Theorem 2.1.1** ([KST18, LT19]). *Let $X$ be a qcqs scheme. Then the natural commutative diagram*

$$\begin{array}{ccc} \mathrm{K}(X) & \longrightarrow & \mathrm{TC}(X) \\ \downarrow & & \downarrow \\ \mathrm{KH}(X) & \longrightarrow & \big(L_{\mathrm{cdh}}\mathrm{TC}\big)(X) \end{array}$$

*is a cartesian square of spectra, where $\mathrm{KH}(X)$ is the homotopy $K$-theory of $X$, $\mathrm{TC}(X)$ is the topological cyclic homology of $X$, $L_{\mathrm{cdh}}$ is the cdh sheafification functor, the top horizontal map is the cyclotomic trace map, and the bottom horizontal map is the cdh sheafified cyclotomic trace map.*

Theorem 2.1.1 states that algebraic $K$-theory of schemes can be reconstructed purely in terms of homotopy $K$-theory (*i.e.*, information coming from $\mathbb{A}^1$-homotopy theory) and topological cyclic homology (*i.e.*, information coming from trace methods). The cdh topology is a Grothendieck topology introduced by Voevodsky [SV00, Voe10], as a way to apply topos theoretic techniques to the study of resolution of singularities. In particular, assuming resolution of singularities, any qcqs scheme would be locally regular in the cdh topology. While homotopy $K$-theory and topological cyclic homology were originally introduced as tools to approximate the existing algebraic $K$-theory, we construct the motivic cohomology of schemes using refinements of homotopy $K$-theory and topological cyclic homology. More precisely, our motivic filtration on algebraic $K$-theory is defined as a pullback of appropriate filtrations on homotopy $K$-theory, topological cyclic homology, and cdh sheafified topological cyclic homology.

On homotopy $K$-theory, we use the recent work of Bachmann–Elmanto–Morrow [BEM24], who construct a functorial multiplicative $\mathbb{N}$-indexed filtration $\mathrm{Fil}^{\star}_{\mathrm{cdh}}\mathrm{KH}(X)$ on the homotopy $K$-theory of qcqs schemes $X$. The shifted graded pieces of this filtration, that we will denote by $\mathbb{Z}(i)^{\mathrm{cdh}}(X)$, provide a good theory of *cdh-local motivic cohomology* for qcqs schemes. Their construction, which we review in Section 4.2, relies on the classical $\mathbb{A}^1$-invariant motivic cohomology of smooth $\mathbb{Z}$-schemes, and extends most of its properties to general qcqs schemes.

Our first construction is that of a functorial multiplicative $\mathbb{Z}$-indexed filtration $\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{TC}(X)$ for qcqs schemes $X$. This filtration recovers the HKR filtration on $\mathrm{HC}^-(X/\mathbb{Q})$ in characteristic zero [Ant19, MRT22, Rak20], and the motivic filtration on $\mathrm{TC}(X;\mathbb{Z}_p)$ after $p$-completion [BMS19, Mor21, BL22, HRW22]. We describe the shifted graded pieces

$$\mathbb{Z}(i)^{\mathrm{TC}}(X) \simeq \mathrm{gr}^i_{\mathrm{mot}}\mathrm{TC}(X)[-2i]$$

of this filtration in the following definition.

**Definition 2.1.2** (See Definition 3.3.3). For every qcqs scheme $X$ and every integer $i \in \mathbb{Z}$, the complex $\mathbb{Z}(i)^{\mathrm{TC}}(X) \in \mathcal{D}(\mathbb{Z})$ is defined by a natural cartesian square

$$\begin{array}{ccc} \mathbb{Z}(i)^{\mathrm{TC}}(X) & \longrightarrow & R\Gamma_{\mathrm{Zar}}\big(X, \widehat{\mathbb{L}\Omega}^{\geq i}_{-/\mathbb{Z}}\big) \\ \downarrow & & \downarrow \\ \prod_{p\in\mathbb{P}} \mathbb{Z}_p(i)^{\mathrm{BMS}}(X) & \longrightarrow & \prod_{p\in\mathbb{P}} R\Gamma_{\mathrm{Zar}}\big(X, (\widehat{\mathbb{L}\Omega}^{\geq i}_{-/\mathbb{Z}})^{\wedge}_p\big). \end{array}$$

It is straightforward from this definition that the presheaf $\mathbb{Z}(i)^{\mathrm{TC}}$ is naturally identified with Bhatt–Morrow–Scholze's syntomic complex $\mathbb{Z}_p(i)^{\mathrm{BMS}}$ in characteristic $p$, and with the Hodge-completed derived de Rham complex $R\Gamma_{\mathrm{Zar}}(-, \widehat{\mathbb{L}\Omega}^{\geq i}_{-/\mathbb{Q}})$ in characteristic zero. Following [EM23], the motivic complex $\mathbb{Z}(i)^{\mathrm{mot}}$ is defined in characteristic $p$ and zero respectively by cartesian squares

$$\begin{array}{ccc} \mathbb{Z}(i)^{\mathrm{mot}}(X) & \longrightarrow & \mathbb{Z}_p(i)^{\mathrm{BMS}}(X) \\ \downarrow & & \downarrow \\ \mathbb{Z}(i)^{\mathrm{cdh}}(X) & \longrightarrow & \big(L_{\mathrm{cdh}}\, \mathbb{Z}_p(i)^{\mathrm{BMS}}\big)(X) \end{array} \qquad \begin{array}{ccc} \mathbb{Z}(i)^{\mathrm{mot}}(X) & \longrightarrow & R\Gamma_{\mathrm{Zar}}\big(X, \widehat{\mathbb{L}\Omega}^{\geq i}_{-/\mathbb{Q}}\big) \\ \downarrow & & \downarrow \\ \mathbb{Z}(i)^{\mathrm{cdh}}(X) & \longrightarrow & R\Gamma_{\mathrm{cdh}}\big(X, \widehat{\mathbb{L}\Omega}^{\geq i}_{-/\mathbb{Q}}\big). \end{array}$$

The following definition is then a natural mixed characteristic generalisation of Elmanto–Morrow's definition over a field.

**Definition 2.1.3** (Motivic cohomology; see Section 4.3). For every qcqs scheme $X$ and every integer $i \in \mathbb{Z}$, the *weight-$i$ motivic complex*

$$\mathbb{Z}(i)^{\mathrm{mot}}(X) \in \mathcal{D}(\mathbb{Z})$$

of $X$ is defined by a natural cartesian square

$$\begin{array}{ccc} \mathbb{Z}(i)^{\mathrm{mot}}(X) & \longrightarrow & \mathbb{Z}(i)^{\mathrm{TC}}(X) \\ \downarrow & & \downarrow \\ \mathbb{Z}(i)^{\mathrm{cdh}}(X) & \longrightarrow & \big(L_{\mathrm{cdh}}\, \mathbb{Z}(i)^{\mathrm{TC}}\big)(X), \end{array}$$

where the bottom horizontal map is induced by a filtered refinement of the cdh sheafified cyclotomic trace map.

However, proving the expected relation between the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$ and algebraic $K$-theory (Theorem C below) requires more efforts in mixed characteristic than over a field. The main foundational obstacle is to prove that the presheaves $\mathbb{Z}(i)^{\mathrm{mot}}$ vanish in weights $i < 0$, as we explain now. Note first that, by construction, the presheaves $\mathbb{Z}(i)^{\mathrm{cdh}}$ do vanish on all qcqs schemes in weights $i < 0$ (Section 4.2).

In characteristic $p$, the presheaves $\mathbb{Z}_p(i)^{\mathrm{BMS}}$ vanish in weights $i < 0$, thus so do the presheaves $\mathbb{Z}(i)^{\mathrm{mot}}$, and the zeroth step of the associated motivic filtration $\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{K}$ recovers algebraic $K$-theory. Ignoring for a moment the completeness issues for this motivic filtration, this means that the presheaves $\mathbb{Z}(i)^{\mathrm{mot}}$ provide a natural cohomological refinement of algebraic $K$-theory on arbitrary characteristic $p$ schemes.

In characteristic zero, the presheaves $R\Gamma_{\mathrm{Zar}}(-,\widehat{\mathbb{L}\Omega}^{\geq i}_{-/\mathbb{Q}})$ do not vanish in weights $i<0$. Instead, they are equal to the presheaf $R\Gamma_{\mathrm{Zar}}(-,\widehat{\mathbb{L}\Omega}_{-/\mathbb{Q}})$, which happens to be a cdh sheaf on qcqs $\mathbb{Q}$-schemes, by results of Cortiñas–Haesemeyer–Schlichting–Weibel [CHSW08], Antieau [Ant19], and Elmanto–Morrow [EM23]. That is, the right vertical map of the previous diagram is an equivalence in weights $i<0$, so the presheaves $\mathbb{Z}(i)^{\mathrm{mot}}$ vanish in weights $i<0$, and the zeroth step of the associated motivic filtration $\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{K}$ recovers algebraic $K$-theory. This means that the presheaves $\mathbb{Z}(i)^{\mathrm{mot}}$ provide a natural cohomological refinement of algebraic $K$-theory on arbitrary characteristic zero schemes.

In mixed characteristic, we prove similarly that in weights $i<0$, the presheaves $\mathbb{Z}(i)^{\mathrm{TC}}$ are cdh sheaves on qcqs schemes, *i.e.*, that the presheaves $\mathbb{Z}(i)^{\mathrm{mot}}$ vanish. The result modulo a prime number $p$ is a consequence, as in characteristic $p$, of the fact that the presheaves $\mathbb{Z}_p(i)^{\mathrm{BMS}}$ vanish in weights $i<0$. The difficulty is to then prove that the presheaves $\mathbb{Q}(i)^{\mathrm{TC}}$ are cdh sheaves in weight $i<0$.

The main cdh descent result used in characteristic zero does not hold in mixed characteristic. That is, the presheaf $R\Gamma_{\mathrm{Zar}}(-,\widehat{\mathbb{L}\Omega}_{-/\mathbb{Z}})$ (or its rationalisation) is not a cdh sheaf on qcqs schemes. We avoid this difficulty by proving a rigid-analytic analogue of this cdh descent result. To formulate this result, denote by

$$R\Gamma_{\mathrm{Zar}}\big(X,\underline{\widehat{\mathbb{L}\Omega}}^{\geq i}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p}\big)$$

the *rigid-analytic derived de Rham cohomology* of a qcqs $\mathbb{Z}_{(p)}$-scheme $X$, which we define as the pushout of the diagram

$$R\Gamma_{\mathrm{Zar}}\big(X,(\widehat{\mathbb{L}\Omega}^{\geq i}_{-/\mathbb{Z}})^{\wedge}_{p}\big)\longleftarrow R\Gamma_{\mathrm{Zar}}\big(X,\widehat{\mathbb{L}\Omega}^{\geq i}_{-/\mathbb{Z}}\big)\longrightarrow R\Gamma_{\mathrm{Zar}}(X,\widehat{\mathbb{L}\Omega}^{\geq i}_{-_{\mathbb{Q}}/\mathbb{Q}})$$

in the derived category $\mathcal{D}(\mathbb{Z})$. Here, we restrict our attention to qcqs $\mathbb{Z}_{(p)}$-schemes for simplicity, and refer to Chapter 5 for the relevant statements over $\mathbb{Z}$. As a consequence of Definition 2.1.2, there is a natural cartesian square

$$\begin{array}{ccc}
\mathbb{Q}(i)^{\mathrm{TC}}(X) & \longrightarrow & R\Gamma_{\mathrm{Zar}}(X,\widehat{\mathbb{L}\Omega}^{\geq i}_{-_{\mathbb{Q}}/\mathbb{Q}}) \\
\big\downarrow & & \big\downarrow \\
\mathbb{Q}_p(i)^{\mathrm{BMS}}(X) & \longrightarrow & R\Gamma_{\mathrm{Zar}}(X,\underline{\widehat{\mathbb{L}\Omega}}^{\geq i}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p})
\end{array}$$

in the derived category $\mathcal{D}(\mathbb{Q})$. In weights $i<0$, the presheaves $\mathbb{Q}_p(i)^{\mathrm{BMS}}$ vanish. As already mentioned, the presheaf $R\Gamma_{\mathrm{Zar}}(-,\widehat{\mathbb{L}\Omega}_{-_{\mathbb{Q}}/\mathbb{Q}})$ is moreover a cdh sheaf on qcqs schemes. So the fact that the presheaves $\mathbb{Q}(i)^{\mathrm{TC}}$ are cdh sheaves in weights $i<0$ reduces to the following result, which can be seen as a rigid-analytic analogue of the latter cdh descent over $\mathbb{Q}$.

**Theorem A** (Cdh descent for rigid-analytic derived de Rham cohomology; see Corollary 5.4.4)**.** *For every prime number $p$, the presheaf $R\Gamma_{\mathrm{Zar}}\big(-,\underline{\widehat{\mathbb{L}\Omega}}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p}\big)$ satisfies cdh descent on qcqs $\mathbb{Z}_{(p)}$-schemes.*

The modern proof of the analogous result over $\mathbb{Q}$ relies on the theory of truncating invariants of Land–Tamme [LT19] and on a theorem of Goodwillie [Goo85], who prove respectively that every truncating invariant is a cdh sheaf on qcqs schemes and that periodic cyclic homology over $\mathbb{Q}$ is a truncating invariant. By definition, a truncating invariant is a localizing invariant $E$ such that for every connective $\mathbb{E}_1$-ring $R$, the natural map $E(R) \to E(\pi_0(R))$ is an equivalence. To prove Theorem A, we then use the condensed mathematics of Clausen–Scholze [CS19], and prove that a suitable rigid-analytic variant of periodic cyclic homology is a truncating invariant. In particular, the proof of Theorem A relies on a result on associative rings (actually, on general solid connective $\mathbb{E}_1$-rings).

As a consequence of Theorem A, we obtain the following cohomological description of rational motivic cohomology.

**Theorem B** ($p$-adic and rational motivic cohomology; see Corollaries 4.3.12 and 5.6.6)**.** *Let $X$ be a qcqs scheme, and $p$ be a prime number. Then for any integers $i \in \mathbb{Z}$ and $k \geq 1$, the natural commutative diagrams*

$$\begin{array}{ccc} \mathbb{Z}/p^k(i)^{\mathrm{mot}}(X) & \longrightarrow & \mathbb{Z}/p^k(i)^{\mathrm{BMS}}(X) \\ \downarrow & & \downarrow \\ \mathbb{Z}/p^k(i)^{\mathrm{cdh}}(X) & \longrightarrow & \big(L_{\mathrm{cdh}}\, \mathbb{Z}/p^k(i)^{\mathrm{BMS}}\big)(X) \end{array} \qquad \begin{array}{ccc} \mathbb{Q}(i)^{\mathrm{mot}}(X) & \longrightarrow & R\Gamma_{\mathrm{Zar}}\big(X, \widehat{\mathbb{L}\Omega}^{\geq i}_{-_{\mathbb{Q}}/\mathbb{Q}}\big) \\ \downarrow & & \downarrow \\ \mathbb{Q}(i)^{\mathrm{cdh}}(X) & \longrightarrow & R\Gamma_{\mathrm{cdh}}\big(X, \widehat{\mathbb{L}\Omega}^{\geq i}_{-_{\mathbb{Q}}/\mathbb{Q}}\big) \end{array}$$

*are cartesian squares in the derived category $\mathcal{D}(\mathbb{Z})$.*

Together, these two cartesian squares recover the cartesian squares of Elmanto–Morrow that define the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$ over a field, and are thus natural mixed characteristic analogues of these. The $p$-adic part of Theorem B is a formal consequence of Definitions 2.1.2 and 2.1.3. The rational part of Theorem B implies that the difference between $\mathbb{Q}(i)^{\mathrm{mot}}(X)$ and $\mathbb{Q}(i)^{\mathrm{cdh}}(X)$ depends only on the rationalisation $X_{\mathbb{Q}}$ of the scheme $X$. If $X$ is regular, this difference should vanish, and this is then more interesting in the presence of singularities. More precisely, Theorem B can be used to capture interesting information about the singularities of an arbitrary commutative ring $R$: cdh sheaves are typically insensitive to singularities, so the singular information in the motivic complex $\mathbb{Z}(i)^{\mathrm{mot}}(R)$ is controlled by the complexes $\mathbb{Z}/p^k(i)^{\mathrm{BMS}}(R)$ and $\mathbb{L}\Omega^{<i}_{(R\otimes_{\mathbb{Z}}\mathbb{Q})/\mathbb{Q}}$, which are accessible to computation.

Theorem B also implies that the presheaves $\mathbb{Q}(i)^{\mathrm{mot}}$ vanish in weights $i < 0$, which was the essential missing part to establish the following fundamental properties of motivic cohomology.

**Theorem C** (Relation to algebraic $K$-theory)**.** *There exists a finitary Nisnevich sheaf of filtered spectra*

$$\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{K}(-) : \mathrm{Sch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

*with the following properties:*

*(1)* (Atiyah–Hirzebruch spectral sequence; see Section 5.5) *For every qcqs scheme $X$, the filtration $\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{K}(X)$ is a multiplicative $\mathbb{N}$-indexed filtration on the non-connective algebraic $K$-theory $\mathrm{K}(X)$, whose graded pieces are naturally given by*

$$\mathrm{gr}^i_{\mathrm{mot}}\mathrm{K}(X) \simeq \mathbb{Z}(i)^{\mathrm{mot}}(X)[2i], \quad i \geq 0.$$

*In particular, writing $\mathrm{H}^j_{\mathrm{mot}}(X, \mathbb{Z}(i)) := \mathrm{H}^j(\mathbb{Z}(i)^{\mathrm{mot}}(X))$ for the corresponding* motivic cohomology groups*, there exists an Atiyah–Hirzebruch spectral sequence*

$$E_2^{i,j} = \mathrm{H}^{i-j}_{\mathrm{mot}}(X, \mathbb{Z}(-j)) \Longrightarrow \mathrm{K}_{-i-j}(X).$$

*If $X$ has finite valuative dimension,*[1] *then the filtration* $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X)$ *is complete, and the Atiyah–Hirzebruch spectral sequence is convergent.*

*(2)* (Adams decomposition; see Corollary 5.5.11) *For every qcqs scheme $X$, the Atiyah–Hirzebruch spectral sequence degenerates rationally and, for every integer $n \in \mathbb{Z}$, there is a natural isomorphism of abelian groups*

$$\mathrm{K}_n(X) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \bigoplus_{i \geq 0} \big(\mathrm{H}^{2i-n}_{\mathrm{mot}}(X, \mathbb{Z}(i)) \otimes_{\mathbb{Z}} \mathbb{Q}\big)$$

*induced by the Adams operations on rationalised algebraic $K$-theory.*

One of the main historical motivations for developing motivic cohomology was to apply cohomological techniques to the study of algebraic $K$-theory [BMS87]. The following theorem summarizes our results on the relations between motivic cohomology and previously studied cohomological invariants. When $X$ is smooth over $\mathbb{Z}$, we denote by

$$\mathbb{Z}(i)^{\mathrm{cla}}(X) := z^i(X, \bullet)[-2i]$$

the *weight-$i$ classical motivic complex*, where $z^i(X, \bullet)$ is Bloch's cycle complex (and $\bullet$ is the cohomological index).

**Theorem D.** *Let $X$ be a qcqs scheme, and $i \geq 0$ be an integer.*

*(1)* (Weight zero; see Example 5.6.8) *There is a natural equivalence*

$$\mathbb{Z}(0)^{\mathrm{mot}}(X) \xrightarrow{\sim} R\Gamma_{\mathrm{cdh}}(X, \mathbb{Z})$$

*in the derived category $\mathcal{D}(\mathbb{Z})$.*

*(2)* (Weight one; see Example 8.1.15) *There is a natural map*

$$R\Gamma_{\mathrm{Nis}}(X, \mathbb{G}_m)[-1] \longrightarrow \mathbb{Z}(1)^{\mathrm{mot}}(X)$$

*in the derived category $\mathcal{D}(\mathbb{Z})$ which is an isomorphism in degrees at most three.*

*(3)* (Étale cohomology; see Corollary 6.1.6) *For every prime number $\ell$ which is invertible in $X$ and every integer $k \geq 1$, there is a natural map*

$$\mathbb{Z}/\ell^k(i)^{\mathrm{mot}}(X) \longrightarrow R\Gamma_{\mathrm{\acute{e}t}}(X, \mu_{\ell^k}^{\otimes i})$$

*in the derived category $\mathcal{D}(\mathbb{Z}/\ell^k)$ which is an isomorphism in degrees at most $i$.*

*(4)* (Syntomic cohomology; see Corollary 6.2.5) *For every prime number $p$ and every integer $k \geq 1$, there is a natural map*

$$\mathbb{Z}/p^k(i)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{syn}}(X)$$

*in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$ which is an isomorphism in degrees at most $i$, where $\mathbb{Z}/p^k(i)^{\mathrm{syn}}(X)$ denotes the weight-$i$ syntomic cohomology of $X$ in the sense of [BL22].*

[1] The valuative dimension of a commutative ring, defined in terms of the ranks of certain valuation rings, was introduced by Jaffard in [Jaf60, Chapter IV], and generalised to schemes in [EHIK21, Section 2.3]. The valuative dimension of a scheme is always at least equal to its Krull dimension, and both notions agree on noetherian schemes. For our purposes, the valuative dimension of a qcqs scheme $X$ will be used as an upper bound on the cohomological dimension of the cdh topos of $X$ ([EHIK21, Theorem 2.4.15]).

*(5)* (Milnor $K$-theory; see Theorem 8.2.6) *If $X = \mathrm{Spec}(A)$ is the spectrum of a henselian local ring $A$, then for every integer $n \geq 1$, there is a natural isomorphism*

$$\widehat{\mathrm{K}}{}^{\mathrm{M}}_i(A)/n \xrightarrow{\cong} \mathrm{H}^i_{\mathrm{mot}}(A, \mathbb{Z}(i))/n$$

*of abelian groups, where $\widehat{\mathrm{K}}{}^{\mathrm{M}}_i(A)$ denotes the $i^{\mathrm{th}}$ improved Milnor $K$-group of $A$ in the sense of [Ker10].*

*(6)* (Classical motivic cohomology; see Chapter7) *If $X$ is smooth over $\mathbb{Z}$, then there is a natural map*

$$\mathbb{Z}(i)^{\mathrm{cla}}(X) \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(X)$$

*in the derived category $\mathcal{D}(\mathbb{Z})$ which is an isomorphism in degrees at most $i+1$ in general, and an isomorphism in all degrees if $X$ has dimension at most one over $\mathbb{Z}$.*

*(7)* (Lisse motivic cohomology; see Corollary 8.1.12) *If $X = \mathrm{Spec}(A)$ is the spectrum of a local ring $A$, then for every integer $i \geq 0$, there is a natural equivalence*

$$\mathbb{Z}(i)^{\mathrm{lisse}}(A) \xrightarrow{\sim} \tau^{\leq i}\, \mathbb{Z}(i)^{\mathrm{mot}}(A)$$

*in the derived category $\mathcal{D}(\mathbb{Z})$, where $\mathbb{Z}(i)^{\mathrm{lisse}}$ denotes the weight-$i$ lisse motivic cohomology of $A$, defined as the left Kan extension from smooth $\mathbb{Z}$-algebras of the classical motivic complex $\mathbb{Z}(i)^{\mathrm{cla}}$. In particular, the functor $\tau^{\leq i}\, \mathbb{Z}(i)^{\mathrm{mot}}$ is left Kan extended on local rings from local essentially smooth $\mathbb{Z}$-algebras.*

*(8)* ($\mathbb{A}^1$-invariant motivic cohomology; see Theorem 12.1.5) *There is a natural equivalence*

$$\big(L_{\mathbb{A}^1}\, \mathbb{Z}(i)^{\mathrm{mot}}\big)(X) \xrightarrow{\sim} \mathbb{Z}(i)^{\mathbb{A}^1}(X)$$

*in the derived category $\mathcal{D}(\mathbb{Z})$, where $\mathbb{Z}(i)^{\mathbb{A}^1}(X)$ denotes the weight-$i$ $\mathbb{A}^1$-invariant motivic cohomology of $X$ in the sense of [BEM24].*

The following result is a consequence of Theorem D (8) and the fact that the $\mathbb{A}^1$-invariant motivic complexes $\mathbb{Z}(i)^{\mathbb{A}^1}$ recover the classical motivic complexes $\mathbb{Z}(i)^{\mathrm{cla}}$ on smooth $\mathbb{Z}$-schemes [BEM24]. In particular, although we expect our motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$ to actually coincide with the classical motivic complexes $\mathbb{Z}(i)^{\mathrm{cla}}$ on smooth $\mathbb{Z}$-schemes, this means that the former at least recover the latter after enforcing $\mathbb{A}^1$-invariance.

**Corollary E** (See Corollary 12.1.9)**.** *Let $X$ be a smooth scheme over $\mathbb{Z}$. Then for every integer $i \geq 0$, there is a natural equivalence*

$$\mathbb{Z}(i)^{\mathrm{cla}}(X) \simeq \big(L_{\mathbb{A}^1}\, \mathbb{Z}(i)^{\mathrm{mot}}\big)(X)$$

*in the derived category $\mathcal{D}(\mathbb{Z})$.*

Assuming the existence of a well-behaved derived category of motives DM, the motivic cohomology groups of a scheme $X$ should be given by

$$\mathrm{H}^j_{\mathrm{mot}}(X, \mathbb{Z}(i)) \cong \mathrm{Hom}_{\mathrm{DM}}(\mathrm{M}(X), \mathbb{Z}(i)[j]),$$

where $\mathrm{M}(X) \in \mathrm{DM}$ is the motive associated to $X$, and $\mathbb{Z}(i) \in \mathrm{DM}$ are the Tate motives, fitting for every integer $r \geq 0$ in a natural decomposition in DM:

$$\mathrm{M}(\mathbb{P}^r_{\mathbb{Z}}) \cong \bigoplus_{j=0}^{r} \mathbb{Z}(j)[2j].$$

In the $\mathbb{A}^1$-invariant framework, Voevodsky constructed such a derived category of motives, in which the classical motivic complexes $\mathbb{Z}(i)^{\text{cla}}$ can be interpreted in terms of these Tate motives. Without assuming $\mathbb{A}^1$-invariance, Annala–Iwasa [AI23] and Annala–Hoyois–Iwasa [AHI23, AHI24] recently introduced a more general derived category of motives, where the decomposition of the motive $\mathrm{M}(\mathbb{P}^r_{\mathbb{Z}})$, *i.e.*, the *projective bundle formula*, is isolated as the key defining property. The following result states that the motivic complexes $\mathbb{Z}(i)^{\text{mot}}$ fit within this theory of non-$\mathbb{A}^1$-invariant motives.

**Theorem F** (Projective bundle formula; see Theorem 9.3.2)**.** *Let $X$ be a qcqs scheme, $i \geq 0$ be an integer, and $\mathcal{E}$ be a vector bundle of rank $r$ on $X$. Then for every integer $i \geq 0$, the powers of the motivic first Chern class $c_1^{\text{mot}}(\mathcal{O}(1)) \in \mathrm{H}^2_{\text{mot}}(\mathbb{P}_X(\mathcal{E}), \mathbb{Z}(1))$ induce a natural equivalence*

$$\bigoplus_{i=0}^{r-1} \mathbb{Z}(i-j)^{\text{mot}}(X)[-2j] \xrightarrow{\sim} \mathbb{Z}(i)^{\text{mot}}(\mathbb{P}_X(\mathcal{E}))$$

*in the derived category $\mathcal{D}(\mathbb{Z})$.*

Theorem F is proved by Elmanto–Morrow in the equicharacteristic case [EM23], where the proof relies on the projective bundle formula for the complexes $\mathbb{Z}(i)^{\text{cdh}}$ [BEM24]. In mixed characteristic, however, the cdh-local motivic complexes $\mathbb{Z}(i)^{\text{cdh}}$ are known to satisfy the projective bundle formula only conditionally on a certain property of valuation rings, called $F$-smoothness [BM23, BEM24]. As we explain now, this condition can be proved in mixed characteristic, for valuation rings over a perfectoid base. The case of general valuation rings remaining open, our proof of Theorem F is different from that of Elmanto–Morrow, and uses in particular our description of motivic cohomology with finite coefficients in terms of syntomic cohomology (Theorem 6.2.4).

## 2.2 Motivic cohomology of valuation rings

In recent years, valuation rings have been used as a way to bypass resolution of singularities, in order to adapt arguments from characteristic zero to positive characteristic [KST21, KM21]. We explain here how to adapt some of these ideas in mixed characteristic, starting with the Cartier isomorphism. Most of the following results on valuation rings have appeared as [Bou23].

Let $p$ be a prime number, which we fix for the rest of the introduction. The Cartier isomorphism states that for any smooth $\mathbb{F}_p$-algebra $A$, the inverse Cartier map

$$C^{-1} : \begin{cases} \Omega^n_{A/\mathbb{F}_p} \longrightarrow \mathrm{H}^n(\Omega^\bullet_{A/\mathbb{F}_p}) \\ f dg_1 \wedge \cdots \wedge dg_n \longmapsto f^p g_1^{p-1} \dots g_n^{p-1} dg_1 \wedge \cdots \wedge dg_n \end{cases}$$

is an isomorphism of $\mathbb{F}_p$-vector spaces for each $n \geq 0$. Assuming resolution of singularities in characteristic $p$, any $\mathbb{F}_p$-algebra would be locally smooth in the cdh topology. The local rings in the cdh topology are valuation rings, so valuation rings of characteristic $p$ should be filtered colimits of smooth $\mathbb{F}_p$-algebras, and in particular should satisfy the Cartier isomorphism. Without assuming resolution of singularities, Gabber proved [KST21, Appendix] that valuation rings of characteristic $p$ actually satisfy the Cartier isomorphism. Motivated by this result and the general lack of understanding of valuation rings, Kelly and Morrow [KM21] introduced *Cartier smooth algebras* as $\mathbb{F}_p$-algebras satisfying the Cartier isomorphism, and study their algebraic $K$-theory.

We develop here an analogue of this story in mixed characteristic. The inverse Cartier map, as an extension of the Frobenius morphism, is specific to characteristic $p$, and we define $p$-Cartier smoothness for general ring homomorphisms essentially in terms of their reduction modulo $p$. This more general notion coincides with that introduced by Kelly–Morrow [KM21] in the special case that the base ring is $\mathbb{F}_p$. For an $\mathbb{F}_p$-algebra $R$, we denote by $\phi_R$ its Frobenius endomorphism.

**Definition 2.2.1** (Cartier smoothness)**.** (1) A morphism $R \to S$ of $\mathbb{F}_p$-algebras is *Cartier smooth*, or $S$ is a *Cartier smooth* $R$-algebra, if it is cotangent smooth, *i.e.*, its cotangent complex $\mathbb{L}_{S/R}$ is a flat $S$-module in degree zero given by $\Omega^1_{S/R}$, and if the inverse Cartier map

$$C^{-1} : \Omega^n_{S/R} \otimes_{R,\phi_R} R \longrightarrow \mathrm{H}^n(\Omega^\bullet_{S/R})$$

is an isomorphism of $R$-modules for each $n \geq 0$.

(2) A morphism $R \to S$ of commutative rings is *$p$-Cartier smooth*, or $S$ is a *$p$-Cartier smooth* $R$-algebra, if the natural map $S \otimes^{\mathbb{L}}_R R/p \longrightarrow S/p[0]$ is an equivalence in the derived category $\mathcal{D}(R/p)$ and the reduction $R/p \to S/p$ modulo $p$ is Cartier smooth.

Said another way, a morphism $R \to S$ of commutative rings is $p$-Cartier smooth if it is $p$-cotangent smooth (Definition 11.1.3) and if its reduction $R/p \to S/p$ modulo $p$ satisfies the Cartier isomorphism. The $p$-cotangent smoothness hypothesis is necessary not to lose control, when specialising to characteristic $p$, on the invariants that we will study. Any smooth morphism of commutative rings is $p$-Cartier smooth (for any prime $p$). Regarding valuation rings, an extension of valuation rings (*i.e.*, an injective morphism of valuation rings) in mixed characteristic is in general not $p$-Cartier smooth. For instance, the morphism $\mathbb{Z}_p \to \overline{\mathbb{Z}}_p$, where $\overline{\mathbb{Z}}_p$ is the ring of integers of an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, is not $p$-cotangent smooth, and does not satisfy the Cartier isomorphism: its cotangent complex is in degree zero, given by the torsion $\overline{\mathbb{Z}}_p$-module $\overline{\mathbb{Q}}_p/\overline{\mathbb{Z}}_p$, and $\Omega^0_{(\overline{\mathbb{Z}}_p/p)/\mathbb{F}_p} = \overline{\mathbb{Z}}_p/p$ contains nilpotent elements; in fact, the de Rham complex of $\mathbb{F}_p \to \overline{\mathbb{Z}}_p/p$ is zero in positive degrees, and thus does not convey a lot of information. However, if the base valuation ring is sufficiently ramified, these obstructions vanish and we can prove the following general result. This generalises a theorem of Gabber in characteristic $p$, which is valid over perfect valuation rings (Theorem 11.2.4 below). We refer the reader to [BMS18, Definition 3.5] for the definition of perfectoid rings.

**Theorem G** (See Theorem 11.2.1)**.** *Let $V_0$ be a valuation ring whose $p$-completion is a perfectoid ring. Let $V$ be a valuation ring extension of $V_0$. Then the morphism $V_0 \to V$ is $p$-Cartier smooth.*

The proof of Theorem G proceeds by reduction to the case of valuation rings of characteristic $p$, and uses deformation theory. We explain now how to apply Theorem G to computations of the motivic cohomology of valuation rings.

Recall that on smooth $\mathbb{F}_p$-schemes, the Beilinson–Lichtenbaum conjecture, proved by Suslin and Voevodsky as a consequence of the Bloch–Kato conjecture [SV00], computes the $\ell$-adic part of motivic cohomology in terms of the cohomology of the étale sheaf $\mu_{\ell^k}$ of $\ell^k$-roots of unity. To describe the $p$-adic part of motivic cohomology, one needs to replace $\mu_{\ell^k}^{\otimes i}$ (which is zero on smooth varieties when $\ell = p$ and $i > 0$) by the logarithmic de Rham–Witt sheaves $W_k\Omega^i_{-,\log}$ [GL00]. The corresponding description of $p$-adic algebraic $K$-theory, in terms of the logarithmic de Rham–Witt sheaves, is generalised in [KM21] to all Cartier smooth $\mathbb{F}_p$-algebras.

On smooth schemes over a mixed characteristic Dedekind domain, the $p$-adic part of classical motivic cohomology is similarly described in low degrees by the étale cohomology of the generic fibre [Gei04]. This result is a consequence of the Gersten conjecture proved by Geisser [Gei04], and is unknown for general regular schemes. Combined with Theorem D (6), the following result extends this description of classical motivic cohomology to the regular case. More precisely, the notion of $F$-smoothness was introduced by Bhatt–Mathew [BM23] as a non-noetherian generalisation of regular schemes, and our result naturally applies to general $p$-torsionfree $F$-smooth schemes.

**Theorem H** (Beilinson–Lichtenbaum conjecture for $F$-smooth schemes; see Corollary 6.2.6)**.** *Let $X$ be a $p$-torsionfree $F$-smooth scheme (e.g., a regular scheme flat over $\mathbb{Z}$). Then for any integers $i \geq 0$ and $k \geq 1$, the Beilinson–Lichtenbaum comparison map*

$$\mathbb{Z}/p^k(i)^{\mathrm{mot}}(X) \longrightarrow R\Gamma_{\mathrm{\acute{e}t}}(X[\tfrac{1}{p}], \mu_{p^k}^{\otimes i})$$

*is an isomorphism in degrees at most $i-1$, and is injective in degree $i$.*

Over a perfectoid base ring, $F$-smoothness and $p$-Cartier smoothness coincide by Theorem 11.1.19. Using Theorems G and H, we then obtain the following complete description of the $p$-adic motivic cohomology of valuation rings over a perfectoid base.

**Corollary I** (Motivic cohomology of valuation rings; see Theorem 11.4.6)**.** *Let $V_0$ be a $p$-torsionfree valuation ring whose $p$-completion is a perfectoid ring, and $V$ be a valuation ring extension of $V_0$. Then for any integers $i \geq 0$ and $k \geq 1$, the motivic complex $\mathbb{Z}/p^k(i)^{\mathrm{mot}}(V)$ is in degrees at most $i$, and the Beilinson–Lichtenbaum comparison map*

$$\mathbb{Z}/p^k(i)^{\mathrm{mot}}(V) \longrightarrow R\Gamma_{\mathrm{\acute{e}t}}(\mathrm{Spec}(V[\tfrac{1}{p}]), \mu_{p^k}^{\otimes i})$$

*is an isomorphism in degrees at most $i-1$. On $\mathrm{H}^i$, this map is injective, with image generated by symbols, via the symbol map*

$$(V^\times)^{\otimes i} \to \mathrm{H}^i_{\mathrm{\acute{e}t}}(\mathrm{Spec}(V[\tfrac{1}{p}]), \mu_{p^k}^{\otimes i}).$$

The proof of Theorem H relies on a syntomic-étale comparison theorem of Bhatt–Mathew [BM23]. We reprove this syntomic-étale comparison theorem in the case of $p$-Cartier smooth algebras over a perfectoid $\mathbb{Z}_p^{\mathrm{cyc}}$-algebra, using a property of $p$-Cartier smooth algebras on the Nygaard filtration of their relative prismatic cohomology (Theorem J ($\mathcal{N}^{\geq}$) below). To prove this property, we extend various results on the relative prismatic cohomology of smooth algebras to general $p$-Cartier smooth algebras. It is *a priori* surprising that these properties are true assuming only a property on the special fibre (namely, the Cartier isomorphism); we prove that some of them even characterise $p$-Cartier smoothness.

**Theorem J** (See Theorem 11.1.16)**.** *Let $(A, I)$ be a bounded prism, and $S$ a $p$-cotangent smooth $A/I$-algebra. The following are equivalent:*

*(CSm) $S$ is $p$-Cartier smooth over $A/I$.*

($\mathbb{L}\Omega = \widehat{\mathbb{L}\Omega}$) *The canonical map $(\mathbb{L}\Omega_{S/(A/I)})^\wedge_p \to (\widehat{\mathbb{L}\Omega}_{S/(A/I)})^\wedge_p$ is an equivalence in the derived category $\mathcal{D}(A/I)$, where $(\widehat{\mathbb{L}\Omega}_{S/(A/I)})^\wedge_p$ is the Hodge-completion of the $p$-completed derived de Rham complex.*

($\mathbb{L}\Omega = \Omega$) *The counit map* $(\mathbb{L}\Omega_{S/(A/I)})^\wedge_p \to (\Omega_{S/(A/I)})^\wedge_p$ *is an equivalence in the derived category* $\mathcal{D}(A/I)$*, where* $(\mathbb{L}\Omega_{S/(A/I)})^\wedge_p$ *is the p-completed derived de Rham complex.*

($dR$) *The de Rham comparison map* $\Delta^{(1)}_{S/A} \otimes^{\mathbb{L}}_A A/I \to (\Omega_{S/(A/I)})^\wedge_p$ *is an equivalence in the derived category* $\mathcal{D}(A/I)$*.*

($\Delta = \widehat{\Delta}$) *The canonical map* $\mathrm{can} : \Delta^{(1)}_{S/A} \to \widehat{\Delta}^{(1)}_{S/A}$ *is an equivalence in the derived category* $\mathcal{D}(A)$*, where* $\widehat{\Delta}^{(1)}_{S/A}$ *is the Nygaard-completion of the prismatic complex.*

($L\eta$) *The Frobenius map* $\tilde{\phi} : \Delta^{(1)}_{S/A} \to L\eta_I \Delta_{S/A}$ *is an equivalence in the derived category* $\mathcal{D}(A)$*.*

($\mathcal{N}^{\geq}$) *The Frobenius map* $\tau^{\leq i}\phi : \tau^{\leq i}\mathcal{N}^{\geq i}\Delta^{(1)}_{S/A} \to \tau^{\leq i} I^i \Delta_{S/A}$ *is an equivalence in the derived category* $\mathcal{D}(A)$ *for all* $i \geq 0$*.*

## 2.3 Motivic cohomology of singular schemes

For the rest of this introduction, we focus on the properties of motivic cohomology that are specific to singular schemes. One of the most interesting, yet mysterious features of the algebraic $K$-theory of singular schemes is the presence of nonzero negative $K$-groups. Most of the current understanding of negative $K$-groups relies on results on the behaviour of algebraic $K$-theory with respect to blowups [CHSW08, KST18]. It was proved in particular by Thomason [Tho93] that algebraic $K$-theory sends the blowup square associated to a regular closed immersion to a long exact sequence of $K$-groups. The following result is a cohomological refinement of Thomason's result.

**Theorem K** (Regular blowup formula; see Theorem 9.3.1)**.** *For every regular closed immersion* $Y \to X$ *of qcqs schemes (i.e., the closed subscheme* $Y$ *is Zariski-locally on* $X$ *defined by a regular sequence) and every integer* $i \geq 0$*, the commutative diagram*

$$\begin{array}{ccc} \mathbb{Z}(i)^{\mathrm{mot}}(X) & \longrightarrow & \mathbb{Z}(i)^{\mathrm{mot}}(Y) \\ \downarrow & & \downarrow \\ \mathbb{Z}(i)^{\mathrm{mot}}(\mathrm{Bl}_Y(X)) & \longrightarrow & \mathbb{Z}(i)^{\mathrm{mot}}(\mathrm{Bl}_Y(X) \times_X Y) \end{array}$$

*is a cartesian square in the derived category* $\mathcal{D}(\mathbb{Z})$*.*

However, algebraic $K$-theory fails to associate long exact sequences to general blowups. Motivated by Grothendieck's theorem on formal functions for quasi-coherent cohomology [Gro61, Theorem 4.1.5], many people hoped for a formal analogue of these long exact sequences in algebraic $K$-theory, that would hold for general blowups. This was finally proved by Kerz–Strunk–Tamme [KST18], in the form of a pro cdh excision property for the algebraic $K$-theory of arbitrary noetherian schemes. Note that the pro cdh topology was recently introduced by Kelly–Saito [KS24], as a way to encode this pro cdh excision property in a descent property for this Grothendieck topology. The following result relies in particular on Theorem B and on Grothendieck's theorem on formal functions, thus shedding some light on the analogy between quasi-coherent and $K$-theoretic techniques.

**Theorem L** (Pro cdh descent; see Theorem 10.2.11)**.** *For every integer $i \geq 0$, the presheaf $\mathbb{Z}(i)^{\mathrm{mot}}$ satisfies pro cdh descent on noetherian schemes. That is, for every abstract blowup square*

$$\begin{array}{ccc} Y' & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

*of noetherian schemes, the associated commutative diagram*

$$\begin{array}{ccc} \mathbb{Z}(i)^{\mathrm{mot}}(X) & \longrightarrow & \mathbb{Z}(i)^{\mathrm{mot}}(X') \\ \downarrow & & \downarrow \\ \{\mathbb{Z}(i)^{\mathrm{mot}}(rY)\}_r & \longrightarrow & \{\mathbb{Z}(i)^{\mathrm{mot}}(rY')\}_r \end{array}$$

*is a weakly cartesian square of pro objects in the derived category $\mathcal{D}(\mathbb{Z})$.*

Kelly–Saito moreover proved that non-connective algebraic $K$-theory not only satisfies pro cdh descent, but is the pro cdh sheafification of connective algebraic $K$-theory [KS24]. Combined with the observation of Bhatt–Lurie that connective algebraic $K$-theory is left Kan extended on commutative rings from smooth $\mathbb{Z}$-algebras [EHK+20], this motivated the definition of the *pro cdh motivic complexes* $\mathbb{Z}(i)^{\mathrm{procdh}}$, as the pro cdh sheafification of the left Kan extension of the classical motivic complexes $\mathbb{Z}(i)^{\mathrm{cla}}$. The following result relies on the comparison to lisse motivic cohomology (Theorem D (7)) and pro cdh descent (Theorem L).

**Corollary M** (Comparison to pro cdh motivic cohomology; see Theorem 10.4.2)**.** *Let $X$ be a noetherian scheme. Then for every integer $i \geq 0$, there is a natural equivalence*

$$\mathbb{Z}(i)^{\mathrm{procdh}}(X) \xrightarrow{\sim} \mathbb{Z}(i)^{\mathrm{mot}}(X)$$

*in the derived category $\mathcal{D}(\mathbb{Z})$.*

Note that the pro cdh motivic complexes $\mathbb{Z}(i)^{\mathrm{procdh}}$ are not finitary, so they cannot coincide with the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$ on general qcqs schemes.

An important conjecture of Weibel [Wei80] states that for every noetherian scheme $X$ of dimension at most $d$, the negative $K$-groups $\mathrm{K}_{-n}(X)$ vanish for integers $n > d$. This conjecture was settled by Kerz–Strunk–Tamme [KST18], as a consequence of pro cdh descent for algebraic $K$-theory. The proof of the following result uses the techniques of Kerz–Strunk–Tamme [KST18] as reformulated by Elmanto–Morrow [EM23], who proved the same result over a field. In particular, Theorem N relies on pro cdh descent for motivic cohomology (Theorem L).

**Theorem N** (Motivic Weibel vanishing; see Theorem 10.3.3)**.** *Let $X$ be a noetherian scheme of finite dimension $d$, and $i \geq 0$ be an integer. Then for every integer $j > i + d$, the motivic cohomology group $\mathrm{H}^j_{\mathrm{mot}}(X, \mathbb{Z}(i))$ is zero.*

Via the Atiyah–Hirzebruch spectral sequence of Theorem C, Theorem N is a motivic refinement of Weibel's vanishing conjecture in $K$-theory.

# Chapter 3

# The motivic filtration on topological cyclic homology

In this chapter, we introduce a motivic filtration on the topological cyclic homology of qcqs derived schemes (Definition 3.3.3), whose shifted graded pieces $\mathbb{Z}(i)^{\mathrm{TC}}$ will serve as a building block for the definition of the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$ (Remark 4.3.7).

We first explain how to express topological cyclic homology in terms of its profinite completion and of negative cyclic homology. Following [NS18], and given a qcqs derived scheme $X$ and a prime number $p$, the $p$-completed topological cyclic homology $\mathrm{TC}(X;\mathbb{Z}_p)$ of $X$ is constructed from its $p$-completed topological negative cyclic homology $\mathrm{TC}^-(X;\mathbb{Z}_p)$ and its $p$-completed topological periodic cyclic homology $\mathrm{TP}(X;\mathbb{Z}_p)$ (see Section 3.2). Following [DGM13, Lemma 6.4.3.2] and [NS18, Section II.4], the topological cyclic homology $\mathrm{TC}(X)$ of $X$ is then defined by a natural cartesian square of spectra

$$\begin{array}{ccc} \mathrm{TC}(X) & \longrightarrow & \mathrm{TC}^-(X) \\ \downarrow & & \downarrow \\ \prod_{p\in\mathbb{P}} \mathrm{TC}(X;\mathbb{Z}_p) & \longrightarrow & \prod_{p\in\mathbb{P}} \mathrm{TC}^-(X;\mathbb{Z}_p). \end{array}$$

The comparison map $\mathrm{THH}(X) \to \mathrm{HH}(X)$, induced by extension of scalars along the map of $\mathbb{E}_\infty$-rings $\mathrm{THH}(\mathbb{Z}) \to \mathbb{Z}$, is $\mathrm{S}^1$-equivariant, and for every integer $n \in \mathbb{Z}$, the kernel and cokernel of the induced map on homotopy groups $\mathrm{THH}_n(X) \to \mathrm{HH}_n(X)$ are killed by an integer depending only on $n$. In particular, the natural commutative diagram

$$\begin{array}{ccc} \mathrm{THH}(X) & \longrightarrow & \mathrm{HH}(X) \\ \downarrow & & \downarrow \\ \prod_{p\in\mathbb{P}} \mathrm{THH}(X;\mathbb{Z}_p) & \longrightarrow & \prod_{p\in\mathbb{P}} \mathrm{HH}(X;\mathbb{Z}_p), \end{array}$$

is a cartesian square of spectra, which in turn defines a natural cartesian square of spectra

$$\begin{array}{ccc} \mathrm{TC}^-(X) & \longrightarrow & \mathrm{HC}^-(X) \\ \downarrow & & \downarrow \\ \prod_{p\in\mathbb{P}} \mathrm{TC}^-(X;\mathbb{Z}_p) & \longrightarrow & \prod_{p\in\mathbb{P}} \mathrm{HC}^-(X;\mathbb{Z}_p) \end{array}$$

by taking homotopy fixed points $(-)^{h\mathrm{S}^1}$. Composing this cartesian square with the cartesian square defining topological cyclic homology then induces a natural cartesian square

$$\begin{array}{ccc} \mathrm{TC}(X) & \longrightarrow & \mathrm{HC}^-(X) \\ \downarrow & & \downarrow \\ \prod_{p\in\mathbb{P}} \mathrm{TC}(X;\mathbb{Z}_p) & \longrightarrow & \prod_{p\in\mathbb{P}} \mathrm{HC}^-(X;\mathbb{Z}_p). \end{array}$$

We will use this cartesian square to define the motivic filtration on $\mathrm{TC}(X)$ (Definition 3.3.3), by glueing existing filtrations on the three other terms; namely, the HKR and BMS filtrations.

## 3.1 The HKR filtrations

In this section, we review the HKR filtrations on Hochschild homology and its variants, as defined, in the generality of qcqs derived schemes, by [Ant19] and [BL22, Section 6.3]. Only the HKR filtration on negative cyclic homology $\mathrm{HC}^-(-)$ (Definition 3.1.4) will be used to define the motivic filtration on topological cyclic homology $\mathrm{TC}(-)$ (Definition 3.3.3). We will use the other HKR filtrations of this section in Chapter 5.

The following result is [BL22, Example 6.1.3 and Remarks 6.1.4 and 6.1.5].

**Proposition 3.1.1** (Tate filtration)**.** *Let $X$ be a spectrum equipped with an $\mathrm{S}^1$-action. Then the Tate construction $X^{t\mathrm{S}^1} \in \mathrm{Sp}$ is naturally equipped with a $\mathbb{Z}$-indexed filtration*

$$\mathrm{Fil}^\star_{\mathrm{T}} X^{t\mathrm{S}^1} \in \mathrm{FilSp}.$$

*This filtration is called the* Tate filtration *on $X^{t\mathrm{S}^1}$, and satisfies the following properties:*

*(1) The filtration $\mathrm{Fil}^\star_{\mathrm{T}} X^{t\mathrm{S}^1} \in \mathrm{FilSp}$ is complete.*

*(2) $\mathrm{Fil}^0_{\mathrm{T}} X^{t\mathrm{S}^1}$ is the homotopy fixed point spectrum $X^{h\mathrm{S}^1}$, which is thus also equipped with an $\mathbb{N}$-indexed complete filtration $\mathrm{Fil}^\star_{\mathrm{T}} X^{h\mathrm{S}^1}$, which we call the Tate filtration on $X^{h\mathrm{S}^1}$.*

*(3) For every integer $n \in \mathbb{Z}$, the graded piece $\mathrm{gr}^n_{\mathrm{T}} X^{t\mathrm{S}^1}$ is naturally identified with the spectrum $X[-2n]$.*

Following [BMS19, Section 5], a filtered spectrum $\mathrm{Fil}^\star X$ is called *connective for the Beilinson $t$-structure* if for every integer $i \in \mathbb{Z}$, the graded piece $\mathrm{gr}^i X \in \mathrm{Sp}$ is in cohomological degrees at most $i$. For every integer $i \in \mathbb{Z}$, also denote by $\tau^{\mathrm{B}}_{\geq i}$ the truncation functor for the Beilinson $t$-structure on filtered spectra.

**Definition 3.1.2** (Décalage filtration)**.** Let $\mathrm{Fil}^\star X \in \mathrm{FilSp}$ be a filtered spectrum. The *décalage filtration* on $\mathrm{Fil}^\star X$ is the bifiltered spectrum

$$\mathrm{Fil}^\star_{\mathrm{B}} \mathrm{Fil}^\star X \in \mathrm{biFilSp}$$

where, for every integer $i \in \mathbb{Z}$, $\mathrm{Fil}^i_{\mathrm{B}} \mathrm{Fil}^\star X$ is the $i$-connective cover of $\mathrm{Fil}^\star X \in \mathrm{FilSp}$ with respect to the Beilinson $t$-structure on the category of filtered spectra:

$$\mathrm{Fil}^i_{\mathrm{B}} \mathrm{Fil}^\star X := \tau^{\mathrm{B}}_{\geq i} \mathrm{Fil}^\star X.$$

**Construction 3.1.3** (HKR filtration on HP)**.** For every integer $i \in \mathbb{Z}$, let

$$\mathrm{Fil}^i_{\mathrm{HKR}}\mathrm{Fil}^\star_{\mathrm{T}}\mathrm{HP}(-) := L_{\mathrm{Zar}}L_{\mathrm{dSch}^{\mathrm{qcqs,op}}/\mathrm{Poly}^{\mathrm{op}}_{\mathbb{Z}}}\mathrm{Fil}^i_{\mathrm{B}}\mathrm{Fil}^\star_{\mathrm{T}}\mathrm{HP}(-),$$

where $\mathrm{Fil}^\star_{\mathrm{T}}\mathrm{HP}(-)$ is the Tate filtration on periodic cyclic homology of qcqs derived schemes, $\mathrm{Fil}^\star_{\mathrm{B}}$ is the décalage filtration of Definition 3.1.2, and the left Kan extension $L_{\mathrm{dSch}^{\mathrm{qcqs,op}}/\mathrm{Poly}^{\mathrm{op}}_{\mathbb{Z}}}$ is taken in the category of filtration-complete filtered spectra. The *HKR filtration on periodic cyclic homology* of qcqs derived schemes is the functor

$$\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HP}(-) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

defined as the underlying filtered object of the bifiltered functor $\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{Fil}^\star_{\mathrm{T}}\mathrm{HP}(-)$:

$$\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HP}(-) := \varinjlim_n \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{Fil}^n_{\mathrm{T}}\mathrm{HP}(-).$$

The following definition is the one which will appear explicitly in the definition of the motivic filtration on topological cyclic homology (Definition 3.3.3).

**Definition 3.1.4** (HKR filtration on $\mathrm{HC}^-$)**.** The *HKR filtration on negative cyclic homology* of qcqs derived schemes is the functor

$$\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(-) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

defined as

$$\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(-) := \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{Fil}^0_{\mathrm{T}}\mathrm{HP}(-).$$

The following definition is motivated by Proposition 3.1.1 (3).

**Definition 3.1.5** (HKR filtration on HH)**.** The *HKR filtration on Hochschild homology* of qcqs derived schemes is the functor

$$\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HH}(-) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

defined as

$$\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HH}(-) := \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{gr}^0_{\mathrm{T}}\mathrm{HP}(-).$$

Cyclic homology $\mathrm{HC}(-)$ is defined as the homotopy orbits $\mathrm{HH}(-)_{h\mathrm{S}^1}$ of the $\mathrm{S}^1$-action on Hochschild homology $\mathrm{HH}(-)$, and is related to negative cyclic homology $\mathrm{HC}^-(-)$ and periodic cyclic homology $\mathrm{HP}(-)$ by a natural fibre sequence

$$\mathrm{HC}^-(-) \longrightarrow \mathrm{HP}(-) \longrightarrow \mathrm{HC}(-)[2].$$

**Definition 3.1.6** (HKR filtration on HC)**.** The *HKR filtration on cyclic homology* of qcqs derived schemes is the functor

$$\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}(-) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

defined, for every integer $i \in \mathbb{Z}$, by

$$\mathrm{Fil}^i_{\mathrm{HKR}}\mathrm{HC}(-) := \mathrm{cofib}\Big(\mathrm{Fil}^{i+1}_{\mathrm{HKR}}\mathrm{HC}^-(-) \longrightarrow \mathrm{Fil}^{i+1}_{\mathrm{HKR}}\mathrm{HP}(-)\Big)[-2],$$

where the map on the right hand side is induced by Construction 3.1.3 and Definition 3.1.4.

**Remark 3.1.7** (Graded pieces of the HKR filtrations)**.** Let $X$ be a qcqs derived scheme. The main result of [Ant19] describes the graded pieces of the HKR filtrations on $\mathrm{HC}^-(X)$, $\mathrm{HP}(X)$, and $\mathrm{HC}(X)$ in terms of the Hodge-completed derived de Rham cohomology of $X$. In particular, Definition 3.1.6 provides a filtered refinement of the fibre sequence

$$\mathrm{HC}^-(-) \longrightarrow \mathrm{HP}(-) \longrightarrow \mathrm{HC}(-)[2],$$

which induces on graded pieces, for every integer $i \in \mathbb{Z}$, a natural fibre sequence

$$R\Gamma_{\mathrm{Zar}}\big(X, \widehat{\mathbb{L}\Omega}^{\geq i}_{-/\mathbb{Z}}\big)[2i] \longrightarrow R\Gamma_{\mathrm{Zar}}\big(X, \widehat{\mathbb{L}\Omega}_{-/\mathbb{Z}}\big)[2i] \longrightarrow R\Gamma_{\mathrm{Zar}}\big(X, \mathbb{L}\Omega^{<i}_{-/\mathbb{Z}}\big)[2i]$$

in the derived category $\mathcal{D}(\mathbb{Z})$.

**Proposition 3.1.8** ([Ant19, BL22])**.** *For every integer $i \in \mathbb{Z}$, the functor $\mathrm{Fil}^i_{\mathrm{HKR}}\mathrm{HC}(-)$, from animated commutative rings to spectra, is left Kan extended from polynomial $\mathbb{Z}$-algebras, commutes with filtered colimits, and its values are in cohomological degrees at most $-i$.*

*Proof.* On animated commutative rings, the Tate filtrations

$$\mathrm{Fil}^{i+1}_{\mathrm{HKR}}\mathrm{Fil}^{\star}_{\mathrm{T}}\mathrm{HC}^-(-) \quad \text{and} \quad \mathrm{Fil}^{i+1}_{\mathrm{HKR}}\mathrm{Fil}^{\star}_{\mathrm{T}}\mathrm{HP}(-)$$

are by definition left Kan extended, as complete filtered objects, from polynomial $\mathbb{Z}$-algebras, thus so is the Tate filtration $\mathrm{Fil}^i_{\mathrm{HKR}}\mathrm{Fil}^{\star}_{\mathrm{T}}\mathrm{HC}(-)$. The Tate filtration $\mathrm{Fil}^i_{\mathrm{HKR}}\mathrm{Fil}^{\star}_{\mathrm{T}}\mathrm{HC}(-)$ is also finite by construction, so the functor $\mathrm{Fil}^i_{\mathrm{HKR}}\mathrm{HC}(-)$ is left Kan extended on animated commutative rings from polynomial $\mathbb{Z}$-algebras. In particular, the functor

$$\mathrm{Fil}^i_{\mathrm{HKR}}\mathrm{HC}(-)$$

commutes with filtered colimits of animated commutative rings. For every integer $j \in \mathbb{Z}$, the $j^{\mathrm{th}}$ graded piece of the filtration $\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}(-)$ is naturally identified with the functor $R\Gamma_{\mathrm{Zar}}\big(-, \mathbb{L}\Omega^{\leq j}_{-/\mathbb{Z}}\big)[2j]$ ([Ant19]), whose values are in degrees at most $-j$ on animated commutative rings. The filtration $\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}(-)$ is moreover complete on animated commutative rings ([BL22, Remark 6.3.5]), hence the desired connectivity result. □

**Lemma 3.1.9** (Completeness of the HKR filtrations, after [BL22])**.** *Let $X$ be a qcqs derived scheme. Then the HKR filtrations*

$$\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HP}(X),\ \mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}^-(X),\ \mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HH}(X),\ \textit{and}\ \mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}(X)$$

*are complete.*

*Proof.* The result for $\mathrm{HC}^-$ and HH is a direct consequence of [BL22, Remark 6.3.5]. The result for HC is a consequence of the connectivity result of Proposition 3.1.8. By Definition 3.1.6, the result for HP is then a consequence of the result for $\mathrm{HC}^-$ and HC. □

**Remark 3.1.10** (Variant over $\mathbb{Q}$)**.** Let $X$ be a qcqs derived scheme. By the base change property for Hochschild homology, the natural map

$$\mathrm{HH}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \mathrm{HH}(X_{\mathbb{Q}}/\mathbb{Q})$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Q})$, where $\mathrm{HH}(-/\mathbb{Q})$ is Hochschild homology relative to $\mathbb{Q}$. Applying the functors $(-)^{h\mathrm{S}^1}$, $(-)^{t\mathrm{S}^1}$, and $(-)_{h\mathrm{S}^1}$ to this Hochschild homology relative to $\mathbb{Q}$ induces relative variants $\mathrm{HC}^-(-_{\mathbb{Q}}/\mathbb{Q})$ of negative cyclic homology, $\mathrm{HP}(-_{\mathbb{Q}}/\mathbb{Q})$ of periodic cyclic homology, and $\mathrm{HC}(-_{\mathbb{Q}}/\mathbb{Q})$ of cylic homology. One can then define similar HKR filtrations

$$\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HH}(X_{\mathbb{Q}}/\mathbb{Q}),\ \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X_{\mathbb{Q}}/\mathbb{Q}),\ \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HP}(-_{\mathbb{Q}}/\mathbb{Q}),\ \text{and } \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}(-_{\mathbb{Q}}/\mathbb{Q}),$$

on these functors, whose graded pieces are versions of derived de Rham cohomology relative to $\mathbb{Q}$.

To introduce and study the motivic filtration on topological cyclic homology (Definition 3.3.3), we will need some $p$-complete variants of the previous HKR filtrations.

**Definition 3.1.11** (HKR filtration on $\mathrm{HC}^-(-;\mathbb{Z}_p)$)**.** Let $p$ be a prime number. The *HKR filtration on $p$-completed negative cyclic homology* of qcqs derived schemes is the functor

$$\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(-;\mathbb{Z}_p) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

defined as

$$\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(-;\mathbb{Z}_p) := \big(\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(-)\big)^\wedge_p.$$

**Remark 3.1.12.** The HKR filtrations on $\mathrm{HP}(-;\mathbb{Z}_p)$, $\mathrm{HC}(-;\mathbb{Z}_p)$, and $\mathrm{HH}(-;\mathbb{Z}_p)$ of qcqs derived schemes are defined as in Definition 3.1.11, where $\mathrm{HC}^-(-;\mathbb{Z}_p)$ is replaced by $\mathrm{HP}(-)$, $\mathrm{HC}(-)$, or $\mathrm{HH}(-)$. In particular, for every qcqs derived scheme $X$, Definition 3.1.6 induces a fibre sequence of filtered spectra

$$\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X;\mathbb{Z}_p) \longrightarrow \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HP}(X;\mathbb{Z}_p) \longrightarrow \mathrm{Fil}^{\star-1}_{\mathrm{HKR}}\mathrm{HC}(X;\mathbb{Z}_p)[2].$$

**Lemma 3.1.13.** *Let $X$ be a qcqs derived scheme. Then the filtrations*

$$\prod_{p\in\mathbb{P}}\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HP}(X;\mathbb{Z}_p),\quad \prod_{p\in\mathbb{P}}\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X;\mathbb{Z}_p),\quad \prod_{p\in\mathbb{P}}\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HH}(X;\mathbb{Z}_p),$$

$$\text{and } \prod_{p\in\mathbb{P}}\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}(X;\mathbb{Z}_p)$$

*are complete.*

*Proof.* The collection of complete filtered spectra is closed under limits in the category of filtered spectra, so this is a consequence of Lemma 3.1.9. □

**Remark 3.1.14** (Exhaustivity of the HKR filtrations)**.** The HKR filtrations $\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(-)$ and $\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HP}(-)$ are not exhaustive on general qcqs derived schemes ([BL22, Remark 6.3.6]). For the purpose of the motivic filtration on algebraic $K$-theory (Definition 4.3.4), we will however only need the fact that the HKR filtration $\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}(-)$ is always $\mathbb{N}$-indexed, and in particular exhaustive.

## 3.2 The BMS filtrations

Let $p$ be a prime number. In this section, we review the BMS filtrations on $p$-completed topological Hochschild homology $\mathrm{THH}(-;\mathbb{Z}_p)$ and its variants, as defined in [BMS19] for $p$-complete $p$-quasisyntomic rings, and generalised in [AMMN22] to $p$-complete rings and in [BL22, Section 6.2] to animated commutative rings. Only the BMS filtration on $p$-completed topological cyclic homology $\mathrm{TC}(-;\mathbb{Z}_p)$ (Definition 3.2.6) will appear in the definition of the motivic filtration on topological cyclic homology $\mathrm{TC}(-)$ (Definition 3.3.3). The other BMS filtrations are necessary to construct the BMS filtration on $p$-completed topological cyclic homology $\mathrm{TC}(-;\mathbb{Z}_p)$.

**Construction 3.2.1** (BMS filtration on $\mathrm{Fil}^\star_{\mathrm{T}}\mathrm{TP}(-;\mathbb{Z}_p)$)**.** Topological Hochschild homology $\mathrm{THH}(-)$ of qcqs derived schemes admits a natural $\mathrm{S}^1$-action, inducing a natural Tate filtration $\mathrm{Fil}^\star_{\mathrm{T}}\mathrm{TP}(-)$ on topological periodic cyclic homology $\mathrm{TP}(-) := \mathrm{THH}(-)^{t\mathrm{S}^1}$ (Proposition 3.1.1). The Tate filtration

$$\mathrm{Fil}^\star_{\mathrm{T}}\mathrm{TP}(-;\mathbb{Z}_p) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

is then defined as the $p$-completion of the Tate filtration $\mathrm{Fil}^\star_{\mathrm{T}}\mathrm{TP}(-)$. For every quasiregular semiperfectoid ring $R$ and every integer $i \in \mathbb{Z}$, define the filtered spectrum

$$\mathrm{Fil}^i_{\mathrm{BMS}}\mathrm{Fil}^\star_{\mathrm{T}}\mathrm{TP}(R;\mathbb{Z}_p) := \tau_{\geq 2i}\mathrm{Fil}^\star_{\mathrm{T}}\mathrm{TP}(R;\mathbb{Z}_p).$$

The filtered object

$$\mathrm{Fil}^i_{\mathrm{BMS}}\mathrm{Fil}^\star_{\mathrm{T}}\mathrm{TP}(-;\mathbb{Z}_p) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

is then first defined on $p$-quasisyntomic rings as the unique such functor satisfying $p$-complete faithfully flat descent (the existence and unicity of such a functor is [BMS19, Proposition 4.31]). In general, polynomial $\mathbb{Z}$-algebras are $p$-quasisyntomic rings, and this filtered object is defined as the Zariski sheafification of its left Kan extension from polynomial $\mathbb{Z}$-algebras

$$\mathrm{Fil}^i_{\mathrm{BMS}}\mathrm{Fil}^\star_{\mathrm{T}}\mathrm{TP}(-;\mathbb{Z}_p) := L_{\mathrm{Zar}}L_{\mathrm{AniRings}/\mathrm{Poly}_{\mathbb{Z}}}\mathrm{Fil}^i_{\mathrm{BMS}}\mathrm{Fil}^\star_{\mathrm{T}}\mathrm{TP}(-;\mathbb{Z}_p),$$

where the left Kan extension is taken in the category of $p$-complete filtration-complete spectra. By [BL22, Theorem 6.2.4], the resulting functor is still given by the double-speed Postnikov filtration on quasiregular semiperfectoid rings and, as a functor from animated commutative rings to $p$-complete filtration-complete spectra, commutes with sifted colimits and satisfies $p$-complete faithfully flat descent.

**Remark 3.2.2.** The BMS filtrations were first defined in [BMS19] in the generality of $p$-complete $p$-quasisyntomic rings. On general animated commutative rings $R$, the BMS filtrations, by construction, depend only on the $p$-completion of $R$ –and in particular vanish on animated commutative $\mathbb{Z}[\frac{1}{p}]$-algebras. Here the $p$-completion is the derived $p$-completion, even on classical commutative rings. On commutative rings with bounded $p$-power torsion (*e.g.*, on $p$-quasisyntomic rings), the derived and classical $p$-completions naturally coincide, and there is no conflict between the two definitions.

**Definition 3.2.3** (BMS filtration on $\mathrm{TP}(-;\mathbb{Z}_p)$)**.** The *BMS filtration on $p$-completed topological periodic cyclic homology* of qcqs derived schemes is the functor

$$\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TP}(-;\mathbb{Z}_p) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

defined as the underlying filtered object of the bifiltered functor $\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{Fil}^\star_{\mathrm{T}}\mathrm{TP}(-;\mathbb{Z}_p)$ of Construction 3.2.1:

$$\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TP}(-;\mathbb{Z}_p) := \varinjlim_n \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{Fil}^n_{\mathrm{T}}\mathrm{TP}(-;\mathbb{Z}_p).$$

Topological negative cyclic homology is to topological periodic cyclic homology what negative cyclic homology is to periodic cyclic homology. Given Definition 3.2.3, the following definition then mimics Definition 3.1.4.

**Definition 3.2.4** (BMS filtration on $\mathrm{TC}^-(-;\mathbb{Z}_p)$)**.** The *BMS filtration on p-completed topological negative cyclic homology* of qcqs derived schemes is the functor

$$\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}^-(-;\mathbb{Z}_p) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

defined as

$$\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}^-(-;\mathbb{Z}_p) := \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{Fil}^0_{\mathrm{T}}\mathrm{TP}(-;\mathbb{Z}_p).$$

Similarly, topological Hochschild homology is to topological periodic and topological negative cyclic homologies what Hochschild homology is to periodic and negative cyclic homologies, and the following definition mimics Definition 3.1.5.

**Definition 3.2.5** (BMS filtration on $\mathrm{THH}(-;\mathbb{Z}_p)$)**.** The *BMS filtration on p-completed topological Hochschild homology* of qcqs derived schemes is the functor

$$\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{THH}(-;\mathbb{Z}_p) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

defined as

$$\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{THH}(-;\mathbb{Z}_p) := \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{gr}^0_{\mathrm{T}}\mathrm{TP}(-;\mathbb{Z}_p).$$

Topological cyclic homology is however not to topological periodic cyclic homology what cyclic homology is to periodic cyclic homology. Following [NS18], it is rather defined, after $p$-completion, by a fibre sequence

$$\mathrm{TC}(-;\mathbb{Z}_p) \longrightarrow \mathrm{TC}^-(-;\mathbb{Z}_p) \xrightarrow{\phi_p - \mathrm{can}} \mathrm{TP}(-;\mathbb{Z}_p).$$

Unwinding the previous definitions, the map $\phi_p - \mathrm{can} : \mathrm{TC}^-(-;\mathbb{Z}_p) \to \mathrm{TP}(-;\mathbb{Z}_p)$ admits a unique refinement as a filtered map

$$\phi_p - \mathrm{can} : \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}^-(-;\mathbb{Z}_p) \longrightarrow \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TP}(-;\mathbb{Z}_p).$$

**Definition 3.2.6** (BMS filtration on $\mathrm{TC}(-;\mathbb{Z}_p)$)**.** The *BMS filtration on p-completed topological cyclic homology* of qcqs derived schemes is the functor

$$\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

defined as

$$\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p) := \mathrm{fib}\Big(\phi_p - \mathrm{can} : \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}^-(-;\mathbb{Z}_p) \longrightarrow \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TP}(-;\mathbb{Z}_p)\Big).$$

The BMS filtration on $p$-completed topological cyclic homology is always complete, as a consequence of a connectivity result of [AMMN22]. We will need the following slightly more precise result when studying the completeness of the motivic filtration on algebraic $K$-theory.

**Lemma 3.2.7.** *Let $X$ be a qcqs derived scheme. Then the filtrations $\prod_{p\in\mathbb{P}} \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(X;\mathbb{Z}_p)$ and $\big(\prod_{p\in\mathbb{P}} \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(X;\mathbb{Z}_p)\big)_{\mathbb{Q}}$ are complete. More precisely, for every integer $i \in \mathbb{Z}$, the values of the presheaves $\prod_{p\in\mathbb{P}} \mathrm{Fil}^i_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)$ and $\big(\prod_{p\in\mathbb{P}} \mathrm{Fil}^i_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)\big)_{\mathbb{Q}}$ are in cohomological degrees at most $-i+1$ on affine derived schemes.*

*Proof.* The presheaves

$$\prod_{p\in\mathbb{P}} \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p) \quad \text{and} \quad \Big(\prod_{p\in\mathbb{P}} \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)\Big)_{\mathbb{Q}}$$

are Zariski sheaves by construction, so it suffices to prove the result for affine derived schemes $X = \mathrm{Spec}(R)$. Let $R$ be an animated commutative ring, and $i \in \mathbb{Z}$ be an integer. The spectrum $\mathrm{Fil}^i_{\mathrm{BMS}}\mathrm{TC}(R;\mathbb{Z}_p)$ is in cohomological degrees at most $-i+1$ for every prime number $p$ ([AMMN22, Theorem 5.1]). Taking the product over all primes $p$ and rationalisation, this implies that the spectra $\prod_{p\in\mathbb{P}} \mathrm{Fil}^i_{\mathrm{BMS}}\mathrm{TC}(R;\mathbb{Z}_p)$ and $\Big(\prod_{p\in\mathbb{P}} \mathrm{Fil}^i_{\mathrm{BMS}}\mathrm{TC}(R;\mathbb{Z}_p)\Big)_{\mathbb{Q}}$ are also in cohomological degrees at most $-i+1$, which implies that the associated filtrations are complete. □

**Remark 3.2.8** (Exhaustivity of the BMS filtrations)**.** The BMS filtrations $\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TP}(-;\mathbb{Z}_p)$ and $\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}^-(-;\mathbb{Z}_p)$ are not exhaustive on general qcqs derived schemes ([BL22, Warning 6.2.7]). For the purpose of the motivic filtration on algebraic $K$-theory (Definition 4.3.4), we will however only need the fact that the BMS filtration $\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)$ is always $\mathbb{N}$-indexed ([BMS19, proof of Proposition 7.16]), and in particular exhaustive.

We refer to [BMS19, BS22, BL22] (see also Sections 11.1.2 and 11.3.1) for the relation between prismatic cohomology and the graded pieces of the BMS filtrations on $\mathrm{TP}(-;\mathbb{Z}_p)$, $\mathrm{TC}^-(-;\mathbb{Z}_p)$, and $\mathrm{THH}(-;\mathbb{Z}_p)$. We only define here the shifted graded pieces of the BMS filtration on $\mathrm{TC}(-;\mathbb{Z}_p)$, which are a version of syntomic cohomology (see Remark 3.2.10), and which will serve as a building block for the $p$-adic motivic complexes (Corollary 4.3.12).

**Definition 3.2.9** (BMS syntomic cohomology)**.** For every integer $i \in \mathbb{Z}$, the *syntomic complex*

$$\mathbb{Z}_p(i)^{\mathrm{BMS}}(-) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathcal{D}(\mathbb{Z})$$

is the functor defined as the shifted graded piece of the BMS filtration on $\mathrm{TC}(-;\mathbb{Z}_p)$:

$$\mathbb{Z}_p(i)^{\mathrm{BMS}}(-) := \mathrm{gr}^i_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)[-2i].$$

**Remark 3.2.10.** Syntomic cohomology $\mathbb{Z}_p(i)^{\mathrm{syn}}(X)$ of qcqs derived (formal) schemes $X$ is defined in [BL22, Section 8.4] (see also Notation 6.2.1), in terms of the syntomic complexes of Definition 3.2.9 and of étale cohomology. From this perspective, the syntomic complexes $\mathbb{Z}_p(i)^{\mathrm{BMS}}(X)$ of Definition 3.2.9 correspond to the syntomic cohomology $\mathbb{Z}_p(i)^{\mathrm{syn}}(\mathfrak{X})$ of the derived $p$-adic formal scheme $\mathfrak{X}$ associated to $X$.

**Theorem 3.2.11.** *(1) ([BMS19, BL22]) The functor* $\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)$*, viewed as a functor from p-quasisyntomic rings to p-complete filtered spectra, satisfies descent for the p-quasisyntomic topology.*

*(2) ([AMMN22, BL22]) The functor* $\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)$*, viewed as a functor from animated commutative rings to p-complete filtered spectra, is left Kan extended from polynomial* $\mathbb{Z}$*-algebras.*

*Proof.* (1) The filtration $\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)$ is complete on $p$-quasisyntomic rings (Lemma 3.2.7), so it suffices to prove the result on graded pieces. The result on graded pieces is a special case of [BL22, Proposition 7.4.7].

(2) By [AMMN22, Theorem 5.1 (2)], the functor $\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)$, viewed as a functor from $p$-complete animated commutative rings to $p$-complete filtered spectra, is left Kan extended from $p$-complete polynomial $\mathbb{Z}$-algebras.[1] Let $R$ be an animated commutative ring, and $R^\wedge_p$ be its (derived) $p$-completion. The natural map

$$\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(R;\mathbb{Z}_p) \longrightarrow \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(R^\wedge_p;\mathbb{Z}_p)$$

is an equivalence of filtered spectra. Indeed, the filtrations

$$\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(R;\mathbb{Z}_p) \quad \text{and} \quad \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(R^\wedge_p;\mathbb{Z}_p)$$

are $\mathbb{N}$-indexed and complete (Lemma 3.2.7 and Remark 3.2.8), so it suffices to prove the result on graded pieces, where this is a direct consequence of [BL22, Corollary 7.4.11]. This implies the desired left Kan extension property. ☐

**Corollary 3.2.12.** *Let $i \in \mathbb{Z}$ be an integer.*

*(1) The functor $\mathbb{Z}_p(i)^{\mathrm{BMS}}(-)$, viewed as a functor from $p$-quasisyntomic rings to $p$-complete objects in the derived category $\mathcal{D}(\mathbb{Z})$, satisfies descent for the $p$-quasisyntomic topology.*

*(2) The functor $\mathbb{Z}_p(i)^{\mathrm{BMS}}(-)$, viewed as a functor from animated commutative rings to $p$-complete objects in the derived category $\mathcal{D}(\mathbb{Z})$, is left Kan extended from polynomial $\mathbb{Z}$-algebras.*

*Proof.* (1) was already part of the proof of Theorem 3.2.11 (1).
(2) is a direct consequence of Theorem 3.2.11 (2). ☐

**Theorem 3.2.13** ([AMMN22])**.** *Let $(A,I)$ be a henselian pair of commutative rings. Then for any integers $i \geq 0$ and $k \geq 1$, the fibre of the natural map*

$$\mathbb{Z}/p^k(i)^{\mathrm{BMS}}(A) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{BMS}}(A/I)$$

*in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$ is in degrees at most $i$.*

*Proof.* By [AMMN22, Theorem 5.2], for every henselian pair $(A,I)$ such that the commutative rings $A$ and $A/I$ are (classically) $p$-complete, the fibre of the natural map

$$\mathbb{Z}/p^k(i)^{\mathrm{BMS}}(A) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{BMS}}(A/I)$$

is in degrees at most $i$. The proof of [AMMN22, Theorem 5.2] proves more generally that for $(A,I)$ a general henselian pair of commutative rings, the fibre of the natural map

$$\mathbb{Z}/p^k(i)^{\mathrm{BMS}}(A^\wedge_p) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{BMS}}((A/I)^\wedge_p)$$

where $(-)^\wedge_p$ is the derived $p$-completion, is in degrees at most $i$. By [BL22, Corollary 7.4.11] (see also the proof of Theorem 3.2.11 (2)), the natural map $\mathbb{Z}/p^k(i)^{\mathrm{BMS}}(-) \to \mathbb{Z}/p^k(i)^{\mathrm{BMS}}((-)^\wedge_p)$ is an equivalence on animated commutative rings, hence for every henselian pair $(A,I)$ of commutative rings, the fibre of the natural map

$$\mathbb{Z}/p^k(i)^{\mathrm{BMS}}(A) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{BMS}}(A/I)$$

is in degrees at most $i$. ☐

[1] More precisely, it is proved to be left Kan extended from $p$-complete polynomial $\mathbb{Z}$-algebras to $p$-complete $p$-quasisyntomic rings. By definition, $p$-quasisyntomic rings have bounded $p$-power torsion. Hence, their derived and classical $p$-completions are naturally identified, and the left Kan extension to $p$-complete animated commutative rings agrees with the left Kan extension to $p$-complete classical rings on $p$-complete $p$-quasisyntomic rings.

## 3.3 The motivic filtration on TC

In this section, we introduce the motivic filtration on topological cyclic homology $\mathrm{TC}(-)$ of general qcqs derived schemes (Definition 3.3.3).

The following proposition is [BL22, Proposition 6.4.1].

**Proposition 3.3.1** ([BL22])**.** *Let $p$ be a prime number. The map*

$$\mathrm{Fil}^\star_{\mathrm{T}}\mathrm{TP}(-;\mathbb{Z}_p) \longrightarrow \mathrm{Fil}^\star_{\mathrm{T}}\mathrm{HP}(-;\mathbb{Z}_p),$$

*viewed as a map of filtered spectra-valued presheaves on the category of qcqs derived schemes, admits a unique, multiplicative extension to a map of bifiltered presheaves of spectra*

$$\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{Fil}^\star_{\mathrm{T}}\mathrm{TP}(-;\mathbb{Z}_p) \longrightarrow \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{Fil}^\star_{\mathrm{T}}\mathrm{HP}(-;\mathbb{Z}_p).$$

**Construction 3.3.2** (BMS-HKR comparison map)**.** Let $p$ be a prime number. The *BMS-HKR comparison map* is the map

$$\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p) \longrightarrow \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(-;\mathbb{Z}_p)$$

of functors from (the opposite category of) qcqs derived schemes to the category of filtered spectra defined as the composite

$$\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p) \longrightarrow \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}^-(-;\mathbb{Z}_p) \longrightarrow \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(-;\mathbb{Z}_p)$$

of the maps given by Definition 3.2.6, and Proposition 3.3.1 after restricting to the zeroth step of the Tate filtration.

**Definition 3.3.3** (Motivic filtration on TC)**.** The *motivic filtration on topological cyclic homology* of qcqs derived schemes

$$\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(-) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

is the functor defined by the cartesian square

$$\begin{array}{ccc} \mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(-) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(-) \\ \downarrow & & \downarrow \\ \prod_{p\in\mathbb{P}} \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p) & \longrightarrow & \prod_{p\in\mathbb{P}} \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(-;\mathbb{Z}_p), \end{array}$$

where the bottom horizontal map is the map of Construction 3.3.2, and the right vertical map is profinite completion. For every integer $i \in \mathbb{Z}$, also define the functor

$$\mathbb{Z}(i)^{\mathrm{TC}}(-) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathcal{D}(\mathbb{Z})^2$$

as the shifted graded piece of this motivic filtration:

$$\mathbb{Z}(i)^{\mathrm{TC}}(-) := \mathrm{gr}^i_{\mathrm{mot}}\mathrm{TC}(-)[-2i].$$

[2]Every value of the functor $\mathbb{Z}(i)^{\mathrm{TC}}(-)$ has a natural module structure over the $\mathbb{E}_\infty$-ring $\mathbb{Z}(0)^{\mathrm{TC}}(\mathbb{Z})$, which, by unwinding the definition, is naturally identified with the $\mathbb{E}_\infty$-ring $H\mathbb{Z}$. This implies that the spectra-valued functor $\mathbb{Z}(i)^{\mathrm{TC}}(-)$ takes values in $H\mathbb{Z}$-linear spectra, *i.e.*, in the derived category $\mathcal{D}(\mathbb{Z})$.

**Remark 3.3.4** (Comparison to [EM23])**.** For every qcqs derived scheme $X$ over $\mathbb{Q}$, the filtered spectrum $\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X)$ is $\mathbb{Q}$-linear by construction, so its profinite completion vanishes. The filtration $\prod_{p\in\mathbb{P}}\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(X;\mathbb{Z}_p)$ also vanishes (Remark 3.2.2), and the natural map

$$\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(X) \longrightarrow \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X/\mathbb{Q})$$

is then an equivalence of filtered spectra.

Similarly, for every prime number $p$ and every qcqs derived scheme $X$ over $\mathbb{F}_p$, the filtered spectrum $\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X)$ is $\mathbb{Z}$-linear and $p$-complete, so it is naturally identified with its profinite completion. Again using Remark 3.2.2, the natural map

$$\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(X) \longrightarrow \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(X;\mathbb{Z}_p)$$

is then an equivalence of filtered spectra.

**Remark 3.3.5** (Comparison to [BL22])**.** In [BL22, Section 6.4], Bhatt–Lurie define filtrations $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TP}(X)$ and $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}^-(X)$ for qcqs derived schemes $X$, with shifted graded pieces called the global prismatic complexes $\widehat{\mathbb{\Delta}}^{\mathrm{gl}}_X\{i\}$ and $\mathcal{N}^{\geq i}\widehat{\mathbb{\Delta}}^{\mathrm{gl}}_X\{i\}$ respectively. These filtrations can be used to obtain an alternative definition of the $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(X)$ of Definition 3.3.3. More precisely, for every prime number $p$ the $p$-completion of Bhatt–Lurie's filtration $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TP}(X)$ is the filtration $\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TP}(X;\mathbb{Z}_p)$ of Definition 3.2.3, and there is a natural fibre sequence

$$\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(X) \longrightarrow \mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}^-(X) \longrightarrow \prod_{p\in\mathbb{P}}\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TP}(X;\mathbb{Z}_p)$$

of filtered spectra. In particular, for every integer $i\in\mathbb{Z}$, this induces a natural fibre sequence

$$\mathbb{Z}(i)^{\mathrm{TC}}(X) \longrightarrow \mathcal{N}^{\geq i}\widehat{\mathbb{\Delta}}^{\mathrm{gl}}_X\{i\} \longrightarrow \prod_{p\in\mathbb{P}}\widehat{\mathbb{\Delta}}_{X,p}\{i\}$$

in the derived category $\mathcal{D}(\mathbb{Z})$, where $\widehat{\mathbb{\Delta}}_{X,p}$ denotes the $p$-adic absolute prismatic cohomology of $X$.

**Proposition 3.3.6.** *Let $X$ be a qcqs derived scheme, and $p$ be a prime number. Then the natural map*

$$\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(X;\mathbb{Z}_p) \longrightarrow \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(X;\mathbb{Z}_p)$$

*is an equivalence of filtered spectra.*

*Proof.* By definition, the natural map

$$\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X) \longrightarrow \prod_{\ell\in\mathbb{P}}\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X;\mathbb{Z}_\ell)$$

in Definition 3.3.3 is profinite completion, so its fibre becomes zero after $p$-completion. □

**Corollary 3.3.7.** *Let $X$ be a qcqs derived scheme, and $p$ be a prime number. Then the natural map*

$$\mathbb{Z}_p(i)^{\mathrm{TC}}(X) \longrightarrow \mathbb{Z}_p(i)^{\mathrm{BMS}}(X)$$

*is an equivalence in the derived category $\mathcal{D}(\mathbb{Z}_p)$.*

*Proof.* This is a direct consequence of Proposition 3.3.6. □

**Proposition 3.3.8.** *Let $X$ be a qcqs derived scheme. Then the filtrations*

$$\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(X) \text{ and } \mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(X;\mathbb{Q})$$

*are complete.*

*Proof.* The filtrations $\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X)$, $\prod_{p\in\mathbb{P}}\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X;\mathbb{Z}_p)$, and $\prod_{p\in\mathbb{P}}\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(X;\mathbb{Z}_p)$ are complete by Lemmas 3.1.9, 3.1.13, and 3.2.7 respectively. By Definition 3.3.3, the filtration $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(X)$ is then complete, as a pullback of three complete filtrations. To prove that the filtration $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(X;\mathbb{Q})$ is complete, consider the cartesian square of filtered spectra

$$\begin{array}{ccc}
\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(X;\mathbb{Q}) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X;\mathbb{Q}) \\
\downarrow & & \downarrow \\
\Big(\prod_{p\in\mathbb{P}}\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(X;\mathbb{Z}_p)\Big)_{\mathbb{Q}} & \longrightarrow & \Big(\prod_{p\in\mathbb{P}}\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X;\mathbb{Z}_p)\Big)_{\mathbb{Q}}
\end{array}$$

induced by taking the rationalisation of Definition 3.3.3. The filtration

$$\big(\prod_{p\in\mathbb{P}}\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(X;\mathbb{Z}_p)\big)_{\mathbb{Q}}$$

is complete by Lemma 3.2.7. The fibre of the natural map

$$\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X) \longrightarrow \prod_{p\in\mathbb{P}}\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X;\mathbb{Z}_p)$$

is complete as an object of the filtered derived category $\mathcal{DF}(\mathbb{Z})$ (Lemmas 3.1.9 and 5.2.9), and is zero modulo $p$ for every prime number $p$ by construction. In particular, it is naturally identified with the fibre of the natural map

$$\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X;\mathbb{Q}) \longrightarrow \Big(\prod_{p\in\mathbb{P}}\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X;\mathbb{Z}_p)\Big)_{\mathbb{Q}},$$

which is thus complete as an object of the filtered derived category $\mathcal{DF}(\mathbb{Q})$. This implies, by the previous cartesian square, that the filtration $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(X;\mathbb{Q})$ is also complete. ☐

**Remark 3.3.9** (Exhaustivity of the motivic filtration on TC)**.** The motivic filtration $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}$ is not exhaustive on general qcqs derived scheme. Although this will not be necessary to prove that the motivic filtration on algebraic $K$-theory is exhaustive (Proposition 5.5.1), one can prove, using [Ant19, Lemma 4.10] and its proof, that if $X$ is a quasi-lci $\mathbb{Z}$-scheme,[3] then the motivic filtration $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(X)$ is exhaustive.

**Proposition 3.3.10.** *For every integer $i \in \mathbb{Z}$, the presheaf*

$$\mathrm{Fil}^i_{\mathrm{mot}}\mathrm{TC}(-) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{Sp}$$

*is an étale sheaf.*

[3] By this, we mean that Zariski-locally on the qcqs scheme $X$, the cotangent complex $\mathbb{L}_{-/\mathbb{Z}}$ has Tor-amplitude in $[-1;0]$.

*Proof.* By Definition 3.3.3, it suffices to prove that the presheaves $\prod_{p \in \mathbb{P}} \mathrm{Fil}^i_{\mathrm{BMS}} \mathrm{TC}(-;\mathbb{Z}_p)$, $\mathrm{Fil}^i_{\mathrm{HKR}} \mathrm{HC}^-(-)$, and $\prod_{p \in \mathbb{P}} \mathrm{Fil}^i_{\mathrm{HKR}} \mathrm{HC}^-(-;\mathbb{Z}_p)$ are étale sheaves. A product of sheaves is a sheaf, and these BMS and HKR filtrations are complete (Lemmas 3.2.7, 3.1.9, and 3.1.13). It then suffices to prove that for every prime number $p$, the presheaves

$$\mathbb{Z}_p(i)^{\mathrm{BMS}}(-), \; R\Gamma_{\mathrm{Zar}}\big(-, \widehat{\mathbb{L}\Omega}^{\geq i}_{-/\mathbb{Z}}\big), \text{ and } R\Gamma_{\mathrm{Zar}}\big(-, (\widehat{\mathbb{L}\Omega}^{\geq i}_{-/\mathbb{Z}})^{\wedge}_p\big)$$

are étale sheaves. The statement for $\mathbb{Z}_p(i)^{\mathrm{BMS}}(-)$ is a consequence of $p$-fpqc descent ([BL22, Proposition 7.4.7]). The statement for the other two presheaves reduces to the fpqc descent for the powers of the cotangent complex ([BMS19, Theorem 3.1]). □

**Corollary 3.3.11.** *For every integer $i \in \mathbb{Z}$, the presheaf*

$$\mathbb{Z}(i)^{\mathrm{TC}}(-) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathcal{D}(\mathbb{Z})$$

*is an étale sheaf.*

*Proof.* This is a direct consequence of Proposition 3.3.10. □

## 3.4 The motivic filtration on $L_{\mathrm{cdh}}\mathrm{TC}$

In this section, we introduce the motivic filtration on the cdh sheafification of topological cyclic homology of general qcqs schemes (Definition 3.4.1).

**Definition 3.4.1** (Motivic filtration on $L_{\mathrm{cdh}}\mathrm{TC}$)**.** The *motivic filtration on cdh sheafified topological cyclic homology* of qcqs schemes

$$\mathrm{Fil}^\star_{\mathrm{mot}} L_{\mathrm{cdh}} \mathrm{TC}(-) : \mathrm{Sch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

is the functor defined as the cdh sheafification of the motivic filtration on topological cyclic homology (Definition 3.3.3)

$$\mathrm{Fil}^\star_{\mathrm{mot}} L_{\mathrm{cdh}} \mathrm{TC}(-) := \big(L_{\mathrm{cdh}} \mathrm{Fil}^\star_{\mathrm{mot}} \mathrm{TC}\big)(-).$$

**Remark 3.4.2** (Graded pieces of $\mathrm{Fil}^\star_{\mathrm{mot}} L_{\mathrm{cdh}} \mathrm{TC}$)**.** Let $X$ be a qcqs scheme. For every integer $i \in \mathbb{Z}$, the canonical map

$$\big(L_{\mathrm{cdh}}\, \mathbb{Z}(i)^{\mathrm{TC}}\big)(X) \longrightarrow \mathrm{gr}^i_{\mathrm{mot}} L_{\mathrm{cdh}} \mathrm{TC}(R)[-2i]$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$. We will usually refer to these shifted graded pieces by the complexes $\big(L_{\mathrm{cdh}}\, \mathbb{Z}(i)^{\mathrm{TC}}\big)(X)$.

**Remark 3.4.3** (Completeness of $\mathrm{Fil}^\star_{\mathrm{mot}} L_{\mathrm{cdh}} \mathrm{TC}$)**.** It is not clear *a priori* that the filtration $\mathrm{Fil}^\star_{\mathrm{mot}} L_{\mathrm{cdh}} \mathrm{TC}(X)$ is complete, even on qcqs schemes of finite valuative dimension. Modulo any prime number $p$, this is a consequence of the connectivity bound [AMMN22, Theorem 5.1 (1)] and [EHIK21, Theorem 2.4.15]. The integral statement will be a consequence of certain cdh descent results in Chapter 5.

**Remark 3.4.4** (Exhaustivity of $\mathrm{Fil}^\star_{\mathrm{mot}} L_{\mathrm{cdh}} \mathrm{TC}$)**.** The filtration $\mathrm{Fil}^\star_{\mathrm{mot}} L_{\mathrm{cdh}} \mathrm{TC}$ is not exhaustive on general qcqs derived schemes. We will prove however, in Chapter 5, that the fibre of the natural map $\mathrm{Fil}^\star_{\mathrm{mot}} \mathrm{TC} \to \mathrm{Fil}^\star_{\mathrm{mot}} L_{\mathrm{cdh}} \mathrm{TC}$ is $\mathbb{N}$-indexed, and in particular exhaustive.

# Chapter 4

# Definition of motivic cohomology

In this chapter, we introduce motivic cohomology of general quasi-compact quasi-separated derived schemes (Definition 4.3.6) and establish some of the fundamental properties of the associated motivic filtration.

## 4.1 Classical motivic cohomology

In this section, we review the classical definition of motivic cohomology of smooth schemes in mixed characteristic. Following [Blo86, Lev01, Gei04], the motivic cohomology of smooth $\mathbb{Z}$-schemes $X$ is classically defined in terms of Bloch's cycle complexes $z^i(X,\bullet)$. Recall that Bloch's cycle complex is a simplicial abelian group defined in terms of algebraic cycles. The homotopy groups of Bloch's cycle complexes, called Bloch's higher Chow groups, are a generalisation of Chow groups that are designed to generalise the relation between the $\mathrm{K}_0$ and the Chow groups of a quasi-projective variety to higher $K$-groups. Via the Dold–Kan correspondence, we view Bloch's cycle complexes as objects of the derived category $\mathcal{D}(\mathbb{Z})$.

**Definition 4.1.1** (Classical motivic cohomology of smooth schemes)**.** Let $B$ be a field or a mixed characteristic Dedekind domain (*e.g.*, $B = \mathbb{Z}$), and $X$ be a smooth $B$-scheme. For any integer $i \in \mathbb{Z}$, the *classical motivic complex*

$$\mathbb{Z}(i)^{\mathrm{cla}}(X) \in \mathcal{D}(\mathbb{Z})$$

is the shift of Bloch's cycle complex $z^i(X,\bullet)$:

$$\mathbb{Z}(i)^{\mathrm{cla}}(X) := z^i(X,\bullet)[-2i],$$

where $\bullet$ is the cohomological index.

Note that, by construction, the classical motivic complexes $\mathbb{Z}(i)^{\mathrm{cla}}$ vanish in degrees more than $2i$, and in all degrees for weights $i < 0$.

In the following definition, we use the slice filtration in stable homotopy theory, as introduced by Voevodsky [Voe02a, Voe02b, BH21].

**Definition 4.1.2** (Motivic filtration on $K$-theory of smooth schemes)**.** Let $B$ be a field or a mixed characteristic Dedekind domain. The *classical motivic filtration* on algebraic $K$-theory of smooth $B$-schemes is the functor

$$\mathrm{Fil}^{\star}_{\mathrm{cla}}\mathrm{K}(-) : \mathrm{Sm}_B^{\mathrm{op}} \longrightarrow \mathrm{FilSp}$$

defined as the image, via the mapping spectrum construction $\omega^{\infty} : \mathrm{SH}(B) \to \mathrm{PSh}(\mathrm{Sm}_B, \mathrm{Sp})$, of the slice filtration $f^{\star}\mathrm{KGL}$ on the $K$-theory motivic spectrum $\mathrm{KGL} \in \mathrm{SH}(B)$.

**Remark 4.1.3.** The pullback of algebraic cycles being well-defined only along flat maps, it is not straightforward to prove that the classical motivic complexes $\mathbb{Z}(i)^{\mathrm{cla}}$ of Definition 4.1.1 are functorial. Over a field, Voevodsky overcomes this technicality by proving that Bloch's cycle complexes are represented in SH by the zeroth slice of the $K$-theory motivic spectrum KGL. Over a mixed characteristic Dedekind domain, this identification is proved by Bachmann [Bac22]. In particular, this means that Bloch's cycle complexes $z^i(-, \bullet)$, when seen as a construction taking values in the derived category $\mathcal{D}(\mathbb{Z})$, is indeed functorial, and multiplicative. In terms of Definitions 4.1.1 and 4.1.2, this implies that for every integer $i \in \mathbb{Z}$, there is an equivalence of $\mathcal{D}(\mathbb{Z})$-valued[1] functors

$$\mathbb{Z}(i)^{\mathrm{cla}}(-) := \mathrm{gr}^i_{\mathrm{cla}}\mathrm{K}(-)[-2i].$$

**Example 4.1.4** (Weight zero classical motivic cohomology)**.** For every smooth scheme $X$ over a field or a mixed characteristic Dedekind domain, there is a natural equivalence

$$\mathbb{Z}(0)^{\mathrm{cla}}(X) \simeq R\Gamma_{\mathrm{Zar}}(X, \mathbb{Z})$$

in the derived category $\mathcal{D}(\mathbb{Z})$ ([Spi18, Proposition 6.1]).

**Example 4.1.5** (Weight one classical motivic cohomology)**.** For every smooth scheme $X$ over a field or a mixed characteristic Dedekind domain, there is a natural equivalence

$$\mathbb{Z}(1)^{\mathrm{cla}}(X) \simeq R\Gamma_{\mathrm{Zar}}(X, \mathbb{G}_m)[-1]$$

in the derived category $\mathcal{D}(\mathbb{Z})$ ([Spi18, Theorem 7.10]). In particular, the complex $\mathbb{Z}(1)^{\mathrm{cla}}(X)$ is concentrated in degrees one and two, where it is given by

$$\mathrm{H}^1(\mathbb{Z}(1)^{\mathrm{cla}}(X)) \cong \mathcal{O}(X)^\times \quad \text{and} \quad \mathrm{H}^2(\mathbb{Z}(1)^{\mathrm{cla}}(X)) \cong \mathrm{Pic}(X).$$

## 4.2 The cdh-local motivic filtration

Following [BEM24], we review in this section the cdh-local motivic filtration on homotopy $K$-theory KH$(-)$ of qcqs schemes (Definition 4.2.5), whose shifted graded pieces $\mathbb{Z}(i)^{\mathrm{cdh}}$ will serve as a building block for the definition of the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$ (Remark 4.3.7). We first define the lisse motivic complexes $\mathbb{Z}(i)^{\mathrm{lisse}}$ as an intermediate construction, and as a practical tool for later chapters.

The following definition is motivated by the observation of Bhatt–Lurie that connective algebraic $K$-theory $\mathrm{K}^{\mathrm{conn}}(-)$ is left Kan extended on animated commutative rings from smooth $\mathbb{Z}$-algebras [EHK+20, Proposition A.0.4].

**Definition 4.2.1** (Motivic filtration on connective $K$-theory of animated rings)**.** The *motivic filtration on connective algebraic $K$-theory* of animated commutative rings is the functor

$$\mathrm{Fil}^\star_{\mathrm{lisse}}\mathrm{K}^{\mathrm{conn}}(-) : \mathrm{AniRings} \longrightarrow \mathrm{FilSp}$$

defined as the left Kan extension of the classical motivic filtration on algebraic $K$-theory of smooth $\mathbb{Z}$-algebras

$$\mathrm{Fil}^\star_{\mathrm{lisse}}\mathrm{K}^{\mathrm{conn}}(-) := \big(L_{\mathrm{AniRings}/\mathrm{Sm}_{\mathbb{Z}}}\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}\big)(-).$$

[1]Every value of the functor $\mathrm{gr}^i_{\mathrm{cla}}\mathrm{K}(-)[-2i]$ has a natural module structure over the $\mathbb{E}_\infty$-ring $\mathrm{gr}^0_{\mathrm{cla}}\mathrm{K}(\mathbb{Z})$, which, by [Spi18, Proposition 6.1] and [Bac22], is naturally identified with the $\mathbb{E}_\infty$-ring $H\mathbb{Z}$. This implies that the spectra-valued functor $\mathrm{gr}^i_{\mathrm{cla}}\mathrm{K}(-)[-2i]$ takes values in $H\mathbb{Z}$-linear spectra, *i.e.*, in the derived category $\mathcal{D}(\mathbb{Z})$.

Note that connective algebraic $K$-theory is not a Zariski sheaf on commutative rings. Most of our results on the following lisse motivic complexes $\mathbb{Z}(i)^{\text{lisse}}$ will be formulated in the generality of local rings.

**Definition 4.2.2** (Lisse motivic cohomology of animated rings)**.** For any integer $i \in \mathbb{Z}$, the *lisse motivic complex*

$$\mathbb{Z}(i)^{\text{lisse}}(-) : \text{AniRings} \longrightarrow \mathcal{D}(\mathbb{Z})$$

is the shifted graded piece of the motivic filtration of Definition 4.2.1:

$$\mathbb{Z}(i)^{\text{lisse}}(-) := \text{gr}^i_{\text{lisse}}\text{K}^{\text{conn}}(-)[-2i].$$

Note that the lisse motivic complexes $\mathbb{Z}(i)^{\text{lisse}}$ are the left Kan extension of the classical motivic complexes $\mathbb{Z}(i)^{\text{cla}}$, and in particular vanish in weights $i < 0$.

**Example 4.2.3** (Weight zero lisse motivic cohomology)**.** For every local ring $R$, there is a natural equivalence

$$\mathbb{Z}(0)^{\text{lisse}}(R) \simeq \mathbb{Z}[0]$$

in the derived category $\mathcal{D}(\mathbb{Z})$. This is a consequence of Example 4.1.4, by using that the functor $\mathbb{Z}(0)^{\text{lisse}}(-)$ is left Kan extended on local rings from its restriction to local essentially smooth $\mathbb{Z}$-algebras.

**Example 4.2.4** (Weight one lisse motivic cohomology)**.** For every commutative ring $R$, there is a natural equivalence

$$\mathbb{Z}(1)^{\text{lisse}}(R) \simeq \big(\tau^{\leq 1} R\Gamma_{\text{Zar}}(R, \mathbb{G}_m)\big)[-1]$$

in the derived category $\mathcal{D}(\mathbb{Z})$. In particular, the complex $\mathbb{Z}(1)^{\text{lisse}}(R) \in \mathcal{D}(\mathbb{Z})$ is concentrated in degrees one and two, where it is given by

$$\text{H}^1(\mathbb{Z}(1)^{\text{lisse}}(R)) \cong \mathcal{O}(R)^\times \quad \text{and} \quad \text{H}^2(\mathbb{Z}(1)^{\text{lisse}}(R)) \cong \text{Pic}(R).$$

This is a consequence of Example 4.1.5, by using that the left Kan extension of a functor taking values in degrees at most two takes values in degrees at most two, and that the functor $\tau^{\leq 1} R\Gamma_{\text{Zar}}(-, \mathbb{G}_m)$ on commutative rings is left Kan extended from its restriction to smooth $\mathbb{Z}$-algebras. Here the latter left Kan extension property is a consequence of the same left Kan extension property for the functors $\mathbb{G}_m(-)$ (which is a special case of Mathew's criterion [EHK+20, Proposition A.0.4]) and $\text{Pic}(-)$ (which is a consequence of rigidity, see [EM23, Lemma 7.6]).

**Definition 4.2.5** (Cdh-local motivic filtration on $KH$-theory of schemes)**.** The *cdh-local motivic filtration on homotopy $K$-theory* of qcqs schemes is the functor

$$\text{Fil}^\star_{\text{cdh}}\text{KH}(-) : \text{Sch}^{\text{qcqs,op}} \longrightarrow \text{FilSp}$$

defined as

$$\text{Fil}^\star_{\text{cdh}}\text{KH}(-) := \big(L_{\text{cdh}}\text{Fil}^\star_{\text{lisse}}\text{K}^{\text{conn}}\big)(-) = \big(L_{\text{cdh}} L_{\text{Sch}^{\text{qcqs,op}}/\text{Sm}^{\text{op}}_{\mathbb{Z}}} \text{Fil}^\star_{\text{cla}}\text{K}\big)(-).$$

**Remark 4.2.6.** By construction of the cdh-local motivic filtration (Definition 4.2.5), there is a natural comparison map of presheaves

$$\text{Fil}^\star_{\text{cla}}\text{K}(-) \longrightarrow \text{Fil}^\star_{\text{cdh}}\text{KH}(-)$$

on smooth $\mathbb{Z}$-schemes.

**Definition 4.2.7** (Cdh-local motivic cohomology of schemes)**.** For any integer $i \in \mathbb{Z}$, the *cdh-local motivic complex*

$$\mathbb{Z}(i)^{\mathrm{cdh}}(-) : \mathrm{Sch}^{\mathrm{qcqs,op}} \longrightarrow \mathcal{D}(\mathbb{Z})$$

is the shifted graded piece of the motivic filtration of Definition 4.2.5:

$$\mathbb{Z}(i)^{\mathrm{cdh}}(-) := \mathrm{gr}^i_{\mathrm{cdh}}\mathrm{KH}(-)[-2i].$$

Although we will refer throughout the text to [BEM24] for the properties of these cdh-local motivic complexes $\mathbb{Z}(i)^{\mathrm{cdh}}$, we already mention the following result, as it will play an important role to establish the completeness of the motivic filtration on algebraic $K$-theory.

**Proposition 4.2.8** (Completeness of the cdh-local motivic filtration, after [BEM24])**.** *Let $d \geq 0$ be an integer, and $X$ be a qcqs scheme of valuative dimension at most $d$. Then for every integer $i \in \mathbb{Z}$, the spectrum $\mathrm{Fil}^i_{\mathrm{cdh}}\mathrm{KH}(X)$ is in cohomological degrees at most $-i+d$. In particular, the filtration $\mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{KH}(X)$, and its rationalisation $\mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{KH}(X;\mathbb{Q})$, are complete.*

## 4.3 Definition of motivic cohomology

In this section, we introduce the motivic filtration on algebraic $K$-theory of qcqs derived schemes (Definitions 4.3.4 and 4.3.5), by constructing a filtered extension of the cartesian square of Theorem 2.1.1. To do so, we first define a filtered cdh-local cyclotomic trace map $\mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{KH}(-) \to \mathrm{Fil}^\star_{\mathrm{mot}}L_{\mathrm{cdh}}\mathrm{TC}(-)$ (Construction 4.3.3).

**Theorem 4.3.1** (Filtered cyclotomic trace in the smooth case)**.** *The cyclotomic trace map*

$$\mathrm{K}(-) \longrightarrow \mathrm{TC}(-),$$

*viewed as a map of spectra-valued presheaves on the category of smooth $\mathbb{Z}$-schemes, admits a unique, multiplicative extension to a map*

$$\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(-) \longrightarrow \mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(-)$$

*of filtered presheaves of spectra.*

*Proof.* By Definition 3.3.3, the natural cartesian square

$$\begin{array}{ccc} \mathrm{TC}(-) & \longrightarrow & \mathrm{HC}^-(-) \\ \downarrow & & \downarrow \\ \prod_{p\in\mathbb{P}} \mathrm{TC}(-;\mathbb{Z}_p) & \longrightarrow & \prod_{p\in\mathbb{P}} \mathrm{HC}^-(-;\mathbb{Z}_p) \end{array}$$

admits a multiplicative filtered extension

$$\begin{array}{ccc} \mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(-) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(-) \\ \downarrow & & \downarrow \\ \prod_{p\in\mathbb{P}} \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p) & \longrightarrow & \prod_{p\in\mathbb{P}} \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(-;\mathbb{Z}_p) \end{array}$$

on qcqs derived schemes. Let $p$ be a prime number. It then suffices to prove that the natural maps

$$\mathrm{K}(-) \longrightarrow \mathrm{HC}^-(-),\ \mathrm{K}(-) \longrightarrow \mathrm{HC}^-(-;\mathbb{Z}_p),\ \text{and}\ \mathrm{K}(-) \longrightarrow \mathrm{TC}(-;\mathbb{Z}_p),$$

viewed as maps of spectra-valued presheaves on the category of smooth $\mathbb{Z}$-schemes, admit unique multiplicative filtered extensions to maps of filtered presheaves of spectra

$$\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(-) \longrightarrow \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(-), \quad \mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(-) \longrightarrow \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(-;\mathbb{Z}_p), \text{ and}$$

$$\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(-) \longrightarrow \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)$$

respectively.

The proof of [EM23, Proposition 4.6], which is stated over a quasismooth map of commutative rings $k_0 \to k$ such that $k$ is a field, applies readily to the case where $k_0 = k = \mathbb{Z}$. More precisely, we use in this proof the Gersten conjecture for classical motivic cohomology on smooth $\mathbb{Z}$-schemes, which is [Gei04, Theorem 1.1]. In particular, the natural map $\mathrm{K}(-) \to \mathrm{HC}^-(-)$, viewed as a map of spectra-valued presheaves on the category of smooth $\mathbb{Z}$-schemes, admits a unique multiplicative filtered extension to a map of filtered presheaves of spectra $\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(-) \to \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(-)$.

Similarly, the $p$-completed cotangent complex $(\mathbb{L}_{R/\mathbb{Z}})^\wedge_p$ of a smooth $\mathbb{Z}$-algebra $R$ is concentrated in degree zero, given by the $p$-flat $R$-module $(\Omega^1_{R/\mathbb{Z}})^\wedge_p$. So for every integer $i \in \mathbb{Z}$, this implies that the object $\mathrm{gr}^i_{\mathrm{HKR}}\mathrm{HC}^-(R;\mathbb{Z}_p) \in \mathcal{D}(\mathbb{Z})$ is in cohomological degrees less than $-i$. The proof of [EM23, Proposition 4.6] then adapts readily to prove that the natural map $\mathrm{K}(-) \to \mathrm{HC}^-(-;\mathbb{Z}_p)$, viewed as a map of spectra-valued presheaves on the category of smooth $\mathbb{Z}$-schemes, admits a unique multiplicative extension to a map of filtered presheaves of spectra $\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(-) \to \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(-;\mathbb{Z}_p)$.

Finally, the natural map $\mathrm{K}(-) \to \mathrm{TC}(-;\mathbb{Z}_p)$, viewed as a map of spectra-valued presheaves on the category of smooth $\mathbb{Z}$-schemes, admits a unique multiplicative extension to a map of filtered presheaves of spectra $\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(-) \to \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)$, by [AHI24, Proposition 6.12]. □

**Remark 4.3.2.** For every prime number $p$, the BMS filtration $\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)$ is determined by its $p$-quasisyntomic-local values (Theorem 3.2.11). By the proof of [AHI24, Lemma 6.10], the $p$-completed left Kan extension of the functor $\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(-)$, from smooth $\mathbb{Z}$-algebras to $p$-quasisyntomic rings, is $p$-quasisyntomic-locally identified, via the map induced by [AHI24, Proposition 6.12], with the functor $\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)$. In particular, one can reconstruct the BMS filtration $\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)$ on qcqs derived schemes (Definition 3.2.6) from the classical motivic filtration $\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(-)$ on smooth $\mathbb{Z}$-schemes (Definition 4.1.2). This will be used in Chapter 5 to construct Adams operations on the BMS filtration $\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)$.

**Construction 4.3.3** (Filtered cdh-local cyclotomic trace)**.** The *filtered cdh-local cyclotomic trace map* is the map

$$\mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{KH}(-) \longrightarrow \mathrm{Fil}^\star_{\mathrm{mot}}L_{\mathrm{cdh}}\mathrm{TC}(-)$$

of functors from (the opposite category of) qcqs schemes to the category of multiplicative filtered spectra defined as the cdh sheafification of the composite

$$\big(L_{\mathrm{Sch}^{\mathrm{qcqs,op}}/\mathrm{Sm}^{\mathrm{op}}_{\mathbb{Z}}}\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}\big)(-) \longrightarrow \big(L_{\mathrm{Sch}^{\mathrm{qcqs,op}}/\mathrm{Sm}^{\mathrm{op}}_{\mathbb{Z}}}\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}\big)(-) \longrightarrow \mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(-),$$

where the first map is the map induced by Theorem 4.3.1 and the second map is the canonical map. Note here that sheafification is a multiplicative operation, and that the compatibility between left Kan extension and multiplicative structures is ensured by [Lur17, Corollary 3.2.3.2] (see also [EM23, 2.3.2]).

**Definition 4.3.4** (Motivic filtration on $K$-theory of schemes)**.** The *motivic filtration on non-connective algebraic $K$-theory* of qcqs schemes is the functor

$$\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{K}(-) : \mathrm{Sch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

defined by the cartesian square of functors of multiplicative filtered spectra

$$\begin{array}{ccc} \mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{K}(-) & \longrightarrow & \mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{TC}(-) \\ \downarrow & & \downarrow \\ \mathrm{Fil}^{\star}_{\mathrm{cdh}}\mathrm{KH}(-) & \longrightarrow & \mathrm{Fil}^{\star}_{\mathrm{mot}}L_{\mathrm{cdh}}\mathrm{TC}(-), \end{array}$$

where the bottom horizontal map is the map of Construction 4.3.3, and the right vertical map is cdh sheafification.

**Definition 4.3.5** (Motivic filtration on $K$-theory of derived schemes)**.** The *motivic filtration on non-connective algebraic $K$-theory* of qcqs derived schemes is the functor

$$\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{K}(-) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

defined by the cartesian square of functors of multiplicative filtered spectra

$$\begin{array}{ccc} \mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{K}(-) & \longrightarrow & \mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{TC}(-) \\ \downarrow & & \downarrow \\ \mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{K}(\pi_0(-)) & \longrightarrow & \mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{TC}(\pi_0(-)) \end{array}$$

where $\pi_0(-) : \mathrm{dSch} \to \mathrm{Sch}$ is restriction to the classical locus, the filtration on $\mathrm{K}(\pi_0(-))$ is given by Definition 4.3.4, and the filtrations on $\mathrm{TC}(-)$ and $\mathrm{TC}(\pi_0(-))$ are given by Definition 3.3.3.

**Definition 4.3.6** (Motivic cohomology of derived schemes)**.** For any integer $i \in \mathbb{Z}$, the *motivic complex*

$$\mathbb{Z}(i)^{\mathrm{mot}} : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathcal{D}(\mathbb{Z})$$

is the shifted graded piece of the motivic filtration of Definition 4.3.5:

$$\mathbb{Z}(i)^{\mathrm{mot}}(-) := \mathrm{gr}^{i}_{\mathrm{mot}}\mathrm{K}(-)[-2i].$$

For every qcqs derived scheme $X$, also denote by

$$\mathrm{H}^n_{\mathrm{mot}}(X, \mathbb{Z}(i)) := \mathrm{H}^n(\mathbb{Z}(i)^{\mathrm{mot}}(X)) \quad n \in \mathbb{Z}$$

the *motivic cohomology groups* of $X$.

**Remark 4.3.7** (Motivic cohomology of schemes)**.** Let $X$ be a qcqs scheme, and $i \in \mathbb{Z}$ be an integer. By Definition 4.3.5, there is a natural cartesian square

$$\begin{array}{ccc} \mathbb{Z}(i)^{\mathrm{mot}}(X) & \longrightarrow & \mathbb{Z}(i)^{\mathrm{TC}}(X) \\ \downarrow & & \downarrow \\ \mathbb{Z}(i)^{\mathrm{cdh}}(X) & \longrightarrow & \big(L_{\mathrm{cdh}}\,\mathbb{Z}(i)^{\mathrm{TC}}\big)(X) \end{array}$$

in the derived category $\mathcal{D}(\mathbb{Z})$, where the bottom horizontal map is induced by Construction 4.3.3 and the right vertical map is cdh sheafification. This cartesian square can serve as a definition for the motivic cohomology of the scheme $X$.

We now construct, for later use, a comparison map from classical motivic cohomology to the motivic cohomology of Definition 4.3.6.

**Definition 4.3.8** (Filtered classical-motivic comparison map)**.** The *filtered classical-motivic comparison map* is the map of presheaves

$$\mathrm{Fil}^{\star}_{\mathrm{cla}}\mathrm{K}(-) \longrightarrow \mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{K}(-)$$

on smooth $\mathbb{Z}$-schemes induced by the maps Remark 4.2.6 and Theorem 4.3.1. Note here that the compatibility between these two maps and Definition 4.3.4 is automatic by Construction 4.3.3.

**Definition 4.3.9** (Classical-motivic comparison map)**.** For any integer $i \in \mathbb{Z}$, the *classical-motivic comparison map* is the map of $\mathcal{D}(\mathbb{Z})$-valued presheaves

$$\mathbb{Z}(i)^{\mathrm{cla}}(-) \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(-)$$

on smooth $\mathbb{Z}$-schemes induced by taking the $i^{\mathrm{th}}$ shifted graded piece of the filtered map of Definition 4.3.8.

In the rest of this chapter, we discuss some of the first properties of the motivic filtration.

**Remark 4.3.10** (Comparison to cdh-local motivic cohomology)**.** By construction (Definition 4.2.5), the cdh-local motivic complex

$$\mathbb{Z}(i)^{\mathrm{cdh}} : \mathrm{Sch}^{\mathrm{qcqs,op}} \longrightarrow \mathcal{D}(\mathbb{Z})$$

is a cdh sheaf, so the common fibre of the horizontal maps in the cartesian square of Remark 4.3.7 is also a cdh sheaf. In particular, for every qcqs scheme $X$, the left vertical map of this cartesian square exhibits cdh-local motivic cohomology as the cdh sheafification of motivic cohomology:

$$\mathbb{Z}(i)^{\mathrm{cdh}}(X) \simeq \big(L_{\mathrm{cdh}}\,\mathbb{Z}(i)^{\mathrm{mot}}\big)(X).$$

**Proposition 4.3.11.** *Let $X$ be a qcqs scheme, and $p$ be a prime number. Then for every integer $k \geq 1$, there is a natural cartesian square*

$$\begin{array}{ccc}
\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{K}(X;\mathbb{Z}/p^k) & \longrightarrow & \mathrm{Fil}^{\star}_{\mathrm{BMS}}\mathrm{TC}(X;\mathbb{Z}/p^k) \\
\downarrow & & \downarrow \\
\mathrm{Fil}^{\star}_{\mathrm{cdh}}\mathrm{KH}(X;\mathbb{Z}/p^k) & \longrightarrow & \mathrm{Fil}^{\star}_{\mathrm{BMS}}L_{\mathrm{cdh}}\mathrm{TC}(X;\mathbb{Z}/p^k)
\end{array}$$

*of filtered spectra.*

*Proof.* This is a consequence of Proposition 3.3.6 and Definition 4.3.5. □

**Corollary 4.3.12.** *Let $X$ be a qcqs scheme, and $p$ be a prime number. Then for any integers $i \in \mathbb{Z}$ and $k \geq 1$, there is natural cartesian square*

$$\begin{array}{ccc}
\mathbb{Z}/p^k(i)^{\mathrm{mot}}(X) & \longrightarrow & \mathbb{Z}/p^k(i)^{\mathrm{BMS}}(X) \\
\downarrow & & \downarrow \\
\mathbb{Z}/p^k(i)^{\mathrm{cdh}}(X) & \longrightarrow & \big(L_{\mathrm{cdh}}\,\mathbb{Z}/p^k(i)^{\mathrm{BMS}}\big)(X)
\end{array}$$

*in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$.*

*Proof.* This is a direct consequence of Proposition 4.3.11. □

**Remark 4.3.13** ($\ell$-adic motivic cohomology)**.** For any prime number $p$ and integer $k \geq 1$, the filtered presheaf $\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}/p^k)$ and its cdh sheafification vanish on qcqs $\mathbb{Z}[\frac{1}{p}]$-schemes. In particular, Proposition 4.3.11 implies that for every qcqs $\mathbb{Z}[\frac{1}{p}]$-scheme $X$, the natural map

$$\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X;\mathbb{Z}/p^k) \longrightarrow \mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{KH}(X;\mathbb{Z}/p^k)$$

is an equivalence of filtered spectra.

**Remark 4.3.14** (Completeness and exhaustivity of $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}$)**.** The filtration $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X)$ of Definition 4.3.5 will be proved to be $\mathbb{N}$-indexed, hence exhaustive, on general qcqs derived schemes $X$ (Proposition 5.5.1), and complete on qcqs schemes of finite valuative dimension (Proposition 5.5.4). Note that these results can already be proved modulo any prime number $p$, as a formal consequence of Proposition 4.3.11 and Chapter 3.

The following result is a filtered version of the classical Dundas–Goodwillie–McCarthy theorem ([DGM13, Theorem 7.0.0.2]).

**Proposition 4.3.15.** *Let $A \to B$ be a map of animated commutative rings such that the induced map $\pi_0(A) \to \pi_0(B)$ of commutative rings is surjective with finitely generated nilpotent kernel. Then the natural commutative diagram*

$$\begin{array}{ccc} \mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(A) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(A) \\ \downarrow & & \downarrow \\ \mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(B) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(B) \end{array}$$

*is a cartesian square of filtered spectra.*

*Proof.* By Definition 4.3.5, it suffices to prove that the natural commutative diagram

$$\begin{array}{ccc} \mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(\pi_0(A)) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(\pi_0(A)) \\ \downarrow & & \downarrow \\ \mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(\pi_0(B)) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(\pi_0(B)) \end{array}$$

is a cartesian square of filtered spectra. For every cdh sheaf $F$ defined on qcqs schemes, the natural map $F(\pi_0(A)) \to F(\pi_0(B))$ is an equivalence. The result is then a consequence of Definition 4.3.4. □

# Chapter 5

# Rational structure of motivic cohomology

In this chapter, we prove some fundamental properties of the motivic filtration that we introduced in the previous chapter: namely, that it is always exhaustive and finitary, and complete on schemes of finite valuative dimension. These properties modulo a prime number $p$ are a formal consequence of the analogous properties for the BMS filtration on $p$-completed topological cyclic homology. Proving these results integrally will however require more understanding of the rational part of this motivic filtration. Our two main results on rational motivic cohomology are as follows.

The first main result is the following generalisation of the rational splitting of the classical motivic filtration.

**Theorem 5.0.1** (The motivic filtration is rationally split)**.** *Let $X$ be a qcqs derived scheme. Then there exists a natural multiplicative equivalence of filtered spectra*

$$\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{K}(X;\mathbb{Q}) \simeq \bigoplus_{j\geq\star} \mathbb{Q}(j)^{\mathrm{mot}}(X)[2j].$$

As for the classical motivic filtration, this splitting is induced by suitable Adams operations, which we construct in the generality of qcqs derived schemes in the first section.

This result is however not enough to prove the exhaustivity, completeness, or finitariness of the motivic filtration. Instead, these will be proved along the way to the following second main result on rational motivic cohomology.

**Theorem 5.0.2.** *Let $X$ be a qcqs scheme. Then there is a natural fibre sequence of filtered spectra*

$$\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{K}(X;\mathbb{Q}) \to \mathrm{Fil}^{\star}_{\mathrm{cdh}}\mathrm{KH}(X;\mathbb{Q}) \to \mathrm{cofib}\big(\mathrm{Fil}^{\star-1}_{\mathrm{HKR}}\mathrm{HC}(X_{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Fil}^{\star-1}_{\mathrm{HKR}}L_{\mathrm{cdh}}\mathrm{HC}(X_{\mathbb{Q}}/\mathbb{Q})\big)[1].$$

To prove this result, we use a classical argument of Weibel at the level of $K$-theory (Lemma 5.6.1) and the rational splitting Theorem 5.0.1 to reduce the statement to the case of characteristic zero, where the result is essentially [EM23, Theorem 4.10 (3)].

The key step in this argument, in order to pass from a statement at the level of $K$-theory to a filtered statement, is to prove beforehand that the motivic filtration $\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{K}$ is $\mathbb{N}$-indexed (Proposition 5.5.1). The strategy to prove this is as follows. We first introduce a new rigid-analytic variant of the HKR filtrations in the generality of qcqs derived schemes, whose graded pieces are rigid-analytic variants of derived de Rham cohomology. We then adapt a theorem

Goodwillie –stating in modern language that periodic cyclic homology is truncating in characteristic zero– to this rigid-analytic variant of periodic cyclic homology. This rigid-analytic Goodwillie theorem implies, by the work of Land–Tamme on truncating invariants, that the rigid-analytic variant of periodic cyclic homology is a cdh sheaf on qcqs schemes. A filtered consequence of this cdh descent result then formally implies the desired result, *i.e.*, that the motivic filtration $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}$ is $\mathbb{N}$-indexed.

## 5.1 Adams operations

In this section we prove Theorem 5.0.1, using Adams operations.

In [EM23, Appendix B], Elmanto–Morrow construct Adams operations $\psi^m$ on the $m$-periodic filtered $K$-theory $\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(X)[\frac{1}{m}]$ of smooth $\mathbb{Z}$-schemes $X$, acting on the $i^{\mathrm{th}}$ graded piece as multiplication by $m^i$. Using this construction, Bachmann–Elmanto–Morrow construct the following Adams operations $\psi^m$ on the filtered $KH$-theory of arbitrary qcqs schemes $X$.

**Proposition 5.1.1** (Adams operations on filtered $KH$-theory, [BEM24])**.** *Let $m \geq 2$ be an integer, and $X$ be a qcqs scheme. Then there exists a natural automorphism $\psi^m$ of the filtered spectrum $\mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{KH}(X)[\frac{1}{m}]$ such that for every integer $i \in \mathbb{Z}$, the induced automorphism on the $i^{\mathrm{th}}$ graded piece $\mathbb{Z}[\frac{1}{m}](i)^{\mathrm{cdh}}(X)[2i]$ is multiplication by $m^i$.*

We now use the Adams operations $\psi^m$ on the $m$-periodic filtered $K$-theory $\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(X)[\frac{1}{m}]$ of smooth $\mathbb{Z}$-schemes $X$ ([EM23, Appendix $B$]) to construct Adams operations $\psi^m$ on the filtered $p$-completed topological cyclic homology $\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(X;\mathbb{Z}_p)$ of qcqs $\mathbb{Z}[\frac{1}{m}]$-schemes $X$.

**Proposition 5.1.2** (Adams operations on filtered $\mathrm{TC}(-;\mathbb{Z}_p)$)**.** *Let $m \geq 2$ be an integer, $X$ be a qcqs derived $\mathbb{Z}[\frac{1}{m}]$-scheme, and $p$ be a prime number. Then there exists a natural multiplicative automorphism $\psi^m$ of the filtered spectrum $\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(X;\mathbb{Z}_p)$ such that for every integer $i \in \mathbb{Z}$, the induced automorphism on the $i^{\mathrm{th}}$ graded piece $\mathbb{Z}_p(i)^{\mathrm{BMS}}(X)[2i]$ is multiplication by $m^i$. Moreover, this automorphism $\psi^m$ is uniquely determined by its naturality and the fact that on smooth $\mathbb{Z}[\frac{1}{m}]$-schemes $X$, the diagram of filtered spectra*

$$\begin{array}{ccc} \mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(X)[\frac{1}{m}] & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(X;\mathbb{Z}_p) \\ \downarrow{\scriptstyle \psi^m} & & \downarrow{\scriptstyle \psi^m} \\ \mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(X)[\frac{1}{m}] & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(X;\mathbb{Z}_p) \end{array}$$

*where the horizontal maps are induced by Remark 4.3.2, and the left vertical map is defined in [EM23, Construction B.4], is commutative.*

*Proof.* By Remark 4.3.2, the functor $\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)$ on $p$-quasisyntomic rings is $p$-quasisyntomic-locally identified with the $p$-completed left Kan extension of the functor $\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(-)$ from smooth $\mathbb{Z}$-algebras to $p$-quasisyntomic rings. The functor $\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)$ on $p$-quasisyntomic rings moreover satisfies $p$-quasisyntomic descent (Theorem 3.2.11 (1)), so the Adams operation $\psi^m$ on the functor $\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(-)[\frac{1}{m}]$ ([EM23, Appendix B]) induces a natural automorphism $\psi^m$ of the presheaf $\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)$, acting as multiplication by $m^i$ on the $i^{\mathrm{th}}$ graded piece $\mathbb{Z}_p(i)^{\mathrm{BMS}}(-)[2i]$. The same result then applies to animated commutative $\mathbb{Z}[\frac{1}{m}]$-algebras by left Kan extending the result on polynomial $\mathbb{Z}[\frac{1}{m}]$-algebras (Theorem 3.2.11 (2)), and to general qcqs derived $\mathbb{Z}[\frac{1}{m}]$-schemes by Zariski sheafifying the result on animated commutative $\mathbb{Z}[\frac{1}{m}]$-algebras. □

The following result is [Rak20, Construction 6.4.8 and Proposition 6.4.12]. Note that if $X$ is a qcqs derived $\mathbb{Z}[\frac{1}{m}]$-scheme, then the filtered spectrum $\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X)$ is naturally $\mathbb{Z}[\frac{1}{m}]$-linear.

**Proposition 5.1.3** (Adams operations on filtered $\mathrm{HC}^-$, [Rak20])**.** *Let $m \geq 2$ be an integer, and $X$ be qcqs derived $\mathbb{Z}[\frac{1}{m}]$-scheme. Then there exists a natural multiplicative automorphism $\psi^m$ of the filtered spectrum $\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X)$ such that for every integer $i \in \mathbb{Z}$, the induced automorphism on the $i^{\mathrm{th}}$ graded piece $\widehat{\mathbb{L}\Omega}^{\geq i}_{X/\mathbb{Z}[\frac{1}{m}]}[2i]$ is multiplication by $m^i$.*

**Lemma 5.1.4.** *Let $m \geq 2$ be an integer, $X$ be a qcqs derived $\mathbb{Z}[\frac{1}{m}]$-scheme, and $p$ be a prime number. Then the natural diagram of filtered spectra*

$$\begin{array}{ccc}
\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(X;\mathbb{Z}_p) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X;\mathbb{Z}_p) \\
\big\downarrow{\scriptstyle \psi^m} & & \big\downarrow{\scriptstyle \psi^m} \\
\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(X;\mathbb{Z}_p) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X;\mathbb{Z}_p)
\end{array}$$

*where the horizontal maps are defined in Construction 3.3.2, the left map is the map defined in Proposition 5.1.2, and the right map is the map induced by Proposition 5.1.3, is commutative.*

*Proof.* By [EM23, Lemma B.8], the natural diagram of filtered spectra

$$\begin{array}{ccc}
\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(X)[\frac{1}{m}] & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X) \\
\big\downarrow{\scriptstyle \psi^m} & & \big\downarrow{\scriptstyle \psi^m} \\
\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(X)[\frac{1}{m}] & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X)
\end{array}$$

is commutative for every smooth $\mathbb{Z}[\frac{1}{m}]$-scheme $X$. The result is then a consequence of Proposition 5.1.2, where the compatibility between the filtered maps is a consequence of the proof of Theorem 4.3.1. ☐

**Construction 5.1.5** (Adams operations on filtered TC)**.** Let $m \geq 2$ be an integer, and $X$ be a qcqs derived $\mathbb{Z}[\frac{1}{m}]$-scheme. The automorphism $\psi^m$ of the filtered spectrum $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(X)$ is the automorphism defined by pullback along the natural cartesian square of filtered spectra

$$\begin{array}{ccc}
\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(X) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X) \\
\big\downarrow & & \big\downarrow \\
\prod_{p\in\mathbb{P}}\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(X;\mathbb{Z}_p) & \longrightarrow & \prod_{p\in\mathbb{P}}\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X;\mathbb{Z}_p),
\end{array}$$

where the automorphism $\psi^m$ of $\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X)$ is the automorphism of Proposition 5.1.3, the automorphism $\psi^m$ of $\prod_{p\in\mathbb{P}}\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X;\mathbb{Z}_p)$ is induced by the automorphism $\psi^m$ of $\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X)$, and the automorphism $\psi^m$ of $\prod_{p\in\mathbb{P}}\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(X;\mathbb{Z}_p)$ is the automorphism of Proposition 5.1.2. Note here that the compatibility between the automorphisms $\psi^m$ and the bottom map is given by Lemma 5.1.4.

Note in the following result that if $X$ is a qcqs derived $\mathbb{Z}[\frac{1}{m}]$-scheme, then the filtered spectrum $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(X)$ is naturally $\mathbb{Z}[\frac{1}{m}]$-linear.

**Corollary 5.1.6.** *Let $m \geq 2$ be an integer, and $X$ be a qcqs derived $\mathbb{Z}[\frac{1}{m}]$-scheme. Then for every integer $i \in \mathbb{Z}$, the automorphism $\psi^m$ induced on the $i^{\text{th}}$ graded piece $\mathbb{Z}(i)^{\mathrm{TC}}(X)[2i]$ of the filtered spectrum $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(X)$ is multiplication by $m^i$. Moreover, if $X$ is smooth over $\mathbb{Z}[\frac{1}{m}]$, then the natural diagram of filtered spectra*

$$\begin{array}{ccc} \mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(X)[\frac{1}{m}] & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(X) \\ \downarrow{\scriptstyle \psi^m} & & \downarrow{\scriptstyle \psi^m} \\ \mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(X)[\frac{1}{m}] & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(X) \end{array}$$

*where the horizontal maps are defined in Theorem 4.3.1, and the left vertical map is defined in [EM23, Construction* B.4*], is commutative.*

*Proof.* The identification of the automorphism $\psi^m$ on the graded pieces is a consequence of Propositions 5.1.2 and 5.1.3. The second statement is a consequence of the analogous compatibilities for $\prod_{p\in\mathbb{P}} \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(X;\mathbb{Z}_p)$ (Proposition 5.1.2) and for $\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X)$ ([EM23, Lemma B.8]), and of Lemma 5.1.4. □

**Lemma 5.1.7.** *Let $m \geq 2$ be an integer, and $X$ be a qcqs $\mathbb{Z}[\frac{1}{m}]$-scheme. Then the natural diagram of filtered spectra*

$$\begin{array}{ccc} \mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{KH}(X)[\frac{1}{m}] & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{mot}}L_{\mathrm{cdh}}\mathrm{TC}(X) \\ \downarrow{\scriptstyle \psi^m} & & \downarrow{\scriptstyle \psi^m} \\ \mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{KH}(X)[\frac{1}{m}] & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{mot}}L_{\mathrm{cdh}}\mathrm{TC}(X) \end{array}$$

*where the horizontal maps are defined in Construction 4.3.3, the left map is the map of Proposition 5.1.1, and the right map is the map induced by Construction 5.1.5, is commutative.*

*Proof.* The left map $\psi^m$ of this diagram is defined by cdh sheafifying the left Kan extension from smooth $\mathbb{Z}[\frac{1}{m}]$-schemes to qcqs $\mathbb{Z}[\frac{1}{m}]$-schemes of the automorphism $\psi^m$ on $\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(-)[\frac{1}{m}]$ ([EM23, Appendix B]). The result then follows from Construction 4.3.3 and Corollary 5.1.6. □

**Construction 5.1.8** (Adams operations on filtered $K$-theory)**.** Let $m \geq 2$ be an integer. Following Definition 4.3.4, if $X$ is a qcqs $\mathbb{Z}[\frac{1}{m}]$-scheme, the automorphism $\psi^m$ of the filtered spectrum $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X)[\frac{1}{m}]$ is the automorphism defined by pullback along the natural cartesian square of filtered spectra

$$\begin{array}{ccc} \mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X)[\frac{1}{m}] & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(X) \\ \downarrow & & \downarrow \\ \mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{KH}(X)[\frac{1}{m}] & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{mot}}L_{\mathrm{cdh}}\mathrm{TC}(X), \end{array}$$

where the automorphism $\psi^m$ of $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(X)$ is the automorphism of Construction 5.1.5, the automorphism $\psi^m$ of $\mathrm{Fil}^\star_{\mathrm{mot}}L_{\mathrm{cdh}}\mathrm{TC}(X)$ is defined by cdh sheafifying the automorphism $\psi^m$ of $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(-)$, and the automorphism $\psi^m$ of $\mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{KH}(X)[\frac{1}{m}]$ is the automorphism of Proposition 5.1.1. Note here that the compatibility between the automorphisms $\psi^m$ and the bottom map is given by Lemma 5.1.7.

Following Definition 4.3.5, if $X$ is a qcqs derived $\mathbb{Z}[\frac{1}{m}]$-scheme, the automorphism $\psi^m$ of the filtered spectrum $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X)[\frac{1}{m}]$ is the automorphism defined by pullback along the natural cartesian square of filtered spectra

$$\begin{array}{ccc} \mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X)[\frac{1}{m}] & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(X) \\ \downarrow & & \downarrow \\ \mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(\pi_0(X))[\frac{1}{m}] & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(\pi_0(X)), \end{array}$$

where the automorphisms $\psi^m$ of $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(X)$ and $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(\pi_0(X))$ are defined in Construction 5.1.5, and the automorphism $\psi^m$ of $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(\pi_0(X))[\frac{1}{m}]$ is the automorphism of the previous paragraph. Note here that the compatibility between the automorphisms $\psi^m$ is automatic by construction.

**Corollary 5.1.9.** *Let $m \geq 2$ be an integer, and $X$ be a qcqs derived $\mathbb{Z}[\frac{1}{m}]$-scheme. Then for every integer $i \in \mathbb{Z}$, the automorphism $\psi^m$ induced on the $i^{\mathrm{th}}$ graded piece $\mathbb{Z}[\frac{1}{m}](i)^{\mathrm{mot}}(X)[2i]$ of the filtered spectrum $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X)[\frac{1}{m}]$ is multiplication by $m^i$.*

*Proof.* This is a consequence of Proposition 5.1.1 and Corollary 5.1.6. □

The following lemma explains how to use Adams operations to deduce splitting results on the rationalisation of certain filtrations.

**Lemma 5.1.10.** *Let*

$$\mathrm{Fil}^\star F(-) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

*be a Zariski sheaf of filtered spectra. For each integer $m \geq 2$, let $\psi^m$ be a natural multiplicative automorphism of the filtered spectrum $\mathrm{Fil}^\star F(X)$ on qcqs derived $\mathbb{Z}[\frac{1}{m}]$-schemes $X$, satisfying the following properties:*

*(i) for every qcqs derived scheme $X$, the rationalised filtration $\mathrm{Fil}^\star F(X;\mathbb{Q})$ is complete;*

*(ii) for any integers $i \in \mathbb{Z}$ and $m \geq 2$, and every qcqs derived $\mathbb{Z}[\frac{1}{m}]$-scheme $X$, the induced automorphism on the $i^{\mathrm{th}}$ graded piece $\mathrm{gr}^i F(X)$ is multiplication by $m^i$;*

*(iii) for any integers $m, m' \geq 2$, and every qcqs derived $\mathbb{Z}[\frac{1}{mm'}]$-scheme $X$, the space of natural transformations from $\psi^m \circ \psi^{m'}$ to $\psi^{mm'}$, as endomorphisms of the filtered spectrum $\mathrm{Fil}^\star F(X)$, is contractible.*

*Then for every qcqs derived scheme $X$, and any integers $i, k \in \mathbb{Z}$ such that $k \geq i$, there exists a natural equivalence of spectra*

$$\mathrm{Fil}^i F(X;\mathbb{Q}) \simeq \big(\bigoplus_{i \leq j < k} \mathrm{gr}^j F(X;\mathbb{Q})\big) \oplus \mathrm{Fil}^k F(X;\mathbb{Q}).$$

*Proof.* For every spectrum $C$ equipped with a map $F : C \to C$, denote by $C^F$ the homotopy fibre of the map $F$. Let $m \geq 2$ be an integer, $i, k \in \mathbb{Z}$ be integers such that $k \geq i$, and $X$ be a qcqs derived $\mathbb{Z}[\frac{1}{m}]$-scheme, for which we first construct the desired equivalence of spectra

$$\mathrm{Fil}^i F(X;\mathbb{Q}) \simeq \big(\bigoplus_{i \leq j < k} \mathrm{gr}^j F(X;\mathbb{Q})\big) \oplus \mathrm{Fil}^k F(X;\mathbb{Q}).$$

We first prove that the spectrum $\big(\mathrm{Fil}^{i+1}F(X;\mathbb{Q})\big)^{\psi^m-m^i}$ is zero. The filtration $\mathrm{Fil}^{i+1+\star}F(X;\mathbb{Q})$ induced on the spectrum $\mathrm{Fil}^{i+1}F(X;\mathbb{Q})$ is complete (hypothesis $(i)$), so it suffices to prove that the natural map

$$\psi^m - m^i : \mathrm{gr}^j F(X;\mathbb{Q}) \longrightarrow \mathrm{gr}^j F(X;\mathbb{Q})$$

is an equivalence of spectra for every integer $j \geq i+1$. For every integer $j \geq i+1$, this map can be identified with multiplication by the nonzero integer $m^i(m^{j-i}-1)$ on the $\mathbb{Q}$-linear spectrum $\mathrm{gr}^j F(X;\mathbb{Q})$ (hypothesis $(ii)$), and is thus an equivalence. Taking the fibre of the natural map $\psi^m - m^i$ on the fibre sequence of spectra

$$\mathrm{Fil}^{i+1}F(X;\mathbb{Q}) \longrightarrow \mathrm{Fil}^{i}F(X;\mathbb{Q}) \longrightarrow \mathrm{gr}^{i}F(X;\mathbb{Q}),$$

this implies that the natural map

$$\big(\mathrm{Fil}^{i}F(X;\mathbb{Q})\big)^{\psi^m-m^i} \longrightarrow \big(\mathrm{gr}^{i}F(X;\mathbb{Q})\big)^{\psi^m-m^i}$$

is an equivalence of spectra. The spectrum $\big(\mathrm{gr}^{i}F(X;\mathbb{Q})\big)^{\psi^m-m^i}$ can be naturally identified with the spectrum $\mathrm{gr}^i F(X;\mathbb{Q}) \oplus \mathrm{gr}^i F(X;\mathbb{Q})[-1]$ (hypothesis $(ii)$), and the induced composite map

$$\mathrm{gr}^{i}F(X;\mathbb{Q}) \longrightarrow \big(\mathrm{gr}^{i}F(X;\mathbb{Q})\big)^{\psi^m-m^i} \xrightarrow{\sim} \big(\mathrm{Fil}^{i}F(X;\mathbb{Q})\big)^{\psi^m-m^i} \xrightarrow{\mathrm{can}} \mathrm{Fil}^{i}F(X;\mathbb{Q})$$

induces a natural splitting of spectra

$$\mathrm{Fil}^{i}F(X;\mathbb{Q}) \simeq \mathrm{gr}^{i}F(X;\mathbb{Q}) \oplus \mathrm{Fil}^{i+1}F(X;\mathbb{Q}).$$

By induction, this implies that for every integer $k \geq i$, there is a natural equivalence of spectra

$$\mathrm{Fil}^{i}F(X;\mathbb{Q}) \simeq \big(\bigoplus_{i\leq j<k} \mathrm{gr}^{j}F(X;\mathbb{Q})\big) \oplus \mathrm{Fil}^{k}F(X;\mathbb{Q}).$$

We now prove the desired equivalence of spectra for a general qcqs derived scheme $X$. The presheaf $\mathrm{Fil}^\star F(-)$ being a Zariski sheaf of filtered spectra, the presheaves $\mathrm{Fil}^i F(-;\mathbb{Q})$ and $\big(\bigoplus_{i\leq j<k} \mathrm{gr}^j F(-;\mathbb{Q})\big) \oplus \mathrm{Fil}^k F(-;\mathbb{Q})$ are Zariski sheaves of spectra. It thus suffices to construct compatible equivalences of spectra

$$\mathrm{Fil}^{i}F(X_{\mathbb{Z}[\frac{1}{m}]};\mathbb{Q}) \simeq \big(\bigoplus_{i\leq j<k} \mathrm{gr}^{j}F(X_{\mathbb{Z}[\frac{1}{m}]};\mathbb{Q})\big) \oplus \mathrm{Fil}^{k}F(X_{\mathbb{Z}[\frac{1}{m}]};\mathbb{Q})$$

for all integers $m \geq 2$. The construction of this equivalence for each integer $m \geq 2$, which depends on the map $\psi^m : \mathrm{Fil}^\star F(X_{\mathbb{Z}[\frac{1}{m}]}) \to \mathrm{Fil}^\star F(X_{\mathbb{Z}[\frac{1}{m}]})$, is the first part of this proof. Let $m, m' \geq 2$ be integers. The compatibility between the constructions of the equivalences for $m$, $m'$, and $mm'$ only depend on the choices, for every integer $i \in \mathbb{Z}$, of the identification of the maps $\psi^m$, $\psi^{m'}$, and $\psi^{mm'}$ on the $i^{\text{th}}$ graded piece with multiplication by $m^i$, $(m')^i$, and $(mm')^i$ respectively. These choices are compatible up to homotopy (hypothesis $(iii)$), which concludes the proof. □

**Remark 5.1.11.** Lemma 5.1.10 can also be proved, with the same proof, for Zariski sheaves of filtered spectra $\mathrm{Fil}^\star F(-)$ that are defined on qcqs schemes, on qcqs schemes of finite valuative dimension, on noetherian schemes of finite dimension, or on smooth schemes over a given commutative ring.

**Remark 5.1.12.** Hypothesis (*iii*) in Lemma 5.1.10 follows from the construction of the Adams operations on the filtrations $\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(-)$ and $\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(-)$, when these are defined. This hypothesis (*iii*) then follows formally for all the other filtrations considered in this section.

**Proposition 5.1.13.** *Let*

$$\mathrm{Fil}^\star F(-) : \mathrm{Sch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

*be a finitary Zariski sheaf of filtered spectra. For each integer $m \geq 2$, let $\psi^m$ be a natural multiplicative automorphism of the filtered spectrum $\mathrm{Fil}^\star F(X)$ on qcqs $\mathbb{Z}[\frac{1}{m}]$-schemes $X$, satisfying the following properties:*

*(i) for every noetherian scheme $X$ of finite dimension, there exists an integer $d \in \mathbb{Z}$ such that for every integer $i \in \mathbb{Z}$, the spectrum $\mathrm{Fil}^i F(X;\mathbb{Q})$ is in cohomological degrees at most $-i+d$;*

*(ii) for any integers $i \in \mathbb{Z}$ and $m \geq 2$, and every qcqs $\mathbb{Z}[\frac{1}{m}]$-scheme $X$, the induced automorphism on the $i^{\mathrm{th}}$ graded piece $\mathrm{gr}^i F(X)$ is multiplication by $m^i$;*

*(iii) for any integers $m, m' \geq 2$, and every qcqs $\mathbb{Z}[\frac{1}{mm'}]$-scheme $X$, the space of natural transformations from $\psi^m \circ \psi^{m'}$ to $\psi^{mm'}$, as endomorphisms of the filtered spectrum $\mathrm{Fil}^\star F(X)$, is contractible.*

*Then for every qcqs scheme $X$, there exists a natural multiplicative equivalence of filtered spectra*

$$\mathrm{Fil}^\star F(X;\mathbb{Q}) \simeq \bigoplus_{j \geq \star} \mathrm{gr}^j F(X;\mathbb{Q}).$$

*Proof.* By finitariness, it suffices to prove the result on noetherian schemes of finite dimension. Hypothesis (*i*) implies that the filtration $\mathrm{Fil}^\star F(X;\mathbb{Q})$ is complete on such schemes $X$. Lemma 5.1.10 and Remark 5.1.11 then imply that there exist natural equivalences of spectra

$$\mathrm{Fil}^i F(X;\mathbb{Q}) \simeq \big(\oplus_{i \leq j < k} \mathrm{gr}^j F(X;\mathbb{Q})\big) \oplus \mathrm{Fil}^k F(X;\mathbb{Q})$$

for all integers $i, k \in \mathbb{Z}$ such that $k \geq i$. Again using completeness, taking the limit over $k \to +\infty$ induces a natural equivalence of spectra

$$\mathrm{Fil}^i F(X;\mathbb{Q}) \simeq \prod_{j \geq i} \mathrm{gr}^j F(X;\mathbb{Q}).$$

Hypothesis (*i*) then implies that, at each cohomological degree, only a finite number of terms in the previous product are nonzero. In particular, the previous equivalence can be rewritten as a natural equivalence of spectra

$$\mathrm{Fil}^i F(X;\mathbb{Q}) \simeq \bigoplus_{j \geq i} \mathrm{gr}^j F(X;\mathbb{Q}),$$

which induces the desired multiplicative equivalence of filtered spectra. □

**Remark 5.1.14.** Let $X$ be a qcqs derived scheme. The argument of Proposition 5.1.13, where the necessary hypotheses are satisfied by Corollary 5.1.6 and Proposition 3.3.8, implies that there is a natural multiplicative equivalence of filtered spectra

$$\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(X;\mathbb{Q}) \simeq \prod_{j \geq \star} \mathbb{Q}(j)^{\mathrm{TC}}(X)[2i].$$

## 5.2 Rigid-analytic HKR filtrations

In this section, we define variants, which we call rigid-analytic, of the HKR filtrations. We start by explaining the relevant objects at the level of Hochschild homology.

For every qcqs derived scheme $X$, there is a natural $\mathrm{S}^1$-equivariant arithmetic fracture square

$$\begin{array}{ccc} \mathrm{HH}(X) & \longrightarrow & \mathrm{HH}(X_{\mathbb{Q}}/\mathbb{Q}) \\ \downarrow & & \downarrow \\ \prod_{p\in\mathbb{P}} \mathrm{HH}(X;\mathbb{Z}_p) & \longrightarrow & \prod'_{p\in\mathbb{P}} \mathrm{HH}(X;\mathbb{Q}_p) \end{array} \tag{5.1}$$

in the derived category $\mathcal{D}(\mathbb{Z})$, where we use base change for Hochschild homology for the top right corner, and the convention adopted in the Notation part for the bottom left and bottom right corners. Applying homotopy fixed points $(-)^{h\mathrm{S}^1}$ for the $\mathrm{S}^1$-action induces a natural cartesian square

$$\begin{array}{ccc} \mathrm{HC}^-(X) & \longrightarrow & \mathrm{HC}^-(X_{\mathbb{Q}}/\mathbb{Q}) \\ \downarrow & & \downarrow \\ \prod_{p\in\mathbb{P}} \mathrm{HC}^-(X;\mathbb{Z}_p) & \longrightarrow & \big(\prod'_{p\in\mathbb{P}} \mathrm{HH}(X;\mathbb{Q}_p)\big)^{h\mathrm{S}^1} \end{array}$$

in the derived category $\mathcal{D}(\mathbb{Z})$, where we use that taking homotopy fixed points $(-)^{h\mathrm{S}^1}$ commutes with limits for the bottom left corner. We call *rigid-analytic variant of Hochschild homology and of negative cyclic homology* the bottom right corners of the previous two cartesian squares, respectively. This terminology should find some justification in Section 5.3. Following Section 3.1, we use the previous cartesian square to introduce a new HKR filtration on this rigid-analytic variant of negative cyclic homology $\big(\prod'_{p\in\mathbb{P}} \mathrm{HH}(X;\mathbb{Q}_p)\big)^{h\mathrm{S}^1}$.

**Definition 5.2.1** (HKR filtration on rigid-analytic $\mathrm{HC}^-$)**.** The *HKR filtration on rigid-analytic negative cyclic homology* of qcqs derived schemes is the functor

$$\mathrm{Fil}^\star_{\mathrm{HKR}}\big(\prod_{p\in\mathbb{P}}{}' \,\mathrm{HH}(-;\mathbb{Q}_p)\big)^{h\mathrm{S}^1} : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

defined by the cocartesian square

$$\begin{array}{ccc} \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(-) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(-_{\mathbb{Q}}/\mathbb{Q}) \\ \downarrow & & \downarrow \\ \prod_{p\in\mathbb{P}} \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(-;\mathbb{Z}_p) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{HKR}}\big(\prod'_{p\in\mathbb{P}} \mathrm{HH}(-;\mathbb{Q}_p)\big)^{h\mathrm{S}^1}. \end{array}$$

**Remark 5.2.2.** Let $p$ be a prime number, and $X$ be a qcqs derived scheme over $\mathbb{Z}_{(p)}$. The natural map

$$\prod_{\ell\in\mathbb{P}} \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X;\mathbb{Z}_\ell) \longrightarrow \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X;\mathbb{Z}_p)$$

is then an equivalence of filtered spectra, and we then denote by $\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HH}(X;\mathbb{Q}_p)^{h\mathrm{S}^1}$ the filtered spectrum $\mathrm{Fil}^\star_{\mathrm{HKR}} \prod'_{\ell\in\mathbb{P}} \mathrm{HH}(X;\mathbb{Q}_\ell)^{h\mathrm{S}^1}$. In particular, the natural commutative diagram

$$\begin{array}{ccc} \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X_{\mathbb{Q}}/\mathbb{Q}) \\ \downarrow & & \downarrow \\ \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X;\mathbb{Z}_p) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HH}(X;\mathbb{Q}_p)^{h\mathrm{S}^1} \end{array}$$

is a cartesian square of filtered spectra.

Similarly, one can apply homotopy orbits $(-)_{h\mathrm{S}^1}$ for the $\mathrm{S}^1$-action to the arithmetic fracture square for Hochschild homology (5.1). The functor $(-)_{h\mathrm{S}^1}$ does not commute with limits, but the $\mathrm{S}^1$-action on the product $\prod_{p\in\mathbb{P}} \mathrm{HH}(X;\mathbb{Z}_p)$ is diagonal. Using the natural fibre sequence

$$\mathrm{HH}(X;\mathbb{Z}_p) \longrightarrow \mathrm{HH}(X;\mathbb{Z}_p)_{h\mathrm{S}^1} \longrightarrow \mathrm{HH}(X;\mathbb{Z}_p)_{h\mathrm{S}^1}[2]$$

in the derived category $\mathcal{D}(\mathbb{Z})$ and the fact that the functors $\mathrm{HH}(-;\mathbb{Z}_p)$ and $\mathrm{HH}(-;\mathbb{Z}_p)_{h\mathrm{S}^1}$ are in non-positive cohomological degrees on animated commutative rings, one can prove that the complex $\mathrm{HH}(X;\mathbb{Z}_p)_{h\mathrm{S}^1} \in \mathcal{D}(\mathbb{Z})$ is derived $p$-complete, hence the natural map

$$\mathrm{HC}(X;\mathbb{Z}_p) \longrightarrow \mathrm{HH}(X;\mathbb{Z}_p)_{h\mathrm{S}^1}$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$. In particular, applying homotopy orbits $(-)_{h\mathrm{S}^1}$ to the arithmetic fracture square for Hochschild homology (5.1) induces a natural cartesian square

$$\begin{array}{ccc} \mathrm{HC}(X) & \longrightarrow & \mathrm{HC}(X_{\mathbb{Q}}/\mathbb{Q}) \\ \downarrow & & \downarrow \\ \prod_{p\in\mathbb{P}} \mathrm{HC}(X;\mathbb{Z}_p) & \longrightarrow & \big(\prod'_{p\in\mathbb{P}} \mathrm{HH}(X;\mathbb{Q}_p)\big)_{h\mathrm{S}^1} \end{array}$$

in the derived category $\mathcal{D}(\mathbb{Z})$. We use this cartesian square to introduce the following HKR filtration on the bottom right corner.

**Definition 5.2.3** (HKR filtration on rigid-analytic HC)**.** The *HKR filtration on rigid-analytic cyclic homology* of qcqs derived schemes is the functor

$$\mathrm{Fil}^\star_{\mathrm{HKR}}\big(\prod_{p\in\mathbb{P}}{}' \,\mathrm{HH}(-;\mathbb{Q}_p)\big)_{h\mathrm{S}^1} : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

defined by the cocartesian square

$$\begin{array}{ccc} \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}(-) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}(-_{\mathbb{Q}}/\mathbb{Q}) \\ \downarrow & & \downarrow \\ \prod_{p\in\mathbb{P}} \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}(-;\mathbb{Z}_p) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{HKR}}\big(\prod'_{p\in\mathbb{P}} \mathrm{HH}(-;\mathbb{Q}_p)\big)_{h\mathrm{S}^1}. \end{array}$$

**Remark 5.2.4.** Taking homotopy orbits $(-)_{h\mathrm{S}^1}$ commutes with colimits, so the natural map

$$\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}(X;\mathbb{Q}) \longrightarrow \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}(X_{\mathbb{Q}}/\mathbb{Q})$$

is an equivalence in the filtered derived category $\mathcal{DF}(\mathbb{Q})$. Upon applying rationalisation to the natural cartesian square

$$\begin{array}{ccc} \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}(X) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}(X_{\mathbb{Q}}/\,\mathbb{Q}) \\ \downarrow & & \downarrow \\ \prod_{p\in\mathbb{P}}\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}(X;\mathbb{Z}_p) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{HKR}}\big(\prod'_{p\in\mathbb{P}}\mathrm{HH}(X;\mathbb{Q}_p)\big)_{h\mathrm{S}^1}, \end{array}$$

this implies that the natural map

$$\prod_{p\in\mathbb{P}}{}' \,\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}(X;\mathbb{Q}_p) := \big(\prod_{p\in\mathbb{P}}\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}(X;\mathbb{Z}_p)\big)_{\mathbb{Q}} \longrightarrow \mathrm{Fil}^\star_{\mathrm{HKR}}\big(\prod_{p\in\mathbb{P}}{}'\,\mathrm{HH}(X;\mathbb{Q}_p)\big)_{h\mathrm{S}^1}$$

is an equivalence in the filtered derived category $\mathcal{DF}(\mathbb{Q})$.

**Remark 5.2.5.** Let $p$ be a prime number, and $X$ be a qcqs derived scheme over $\mathbb{Z}_{(p)}$. As in Remark 5.2.2, we denote the filtered complex $\mathrm{Fil}^\star_{\mathrm{HKR}}\big(\prod'_{\ell\in\mathbb{P}}\mathrm{HH}(X;\mathbb{Q}_\ell)\big)_{h\mathrm{S}^1}$ by

$$\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HH}(X;\mathbb{Q}_p)_{h\mathrm{S}^1} \in \mathcal{DF}(\mathbb{Q}).$$

In particular, the natural map

$$\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}(X;\mathbb{Q}_p) \longrightarrow \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HH}(X;\mathbb{Q}_p)_{h\mathrm{S}^1}$$

is an equivalence in the filtered derived category $\mathcal{DF}(\mathbb{Q})$ by Remark 5.2.4.

The Tate construction $(-)^{t\mathrm{S}^1}$ is by definition the cofibre of the norm map

$$(-)_{h\mathrm{S}^1}[1] \to (-)^{h\mathrm{S}^1}.$$

Applying the Tate construction $(-)^{t\mathrm{S}^1}$ to the arithmetic fracture square for Hochschild homology (5.1) then induces a natural cartesian square

$$\begin{array}{ccc} \mathrm{HP}(X) & \longrightarrow & \mathrm{HP}(X_{\mathbb{Q}}/\,\mathbb{Q}) \\ \downarrow & & \downarrow \\ \prod_{p\in\mathbb{P}}\mathrm{HP}(X;\mathbb{Z}_p) & \longrightarrow & \big(\prod'_{p\in\mathbb{P}}\mathrm{HH}(X;\mathbb{Q}_p)\big)^{t\mathrm{S}^1} \end{array}$$

in the derived category $\mathcal{D}(\mathbb{Z})$, where we use the analogous cartesian squares for $\mathrm{HC}^-$ and HC to identify the bottom left corner.

**Definition 5.2.6** (HKR filtration on rigid-analytic HP)**.** The *HKR filtration on rigid-analytic periodic cyclic homology* of qcqs derived schemes is the functor

$$\mathrm{Fil}^\star_{\mathrm{HKR}}\big(\prod_{p\in\mathbb{P}}{}'\,\mathrm{HH}(-;\mathbb{Q}_p)\big)^{t\mathrm{S}^1} : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

defined by the cocartesian square

$$\begin{array}{ccc} \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HP}(-) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HP}(-_{\mathbb{Q}}/\,\mathbb{Q}) \\ \downarrow & & \downarrow \\ \prod_{p\in\mathbb{P}}\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HP}(-;\mathbb{Z}_p) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{HKR}}\big(\prod'_{p\in\mathbb{P}}\mathrm{HH}(-;\mathbb{Q}_p)\big)^{t\mathrm{S}^1}. \end{array}$$

**Remark 5.2.7.** Let $p$ be a prime number, and $X$ be a qcqs derived scheme over $\mathbb{Z}_{(p)}$. As in Remarks 5.2.2 and 5.2.5, we denote the filtered complex $\mathrm{Fil}^\star_{\mathrm{HKR}}\big(\prod_{\ell\in\mathbb{P}}\mathrm{HH}(X;\mathbb{Q}_\ell)\big)^{t\mathrm{S}^1}$ by

$$\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HH}(X;\mathbb{Q}_p)^{t\mathrm{S}^1} \in \mathcal{DF}(\mathbb{Q}).$$

In particular, the natural commutative diagram

$$\begin{array}{ccc} \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HP}(X) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HP}(X_{\mathbb{Q}}/\mathbb{Q}) \\ \downarrow & & \downarrow \\ \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HP}(X;\mathbb{Z}_p) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HH}(X;\mathbb{Q}_p)^{t\mathrm{S}^1} \end{array}$$

is a cartesian square of filtered spectra.

**Lemma 5.2.8.** *Let $X$ be a qcqs derived scheme. Then the natural commutative diagram*

$$\begin{array}{ccc} \prod'_{p\in\mathbb{P}}\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X;\mathbb{Q}_p) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{HKR}}\big(\prod'_{p\in\mathbb{P}}\mathrm{HH}(X;\mathbb{Q}_p)\big)^{h\mathrm{S}^1} \\ \downarrow & & \downarrow \\ \prod'_{p\in\mathbb{P}}\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HP}(X;\mathbb{Q}_p) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{HKR}}\big(\prod'_{p\in\mathbb{P}}\mathrm{HH}(X;\mathbb{Q}_p)\big)^{t\mathrm{S}^1} \end{array}$$

*is a cartesian square of filtered spectra.*

*Proof.* There is a natural commutative diagram of filtered spectra

$$\begin{array}{ccccc} \prod'_{p\in\mathbb{P}}\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X;\mathbb{Q}_p) & \longrightarrow & \prod'_{p\in\mathbb{P}}\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HP}(X;\mathbb{Q}_p) & \longrightarrow & \prod'_{p\in\mathbb{P}}\mathrm{Fil}^{\star-1}_{\mathrm{HKR}}\mathrm{HC}(X;\mathbb{Q}_p)[2] \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Fil}^\star_{\mathrm{HKR}}\big(\prod'_{p\in\mathbb{P}}\mathrm{HH}(X;\mathbb{Q}_p)\big)^{h\mathrm{S}^1} & \to & \mathrm{Fil}^\star_{\mathrm{HKR}}\big(\prod'_{p\in\mathbb{P}}\mathrm{HH}(X;\mathbb{Q}_p)\big)^{t\mathrm{S}^1} & \to & \mathrm{Fil}^{\star-1}_{\mathrm{HKR}}\big(\prod'_{p\in\mathbb{P}}\mathrm{HH}(X;\mathbb{Q}_p)\big)_{h\mathrm{S}^1}[2] \end{array}$$

where, by definition of the bottom terms (Definitions 5.2.1, 5.2.6, and 5.2.3), the horizontal lines are fibre sequences. In this diagram, the right vertical map is an equivalence (Remark 5.2.4), so the left square is a cartesian square. ☐

**Lemma 5.2.9.** *Let $X$ be a qcqs derived scheme. Then the filtrations*

$$\prod_{p\in\mathbb{P}}{}'\big(\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HH}(X;\mathbb{Q}_p)\big)^{h\mathrm{S}^1},\ \prod_{p\in\mathbb{P}}{}'\big(\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HH}(X;\mathbb{Q}_p)\big)_{h\mathrm{S}^1},\ \textit{and}\ \prod_{p\in\mathbb{P}}{}'\big(\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HH}(X;\mathbb{Q}_p)\big)^{t\mathrm{S}^1}$$

*are complete.*

*Proof.* The HKR filtrations on $\mathrm{HC}^-(X)$, $\mathrm{HC}(X)$, and $\mathrm{HP}(X)$ are complete by Lemma 3.1.9. The product on prime numbers $p$ of the $p$-completions of these filtrations are also complete, since $p$-completions and products commute with limits. Similarly, the HKR filtrations on $\mathrm{HC}^-(X_{\mathbb{Q}}/\mathbb{Q})$, $\mathrm{HC}(X_{\mathbb{Q}}/\mathbb{Q})$, and $\mathrm{HP}(X_{\mathbb{Q}}/\mathbb{Q})$ are complete by [Ant19, Theorem 1.1]. The rigid-analytic HKR filtrations on $\mathrm{HC}^-$, HC, and HP are thus also complete, as pushouts of three complete filtrations. ☐

We now describe the graded pieces of these rigid-analytic HKR filtrations, by analogy with the classical HKR filtrations.

**Definition 5.2.10** (Rigid-analytic Hodge-complete derived de Rham cohomology)**.** For every integer $i \in \mathbb{Z}$, the functor

$$R\Gamma_{\mathrm{Zar}}\Big(-, \prod_{p\in\mathbb{P}}{}' \, \underline{\widehat{\mathbb{L}\Omega}}^{\geq i}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p}\Big) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathcal{D}(\mathbb{Q})$$

is defined as the shifted graded piece of the HKR filtration on $\prod'_{p\in\mathbb{P}} \mathrm{HH}(-;\mathbb{Q}_p)^{h\mathrm{S}^1}$:

$$R\Gamma_{\mathrm{Zar}}\Big(-, \prod_{p\in\mathbb{P}}{}' \, \underline{\widehat{\mathbb{L}\Omega}}^{\geq i}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p}\Big) := \mathrm{gr}^i_{\mathrm{HKR}}\big(\prod_{p\in\mathbb{P}}{}' \, \mathrm{HH}(-;\mathbb{Q}_p)\big)^{h\mathrm{S}^1}[-2i].$$

**Remark 5.2.11.** Let $X$ be a qcqs derived scheme, and $i \in \mathbb{Z}$ be an integer. By Definition 5.2.1, there is a natural cartesian square

$$\begin{array}{ccc} R\Gamma_{\mathrm{Zar}}\Big(X, \widehat{\mathbb{L}\Omega}^{\geq i}_{-/\mathbb{Z}}\Big) & \longrightarrow & R\Gamma_{\mathrm{Zar}}\Big(X, \widehat{\mathbb{L}\Omega}^{\geq i}_{-_{\mathbb{Q}}/\mathbb{Q}}\Big) \\ \downarrow & & \downarrow \\ R\Gamma_{\mathrm{Zar}}\Big(X, \prod_{p\in\mathbb{P}} \big(\widehat{\mathbb{L}\Omega}^{\geq i}_{-/\mathbb{Z}}\big)^{\wedge}_p\Big) & \longrightarrow & R\Gamma_{\mathrm{Zar}}\Big(X, \prod'_{p\in\mathbb{P}} \underline{\widehat{\mathbb{L}\Omega}}^{\geq i}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p}\Big) \end{array}$$

in the derived category $\mathcal{D}(\mathbb{Z})$, which can serve as an alternative definition for the bottom right term.

**Remark 5.2.12.** Let $X$ be a qcqs derived scheme, and $i \in \mathbb{Z}$ be an integer. The complexes

$$R\Gamma_{\mathrm{Zar}}\Big(X, \prod_{p\in\mathbb{P}}{}' \, \underline{\widehat{\mathbb{L}\Omega}}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p}\Big) \quad \text{and} \quad R\Gamma_{\mathrm{Zar}}\Big(X, \prod_{p\in\mathbb{P}}{}' \, \underline{\mathbb{L}\Omega}^{\leq i}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p}\Big)$$

are defined as in Definition 5.2.10, where $(-)^{h\mathrm{S}^1}$ is replaced by $(-)^{t\mathrm{S}^1}$ and $(-)_{h\mathrm{S}^1}$ respectively. In particular, the natural fibre sequence

$$\mathrm{Fil}^{\star}_{\mathrm{HKR}}\big(\prod_{p\in\mathbb{P}}{}' \, \mathrm{HH}(X;\mathbb{Q}_p)\big)^{h\mathrm{S}^1} \to \mathrm{Fil}^{\star}_{\mathrm{HKR}}\big(\prod_{p\in\mathbb{P}}{}' \, \mathrm{HH}(X;\mathbb{Q}_p)\big)^{t\mathrm{S}^1} \to \mathrm{Fil}^{\star-1}_{\mathrm{HKR}}\big(\prod_{p\in\mathbb{P}}{}' \, \mathrm{HH}(X;\mathbb{Q}_p)\big)_{h\mathrm{S}^1}[2]$$

induces a natural fibre sequence

$$R\Gamma_{\mathrm{Zar}}\Big(X, \prod_{p\in\mathbb{P}}{}' \, \underline{\widehat{\mathbb{L}\Omega}}^{\geq i}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p}\Big) \longrightarrow R\Gamma_{\mathrm{Zar}}\Big(X, \prod_{p\in\mathbb{P}}{}' \, \underline{\widehat{\mathbb{L}\Omega}}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p}\Big) \longrightarrow R\Gamma_{\mathrm{Zar}}\Big(X, \prod_{p\in\mathbb{P}}{}' \, \underline{\mathbb{L}\Omega}^{<i}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p}\Big),$$

where the right term is naturally identified with the complex

$$\Big(\prod_{p\in\mathbb{P}} R\Gamma_{\mathrm{Zar}}\big(X, \big(L\Omega^{<i}_{-/\mathbb{Z}}\big)^{\wedge}_p\big)\Big)_{\mathbb{Q}} \in \mathcal{D}(\mathbb{Q})$$

by Remark 5.2.4.

**Remark 5.2.13.** Let $p$ be a prime number, $X$ be a qcqs derived scheme over $\mathbb{Z}_{(p)}$, and $i \in \mathbb{Z}$ be an integer. Following Remarks 5.2.2, 5.2.5, and 5.2.7, we denote the complexes

$$R\Gamma_{\mathrm{Zar}}\Big(X, \prod_{\ell\in\mathbb{P}}{}' \underline{\widehat{\mathbb{L}\Omega}}^{\geq i}_{-_{\mathbb{Q}_\ell}/\mathbb{Q}_\ell}\Big),\ R\Gamma_{\mathrm{Zar}}\Big(X, \prod_{\ell\in\mathbb{P}}{}' \underline{\widehat{\mathbb{L}\Omega}}_{-_{\mathbb{Q}_\ell}/\mathbb{Q}_\ell}\Big),\ \text{and}\ R\Gamma_{\mathrm{Zar}}\Big(X, \prod_{\ell\in\mathbb{P}}{}' \underline{\mathbb{L}\Omega}^{<i}_{-_{\mathbb{Q}_\ell}/\mathbb{Q}_\ell}\Big)$$

by

$$R\Gamma_{\mathrm{Zar}}\big(X, \widehat{\underline{\mathbb{L}\Omega}}^{\geq i}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p}\big),\ R\Gamma_{\mathrm{Zar}}\big(X, \widehat{\underline{\mathbb{L}\Omega}}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p}\big),\ \text{and } R\Gamma_{\mathrm{Zar}}\big(X, \underline{\mathbb{L}\Omega}^{<i}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p}\big)$$

respectively. In particular, by Remark 5.2.12, there is a natural fibre sequence

$$R\Gamma_{\mathrm{Zar}}\big(X, \widehat{\underline{\mathbb{L}\Omega}}^{\geq i}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p}\big) \longrightarrow R\Gamma_{\mathrm{Zar}}\big(X, \widehat{\underline{\mathbb{L}\Omega}}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p}\big) \longrightarrow R\Gamma_{\mathrm{Zar}}\big(X, \underline{\mathbb{L}\Omega}^{<i}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p}\big)$$

in the derived category $\mathcal{D}(\mathbb{Q}_p)$, where the right term is naturally identified with the complex

$$R\Gamma_{\mathrm{Zar}}\big(X, \big(\mathbb{L}\Omega^{<i}_{-/\mathbb{Z}}\big)^{\wedge}_{p}[\tfrac{1}{p}]\big) \in \mathcal{D}(\mathbb{Q}_p).$$

In the following result, we reformulate the motivic filtration on topological cyclic homology of Definition 3.3.3 in terms of the rigid-analytic HKR filtration on negative cyclic homology. This can be interpreted as a filtered arithmetic fracture square for topological cyclic homology.

**Proposition 5.2.14.** *Let $X$ be qcqs derived scheme. Then the natural commutative diagram*

$$\begin{array}{ccc} \mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{TC}(X) & \longrightarrow & \mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}^{-}(X_{\mathbb{Q}}/\mathbb{Q}) \\ \downarrow & & \downarrow \\ \prod_{p\in\mathbb{P}} \mathrm{Fil}^{\star}_{\mathrm{BMS}}\mathrm{TC}(X;\mathbb{Z}_p) & \longrightarrow & \mathrm{Fil}^{\star}_{\mathrm{HKR}}\Big(\prod'_{p\in\mathbb{P}} \mathrm{HH}(X;\mathbb{Q}_p)\Big)^{h\mathrm{S}^1} \end{array}$$

*is a cartesian square of filtered spectra.*

*Proof.* This is a consequence of Definitions 3.3.3 and 5.2.1. ☐

**Corollary 5.2.15.** *Let $X$ be a qcqs scheme. Then the natural commutative diagram*

$$\begin{array}{ccc} \big(L_{\mathrm{cdh}}\,\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{TC}(-)\big)(X) & \longrightarrow & \big(L_{\mathrm{cdh}}\,\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}^{-}(-_{\mathbb{Q}}/\mathbb{Q})\big)(X) \\ \downarrow & & \downarrow \\ \Big(L_{\mathrm{cdh}}\prod_{p\in\mathbb{P}} \mathrm{Fil}^{\star}_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)\Big)(X) & \longrightarrow & \Big(L_{\mathrm{cdh}}\,\mathrm{Fil}^{\star}_{\mathrm{HKR}}\big(\prod'_{p\in\mathbb{P}} \mathrm{HH}(-;\mathbb{Q}_p)\big)^{h\mathrm{S}^1}\Big)(X), \end{array}$$

*is a cartesian square of filtered spectra.*

*Proof.* This is a consequence of Proposition 5.2.14, which we restrict to qcqs schemes and then sheafify for the cdh topology. ☐

**Corollary 5.2.16.** *Let $X$ be a qcqs derived scheme. Then for every integer $i \in \mathbb{Z}$, the natural commutative diagram*

$$\begin{array}{ccc} \mathbb{Z}(i)^{\mathrm{TC}}(X) & \longrightarrow & R\Gamma_{\mathrm{Zar}}\Big(X, \widehat{\mathbb{L}\Omega}^{\geq i}_{-_{\mathbb{Q}}/\mathbb{Q}}\Big) \\ \downarrow & & \downarrow \\ \prod_{p\in\mathbb{P}} \mathbb{Z}_p(i)^{\mathrm{BMS}}(X) & \longrightarrow & R\Gamma_{\mathrm{Zar}}\Big(X, \prod'_{p\in\mathbb{P}} \widehat{\underline{\mathbb{L}\Omega}}^{\geq i}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p}\Big), \end{array}$$

*is a cartesian square in the derived category $\mathcal{D}(\mathbb{Z})$.*

*Proof.* This is a direct consequence of Proposition 5.2.14. ☐

## 5.3 A rigid-analytic Goodwillie theorem

In this section, we prove that the rigid-analytic version of periodic cyclic homology is a truncating invariant (Theorem 5.3.9). We first recall the definition of Hochschild, cyclic, negative cyclic, and periodic cyclic homologies of a general cyclic object.

**Definition 5.3.1** (Cyclic object)**.** Let $\mathcal{C}$ be a category, or an $\infty$-category. The *cyclic category* $\Lambda$ is the category with objects $[n]$ indexed by non-negative integers $n$, and morphisms $[m] \to [n]$ given by homotopy classes of degree one increasing maps from $\mathrm{S}^1$ to itself that map the subgroup $\mathbb{Z}/(m+1)$ to $\mathbb{Z}/(n+1)$. A *cyclic object* in $\mathcal{C}$ is then a contravariant functor from the cyclic category $\Lambda$ to $\mathcal{C}$.

**Notation 5.3.2** (Hochschild homology of a cyclic object)**.** Let $\mathcal{A}$ be an abelian category with exact infinite products, and $X_\bullet$ be a cyclic object in $\mathcal{A}$. Following [Goo85, Section II] (see also [Mor18b, Section 2.2]), one can define the Hochschild homology $\mathrm{HH}(X_\bullet)$ of $X_\bullet$, as an object of the derived category $\mathcal{D}(\mathcal{A})$ equipped with a natural $\mathrm{S}^1$-action. The cyclic, negative cyclic, and periodic cyclic homologies of the cyclic object $X_\bullet$ are then defined by

$$\mathrm{HC}(X_\bullet) := \mathrm{HH}(X_\bullet)_{h\mathrm{S}^1},\ \mathrm{HC}^-(X_\bullet) := \mathrm{HH}(X_\bullet)^{h\mathrm{S}^1},\ \text{and } \mathrm{HP}(X_\bullet) := \mathrm{HH}(X_\bullet)^{t\mathrm{S}^1}.$$

In this context, there is moreover a natural map

$$s : \mathrm{HC}(X_\bullet)[-2] \longrightarrow \mathrm{HC}(X_\bullet)$$

in the derived category $\mathcal{D}(\mathcal{A})$, from which periodic cyclic homology $\mathrm{HP}(X_\bullet) \in \mathcal{D}(\mathcal{A})$ can be recovered by the formula

$$\mathrm{HP}(X_\bullet) \simeq \varprojlim \Big( \cdots \xrightarrow{s} \mathrm{HC}(X_\bullet)[-4] \xrightarrow{s} \mathrm{HC}(X_\bullet)[-2] \xrightarrow{s} \mathrm{HC}(X_\bullet) \Big).$$

We now apply the previous general construction to define Hochschild homology and its variants on solid associative derived algebras over a solid commutative ring.

**Definition 5.3.3** (Solid Hochschild homology)**.** Let $k$ be a solid commutative ring, and $R$ be a connective solid $k$-$\mathbb{E}_1$-algebra. The simplicial object

$$\cdots \ \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} \ R \otimes_k R \otimes_k R \ \substack{\rightarrow\\\rightarrow\\\rightarrow} \ R \otimes_k R \rightrightarrows R$$

has a natural structure of cyclic object in the derived $\infty$-category of solid $k$-modules (Definition 5.3.1), induced by permutation of the tensor factors. We write $\underline{\mathrm{HH}}(R/k)$, $\underline{\mathrm{HC}}(R/k)$, $\underline{\mathrm{HC}^-}(R/k)$ and $\underline{\mathrm{HP}}(R/k)$ for the Hochschild, cyclic, negative cyclic and periodic cyclic homologies of this cyclic object (Notation 5.3.2). If $k = \mathbb{Z}$, we simply denote these by $\underline{\mathrm{HH}}(R)$, $\underline{\mathrm{HC}}(R)$, $\underline{\mathrm{HC}^-}(R)$ and $\underline{\mathrm{HP}}(R)$.

For $f : R \to R'$ a map of $\mathbb{Z}$-$\mathbb{E}_1$-algebras and $F : \mathbb{E}_1\text{-}\mathrm{Alg}_{\mathbb{Z}} \to \mathcal{D}(\mathbb{Z})$ a functor, we denote by $F(f) \in \mathcal{D}(\mathbb{Z})$ the cofibre of the map $F(R) \to F(R')$. More generally, for $f : R \to R'$ a map of solid $\mathbb{Z}$-$\mathbb{E}_1$-algebras and $F$ a $\mathcal{D}(\mathrm{Solid})$-valued functor on solid $\mathbb{Z}$-$\mathbb{E}_1$-algebras, denote by $F(f) \in \mathcal{D}(\mathrm{Solid})$ the cofibre of the map $F(R) \to F(R')$.

The following result is [Goo85, Theorem IV.2.6]. More precisely, this result of Goodwillie is for maps of simplicial $\mathbb{Z}$-algebras, and the underlying $\infty$-category of simplicial $\mathbb{Z}$-algebras is naturally identified with the $\infty$-category of connective $\mathbb{Z}$-$\mathbb{E}_1$-algebras, by the monoidal Dold–Kan correspondence and [Lur17, Proposition 7.1.4.6].

**Theorem 5.3.4** ([Goo85])**.** *Let $f : R \to R'$ be a 1-connected map of connective $\mathbb{Z}$-$\mathbb{E}_1$-algebras. For every integer $n \geq 0$, the natural map*

$$n!s^n : \mathrm{HC}_{*+2n}(f) \longrightarrow \mathrm{HC}_*(f)$$

*is the zero map for $* \leq n-1$.*

Goodwillie's proof of Theorem 5.3.4, although stated with respect to the abelian category of $\mathbb{Z}$-modules (or $k$-modules, for $k$ an arbitrary discrete commutative ring), is valid for any abelian symmetric monoidal category with exact infinite products (see Notation 5.3.2 and Definition 5.3.3). One can thus prove the following generalisation of the previous result.

**Theorem 5.3.5.** *Let $f : R \to R'$ be a 1-connected map of connective solid $\mathbb{Z}$-$\mathbb{E}_1$-algebras. Then for every integer $n \geq 0$, the natural map*

$$n!s^n : \underline{\mathrm{HC}}(f)[-2n] \longrightarrow \underline{\mathrm{HC}}(f),$$

*in the derived category $\mathcal{D}(\mathrm{Solid})$, is the zero map on cohomology groups*[1] *in degrees at least $-n+1$.*

*Proof.* To prove the result for simplicial solid $\mathbb{Z}$-algebras, it suffices to prove that the abelian category of solid abelian groups is symmetric monoidal, and has exact infinite products. The first claim is [CS19, Theorem 6.2 (i)]. The second claim is a consequence of the fact that the abelian category of condensed abelian groups has exact arbitrary products ([CS19, Theorem 1.10]), and that the category of solid abelian groups, as a subcategory of the abelian category of condensed abelian groups, is stable under all limits ([CS19, Theorem 5.8 (*i*)]). We omit the proof of the analogue of [Lur17, Proposition 7.1.4.6] to pass from simplicial solid $\mathbb{Z}$-algebras to connective solid $\mathbb{Z}$-$\mathbb{E}_1$-algebras. □

For the rest of this section, and following the convention of condensed mathematics, a condensed object is called *discrete* if its condensed structure is trivial. Given a classical object $X$, we denote by $\underline{X}$ the associated discrete condensed object.

**Lemma 5.3.6.** *Let $R$ be a connective $\mathbb{Z}$-$\mathbb{E}_1$-algebra, and $p$ be a prime number. Then the natural map*

$$\underline{\mathrm{HH}(R)} \longrightarrow \underline{\mathrm{HH}}(\underline{R}^\wedge_p),$$

*in the derived category $\mathcal{D}(\mathrm{Solid})$, exhibits the target as the $p$-completion of the source. In particular, there is a natural equivalence*

$$\underline{\mathrm{HH}(R)}^\wedge_p[\tfrac{1}{p}] \xrightarrow{\sim} \underline{\mathrm{HH}}(\underline{R}^\wedge_p[\tfrac{1}{p}]/\,\mathbb{Q}_p)$$

*in the derived category $\mathcal{D}(\mathrm{Solid})$.*

*Proof.* The solid tensor product of $p$-complete solid connective $\mathbb{Z}$-$\mathbb{E}_1$-algebras is $p$-complete, and so is the totalisation of a complex of $p$-complete solid connective $\mathbb{Z}$-modules ([CS21]). In particular, the complex $\underline{\mathrm{HH}}(\underline{R}^\wedge_p)$, as a totalisation of tensor powers of the $p$-complete solid

[1] By this, we mean that it is the zero map as a map of solid abelian groups, *i.e.*, that it factors through the zero object of the abelian category of solid abelian groups. Note that the underlying abelian group of a nonzero solid abelian group can be zero (*e.g.*, the quotient of $\mathbb{Z}_p$ with the $p$-adic topology by $\mathbb{Z}_p$ with the discrete topology). In particular, being zero for a map of solid abelian groups cannot be detected on the underlying map of abelian groups.

connective $\mathbb{Z}$-$\mathbb{E}_1$-algebra $\underline{R}_p^\wedge$, is $p$-complete. By the derived Nakayama lemma ([Sta19, 091N], see also [CS21]), it thus suffices to prove the first statement modulo $p$. By base change for Hochschild homology (resp. solid Hochschild homology), this is equivalent to proving that the natural map

$$\underline{\mathrm{HH}((R/p)/\mathbb{F}_p)} \longrightarrow \underline{\mathrm{HH}}((\underline{R}/p)/\underline{\mathbb{F}_p})$$

is an equivalence in the derived category $\mathcal{D}(\mathrm{Solid})$. The desired equivalence is then a consequence of the fact that reduction modulo $p$ and tensor products commute with the functor $\underline{(-)}$ from $\mathbb{Z}$-$\mathbb{E}_1$-algebras to solid $\mathbb{Z}$-$\mathbb{E}_1$-algebras. The second statement follows from the fact that rationalisation commutes with the functor

$$\underline{(-)} : \mathcal{D}(\mathbb{Z}) \longrightarrow \mathcal{D}(\mathrm{Solid}),$$

and base change for solid Hochschild homology. □

**Proposition 5.3.7.** *Let $R$ be a connective $\mathbb{Z}$-$\mathbb{E}_1$-algebra, and $p$ be a prime number. Then the natural map*

$$\underline{\mathrm{HC}(R)} \longrightarrow \underline{\mathrm{HC}}(\underline{R}_p^\wedge),$$

*in the derived category $\mathcal{D}(\mathrm{Solid})$, exhibits the target as the $p$-completion of the source. In particular, there is a natural equivalence*

$$\underline{\mathrm{HC}(R)}_p^\wedge[\tfrac{1}{p}] \xrightarrow{\sim} \underline{\mathrm{HC}}(\underline{R}_p^\wedge[\tfrac{1}{p}]/\,\mathbb{Q}_p)$$

*in the derived category $\mathcal{D}(\mathrm{Solid})$.*

*Proof.* The first statement is a consequence of Lemma 5.3.6, and the description [Mor18b, Definition 2.19] of (solid) cyclic homology in terms of the cyclic object $\mathrm{HH}(R)$ (resp. $\underline{\mathrm{HH}}(\underline{R}_p^\wedge)$) in the stable $\infty$-category $\mathcal{D}(\mathbb{Z})$ (resp. $\mathcal{D}(\mathrm{Solid})$). The second statement follows from the fact that rationalisation commutes direct sums (or equivalently, with the functor $(-)_{h\mathbb{S}^1}$) and with the functor $\underline{(-)}$ from the derived category $\mathcal{D}(\mathbb{Z})$ to the derived category $\mathcal{D}(\mathrm{Solid})$. □

**Corollary 5.3.8.** *Let $f : R \to R'$ a 1-connected map of connective $\mathbb{Z}$-$\mathbb{E}_1$-algebras. Then for every integer $n \geq 0$, the map*

$$n!s^n : \prod_{p\in\mathbb{P}}{}' \,\mathrm{HC}(f;\mathbb{Q}_p)[-2n] \longrightarrow \prod_{p\in\mathbb{P}}{}' \,\mathrm{HC}(f;\mathbb{Q}_p),$$

*in the derived category $\mathcal{D}(\mathbb{Q})$, is the zero map on cohomology groups in degrees at least $-n+1$.*

*Proof.* For every prime number $p$, the natural map

$$n!s^n : \underline{\mathrm{HC}}(\underline{f}_p^\wedge)[-2n] \longrightarrow \underline{\mathrm{HC}}(\underline{f}_p^\wedge),$$

in the derived category $\mathcal{D}(\mathrm{Solid})$, is zero in cohomological degrees at least $-n+1$ (Theorem 5.3.5). Forgetting the condensed structure and taking the product over all primes $p$, this implies that the natural map

$$n!s^n : \prod_{p\in\mathbb{P}} \mathrm{HC}(f;\mathbb{Z}_p)[-2n] \longrightarrow \prod_{p\in\mathbb{P}} \mathrm{HC}(f;\mathbb{Z}_p),$$

in the derived category $\mathcal{D}(\mathbb{Z})$, is zero in cohomological degrees at least $-n+1$ (Proposition 5.3.7). Taking rationalisation then implies the desired result. □

**Theorem 5.3.9** (Rigid-analytic HP is truncating)**.** *The construction*

$$R \longmapsto \Big(\prod_{p\in\mathbb{P}}{}' \operatorname{HH}(R;\mathbb{Q}_p)\Big)^{t\mathrm{S}^1} := \Big(\Big(\prod_{p\in\mathbb{P}} \operatorname{HH}(R;\mathbb{Z}_p)\Big)_{\mathbb{Q}}\Big)^{t\mathrm{S}^1},$$

*from connective $\mathbb{Z}$-$\mathbb{E}_1$-algebras $R$ to the derived category $\mathcal{D}(\mathbb{Q})$, is truncating. More precisely, there exists a truncating invariant $E : \mathrm{Cat}_\infty^{\mathrm{perf}} \to \mathcal{D}(\mathbb{Q})$ such that the previous construction is the composite $R \mapsto \operatorname{Perf}(R) \mapsto E(\operatorname{Perf}(R))$.*

*Proof.* Let $f : R \to R'$ be a 1-connected map of connective $\mathbb{Z}$-$\mathbb{E}_1$-algebras. We want to prove that the natural map

$$\Big(\prod_{p\in\mathbb{P}}{}' \operatorname{HH}(R;\mathbb{Q}_p)\Big)^{t\mathrm{S}^1} \longrightarrow \Big(\prod_{p\in\mathbb{P}}{}' \operatorname{HH}(R';\mathbb{Q}_p)\Big)^{t\mathrm{S}^1}$$

is an equivalence, or equivalently that its homotopy cofibre $\big(\prod'_{p\in\mathbb{P}} \operatorname{HH}(f;\mathbb{Q}_p)\big)^{t\mathrm{S}^1}$ vanishes in the derived category $\mathcal{D}(\mathbb{Q})$. For every integer $n \geq 0$, the natural map

$$n!s^n : \prod_{p\in\mathbb{P}}{}' \operatorname{HC}(f;\mathbb{Q}_p)[-2n] \longrightarrow \prod_{p\in\mathbb{P}}{}' \operatorname{HC}(f;\mathbb{Q}_p)$$

is the zero map in cohomological degrees at least $-n+1$ (Corollary 5.3.8). This map is $\mathbb{Q}$-linear, so the map

$$s^n : \prod_{p\in\mathbb{P}}{}' \operatorname{HC}(f;\mathbb{Q}_p)[-2n] \longrightarrow \prod_{p\in\mathbb{P}}{}' \operatorname{HC}(f;\mathbb{Q}_p)$$

is also the zero map in cohomological degrees at least $-n+1$. Taking the inverse limit over integers $n \geq 0$ and using the equivalence at the end of Notation 5.3.2 then implies that the complex $\big(\prod'_{p\in\mathbb{P}} \operatorname{HH}(f;\mathbb{Q}_p)\big)^{t\mathrm{S}^1}$ is zero in the derived category $\mathcal{D}(\mathbb{Q})$. □

## 5.4 Rigid-analytic derived de Rham cohomology is a cdh sheaf

The aim of this section is to prove the cdh descent result Corollary 5.4.5, which is a rigid-analytic analogue of the following result, used in [EM23] to prove Theorem 5.0.2 in characteristic zero.

**Proposition 5.4.1** ([CHSW08, Bal23])**.** *For every integer $i \in \mathbb{Z}$, the presheaf*

$$\operatorname{Fil}^i_{\mathrm{HKR}}\operatorname{HP}(-_{\mathbb{Q}}/\,\mathbb{Q}) : \mathrm{Sch}^{\mathrm{qcqs,op}} \longrightarrow \mathcal{D}(\mathbb{Q})$$

*is a cdh sheaf.*

*Proof.* This is a consequence of [LT19, Corollary A.6] and [Bal23, Theorem 1.3]. □

We first extract the following cdh descent result from Theorem 5.3.9. Note that this argument uses the theory of truncating invariants, as developed by Land–Tamme [LT19], in a crucial way.

**Corollary 5.4.2.** *The presheaf*

$$\Big(\prod_{p\in\mathbb{P}}{}' \operatorname{HH}(-;\mathbb{Q}_p)\Big)^{t\mathrm{S}^1} : \mathrm{Sch}^{\mathrm{qcqs},\mathrm{op}} \longrightarrow \mathcal{D}(\mathbb{Q})$$

*is a cdh sheaf.*

*Proof.* This is a consequence of Theorem 5.3.9 and [LT19, Theorem E]. □

We then adapt the main splitting result of [Bal23] on periodic cyclic homology over $\mathbb{Q}$ to our rigid-analytic setting.

**Proposition 5.4.3.** *Let $X$ be a qcqs derived scheme. Then there exists a natural equivalence*

$$\Big(\prod_{p\in\mathbb{P}}{}' \operatorname{HH}(X;\mathbb{Q}_p)\Big)^{t\mathrm{S}^1} \simeq \prod_{i\in\mathbb{Z}} R\Gamma_{\mathrm{Zar}}\Big(X, \prod_{p\in\mathbb{P}}{}' \underline{\widehat{\mathbb{L}\Omega}}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p}\Big)[2i]$$

*in the derived category $\mathcal{D}(\mathbb{Q})$.*

*Proof.* We adapt the proof of [Bal23, Theorem 3.4], which proves a similar decomposition for periodic cyclic homology over $\mathbb{Q}$. The crucial point to adapt this proof is to note that there is a natural equivalence

$$\prod_{p\in\mathbb{P}}{}' \operatorname{HH}(X;\mathbb{Q}_p) \simeq \bigoplus_{i\in\mathbb{N}} R\Gamma_{\mathrm{Zar}}\Big(X, \big(\prod_{p\in\mathbb{P}}(\mathbb{L}^i_{-/\mathbb{Z}})^{\wedge}_p\big)_{\mathbb{Q}}\Big)[i]$$

in the derived category $\mathcal{D}(\mathbb{Q})$ (see [Bal23, Remark 2.8]). This is the rigid-analytic analogue of [Bal23, Proposition 2.7], and it is for instance a consequence of Lemma 5.1.10 applied to the $\mathbb{N}$-indexed filtration $\mathrm{Fil}^\star_{\mathrm{HKR}} \prod'_{p\in\mathbb{P}} \operatorname{HH}(-;\mathbb{Q}_p)$. By [Bal23, Theorem 3.2], this implies that there is a natural equivalence

$$\Big(\mathrm{Fil}^\star_{\mathrm{H}} R\Gamma_{\mathrm{Zar}}\big(X, \prod_{p\in\mathbb{P}}{}'\underline{\widehat{\mathbb{L}\Omega}}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p}\big) \otimes \mathrm{Fil}^\star_{\mathrm{T}}\big(\prod_{p\in\mathbb{P}}{}' \mathbb{Q}_p\big)^{t\mathrm{S}^1}\Big)^{\wedge} \xrightarrow{\sim} \mathrm{Fil}^\star_{\mathrm{T}}\Big(\prod_{p\in\mathbb{P}}{}' \operatorname{HH}(X;\mathbb{Q}_p)\Big)^{t\mathrm{S}^1}$$

in the filtered derived category $\mathcal{DF}(\mathbb{Q})$, where the tensor product $\otimes$ is the Day convolution tensor product, $\mathrm{Fil}^\star_{\mathrm{H}}$ is the Hodge filtration on derived de Rham cohomology, and $(-)^{\wedge}$ is the completion with respect to the associated filtration. By a degree argument explained in [Bal23, proof of Theorem 3.4], the filtered object $\mathrm{Fil}^\star_{\mathrm{T}}\big(\prod'_{p\in\mathbb{P}}\mathbb{Q}_p\big)^{t\mathrm{S}^1}$ carries a canonical splitting, which induces an equivalence

$$\prod_{i\in\mathbb{Z}}\Big(\mathrm{Fil}^{\star+i}_{\mathrm{H}} R\Gamma_{\mathrm{Zar}}\big(X, \prod_{p\in\mathbb{P}}{}'\underline{\widehat{\mathbb{L}\Omega}}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p}\big)\Big)[2i] \xrightarrow{\sim} \mathrm{Fil}^\star_{\mathrm{T}}\Big(\prod_{p\in\mathbb{P}}{}' \operatorname{HH}(X;\mathbb{Q}_p)\Big)^{t\mathrm{S}^1}$$

in the filtered derived category $\mathcal{DF}(\mathbb{Q})$.

It then suffices to prove that the desired result is indeed obtained by taking the colimit over $\star \to -\infty$ of the previous equivalence. Following [Bal23, proof of Theorem 3.4], the result for the source is a formal consequence of the connectivity estimate for the functor

$$R\Gamma_{\mathrm{Zar}}\Big(-, \prod_{p\in\mathbb{P}}{}'\underline{\mathbb{L}\Omega}^{\leq i}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p}\Big)$$

on animated commutative ring, which is itself a consequence of Remark 5.2.4, and of the classical connectivity estimate for the HKR filtration on cyclic homology (Proposition 3.1.8). The result for the target is a consequence of the fact that the Tate filtration is always exhaustive ([Bal23, Proposition B.6]). □

**Corollary 5.4.4.** *The presheaf*

$$R\Gamma_{\mathrm{Zar}}\Big(-, \prod_{p\in\mathbb{P}}{}' \, \widehat{\underline{\mathbb{L}\Omega}}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p}\Big) : \mathrm{Sch}^{\mathrm{qcqs,op}} \longrightarrow \mathcal{D}(\mathbb{Q})$$

*is a cdh sheaf.*

*Proof.* This is a consequence of the natural splitting Proposition 5.4.3, and of the cdh descent result Corollary 5.4.2. □

**Corollary 5.4.5.** *For every integer $i \in \mathbb{Z}$, the presheaf*

$$\mathrm{Fil}^i_{\mathrm{HKR}}\Big(\prod_{p\in\mathbb{P}}{}' \, \mathrm{HH}(-;\mathbb{Q}_p)\Big)^{tS^1} : \mathrm{Sch}^{\mathrm{qcqs,op}} \longrightarrow \mathcal{D}(\mathbb{Q})$$

*is a cdh sheaf.*

*Proof.* The HKR filtration on $\big(\prod'_{p\in\mathbb{P}} \mathrm{HH}(-;\mathbb{Q}_p)\big)^{tS^1}$ is complete by Lemma 5.2.9, so it suffices to prove the result on graded pieces. The result is then Corollary 5.4.4. □

## 5.5 The Atiyah–Hirzebruch spectral sequence

In this section, we use the results of the previous sections to prove Theorem C.

**Proposition 5.5.1** (The motivic filtration is $\mathbb{N}$-indexed)**.** *Let $X$ be a qcqs derived scheme. Then for every integer $i \leq 0$, the natural map*

$$\mathrm{Fil}^i_{\mathrm{mot}}\mathrm{K}(X) \longrightarrow \mathrm{K}(X)$$

*is an equivalence of spectra. In particular, the motivic filtration $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X)$ on $\mathrm{K}(X)$ is exhaustive.*

*Proof.* First assume that $X$ is a qcqs classical scheme. The filtration $\mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{KH}(X)$ is $\mathbb{N}$-indexed by [BEM24], so it suffices to prove that the filtration

$$\mathrm{cofib}\Big(\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(X) \longrightarrow \big(L_{\mathrm{cdh}}\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}\big)(X)\Big)$$

is $\mathbb{N}$-indexed (Definition 4.3.4). To prove this, we use Proposition 5.2.14 and Corollary 5.2.15. For every prime number $p$, the filtration $\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)$ is $\mathbb{N}$-indexed on qcqs schemes ([BMS19, Theorem 7.2 (1)] and Theorem 3.2.11 (2)). In particular, the filtration

$$\mathrm{cofib}\Big(\prod_{p\in\mathbb{P}}\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(X;\mathbb{Z}_p) \longrightarrow \big(L_{\mathrm{cdh}}\prod_{p\in\mathbb{P}}\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)\big)(X)\Big)$$

is $\mathbb{N}$-indexed. The presheaf $\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HP}(-_{\mathbb{Q}}/\mathbb{Q})$ is a cdh sheaf on qcqs schemes (Proposition 5.4.1), and the filtration $\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}(X_{\mathbb{Q}}/\mathbb{Q})$ is $\mathbb{N}$-indexed by construction. In particular, the filtration

$$\mathrm{cofib}\Big(\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X_{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \big(L_{\mathrm{cdh}}\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(-_{\mathbb{Q}}/\mathbb{Q})\big)(X)\Big)$$

is naturally identified with the filtration

$$\mathrm{cofib}\Big(\mathrm{Fil}^{\star-1}_{\mathrm{HKR}}\mathrm{HC}(X_{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \big(L_{\mathrm{cdh}}\mathrm{Fil}^{\star-1}_{\mathrm{HKR}}\mathrm{HC}(-_{\mathbb{Q}}/\mathbb{Q})\big)(X)\Big)[1],$$

which is $\mathbb{N}$-indexed. Similarly, the presheaf $\mathrm{Fil}^\star_{\mathrm{HKR}}\big(\prod'_{p\in\mathbb{P}}\mathrm{HH}(-;\mathbb{Q}_p)\big)^{t\mathrm{S}^1}$ is a cdh sheaf on qcqs schemes (Corollary 5.4.5), and the filtration $\mathrm{Fil}^\star_{\mathrm{HKR}}\big(\prod'_{p\in\mathbb{P}}\mathrm{HH}(-;\mathbb{Q}_p)\big)_{h\mathrm{S}^1}$ is $\mathbb{N}$-indexed (Remark 5.2.4). In particular, the filtration

$$\mathrm{cofib}\Big(\mathrm{Fil}^\star_{\mathrm{HKR}}\big(\prod_{p\in\mathbb{P}}{}'\mathrm{HH}(X;\mathbb{Q}_p)\big)^{h\mathrm{S}^1}\longrightarrow\big(L_{\mathrm{cdh}}\mathrm{Fil}^\star_{\mathrm{HKR}}\big(\prod_{p\in\mathbb{P}}{}'\mathrm{HH}(-;\mathbb{Q}_p)\big)^{t\mathrm{S}^1}\big)(X)\Big)$$

is naturally identified with the filtration

$$\mathrm{cofib}\Big(\mathrm{Fil}^{\star-1}_{\mathrm{HKR}}\big(\prod_{p\in\mathbb{P}}{}'\mathrm{HH}(X;\mathbb{Q}_p)\big)_{h\mathrm{S}^1}\longrightarrow\big(L_{\mathrm{cdh}}\mathrm{Fil}^{\star-1}_{\mathrm{HKR}}\big(\prod_{p\in\mathbb{P}}{}'\mathrm{HH}(-;\mathbb{Q}_p)\big)_{h\mathrm{S}^1}\big)(X)\Big)[1],$$

which is $\mathbb{N}$-indexed.

Assume now that $X$ is a general qcqs derived scheme. By Definition 4.3.5 and the previous paragraph, it suffices to prove that the filtration

$$\mathrm{cofib}\Big(\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(X)\longrightarrow\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(\pi_0(X))\Big)$$

is $\mathbb{N}$-indexed. To prove this, we use Proposition 5.2.14. For every prime number $p$, the filtration $\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(X;\mathbb{Z}_p)$ is $\mathbb{N}$-indexed on polynomial $\mathbb{Z}$-algebras by the previous paragraph, and hence on general qcqs derived schemes by Zariski descent and Theorem 3.2.11 (2). In particular, the filtration

$$\mathrm{cofib}\Big(\prod_{p\in\mathbb{P}}\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(X;\mathbb{Z}_p)\longrightarrow\prod_{p\in\mathbb{P}}\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(\pi_0(X);\mathbb{Z}_p)\Big)$$

is $\mathbb{N}$-indexed. The filtration

$$\mathrm{cofib}\Big(\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X_{\mathbb{Q}}/\mathbb{Q})\longrightarrow\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(\pi_0(X)_{\mathbb{Q}}/\mathbb{Q})\Big)$$

is $\mathbb{N}$-indexed by [EM23, Theorem 4.39]. Similarly, using Theorem 5.3.9 and Proposition 5.4.3 as in the proof of Corollary 5.4.5, the natural map

$$\mathrm{Fil}^\star_{\mathrm{HKR}}\Big(\prod_{p\in\mathbb{P}}{}'\mathrm{HH}(X;\mathbb{Q}_p)\Big)^{t\mathrm{S}^1}\longrightarrow\mathrm{Fil}^\star_{\mathrm{HKR}}\Big(\prod_{p\in\mathbb{P}}{}'\mathrm{HH}(\pi_0(X);\mathbb{Q}_p)\Big)^{t\mathrm{S}^1}$$

is an equivalence of filtered spectra. In particular, the filtration

$$\mathrm{cofib}\Big(\mathrm{Fil}^\star_{\mathrm{HKR}}\Big(\prod_{p\in\mathbb{P}}{}'\mathrm{HH}(X;\mathbb{Q}_p)\Big)^{h\mathrm{S}^1}\longrightarrow\mathrm{Fil}^\star_{\mathrm{HKR}}\Big(\prod_{p\in\mathbb{P}}{}'\mathrm{HH}(\pi_0(X);\mathbb{Q}_p)\Big)^{h\mathrm{S}^1}\Big)$$

is naturally identified with the filtration

$$\mathrm{cofib}\Big(\mathrm{Fil}^\star_{\mathrm{HKR}}\Big(\prod_{p\in\mathbb{P}}{}'\mathrm{HH}(X;\mathbb{Q}_p)\Big)_{h\mathrm{S}^1}\longrightarrow\mathrm{Fil}^\star_{\mathrm{HKR}}\Big(\prod_{p\in\mathbb{P}}{}'\mathrm{HH}(\pi_0(X);\mathbb{Q}_p)\Big)_{h\mathrm{S}^1}\Big)[1],$$

which is $\mathbb{N}$-indexed by Remark 5.2.4. □

**Corollary 5.5.2.** *Let $X$ be a qcqs scheme. Then for every integer $i<0$, the motivic complex $\mathbb{Z}(i)^{\mathrm{mot}}(X)\in\mathcal{D}(X)$ is zero.*

*Proof.* This is a direct consequence of Proposition 5.5.1. □

**Proposition 5.5.3.** *Let $d \geq 0$ be an integer, and $X$ be a qcqs scheme of valuative dimension at most $d$. Then for every integer $i \in \mathbb{Z}$, the fibre of the natural map of spectra*

$$\mathrm{Fil}^i_{\mathrm{mot}}\mathrm{TC}(X) \longrightarrow \mathrm{Fil}^i_{\mathrm{mot}} L_{\mathrm{cdh}}\mathrm{TC}(X)$$

*is in cohomological degrees at most $-i+d+2$. In particular, the filtration formed by these spectra for all integers $i \in \mathbb{Z}$, and the rationalisation of this filtration, are complete.*

*Proof.* The last statement is a consequence of the first, as the connectivity bound for a given filtration induces the same connectivity bound for its rationalisation. For the connectivity bound, we use Proposition 5.2.14 and Corollary 5.2.15 to compare the spectra $\mathrm{Fil}^i_{\mathrm{mot}}\mathrm{TC}(X)$ and $\mathrm{Fil}^i_{\mathrm{mot}} L_{\mathrm{cdh}}\mathrm{TC}(X)$.

The presheaf $\prod_{p \in \mathbb{P}} \mathrm{Fil}^i_{\mathrm{BMS}}\mathrm{TC}(X;\mathbb{Z}_p)$ takes values in cohomological degrees at most $-i+1$ on affine schemes (Lemma 3.2.7). In particular, the fibre of the natural map of spectra

$$\prod_{p \in \mathbb{P}} \mathrm{Fil}^i_{\mathrm{BMS}}\mathrm{TC}(X;\mathbb{Z}_p) \longrightarrow \big(L_{\mathrm{cdh}} \prod_{p \in \mathbb{P}} \mathrm{Fil}^i_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)\big)(X)$$

is in cohomological degrees at most $-i+d+2$, as each term is in cohomological degrees at most $-i+d+1$ ([CM21, Theorem 3.12] and [EHIK21, Theorem 2.4.15]).

By Proposition 5.4.1, the fibre of the natural map of spectra

$$\mathrm{Fil}^i_{\mathrm{HKR}}\mathrm{HC}^-(X_{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \big(L_{\mathrm{cdh}}\mathrm{Fil}^i_{\mathrm{HKR}}\mathrm{HC}^-(-_{\mathbb{Q}}/\mathbb{Q})\big)(X)$$

is naturally identified with the fibre of the natural map of spectra

$$\mathrm{Fil}^{i-1}_{\mathrm{HKR}}\mathrm{HC}(X_{\mathbb{Q}}/\mathbb{Q})[1] \longrightarrow \big(L_{\mathrm{cdh}}\mathrm{Fil}^{i-1}_{\mathrm{HKR}}\mathrm{HC}(-_{\mathbb{Q}}/\mathbb{Q})[1]\big)(X).$$

The presheaf $\mathrm{Fil}^{i-1}_{\mathrm{HKR}}\mathrm{HC}(-_{\mathbb{Q}}/\mathbb{Q})$ takes values in cohomological degrees at most $-i-1$ (*e.g.*, by Lemma 3.1.9 and the description of the graded pieces Remark 3.1.7), so the previous fibre is in cohomological degrees at most $-i+d-1$ ([CM21, Theorem 3.12] and [EHIK21, Theorem 2.4.15]).

Similarly, by Corollary 5.4.5, the fibre of the natural map of spectra

$$\mathrm{Fil}^i_{\mathrm{HKR}}\big(\prod_{p \in \mathbb{P}}{}' \mathrm{HH}(X;\mathbb{Q}_p)\big)^{h\mathrm{S}^1} \longrightarrow \Big(L_{\mathrm{cdh}}\mathrm{Fil}^i_{\mathrm{HKR}}\big(\prod_{p \in \mathbb{P}}{}' \mathrm{HH}(-;\mathbb{Q}_p)\big)^{h\mathrm{S}^1}\Big)(X)$$

is naturally identified with the fibre of the natural map of spectra

$$\mathrm{Fil}^{i-1}_{\mathrm{HKR}}\big(\prod_{p \in \mathbb{P}}{}' \mathrm{HH}(X;\mathbb{Q}_p)\big)_{h\mathrm{S}^1}[1] \longrightarrow \Big(L_{\mathrm{cdh}}\mathrm{Fil}^{i-1}_{\mathrm{HKR}}\big(\prod_{p \in \mathbb{P}}{}' \mathrm{HH}(-;\mathbb{Q}_p)\big)_{h\mathrm{S}^1}[1]\Big)(X)$$

The presheaf $\mathrm{Fil}^{i-1}_{\mathrm{HKR}}\big(\prod'_{p \in \mathbb{P}} \mathrm{HH}(-;\mathbb{Q}_p)\big)_{h\mathrm{S}^1}$ takes values in cohomological degrees at most $-i-1$ (Remark 5.2.4), so the previous fibre is in cohomological degrees at most $-i+d-1$ ([CM21, Theorem 3.12] and [EHIK21, Theorem 2.4.15]).

The previous three connectivity results imply the desired result. □

**Proposition 5.5.4** (Completeness of the motivic filtration)**.** *Let $d \geq 0$ be an integer, and $X$ be a qcqs scheme of valuative dimension at most $d$. Then for every integer $i \in \mathbb{Z}$, the spectrum $\mathrm{Fil}^i_{\mathrm{mot}}\mathrm{K}(X)$ is in cohomological degrees at most $-i+d+2$. In particular, the motivic filtration $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X)$, and its rationalisation $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X;\mathbb{Q})$, are complete.*

*Proof.* As in the proof of Proposition 5.5.3, the last statement is a consequence of the first. By Definition 4.3.4, there is a natural fibre sequence of spectra

$$\mathrm{fib}\Big(\mathrm{Fil}^i_{\mathrm{mot}}\mathrm{TC}(X) \longrightarrow \mathrm{Fil}^i_{\mathrm{mot}}L_{\mathrm{cdh}}\mathrm{TC}(X)\Big) \longrightarrow \mathrm{Fil}^i_{\mathrm{mot}}\mathrm{K}(X) \longrightarrow \mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{KH}(X).$$

The left term of this fibre sequence is in cohomological degrees at most $-i+d+2$ by Proposition 5.5.3, and the right term is in cohomological degrees at most $-i+d$ by [BEM24], hence the desired result. □

**Remark 5.5.5** (Non-noetherian Weibel vanishing)**.** Let $X$ be a qcqs scheme of valuative dimension at most $d$. The same argument as in Proposition 5.5.4 implies that the negative $K$-groups $\mathrm{K}_{-n}(X)$ vanish for integers $n > d+2$. This is a weak form of Weibel vanishing in the non-noetherian case.

**Remark 5.5.6** (Motivic Weibel vanishing)**.** Let $X$ be a qcqs scheme of dimension at most $d$. Proposition 5.5.4 implies that for every integer $i \in \mathbb{Z}$, the motivic complex $\mathbb{Z}(i)^{\mathrm{mot}}(X) \in \mathcal{D}(\mathbb{Z})$ is in degrees at most $i+d+2$. When $X$ is noetherian (in which case the valuative and Krull dimensions coincide), we will prove that it is even in degrees at most $i+d$ (Theorem 10.3.3).

**Remark 5.5.7.** A map of finitary presheaves of filtered spectra on qcqs schemes, which are filtration-complete on finite-dimensional noetherian schemes, is an equivalence if and only if it is an equivalence on graded pieces. In light of Propositions 5.5.13 and 5.5.4, we will formulate most of our results at the level of motivic cohomology, although they can often be promoted to results on the associated filtered spectra.

**Corollary 5.5.8** (Completeness of $\mathrm{Fil}^\star_{\mathrm{mot}}L_{\mathrm{cdh}}\mathrm{TC}$)**.** *Let $X$ be a qcqs scheme of finite valuative dimension. Then the filtrations $\mathrm{Fil}^\star_{\mathrm{mot}}L_{\mathrm{cdh}}\mathrm{TC}(X)$ and $\mathrm{Fil}^\star_{\mathrm{mot}}L_{\mathrm{cdh}}\mathrm{TC}(X;\mathbb{Q})$ are complete.*

*Proof.* This is a consequence of Propositions 3.3.8 and 5.5.3. □

**Remark 5.5.9.** The filtrations $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(X)$ and $\mathrm{Fil}^\star_{\mathrm{mot}}L_{\mathrm{cdh}}\mathrm{TC}(X)$ do not satisfy separately a connectivity bound similar to that of Proposition 5.5.3.

**Corollary 5.5.10** (Atiyah–Hirzebruch spectral sequence)**.** *Let $X$ be a qcqs derived scheme. The motivic filtration $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X)$ on non-connective algebraic $K$-theory $\mathrm{K}(X)$ induces a natural Atiyah–Hirzebruch spectral sequence*

$$E_2^{i,j} = \mathrm{H}^{i-j}_{\mathrm{mot}}(X,\mathbb{Z}(-j)) \Longrightarrow \mathrm{K}_{-i-j}(X).$$

*If $X$ is a qcqs classical scheme of finite valuative dimension, then this Atiyah–Hirzebruch spectral sequence is convergent.*

*Proof.* The first statement is a consequence of the fact the motivic filtration $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X)$ is $\mathbb{N}$-indexed (Proposition 5.5.1). The second statement is a consequence of the connectivity bound for this motivic filtration (Proposition 5.5.4). □

The main consequence of Propositions 5.5.1 and 5.5.4 that we will use is the following result.

**Corollary 5.5.11.** *Let $X$ be a qcqs derived scheme. Then for every integer $i \geq 0$, there exists a natural equivalence of spectra*

$$\mathrm{K}(X;\mathbb{Q}) \simeq \big(\bigoplus_{0 \leq j < i} \mathbb{Q}(i)^{\mathrm{mot}}(X)[2i]\big) \oplus \mathrm{Fil}^i_{\mathrm{mot}}\mathrm{K}(X;\mathbb{Q}).$$

*Proof.* The motivic filtration $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X;\mathbb{Q})$ is $\mathbb{N}$-indexed by Proposition 5.5.1. The result on qcqs classical schemes is then a consequence of Lemma 5.1.10 and Remark 5.1.11, where the necessary hypotheses are satisfied by Proposition 5.5.4 and Corollary 5.1.9.

Assume now that $X$ is a general qcqs derived scheme. Again by Lemma 5.1.10, where the necessary hypotheses are satisfied for the filtration $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(-)$ by Proposition 3.3.8 and Corollary 5.1.6, there is a natural equivalence of spectra

$$\mathrm{Fil}^0_{\mathrm{mot}}\mathrm{TC}(X;\mathbb{Q}) \simeq \big(\bigoplus_{0\leq j<i} \mathbb{Q}(j)^{\mathrm{TC}}(X)[2j]\big) \oplus \mathrm{Fil}^i_{\mathrm{mot}}\mathrm{TC}(X;\mathbb{Q}).$$

The result is then a consequence of Definition 4.3.5 and the previous case, where the equivalences

$$\mathrm{Fil}^0_{\mathrm{mot}}\mathrm{K}(\pi_0(X);\mathbb{Q}) \simeq \bigoplus_{0\leq j<i} \mathbb{Q}(j)^{\mathrm{mot}}(\pi_0(X))[2j]\big) \oplus \mathrm{Fil}^i_{\mathrm{mot}}\mathrm{K}(\pi_0(X);\mathbb{Q})$$

and

$$\mathrm{Fil}^0_{\mathrm{mot}}\mathrm{TC}(\pi_0(X);\mathbb{Q}) \simeq \big(\bigoplus_{0\leq j<i} \mathbb{Q}(j)^{\mathrm{TC}}(\pi_0(X))[2j]\big) \oplus \mathrm{Fil}^i_{\mathrm{mot}}\mathrm{TC}(X;\mathbb{Q})$$

are compatible, by construction, with the natural map

$$\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(\pi_0(X);\mathbb{Q}) \longrightarrow \mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(\pi_0(X);\mathbb{Q})$$

of Definition 4.3.4. □

**Lemma 5.5.12.** *Let $\tau$ be the Zariski, Nisnevich, or cdh topology, and $(F_i)_{i\in I}$ be a direct system of presheaves. Then the natural map of presheaves*

$$\varinjlim_i L_\tau F_i \longrightarrow L_\tau \varinjlim_i F_i$$

*is an equivalence. In particular, the sheafification functor $L_\tau$ sends finitary Zariski sheaves to finitary $\tau$ sheaves.*

*Proof.* As a left adjoint, the sheafification functor $L_\tau$, from presheaves to $\tau$ sheaves, commutes with all colimits. Being a sheaf for the topology $\tau$ is detected using only finite limits, so the inclusion functor from $\tau$ sheaves to presheaves commutes with filtered colimits. Composing the previous two functors then implies that the functor $L_\tau$, from presheaves to presheaves, commutes with filtered colimits.

To prove the second statement, let $F$ be a finitary Zariski sheaf, and $(R_i)_{i\in I}$ be a direct system of commutative rings. The fact that the natural map

$$\varinjlim_i L_\tau F(R_i) \longrightarrow L_\tau F(\varinjlim_i R_i)$$

is an equivalence is a consequence of the finitariness of $F$, and of the first statement applied to the direct system of presheaves $\big(F(-_{R_i})\big)_{i\in I}$. □

**Proposition 5.5.13** (The motivic filtration is finitary)**.** *Let $i\in\mathbb{Z}$ be an integer. Then the presheaf*

$$\mathrm{Fil}^i_{\mathrm{mot}}\mathrm{K}(-) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{Sp}$$

*is a finitary Nisnevich sheaf,* i.e., *it satisfies descent for the Nisnevich topology and commutes with filtered colimits of rings.*

*Proof.* It suffices to prove the result modulo $p$ for every prime number $p$, and rationally. Algebraic $K$-theory is a finitary Nisnevich sheaf on qcqs derived schemes ([CMNN20, Proposition A.15]). The presheaf $\mathrm{Fil}^i_{\mathrm{mot}}\mathrm{K}(-;\mathbb{Q})$ is a natural direct summand of rationalised algebraic $K$-theory (Corollary 5.5.11), so it also is a finitary Nisnevich sheaf. To prove the result modulo a prime number $p$, note that the presheaf $\mathrm{Fil}^i_{\mathrm{cdh}}\mathrm{KH}(-)$ is a finitary cdh sheaf ([BEM24]), and in particular a finitary Nisnevich sheaf. By Theorem 3.2.11, the presheaf $\mathrm{Fil}^i_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{F}_p)$ is a finitary Nisnevich sheaf. By Lemma 5.5.12, this implies that the presheaf $L_{\mathrm{cdh}}\mathrm{Fil}^i_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{F}_p)$ is a finitary Nisnevich sheaf, and the result modulo $p$ is then a consequence of Proposition 4.3.11. □

**Corollary 5.5.14.** *Let $i \in \mathbb{Z}$ be an integer. Then the presheaf*

$$\mathbb{Z}(i)^{\mathrm{mot}}(-) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathcal{D}(\mathbb{Z})$$

*is a finitary Nisnevich sheaf.*

*Proof.* This is a direct consequence of Proposition 5.5.13. □

*Proof of Theorem 5.0.1.* The motivic filtration $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(-)$ is finitary on qcqs schemes (Proposition 5.5.13), and, for every noetherian scheme $X$ of finite dimension $d$ and every integer $i \in \mathbb{Z}$, the spectrum $\mathrm{Fil}^i_{\mathrm{mot}}\mathrm{K}(X;\mathbb{Q})$ is in cohomological degrees at most $-i+d+2$ (Proposition 5.5.4). Proposition 5.1.13 then implies that there exists a natural multiplicative equivalence of filtered spectra

$$\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X;\mathbb{Q}) \simeq \bigoplus_{j\geq\star} \mathbb{Q}(j)^{\mathrm{mot}}(X)[2j].$$

The same argument as in Corollary 5.5.11 then implies the result for general qcqs derived schemes.

□

**Corollary 5.5.15.** *Let $X$ be a qcqs derived scheme. Then there exists a natural equivalence of spectra*

$$\mathrm{K}(X;\mathbb{Q}) \simeq \bigoplus_{i\geq 0} \mathbb{Q}(i)^{\mathrm{mot}}(X)[2i].$$

*Proof.* The motivic filtration $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X;\mathbb{Q})$ is $\mathbb{N}$-indexed by Proposition 5.5.1. The result is then a consequence of the rational splitting Theorem 5.0.1. □

## 5.6 Rational structure of motivic cohomology

In this section, we finish the proof of Theorem 5.0.2. We first use an argument of Weibel to prove the following result at the level of $K$-theory. We then use the rational splitting Corollary 5.5.11 to prove a filtered version of this result, which reduces the proof of Theorem 5.0.2 to the case of characteristic zero.

**Lemma 5.6.1.** *Let $X$ be a qcqs scheme. Then the natural commutative diagram*

$$\begin{array}{ccc} \mathrm{K}(X;\mathbb{Q}) & \longrightarrow & \mathrm{K}(X_{\mathbb{Q}};\mathbb{Q}) \\ \downarrow & & \downarrow \\ \mathrm{KH}(X;\mathbb{Q}) & \longrightarrow & \mathrm{KH}(X_{\mathbb{Q}};\mathbb{Q}) \end{array}$$

*is a cartesian square of spectra.*

*Proof.* By Zariski descent, it suffices to prove the result for affine schemes $X = \mathrm{Spec}(R)$. For every integer $n \in \mathbb{Z}$, let

$$\mathrm{NK}_n(R) := \mathrm{coker}\big(\mathrm{K}_n(R) \longrightarrow \mathrm{K}_n(R[T])\big)$$

be the $n^{\text{th}}$ $NK$-group of $R$. By [Wei81, Corollary 6.4] (see also [TT90, Exercise 9.12], where some unnecessary hypotheses in Weibel's result are removed), there is a natural isomorphism of abelian groups

$$\mathrm{NK}_n(R) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} \mathrm{NK}_n(R \otimes_{\mathbb{Z}} \mathbb{Q})$$

for every integer $n \in \mathbb{Z}$ and every commutative ring $R$. For every commutative ring $R$, the homotopy groups of the fibre $\mathrm{K}^{\mathrm{W}}(R)$ of the map of spectra $\mathrm{K}(R) \to \mathrm{KH}(R)$ have a natural exhaustive complete filtration with graded pieces given by the iterated $NK$-groups of $R$. In particular, for every integer $n \in \mathbb{Z}$ and every commutative ring $R$, the natural map

$$\mathrm{K}^{\mathrm{W}}(R; \mathbb{Q}) \longrightarrow \mathrm{K}^{\mathrm{W}}(R \otimes_{\mathbb{Z}} \mathbb{Q}; \mathbb{Q})$$

is an equivalence of spectra, which implies the desired result. □

**Corollary 5.6.2.** *Let $X$ be a qcqs scheme. Then the natural commutative diagram*

$$\begin{array}{ccc}
\mathrm{K}(X;\mathbb{Q}) & \longrightarrow & \mathrm{HC}^-(X_{\mathbb{Q}}/\mathbb{Q}) \\
\downarrow & & \downarrow \\
\mathrm{KH}(X;\mathbb{Q}) & \longrightarrow & L_{\mathrm{cdh}}\mathrm{HC}^-(-_{\mathbb{Q}}/\mathbb{Q})(X)
\end{array}$$

*is a cartesian square of spectra.*

*Proof.* The result for qcqs $\mathbb{Q}$-schemes $X$ is due to Cortiñas–Haesemeyer–Schlichting–Weibel [CHSW08, CHW08] (see also Theorem 2.1.1). The general result is then a consequence of Lemma 5.6.1. □

We record for completeness the following result, which is well-known in characteristic zero ([CHSW08, CHW08]).

**Corollary 5.6.3.** *Let $X$ be a qcqs scheme. Then there is a natural fibre sequence of spectra*

$$\mathrm{K}(X;\mathbb{Q}) \longrightarrow \mathrm{KH}(X;\mathbb{Q}) \longrightarrow \mathrm{cofib}\big(\mathrm{HC}(X_{\mathbb{Q}}/\mathbb{Q}) \longrightarrow L_{\mathrm{cdh}}\mathrm{HC}(-_{\mathbb{Q}}/\mathbb{Q})(X)\big)[1].$$

*Proof.* By Theorem 2.1.1 and Lemma 5.6.1, the natural commutative diagram

$$\begin{array}{ccc}
\mathrm{K}(X;\mathbb{Q}) & \longrightarrow & \mathrm{HC}^-(X_{\mathbb{Q}}/\mathbb{Q}) \\
\downarrow & & \downarrow \\
\mathrm{KH}(X;\mathbb{Q}) & \longrightarrow & L_{\mathrm{cdh}}\mathrm{HC}^-(-_{\mathbb{Q}}/\mathbb{Q})(X)
\end{array}$$

is a cartesian square of spectra. By construction, there is moreover a natural commutative diagram of spectra

$$\begin{array}{ccccc}
\mathrm{HC}^-(X_{\mathbb{Q}}/\mathbb{Q}) & \longrightarrow & \mathrm{HP}(X_{\mathbb{Q}}/\mathbb{Q}) & \longrightarrow & \mathrm{HC}(X_{\mathbb{Q}}/\mathbb{Q})[2] \\
\downarrow & & \downarrow & & \downarrow \\
L_{\mathrm{cdh}}\mathrm{HC}^-(-_{\mathbb{Q}}/\mathbb{Q})(X) & \longrightarrow & L_{\mathrm{cdh}}\mathrm{HP}(-_{\mathbb{Q}}/\mathbb{Q})(X) & \longrightarrow & L_{\mathrm{cdh}}\mathrm{HC}(-_{\mathbb{Q}}/\mathbb{Q})(X)[2],
\end{array}$$

where the horizontal lines are fibre sequences. The middle vertical line of this diagram is an equivalence ([CHSW08, Corollary 3.13], see also [LT19, Corollary A.6]), so the cofibre of the left vertical map is naturally identified with the spectrum

$$\mathrm{cofib}\big(\mathrm{HC}(X_{\mathbb{Q}}/\mathbb{Q}) \longrightarrow L_{\mathrm{cdh}}\mathrm{HC}(-_{\mathbb{Q}}/\mathbb{Q})(X)\big)[1].$$

□

The following result is a filtered refinement of Lemma 5.6.1.

**Corollary 5.6.4.** *Let $X$ be a qcqs scheme. Then the natural commutative diagram*

$$\begin{array}{ccc} \mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{K}(X;\mathbb{Q}) & \longrightarrow & \mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{K}(X_{\mathbb{Q}};\mathbb{Q}) \\ \big\downarrow & & \big\downarrow \\ \mathrm{Fil}^{\star}_{\mathrm{cdh}}\mathrm{KH}(X;\mathbb{Q}) & \longrightarrow & \mathrm{Fil}^{\star}_{\mathrm{cdh}}\mathrm{KH}(X_{\mathbb{Q}};\mathbb{Q}) \end{array}$$

*is a cartesian square of filtered spectra.*

*Proof.* By Corollary 5.5.11, for every integer $i \geq 0$, the spectrum $\mathrm{Fil}^{i}_{\mathrm{mot}}\mathrm{K}(X;\mathbb{Q})$ is naturally a direct summand of the spectrum $\mathrm{K}(X;\mathbb{Q})$. The same applies to the other three filtrations of the cartesian square, by noting that the motivic filtration $\mathrm{Fil}^{\star}_{\mathrm{cdh}}\mathrm{KH}(-)$ also is $\mathbb{N}$-indexed ([BEM24]). The compatibility between the several filtrations is automatic from the construction of the splittings. So the result is a consequence of Lemma 5.6.1. □

The following result is a filtered refinement of Corollary 5.6.2.

**Theorem 5.6.5.** *Let $X$ be a qcqs scheme. Then the natural commutative diagram*

$$\begin{array}{ccc} \mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{K}(X;\mathbb{Q}) & \longrightarrow & \mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}^{-}(X_{\mathbb{Q}}/\mathbb{Q}) \\ \big\downarrow & & \big\downarrow \\ \mathrm{Fil}^{\star}_{\mathrm{cdh}}\mathrm{KH}(X;\mathbb{Q}) & \longrightarrow & L_{\mathrm{cdh}}\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}^{-}(-_{\mathbb{Q}}/\mathbb{Q})(X) \end{array}$$

*is a cartesian square of filtered spectra.*

*Proof.* This is a consequence of Corollary 5.6.4, where the filtration $\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{TC}(-)$ of qcqs $\mathbb{Q}$-schemes is naturally identified with the filtration $\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}^{-}(-/\mathbb{Q})$ (Remark 3.3.4). □

The following result is the rational part of Theorem B.

**Corollary 5.6.6.** *Let $X$ be a qcqs scheme. Then for every integer $i \in \mathbb{Z}$, the natural commutative diagram*

$$\begin{array}{ccc} \mathbb{Q}(i)^{\mathrm{mot}}(X) & \longrightarrow & R\Gamma_{\mathrm{Zar}}\big(X, \widehat{\mathbb{L}\Omega}^{\geq i}_{-_{\mathbb{Q}}/\mathbb{Q}}\big) \\ \big\downarrow & & \big\downarrow \\ \mathbb{Q}(i)^{\mathrm{cdh}}(X) & \longrightarrow & R\Gamma_{\mathrm{cdh}}\big(X, \widehat{\mathbb{L}\Omega}^{\geq i}_{-_{\mathbb{Q}}/\mathbb{Q}}\big) \end{array}$$

*is a cartesian square in the derived category $\mathcal{D}(\mathbb{Q})$.*

*Proof.* This is a direct consequence of Theorem 5.6.5. □

*Proof of Theorem 5.0.2.* By Theorem 5.6.5, the natural commutative diagram

$$\begin{array}{ccc} \mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X;\mathbb{Q}) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X_{\mathbb{Q}}/\mathbb{Q}) \\ \downarrow & & \downarrow \\ \mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{KH}(X;\mathbb{Q}) & \longrightarrow & L_{\mathrm{cdh}}\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(-_{\mathbb{Q}}/\mathbb{Q})(X) \end{array}$$

is a cartesian square of filtered spectra. By Definition 3.1.6 and Remark 3.1.10, there is a natural commutative diagram of filtered spectra

$$\begin{array}{ccccc} \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X_{\mathbb{Q}}/\mathbb{Q}) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HP}(X_{\mathbb{Q}}/\mathbb{Q}) & \longrightarrow & \mathrm{Fil}^{\star-1}_{\mathrm{HKR}}\mathrm{HC}(X_{\mathbb{Q}}/\mathbb{Q})[2] \\ \downarrow & & \downarrow & & \downarrow \\ L_{\mathrm{cdh}}\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(-_{\mathbb{Q}}/\mathbb{Q})(X) & \longrightarrow & L_{\mathrm{cdh}}\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HP}(-_{\mathbb{Q}}/\mathbb{Q})(X) & \longrightarrow & L_{\mathrm{cdh}}\mathrm{Fil}^{\star-1}_{\mathrm{HKR}}\mathrm{HC}(-_{\mathbb{Q}}/\mathbb{Q})(X)[2] \end{array}$$

where the horizontal lines are fibre sequences. The middle vertical map of this diagram is an equivalence (Proposition 5.4.1), so the cofibre of the left vertical map is naturally identified with the filtered spectrum

$$\mathrm{cofib}\big(\mathrm{Fil}^{\star-1}_{\mathrm{HKR}}\mathrm{HC}(X_{\mathbb{Q}}/\mathbb{Q}) \longrightarrow L_{\mathrm{cdh}}\mathrm{Fil}^{\star-1}_{\mathrm{HKR}}\mathrm{HC}(-_{\mathbb{Q}}/\mathbb{Q})(X)\big)[1].$$

□

The following result was proved by Elmanto–Morrow [EM23] for qcqs schemes $X$ over $\mathbb{Q}$.

**Corollary 5.6.7** (Rational motivic cohomology)**.** *Let $X$ be qcqs scheme. Then for every integer $i \in \mathbb{Z}$, there is a natural fibre sequence*

$$\mathbb{Q}(i)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Q}(i)^{\mathrm{cdh}}(X) \longrightarrow \mathrm{cofib}\Big(R\Gamma_{\mathrm{Zar}}\big(X, \mathbb{L}\Omega^{<i}_{-_{\mathbb{Q}}/\mathbb{Q}}\big) \longrightarrow R\Gamma_{\mathrm{cdh}}\big(X, \Omega^{<i}_{-_{\mathbb{Q}}/\mathbb{Q}}\big)\Big)[-1]$$

*in the derived category $\mathcal{D}(\mathbb{Q})$.*

*Proof.* For every valuation ring extension $V$ of $\mathbb{Q}$, the natural map

$$\mathbb{L}\Omega^{<i}_{V/\mathbb{Q}} \longrightarrow \Omega^{<i}_{V/\mathbb{Q}}$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Q})$ by results of Gabber–Ramero ([GR03, Theorem 6.5.8 (*ii*) and Corollary 6.5.21]). The presheaves $R\Gamma_{\mathrm{cdh}}(-, \mathbb{L}\Omega^{<i}_{-_{\mathbb{Q}}/\mathbb{Q}})$ and $R\Gamma_{\mathrm{cdh}}(-, \Omega^{<i}_{-_{\mathbb{Q}}/\mathbb{Q}})$ are finitary cdh sheaves on qcqs schemes, so the natural map

$$R\Gamma_{\mathrm{cdh}}\big(-, \mathbb{L}\Omega^{<i}_{-_{\mathbb{Q}}/\mathbb{Q}}\big) \longrightarrow R\Gamma_{\mathrm{cdh}}\big(-, \Omega^{<i}_{-_{\mathbb{Q}}/\mathbb{Q}}\big)$$

is an equivalence ([EHIK21, Corollary 2.4.19]). The result then follows from Theorem 5.0.2. □

**Example 5.6.8** (Weight zero motivic cohomology)**.** For every qcqs scheme $X$, the natural map

$$\mathbb{Z}(0)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Z}(0)^{\mathrm{cdh}}(X) \simeq R\Gamma_{\mathrm{cdh}}(X, \mathbb{Z}),$$

where the last idenfication is [BEM24], is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$. Indeed, it suffices to prove the result rationally, and modulo $p$ for every prime number $p$. The result rationally is a consequence of Corollary 5.6.7. For every prime number $p$, the presheaf $\mathbb{F}_p(0)^{\mathrm{BMS}}$ is naturally identified with the presheaf $R\Gamma_{\text{ét}}(-, \mathbb{F}_p)$ ([BMS19, Proposition 7.16]) which is a cdh sheaf on qcqs schemes ([BM21, Theorem 5.4]), so the result modulo $p$ is a consequence of Corollary 4.3.12.

## 5.7 A global Beilinson fibre square

In this section we prove Theorem 5.7.3, which is a rewriting of the Beilinson fibre square of Antieau–Mathew–Morrow–Nikolaus [AMMN22, Theorem 6.17] in terms of the rigid-analytic derived de Rham cohomology of Section 5.2. Note that our statements are formulated in the generality of derived schemes, and that the functor $-_{\mathbb{F}_p}$ is then the derived base change from $\mathbb{Z}$ to $\mathbb{F}_p$. The results in [AMMN22] are stated in the generality of $p$-torsionfree commutative rings, on which derived and classical reduction modulo $p$ coincide.

**Construction 5.7.1** (The map $\chi$)**.** Let $i \in \mathbb{Z}$ be an integer. Following [AMMN22], we construct for every qcqs derived scheme $X$ a natural map

$$\chi : \Big(\prod_{p\in\mathbb{P}} \mathbb{Z}_p(i)^{\mathrm{BMS}}(X_{\mathbb{F}_p})\Big)_{\mathbb{Q}} \longrightarrow \Big(\prod_{p\in\mathbb{P}} R\Gamma_{\mathrm{Zar}}\big(X, (\mathbb{L}\Omega_{-/\mathbb{Z}})^{\wedge}_p\big)\Big)_{\mathbb{Q}}$$

in the derived category $\mathcal{D}(\mathbb{Q})$.

(1) ($p \leq i+1$) Let $p$ be a prime number. By [AMMN22, Theorem 6.17], there exists an integer $N \geq 0$ depending only on $i$ and a natural map

$$\chi : \mathbb{Z}_p(i)^{\mathrm{BMS}}(R/p) \longrightarrow \frac{1}{p^N}\big(\mathbb{L}\Omega_{R/\mathbb{Z}}\big)^{\wedge}_p$$

on $p$-torsionfree $p$-quasisyntomic rings $R$, and in particular on polynomial $\mathbb{Z}$-algebras $R$. The functors $\mathbb{Z}_p(i)^{\mathrm{BMS}}(-/p)$ and $\frac{1}{p^N}(\mathbb{L}\Omega_{-/\mathbb{Z}})^{\wedge}_p$, as functors from animated commutative rings to $p$-complete objects in the derived category $\mathcal{D}(\mathbb{Z})$, are left Kan extended from polynomial $\mathbb{Z}$-algebras (Corollary 3.2.12 (2) and by construction, respectively). Left Kan extending the previous map then induces a natural map

$$\chi : \mathbb{Z}_p(i)^{\mathrm{BMS}}(R/p) \longrightarrow \frac{1}{p^N}\big(\mathbb{L}\Omega_{R/\mathbb{Z}}\big)^{\wedge}_p$$

on animated commutative rings $R$, where the reduction modulo $p$ is the derived one. Inverting $p$ and Zariski sheafifying induces a natural map

$$\chi : \mathbb{Q}_p(i)^{\mathrm{BMS}}(X_{\mathbb{F}_p}) \longrightarrow R\Gamma_{\mathrm{Zar}}\big(X, (\mathbb{L}\Omega_{-/\mathbb{Z}})^{\wedge}_p[\tfrac{1}{p}]\big)$$

in the derived category $\mathcal{D}(\mathbb{Q})$, on general qcqs derived schemes $X$.

(2) ($p \geq i+2$) For prime numbers $p$ such that $p \geq i+2$, there actually exists a natural map

$$\chi : \mathbb{Z}_p(i)^{\mathrm{BMS}}(R/p) \longrightarrow \big(\mathbb{L}\Omega_{R/\mathbb{Z}}\big)^{\wedge}_p$$

on $p$-torsionfree $p$-quasisyntomic rings $R$ ([AMMN22, Theorem 6.17]), and in particular on polynomial $\mathbb{Z}$-algebras $R$. Left Kan extending this map again induces a natural map

$$\chi : \mathbb{Z}_p(i)^{\mathrm{BMS}}(R/p) \longrightarrow \big(\mathbb{L}\Omega_{R/\mathbb{Z}}\big)^{\wedge}_p$$

on animated commutative rings $R$. Taking the product over all primes $p \geq i+2$, and then rationalisation and Zariski sheafification, induces a natural map

$$\chi : \Big(\prod_{p\in\mathbb{P}_{\geq i+2}} \mathbb{Z}_p(i)^{\mathrm{BMS}}(X_{\mathbb{F}_p})\Big)_{\mathbb{Q}} \longrightarrow \Big(\prod_{p\in\mathbb{P}_{\geq i+2}} R\Gamma_{\mathrm{Zar}}\big(X, (\mathbb{L}\Omega_{-/\mathbb{Z}})^{\wedge}_p\big)\Big)_{\mathbb{Q}}$$

in the derived category $\mathcal{D}(\mathbb{Q})$, on general qcqs derived schemes $X$.

(3) (general construction) For every qcqs derived scheme $X$, define the desired natural map $\chi$ as the product of the map $\chi$ of (2) with the finite product over prime numbers $p \leq i+1$ of the map $\chi$ of (1).

**Theorem 5.7.2** (Beilinson fibre square, after [AMMN22])**.** *Let $X$ be a qcqs derived scheme. Then for every integer $i \in \mathbb{Z}$, the natural diagram*

$$\begin{array}{ccc} \Big(\prod_{p\in\mathbb{P}} \mathbb{Z}_p(i)^{\mathrm{BMS}}(X)\Big)_{\mathbb{Q}} & \longrightarrow & \Big(\prod_{p\in\mathbb{P}} R\Gamma_{\mathrm{Zar}}\big(X,(\mathbb{L}\Omega^{\geq i}_{-/\mathbb{Z}})^{\wedge}_p\big)\Big)_{\mathbb{Q}} \\ \downarrow & & \downarrow \\ \Big(\prod_{p\in\mathbb{P}} \mathbb{Z}_p(i)^{\mathrm{BMS}}(X_{\mathbb{F}_p})\Big)_{\mathbb{Q}} & \xrightarrow{\chi} & \Big(\prod_{p\in\mathbb{P}} R\Gamma_{\mathrm{Zar}}\big(X,(\mathbb{L}\Omega_{-/\mathbb{Z}})^{\wedge}_p\big)\Big)_{\mathbb{Q}} \end{array}$$

*in the derived category $\mathcal{D}(\mathbb{Q})$ is commutative, with total cofibre naturally identified with the complex*

$$\Big(\prod_{p\in\mathbb{P}} R\Gamma_{\mathrm{Zar}}\big(X,\mathbb{L}\Omega^{<i}_{-_{\mathbb{F}_p}/\mathbb{Z}}\big)\Big)_{\mathbb{Q}} \in \mathcal{D}(\mathbb{Q}).$$

*Proof.* This is a consequence of Construction 5.7.1 and [AMMN22, Theorem 6.17]. □

In the following result, the map $\underline{\widehat{\chi}}$ is defined as the composite of the map $\chi$ of Construction 5.7.1 with the natural maps

$$\big(\prod_{p\in\mathbb{P}} R\Gamma_{\mathrm{Zar}}\big(X,(\mathbb{L}\Omega_{-/\mathbb{Z}})^{\wedge}_p\big)\big)_{\mathbb{Q}} \to \big(\prod_{p\in\mathbb{P}} R\Gamma_{\mathrm{Zar}}\big(X,(\widehat{\mathbb{L}\Omega}_{-/\mathbb{Z}})^{\wedge}_p\big)\big)_{\mathbb{Q}} \to R\Gamma_{\mathrm{Zar}}\big(X,{\prod_{p\in\mathbb{P}}}' \underline{\widehat{\mathbb{L}\Omega}}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p}\big)$$

induced by Hodge-completion and Remark 5.2.11.

**Theorem 5.7.3.** *Let $X$ be a qcqs derived scheme. Then for every integer $i \in \mathbb{Z}$, there exists a natural commutative square*

$$\begin{array}{ccc} \mathbb{Q}(i)^{\mathrm{TC}}(X) & \longrightarrow & R\Gamma_{\mathrm{Zar}}\big(X,\widehat{\mathbb{L}\Omega}^{\geq i}_{-_{\mathbb{Q}}/\mathbb{Q}}\big) \\ \downarrow & & \downarrow \\ \big(\prod_{p\in\mathbb{P}} \mathbb{Z}_p(i)^{\mathrm{BMS}}(X_{\mathbb{F}_p})\big)_{\mathbb{Q}} & \xrightarrow{\underline{\widehat{\chi}}} & R\Gamma_{\mathrm{Zar}}\big(X,\prod'_{p\in\mathbb{P}} \underline{\widehat{\mathbb{L}\Omega}}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p}\big) \end{array}$$

*in the derived category $\mathcal{D}(\mathbb{Q})$, whose total cofibre is naturally identified with the complex*

$$\Big(\prod_{p\in\mathbb{P}} R\Gamma_{\mathrm{Zar}}\big(X,\mathbb{L}\Omega^{<i}_{-_{\mathbb{F}_p}/\mathbb{Z}}\big)\Big)_{\mathbb{Q}} \in \mathcal{D}(\mathbb{Q}).$$

*Proof.* By construction, there exists a natural commutative diagram

$$\begin{array}{ccccc} \mathbb{Q}(i)^{\mathrm{TC}}(X) & \longrightarrow & R\Gamma_{\mathrm{Zar}}\big(X,(\widehat{\mathbb{L}\Omega}^{\geq i}_{-/\mathbb{Z}})_{\mathbb{Q}}\big) & \longrightarrow & R\Gamma_{\mathrm{Zar}}\big(X,\widehat{\mathbb{L}\Omega}^{\geq i}_{-_{\mathbb{Q}}/\mathbb{Q}}\big) \\ \downarrow & & \downarrow & & \downarrow \\ \Big(\prod_{p\in\mathbb{P}} \mathbb{Z}_p(i)^{\mathrm{BMS}}(X)\Big)_{\mathbb{Q}} & \longrightarrow & \Big(\prod_{p\in\mathbb{P}} R\Gamma_{\mathrm{Zar}}\big(X,(\widehat{\mathbb{L}\Omega}^{\geq i}_{-/\mathbb{Z}})^{\wedge}_p\big)\Big)_{\mathbb{Q}} & \to & R\Gamma_{\mathrm{Zar}}\big(X,\prod'_{p\in\mathbb{P}} \underline{\widehat{\mathbb{L}\Omega}}^{\geq i}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p}\big) \\ \downarrow & & \downarrow & & \downarrow \\ \Big(\prod_{p\in\mathbb{P}} \mathbb{Z}_p(i)^{\mathrm{BMS}}(X_{\mathbb{F}_p})\Big)_{\mathbb{Q}} & \to & \Big(\prod_{p\in\mathbb{P}} R\Gamma_{\mathrm{Zar}}\big(X,(\widehat{\mathbb{L}\Omega}_{-/\mathbb{Z}})^{\wedge}_p\big)\Big)_{\mathbb{Q}} & \to & R\Gamma_{\mathrm{Zar}}\big(X,\prod'_{p\in\mathbb{P}} \underline{\widehat{\mathbb{L}\Omega}}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p}\big) \end{array}$$

in the derived category $\mathcal{D}(\mathbb{Q})$, where all the inner squares are cartesian expect the left bottom one, and where the commutativity for the left bottom square is part of Theorem 5.7.2. The desired total cofibre is then naturally identified with the total cofibre of the left bottom square, and the result is then a consequence of Theorem 5.7.2. □

Theorem 5.7.3 means in particular that the complex $\mathbb{Q}(i)^{\mathrm{TC}}(X)$ can be expressed purely in terms of characteristic zero, characteristic $p$, and rigid-analytic information.

**Corollary 5.7.4.** *Let $p$ be a prime number, and $X$ be a qcqs derived $\mathbb{Z}_{(p)}$-scheme. Then for every integer $i \in \mathbb{Z}$, there exists a natural cartesian square*

$$\begin{array}{ccc} \mathbb{Q}(i)^{\mathrm{TC}}(X) & \longrightarrow & R\Gamma_{\mathrm{Zar}}\big(X, \widehat{\mathbb{L}\Omega}^{\geq i}_{-_{\mathbb{Q}}/\mathbb{Q}}\big) \\ \downarrow & & \downarrow \\ \mathbb{Q}_p(i)^{\mathrm{BMS}}(X_{\mathbb{F}_p}) & \xrightarrow{\widehat{\chi}} & R\Gamma_{\mathrm{Zar}}\big(X, \underline{\widehat{\mathbb{L}\Omega}}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p}\big) \end{array}$$

*in the derived category $\mathcal{D}(\mathbb{Q})$.*

*Proof.* The base change $X_{\mathbb{F}_\ell}$ is zero for every prime number $\ell$ different from $p$. The complex $\mathbb{L}\Omega^{<i}_{X_{\mathbb{F}_p}/\mathbb{Z}}$ is $\mathbb{F}_p$-linear, so its rationalisation vanishes. The result is then a consequence of Theorem 5.7.3 and Remark 5.2.7. □

The following result is an analogue of Theorem 5.7.3 at the level of filtered objects.

**Theorem 5.7.5.** *Let $X$ be qcqs derived scheme. Then there exists a natural commutative square of filtered spectra*

$$\begin{array}{ccc} \mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(X;\mathbb{Q}) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X_{\mathbb{Q}}/\mathbb{Q}) \\ \downarrow & & \downarrow \\ \Big(\prod_{p\in\mathbb{P}} \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(X_{\mathbb{F}_p})\Big)_{\mathbb{Q}} & \xrightarrow{\widehat{\chi}} & \mathrm{Fil}^\star_{\mathrm{HKR}}\Big(\prod'_{p\in\mathbb{P}} \mathrm{HH}(X;\mathbb{Q}_p)\Big)^{t\mathrm{S}^1}, \end{array}$$

*whose total cofibre is naturally identified with the filtered spectrum $\big(\prod_{p\in\mathbb{P}} \mathrm{Fil}^{\star-1}_{\mathrm{HKR}}\mathrm{HC}(X_{\mathbb{F}_p})\big)_{\mathbb{Q}}[2]$.*

*Proof.* The construction of the map $\chi$ in the proof of [AMMN22, Theorem 6.17] adapts readily to define a map at the filtered level instead of graded pieces. The proof is then the same as in Construction 5.7.1 and Theorem 5.7.3. □

**Question 5.7.6.** Given the results of the previous sections, and in particular Corollary 5.4.5 and Theorem 5.7.5, it is a natural question –to which we do not know the answer– to ask whether the presheaf

$$\Big(\prod_{p\in\mathbb{P}} \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-_{\mathbb{F}_p})\Big)_{\mathbb{Q}} : \mathrm{Sch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

is a cdh sheaf, where $-_{\mathbb{F}_p}$ again means derived base change from $\mathbb{Z}$ to $\mathbb{F}_p$.

# Chapter 6

# $p$-adic structure of motivic cohomology

In this chapter, we give a description of motivic cohomology with finite coefficients in terms of Bhatt–Lurie's syntomic cohomology (Theorem 6.2.4).

## 6.1 Comparison to étale cohomology

In this section, we construct a natural comparison map, called the Beilinson–Lichtenbaum comparison map, from the motivic cohomology of a scheme to the étale cohomology of its generic fibre (Definition 6.1.3). We then use this comparison map to establish a complete description of $\ell$-adic motivic cohomology in terms of étale cohomology (Theorem 6.1.5).

We use the following important result of Deligne to construct the Beilinson–Lichtenbaum comparison map.

**Theorem 6.1.1** ([BM21]). *Let $p$ be a prime number, and $k \geq 1$ be an integer. Then for every integer $i \geq 0$, the presheaf*

$$R\Gamma_{\text{ét}}(-[\tfrac{1}{p}], \mu_{p^k}^{\otimes i}) : \mathrm{Sch}^{\mathrm{qcqs,op}} \longrightarrow \mathcal{D}(\mathbb{Z}/p^k)$$

*is a cdh sheaf.*

*Proof.* By [BM21, Theorem 5.4], the presheaf $R\Gamma_{\text{ét}}(-[\frac{1}{p}], \mu_{p^k}^{\otimes i})$ is an arc sheaf on qcqs schemes, and the arc topology is finer than the cdh topology.[1] □

**Definition 6.1.2** (Cdh-local Beilinson–Lichtenbaum comparison map)**.** Let $p$ be a prime number, and $k \geq 1$ be an integer. For any integer $i \geq 0$, the *cdh-local Beilinson–Lichtenbaum comparison map* is the map

$$\mathbb{Z}/p^k(i)^{\mathrm{cdh}}(-) \longrightarrow R\Gamma_{\text{ét}}(-[\tfrac{1}{p}], \mu_{p^k}^{\otimes i})$$

of functors from (the opposite category of) qcqs derived schemes to the derived category $\mathcal{D}(\mathbb{Z}/p^k)$ defined as the composite

$$\mathbb{Z}/p^k(i)^{\mathrm{cdh}}(-) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{cdh}}(-[\tfrac{1}{p}]) \simeq \big(L_{\mathrm{cdh}}\tau^{\leq i} R\Gamma_{\text{ét}}(-, \mu_{p^k}^{\otimes i})\big)(-[\tfrac{1}{p}]) \longrightarrow R\Gamma_{\text{ét}}(-[\tfrac{1}{p}], \mu_{p^k}^{\otimes i})$$

where the first map is induced by base change from $\mathbb{Z}$ to $\mathbb{Z}[\frac{1}{p}]$, the equivalence is [BEM24], and the last map is induced by the natural transformation $\tau^{\leq i} \to \mathrm{id}$ and Theorem 6.1.1.

[1] More precisely, the arc topology is finer than the $v$ topology, which is finer than the h topology, which is finer than the cdh topology (see [BM21, EHIK21]).

**Definition 6.1.3** (Beilinson–Lichtenbaum comparison map)**.** Let $p$ be a prime number, and $k \geq 1$ be an integer. For any integer $i \geq 0$, the *Beilinson–Lichtenbaum comparison map* (or *motivic-étale comparison map*) is the map

$$\mathbb{Z}/p^k(i)^{\mathrm{mot}}(-) \longrightarrow R\Gamma_{\text{ét}}(-[\tfrac{1}{p}], \mu_{p^k}^{\otimes i})$$

of functors from (the opposite category of) qcqs schemes to the category $\mathcal{D}(\mathbb{Z}/p^k)$ defined as the composite

$$\mathbb{Z}/p^k(i)^{\mathrm{mot}}(-) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{cdh}}(-) \longrightarrow R\Gamma_{\text{ét}}(-[\tfrac{1}{p}], \mu_{p^k}^{\otimes i})$$

where the first map is cdh sheafification and the second map is the cdh-local Beilinson–Lichtenbaum comparison map of Definition 6.1.2.

**Remark 6.1.4.** Let $p$ be a prime number, $k \geq 1$ be an integer, $R$ be a commutative ring, and $R^h_p$ be the $p$-henselisation of $R$. Then for every integer $i \geq 0$, the natural diagram

$$\begin{array}{ccccc}
\mathbb{Z}/p^k(i)^{\mathrm{mot}}(\mathrm{Spec}(R)) & \longrightarrow & \mathbb{Z}/p^k(i)^{\mathrm{cdh}}(\mathrm{Spec}(R)) & \longrightarrow & R\Gamma_{\text{ét}}\big(\mathrm{Spec}(R[\tfrac{1}{p}]), \mu_{p^k}^{\otimes i}\big) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{Z}/p^k(i)^{\mathrm{mot}}(\mathrm{Spec}(R^h_p)) & \longrightarrow & \mathbb{Z}/p^k(i)^{\mathrm{cdh}}(\mathrm{Spec}(R^h_p)) & \longrightarrow & R\Gamma_{\text{ét}}\big(\mathrm{Spec}(R^h_p[\tfrac{1}{p}]), \mu_{p^k}^{\otimes i}\big) \\
\downarrow & & \downarrow & & \downarrow \mathrm{id} \\
\mathbb{Z}/p^k(i)^{\mathrm{BMS}}(\mathrm{Spec}(R^h_p)) & \longrightarrow & \big(L_{\mathrm{cdh}}\, \mathbb{Z}/p^k(i)^{\mathrm{BMS}}\big)(\mathrm{Spec}(R^h_p)) & \longrightarrow & R\Gamma_{\text{ét}}\big(\mathrm{Spec}(R^h_p[\tfrac{1}{p}]), \mu_{p^k}^{\otimes i}\big),
\end{array}$$

is a commutative diagram in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$, where the top horizontal right map and the middle horizontal right map are given by Definition 6.1.2, and the bottom horizontal right map is induced by [BL22, Theorem 8.3.1]. This statement is a consequence of the naturality of the constructions, except for the commutativity of the bottom right square, which is proved in [BEM24].

**Theorem 6.1.5** ($\ell$-adic motivic cohomology)**.** *Let $p$ be a prime number, $X$ be a qcqs scheme over $\mathbb{Z}[\frac{1}{p}]$, and $k \geq 1$ be an integer. Then for every integer $i \geq 0$, the Beilinson–Lichtenbaum comparison for classical motivic cohomology induces a natural equivalence*

$$\mathbb{Z}/p^k(i)^{\mathrm{mot}}(X) \simeq \big(L_{\mathrm{cdh}} \tau^{\leq i} R\Gamma_{\text{ét}}(-, \mu_{p^k}^{\otimes i})\big)(X)$$

*in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$.*

*Proof.* The syntomic complex $\mathbb{Z}/p^k(i)^{\mathrm{BMS}}$ and its cdh sheafification $L_{\mathrm{cdh}}\,\mathbb{Z}/p^k(i)^{\mathrm{BMS}}$ vanish on qcqs derived $\mathbb{Z}[\frac{1}{p}]$-schemes. In particular, the natural map

$$\mathbb{Z}/p^k(i)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{cdh}}(X)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$ (Proposition 4.3.11). The Beilinson–Lichtenbaum comparison for classical motivic cohomology induces a natural equivalence

$$\mathbb{Z}/p^k(i)^{\mathrm{cdh}}(X) \xleftarrow{\sim} \big(L_{\mathrm{cdh}} \tau^{\leq i} R\Gamma_{\text{ét}}(-, \mu_{p^k}^{\otimes i})\big)(X)$$

in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$ ([BEM24]). The desired equivalence is the composite of the previous two equivalences. □

**Corollary 6.1.6.** *Let $p$ be a prime number, $X$ be a qcqs $\mathbb{Z}[\frac{1}{p}]$-scheme, and $k \geq 1$ be an integer. Then for every integer $i \geq 0$, there is a natural equivalence*

$$\tau^{\leq i} \mathbb{Z}/p^k(i)^{\mathrm{mot}}(X) \simeq \tau^{\leq i} R\Gamma_{\text{ét}}(X, \mu_{p^k}^{\otimes i})$$

*in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$.*

*Proof.* The natural map

$$\tau^{\leq i}\big(L_{\mathrm{cdh}} \tau^{\leq i} R\Gamma_{\text{ét}}(-, \mu_{p^k}^{\otimes i})\big)(X) \longrightarrow \tau^{\leq i}\big(L_{\mathrm{cdh}} R\Gamma_{\text{ét}}(-, \mu_{p^k}^{\otimes i})\big)(X)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$. The result is then a consequence of Theorems 6.1.1 and 6.1.5. □

## 6.2 Comparison to syntomic cohomology

In this section, we study motivic cohomology with finite coefficients. Our main result is a computation of $p$-adic motivic cohomology in terms of syntomic cohomology (Theorem 6.2.4).

**Notation 6.2.1** (Syntomic cohomology of derived scheme, after Bhatt–Lurie [BL22])**.** Let $X$ be a qcqs derived scheme, $p$ be a prime number, and $i \in \mathbb{Z}$ be an integer. We denote by

$$\mathbb{Z}_p(i)^{\mathrm{syn}}(X) \in \mathcal{D}(\mathbb{Z}_p)$$

the *syntomic cohomology* of $X$, as defined in [BL22, Section 8.4]. For every integer $k \geq 1$, we also denote by $\mathbb{Z}/p^k(i)^{\mathrm{syn}}(X)$ the derived reduction modulo $p^k$ of the previous complex. In particular, the presheaf $\mathbb{Z}/p^k(i)^{\mathrm{syn}}$ is a Zariski sheaf, whose restriction to animated commutative rings is left Kan extended from smooth $\mathbb{Z}$-algebras, and such that on classical commutative rings $R$, there is, by definition, a natural cartesian square

$$\begin{array}{ccc} \mathbb{Z}/p^k(i)^{\mathrm{syn}}(\mathrm{Spec}(R)) & \longrightarrow & R\Gamma_{\text{ét}}(\mathrm{Spec}(R[\frac{1}{p}]), \mu_{p^k}^{\otimes i}) \\ \downarrow & & \downarrow \\ \mathbb{Z}/p^k(i)^{\mathrm{BMS}}(\mathrm{Spec}(R^h_p)) & \longrightarrow & R\Gamma_{\text{ét}}(\mathrm{Spec}(R^h_p[\frac{1}{p}]), \mu_{p^k}^{\otimes i}) \end{array}$$

in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$, where $R^h_p$ is the $p$-henselisation of the commutative ring $R$, and the bottom map is the map of [BL22, Theorem 8.3.1].

**Construction 6.2.2** (Motivic-syntomic comparison map)**.** Let $p$ be a prime number, and $k \geq 1$ be an integer. For any integer $i \geq 0$, the *motivic-syntomic comparison map* is the map

$$\mathbb{Z}/p^k(i)^{\mathrm{mot}}(-) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{syn}}(-)$$

of functors from (the opposite category of) qcqs schemes to the derived category $\mathcal{D}(\mathbb{Z}/p^k)$ defined as the Zariski sheafification of the map on commutative rings $R$ induced by the natural commutative diagram

$$\begin{array}{ccc} \mathbb{Z}/p^k(i)^{\mathrm{mot}}(\mathrm{Spec}(R)) & \longrightarrow & R\Gamma_{\text{ét}}\big(\mathrm{Spec}(R[\frac{1}{p}]), \mu_{p^k}^{\otimes i}\big) \\ \downarrow & & \downarrow \\ \mathbb{Z}/p^k(i)^{\mathrm{BMS}}(\mathrm{Spec}(R^h_p)) & \longrightarrow & R\Gamma_{\text{ét}}\big(\mathrm{Spec}(R^h_p[\frac{1}{p}]), \mu_{p^k}^{\otimes i}\big) \end{array}$$

of Remark 6.1.4.

The following cartesian square, where the bottom horizontal map was described independently in [BEM24], can be seen as an alternative definition of $p$-adic motivic cohomology of qcqs schemes (see Corollary 4.3.12).

**Proposition 6.2.3.** *Let $X$ be a qcqs scheme, $p$ be a prime number, and $k \geq 1$ be an integer. Then for every integer $i \geq 0$, the commutative diagram*

$$\begin{array}{ccc} \mathbb{Z}/p^k(i)^{\mathrm{mot}}(X) & \longrightarrow & \mathbb{Z}/p^k(i)^{\mathrm{syn}}(X) \\ \downarrow & & \downarrow \\ \mathbb{Z}/p^k(i)^{\mathrm{cdh}}(X) & \longrightarrow & \big(L_{\mathrm{cdh}}\,\mathbb{Z}/p^k(i)^{\mathrm{syn}}\big)(X) \end{array}$$

*where the top horizontal map is the motivic-syntomic comparison map of Construction 6.2.2 and the vertical maps are cdh sheafification, is a cartesian square in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$.*

*Proof.* By [BL22, Remark 8.4.4], there is a natural fibre sequence

$$R\Gamma_{\text{ét}}(X, j_!\mu_{p^k}^{\otimes i}) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{syn}}(X) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{BMS}}(X)$$

in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$. The first term of this fibre sequence satisfies arc descent by [BM21, Theorem 5.4], hence in particular cdh descent. The result is then a consequence of Corollary 4.3.12. ☐

The following result is a mixed characteristic generalisation of Elmanto–Morrow's fundamental fibre sequence for motivic cohomology of characteristic $p$ schemes ([EM23, Corollary 4.32]).

**Theorem 6.2.4** ($p$-adic motivic cohomology)**.** *Let $X$ be a qcqs scheme, $p$ be a prime number, and $k \geq 1$ be an integer. Then for every integer $i \geq 0$, there is a natural fibre sequence*

$$\mathbb{Z}/p^k(i)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{syn}}(X) \longrightarrow \big(L_{\mathrm{cdh}}\tau^{>i}\,\mathbb{Z}/p^k(i)^{\mathrm{syn}}\big)(X)$$

*in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$. In particular, the fibre of the motivic-syntomic comparison map is in degrees at least $i+2$.*

*Proof.* By [BEM24], there is a natural equivalence

$$\mathbb{Z}/p^k(i)^{\mathrm{cdh}}(X) \simeq \big(L_{\mathrm{cdh}}\tau^{\leq i}\,\mathbb{Z}/p^k(i)^{\mathrm{syn}}\big)(X)$$

in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$. The result is then a consequence of Proposition 6.2.3. ☐

**Corollary 6.2.5.** *Let $X$ be a qcqs scheme, $p$ be a prime number, and $k \geq 1$ be an integer. Then for every integer $i \geq 0$, the motivic-syntomic comparison map induces a natural equivalence*

$$\tau^{\leq i}\,\mathbb{Z}/p^k(i)^{\mathrm{mot}}(X) \xrightarrow{\sim} \tau^{\leq i}\,\mathbb{Z}/p^k(i)^{\mathrm{syn}}(X)$$

*in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$.*

*Proof.* This is a consequence of Theorem 6.2.4. ☐

Algebraic $K$-theory being $\mathbb{A}^1$-invariant on regular schemes, one could expect the classical $\mathbb{A}^1$-invariant motivic cohomology to be a good theory for general regular schemes, not only schemes that are smooth over a Dedekind domain. However, most of the results on classical motivic cohomology in mixed characteristic are proved only in the smooth case, as consequences of the Gersten conjecture proved by Geisser [Gei04]. This is the case for the Beilinson–Lichtenbaum conjecture, which compares motivic cohomology with finite coefficients to étale cohomology. Combined with Theorem D (6), the following result extends the analogous result for classical motivic cohomology to the regular case.

**Corollary 6.2.6** (Beilinson–Lichtenbaum conjecture for $F$-smooth schemes)**.** *Let $p$ be a prime number, $X$ be a $p$-torsionfree $F$-smooth scheme (e.g., a regular scheme flat over $\mathbb{Z}$), and $k \geq 1$ be an integer. Then for every integer $i \geq 0$, the fibre of the Beilinson–Lichtenbaum comparison map*

$$\mathbb{Z}/p^k(i)^{\mathrm{mot}}(X) \longrightarrow R\Gamma_{\text{ét}}\big(X[\tfrac{1}{p}], \mu_{p^k}^{\otimes i}\big)$$

*is in degrees at least $i+1$.*

*Proof.* By [BM23, Theorem 1.8], the fibre of the natural map

$$\mathbb{Z}/p^k(i)^{\mathrm{syn}}(X) \longrightarrow R\Gamma_{\text{ét}}\big(X[\tfrac{1}{p}], \mu_{p^k}^{\otimes i}\big)$$

is in degrees at least $i+1$. The result is then a consequence of Theorem 6.2.4. □

**Corollary 6.2.7.** *Let $p$ be a prime number, $X$ be a $p$-torsionfree $F$-smooth scheme, and $k \geq 1$ be an integer. Then the fibre of the natural map*

$$\mathbb{Z}/p^k(i)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{mot}}\big(X[\tfrac{1}{p}]\big)$$

*is in degrees at least $i+1$.*

*Proof.* By construction, the Beilinson–Lichtenbaum comparison map

$$\mathbb{Z}/p^k(i)^{\mathrm{mot}}(X) \longrightarrow R\Gamma_{\text{ét}}\big(X[\tfrac{1}{p}], \mu_{p^k}^{\otimes i}\big)$$

naturally factors as the composite

$$\mathbb{Z}/p^k(i)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{mot}}\big(X[\tfrac{1}{p}]\big) \longrightarrow R\Gamma_{\text{ét}}\big(X[\tfrac{1}{p}], \mu_{p^k}^{\otimes i}\big).$$

The fibre of the second map is in degrees at least $i+1$ by Corollary 6.1.6, and the fibre of the composite is in degrees at least $i+1$ by Corollary 6.2.6, so the fibre of the first map is in degrees at least $i+1$. □

# Chapter 7

# Comparison to classical motivic cohomology

For smooth schemes over a field, Elmanto–Morrow proved that the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$ coincide with the classical motivic complexes $\mathbb{Z}(i)^{\mathrm{cla}}$ ([EM23, Corollary 6.4]). Their proof uses Gersten injectivity and the projective bundle formula to reduce the statement to the case of fields, and relies on Gabber's presentation lemma ([CTHK97, Theorem 3.1.1]), which is unknown in mixed characteristic. In this chapter, we prove partial results comparing the complexes $\mathbb{Z}(i)^{\mathrm{mot}}$ and $\mathbb{Z}(i)^{\mathrm{cla}}$ in mixed characteristic.

## 7.1 Comparison to classical motivic cohomology in low degrees

In this section, we prove that the classical-motivic comparison map $\mathbb{Z}(i)^{\mathrm{cla}} \to \mathbb{Z}(i)^{\mathrm{mot}}$ (Definition 4.3.9) is an equivalence with rational or $\ell$-adic coefficients, and integrally in degrees at most $i+1$.

**Proposition 7.1.1.** *Let $p$ be a prime number, $B$ be a Dedekind domain such that every characteristic $p$ residue field of $B$ is perfect, and $X$ be a smooth scheme over $B$. Then for any integers $i \geq 0$ and $k \geq 1$, the fibre of the classical-motivic comparison map*

$$\mathbb{Z}/p^k(i)^{\mathrm{cla}}(X) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{mot}}(X)$$

*is in degrees at least $i+2$ in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$. If $p$ is moreover invertible in $X$, then this fibre vanishes, i.e., the previous classical-motivic comparison map is an equivalence.*

*Proof.* If $p$ is invertible in the scheme $X$, then the composite map

$$\mathbb{Z}/p^k(i)^{\mathrm{cla}}(X) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{cdh}}(X)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$ by [BEM24]. The right map is also an equivalence by Remark 4.3.13, so the left map is an equivalence. In general, consider the natural commutative diagram

$$\begin{array}{ccc} \mathbb{Z}/p^k(i)^{\mathrm{cla}}(X) & \longrightarrow & \big(L_{\mathrm{Nis}}\tau^{\leq i}\,\mathbb{Z}/p^k(i)^{\mathrm{syn}}\big)(X) \\ \downarrow & & \downarrow \\ \mathbb{Z}/p^k(i)^{\mathrm{mot}}(X) & \longrightarrow & \mathbb{Z}/p^k(i)^{\mathrm{syn}}(X) \end{array}$$

in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$. The top horizontal map is an equivalence by [Gei04, Theorems 1.2 (2) and 1.3] and [BM23, Theorem 5.8]. The fibre of the right vertical map is naturally identified with $\big(L_{\mathrm{Nis}}\tau^{>i}\,\mathbb{Z}/p^k(i)^{\mathrm{syn}}\big)(X)[-1]$, and is thus in degrees at least $i+2$. The fibre of the bottom horizontal map is in degrees at least $i+2$ by Theorem 6.2.4. So the fibre of the left vertical map is in degrees at least $i+2$. □

**Proposition 7.1.2.** *Let $B$ be a mixed characteristic Dedekind domain, and $X$ be a smooth scheme over $B$. Then for every integer $i \geq 0$, the classical-motivic comparison map*

$$\mathbb{Q}(i)^{\mathrm{cla}}(X) \longrightarrow \mathbb{Q}(i)^{\mathrm{mot}}(X)$$

*is an equivalence in the derived category $\mathcal{D}(\mathbb{Q})$.*

*Proof.* This is a consequence of the rational splitting of algebraic $K$-theory induced by Adams operations. More precisely, we use the splitting induced by Lemma 5.1.10 for the filtrations $\mathrm{Fil}^{\star}_{\mathrm{cla}}\mathrm{K}(-;\mathbb{Q})$ (which is $\mathbb{N}$-indexed by construction) and $\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{K}(-;\mathbb{Q})$ (which is $\mathbb{N}$-indexed by Proposition 5.5.1). These decompositions are compatible with the classical-motivic comparison map because of the compatibility between the associated Adams operations (Section 5.1). □

**Theorem 7.1.3** (Comparison to classical motivic cohomology)**.** *Let $B$ be a mixed characteristic Dedekind domain such that every residue field of $B$ is perfect, and $X$ be a smooth scheme over $B$. Then for every integer $i \geq 0$, the fibre of the classical-motivic comparison map*

$$\mathbb{Z}(i)^{\mathrm{cla}}(X) \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(X)$$

*is in degrees at least $i+3$.*

*Proof.* Let $F(X) \in \mathcal{D}(\mathbb{Z})$ be the fibre of the classical-motivic comparison map

$$\mathbb{Z}(i)^{\mathrm{cla}}(X) \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(X).$$

We want to prove that for every integer $k \leq i+2$, the abelian group $\mathrm{H}^k(F(X))$ is zero. By Proposition 7.1.2, the abelian group $\mathrm{H}^k(F(X))$ is torsion for every integer $k \in \mathbb{Z}$. For every prime number $p$ and every integer $k \in \mathbb{Z}$, there is a natural short exact sequence of abelian groups

$$0 \longrightarrow \mathrm{H}^k(F(X))/p \longrightarrow \mathrm{H}^k(F(X)/p) \longrightarrow \mathrm{H}^{k+1}(F(X))[p] \longrightarrow 0.$$

By Proposition 7.1.1, for every prime number $p$, the abelian group $\mathrm{H}^k(F(X)/p)$ is zero if $k \leq i+1$, hence the abelian group $\mathrm{H}^k(F(X))[p]$ is zero for every integer $k \leq i+2$. This implies the desired result. □

**Corollary 7.1.4.** *Let $B$ be a mixed characteristic Dedekind domain such that every residue field of $B$ is perfect. Then for every integer $i \geq 0$, the classical-motivic comparison map induces an equivalence of $\mathcal{D}(\mathbb{Z})$-valued presheaves*

$$\mathbb{Z}(i)^{\mathrm{cla}}(-) \longrightarrow \big(L_{\mathrm{Zar}}\tau^{\leq i}\,\mathbb{Z}(i)^{\mathrm{mot}}\big)(-)$$

*on essentially smooth $B$-schemes.*

*Proof.* The classical motivic complex $\mathbb{Z}(i)^{\mathrm{cla}}$ is a Zariski sheaf which is locally supported in degrees at most $i$ ([Gei04, Corollary 4.4]), so the result is a consequence of Theorem 7.1.3. □

## 7.2 Comparison to classical motivic cohomology in low dimensions

In this section, we study in more detail the defect for the classical-motivic comparison map to be an equivalence (Theorem 7.2.5), and prove that this defect vanishes on smooth schemes of dimension at most one over a mixed characteristic Dedekind domain (Corollary 7.2.6).

**Lemma 7.2.1.** *Let $B$ be a commutative ring, $\pi$ be an element of $B$, $\mathcal{C}$ be a presentable $\infty$-category, and $F : \mathrm{Alg}_B \to \mathcal{C}$ be a finitary and rigid functor. If the functor $F$ is zero on $B[\frac{1}{\pi}]$-algebas, then for every qcqs $B$-scheme $X$, the natural map*

$$\big(L_{\mathrm{cdh}} F\big)(X) \longrightarrow \big(L_{\mathrm{cdh}} F\big)(X_{B/\pi})$$

*is an equivalence in the $\infty$-category $\mathcal{C}$.*

*Proof.* Covers in a site are stable under base change, and the cdh sheafification of a finitary presheaf is finitary, so the presheaf

$$(L_{\mathrm{cdh}} F)(-_{B/\pi})$$

is a finitary cdh sheaf on qcqs $B$-schemes. It then suffices to prove that for every henselian valuation ring $V$ which is a $B$-algebra, the natural map

$$F(V) \longrightarrow \big(L_{\mathrm{cdh}} F\big)(V/\pi)$$

is an equivalence in the $\infty$-category $\mathcal{C}$. The presheaf $L_{\mathrm{cdh}} F$ is a finitary cdh sheaf, so it is invariant under nilpotent extensions. In particular, the natural map

$$\big(L_{\mathrm{cdh}} F\big)(V/\pi) \longrightarrow \big(L_{\mathrm{cdh}} F\big)(V/\sqrt{(\pi)})$$

is an equivalence in the $\infty$-category $\mathcal{C}$. The quotient of a henselian valuation ring by one of its prime ideals is a henselian valuation ring, so the target of the previous map is naturally identified with the object $F(V/\sqrt{(\pi)}) \in \mathcal{C}$. We finally prove that the natural map

$$F(V) \longrightarrow F(V/\sqrt{(\pi)})$$

is an equivalence in the $\infty$-category $\mathcal{C}$. If $\pi$ is invertible in the henselian valuation ring $V$, then both terms are zero by hypothesis on the functor $F$. And if $\pi$ is not invertible in $V$, then in particular the henselian local ring $V$ is $\pi$-henselian, and the result is a consequence of rigidity for the functor $F$. □

**Corollary 7.2.2.** *Let $p$ be a prime number, $B$ be a discrete valuation ring of mixed characteristic $(0, p)$, $\pi$ be a uniformizer of $B$, and $X$ be a qcqs $B$-scheme. Then for any integers $i \geq 0$ and $k \geq 1$, the natural map*

$$\big(L_{\mathrm{cdh}} \tau^{>i}\, \mathbb{Z}/p^k(i)^{\mathrm{BMS}}\big)(X) \longrightarrow \big(L_{\mathrm{cdh}} \tau^{>i}\, \mathbb{Z}/p^k(i)^{\mathrm{BMS}}\big)(X_{B/\pi})$$

*is an equivalence in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$.*

*Proof.* The functor

$$\tau^{>i}\, \mathbb{Z}/p^k(i)^{\mathrm{BMS}} : \mathrm{Alg}_B \longrightarrow \mathcal{D}(\mathbb{Z}/p^k)$$

is finitary (Theorem 3.2.11 (2)) and rigid (Theorem 3.2.13). The functor $\mathbb{Z}/p^k(i)^{\mathrm{BMS}}$ is moreover zero on $\mathbb{Z}[\frac{1}{p}]$-algebras (Remark 3.2.2), so the result is a consequence of Lemma 7.2.1. □

**Remark 7.2.3.** One can prove similarly that Corollary 7.2.2 holds for $B$ a general valuation ring of mixed characteristic $(0,p)$, where the base change to the characteristic $p$ field $B/\pi$ is replaced by the base change to the characteristic $p$ valuation ring $B/\sqrt{(p)}$.

**Proposition 7.2.4.** *Let $p$ be a prime number, $B$ be a discrete valuation ring of mixed characteristic $(0,p)$, and $R$ be a henselian local ind-smooth $B$-algebra of residue characteristic $p$. Then for any integers $i \geq 0$ and $k \geq 1$, the natural map*

$$\tau^{>i}\,\mathbb{Z}/p^k(i)^{\mathrm{BMS}}(R) \longrightarrow \big(L_{\mathrm{cdh}}\tau^{>i}\,\mathbb{Z}/p^k(i)^{\mathrm{BMS}}\big)(R)$$

*is an equivalence in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$.*

*Proof.* Let $\pi$ be a uniformizer of the discrete valuation ring $B$, and consider the commutative diagram

$$\begin{array}{ccc} \tau^{>i}\,\mathbb{Z}/p^k(i)^{\mathrm{BMS}}(R) & \longrightarrow & \big(L_{\mathrm{cdh}}\tau^{>i}\,\mathbb{Z}/p^k(i)^{\mathrm{BMS}}\big)(R) \\ \downarrow & & \downarrow \\ \tau^{>i}\,\mathbb{Z}/p^k(i)^{\mathrm{BMS}}(R/\pi) & \longrightarrow & \big(L_{\mathrm{cdh}}\tau^{>i}\,\mathbb{Z}/p^k(i)^{\mathrm{BMS}}\big)(R/\pi) \end{array}$$

in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$. The commutative ring $R$ is $p$-henselian, hence $\pi$-henselian, so the left vertical map is an equivalence by Theorem 3.2.13. The right vertical map is an equivalence by Corollary 7.2.2. We prove now that the bottom horizontal map is an equivalence. The commutative ring $R/\pi$ is a henselian local ind-smooth algebra over the field $B/\pi$, so the classical-motivic comparison map

$$\mathbb{Z}/p^k(i)^{\mathrm{cla}}(R/\pi) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{mot}}(R/\pi)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$ ([EM23, Corollary 6.4]). The complex $\mathbb{Z}/p^k(i)^{\mathrm{cla}}(R) \in \mathcal{D}(\mathbb{Z}/p^k)$ is moreover in degrees at most $i$ since the commutative ring $R$ is henselian local, so this is equivalent to the fact that the natural map

$$\tau^{>i}\,\mathbb{Z}/p^k(i)^{\mathrm{BMS}}(R/\pi) \longrightarrow \big(L_{\mathrm{cdh}}\tau^{>i}\,\mathbb{Z}/p^k(i)^{\mathrm{BMS}}\big)(R/\pi)$$

is an equivalence in the derived $\mathcal{D}(\mathbb{Z}/p^k)$ (Theorem 6.2.4). □

**Theorem 7.2.5.** *Let $p$ be a prime number, $B$ be a mixed characteristic Dedekind domain such that every characteristic $p$ residue field of $B$ is perfect, and $R$ be a henselian local ind-smooth $B$-algebra of residue characteristic $p$. Then for any integers $i \geq 0$ and $k \geq 1$, the fibre of the classical-motivic comparison map*

$$\mathbb{Z}/p^k(i)^{\mathrm{cla}}(R) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{mot}}(R)$$

*is in degrees at least $i+2$, given by the complex*

$$\big(L_{\mathrm{cdh}}\tau^{\leq i}R\Gamma_{\mathrm{\acute{e}t}}(-, j_!\mu_{p^k}^{\otimes i})\big)(R)[-1] \in \mathcal{D}(\mathbb{Z}/p^k).$$

*Proof.* The fact that the fibre of the classical-motivic comparison map is in degrees at least $i+2$ is a special case of Proposition 7.1.1. The henselian local ring $R$ is local for the Nisnevich topology, so the proof of Proposition 7.1.1 implies that the complex $\mathbb{Z}/p^k(i)^{\mathrm{cla}}(R) \in \mathcal{D}(\mathbb{Z}/p^k)$ is moreover in degrees at most $i$, and this fibre is naturally identified with the complex

$\big(\tau^{>i}\mathbb{Z}/p^k(i)^{\mathrm{mot}}(R)\big)[-1] \in \mathcal{D}(\mathbb{Z}/p^k)$. By Theorem 6.2.4, this complex is in turn naturally identified with the complex

$$\mathrm{fib}\big(\tau^{>i}\mathbb{Z}/p^k(i)^{\mathrm{syn}}(R) \longrightarrow \big(L_{\mathrm{cdh}}\tau^{>i}\mathbb{Z}/p^k(i)^{\mathrm{syn}}(R)\big)[-1] \in \mathcal{D}(\mathbb{Z}/p^k).$$

Consider the commutative diagram

$$\begin{array}{ccc} \tau^{>i}\mathbb{Z}/p^k(i)^{\mathrm{syn}}(R) & \longrightarrow & \big(L_{\mathrm{cdh}}\tau^{>i}\mathbb{Z}/p^k(i)^{\mathrm{syn}}\big)(R) \\ \big\downarrow & & \big\downarrow \\ \tau^{>i}\mathbb{Z}/p^k(i)^{\mathrm{BMS}}(R) & \longrightarrow & \big(L_{\mathrm{cdh}}\tau^{>i}\mathbb{Z}/p^k(i)^{\mathrm{BMS}}\big)(R) \end{array}$$

in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$. The commutative ring $R$ is $p$-henselian, so the left vertical map is an equivalence (Notation 6.2.1). The commutative ring $R$ is local, so there exists a prime ideal $\mathfrak{p}$ of the Dedekind domain $B$ such that $p \in \mathfrak{p}$ and $R$ is essentially smooth over the localisation $B_{\mathfrak{p}}$. The local ring $B_{\mathfrak{p}}$ is a discrete valuation ring of mixed characteristic $(0, p)$, so the bottom horizontal map is an equivalence (Proposition 7.2.4). The fibre of the top horizontal map is then naturally identified with the $(-1)$-cohomological shift of the fibre of the right vertical map. By Lemma 9.2.2 (applied for $j = i$ and after cdh sheafification), this fibre is naturally identified with the complex

$$\big(L_{\mathrm{cdh}}\tau^{>i}R\Gamma_{\text{ét}}(-, j_!\mu_{p^k}^{\otimes i})\big)(R) \in \mathcal{D}(\mathbb{Z}/p^k).$$

The functor $R\Gamma_{\text{ét}}(-, j_!\mu_{p^k}^{\otimes i})$ is rigid ([Gab94], see also [BM21, Corollary 1.18 (1)]) and satisfies cdh descent ([BM21, Theorem 5.4]), so the object

$$\big(L_{\mathrm{cdh}}R\Gamma_{\text{ét}}(-, j_!\mu_{p^k}^{\otimes i})\big)(R)$$

is zero in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$ for the $p$-henselian commutative ring $R$. In particular, there is a natural equivalence

$$\big(L_{\mathrm{cdh}}\tau^{>i}R\Gamma_{\text{ét}}(-, j_!\mu_{p^k}^{\otimes i})\big)(R) \simeq \big(L_{\mathrm{cdh}}\tau^{\leq i}R\Gamma_{\text{ét}}(-, j_!\mu_{p^k}^{\otimes i})\big)(R)[1]$$

in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$, which implies the desired equivalence. □

**Corollary 7.2.6.** *Let $B$ be a mixed characteristic Dedekind domain such that every residue field of $B$ is perfect, and $X$ be a smooth $B$-scheme of dimension at most one over $B$. Then for every integer $i \geq 0$, the classical-motivic comparison map*

$$\mathbb{Z}(i)^{\mathrm{cla}}(X) \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(X)$$

*is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$.*

*Proof.* The presheaves $\mathbb{Z}(i)^{\mathrm{cla}}$ and $\mathbb{Z}(i)^{\mathrm{mot}}$ are finitary Nisnevich sheaves, hence it suffices to prove the result on the henselisation of every local ring of the scheme $X$. Let $R$ be such a henselian local ring, which is then a henselian local ind-smooth $B$-algebra of Krull dimension at most 3. The result rationally is a special case of Proposition 7.1.2, so it suffices to prove the result modulo $p$, for every prime number $p$. Let $p$ be a prime number. If $p$ is invertible in the local ring $R$, this is Proposition 7.1.1. Assume now that $p$ is not invertible in the henselian local ring $R$, and in particular that the ring $R$ is $p$-henselian. By Theorem 7.2.5, the fibre of the classical-motivic comparison map

$$\mathbb{F}_p(i)^{\mathrm{cla}}(R) \longrightarrow \mathbb{F}_p(i)^{\mathrm{mot}}(R)$$

is in degrees at least $i+2$, given by the complex

$$\big(L_{\mathrm{cdh}} \tau^{\leq i} R\Gamma_{\text{ét}}(-, j_! \mu_p^{\otimes i})\big)(R)[-1] \in \mathcal{D}(\mathbb{F}_p).$$

The commutative ring $R$ is a noetherian ring of Krull dimension at most 2, hence of valuative dimension at most 2. So this complex is also in degrees at most $i+3$ ([EHIK21, Theorem 2.4.15]), and it is thus only in degrees $i+2$ and $i+3$. The complexes $\mathbb{F}_p(i)^{\mathrm{cla}}(R)$ and $\mathbb{F}_p(i)^{\mathrm{mot}}(R)$ (for $i \geq 0$) are the graded pieces of the $\mathbb{N}$-indexed complete filtrations $\mathrm{Fil}^{\star}_{\mathrm{cla}}\mathrm{K}(R; \mathbb{F}_p)$ and $\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{K}(X; \mathbb{F}_p)$ on the algebraic $K$-theory of $R$. In particular, there is natural spectral sequence

$$E_2^{i,j} = \mathrm{H}^{i-j}\big(\big(L_{\mathrm{cdh}} \tau^{\leq -j} R\Gamma_{\text{ét}}(-, j_! \mu_p^{\otimes(-j)})\big)(R)\big) \Longrightarrow 0.$$

The previous cohomological bound implies that all the differentials in this spectral sequence are zero, so this spectral sequence degenerates. This implies the desired equivalence. □

# Chapter 8

# Comparison to Milnor $K$-theory and lisse motivic cohomology

In this chapter, we study the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$ on local rings. We prove that these are left Kan extended, in degrees at most $i$, from local essentially smooth $\mathbb{Z}$-algebras (Theorem 8.1.11). Via the comparison to classical motivic cohomology in the smooth case (Theorem 7.1.3), this means that the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$ are controlled, up to degree $i$, by classical motivic cohomology (Corollary 8.1.12). We use the latter result to construct a comparison map from the $i^{\mathrm{th}}$ (improved) Milnor $K$-group of a general local ring $A$ to the motivic cohomology group $\mathrm{H}^i_{\mathrm{mot}}(A, \mathbb{Z}(i))$, and prove that this map is an isomorphism with finite coefficients (Theorem 8.2.6).

## 8.1 Comparison to lisse motivic cohomology

In this section, we prove a comparison between motivic cohomology and lisse motivic cohomology on general local rings (Corollary 8.1.12), which generalises to mixed characteristic the analogous comparison result of Elmanto–Morrow over a field ([EM23, Theorem 7.7]). To do so, we use the following comparison map.

**Definition 8.1.1** (Lisse-motivic comparison map)**.** For every integer $i \in \mathbb{Z}$, the *lisse-motivic comparison map* is the map

$$\mathbb{Z}(i)^{\mathrm{lisse}}(-) \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(-)$$

of functors from animated commutative rings to the derived category $\mathcal{D}(\mathbb{Z})$ defined as the composite

$$\big(L_{\mathrm{AniRings}/\mathrm{Sm}_{\mathbb{Z}}}\, \mathbb{Z}(i)^{\mathrm{cla}}\big)(-) \longrightarrow \big(L_{\mathrm{AniRings}/\mathrm{Sm}_{\mathbb{Z}}}\, \mathbb{Z}(i)^{\mathrm{mot}}\big)(-) \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(-),$$

where the first map is the map induced by Definition 4.3.9 and the second map is the canonical map.

**Lemma 8.1.2.** *For every integer $i \geq 0$, the functor*

$$\tau^{\leq 2i}\, \mathbb{Q}(i)^{\mathrm{mot}}(-) : \mathrm{AniRings} \longrightarrow \mathcal{D}(\mathbb{Q})$$

*is left Kan extended from smooth $\mathbb{Z}$-algebras.*

*Proof.* By [EHK+20, Example 1.0.6], connective algebraic $K$-theory

$$\tau^{\leq 0}\mathrm{K}(-;\mathbb{Q}) : \mathrm{AniRings} \longrightarrow \mathcal{D}(\mathbb{Q})$$

is left Kan extended from smooth $\mathbb{Z}$-algebras. By Corollary 5.5.11, this implies that the functor

$$\bigoplus_{i \geq 0} \tau^{\leq 0}\big(\,\mathbb{Q}(i)^{\mathrm{mot}}(-)[2i]\big) : \mathrm{AniRings} \longrightarrow \mathcal{D}(\mathbb{Q})$$

is left Kan extended from smooth $\mathbb{Z}$-algebras, which is equivalent to the desired result. □

**Corollary 8.1.3.** *Let $R$ be an animated commutative ring. Then for every integer $i \geq 0$, the lisse-motivic comparison map induces a natural equivalence*

$$\mathbb{Q}(i)^{\mathrm{lisse}}(R) \xrightarrow{\sim} \tau^{\leq 2i}\,\mathbb{Q}(i)^{\mathrm{mot}}(R)$$

*in the derived category $\mathcal{D}(\mathbb{Q})$.*

*Proof.* If $R$ is a smooth $\mathbb{Z}$-algebra, the result is a consequence of Proposition 7.1.2 and the fact that, by construction, the classical motivic complex $\mathbb{Z}(i)^{\mathrm{cla}}(R) \in \mathcal{D}(\mathbb{Z})$ is in degrees at most $2i$. In general, this is then a consequence of Lemma 8.1.2. □

**Proposition 8.1.4.** *Let $R$ be a local ring. Then for every integer $i \geq 0$, the lisse-motivic comparison map induces a natural equivalence*

$$\mathbb{Q}(i)^{\mathrm{lisse}}(R) \longrightarrow \tau^{\leq i}\,\mathbb{Q}(i)^{\mathrm{mot}}(R)$$

*in the derived category $\mathcal{D}(\mathbb{Q})$. Moreover, the motivic cohomology group $\mathrm{H}^j_{\mathrm{mot}}(R,\mathbb{Q}(i))$ is zero for $i < j \leq 2i$.*

*Proof.* The classical motivic complex $\mathbb{Z}(i)^{\mathrm{cla}}(-)$ is Zariski-locally in degrees at most $i$ ([Gei04, Corollary 4.4]). By taking left Kan extension, this implies that the lisse motivic complex $\mathbb{Z}(i)^{\mathrm{lisse}}(-)$ is also Zariski-locally in degrees at most $i$. In particular, the lisse motivic complex $\mathbb{Q}(i)^{\mathrm{lisse}}(R)$ is in degrees at most $i$. The result is then a consequence of Corollary 8.1.3. □

**Remark 8.1.5.** By Drinfeld's theorem ([Dri06, Theorem 3.7]), the $K$-group $\mathrm{K}_{-1}(R)$ vanishes for every henselian local ring $R$. By Corollary 5.5.11, this implies that for every integer $i \geq 0$, the motivic cohomology group $\mathrm{H}^{2i+1}_{\mathrm{mot}}(R,\mathbb{Q}(i))$ is zero, *i.e.*, that the motivic cohomology group $\mathrm{H}^{2i+1}_{\mathrm{mot}}(R,\mathbb{Z}(i))$ is torsion.

**Corollary 8.1.6.** *For every integer $i \geq 0$, the functor $\tau^{\leq i}\,\mathbb{Q}(i)^{\mathrm{mot}}$, from local rings to the derived category $\mathcal{D}(\mathbb{Q})$, is left Kan extended from local essentially smooth $\mathbb{Z}$-algebras.*

*Proof.* This is a consequence of Lemma 8.1.2 and Proposition 8.1.4. □

**Proposition 8.1.7.** *Let $p$ be a prime number, and $k$ be an integer. Then for every integer $i \geq 0$, the functor*

$$\tau^{\leq i}\,\mathbb{Z}/p^k(i)^{\mathrm{mot}}(-) : \mathrm{Rings} \longrightarrow \mathcal{D}(\mathbb{Z}/p^k)$$

*is left Kan extended from smooth $\mathbb{Z}$-algebras.*

*Proof.* By Theorem 6.2.4, this is equivalent to the fact that the functor $\tau^{\leq i}\,\mathbb{Z}/p^k(i)^{\mathrm{syn}}(-)$ on commutative rings is left Kan extended from smooth $\mathbb{Z}$-algebras. The functor $\mathbb{Z}/p^k(i)^{\mathrm{syn}}(-)$ is left Kan extended from smooth $\mathbb{Z}$-algebras (Notation 6.2.1), so this is equivalent to the fact that the functor $\tau^{>i}\,\mathbb{Z}/p^k(i)^{\mathrm{syn}}(-)$ on commutative rings is left Kan extended from smooth $\mathbb{Z}$-algebras. By [EM23, Lemma 7.6], it then suffices to prove that the functor $\tau^{>i}\,\mathbb{Z}/p^k(i)^{\mathrm{syn}}(-)$ is rigid. To prove this, consider the fibre sequence of $\mathcal{D}(\mathbb{Z}/p^k)$-valued functors

$$R\Gamma_{\text{ét}}(-, j_!\mu_{p^k}^{\otimes i}) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{syn}}(-) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{BMS}}(-)$$

on commutative rings ([BL22, Remark 8.4.4]). By rigidity for étale cohomology ([Gab94], see also [BM21, Corollary 1.18 (1)]), the first term of this fibre sequence is rigid. The desired result is then a consequence of Theorem 3.2.13. □

**Corollary 8.1.8.** *Let $p$ be a prime number, and $k \geq 1$ be an integer. Then for every integer $i \geq 0$, the functor $\tau^{\leq i}\,\mathbb{Z}/p^k(i)^{\mathrm{mot}}$, from local rings to the derived category $\mathcal{D}(\mathbb{Z}/p^k)$, is left Kan extended from local essentially smooth $\mathbb{Z}$-algebras.*

*Proof.* This is a consequence of Proposition 8.1.7. □

**Lemma 8.1.9.** *Let $R$ be a local ring, $p$ be a prime number, and $k \geq 1$ be an integer. Then for every integer $i \geq 0$, the natural map of abelian groups*

$$\mathrm{H}^i_{\mathrm{mot}}(R, \mathbb{Z}(i)) \longrightarrow \mathrm{H}^i_{\mathrm{mot}}(R, \mathbb{Z}/p^k(i))$$

*is surjective.*

*Proof.* Let $P \to R$ be a henselian surjection, where $P$ is a local ind-smooth $\mathbb{Z}$-algebra. By Corollary 8.1.8, the functor $\tau^{\leq i}\,\mathbb{Z}/p^k(i)^{\mathrm{mot}}$ is left Kan extended on local rings from local essentially smooth $\mathbb{Z}$-algebras, so the natural map of abelian groups

$$\mathrm{H}^i_{\mathrm{mot}}(P, \mathbb{Z}/p^k(i)) \longrightarrow \mathrm{H}^i_{\mathrm{mot}}(R, \mathbb{Z}/p^k(i))$$

is surjective. That is, the right vertical map in the commutative diagram of abelian groups

$$\begin{array}{ccc} \mathrm{H}^i_{\mathrm{mot}}(P, \mathbb{Z}(i)) & \longrightarrow & \mathrm{H}^i_{\mathrm{mot}}(P, \mathbb{Z}/p^k(i)) \\ \downarrow & & \downarrow \\ \mathrm{H}^i_{\mathrm{mot}}(R, \mathbb{Z}(i)) & \longrightarrow & \mathrm{H}^i_{\mathrm{mot}}(R, \mathbb{Z}/p^k(i)) \end{array}$$

is surjective. To prove that the bottom horizontal map is surjective, it thus suffices to prove that the top vertical map is surjective. The local ring $P$ is a filtered colimit of local essentially smooth $\mathbb{Z}$-algebras, so it suffices to prove that this top vertical map is surjective for local essentially smooth $\mathbb{Z}$-algebras. To prove this, it suffices to prove that the motivic complex $\mathbb{Z}(i)^{\mathrm{mot}}(-)$ is zero in degree $i+1$ on local essentially smooth $\mathbb{Z}$-algebras. The classical motivic complex $\mathbb{Z}(i)^{\mathrm{cla}}(-)$ is Zariski-locally in degrees at most $i$ ([Gei04, Corollary 4.4]), so this is a consequence of Theorem 7.1.3. □

**Corollary 8.1.10.** *Let $R$ be a local ring. Then for every integer $i \geq 1$, the motivic cohomology group $\mathrm{H}^{i+1}_{\mathrm{mot}}(R, \mathbb{Z}(i))$ is zero. If the local ring $R$ is moreover henselian, then the motivic cohomology group $\mathrm{H}^1_{\mathrm{mot}}(R, \mathbb{Z}(0))$ is zero.*

*Proof.* By Lemma 8.1.9 and the short exact sequence of abelian groups

$$0 \longrightarrow \mathrm{H}^i_{\mathrm{mot}}(R, \mathbb{Z}(i))/p \longrightarrow \mathrm{H}^i_{\mathrm{mot}}(R, \mathbb{F}_p(i)) \longrightarrow \mathrm{H}^{i+1}_{\mathrm{mot}}(R, \mathbb{Z}(i))[p] \longrightarrow 0$$

for every prime number $p$ and every integer $i \geq 0$, the abelian group $\mathrm{H}^{i+1}_{\mathrm{mot}}(R, \mathbb{Z}(i))$ is torsion-free. By Proposition 8.1.4 if $i \geq 1$, and by Remark 8.1.5 if $i = 0$ and $R$ is henselian, it is also torsion, so it is zero. □

**Theorem 8.1.11.** *For every integer $i \geq 0$, the functor $\tau^{\leq i}\,\mathbb{Z}(i)^{\mathrm{mot}}$, from local rings to the derived category $\mathcal{D}(\mathbb{Z})$, is left Kan extended from local essentially smooth $\mathbb{Z}$-algebras.*

*Proof.* It suffices to prove the result rationally, and modulo $p$ for every prime number $p$. The result rationally is Corollary 8.1.6. Let $p$ be a prime number. For every local ring $R$, the natural map of abelian groups

$$\mathrm{H}^i_{\mathrm{mot}}(R, \mathbb{Z}(i)) \longrightarrow \mathrm{H}^i_{\mathrm{mot}}(R, \mathbb{F}_p(i))$$

is surjective by Lemma 8.1.9, so the natural map

$$\big(\tau^{\leq i}\,\mathbb{Z}(i)^{\mathrm{mot}}(R)\big)/p \longrightarrow \tau^{\leq i}\,\mathbb{F}_p(i)^{\mathrm{mot}}(R)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{F}_p)$. The result modulo $p$ is then Corollary 8.1.8. □

Note that the proof of Theorem 8.1.11 is similar to the proof of Elmanto–Morrow in equicharacteristic. The following consequence, however, uses the comparison to classical motivic cohomology Theorem 7.1.3. The proof of the latter, in any characteristic, is somehow simpler than the proof of Elmanto–Morrow's (stronger) comparison result to classical motivic cohomology: in particular, it does not use a presentation lemma, or the projective bundle formula. The proof of Corollary 8.1.12 then provides an alternative argument to the proof of [EM23, Theorem 7.7].

**Corollary 8.1.12** (Comparison to lisse motivic cohomology)**.** *Let $R$ be a local ring. Then for every integer $i \geq 0$, the lisse-motivic comparison map induces a natural equivalence*

$$\mathbb{Z}(i)^{\mathrm{lisse}}(R) \xrightarrow{\sim} \tau^{\leq i}\,\mathbb{Z}(i)^{\mathrm{mot}}(R)$$

*in the derived category $\mathcal{D}(\mathbb{Z})$.*

*Proof.* The classical motivic complex $\mathbb{Z}(i)^{\mathrm{cla}}(-)$ is Zariski-locally in degrees at most $i$ ([Gei04, Corollary 4.4]). The result is then a consequence of Theorems 7.1.3 and 8.1.11. □

In the rest of this section, we restrict our attention to henselian local rings, in order to describe the motivic cohomology group $\mathrm{H}^2_{\mathrm{mot}}(-, \mathbb{Z}(1))$.

**Lemma 8.1.13.** *Let $R$ be a henselian local ring, and $p$ be a prime number. Then for any integers $i \geq 0$ and $k \geq 1$, the motivic cohomology group $\mathrm{H}^{i+1}_{\mathrm{mot}}(R, \mathbb{Z}/p^k(i))$ is zero.*

*Proof.* By Theorem 6.2.4, the motivic cohomology group $\mathrm{H}^{i+1}_{\mathrm{mot}}(R, \mathbb{Z}/p^k(i))$ is naturally identified with the kernel of the natural map of abelian groups

$$\mathrm{H}^{i+1}(\mathbb{Z}/p^k(i)^{\mathrm{syn}}(R)) \longrightarrow \mathrm{H}^{i+1}((L_{\mathrm{cdh}}\tau^{>i}\,\mathbb{Z}/p^k(i)^{\mathrm{syn}})(R))$$

for every commutative ring $R$. If $R$ is henselian local, let $\mathfrak{m}$ be its maximal ideal, and consider the natural commutative diagram

$$\begin{array}{ccc} \mathrm{H}^{i+1}(\mathbb{Z}/p^k(i)^{\mathrm{syn}}(R)) & \longrightarrow & \mathrm{H}^{i+1}((L_{\mathrm{cdh}}\tau^{>i}\mathbb{Z}/p^k(i)^{\mathrm{syn}})(R)) \\ \downarrow & & \downarrow \\ \mathrm{H}^{i+1}(\mathbb{Z}/p^k(i)^{\mathrm{syn}}(R/\mathfrak{m})) & \longrightarrow & \mathrm{H}^{i+1}((L_{\mathrm{cdh}}\tau^{>i}\mathbb{Z}/p^k(i)^{\mathrm{syn}})(R/\mathfrak{m})) \end{array}$$

of abelian groups. The functor $\tau^{>i}\mathbb{Z}/p^k(i)^{\mathrm{syn}}$ is rigid (proof of Proposition 8.1.7), so the left vertical map is an isomorphism. The field $R/\mathfrak{m}$ is a local ring for the cdh topology, so the bottom horizontal map is an isomorphism. In particular, the top horizontal map is injective. □

**Proposition 8.1.14.** *Let $R$ be a henselian local ring. Then for every integer $i \geq 1$, the motivic cohomology group $\mathrm{H}^{i+2}_{\mathrm{mot}}(R, \mathbb{Z}(i))$ is zero.*

*Proof.* By Lemma 8.1.13 and the short exact sequence of abelian groups

$$0 \longrightarrow \mathrm{H}^{i+1}_{\mathrm{mot}}(R, \mathbb{Z}(i))/p \longrightarrow \mathrm{H}^{i+1}_{\mathrm{mot}}(R, \mathbb{F}_p(i)) \longrightarrow \mathrm{H}^{i+2}_{\mathrm{mot}}(R, \mathbb{Z}(i))[p] \longrightarrow 0$$

for every prime number $p$, the abelian group $\mathrm{H}^{i+2}_{\mathrm{mot}}(R, \mathbb{Z}(i))$ is torsionfree. By Proposition 8.1.4 if $i \geq 2$, and by Remark 8.1.5 if $i = 1$, it is also torsion, so it is zero. □

The following example is a consequence of Example 4.2.4, Corollary 8.1.12, and Proposition 8.1.14.

**Example 8.1.15.** For every qcqs scheme $X$, the natural map

$$R\Gamma_{\mathrm{Nis}}(X, \mathbb{G}_m)[-1] \longrightarrow \mathbb{Z}(1)^{\mathrm{mot}}(X),$$

defined as the Nisnevich sheafification of the lisse-motivic comparison map (see also Definition 9.1.1), is an isomorphism in degrees at most three. That is, the motivic complex $\mathbb{Z}(1)^{\mathrm{mot}}(X)$ vanishes in degrees at most zero, and there are natural isomorphisms of abelian groups

$$\mathrm{H}^1_{\mathrm{mot}}(X, \mathbb{Z}(1)) \cong \mathcal{O}(X)^{\times}, \quad \mathrm{H}^2_{\mathrm{mot}}(X, \mathbb{Z}(1)) \cong \mathrm{Pic}(X), \quad \mathrm{H}^3_{\mathrm{mot}}(X, \mathbb{Z}(1)) \cong \mathrm{H}^2_{\mathrm{Nis}}(X, \mathbb{G}_m).$$

## 8.2 Comparison to Milnor $K$-theory

In this section, we construct, for every integer $i \geq 1$, a symbol map

$$\mathrm{K}^{\mathrm{M}}_i(A) \longrightarrow \mathrm{H}^i_{\mathrm{mot}}(A, \mathbb{Z}(i))$$

for local rings $A$ (Definition 8.2.3), through which we compare the Milnor $K$-groups to motivic cohomology (Theorem 8.2.6). Note that the arguments in Lemmas 8.2.2 and 8.2.4 are very similar to that of [EM23, Section 7], except for the Gersten injectivity for classical motivic cohomology, which is unknown integrally in mixed characteristic.

For every commutative ring $A$, the lisse-motivic comparison map (Definition 8.1.1 and Example 4.2.4) induces on $\mathrm{H}^1$ a natural isomorphism of abelian groups

$$A^{\times} \xrightarrow{\cong} \mathrm{H}^1_{\mathrm{mot}}(A, \mathbb{Z}(1)).$$

By multiplicativity of the motivic complexes, this induces, for every integer $i \geq 0$, a symbol map of abelian groups

$$(A^{\times})^{\otimes i} \longrightarrow \mathrm{H}^i_{\mathrm{mot}}(A, \mathbb{Z}(i)).$$

**Lemma 8.2.1.** *For every local essentially smooth $\mathbb{Z}$-algebra $A$, there is a natural isomorphism*

$$\mathrm{H}^2_{\mathrm{mot}}(A, \mathbb{Z}(2)) \cong \mathrm{K}_2(A)$$

*of abelian groups.*

*Proof.* By Theorem 7.1.3, the classical-motivic comparison map

$$\mathrm{H}^2_{\mathrm{cla}}(A, \mathbb{Z}(2)) \longrightarrow \mathrm{H}^2_{\mathrm{mot}}(A, \mathbb{Z}(2))$$

is an isomorphism of abelian groups. The result is then a consequence of the Atiyah–Hirzebruch spectral sequence for classical motivic cohomology (Remark 4.1.3), where we use that the classical motivic complex $\mathbb{Z}(1)^{\mathrm{cla}}(A) \in \mathcal{D}(\mathbb{Z})$ is concentrated in degree one (Example 4.1.5 and [Gei04, Corollary 4.4]). □

**Lemma 8.2.2.** *Let $A$ be a local ring. Then for every integer $i \geq 0$, the natural map of abelian groups*

$$(A^\times)^{\otimes i} \longrightarrow \mathrm{H}^i_{\mathrm{mot}}(A, \mathbb{Z}(i))$$

*induced by the lisse-motivic comparison map factors through the Milnor $K$-group $\mathrm{K}^{\mathrm{M}}_i(A)$.*

*Proof.* By definition of the Milnor $K$-groups, it suffices to prove that the symbol map respects the Steinberg relations. Let $a \in A$ be an element such that $a$ and $1-a$ are units in $A$. By multiplicativity of the motivic complexes, it suffices to consider the case $i = 2$ and to prove that $a \otimes (1-a)$ is sent to zero via the symbol map. Let $\mathbb{Z}[t] \to A$ be the ring homomorphism sending $t$ to $a$, and let $\mathfrak{p} \subset \mathbb{Z}[t]$ be the prime ideal defined as the inverse image of the maximal ideal of $A$ via this ring homomorphism. By naturality of the symbol map, the diagram of abelian groups

$$\begin{array}{ccc} (\mathbb{Z}[t]_{\mathfrak{p}})^\times \otimes_{\mathbb{Z}} (\mathbb{Z}[t]_{\mathfrak{p}})^\times & \longrightarrow & \mathrm{H}^2_{\mathrm{mot}}(\mathbb{Z}[t]_{\mathfrak{p}}, \mathbb{Z}(2)) \\ \downarrow & & \downarrow \\ A^\times \otimes_{\mathbb{Z}} A^\times & \longrightarrow & \mathrm{H}^2_{\mathrm{mot}}(A, \mathbb{Z}(2)) \end{array}$$

is commutative. It then suffices to prove that the top horizontal arrow of this diagram sends $t \otimes (1-t)$ to zero. The local ring $\mathbb{Z}[t]_{\mathfrak{p}}$ is essentially smooth over $\mathbb{Z}$, so the right vertical map of the commutative diagram of abelian groups

$$\begin{array}{ccc} (\mathbb{Z}[t]_{\mathfrak{p}})^\times \otimes_{\mathbb{Z}} (\mathbb{Z}[t]_{\mathfrak{p}})^\times & \longrightarrow & \mathrm{H}^2_{\mathrm{mot}}(\mathbb{Z}[t]_{\mathfrak{p}}, \mathbb{Z}(2)) \\ \downarrow & & \downarrow \\ (\mathrm{Frac}(\mathbb{Z}[t]_{\mathfrak{p}}))^\times \otimes_{\mathbb{Z}} (\mathrm{Frac}(\mathbb{Z}[t]_{\mathfrak{p}}))^\times & \longrightarrow & \mathrm{H}^2_{\mathrm{mot}}(\mathrm{Frac}(\mathbb{Z}[t]_{\mathfrak{p}}), \mathbb{Z}(2)) \end{array}$$

is injective. Indeed, by Lemma 8.2.1 this is equivalent to the fact that the natural map

$$\mathrm{K}_2(A) \longrightarrow \mathrm{K}_2(\mathrm{Frac}(A))$$

is injective, and this is the Gersten injectivity for $\mathrm{K}_2$ ([GL87, Corollary 6] and [DS75, Theorem 2.2]). It then suffices to prove that the bottom horizontal map of this diagram sends $t \otimes (1-t)$ to zero. This is a consequence of the fact that the symbol map to classical motivic cohomology respects the Steinberg relation for fields ([NS89], see also [Tot92]). □

**Definition 8.2.3** (Symbol map)**.** Let $A$ be a local ring. For every integer $i \geq 0$, the *symbol map*

$$\mathrm{K}_{\mathrm{i}}^{\mathrm{M}}(A) \longrightarrow \mathrm{H}^i_{\mathrm{mot}}(A, \mathbb{Z}(i))$$

is the natural map of abelian groups of Lemma 8.2.2.

Following [Ker10], for $A$ a local ring and $i \geq 0$ an integer, we denote by $\widehat{\mathrm{K}}_i^{\mathrm{M}}(A)$ the $i^{\text{th}}$ *improved Milnor K-group* of $A$.

**Lemma 8.2.4.** *Let $A$ be a local ring. Then for every integer $i \geq 0$, the symbol map*

$$\mathrm{K}_i^{\mathrm{M}}(A) \longrightarrow \mathrm{H}^i_{\mathrm{mot}}(A, \mathbb{Z}(i))$$

*factors through the improved Milnor K-group $\widehat{\mathrm{K}}_i^{\mathrm{M}}(A)$.*

*Proof.* Let $M_i \geq 1$ be the integer defined in [Ker10]. If the residue field of the local ring $A$ has at least $M_i$ elements, then the natural map

$$\mathrm{K}_i^{\mathrm{M}}(A) \longrightarrow \widehat{\mathrm{K}}_i^{\mathrm{M}}(A)$$

is an isomorphism of abelian groups ([Ker10, Proposition 10 (5)]). Assume now that the residue field of the local ring $A$ has less than $M_i$ elements. We want to prove that the symbol map

$$\mathrm{K}_i^{\mathrm{M}}(A) \longrightarrow \mathrm{H}^i_{\mathrm{mot}}(A, \mathbb{Z}(i))$$

factors through the surjective map $\mathrm{K}_i^{\mathrm{M}}(A) \to \widehat{\mathrm{K}}_i^{\mathrm{M}}(A)$, *i.e.*, that every element of the abelian group $\ker(\mathrm{K}_i^{\mathrm{M}}(A) \to \widehat{\mathrm{K}}_i^{\mathrm{M}}(A))$ is sent to zero by the previous symbol map. Let $\mathfrak{m}$ be the maximal ideal of the local ring $A$, and $p$ be its residue characteristic. The residue field $A/\mathfrak{m}$ of the local ring $A$ is isomorphic to a finite extension $\mathbb{F}_q$ of $\mathbb{F}_p$. Let $\ell \geq 1$ be an integer which is coprime to the degree of this extension, and such that $p^\ell \geq M_i$. As a tensor product of finite field extensions of coprime degree, the commutative ring $\mathbb{F}_q \otimes_{\mathbb{F}_p} \mathbb{F}_{p^\ell}$ is a field. Let $V$ be the finite étale extension of $\mathbb{Z}_{(p)}$ corresponding to the field extension $\mathbb{F}_{p^\ell}$ of $\mathbb{F}_p$. The commutative ring $A' := A \otimes_{\mathbb{Z}_{(p)}} V$ is finite over the local ring $A$, and the quotient $A'/\mathfrak{m}A'$ is a field, so the commutative ring $A'$ is a local $A$-algebra, whose residue field has at least $M_i$ elements.

Let $P_\bullet \to A$ be a simplicial resolution of the local ring $A$ where each term $P_m$ is a local ind-smooth $\mathbb{Z}$-algebra, and each face map $P_{m+1} \to P_m$ is a henselian surjection. By Theorem 8.1.11, there is then a natural equivalence

$$\operatorname*{colim}_m \tau^{\leq i}\, \mathbb{Z}(i)^{\mathrm{mot}}(P_m) \xrightarrow{\sim} \tau^{\leq i}\, \mathbb{Z}(i)^{\mathrm{mot}}(A)$$

in the derived category $\mathcal{D}(\mathbb{Z})$. In particular, this equivalence induces a natural isomorphism

$$\mathrm{coeq}\big(\mathrm{H}^i_{\mathrm{mot}}(P_1, \mathbb{Z}(i)) \rightrightarrows \mathrm{H}^i_{\mathrm{mot}}(P_0, \mathbb{Z}(i))\big) \xrightarrow{\cong} \mathrm{H}^i_{\mathrm{mot}}(A, \mathbb{Z}(i))$$

of abelian groups, where the motivic cohomology groups in the left term are naturally identified with classical motivic cohomology groups by Theorem 7.1.3. Similarly, $P_\bullet \otimes_{\mathbb{Z}_{(p)}} V \to A'$ is a simplicial resolution of the local ring $A'$ where each term $P_m \otimes_{\mathbb{Z}_{(p)}} V$ is an ind-smooth $V$-algebra, and each face map $P_{m+1} \otimes_{\mathbb{Z}_{(p)}} V \to P_m \otimes_{\mathbb{Z}_{(p)}} V$ is a henselian surjection, so there is a natural isomorphism

$$\mathrm{coeq}\big(\mathrm{H}^i_{\mathrm{mot}}(P_1 \otimes_{\mathbb{Z}_{(p)}} V, \mathbb{Z}(i)) \rightrightarrows \mathrm{H}^i_{\mathrm{mot}}(P_0 \otimes_{\mathbb{Z}_{(p)}} V, \mathbb{Z}(i))\big) \xrightarrow{\cong} \mathrm{H}^i_{\mathrm{mot}}(A', \mathbb{Z}(i))$$

of abelian groups, where the motivic cohomology groups of the left term are naturally identified with classical motivic cohomology groups. Classical motivic cohomology of smooth schemes over a mixed characteristic Dedekind domain admits functorial transfer maps along finite étale morphisms, so the previous two isomorphisms induce a transfer map

$$N_\ell : \mathrm{H}^i_{\mathrm{mot}}(A', \mathbb{Z}(i)) \longrightarrow \mathrm{H}^i_{\mathrm{mot}}(A, \mathbb{Z}(i))$$

such that pre-composition with the natural map $\mathrm{H}^i_{\mathrm{mot}}(A, \mathbb{Z}(i)) \to \mathrm{H}^i_{\mathrm{mot}}(A', \mathbb{Z}(i))$ is multiplication by $\ell$. In particular, the kernel of the natural map $\mathrm{H}^i_{\mathrm{mot}}(A, \mathbb{Z}(i)) \to \mathrm{H}^i_{\mathrm{mot}}(A', \mathbb{Z}(i))$ is $\ell$-torsion.

Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{K}^{\mathrm{M}}_i(A) & \longrightarrow & \mathrm{K}^{\mathrm{M}}_i(A') \\ \downarrow & & \downarrow \\ \mathrm{H}^i_{\mathrm{mot}}(A, \mathbb{Z}(i)) & \longrightarrow & \mathrm{H}^i_{\mathrm{mot}}(A', \mathbb{Z}(i)) \end{array}$$

of abelian groups, and let $x$ be an element of the abelian group $\ker(\mathrm{K}^{\mathrm{M}}_i(A) \to \widehat{\mathrm{K}}^{\mathrm{M}}_i(A))$. The residue field of the local ring $A'$ has at least $M_i$ elements, so the natural map $\mathrm{K}^{\mathrm{M}}_i(A') \to \widehat{\mathrm{K}}^{\mathrm{M}}_i(A')$ is an isomorphism ([Ker10, Proposition 10 (5)]), and $x$ is sent to zero by the top horizontal map. In particular, the image of $x$ by the left vertical map is in the kernel of the bottom horizontal map, and is thus $\ell$-torsion by the previous paragraph. Let $\ell' \geq 1$ be an integer which is coprime to $\ell$ and to the degree of $\mathbb{F}_q$ over $\mathbb{F}_p$, and such that $p^{\ell'} \geq M_i$. The previous argument for this integer $\ell'$ implies that the image of $x$ by the left vertical map is also $\ell'$-torsion, hence it is zero. □

**Conjecture 8.2.5.** Let $A$ be a local ring. Then for every integer $i \geq 0$, the natural map

$$\widehat{\mathrm{K}}^{\mathrm{M}}_i(A) \longrightarrow \mathrm{H}^i_{\mathrm{mot}}(A, \mathbb{Z}(i))$$

induced by Lemma 8.2.4 is an isomorphism of abelian groups.

The previous conjecture was proved by Elmanto–Morrow for equicharacteristic local rings ([EM23, Theorem 7.12]). Their proof uses as an input the analogous result in the smooth case for classical motivic cohomology, which is unknown in mixed characteristic (see Remark 8.2.7).

**Theorem 8.2.6** (Singular Nesterenko–Suslin isomorphism with finite coefficients)**.** *Let $A$ be a henselian local ring. Then for any integers $i \geq 0$ and $n \geq 1$, the natural map*

$$\widehat{\mathrm{K}}^{\mathrm{M}}_i(A)/n \longrightarrow \mathrm{H}^i_{\mathrm{mot}}(A, \mathbb{Z}(i))/n$$

*is an isomorphism of abelian groups.*

*Proof.* If the local ring $A$ contains a field, then the natural map

$$\widehat{\mathrm{K}}^{\mathrm{M}}_i(A) \longrightarrow \mathrm{H}^i_{\mathrm{mot}}(A, \mathbb{Z}(i))$$

is an isomorphism of abelian groups ([EM23, Theorem 7.12]). Otherwise, $A$ is a henselian local ring of mixed characteristic $(0, p)$ for some prime number $p$. In particular, the local ring $A$ is $p$-henselian. If $p$ does not divide the integer $n$, then consider the commutative diagram

$$\begin{array}{ccc} \widehat{\mathrm{K}}^{\mathrm{M}}_i(A)/n & \longrightarrow & \mathrm{H}^i_{\mathrm{mot}}(A, \mathbb{Z}(i))/n \\ \downarrow & & \downarrow \\ \widehat{\mathrm{K}}^{\mathrm{M}}_i(A/p)/n & \longrightarrow & \mathrm{H}^i_{\mathrm{mot}}(A/p, \mathbb{Z}(i))/n \end{array}$$

of abelian groups. The left vertical map is an isomorphism by [Ker10, Proposition 10 (7)]. By Lemma 8.1.9 and Corollary 6.1.6, the right vertical map is naturally identified with the natural map of abelian groups

$$\mathrm{H}^i_{\text{ét}}(A, \mu_n^{\otimes i}) \longrightarrow \mathrm{H}^i_{\text{ét}}(A/p, \mu_n^{\otimes i}),$$

which is an isomorphism by rigidity of étale cohomology ([Gab94], see also [BM21, Corollary 1.18 (1)]). The local ring $A/p$ is an $\mathbb{F}_p$-algebra, so the bottom horizontal map is an isomorphism ([EM23, Theorem 7.12]), and the result is true in this case. It then suffices to prove that for every integer $k \geq 1$, the natural map

$$\widehat{\mathrm{K}}^{\mathrm{M}}_i(A)/p^k \longrightarrow \mathrm{H}^i_{\mathrm{mot}}(A, \mathbb{Z}(i))/p^k$$

is an isomorphism of abelian groups. By Lemma 8.1.9 and Corollary 6.2.5, the natural map

$$\mathrm{H}^i_{\mathrm{mot}}(A, \mathbb{Z}(i))/p^k \longrightarrow \mathrm{H}^i(\mathbb{Z}/p^k(i)^{\mathrm{BMS}}(A))$$

is an isomorphism of abelian groups. By [LM23, Theorem 3.1], the composite map

$$\widehat{\mathrm{K}}^{\mathrm{M}}_i(A)/p^k \longrightarrow \mathrm{H}^i_{\mathrm{mot}}(A, \mathbb{Z}(i))/p^k \longrightarrow \mathrm{H}^i(\mathbb{Z}/p^k(i)^{\mathrm{BMS}}(A))$$

is an isomorphism of abelian groups, hence the desired result. □

**Remark 8.2.7.** If Conjecture 8.2.5 is true for local ind-smooth $\mathbb{Z}$-algebras, then the left Kan extension properties [LM23, Proposition 1.17] and Theorem 8.1.11 imply that Conjecture 8.2.5 is true for all local rings. See the proof of [EM23, Theorem 7.12] for more details.

**Remark 8.2.8.** Let $A$ be a local essentially smooth $\mathbb{Z}$-algebra, $i \geq 0$ be an integer, and consider the commutative diagram

$$\begin{array}{ccc} \widehat{\mathrm{K}}^{\mathrm{M}}_i(A) & \longrightarrow & \mathrm{H}^i_{\mathrm{mot}}(A, \mathbb{Z}(i)) \\ \downarrow & & \downarrow \\ \widehat{\mathrm{K}}^{\mathrm{M}}_i(\mathrm{Frac}(A)) & \longrightarrow & \mathrm{H}^i_{\mathrm{mot}}(\mathrm{Frac}(A), \mathbb{Z}(i)) \end{array}$$

of abelian groups. The bottom horizontal map is an isomorphism by the Nesterenko–Suslin isomorphism for fields ([NS89], see also [EM23, Theorem 7.12]). The left vertical being injective then implies that the top horizontal map is injective. That is, the Gersten injectivity conjecture for the improved Milnor $K$-groups would imply the injectivity part of Conjecture 8.2.5. Knowing the Gersten injectivity conjecture for the motivic cohomology group $\mathrm{H}^i_{\mathrm{mot}}(-, \mathbb{Z}(i))$ would imply that these two facts are equivalent. See [Lüd22] for related results on the Gersten conjecture for improved Milnor $K$-groups.

# Chapter 9

# The projective bundle formula

In this chapter, we prove that the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$ satisfy the projective bundle formula (Theorem 9.3.2) and regular blowup excision (Theorem 9.3.1). This implies in particular that the presheaves $\mathbb{Z}(i)^{\mathrm{mot}}$ fit within the recent theory of non-$\mathbb{A}^1$-invariant motives of Annala–Iwasa [AI23] and Annala–Hoyois–Iwasa [AHI23, AHI24].

## 9.1 First Chern classes

In this section, we construct the motivic first Chern class (Definition 9.1.1) in order to formulate the projective bundle formula (Theorem 9.3.2).

**Definition 9.1.1** (Motivic first Chern class)**.** Let $X$ be a qcqs derived scheme. The *motivic first Chern class* is the natural map

$$c_1^{\mathrm{mot}} : R\Gamma_{\mathrm{Nis}}(X, \mathbb{G}_m)[-1] \longrightarrow \mathbb{Z}(1)^{\mathrm{mot}}(X),$$

in the derived category $\mathcal{D}(\mathbb{Z})$, defined as the Nisnevich sheafification of the natural map of presheaves

$$\big(\tau^{\leq 1} R\Gamma_{\mathrm{Zar}}(-, \mathbb{G}_m)\big)[-1] \longrightarrow \mathbb{Z}(1)^{\mathrm{mot}}(-)$$

induced by Definition 8.1.1 and Example 4.2.4. We also denote by

$$c_1^{\mathrm{mot}} : \mathrm{Pic}(X) \longrightarrow \mathrm{H}^2_{\mathrm{mot}}(X, \mathbb{Z}(1))$$

the map induced on $\mathrm{H}^2$.

**Remark 9.1.2.** The motivic first Chern class of Definition 9.1.1 is uniquely determined by its naturality, and the fact that it is given by the map of Definition 4.3.9 on smooth $\mathbb{Z}$-schemes.

For every qcqs scheme $X$, the line bundle $\mathcal{O}(1) \in \mathrm{Pic}(\mathbb{P}^1_X)$, via the multiplicative structure of the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$, induces, for every integer $i \in \mathbb{Z}$, a natural map

$$c_1^{\mathrm{mot}}(\mathcal{O}(1)) : \mathbb{Z}(i-1)^{\mathrm{mot}}(\mathbb{P}^1_X)[-2] \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(\mathbb{P}^1_X)$$

in the derived category $\mathcal{D}(\mathbb{Z})$. If $\pi : \mathbb{P}^1_X \to X$ is the canonical projection map, this in turn induces a natural map

$$\pi^* \oplus c_1^{\mathrm{mot}}(\mathcal{O}(1))\pi^* : \mathbb{Z}(i)^{\mathrm{mot}}(X) \oplus \mathbb{Z}(i-1)^{\mathrm{mot}}(X)[-2] \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(\mathbb{P}^1_X) \qquad (9.1)$$

in the derived category $\mathcal{D}(\mathbb{Z})$. The aim of the following section is to prove that this map is an equivalence (Theorem 9.2.1). To prove such an equivalence, we will need compatibilities with other first Chern classes.

**Construction 9.1.3** ($\mathbb{P}^1$-bundle formula for additive invariants)**.** Following [EM23, Section 5.1], every additive invariant $E$ of $\mathbb{Z}$-linear $\infty$-categories in the sense of [HSS17, Definition 5.11] admits a natural first Chern class, inducing a natural map of spectra

$$\pi^* \oplus (1 - c_1)(\mathcal{O}(-1))\pi^* : E(X) \oplus E(X) \longrightarrow E(\mathbb{P}^1_X).$$

For every additive invariant of $\mathbb{Z}$-linear $\infty$-categories, this map is an equivalence ([EM23, Lemma 5.6]).

**Remark 9.1.4** (Compatibility with filtrations)**.** If the additive invariant $E$ of $\mathbb{Z}$-linear $\infty$-categories, seen as a presheaf of spectra on qcqs schemes, admits a multiplicative filtered refinement $\mathrm{Fil}^\star E$ which is a multiplicative filtered module over the lisse motivic filtration $\mathrm{Fil}^\star_{\mathrm{lisse}}\mathrm{K}^{\mathrm{conn}}$ (Definition 4.2.2),[1] then this map has a natural filtered refinement

$$\pi^* \oplus (1 - c_1)(\mathcal{O}(-1))\pi^* : \mathrm{Fil}^\star E(X) \oplus \mathrm{Fil}^{\star-1} E(X) \longrightarrow \mathrm{Fil}^\star E(\mathbb{P}^1_X)$$

by [EM23, Construction 5.11 and Lemma 5.12]. If $E$ is algebraic $K$-theory, equipped with the motivic filtration $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}$ (Definition 4.3.6), the argument of [EM23, Lemma 5.12] and Remark 9.1.2 imply that this map recovers, up to a shift, the map (9.1) on graded pieces.

**Example 9.1.5** (Compatibility with cdh-local motivic cohomology)**.** The multiplicative filtration $\mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{KH}$ (Definition 4.2.5) is naturally a module over the multiplicative filtration $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}$ (*e.g.*, because cdh sheafification preserves multiplicative structures), so the first Chern class on cdh-local motivic complexes of [BEM24] is compatible with the motivic first Chern class of Definition 9.1.1.

**Example 9.1.6** (Compatibility with syntomic cohomology)**.** Let $X$ be a qcqs scheme, and $p$ be a prime number. By [BEM24], the syntomic first Chern class of [BL22, Section 7] is compatible with the motivic first Chern class of Definition 9.1.1 via the motivic-syntomic comparison map (Construction 6.2.2). Note here that the motivic-syntomic comparison map can be seen as the map induced on graded pieces from a multiplicative map of filtered spectra

$$\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X;\mathbb{Z}_p) \longrightarrow \mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}^{\mathrm{Sel}}(X;\mathbb{Z}_p),$$

where the target is the filtration on $p$-completed Selmer $K$-theory ([BL22, Remark 8.4.3]). These motivic and syntomic first Chern classes then coincide with the first Chern classes coming from the additive invariants $\mathrm{K}(-;\mathbb{Z}_p)$ and $\mathrm{K}^{\mathrm{Sel}}(-;\mathbb{Z}_p)$ (Remark 9.1.4).

## 9.2 $\mathbb{P}^1$-bundle formula

In this section, we prove that the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$ satisfy the $\mathbb{P}^1$-bundle formula on qcqs schemes (Theorem 9.2.1). Note that the $\mathbb{P}^1$-bundle formula is unknown for the cdh-local motivic complexes $\mathbb{Z}(i)^{\mathrm{cdh}}$ on general qcqs schemes.[2] The cartesian square of Remark 4.3.7 thus cannot be used directly to prove the $\mathbb{P}^1$-bundle formula for the motivic

[1] The lisse motivic filtration $\mathrm{Fil}^\star_{\mathrm{lisse}}\mathrm{K}^{\mathrm{conn}}$ is usually defined only on affine schemes. Here the argument works if $\mathrm{Fil}^\star E$ is a Zariski sheaf of filtered spectra on qcqs schemes, and if its restriction to affine schemes is a multiplicative filtered module over the lisse motivic filtration $\mathrm{Fil}^\star_{\mathrm{lisse}}\mathrm{K}^{\mathrm{conn}}$.

[2] More precisely, Bachmann–Elmanto–Morrow proved in [BEM24] that if a qcqs scheme $X$ satisfies the condition $\mathrm{Val}(X)$, then the cdh-local motivic complexes $\mathbb{Z}(i)^{\mathrm{cdh}}$ satisfy the $\mathbb{P}^1$-bundle formula at $X$. In particular, the $\mathbb{P}^1$-bundle formula is known for these complexes over a field, and over a mixed characteristic perfectoid valuation ring by the results of Chapter 11 (see also [Bou23]), but not on general qcqs schemes.

complexes $\mathbb{Z}(i)^{\mathrm{mot}}$, as was done by Elmanto–Morrow in equicharacteristic ([EM23, Section 5]). Instead, we use in a crucial way our main result on $p$-adic motivic cohomology (Theorem 6.2.4), and a degeneration argument using Selmer $K$-theory.

**Theorem 9.2.1** ($\mathbb{P}^1$-bundle formula)**.** *Let $X$ be a qcqs scheme, and $\pi : \mathbb{P}^1_X \to X$ be the canonical projection map. Then for every integer $i \in \mathbb{Z}$, the natural map*

$$\pi^* \oplus c_1^{\mathrm{mot}}(\mathcal{O}(1))\pi^* : \mathbb{Z}(i)^{\mathrm{mot}}(X) \oplus \mathbb{Z}(i-1)^{\mathrm{mot}}(X)[-2] \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(\mathbb{P}^1_X)$$

*is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$.*

Using Theorem 6.2.4, the proof of Theorem 9.2.1 will reduce to the proof of a similar equivalence for the cdh sheaves $\big(L_{\mathrm{cdh}}\tau^{>i}\,\mathbb{F}_p(i)^{\mathrm{syn}}\big)(-)$ (Proposition 9.2.8). Most of this section is devoted to the study of these cdh sheaves.

**Lemma 9.2.2.** *Let $p$ be a prime number. Then for any integers $i, j \geq 0$ and $k \geq 1$, the natural sequence*

$$L_{\mathrm{Nis}}\tau^{>j}R\Gamma_{\mathrm{\acute{e}t}}(-, j_!\mu_{p^k}^{\otimes i}) \longrightarrow \big(L_{\mathrm{Nis}}\tau^{>j}\,\mathbb{Z}/p^k(i)^{\mathrm{syn}}\big)(-) \longrightarrow \big(L_{\mathrm{Nis}}\tau^{>j}\,\mathbb{Z}/p^k(i)^{\mathrm{BMS}}\big)(-)$$

*is a fibre sequence of $\mathcal{D}(\mathbb{Z}/p^k)$-valued presheaves on qcqs schemes.*

*Proof.* The three presheaves are finitary Nisnevich sheaves, so it suffices to prove the result on henselian local rings ([CM21, Corollary 3.27 and Example 4.31]). Let $A$ be a henselian local ring. By [BL22, Remark 8.4.4], there is a natural fibre sequence

$$R\Gamma_{\mathrm{\acute{e}t}}(A, j_!\mu_{p^k}^{\otimes i}) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{syn}}(A) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{BMS}}(A)$$

in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$, so it suffices to prove that the natural map

$$\mathbb{Z}/p^k(i)^{\mathrm{syn}}(A) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{BMS}}(A)$$

is surjective in degree $j$. If $p$ is invertible in the henselian valuation ring $A$, the target of this map is zero. If $p$ is not invertible in $A$, then the valuation ring $A$ is $p$-henselian, and this map is an equivalence (Notation 6.2.1). □

**Lemma 9.2.3.** *Let $p$ be a prime number, and $V$ be a rank one henselian valuation ring of mixed characteristic $(0,p)$. Then for any integers $i \geq 0$ and $k \geq 1$, the complex*

$$\big(L_{\mathrm{cdh}}\tau^{>i}R\Gamma_{\mathrm{\acute{e}t}}(-, j_!\mu_{p^k}^{\otimes i})\big)(\mathbb{P}^1_V) \in \mathcal{D}(\mathbb{Z}/p^k)$$

*is concentrated in degree $i+1$.*[3]

*Proof.* The presheaf $L_{\mathrm{cdh}}\tau^{>i}R\Gamma_{\mathrm{\acute{e}t}}(-, j_!\mu_{p^k}^{\otimes i})$ is the cdh sheafification of a presheaf taking values in degrees at least $i+1$, so it takes values in degrees at least $i+1$. To prove that the complex

$$\big(L_{\mathrm{cdh}}\tau^{>i}R\Gamma_{\mathrm{\acute{e}t}}(-, j_!\mu_{p^k}^{\otimes i})\big)(\mathbb{P}^1_V) \in \mathcal{D}(\mathbb{Z}/p^k)$$

is in degrees at most $i+1$, consider the fibre sequence

$$\big(L_{\mathrm{cdh}}\tau^{\leq i}R\Gamma_{\mathrm{\acute{e}t}}(-, j_!\mu_{p^k}^{\otimes i})\big)(\mathbb{P}^1_V) \longrightarrow R\Gamma_{\mathrm{\acute{e}t}}(\mathbb{P}^1_V, j_!\mu_{p^k}^{\otimes i}) \longrightarrow \big(L_{\mathrm{cdh}}\tau^{>i}R\Gamma_{\mathrm{\acute{e}t}}(-, j_!\mu_{p^k}^{\otimes i})\big)(\mathbb{P}^1_V)$$

[3] We will prove, at the end of Proposition 9.2.6, that this complex is actually zero.

in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$, which is a consequence of arc descent for the presheaf $R\Gamma_{\text{ét}}(-, j_!\mu_{p^k}^{\otimes i})$ ([BM21, Theorem 1.8]). The scheme $\mathbb{P}^1_V$ has valuative dimension two, so the complex

$$\big(L_{\text{cdh}}\tau^{\leq i} R\Gamma_{\text{ét}}(-, j_!\mu_{p^k}^{\otimes i})\big)(\mathbb{P}^1_V) \in \mathcal{D}(\mathbb{Z}/p^k)$$

is in degrees at most $i+2$ ([EHIK21, Theorem 2.4.15]). By the $\mathbb{P}^1$-bundle formula for the presheaves $R\Gamma_{\text{ét}}(-, j_!\mu_{p^k}^{\otimes i})$ ([BL22, proof of Theorem 9.1.1]), there is a natural equivalence

$$R\Gamma_{\text{ét}}(V, j_!\mu_{p^k}^{\otimes i}) \oplus R\Gamma_{\text{ét}}(V, j_!\mu_{p^k}^{\otimes (i-1)})[-2] \longrightarrow R\Gamma_{\text{ét}}(\mathbb{P}^1_V, j_!\mu_{p^k}^{\otimes i})$$

in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$. The functors $R\Gamma_{\text{ét}}(-, j_!\mu_{p^k}^{\otimes i})$ and $R\Gamma_{\text{ét}}(-, j_!\mu_{p^k}^{\otimes (i-1)})$ are moreover rigid ([Gab94], see also [BM21, Corollary 1.18 (1)]) and the valuation ring $V$ is $p$-henselian, so the complex $R\Gamma_{\text{ét}}(\mathbb{P}^1_V, j_!\mu_{p^k}^{\otimes i}) \in \mathcal{D}(\mathbb{Z}/p^k)$ is zero. This implies that the complex $\big(L_{\text{cdh}}\tau^{>i} R\Gamma_{\text{ét}}(-, j_!\mu_{p^k}^{\otimes i})\big)(\mathbb{P}^1_V)$ is naturally identified with the complex

$$\big(L_{\text{cdh}}\tau^{\leq i} R\Gamma_{\text{ét}}(-, j_!\mu_{p^k}^{\otimes i})\big)(\mathbb{P}^1_V)[1] \in \mathcal{D}(\mathbb{Z}/p^k),$$

and is thus in degrees at most $i+1$. ☐

Following [EHIK21], we say that a $\mathcal{D}(\mathbb{Z})$-valued presheaf on qcqs schemes satisfies *henselian v-excision* if for every henselian valuation ring $V$ and every prime ideal $\mathfrak{p}$ of $V$, this presheaf sends the bicartesian square of commutative rings

$$\begin{array}{ccc} V & \longrightarrow & V_{\mathfrak{p}} \\ \downarrow & & \downarrow \\ V/\mathfrak{p} & \longrightarrow & V_{\mathfrak{p}}/\mathfrak{p}V_{\mathfrak{p}} \end{array}$$

to a cartesian square. Note that in the previous bicartesian square, all the commutative rings are henselian valuation rings by [EHIK21, Lemma 3.3.5]. The following lemma explains how to use henselian $v$-excision to prove that a map of cdh sheaves is an equivalence.

**Lemma 9.2.4.** *Let $S$ be a qcqs scheme of finite valuative dimension, $\mathcal{C}$ be a $\infty$-category which is compactly generated by cotruncated objects, and $F, G : \mathrm{Sch}_S^{\mathrm{qcqs,op}} \to \mathcal{C}$ be finitary cdh sheaves satisfying henselian v-excision. Then a map of presheaves $F \to G$ is an equivalence of presheaves if and only if the map $F(V) \to G(V)$ is an equivalence in $\mathcal{C}$ for every henselian valuation ring $V$ of rank at most one with a map $\mathrm{Spec}(V) \to S$.*

*Proof.* By [EHIK21, Proposition 3.1.8 (2)], a map $F \to G$ is an equivalence of presheaves if and only if it is an equivalence on henselian valuation rings over $S$. The presheaves $F$ and $G$ being finitary, this is equivalent to the fact that it is an equivalence on henselian valuation rings of finite rank over $S$. By induction, and using henselian $v$-excision, this is in turn equivalent to the fact that it is an equivalence on henselian valuation rings of rank at most one over $S$. ☐

**Lemma 9.2.5.** *Let $p$ be a prime number. Then for any integers $i \geq 0$ and $k \geq 1$, the $\mathcal{D}(\mathbb{Z}/p^k)$-valued presheaves $\big(L_{\text{cdh}}\tau^{>i}\,\mathbb{Z}/p^k(i)^{\text{syn}}\big)(-)$ and $\big(L_{\text{cdh}}\tau^{>i}\,\mathbb{Z}/p^k(i)^{\text{syn}}\big)(\mathbb{P}^1_-)$ are finitary cdh sheaves on qcqs schemes, and satisfy henselian v-excision.*

*Proof.* The presheaf $\mathbb{Z}/p^k(i)^{\mathrm{syn}}$ is finitary, and the cdh sheafification of a finitary presheaf is a finitary cdh sheaf (Lemma 5.5.12), so the presheaf $\big(L_{\mathrm{cdh}}\tau^{>i}\mathbb{Z}/p^k(i)^{\mathrm{syn}}\big)(-)$ is a finitary cdh sheaf. Covers in a site are stable under base change, so the presheaf

$$\big(L_{\mathrm{cdh}}\tau^{>i}\mathbb{Z}/p^k(i)^{\mathrm{syn}}\big)(\mathbb{P}^1_-)$$

is also a finitary cdh sheaf. Henselian valuation rings are local rings for the cdh topology, and the presheaves $\tau^{>i}R\Gamma_{\text{ét}}(-,j_!\mu_{p^k}^{\otimes i})$ and $\tau^{>i}\mathbb{Z}/p^k(i)^{\mathrm{BMS}}$ are rigid ([Gab94] and Theorem 3.2.13), so the presheaves

$$\big(L_{\mathrm{cdh}}\tau^{>i}R\Gamma_{\text{ét}}(-,j_!\mu_{p^k}^{\otimes i})\big)(-) \quad \text{and} \quad \big(L_{\mathrm{cdh}}\tau^{>i}\mathbb{Z}/p^k(i)^{\mathrm{BMS}}\big)(-)$$

satisfy henselian $v$-excision. By Lemma 9.2.2 (applied for $j=i$ and after cdh sheafification), the presheaf $\big(L_{\mathrm{cdh}}\tau^{>i}\mathbb{Z}/p^k(i)^{\mathrm{syn}}\big)(-)$ then satisfies henselian $v$-excision. Finally, the presheaf $\big(L_{\mathrm{cdh}}\tau^{>i}\mathbb{Z}/p^k(i)^{\mathrm{syn}}\big)(\mathbb{P}^1_-)$ satisfies henselian $v$-excision, as a consequence of [EHIK21, Lemma 3.3.7], and henselian $v$-excision for the presheaf $\big(L_{\mathrm{cdh}}\tau^{>i}\mathbb{Z}/p^k(i)^{\mathrm{syn}}\big)(-)$. □

For every qcqs scheme $X$, the compatibilities between the motivic and syntomic first Chern classes of Section 9.1 imply that the natural diagram

$$\begin{array}{ccc}
\mathbb{Z}/p^k(i)^{\mathrm{mot}}(X)\oplus\mathbb{Z}/p^k(i-1)^{\mathrm{mot}}(X)[-2] & \xrightarrow{\pi^*\oplus c_1^{\mathrm{mot}}(\mathcal{O}(1))\pi^*} & \mathbb{Z}/p^k(i)^{\mathrm{mot}}(\mathbb{P}^1_X) \\
\downarrow & & \downarrow \\
\mathbb{Z}/p^k(i)^{\mathrm{syn}}(X)\oplus\mathbb{Z}/p^k(i-1)^{\mathrm{syn}}(X)[-2] & \xrightarrow{\pi^*\oplus c_1^{\mathrm{syn}}(\mathcal{O}(1))\pi^*} & \mathbb{Z}/p^k(i)^{\mathrm{syn}}(\mathbb{P}^1_X)
\end{array}$$

is commutative. We define the natural map

$$\big(L_{\mathrm{cdh}}\tau^{>i}\mathbb{Z}/p^k(i)^{\mathrm{syn}}\big)(X)\oplus\big(L_{\mathrm{cdh}}\tau^{>i-1}\mathbb{Z}/p^k(i-1)^{\mathrm{syn}}\big)(X)[-2]\longrightarrow\big(L_{\mathrm{cdh}}\tau^{>i}\mathbb{Z}/p^k(i)^{\mathrm{syn}}\big)(\mathbb{P}^1_X)$$

in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$ as the map induced, via Theorem 6.2.4, by taking cofibres along the vertical maps of this commutative diagram.

**Proposition 9.2.6.** *Let $p$ be a prime number, and $V$ be a rank one henselian valuation ring of mixed characteristic $(0,p)$. Then for any integers $i\geq 0$ and $k\geq 1$, the natural map*

$$\tau^{>i}\mathbb{Z}/p^k(i)^{\mathrm{syn}}(V)\oplus\big(\tau^{>i-1}\mathbb{Z}/p^k(i-1)^{\mathrm{syn}}(V)\big)[-2]\longrightarrow\big(L_{\mathrm{cdh}}\tau^{>i}\mathbb{Z}/p^k(i)^{\mathrm{syn}}\big)(\mathbb{P}^1_V)$$

*is an equivalence in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$.*

*Proof.* The valuation ring $V$ is $p$-henselian, so the natural maps

$$\tau^{>i}\mathbb{Z}/p^k(i)^{\mathrm{syn}}(V)\longrightarrow\tau^{>i}\mathbb{Z}/p^k(i)^{\mathrm{BMS}}(V)$$

and

$$\big(\tau^{>i-1}\mathbb{Z}/p^k(i-1)^{\mathrm{syn}}(V)\big)[-2]\longrightarrow\big(\tau^{>i-1}\mathbb{Z}/p^k(i)^{\mathrm{BMS}}(V)\big)[-2]$$

are equivalences in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$. We first prove that the induced map

$$\tau^{>i}\mathbb{Z}/p^k(i)^{\mathrm{BMS}}(V)\oplus\big(\tau^{>i-1}\mathbb{Z}/p^k(i-1)^{\mathrm{BMS}}(V)\big)[-2]\longrightarrow\big(L_{\mathrm{cdh}}\tau^{>i}\mathbb{Z}/p^k(i)^{\mathrm{BMS}}\big)(\mathbb{P}^1_V)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$. Let $\kappa$ be the residue field of $V$. By the rigidity property Theorem 3.2.13, the natural maps

$$\tau^{>i}\mathbb{Z}/p^k(i)^{\mathrm{BMS}}(V)\longrightarrow\tau^{>i}\mathbb{Z}/p^k(i)^{\mathrm{BMS}}(\kappa)$$

and

$$\big(\tau^{>i-1}\,\mathbb{Z}/p^k(i-1)^{\mathrm{BMS}}(V)\big)[-2] \longrightarrow \big(\tau^{>i-1}\,\mathbb{Z}/p^k(i-1)^{\mathrm{BMS}}(\kappa)\big)[-2]$$

are equivalences in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$. By Corollary 7.2.2, the natural map

$$\big(L_{\mathrm{cdh}}\tau^{>i}\,\mathbb{Z}/p^k(i)^{\mathrm{BMS}}\big)(\mathbb{P}^1_V) \longrightarrow \big(L_{\mathrm{cdh}}\tau^{>i}\,\mathbb{Z}/p^k(i)^{\mathrm{BMS}}\big)(\mathbb{P}^1_{V/p})$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$. The presheaf $(L_{\mathrm{cdh}}\tau^{>i}\,\mathbb{Z}/p^k(i)^{\mathrm{BMS}})(\mathbb{P}^1_{-})$ is moreover a finitary cdh sheaf, so it is invariant under nilpotent extensions. In particular, the natural map

$$\big(L_{\mathrm{cdh}}\tau^{>i}\,\mathbb{Z}/p^k(i)^{\mathrm{BMS}}\big)(\mathbb{P}^1_{V/p}) \longrightarrow \big(L_{\mathrm{cdh}}\tau^{>i}\,\mathbb{Z}/p^k(i)^{\mathrm{BMS}}\big)(\mathbb{P}^1_{\kappa})$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$. It then suffices to prove that the natural map

$$\tau^{>i}\,\mathbb{Z}/p^k(i)^{\mathrm{BMS}}(\kappa) \oplus \big(\tau^{>i-1}\,\mathbb{Z}/p^k(i-1)^{\mathrm{BMS}}(\kappa)\big)[-2] \longrightarrow \big(L_{\mathrm{cdh}}\tau^{>i}\,\mathbb{Z}/p^k(i)^{\mathrm{BMS}}\big)(\mathbb{P}^1_{\kappa})$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$, and this is a consequence of the $\mathbb{P}^1$-bundle formula on characteristic $p$ fields for the presheaves $\mathbb{Z}/p^k(i)^{\mathrm{cdh}}$ ([BEM24]) and $L_{\mathrm{cdh}}\,\mathbb{Z}/p^k(i)^{\mathrm{BMS}}$ ([EM23, Lemma 5.17]).

We prove now that the natural map

$$\tau^{>i}\,\mathbb{Z}/p^k(i)^{\mathrm{syn}}(V) \oplus \big(\tau^{>i-1}\,\mathbb{Z}/p^k(i-1)^{\mathrm{syn}}(V)\big)[-2] \longrightarrow \big(L_{\mathrm{cdh}}\tau^{>i}\,\mathbb{Z}/p^k(i)^{\mathrm{syn}}\big)(\mathbb{P}^1_V)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$. By Lemma 9.2.2 (applied for $j=i$ and after cdh sheafification), we just proved that the cofibre of this map is naturally identified with the complex

$$\big(L_{\mathrm{cdh}}\tau^{>i}R\Gamma_{\text{ét}}(-,j_!\mu_{p^k}^{\otimes i})\big)(\mathbb{P}^1_V) \in \mathcal{D}(\mathbb{Z}/p^k).$$

By Example 9.1.6, these complexes, indexed by integers $i \geq 0$, form the graded pieces of the filtered spectrum defined as the cofibre of the natural map of filtered spectra

$$\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{K}(X;\mathbb{Z}/p^k) \longrightarrow \mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{K}^{\mathrm{Sel}}(X;\mathbb{Z}/p^k).$$

The cofibre of the natural map $\mathrm{K}(-;\mathbb{Z}/p^k) \to \mathrm{K}^{\mathrm{Sel}}(-;\mathbb{Z}/p^k)$, as a cofibre of two additive invariants of $\mathbb{Z}$-linear $\infty$-categories, is an additive invariant of $\mathbb{Z}$-linear $\infty$-categories and, as such, satisfies the $\mathbb{P}^1$-bundle formula ([EM23, Lemma 5.6]). This filtration then induces a spectral sequence

$$E_2^{i,j} = \mathrm{H}^{i-j}\Big(\big(L_{\mathrm{cdh}}\tau^{>-j}R\Gamma_{\text{ét}}(-,j_!\mu_{p^k}^{\otimes(-j)})\big)(\mathbb{P}^1_V)\Big) \Longrightarrow 0.$$

By Lemma 9.2.3, for every integer $i \geq 0$, the complex $\big(L_{\mathrm{cdh}}\tau^{>i}R\Gamma_{\text{ét}}(-,j_!\mu_{p^k}^{\otimes i})\big)(\mathbb{P}^1_V) \in \mathcal{D}(\mathbb{Z}/p^k)$ is concentrated in degree $i+1$, so this spectral sequence degenerates. This implies the desired equivalence. □

**Remark 9.2.7.** The proof of Proposition 9.2.6 uses a reduction to the case of fields of characteristic $p$, where the result is a consequence of the $\mathbb{P}^1$-bundle formula for the presheaves $\mathbb{Z}/p^k(i)^{\mathrm{cdh}}$ ([BEM24]) and $L_{\mathrm{cdh}}\,\mathbb{Z}/p^k(i)^{\mathrm{BMS}}$ ([EM23, Lemma 5.17]). It is however possible to bypass these two results and prove directly the $\mathbb{P}^1$-bundle formula on fields of characteristic $p$ for the presheaves $L_{\mathrm{cdh}}\tau^{>i}\,\mathbb{Z}/p^k(i)^{\mathrm{BMS}}$, by imitating the degeneration argument of [EM23, Lemma 5.17].

**Proposition 9.2.8.** *Let $X$ be a qcqs scheme, and $p$ be a prime number. Then for any integers $i \geq 0$ and $k \geq 1$, the natural map*

$$\big(L_{\mathrm{cdh}}\tau^{>i}\,\mathbb{Z}/p^k(i)^{\mathrm{syn}}\big)(X) \oplus \big(L_{\mathrm{cdh}}\tau^{>i-1}\,\mathbb{Z}/p^k(i-1)^{\mathrm{syn}}\big)(X)[-2] \longrightarrow \big(L_{\mathrm{cdh}}\tau^{>i}\,\mathbb{Z}/p^k(i)^{\mathrm{syn}}\big)(\mathbb{P}^1_X)$$

*is an equivalence in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$.*

*Proof.* The presheaves $L_{\mathrm{cdh}}\tau^{>i}\,\mathbb{Z}/p^k(i)^{\mathrm{syn}}$ and $\big(L_{\mathrm{cdh}}\tau^{>i}\,\mathbb{Z}/p^k(i)^{\mathrm{syn}}\big)(\mathbb{P}^1_-)$ are finitary cdh sheaves on qcqs schemes, and satisfy henselian $v$-excision (Lemma 9.2.5). It then suffices to prove the desired equivalence for henselian valuation rings of rank at most one (Lemma 9.2.4). Let $V$ be a henselian valuation ring of rank at most one.

If $p$ is invertible in the valuation ring $V$, then this is equivalent to proving that the natural map

$$\tau^{>i}R\Gamma_{\text{ét}}(V,\mu_{p^k}^{\otimes i}) \oplus \big(\tau^{>i-1}R\Gamma_{\text{ét}}(V,\mu_{p^k}^{\otimes(i-1)})\big)[-2] \longrightarrow \big(L_{\mathrm{cdh}}\tau^{>i}R\Gamma_{\text{ét}}(-,\mu_{p^k}^{\otimes i})\big)(\mathbb{P}^1_V)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$. For every integer $i \geq 0$, there is a fibre sequence of $\mathcal{D}(\mathbb{Z}/p^k)$-valued presheaves

$$\mathbb{Z}/p^k(i)^{\mathrm{cdh}}(-) \longrightarrow R\Gamma_{\text{ét}}(-,\mu_{p^k}^{\otimes i}) \longrightarrow L_{\mathrm{cdh}}\tau^{>i}R\Gamma_{\text{ét}}(-,\mu_{p^k}^{\otimes i})$$

on qcqs $\mathbb{Z}[\frac{1}{p}]$-schemes ([BEM24]). The desired equivalence is then a consequence of the $\mathbb{P}^1$-bundle formula on qcqs $\mathbb{Z}[\frac{1}{p}]$-schemes for the presheaf $\mathbb{Z}/p^k(i)^{\mathrm{cdh}}$ ([BEM24]) and the presheaf $R\Gamma_{\text{ét}}(-,\mu_{p^k}^{\otimes i})$ ([BL22, proof of Theorem 9.1.1]).

If $p$ is zero in the valuation ring $V$, then this is a consequence of the $\mathbb{P}^1$-bundle formula on qcqs $\mathbb{F}_p$-schemes for the presheaves $\mathbb{Z}/p^k(i)^{\mathrm{cdh}}$ ([BEM24]) and $L_{\mathrm{cdh}}\,\mathbb{Z}/p^k(i)^{\mathrm{BMS}}$ ([EM23, Theorem 5.14]).

If $p$ is neither invertible nor zero in the valuation ring $V$, then $V$ is a rank one henselian valuation ring of mixed characteristic $(0,p)$, and the result is Proposition 9.2.6. □

*Proof of Theorem 9.2.1.* It suffices to prove the result rationally, and modulo $p$ for every prime number $p$. Rationally, the Atiyah–Hirzebruch spectral sequence degenerates (Theorem 5.0.1), so the result is a consequence of the $\mathbb{P}^1$-bundle formula for algebraic $K$-theory (Section 9.1). Let $p$ be a prime number. By Theorem 6.2.4, for every integer $i \in \mathbb{Z}$, there is a fibre sequence of $\mathcal{D}(\mathbb{F}_p)$-valued presheaves on qcqs schemes

$$\mathbb{F}_p(i)^{\mathrm{mot}}(-) \longrightarrow \mathbb{F}_p(i)^{\mathrm{syn}}(-) \longrightarrow \big(L_{\mathrm{cdh}}\tau^{>i}\,\mathbb{F}_p(i)^{\mathrm{syn}}\big)(-).$$

By [BL22, Theorem 9.1.1], the natural map

$$\pi^* \oplus c_1^{\mathrm{syn}}(\mathcal{O}(1))\pi^* : \mathbb{F}_p(i)^{\mathrm{syn}}(X) \oplus \mathbb{F}_p(i-1)^{\mathrm{syn}}(X)[-2] \longrightarrow \mathbb{F}_p(i)^{\mathrm{syn}}(\mathbb{P}^1_X)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{F}_p)$. The result modulo $p$ is then a consequence of Proposition 9.2.8. □

## 9.3 Regular blowup and projective bundle formulae

In this section, we prove the regular blowup formula for the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$ (Theorem 9.3.1). By an argument of Annala–Iwasa, this and the $\mathbb{P}^1$-bundle formula imply the general projective bundle formula for the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$ (Theorem 9.3.2).

**Theorem 9.3.1** (Regular blowup formula). *Let $Y \to Z$ be a regular closed immersion of qcqs schemes.*[4] *Then for every integer $i \geq 0$, the commutative diagram*

$$\begin{array}{ccc} \mathbb{Z}(i)^{\mathrm{mot}}(Z) & \longrightarrow & \mathbb{Z}(i)^{\mathrm{mot}}(Y) \\ \downarrow & & \downarrow \\ \mathbb{Z}(i)^{\mathrm{mot}}(\mathrm{Bl}_Y(Z)) & \longrightarrow & \mathbb{Z}(i)^{\mathrm{mot}}(\mathrm{Bl}_Y(Z) \times_Z Y) \end{array}$$

*is a cartesian square in the derived category $\mathcal{D}(\mathbb{Z})$.*

*Proof.* It suffices to prove the result rationally, and modulo $p$ for every prime number $p$. By definition, a cdh sheaf sends an abstract blowup square to a cartesian square, and in particular satisfies the regular blowup formula. By Corollary 5.6.7, the regular blowup formula for the presheaf $\mathbb{Q}(i)^{\mathrm{mot}}$ is then equivalent to the regular blowup formula for the presheaf $R\Gamma_{\mathrm{Zar}}(-, \mathbb{L}\Omega^{<i}_{-_{\mathbb{Q}}/\mathbb{Q}})$. And the regular blowup formula for the presheaf $R\Gamma_{\mathrm{Zar}}(-, \mathbb{L}\Omega^{<i}_{-_{\mathbb{Q}}/\mathbb{Q}})$ is a consequence of the fact that for every integer $j \geq 0$, the presheaf $R\Gamma_{\mathrm{Zar}}(-, \mathbb{L}^j_{-/\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q})$ satisfies the regular blowup formula ([BL22, Lemma 9.4.3]).

Let $p$ be a prime number. Similarly, Corollary 4.3.12 implies that the regular blowup formula for the presheaf $\mathbb{F}_p(i)^{\mathrm{mot}}$ is equivalent to the regular blowup formula for the presheaf $\mathbb{F}_p(i)^{\mathrm{BMS}}$. By [AMMN22, Corollary 5.31], there exists an integer $m \geq 0$ and an equivalence of presheaves

$$\mathbb{F}_p(i)^{\mathrm{BMS}}(-) \xrightarrow{\sim} \mathrm{fib}\Big(\mathrm{can} - \phi_i : (\mathcal{N}^{\geq i}\mathbb{\Delta}_-\{i\}/\mathcal{N}^{\geq i+m}\mathbb{\Delta}_-\{i\})/p \longrightarrow (\mathbb{\Delta}_-\{i\}/\mathcal{N}^{\geq i+m}\mathbb{\Delta}_-\{i\})/p\Big).$$

In particular, it suffices to prove that for every integer $j \geq 0$, the presheaf $\mathcal{N}^j\mathbb{\Delta}_-/p$ satisfies the regular blowup formula. By [BL22, Remark 5.5.8 and Example 4.7.8], there is a fibre sequence of presheaves

$$\mathcal{N}^j\mathbb{\Delta}_-\{i\}/p \longrightarrow \mathrm{Fil}^{\mathrm{conj}}_j\overline{\mathbb{\Delta}}_{-/\mathbb{Z}_p[\![\widetilde{p}]\!]}/p \xrightarrow{\Theta+j} \mathrm{Fil}^{\mathrm{conj}}_{j-1}\overline{\mathbb{\Delta}}_{-/\mathbb{Z}_p[\![\widetilde{p}]\!]}/p.$$

The presheaves $\mathrm{Fil}^{\mathrm{conj}}_j\overline{\mathbb{\Delta}}_{-/\mathbb{Z}_p[\![\widetilde{p}]\!]}/p$ and $\mathrm{Fil}^{\mathrm{conj}}_{j-1}\overline{\mathbb{\Delta}}_{-/\mathbb{Z}_p[\![\widetilde{p}]\!]}/p$ have finite filtrations with graded pieces given by modulo $p$ powers of the cotangent complex, and the result is then a consequence of the regular blowup formula for powers of the cotangent complex ([BL22, Lemma 9.4.3]). $\square$

**Theorem 9.3.2** (Projective bundle formula). *Let $X$ be a qcqs scheme, $r \geq 1$ be an integer, $\mathcal{E}$ be a vector bundle of rank $r+1$ on $X$, and $\pi : \mathbb{P}_X(\mathcal{E}) \to X$ be the projectivisation of $\mathcal{E}$. Then for every integer $i \in \mathbb{Z}$, the natural map*

$$\sum_{j=0}^{r} c_1^{\mathrm{mot}}(\mathcal{O}(1))^j \pi^* : \bigoplus_{j=0}^{r} \mathbb{Z}(i-j)^{\mathrm{mot}}(X)[-2j] \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(\mathbb{P}_X(\mathcal{E}))$$

*is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$.*

*Proof.* By Zariski descent, it suffices to consider the case where the vector bundle $\mathcal{E}$ is given by $\mathbb{A}^{r+1}_X$, *i.e.*, to prove that the natural map

$$\sum_{j=0}^{r} c_1^{\mathrm{mot}}(\mathcal{O}(1))^j \pi^* : \bigoplus_{j=0}^{r} \mathbb{Z}(i-j)^{\mathrm{mot}}(X)[-2j] \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(\mathbb{P}^r_X)$$

[4]A morphism $Y \to Z$ is a regular closed immersion if it is a closed immersion, and if $Z$ admits an affine open cover such that $Y$ is defined by a regular sequence on each of the corresponding affine schemes.

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$. The presheaves $\mathbb{Z}(i)^{\mathrm{mot}}$ satisfy the $\mathbb{P}^1$-bundle formula (Theorem 9.2.1). Moreover, for every qcqs scheme $X$ and every integer $m \geq 0$, they send the blowup square

$$\begin{array}{ccc} \mathbb{P}^m_X & \longrightarrow & \mathrm{Bl}_X(\mathbb{A}^{m+1}_X) \\ \downarrow & & \downarrow \\ X & \xrightarrow{0} & \mathbb{A}^{m+1}_X \end{array}$$

to a cartesian square in the derived category $\mathcal{D}(\mathbb{Z})$ (Theorem 9.3.1, in the special case where the regular closed immersion $Y \to Z$ is the zero section $X \to \mathbb{A}^{m+1}_X$). By the argument of [AI23, Lemma 3.3.5], these two properties imply, by induction, the desired projective bundle formula. □

In the following result, denote by $\mathbb{Z}(i)^{\mathrm{mot}}_X : \mathrm{Sm}^{\mathrm{op}}_X \to \mathcal{D}(\mathbb{Z})$ the Zariski sheaves on smooth schemes over $X$ induced by restriction of the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$.

**Corollary 9.3.3** (Motivic cohomology is represented in motivic spectra)**.** *For every qcqs scheme $X$, the motivic complexes $\{\mathbb{Z}(i)^{\mathrm{mot}}_X\}_{i\in\mathbb{Z}}$ are represented by a $\mathbb{P}^1$-motivic spectrum in the sense of [AI23].*

*Proof.* By definition of $\mathbb{P}^1$-motivic spectra, this is a consequence of elementary blowup excision (which is a special case of Theorem 9.3.1) and the $\mathbb{P}^1$-bundle formula (Theorem 9.2.1). □

# Chapter 10

# Motivic Weibel vanishing and pro cdh descent

In this chapter, we study the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$ on noetherian schemes. We prove a general vanishing result which refines Weibel's vanishing conjecture on negative $K$-groups (Theorem 10.3.3), and prove that they coincide with Kelly–Saito's pro cdh motivic complexes $\mathbb{Z}(i)^{\mathrm{procdh}}$ (Theorem 10.4.2). The key input for both these results is the fact that the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$ satisfy pro cdh excision (Theorem 10.2.11), *i.e.*, that they send abstract blowup squares to pro cartesian squares.

**Notation 10.0.1** (Abstract blowup square)**.** An *abstract blowup square* (of noetherian schemes) is a cartesian square

$$\begin{array}{ccc} Y' & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array} \tag{10.1}$$

of qcqs schemes (resp. of noetherian schemes) such that $X' \to X$ is proper and finitely presented, $Y \to X$ is a finitely presented closed immersion, and the induced map $X' \setminus Y' \to X \setminus Y$ is an isomorphism. In this context, we also denote, for every integer $r \geq 0$, by $rY$ (resp. $rY'$) the $r-1^{\mathrm{st}}$ infinitesimal thickening of $Y$ inside $X$ (resp. of $Y'$ inside $X'$).

We use Kelly–Saito's recent definition in [KS24] of the pro cdh topology to encode the fact that a Nisnevich sheaf (*e.g.*, the motivic complex $\mathbb{Z}(i)^{\mathrm{mot}}$) satisfies pro cdh excision. Kelly–Saito proved in particular that if $S$ has finite valuative dimension and noetherian topological space, then the pro cdh topos of $S$ is hypercomplete and has enough points. For our purposes, the following definition will be used only for noetherian schemes $S$.

**Definition 10.0.2** (Pro cdh descent, after [KS24])**.** Let $S$ be a qcqs scheme. A *pro cdh sheaf* on finitely presented $S$-schemes is a presheaf

$$F : \mathrm{Sch}_S^{\mathrm{fp,op}} \longrightarrow \mathcal{D}(\mathbb{Z})$$

satisfying Nisnevich descent, and such that for every abstract blowup square of finitely presented $S$-schemes (10.1), the natural commutative diagram

$$\begin{array}{ccc} F(X) & \longrightarrow & F(X') \\ \downarrow & & \downarrow \\ \{F(rY)\}_r & \longrightarrow & \{F(rY')\}_r \end{array}$$

is a weakly cartesian square of pro objects in the derived category $\mathcal{D}(\mathbb{Z})$.[1]

## 10.1 Pro cdh descent for the cotangent complex

In this section, we review the pro cdh descent for powers of the cotangent complex on noetherian schemes (Proposition 10.1.7). On finite-dimensional noetherian schemes, this is [Mor16, Theorem 2.10]. On general noetherian schemes, the proof follows the sketch presented in [EM23, proof of Lemma 8.5]. In particular, the arguments are exactly as in [Mor16], except for the following generalisation of Grothendieck's formal functions theorem ([Gro61, Corollary 4.1.7]), where the finite dimensionality hypothesis is removed. We give some details for the sake of completeness.

For every commutative ring $A$, recall that a pro $A$-module $\{M_r\}_r$ is zero if for every index $r$, there exists an index $r' \geq r$ such that the map $M_{r'} \to M_r$ is the zero map. Similarly, a map $\{M_r\}_r \to \{N_r\}_r$ of pro $A$-modules is an isomorphism if its kernel and cokernel are zero pro $A$-modules. We say that a pro object $\{C_r\}_r$ in the derived category $\mathcal{D}(A)$ is weakly zero if all its cohomology groups are zero pro $A$-modules. Note that all the pro complexes that we will consider are uniformly bounded above, so this definition is equivalent to being weakly zero in the stable $\infty$-category of pro objects in the derived category $\mathcal{D}(A)$ ([LT19, Definition 2.27]). Similarly, we say that a map $\{C_r\}_r \to \{C'_r\}_r$ of pro objects in the derived category $\mathcal{D}(A)$ is a weak equivalence if its fibre is weakly zero as a pro object in the derived category $\mathcal{D}(A)$.

**Lemma 10.1.1** (Formal functions theorem, after Lurie [Lur19])**.** *Let $A$ be a noetherian commutative ring, $I$ be an ideal of $A$, $X$ be a proper scheme over $\operatorname{Spec}(A)$, and $X^\wedge_I$ be the formal completion of $X$ along the vanishing locus of $I$. Then for every coherent sheaf $\mathcal{F}$ over $X$, the natural map*

$$R\Gamma_{\mathrm{Zar}}(X,\mathcal{F}) \longrightarrow R\Gamma_{\mathrm{Zar}}(X^\wedge_I,\mathcal{F}^\wedge_I)$$

*where $\mathcal{F}^\wedge_I$ is the pullback of $\mathcal{F}$ along the natural map $X^\wedge_I \to X$, exhibits the target as the $I$-adic completion[2] of the source in the derived category $\mathcal{D}(A)$. More precisely, the natural map*

$$\{R\Gamma_{\mathrm{Zar}}(X,\mathcal{F})/I^r\}_r \longrightarrow \{R\Gamma_{\mathrm{Zar}}(X \times_{\operatorname{Spec}(A)} \operatorname{Spec}(A/I^r), \mathcal{F} \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{O}_X/I^r\mathcal{O}_X)\}_r$$

*is a weak equivalence of pro objects in the derived category $\mathcal{D}(A)$.*

*Proof.* The first statement is a special case of [Lur19, Lemma 8.5.1.1]. The second statement, although *a priori* stronger, follows by an examination of the previous proof (and in particular, the proof of [Lur19, Lemma 8.1.2.3]). □

**Lemma 10.1.2.** *Let $A$ be a noetherian commutative ring, and $X$ be a proper scheme over $\operatorname{Spec}(A)$. Then for any integers $j \geq 0$ and $n \in \mathbb{Z}$, the $A$-module $\mathrm{H}^n_{\mathrm{Zar}}(X,\mathbb{L}^j_{-/A})$ is finitely generated.*

[1] By this, we mean that all the cohomology groups of the total fibre of this commutative square are zero as pro abelian groups. All the presheaves $F$ that we will consider (most importantly, the presheaves $\mathbb{Z}(i)^{\mathrm{mot}}$) are bounded above on noetherian schemes, by a constant depending only on the dimension of their input (for the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$, this is Proposition 5.5.4); this definition of weakly cartesian square will then be equivalent to being weakly cartesian in the stable $\infty$-category of pro objects in the derived category $\mathcal{D}(\mathbb{Z})$, in the sense of [LT19, Definition 2.27].

[2] The cohomology groups of these coherent sheaves are finitely generated $A$-modules (because $X$ is proper over $\operatorname{Spec}(A)$), so the derived $I$-adic completion and the classical $I$-adic completion coincide in this context.

*Proof.* The scheme $X$ is of finite type over $\mathrm{Spec}(A)$, so the $\mathcal{O}_X$-module $\mathcal{H}^n_{\mathrm{Zar}}(-,\mathbb{L}^j_{-/A})$ is coherent. Because $X$ is proper over $\mathrm{Spec}(A)$, its cohomology groups are thus finitely generated $A$-modules. □

**Corollary 10.1.3.** *Let $A$ be a noetherian commutative ring, $I$ be an ideal of $A$, and $X$ be a noetherian scheme which is proper over* $\mathrm{Spec}(A)$. *Then for any integers $j \geq 0$ and $n \in \mathbb{Z}$, the natural map*

$$\{\mathrm{H}^n_{\mathrm{Zar}}(X,\mathbb{L}^j_{-/A})/I^r\}_r \longrightarrow \{\mathrm{H}^n_{\mathrm{Zar}}(X,\mathbb{L}^j_{-/A}\otimes^{\mathbb{L}}_{\mathcal{O}_X}\mathcal{O}_X/I^r\mathcal{O}_X)\}_r$$

*is an isomorphism of pro $A$-modules.*

*Proof.* By Lemma 10.1.2 and its proof, all the terms in the hypercohomology spectral sequence

$$E_2^{p,q} = \mathrm{H}^p_{\mathrm{Zar}}(X,\mathcal{H}^q_{\mathrm{Zar}}(-,\mathbb{L}^j_{-/R})) \Longrightarrow \mathrm{H}^{p+q}_{\mathrm{Zar}}(X,\mathbb{L}^j_{-/A})$$

are finitely generated $A$-modules. The functor $\{-\otimes_A A/I^r\}_r$ is exact on the category of finitely generated $A$-modules ([Mor16, Theorem 1.1 (ii)]), so it induces a spectral sequence of pro $A$-modules

$$E_2^{p,q} = \{\mathrm{H}^p_{\mathrm{Zar}}(X,\mathcal{H}^q_{\mathrm{Zar}}(-,\mathbb{L}^j_{-/A}))/I^r\}_r \Longrightarrow \{\mathrm{H}^{p+q}_{\mathrm{Zar}}(X,\mathbb{L}^j_{-/A})/I^r\}_r.$$

It then suffices to prove that the natural map of pro $A$-modules

$$\{\mathrm{H}^p_{\mathrm{Zar}}(X,\mathcal{H}^q_{\mathrm{Zar}}(-,\mathbb{L}^j_{-/A}))/I^r\}_r \longrightarrow \{\mathrm{H}^p_{\mathrm{Zar}}(X,\mathcal{H}^q_{\mathrm{Zar}}(-,\mathbb{L}^j_{-/A}\otimes^{\mathbb{L}}_{\mathcal{O}_X}\mathcal{O}_X/I^r\mathcal{O}_X))\}_r$$

is an isomorphism for all integers $p,q \geq 0$. The natural map

$$\{\mathrm{H}^p_{\mathrm{Zar}}(X,\mathcal{H}^q_{\mathrm{Zar}}(-,\mathbb{L}^j_{-/A}))/I^r\}_r \longrightarrow \{\mathrm{H}^p_{\mathrm{Zar}}(X,\mathcal{H}^q_{\mathrm{Zar}}(-,\mathbb{L}^j_{-/A})\otimes^{\mathbb{L}}_A A/I^r)\}_r$$

is an isomorphism by Lemma 10.1.1 applied to the coherent sheaf $\mathcal{H}^q_{\mathrm{Zar}}(-,\mathbb{L}_{-/A})$ on $X$, and the natural map

$$\{\mathrm{H}^p_{\mathrm{Zar}}(X,\mathcal{H}^q_{\mathrm{Zar}}(-,\mathbb{L}^j_{-/A})\otimes^{\mathbb{L}}_A A/I^r)\}_r \longrightarrow \{\mathrm{H}^p_{\mathrm{Zar}}(X,\mathcal{H}^q_{\mathrm{Zar}}(-,\mathbb{L}^j_{-/A}\otimes^{\mathbb{L}}_{\mathcal{O}_X}\mathcal{O}_X/I^r\mathcal{O}_X))\}_r$$

is an isomorphism by [Mor16, Lemma 2.3]. □

**Lemma 10.1.4.** *Let $A$ be a noetherian commutative ring, $I$ be an ideal of $A$, and $X$ be a proper scheme over* $\mathrm{Spec}(A)$ *such that the induced map $X \setminus V(I\mathcal{O}_X) \to \mathrm{Spec}(A) \setminus V(I)$ is an isomorphism.*

*(1) For any integers $j \geq 0$ and $n \in \mathbb{Z}$, the natural map*

$$\{\mathrm{H}^n_{\mathrm{Zar}}(X,\mathbb{L}^j_{-/A}\otimes^{\mathbb{L}}_{\mathcal{O}_X} I^r\mathcal{O}_X)\}_r \longrightarrow \{I^r\mathrm{H}^n_{\mathrm{Zar}}(X,\mathbb{L}^j_{-/A})\}_r$$

*is an isomorphism of pro $A$-modules.*

*(2) For any integers $j \geq 0$ and $n \in \mathbb{Z}$ such that $(j,n) \neq (0,0)$, the $A$-module $\mathrm{H}^n_{\mathrm{Zar}}(X,\mathbb{L}^j_{-/A})$ is killed by a power of $I$; in particular, the pro $A$-module $\{I^r\mathrm{H}^n_{\mathrm{Zar}}(X,\mathbb{L}^j_{-/A})\}_r$ is zero.*

*Proof.* (1) The short exact sequence $0 \to \{I^r\mathcal{O}_X\}_r \to \mathcal{O}_X \to \{\mathcal{O}_X/I^r\mathcal{O}_X\}_r \to 0$ of pro $\mathcal{O}_X$-modules induces a long exact sequence

$$\cdots \to \mathrm{H}^n_{\mathrm{Zar}}(X, \mathbb{L}^j_{-/A} \otimes^{\mathbb{L}}_{\mathcal{O}_X} I^r\mathcal{O}_X)\}_r \to \mathrm{H}^n_{\mathrm{Zar}}(X, \mathbb{L}^j_{-/A}) \to \{\mathrm{H}^n_{\mathrm{Zar}}(X, \mathbb{L}^j_{-/A} \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{O}_X/I^r\mathcal{O}_X)\}_r \to$$

of pro $A$-modules. By Corollary 10.1.3, the boundary maps of this long exact sequence vanish, hence the natural map

$$\{\mathrm{H}^n_{\mathrm{Zar}}(X, \mathbb{L}^j_{-/A} \otimes^{\mathbb{L}}_{\mathcal{O}_X} I^r\mathcal{O}_X)\}_r \longrightarrow \{I^r\mathrm{H}^n_{\mathrm{Zar}}(X, \mathbb{L}^j_{-/A})\}_r$$

is an isomorphism of pro $A$-modules.
(2) By Lemma 10.1.2, the $A$-module $\mathrm{H}^n_{\mathrm{Zar}}(X, \mathbb{L}^j_{-/A})$ is finitely generated. Because the map $X \setminus V(I\mathcal{O}_X) \to \mathrm{Spec}(A) \setminus V(I)$ is an isomorphism, this $A$-module is moreover supported on $V(I)$ if $(j,n) \neq (0,0)$. If $(j,n) \neq (0,0)$, this implies that the $A$-module $\mathrm{H}^n_{\mathrm{Zar}}(X, \mathbb{L}^j_{-/A})$ is killed by a power of $I$. □

**Corollary 10.1.5.** *Let $A$ be a noetherian commutative ring, $I$ be an ideal of $A$, and $X$ be a proper scheme over* $\mathrm{Spec}(A)$ *such that the induced map $X \setminus V(I\mathcal{O}_X) \to \mathrm{Spec}(A) \setminus V(I)$ is an isomorphism. Then for every integer $j \geq 0$, the natural map*

$$\{\mathbb{L}^j_{A/\mathbb{Z}} \otimes^{\mathbb{L}}_A I^r\}_r \longrightarrow \{R\Gamma_{\mathrm{Zar}}(X, \mathbb{L}^j_{-/\mathbb{Z}} \otimes^{\mathbb{L}}_{\mathcal{O}_X} I^r\mathcal{O}_X)\}_r$$

*is a weak equivalence of pro objects in the derived category $\mathcal{D}(A)$.*

*Proof.* By Lemma 10.1.4, and for any integers $n, a, b \in \mathbb{Z}$, the pro $A$-module

$$\{\mathrm{H}^n(\mathbb{L}^j_{A/\mathbb{Z}} \otimes^{\mathbb{L}}_A \mathrm{H}^a_{\mathrm{Zar}}(X, \mathbb{L}^b_{-/A} \otimes^{\mathbb{L}}_{\mathcal{O}_X} I^r\mathcal{O}_X))\}_r$$

is zero, except if $(a,b) = (0,0)$. By transitivity for the powers of the cotangent complex (see the proof of [Mor16, Lemma 2.8 (*ii*)] for more details), this implies that the natural map

$$\{\mathrm{H}^n(\mathbb{L}^j_{A/\mathbb{Z}} \otimes^{\mathbb{L}}_A \mathrm{H}^0_{\mathrm{Zar}}(X, I^r\mathcal{O}_X))\}_r \longrightarrow \{\mathrm{H}^n_{\mathrm{Zar}}(X, \mathbb{L}^j_{-/\mathbb{Z}} \otimes^{\mathbb{L}}_{\mathcal{O}_X} I^r\mathcal{O}_X)\}_r$$

is an isomorphism of pro $A$-modules. Let $B$ be the $A$-algebra $\mathrm{H}^0_{\mathrm{Zar}}(X, \mathcal{O}_X)$. Applying Lemma 10.1.4 (1) for $j = n = 0$, it then suffices to prove that the natural map $\{I^r\}_r \to \{I^rB\}_r$ is an isomorphism of pro $A$-modules. The $A$-algebra $B$ is finite and isomorphic to $A$ away from the vanishing locus of $I$, so the kernel and cokernel of the structure map $A \to B$ are killed by a power of $I$. The result is then a formal consequence of [Mor16, Theorem 1.1 (ii)]. □

**Lemma 10.1.6.** *Let $Y \to X$ be a closed immersion of noetherian schemes, and $\mathcal{I}$ be the associated ideal sheaf on $X$. Then for every integer $j \geq 0$, the natural map*

$$\{R\Gamma_{\mathrm{Zar}}(X, \mathbb{L}^j_{-/\mathbb{Z}} \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{O}_X/\mathcal{I}^r)\}_r \longrightarrow \{R\Gamma_{\mathrm{Zar}}(rY, \mathbb{L}^j_{-/\mathbb{Z}})\}_r$$

*is a weak equivalence of pro objects in the derived category $\mathcal{D}(A)$.*

*Proof.* The scheme $X$ is noetherian, hence quasi-separated, so we may assume by induction that $X$ is affine. In this case, the result is [Mor18a, Corollary 4.5 (*ii*)]. □

**Proposition 10.1.7.** *Let $j \geq 0$ be an integer. Then for every abstract blowup square of noetherian schemes (10.1), the natural commutative diagram*

$$\begin{array}{ccc} R\Gamma_{\mathrm{Zar}}(X, \mathbb{L}^j_{-/\mathbb{Z}}) & \longrightarrow & R\Gamma_{\mathrm{Zar}}(X', \mathbb{L}^j_{-/\mathbb{Z}}) \\ \downarrow & & \downarrow \\ \{R\Gamma_{\mathrm{Zar}}(rY, \mathbb{L}^j_{-/\mathbb{Z}})\}_r & \longrightarrow & \{R\Gamma_{\mathrm{Zar}}(rY', \mathbb{L}^j_{-/\mathbb{Z}})\}_r \end{array}$$

*is a weakly cartesian square of pro objects in the derived category $\mathcal{D}(\mathbb{Z})$. In particular, the presheaf $R\Gamma_{\mathrm{Zar}}(-, \mathbb{L}^j_{-/\mathbb{Z}})$ is a pro cdh sheaf on noetherian schemes.*

*Proof.* The scheme $X$ is noetherian, hence quasi-separated, so we may assume by induction that $X$ is affine, given by the spectrum of a noetherian commutative ring $A$. Let $I$ be the ideal of $A$ defining the closed subscheme $Y$ of $\mathrm{Spec}(A)$. By Lemma 10.1.6, the desired statement is equivalent to the fact that the commutative diagram

$$\begin{array}{ccc} \mathbb{L}^j_{A/\mathbb{Z}} & \longrightarrow & R\Gamma_{\mathrm{Zar}}(X', \mathbb{L}^j_{-/\mathbb{Z}}) \\ \downarrow & & \downarrow \\ \{\mathbb{L}^j_{A/\mathbb{Z}} \otimes^{\mathbb{L}}_A A/I^r\}_r & \longrightarrow & \{R\Gamma_{\mathrm{Zar}}(X', \mathbb{L}^j_{-/\mathbb{Z}} \otimes^{\mathbb{L}}_{\mathcal{O}_{X'}} \mathcal{O}_{X'}/I^r\mathcal{O}_{X'})\}_r \end{array}$$

is a weakly cartesian square of pro objects in the derived category $\mathcal{D}(\mathbb{Z})$. Taking fibres along the vertical maps, this is exactly Corollary 10.1.5. □

We now use Proposition 10.1.7 to prove pro cdh descent for variants of the cotangent complex (Corollary 10.1.10). In the following two lemmas, we consider inverse systems of objects in the derived category $\mathcal{D}(\mathbb{Z})$. We say that an inverse system $(C_r)_r$ in the derived category $\mathcal{D}(\mathbb{Z})$ is *essentially zero* if for every index $r$ and every integer $n \in \mathbb{Z}$, there exists an index $r' \geq r$ such that the map $\mathrm{H}^n(C_{r'}) \to \mathrm{H}^n(C_r)$ is the zero map. In particular, an inverse system $(C_r)_r$ in the derived category $\mathcal{D}(\mathbb{Z})$ is essentially zero if and only if the associated pro object $\{C_r\}_r$ in the derived category $\mathcal{D}(\mathbb{Z})$ is weakly zero.

**Lemma 10.1.8.** *Let $(C_r)_r$ be an inverse system in the derived category $\mathcal{D}(\mathbb{Z})$. If $(C_r)_r$ is essentially zero, then $\left(\prod_{p\in\mathbb{P}} C_r/p\right)_r$ is essentially zero.*

*Proof.* Assume that the inverse system $(C_r)_r$ is essentially zero. Let $r_0$ be an index of this inverse system, $n \in \mathbb{Z}$ be an integer, and $p$ be a prime number. We will use repeatedly that for every index $r$, there is a natural short exact sequence

$$0 \longrightarrow \mathrm{H}^n(C_r)/p \longrightarrow \mathrm{H}^n(C_r/p) \longrightarrow \mathrm{H}^{n+1}(C_r)[p] \longrightarrow 0$$

of abelian groups. Let $r_1 \geq r_0$ be an index such that the map $\mathrm{H}^n(C_{r_1}) \to \mathrm{H}^n(C_{r_0})$ is the zero map. Then for every index $r \geq r_1$, the map $\mathrm{H}^n(C_r)/p \to \mathrm{H}^n(C_{r_0})/p$ is the zero map, and the map $\mathrm{H}^n(C_r/p) \to \mathrm{H}^n(C_{r_0}/p)$ thus factors through the map $\mathrm{H}^n(C_r/p) \to \mathrm{H}^{n+1}(C_r)[p]$. Let $r_2 \geq r_1$ be an index such that the map $\mathrm{H}^{n+1}(C_{r_2}) \to \mathrm{H}^{n+1}(C_{r_1})$ is the zero map. Then the map $\mathrm{H}^{n+1}(C_{r_2})[p] \to \mathrm{H}^{n+1}(C_{r_1})[p]$ is the zero map. By construction, the map

$$\mathrm{H}^n(C_{r_2}/p) \longrightarrow \mathrm{H}^n(C_{r_0}/p)$$

factors as

$$\mathrm{H}^n(C_{r_2}/p) \longrightarrow \mathrm{H}^{n+1}(C_{r_2})[p] \xrightarrow{0} \mathrm{H}^{n+1}(C_{r_1})[p] \longrightarrow \mathrm{H}^n(C_{r_0}/p),$$

and is thus also the zero map. The index $r_2$ does not depend on the prime number $p$, so the map

$$\prod_{p\in\mathbb{P}} \mathrm{H}^n(C_{r_2}/p) \longrightarrow \prod_{p\in\mathbb{P}} \mathrm{H}^n(C_{r_0}/p)$$

is the zero map, and the inverse system $\big(\prod_{p\in\mathbb{P}} C_r/p\big)_r$ is essentially zero. □

**Lemma 10.1.9.** *Let $(C_r)_r$ be an inverse system in the derived category $\mathcal{D}(\mathbb{Z})$. If $(C_r)_r$ is essentially zero, then $\big(\prod_{p\in\mathbb{P}}(C_r)^\wedge_p\big)_r$ is essentially zero.*

*Proof.* Assume that the inverse system $(C_r)_r$ is essentially zero. Let $r_0$ be an index of this inverse system, $n \in \mathbb{Z}$ be an integer, and $p$ be a prime number. We will use repeatedly that for every index $r \geq 0$, there is a short exact sequence

$$0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}_p}(\mathbb{Q}_p\,/\,\mathbb{Z}_p, \mathrm{H}^n(C_r)) \longrightarrow \mathrm{H}^n((C_r)^\wedge_p) \longrightarrow \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p\,/\,\mathbb{Z}_p, \mathrm{H}^{n+1}(C_r)) \longrightarrow 0$$

of abelian groups. Let $r_1 \geq r_0$ be an index such that the map $\mathrm{H}^n(C_{r_1}) \to \mathrm{H}^n(C_{r_0})$ is the zero map. Then for every index $r \geq r_1$, the map $\mathrm{Ext}^1_{\mathbb{Z}_p}(\mathbb{Q}_p\,/\,\mathbb{Z}_p, \mathrm{H}^n(C_r)) \to \mathrm{Ext}^1_{\mathbb{Z}_p}(\mathbb{Q}_p\,/\,\mathbb{Z}_p, \mathrm{H}^n(C_{r_0}))$ is the zero map, and the map $\mathrm{H}^n((C_r)^\wedge_p) \to \mathrm{H}^n((C_{r_0})^\wedge_p)$ thus factors through the map

$$\mathrm{H}^n((C_r)^\wedge_p) \longrightarrow \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p\,/\,\mathbb{Z}_p, \mathrm{H}^{n+1}(C_r)).$$

Let $r_2 \geq r_1$ be an index such that the map $\mathrm{H}^{n+1}(C_{r_2}) \to \mathrm{H}^{n+1}(C_{r_1})$ is the zero map. Then the map $\mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p\,/\,\mathbb{Z}_p, \mathrm{H}^{n+1}(C_{r_2})) \to \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p\,/\,\mathbb{Z}_p, \mathrm{H}^{n+1}(C_{r_1}))$ is the zero map. By construction, the map

$$\mathrm{H}^n((C_{r_2})^\wedge_p) \longrightarrow \mathrm{H}^n((C_{r_0})^\wedge_p)$$

factors as

$$\mathrm{H}^n((C_{r_2})^\wedge_p) \longrightarrow \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p\,/\,\mathbb{Z}_p, \mathrm{H}^{n+1}(C_{r_2})) \xrightarrow{0} \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p\,/\,\mathbb{Z}_p, \mathrm{H}^{n+1}(C_{r_1})) \longrightarrow \mathrm{H}^n((C_{r_0})^\wedge_p),$$

and is thus the zero map. The index $r_2$ does not depend on the prime number $p$, so the map

$$\prod_{p\in\mathbb{P}} \mathrm{H}^n((C_{r_2})^\wedge_p) \longrightarrow \prod_{p\in\mathbb{P}} \mathrm{H}^n((C_{r_0})^\wedge_p)$$

is the zero map, and the inverse system $\big(\prod_{p\in\mathbb{P}}(C_r)^\wedge_p\big)_r$ is essentially zero. □

**Corollary 10.1.10.** *Let $j \geq 0$ be an integer, and let $F$ be one of the presheaves*

$$R\Gamma_{\mathrm{Zar}}\big(-, \mathbb{L}^j_{-/\mathbb{Z}}\big),\ R\Gamma_{\mathrm{Zar}}\big(-, \prod_{p\in\mathbb{P}} \mathbb{L}^j_{-_{\mathbb{F}_p}/\mathbb{F}_p}\big),\ R\Gamma_{\mathrm{Zar}}\big(-, \prod_{p\in\mathbb{P}} (\mathbb{L}^j_{-/\mathbb{Z}})^\wedge_p\big),\ \textit{and}\ R\Gamma_{\mathrm{Zar}}\big(-, \mathbb{L}^j_{-_{\mathbb{Q}}/\mathbb{Q}}\big),$$

*where $-_{\mathbb{F}_p}$ is the derived base change from $\mathbb{Z}$ to $\mathbb{F}_p$. Then the presheaf $F$ is a pro cdh sheaf on noetherian schemes.*

*Proof.* The presheaf $F$ is a Nisnevich sheaf, so the result is equivalent to proving that $F$ sends an abstract blowup square of noetherian schemes to a weakly cartesian square of pro objects in the derived category $\mathcal{D}(\mathbb{Z})$. For $R\Gamma_{\mathrm{Zar}}\big(-, \mathbb{L}^j_{-/\mathbb{Z}}\big)$, this is Proposition 10.1.7. For $R\Gamma_{\mathrm{Zar}}\big(-, \prod_{p\in\mathbb{P}} \mathbb{L}^j_{-_{\mathbb{F}_p}/\mathbb{F}_p}\big)$, this is a formal consequence of Proposition 10.1.7 and Lemma 10.1.8. For $R\Gamma_{\mathrm{Zar}}\big(-, \prod_{p\in\mathbb{P}}(\mathbb{L}^j_{-/\mathbb{Z}})^\wedge_p\big)$, this is similarly a formal consequence of Proposition 10.1.7 and Lemma 10.1.9. And for $R\Gamma_{\mathrm{Zar}}\big(-, \mathbb{L}^j_{-_{\mathbb{Q}}/\mathbb{Q}}\big)$, this is a consequence of Proposition 10.1.7 and the fact that the rationalisation of a zero pro system of abelian groups is zero. □

## 10.2 Pro cdh descent for motivic cohomology

In this section, we prove pro cdh descent for the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$ (Theorem 10.2.11). We use Corollary 5.2.16 to decompose the proof into several steps, which ultimately all rely on Corollary 10.1.10. We start with the following rational results.

**Proposition 10.2.1** ([EM23]). *For every integer $i \geq 0$, the presheaf $R\Gamma_{\mathrm{Zar}}\big(-, \widehat{\mathbb{L}\Omega}^{\geq i}_{-_{\mathbb{Q}}/\mathbb{Q}}\big)$ is a pro cdh sheaf on noetherian schemes.*

*Proof.* This is a part of [EM23, proof of Theorem 8.2]. More precisely, one uses Proposition 5.4.1 to reduce the proof to a finite number of powers of the cotangent complex relative to $\mathbb{Q}$, where this is Corollary 10.1.10. □

The following result is a rigid-analytic variant of Proposition 10.2.1.

**Proposition 10.2.2.** *For every integer $i \geq 0$, the presheaf $R\Gamma_{\mathrm{Zar}}\big(-, \prod'_{p\in\mathbb{P}} \underline{\widehat{\mathbb{L}\Omega}}^{\geq i}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p}\big)$ is a pro cdh sheaf on noetherian schemes.*

*Proof.* By Remark 5.2.12, there is a fibre sequence of presheaves

$$R\Gamma_{\mathrm{Zar}}\Big(-, \prod_{p\in\mathbb{P}}{}' \underline{\widehat{\mathbb{L}\Omega}}^{\geq i}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p}\Big) \longrightarrow R\Gamma_{\mathrm{Zar}}\Big(-, \prod_{p\in\mathbb{P}}{}' \underline{\widehat{\mathbb{L}\Omega}}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p}\Big) \longrightarrow R\Gamma_{\mathrm{Zar}}\Big(-, \big(\prod_{p\in\mathbb{P}} \big(\mathbb{L}\Omega^{<i}_{-/\mathbb{Z}}\big)^{\wedge}_{p}\big)_{\mathbb{Q}}\Big)$$

on qcqs derived schemes, and in particular on noetherian schemes. By Corollary 5.4.2, the presheaf $R\Gamma_{\mathrm{Zar}}\big(-, \prod'_{p\in\mathbb{P}} \underline{\widehat{\mathbb{L}\Omega}}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p}\big)$ is a cdh sheaf on noetherian schemes, so it is a pro cdh sheaf on noetherian schemes. The presheaf $R\Gamma_{\mathrm{Zar}}\big(-, \big(\prod_{p\in\mathbb{P}} \big(\mathbb{L}\Omega^{<i}_{-/\mathbb{Z}}\big)^{\wedge}_{p}\big)_{\mathbb{Q}}\big)$ has a finite filtration with graded pieces given by the presheaves $R\Gamma_{\mathrm{Zar}}\big(-, \big(\prod_{p\in\mathbb{P}} \big(\mathbb{L}^{j}_{-/\mathbb{Z}}\big)^{\wedge}_{p}\big)_{\mathbb{Q}}\big)$ $(0 \leq j < i)$. These presheaves are pro cdh sheaves on noetherian schemes by Corollary 10.1.10, so the presheaf $R\Gamma_{\mathrm{Zar}}\big(-, \big(\prod_{p\in\mathbb{P}} \big(\mathbb{L}\Omega^{<i}_{-/\mathbb{Z}}\big)^{\wedge}_{p}\big)_{\mathbb{Q}}\big)$ is a pro cdh sheaf on noetherian schemes. This implies that the presheaf $R\Gamma_{\mathrm{Zar}}\big(-, \prod'_{p\in\mathbb{P}} \underline{\widehat{\mathbb{L}\Omega}}^{\geq i}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p}\big)$ is a pro cdh sheaf on noetherian schemes. □

**Proposition 10.2.3.** *For every integer $i \geq 0$, the presheaf $\mathbb{Q}(i)^{\mathrm{mot}}$ is a pro cdh sheaf on noetherian schemes.*

*Proof.* By Corollary 5.6.7, there is a fibre sequence of presheaves

$$\mathbb{Q}(i)^{\mathrm{mot}}(-) \longrightarrow \mathbb{Q}(i)^{\mathrm{cdh}}(-) \longrightarrow \mathrm{cofib}\Big(R\Gamma_{\mathrm{Zar}}\big(-, \mathbb{L}\Omega^{<i}_{-_{\mathbb{Q}}/\mathbb{Q}}\big) \longrightarrow R\Gamma_{\mathrm{cdh}}\big(-, \Omega^{<i}_{-_{\mathbb{Q}}/\mathbb{Q}}\big)\Big)[-1]$$

on qcqs derived schemes, and in particular on noetherian schemes. Cdh sheaves are in particular pro cdh sheaves, so it suffices to prove that the presheaf $R\Gamma_{\mathrm{Zar}}\big(X, \mathbb{L}\Omega^{<i}_{-_{\mathbb{Q}}/\mathbb{Q}}\big)$ is a pro cdh sheaf on noetherian schemes. This presheaf has a finite filtration with graded pieces given by the presheaves $R\Gamma_{\mathrm{Zar}}\big(-, \mathbb{L}^{j}_{-_{\mathbb{Q}}/\mathbb{Q}}\big)$ $(0 \leq j < i)$, so the result is a consequence of Corollary 10.1.10.

Alternatively, one can prove this result by using Corollary 5.5.11 and pro cdh descent for algebraic $K$-theory ([KST18, Theorem A]). □

**Corollary 10.2.4.** *For every integer $i \geq 0$, the presheaf $\mathbb{Q}(i)^{\mathrm{TC}}$ is a pro cdh sheaf on noetherian schemes.*

*Proof.* By Remark 4.3.7, the presheaf $\mathbb{Q}(i)^{\mathrm{TC}}$ is a pro cdh sheaf on noetherian schemes if and only if the presheaf $\mathbb{Q}(i)^{\mathrm{mot}}$ is a pro cdh sheaf on noetherian schemes. The result is then a consequence of Proposition 10.2.3. $\square$

By Remark 4.3.7, the presheaf $\mathbb{Q}(i)^{\mathrm{TC}}$ is a pro cdh sheaf on noetherian schemes if and only if the presheaf $\mathbb{Q}(i)^{\mathrm{mot}}$ is a pro cdh sheaf on noetherian schemes. One can then prove Proposition 10.2.3 alternatively by using Corollary 5.5.11 and pro cdh descent for algebraic $K$-theory ([KST18, Theorem A]).

We now turn our attention to Bhatt–Morrow–Scholze's syntomic complexes $\mathbb{Z}_p(i)^{\mathrm{BMS}}$.

**Corollary 10.2.5.** *For every integer $i \geq 0$, the presheaf $\big(\prod_{p\in\mathbb{P}} \mathbb{Z}_p(i)^{\mathrm{BMS}}\big)_{\mathbb{Q}}$ is a pro cdh sheaf on noetherian schemes.*

*Proof.* Rationalising the cartesian square of Corollary 5.2.16 yields a cartesian square of presheaves

$$\begin{array}{ccc}
\mathbb{Q}(i)^{\mathrm{TC}}(-) & \longrightarrow & R\Gamma_{\mathrm{Zar}}\Big(-, \widehat{\mathbb{L}\Omega}^{\geq i}_{-_{\mathbb{Q}}/\mathbb{Q}}\Big) \\
\downarrow & & \downarrow \\
\big(\prod_{p\in\mathbb{P}} \mathbb{Z}_p(i)^{\mathrm{BMS}}(-)\big)_{\mathbb{Q}} & \longrightarrow & R\Gamma_{\mathrm{Zar}}\Big(-, \prod'_{p\in\mathbb{P}} \underline{\widehat{\mathbb{L}\Omega}}^{\geq i}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p}\Big)
\end{array}$$

on qcqs derived schemes, and in particular on noetherian schemes. The other three presheaves of this cartesian square being pro cdh sheaves on noetherian schemes (Propositions 10.2.1, 10.2.2, and 10.2.3), the bottom left presheaf is also a pro cdh sheaf on noetherian schemes. $\square$

**Lemma 10.2.6.** *Let $p$ be a prime number. Then for every integer $i \geq 0$, the presheaf $\mathbb{F}_p(i)^{\mathrm{BMS}}$ is a pro cdh sheaf on noetherian schemes.*

*Proof.* By [AMMN22, Corollary 5.31], there exists an integer $m \geq 0$ and an equivalence of presheaves[3]

$$\mathbb{F}_p(i)^{\mathrm{BMS}}(-) \xrightarrow{\sim} \mathrm{fib}\Big(\mathrm{can} - \phi_i : (\mathcal{N}^{\geq i}\Delta_-\{i\}/\mathcal{N}^{\geq i+m}\Delta_-\{i\})/p \longrightarrow (\Delta_-\{i\}/\mathcal{N}^{\geq i+m}\Delta_-\{i\})/p\Big).$$

In particular, it suffices to prove that for every integer $j \geq 0$, the presheaf $\mathcal{N}^j\Delta_-/p$ is a pro cdh sheaf on noetherian schemes. By [BL22, Remark 5.5.8 and Example 4.7.8], there is a fibre sequence of presheaves

$$\mathcal{N}^j\Delta_-\{i\}/p \longrightarrow \mathrm{Fil}_j^{\mathrm{conj}}\overline{\Delta}_{-/\mathbb{Z}_p[\![\widetilde{p}]\!]}/p \xrightarrow{\Theta+j} \mathrm{Fil}_{j-1}^{\mathrm{conj}}\overline{\Delta}_{-/\mathbb{Z}_p[\![\widetilde{p}]\!]}/p.$$

The presheaves $\mathrm{Fil}_j^{\mathrm{conj}}\overline{\Delta}_{-/\mathbb{Z}_p[\![\widetilde{p}]\!]}/p$ and $\mathrm{Fil}_{j-1}^{\mathrm{conj}}\overline{\Delta}_{-/\mathbb{Z}_p[\![\widetilde{p}]\!]}/p$ have finite filtrations with graded pieces given by modulo $p$ powers of the cotangent complex. The result is then a consequence of Corollary 10.1.10. $\square$

**Lemma 10.2.7.** *Let $A$ be an abelian group of the form $A = \prod_{p\in\mathbb{P}} A_p$, where $A_p$ is a derived $p$-complete abelian group. If $A$ is torsion, then $A$ is bounded torsion (i.e., there exists an integer $N \geq 1$ such that $A$ is $N$-torsion).*

[3]Prismatic cohomology was first defined on $p$-complete $p$-quasisyntomic rings ([BMS19, BS22]), and then generalised to arbitrary animated commutative rings by taking the left Kan extension from polynomial $\mathbb{Z}$-algebras, and imposing that it depends only on the derived $p$-completion of its input ([AMMN22, BL22]). On noetherian rings $R$, the derived and classical $p$-completions agree, so the prismatic cohomology of $R$ is naturally identified with the prismatic cohomology of the classical $p$-completion of $R$.

*Proof.* Assume that the abelian group $A$ is torsion. Then for every prime number $p$, the abelian group $A_p$ is torsion and derived $p$-complete, hence it is bounded $p$-power torsion by [Bha19, Theorem 1.1]. Let $S$ be the set of prime numbers $p$ such that $A_p$ is not the zero group. Then there exists an inclusion of abelian groups $\prod_{p\in S}\mathbb{F}_p \subseteq A$, and, if $S$ is infinite, then $\prod_{p\in S}\mathbb{F}_p$ is not torsion. So $S$ is finite, and, as a finite product of bounded torsion abelian groups, the abelian group $A$ is bounded torsion. □

**Proposition 10.2.8.** *For every integer $i \geq 0$, the presheaf $\prod_{p\in\mathbb{P}}\mathbb{Z}_p(i)^{\mathrm{BMS}}$ is a pro cdh sheaf on noetherian schemes.*

*Proof.* Fix an abstract blowup square of noetherian schemes (10.1). Let $\{C_r\}_r$ be the pro object in the derived category $\mathcal{D}(\mathbb{Z})$ defined as the total fibre of the commutative square obtained by applying the presheaf $\prod_{p\in\mathbb{P}}\mathbb{Z}_p(i)^{\mathrm{BMS}}$ to this abstract blowup square. We want to prove that $\{C_r\}_r$ is weakly zero. By Corollary 10.2.5, its rationalisation $\{C_r\otimes_{\mathbb{Z}}\mathbb{Q}\}_r$ is weakly zero.

Let $r_0 \geq 0$ and $n\in\mathbb{Z}$ be integers. Let $r_1 \geq r_0$ be an integer such that the map

$$\mathrm{H}^n(C_{r_1})\otimes_{\mathbb{Z}}\mathbb{Q} \longrightarrow \mathrm{H}^n(C_{r_0})\otimes_{\mathbb{Z}}\mathbb{Q}$$

is the zero map. We now construct an integer $r_2 \geq r_1$ such that the map

$$\mathrm{H}^n(C_{r_2}) \longrightarrow \mathrm{H}^n(C_{r_0})$$

is the zero map. By Lemma 10.2.6, and for every prime number $p$, the pro abelian group $\{\mathrm{H}^n(C_r/p)\}_r$ is zero, which implies that the pro abelian group $\{\mathrm{H}^n(C_r)/p\}_r$ is zero. By induction, this implies that for every integer $N \geq 1$, the pro abelian group $\{\mathrm{H}^n(C_r)/N\}_r$ is zero. By construction, the cohomology groups $\mathrm{H}^n(C_r)$ ($r \geq 0$) are naturally products, indexed by prime numbers $p$, of derived $p$-complete abelian groups. The kernel and cokernel of a map of derived $p$-complete abelian groups are derived $p$-complete abelian groups. So the image $A_{r_0}$ of the map $\mathrm{H}^n(C_{r_1}) \to \mathrm{H}^n(C_{r_0})$ is a product, indexed by prime numbers $p$, of derived $p$-complete abelian groups. This abelian group $A_{r_0}$ is also torsion by definition of the integer $r_1$, so Lemma 10.2.7 implies that there exists an integer $N \geq 1$ such that $A_{r_0}$ is $N$-torsion. Let $r_2 \geq r_1$ be an integer such that the map $\mathrm{H}^n(C_{r_2})/N \to \mathrm{H}^n(C_{r_1})/N$ is the zero map. Then the map

$$\mathrm{H}^n(C_{r_2}) \longrightarrow \mathrm{H}^n(C_{r_0})$$

factors as

$$\mathrm{H}^n(C_{r_2}) \longrightarrow \mathrm{H}^n(C_{r_2})/N \xrightarrow{0} \mathrm{H}^n(C_{r_1})/N \longrightarrow A_{r_0} \subseteq \mathrm{H}^n(C_{r_0}),$$

and is thus the zero map, which concludes the proof. □

**Corollary 10.2.9.** *Let $p$ be a prime number. Then for every integer $i \geq 0$, the presheaf $\mathbb{Z}_p(i)^{\mathrm{BMS}}$ is a pro cdh sheaf on noetherian schemes.*

*Proof.* The presheaf $\mathbb{Z}_p(i)^{\mathrm{BMS}}$ is a direct summand of the presheaf $\prod_{\ell\in\mathbb{P}}\mathbb{Z}_\ell(i)^{\mathrm{BMS}}$, so the result is a consequence of Proposition 10.2.8. □

**Proposition 10.2.10.** *For every integer $i \geq 0$, the presheaf $\mathbb{Z}(i)^{\mathrm{TC}}$ is a pro cdh sheaf on noetherian schemes.*

*Proof.* By Corollary 5.2.16, there is a cartesian square of presheaves

$$\begin{array}{ccc} \mathbb{Z}(i)^{\mathrm{TC}} & \longrightarrow & R\Gamma_{\mathrm{Zar}}\big(-, \widehat{\mathbb{L}\Omega}^{\geq i}_{-_{\mathbb{Q}}/\mathbb{Q}}\big) \\ \downarrow & & \downarrow \\ \prod_{p\in\mathbb{P}} \mathbb{Z}_p(i)^{\mathrm{BMS}} & \longrightarrow & R\Gamma_{\mathrm{Zar}}\big(-, \prod'_{p\in\mathbb{P}} \underline{\widehat{\mathbb{L}\Omega}}^{\geq i}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p}\big) \end{array}$$

on qcqs derived schemes, and in particular on noetherian schemes. The presheaves

$$R\Gamma_{\mathrm{Zar}}\big(-, \widehat{\mathbb{L}\Omega}^{\geq i}_{-_{\mathbb{Q}}/\mathbb{Q}}\big),\ R\Gamma_{\mathrm{Zar}}\big(-, \prod_{p\in\mathbb{P}}{}' \underline{\widehat{\mathbb{L}\Omega}}^{\geq i}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p}\big),\ \text{and}\ \prod_{p\in\mathbb{P}} \mathbb{Z}_p(i)^{\mathrm{BMS}}$$

are pro cdh sheaves on noetherian schemes by Propositions 10.2.1, 10.2.2, and 10.2.8 respectively. So the presheaf $\mathbb{Z}(i)^{\mathrm{TC}}$ is a pro cdh sheaf on noetherian schemes. ☐

The following result was proved on noetherian schemes over a field by Elmanto–Morrow [EM23].

**Theorem 10.2.11** (Pro cdh descent)**.** *For every integer $i \geq 0$, the motivic complex $\mathbb{Z}(i)^{\mathrm{mot}}$ is a pro cdh sheaf on noetherian schemes.*

*Proof.* By Remark 4.3.7, there is a cartesian square of presheaves

$$\begin{array}{ccc} \mathbb{Z}(i)^{\mathrm{mot}} & \longrightarrow & \mathbb{Z}(i)^{\mathrm{TC}} \\ \downarrow & & \downarrow \\ \mathbb{Z}(i)^{\mathrm{cdh}} & \longrightarrow & L_{\mathrm{cdh}}\, \mathbb{Z}(i)^{\mathrm{TC}} \end{array}$$

on qcqs schemes, and in particular on noetherian schemes. The presheaf $\mathbb{Z}(i)^{\mathrm{TC}}$ is a pro cdh sheaf on noetherian schemes by Proposition 10.2.10. The presheaves $\mathbb{Z}(i)^{\mathrm{cdh}}$ and $L_{\mathrm{cdh}}\, \mathbb{Z}(i)^{\mathrm{TC}}$ are cdh sheaves on noetherian schemes by construction, hence pro cdh sheaves on noetherian schemes. So the presheaf $\mathbb{Z}(i)^{\mathrm{mot}}$ is a pro cdh sheaf. ☐

**Remark 10.2.12** (Pro cdh descent for algebraic $K$-theory)**.** The arguments to prove Theorem 10.2.11 can be adapted to give a new proof of the pro cdh descent for algebraic $K$-theory of Kerz–Strunk–Tamme [KST18]. More precisely, by Theorem 2.1.1, pro cdh descent for algebraic $K$-theory is equivalent to pro cdh descent for TC. By Corollary 5.6.3, the result rationally reduces to the pro cdh descent for HC, which is proved by Morrow ([Mor16, Theorem 0.2]). The result mod $p$ is similar to that of Lemma 10.2.6, where the Nygaard filtration and the relative prismatic cohomology are replaced by the Tate filtration and by relative THH; the pro cdh descent for relative THH then reduces to the pro cdh descent for powers of the cotangent complex by [AMMN22, Section 5.2]. Following Section 5.2, there is a natural cartesian square

$$\begin{array}{ccc} \mathrm{TC}(-) & \longrightarrow & \mathrm{HC}^-(-_{\mathbb{Q}}/\mathbb{Q}) \\ \downarrow & & \downarrow \\ \prod_{p\in\mathbb{P}} \mathrm{TC}(-;\mathbb{Z}_p) & \longrightarrow & \Big(\prod'_{p\in\mathbb{P}} \mathrm{HH}(-;\mathbb{Q}_p)\Big)^{h\mathrm{S}^1}. \end{array}$$

Using the cdh descent for the presheaves $\mathrm{HP}(-_{\mathbb{Q}}/\mathbb{Q})$ ([LT19]) and $\big(\prod'_{p\in\mathbb{P}} \mathrm{HH}(-;\mathbb{Q}_p)\big)^{t\mathrm{S}^1}$ (Corollary 5.4.2), the pro cdh descent of the two right terms reduces to the pro cdh descent for HC. The integral statement is then similarly a consequence of Lemma 10.2.7.

## 10.3 Motivic Weibel vanishing

In this section, we prove Theorem 10.3.3, which is a motivic refinement of Weibel's vanishing conjecture on negative $K$-groups ([KST18, Theorem B $(i)$]).

**Lemma 10.3.1.** *Let $V$ be a henselian valuation ring. Then for every integer $i \geq 0$, the motivic complex $\mathbb{Z}(i)^{\mathrm{mot}}(V) \in \mathcal{D}(\mathbb{Z})$ is in degrees at most $i$.*

*Proof.* Henselian valuation rings are local rings for the cdh topology, so the natural maps

$$\mathbb{Z}(i)^{\mathrm{mot}}(V) \longrightarrow \mathbb{Z}(i)^{\mathrm{cdh}}(V) \longleftarrow \mathbb{Z}(i)^{\mathrm{lisse}}(V)$$

are equivalences in the derived category $\mathcal{D}(\mathbb{Z})$ (Remark 4.3.10 and Definition 4.2.5). □

**Lemma 10.3.2.** *Let $A$ be a local ring, and $I$ be a nil ideal of $A$. Then for every integer $i \geq 0$, the fibre of the natural map*

$$\mathbb{Z}(i)^{\mathrm{mot}}(A) \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(A/I)$$

*is in degrees at most $i$.*

*Proof.* We first prove the result rationally, and modulo $p$ for every prime number $p$. Any finitary cdh sheaf is invariant under nil extensions. By Corollary 5.6.7, the result after rationalisation is thus equivalent to the fact that the fibre of the natural map

$$\mathbb{L}\Omega^{<i}_{(A_{\mathbb{Q}})/\mathbb{Q}}[-1] \longrightarrow \mathbb{L}\Omega^{<i}_{((A/I)_{\mathbb{Q}})/\mathbb{Q}}[-1]$$

is in degrees at most $i$. Both terms of this map are in degrees at most $i$. In degree $i$, this map is given by the natural map

$$\Omega^{i-1}_{(A_{\mathbb{Q}})/\mathbb{Q}} \longrightarrow \Omega^{i-1}_{((A/I)_{\mathbb{Q}})/\mathbb{Q}},$$

which is surjective as the $\mathbb{Q}$-algebra $(A/I)_{\mathbb{Q}}$ is a quotient of the $\mathbb{Q}$-algebra $A_{\mathbb{Q}}$. Let $p$ be a prime number. By Corollary 4.3.12, the result modulo $p$ is equivalent to the fact that the fibre of the natural map

$$\mathbb{F}_p(i)^{\mathrm{BMS}}(A) \longrightarrow \mathbb{F}_p(i)^{\mathrm{BMS}}(A/I)$$

is in degrees at most $i$. The pair $(A,I)$ is henselian, so this is a consequence of Theorem 3.2.13.

By the previous rational statement, the fibre $F \in \mathcal{D}(\mathbb{Z})$ of the natural map

$$\mathbb{Z}(i)^{\mathrm{mot}}(A) \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(A/I)$$

has torsion cohomology groups in degrees at least $i+1$. By the short exact sequence of abelian groups

$$0 \longrightarrow \mathrm{H}^j(F)/p \longrightarrow \mathrm{H}^j(F/p) \longrightarrow \mathrm{H}^{j+1}(F)[p] \longrightarrow 0$$

for every prime number $p$ and every integer $j \geq i+1$, the previous torsion statement implies that these cohomology groups are also torsionfree, hence zero, in degrees at least $i+2$. It then remains to prove that the abelian group $\mathrm{H}^{i+1}(F)$ is zero. By Corollary 8.1.10 and its proof, the abelian group $\mathrm{H}^{i+1}_{\mathrm{mot}}(A, \mathbb{Z}(i))$ is torsionfree, so it suffices to prove that the natural map of abelian groups $\mathrm{H}^{i}_{\mathrm{mot}}(A, \mathbb{Z}(i)) \to \mathrm{H}^{i}_{\mathrm{mot}}(A/I, \mathbb{Z}(i))$ is surjective. Let $P$ be a local ind-smooth $\mathbb{Z}$-algebra with a surjective map $P \to A$. By Theorem 8.1.11 (see also Lemma 8.2.4 for a related argument), and because $P \to A/I$ is also a surjection from a local ind-smooth $\mathbb{Z}$-algebra, the composite map of abelian groups

$$\mathrm{H}^{i}_{\mathrm{mot}}(P, \mathbb{Z}(i)) \longrightarrow \mathrm{H}^{i}_{\mathrm{mot}}(A, \mathbb{Z}(i)) \longrightarrow \mathrm{H}^{i}_{\mathrm{mot}}(A/I, \mathbb{Z}(i))$$

is surjective, so the right map is surjective, as desired. □

**Theorem 10.3.3** (Motivic Weibel vanishing)**.** *Let $d \geq 0$ be an integer, and $X$ be a noetherian scheme of dimension at most $d$. Then for every integer $i \geq 0$, the motivic complex $\mathbb{Z}(i)^{\mathrm{mot}}(X) \in \mathcal{D}(\mathbb{Z})$ is in degrees at most $i + d$.*

*Proof.* The presheaf $\mathbb{Z}(i)^{\mathrm{mot}} : \mathrm{Sch}^{\mathrm{qcqs,op}} \to \mathcal{D}(\mathbb{Z})$ satisfies the following properties:

(1) it is finitary (Corollary 5.5.14);

(2) it satisfies pro cdh descent on noetherian schemes (Theorem 10.2.11);

(3) for every henselian valuation ring $V$, the complex $\mathbb{Z}(i)^{\mathrm{mot}}(V)$ is in degrees at most $i$ (Lemma 10.3.1);

(4) for every noetherian local ring $A$ and every nilpotent ideal $I$ of $A$, the fibre of the natural map $\mathbb{Z}(i)^{\mathrm{mot}}(A) \to \mathbb{Z}(i)^{\mathrm{mot}}(A/I)$ is in degrees at most $i$ (Lemma 10.3.2).

By [BEM24], the presheaf $\mathbb{Z}(i)^{\mathrm{cdh}} : \mathrm{Sch}^{\mathrm{qcqs,op}} \to \mathcal{D}(\mathbb{Z})$ is a finitary cdh sheaf which is in degrees at most $i$ on henselian valuation rings, hence it also satisfies the previous properties.

By [EM23, Proposition 8.10] applied to the presheaf $\mathrm{fib}\big(\mathbb{Z}(i)^{\mathrm{mot}} \to \mathbb{Z}(i)^{\mathrm{cdh}}\big)[i]$, this implies that for every noetherian scheme $X$ of dimension at most $d$, the complex

$$\mathrm{fib}\big(\mathbb{Z}(i)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Z}(i)^{\mathrm{cdh}}(X)\big)$$

is in degrees at most $i+d$. The complex $\mathbb{Z}(i)^{\mathrm{cdh}}(X)$ is also in degrees at most $i+d$ ([BEM24]), so the complex $\mathbb{Z}(i)^{\mathrm{mot}}(X)$ is in degrees at most $i+d$. □

**Remark 10.3.4** (Relation to Weibel's $K$-theoretic vanishing conjecture)**.** Let $X$ be a noetherian scheme of dimension at most $d$. Theorem 10.3.3 states that the Atiyah–Hirzebruch spectral sequence

$$E_2^{i,j} = \mathrm{H}^{i-j}_{\mathrm{mot}}(X, \mathbb{Z}(-j)) \Longrightarrow \mathrm{K}_{-i-j}(X)$$

is supported in the left half plane $x \leq d$: see the following representation of the $E_2$ page, where $\mathrm{H}^j(i)$ denotes the motivic cohomology group $\mathrm{H}^j_{\mathrm{mot}}(X, \mathbb{Z}(i))$.

$$\begin{array}{cccccccccc}
\cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\cdots & 0 & 0 & \mathrm{H}^0(0) & \mathrm{H}^1(0) & \cdots & \mathrm{H}^{d-2}(0) & \mathrm{H}^{d-1}(0) & \mathrm{H}^d(0) & 0 \\
\cdots & 0 & \mathrm{H}^0(1) & \mathrm{H}^1(1) & \mathrm{H}^2(1) & \cdots & \mathrm{H}^{d-1}(1) & \mathrm{H}^d(1) & \mathrm{H}^{d+1}(1) & 0 \\
\cdots & \mathrm{H}^0(2) & \mathrm{H}^1(2) & \mathrm{H}^2(2) & \mathrm{H}^3(2) & \cdots & \mathrm{H}^d(2) & \mathrm{H}^{d+1}(2) & \mathrm{H}^{d+2}(2) & 0 \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}$$

In particular, the negative $K$-groups $\mathrm{K}_{-i-j}(X)$ vanish for $-i-j < -d$ (this is Weibel's vanishing conjecture on algebraic $K$-theory), and there is a natural edge map isomorphism

$$\mathrm{K}_{-d}(X) \cong \mathrm{H}^d_{\mathrm{mot}}(X, \mathbb{Z}(0))$$

of abelian groups. Using the description of weight zero motivic cohomology (Example 5.6.8), the latter result recovers the known description of $\mathrm{K}_{-d}(X)$ ([KST18, Corollary D]). Note that Theorem 10.3.3 is however not a new proof of these results of Kerz–Strunk–Tamme, as our Atiyah–Hirzebruch spectral sequence relating motivic cohomology and algebraic $K$-theory relies on Theorem 2.1.1, which itself relies on the results in [KST18].

**Remark 10.3.5.** Let $X$ be a noetherian scheme of dimension at most $d$. Then for every integer $i \geq 0$, the proof of Theorem 10.3.3 also implies that the natural map

$$\mathbb{Z}(i)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Z}(i)^{\mathrm{cdh}}(X)$$

is surjective on $\mathrm{H}^{i+d}$. For $i = 0$, this map is even an isomorphism on $\mathrm{H}^d$ (actually on all cohomology groups, by Example 5.6.8), thus recovering Weibel's conjecture that the map $\mathrm{K}_{-d}(X) \to \mathrm{KH}_{-d}(X)$ is an isomorphism [Wei80, KST18].

The following result is a description of the group $\mathrm{K}_{-d+1}$, similar to the description of the group $\mathrm{K}_{-d}$ predicted by Weibel (Remark 10.3.4).

**Corollary 10.3.6.** *Let $d \geq 0$ be an integer, and $X$ be a noetherian scheme of dimension at most $d$. Then there is a natural exact sequence*

$$\mathrm{H}^{d-2}_{\mathrm{cdh}}(X, \mathbb{Z}) \xrightarrow{\delta} \mathrm{H}^{d+1}_{\mathrm{mot}}(X, \mathbb{Z}(1)) \longrightarrow \mathrm{K}_{-d+1}(X) \longrightarrow \mathrm{H}^{d-1}_{\mathrm{cdh}}(X, \mathbb{Z}) \longrightarrow 0$$

*of abelian groups, where $\delta$ is the differential map coming from the $E_2$-page of the Atiyah–Hirzebruch spectral sequence (Corollary 5.5.10). Moreover, for every integer $m \geq 2$, if $m$ is invertible in $X$, then the image of the map $(m-1)\delta$ is $m$-power torsion.*

*Proof.* The motivic complex $\mathbb{Z}(i)^{\mathrm{mot}}(X)$ is zero for $i < 0$ (Corollary 5.5.2), and is naturally identified with the complex $R\Gamma_{\mathrm{cdh}}(X, \mathbb{Z})$ for $i = 0$ (Example 5.6.8). The first statement is then a consequence of the Atiyah–Hirzebruch spectral sequence (Corollary 5.5.10) and of motivic Weibel's vanishing (Theorem 10.3.3). The second statement is a consequence of the compatibility of the map $\delta$ with the Adams operation $\psi^m$ (Construction 5.1.8). More precisely, Corollary 5.1.9 implies that the induced map

$$\delta : \mathrm{H}^{d-2}_{\mathrm{cdh}}(X, \mathbb{Z})[\tfrac{1}{m}] \longrightarrow \mathrm{H}^{d+1}_{\mathrm{mot}}(X, \mathbb{Z}(1))[\tfrac{1}{m}]$$

satisfies the equation $\delta = m\delta$, *i.e.*, $(m-1)\delta = 0$, which implies the desired result. □

## 10.4 Comparison to pro cdh motivic cohomology

In this section, we compare the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$ to Kelly–Saito's pro cdh motivic complexes $\mathbb{Z}(i)^{\mathrm{procdh}}$ (Theorem 10.4.2). In equicharacteristic, this is [KS24, Corollary 1.11] (see also [EM23, Theorem 1.15]). Our proof is structurally the same, although finitariness and pro cdh descent in mixed characteristic rely on the main results of Chapter 5, and our proof of the comparison to lisse motivic cohomology is different in mixed characteristic (see comment before Corollary 8.1.12).

**Lemma 10.4.1.** *Let $R$ be a nil extension of a henselian valuation ring, i.e., a commutative ring $R$ whose quotient $R/I$ by its ideal of nilpotent elements $I$ is a henselian valuation ring. Then for every integer $i \geq 0$, the lisse-motivic comparison map*

$$\mathbb{Z}(i)^{\mathrm{lisse}}(R) \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(R)$$

*is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$.*

*Proof.* By Corollary 8.1.12, it suffices to prove that the complex $\mathbb{Z}(i)^{\mathrm{mot}}(R) \in \mathcal{D}(\mathbb{Z})$ is in degrees at most $i$. Let $I$ be the ideal of nilpotent elements of the commutative ring $R$. Using the natural fibre sequence

$$\mathrm{fib}\big(\mathbb{Z}(i)^{\mathrm{mot}}(R) \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(R/I)\big) \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(R) \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(R/I)$$

in the derived category $\mathcal{D}(\mathbb{Z})$, the result is then a consequence of Lemmas 10.3.1 and 10.3.2. □

**Theorem 10.4.2** (Comparison to pro cdh motivic cohomology)**.** *Let $X$ be a noetherian scheme. Then for every integer $i \geq 0$, the lisse-motivic comparison map induces a natural equivalence*

$$\mathbb{Z}(i)^{\mathrm{procdh}}(X) \xrightarrow{\sim} \mathbb{Z}(i)^{\mathrm{mot}}(X)$$

*in the derived category $\mathcal{D}(\mathbb{Z})$.*

*Proof.* The presheaf $\mathbb{Z}(i)^{\mathrm{mot}} : \mathrm{Sch}^{\mathrm{qcqs,op}} \to \mathcal{D}(\mathbb{Z})$ satisfies the following properties:

(1) it is finitary (Corollary 5.5.14);

(2) it satisfies pro cdh descent on noetherian schemes (Theorem 10.2.11);

(3) for every pro cdh local ring $R$, the lisse-motivic comparison map

$$\mathbb{Z}(i)^{\mathrm{lisse}}(R) \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(R)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$ (Lemma 10.4.1 and [KS24, Proposition 1.7]).

By [KS24, Theorem 9.7],[4] this implies that for every noetherian scheme $X$, the lisse-motivic comparison map induces a natural equivalence

$$\mathbb{Z}(i)^{\mathrm{procdh}}(X) \xrightarrow{\sim} \mathbb{Z}(i)^{\mathrm{mot}}(X)$$

in the derived category $\mathcal{D}(\mathbb{Z})$. □

[4]This theorem is stated for schemes over a field, but the proof works over any noetherian commutative ring.

# Chapter 11

# Motivic cohomology of valuation rings

In this chapter, we describe the motivic cohomology of valuation rings. In particular, we give a complete description of the $p$-adic motivic cohomology of valuation rings over a perfectoid base (Theorem 11.4.6). Our proof relies on the study of the prismatic cohomology of these valuation rings, which applies naturally to a broader class of algebras that we call $p$-Cartier smooth. Note that the results of this chapter have appeared as [Bou23].

## 11.1 Prismatic cohomology of Cartier smooth algebras

Let $p$ be a prime number. In this section, we introduce the notion of $p$-Cartier smoothness for morphisms of commutative rings (Definition 11.1.5), and give several characterisations in terms of prismatic cohomology (Theorem 11.1.16).

The $p$-Cartier smooth morphisms generalise smooth morphisms, and behave as if they were smooth from the point of view of syntomic cohomology. Syntomic cohomology can be defined in terms of Frobenius eigenspaces of prismatic cohomology, and this section is devoted to the prismatic cohomology of $p$-Cartier smooth algebras. Our main result is Theorem 11.1.16. We also give a comparison with the notion of $F$-smoothness introduced in [BM23] (see Section 11.1.4).

### 11.1.1 Definitions

For a commutative ring $R$, the cotangent complex

$$\mathbb{L}_{-/R} : R\text{-Alg} \longrightarrow \mathcal{D}(R)$$

is the natural derived version of the module of Kähler differentials

$$\Omega^1_{-/R} : R\text{-Alg} \longrightarrow R\text{-Mod}.$$

It controls information on the relative prismatic complex via the Hodge–Tate comparison theorem [BS22, Theorem 4.11]. The condition that $S \otimes^{\mathbb{L}}_R R/p \in \mathcal{D}(R/p)$ is in degree zero and base change for the cotangent complex imply that the morphism $\mathbb{L}_{S/R} \otimes^{\mathbb{L}}_R R/p \to \mathbb{L}_{(S/p)/(R/p)}$ is an equivalence. This allows one to lift properties of the cotangent complex of $R/p \to S/p$ to similar properties on the $p$-completed cotangent complex of $R \to S$.

**Definition 11.1.1** ($p$-discreteness)**.** A morphism $R \to S$ of commutative rings is *$p$-discrete* if the derived tensor product $S \otimes^{\mathbb{L}}_R R/p \in \mathcal{D}(R/p)$ is concentrated in degree zero, where it is given by $S/p$.

**Examples 11.1.2.** A morphism $R \to S$ of commutative rings is $p$-discrete in the following cases.

(1) $R$ and $S$ are $\mathbb{F}_p$-algebras.

(2) The morphism $R \to S$ is flat.

(3) $R$ and $S$ are $p$-torsionfree. More precisely, if $R$ is $p$-torsionfree, then the morphism $R \to S$ is $p$-discrete if and only if $S$ is $p$-torsionfree.

The following definition axiomatises some properties satisfied by the cotangent complex in the smooth case.

**Definition 11.1.3** (Cotangent smoothness, [BMS19]). (1) A morphism $R \to S$ of commutative rings is *cotangent smooth*, or $S$ is a *cotangent smooth* $R$-algebra, if the cotangent complex $\mathbb{L}_{S/R}$ has Tor-amplitude in $[0;0]$, i.e., $\Omega^1_{S/R}$ is a flat $S$-module and $\mathrm{H}_n(\mathbb{L}_{S/R}) \cong 0$ for all $n > 0$.

(2) A morphism $R \to S$ of commutative rings is *$p$-cotangent smooth*, or $S$ is a *$p$-cotangent smooth* $R$-algebra, if it is $p$-discrete and its reduction $R/p \to S/p$ modulo $p$ is cotangent smooth.[1]

**Examples 11.1.4.** (1) A smooth morphism of commutative rings is cotangent smooth, because its cotangent complex is in degree zero, where it is given by the locally free module of differential forms.

(2) A morphism of perfect $\mathbb{F}_p$-algebras is cotangent smooth, because its cotangent complex is zero. More generally, a morphism $R \to S$ of perfectoid rings is $p$-cotangent smooth. Indeed, it is $p$-discrete as a base change of the morphism $A_{\mathrm{inf}}(R) \to A_{\mathrm{inf}}(S)$ of $p$-torsionfree rings. And the cotangent complex $\mathbb{L}_{(S/p)/(R/p)}$ is zero, as a base change of the cotangent complex $\mathbb{L}_{S^\flat/R^\flat}$ of the morphism $R^\flat \to S^\flat$ of perfect $\mathbb{F}_p$-algebras.

(3) A filtered colimit of ($p$-)cotangent smooth algebras is ($p$-)cotangent smooth, because the cotangent complex commutes with filtered colimits.

(4) Let $V$ be a valuation ring with perfect fraction field, and $V'$ a valuation ring extension of $V$ (see section 11.2). The morphism $V \to V'$ has cotangent complex concentrated in degree zero ([GR03, Theorem 6.5.8 (*ii*)]). If $V$ contains a perfect field, then the morphism $V \to V'$ is cotangent smooth ([GR03, Corollary 6.5.21]). If the $p$-completion of $V$ is a perfectoid ring, then the morphism $V \to V'$ is $p$-cotangent smooth (Proposition 11.2.8 below).

Several properties of prismatic cohomology in the smooth case –such as the comparison between prismatic cohomology and de Rham cohomology [BS22, Corollary 15.4]– do not hold for general $p$-cotangent smooth morphisms $R \to S$. We will prove (Theorem 11.1.16) that the obstruction to satisfy these properties vanishes exactly when the morphism $R/p \to S/p$ satisfies the Cartier isomorphism. For an $\mathbb{F}_p$-algebra $R$, we denote by $\phi_R$ its Frobenius endomorphism.

[1] More generally, a morphism $R \to S$ of commutative rings, possibly with bounded $p$-power torsion, will be called $p$-something if it is $p$-discrete and if its reduction $R/p \to S/p$ modulo $p$ is something. We distinguish this from similar notions on $p$-complete rings, often called $p$-completely something.

**Definition 11.1.5** (Cartier smoothness). (1) A morphism $R \to S$ of $\mathbb{F}_p$-algebras is *Cartier smooth*, or $S$ is a *Cartier smooth* $R$-algebra, if it is cotangent smooth and if the inverse Cartier map

$$C^{-1} : \Omega^n_{S/R} \otimes_{R,\phi_R} R \longrightarrow \mathrm{H}^n(\Omega^\bullet_{S/R})$$

is an isomorphism of $R$-modules for each $n \geq 0$.

(2) A morphism $R \to S$ of commutative rings is *$p$-Cartier smooth*, or $S$ is a *$p$-Cartier smooth* $R$-algebra, if it is $p$-discrete and its reduction $R/p \to S/p$ modulo $p$ is Cartier smooth.

**Remark 11.1.6.** If a morphism $R \to S$ of commutative rings is $p$-Cartier smooth (resp. $p$-cotangent smooth), then its $p$-completion $R^\wedge_p \to S^\wedge_p$ is $p$-Cartier smooth (resp. $p$-cotangent smooth). Moreover, if the morphism $R \to S$ is $p$-discrete and its $p$-completion $R^\wedge_p \to S^\wedge_p$ is $p$-Cartier smooth (resp. $p$-cotangent smooth), then so is $R \to S$.

Equivalently, a morphism $R \to S$ of commutative rings is $p$-Cartier smooth if it is $p$-cotangent smooth and if the morphism $R/p \to S/p$ satisfies the Cartier isomorphism. For a morphism $R \to S$ of $\mathbb{F}_p$-algebras, recall that the inverse Cartier map

$$C^{-1} : \Omega^n_{S/R} \otimes_{R,\phi_R} R \longrightarrow \mathrm{H}^n(\Omega^\bullet_{S/R}),$$

for any integer $n \geq 0$, is defined as the unique $R$-linear map satisfying

$$C^{-1}(f dg_1 \wedge \cdots \wedge dg_n \otimes 1) = f^p g_1^{p-1} \ldots g_n^{p-1} dg_1 \wedge \cdots \wedge dg_n.$$

Cartier smoothness when $R = \mathbb{F}_p$ was already studied in [KM21].

**Examples 11.1.7.** A morphism $R \to S$ of commutative rings is $p$-Cartier smooth in the following cases.

(1) The morphism $R \to S$ is smooth. Indeed, $S$ is a flat $R$-algebra and the cotangent complex $\mathbb{L}_{S/R}$ is a flat $S$-module in degree zero, so the morphism $R \to S$ is $p$-cotangent smooth. The Cartier isomorphism is a standard property of smooth morphisms in characteristic $p$ (see for instance [DI87, Theorem 1.2]).

(2) $S$ is a filtered colimit of smooth $R$-algebras. Indeed, filtered colimits of $p$-cotangent smooth algebras are $p$-cotangent smooth, and the inverse Cartier map commutes with filtered colimits.

(3) $R = \mathbb{F}_p$ and $S$ is a valuation ring. The cotangent smoothness is a result of Gabber–Ramero (Example 11.1.4 (4)). The Cartier isomorphism is a result of Gabber ([KST21, Corollary A.4]).

(4) $R \to S$ is a morphism of perfectoid rings. The cotangent complex of the morphism $R/p \to S/p$ is zero (Example 11.1.4 (2)), so the Cartier isomorphism is trivial for $n \geq 1$. If $R$ and $S$ are perfect $\mathbb{F}_p$-algebras and $n = 0$ the Cartier isomorphism holds; in general the inverse Cartier map for $n = 0$ is the reduction modulo $p^\flat$, for some element $p^\flat \in R^\flat$, of the inverse Cartier map associated to the morphism $R^\flat \to S^\flat$ of perfect $\mathbb{F}_p$-algebras, and is thus an isomorphism.

(5) $R$ is a valuation ring whose $p$-completion is a perfectoid ring, and $S$ is a valuation ring extension of $R$ (Theorem 11.2.1 below).

**Example 11.1.8.** An imperfect $\mathbb{F}_p$-algebra $S$ whose cotangent complex $\mathbb{L}_{S/\mathbb{F}_p}$ is trivial, is cotangent smooth but not Cartier smooth. Indeed, it is cotangent smooth because any algebra over the field $\mathbb{F}_p$ is flat and because the zero complex of $S$-modules has Tor-amplitude in $[0;0]$. In degree zero, the inverse Cartier map for the map $\mathbb{F}_p \to S$ is the Frobenius map $\phi_S : S \to S$, and is thus not an isomorphism; so the $\mathbb{F}_p$-algebra $S$ is not Cartier smooth. Remark that imperfect $\mathbb{F}_p$-algebras with trivial cotangent complex exist, by [Bha13].

**Lemma 11.1.9** (Base change for Cartier smoothness)**.** *Let $R \to S$ be a $p$-Cartier smooth morphism, and $R'$ an $R$-algebra. If the morphism $R' \to S \otimes_R R'$ is $p$-discrete (*e.g.*, if $R'$ is a flat $R$-algebra, or if $R$ is an $\mathbb{F}_p$-algebra), then it is $p$-Cartier smooth.*

*Proof.* Assuming $p$-discreteness, the $p$-cotangent smoothness is preserved by base change. The Cartier isomorphism depends only on the reduction modulo $p$ of the morphism $R' \to S \otimes_R R'$, so we can assume that $R$, $R'$ and $S$ are $\mathbb{F}_p$-algebras. Let $n \geq 0$ be an integer. There is a canonical isomorphism of $R'$-modules

$$\Omega^n_{(S \otimes_R R')/R'} \cong \Omega^n_{S/R} \otimes_R R'.$$

By Cartier smoothness of the morphism $R \to S$, the $R$-module $\mathrm{H}^n(\Omega^\bullet_{S/R})$ is flat, so there is a canonical isomorphism of $R'$-modules

$$\mathrm{H}^n(\Omega^\bullet_{S/R}) \otimes_R R' \cong \mathrm{H}^n(\Omega^\bullet_{S/R} \otimes^{\mathbb{L}}_R R').$$

Moreover, $\Omega^\bullet_{S/R}$ is a complex of flat $R$-modules by cotangent smoothness of the morphism $R \to S$, so there is a canonical isomorphism of $R'$-modules

$$\mathrm{H}^n(\Omega^\bullet_{S/R} \otimes^{\mathbb{L}}_R R') \cong \mathrm{H}^n(\Omega^\bullet_{(S \otimes_R R')/R'})$$

and the morphism $R' \to S \otimes_R R'$ satisfies the Cartier isomorphism. □

**Remark 11.1.10** (Transitivity of Cartier smoothness)**.** One can also prove that the composite of two $p$-Cartier smooth morphisms of commutative rings is $p$-Cartier smooth, using the transitivity property of the derived de Rham complex ([Bha12, Proposition 3.22]) and Theorem 11.1.16 ($\mathbb{L}\Omega = \Omega$).

### 11.1.2 Review of relative prismatic cohomology

Prisms are defined in [BS22] as pairs $(A, I)$, where $A$ is a $\delta$-ring (roughly,[2] a $\mathbb{Z}_{(p)}$-algebra with a lift of Frobenius $\phi_A : A \to A$) and $I \subset A$ is an ideal defining a Cartier divisor in $\mathrm{Spec}(A)$, such that $A$ is derived $(p, I)$-complete and $p \in I + \phi_A(I)A$. In all the cases of interest the ideal $I \subset A$ will be a principal ideal generated by a nonzerodivisor. A prism $(A, I)$ is called bounded if the ring $A/I$ has bounded $p$-power torsion. Restricting to bounded prisms avoids complications involving derived completions: for instance, the underlying ring $A$ of a bounded prism $(A, I)$ is $(p, I)$-complete in the classical sense.

Following [BMS19, Section 4] or [BL22, Appendix C], a morphism $R \to S$ of commutative rings is *$p$-quasisyntomic* if it is $p$-flat and $\mathbb{L}_{(S/p)/(R/p)}$ has Tor-amplitude in $[-1;0]$. For instance, any noetherian local complete intersection ring $S$ is $p$-quasisyntomic over $\mathbb{Z}$ by [Avr99, Theorem 1.2].

[2] More precisely, a $\delta$-ring is a $\mathbb{Z}_{(p)}$-algebra $A$ equipped with a map $\delta_A : A \to A$ such that $\phi_A : A \to A, x \mapsto x^p + p\delta_A(x)$ is a ring homomorphism. If $A$ is a $p$-torsionfree $\mathbb{Z}_{(p)}$-algebra, a $\delta$-ring structure on $A$ is the same as a ring endomorphism lifting the Frobenius on $A/p$. If $A$ is a general $\mathbb{Z}_{(p)}$-algebra, a $\delta$-ring structure on $A$ is the same as a lift of Frobenius in the derived sense ([BS22, Remark 2.5]).

Given a bounded prism $(A, I)$ and a $p$-quasisyntomic $A/I$-algebra $S$, [BS22, Section 4.1] defines the prismatic site $(S/A)_{\Delta}$ as the site having as objects the prisms $(B, IB)$ over $(A, I)$ with a map $S \to B/IB$, and covers given by flat covers. The relative prismatic complex $\Delta_{S/A} \in \mathcal{D}(A)$ is defined as the cohomology of the sheaf

$$\mathcal{O}_{\Delta} : (S/A)_{\Delta} \longrightarrow A\text{-Alg}, \ (B, IB) \longmapsto B.$$

Similarly, the Hodge–Tate complex $\overline{\Delta}_{S/A} \in \mathcal{D}(A/I)$ is defined as the cohomology of the sheaf

$$\overline{\mathcal{O}}_{\Delta} : (S/A)_{\Delta} \longrightarrow A/I\text{-Alg}, \ (B, IB) \longmapsto B/IB.$$

For instance, if $k$ is a perfect field of characteristic $p$ and $S$ is a ($p$-)quasisyntomic $k$-algebra, then $(W(k), (p))$ is an example of bounded prism and $\Delta_{S/W(k)}$ recovers the crystalline cohomology of $S$ ([BL22, Section 4.6]). For a general $A/I$-algebra $S$, prismatic and Hodge–Tate complexes are defined by left Kan extension from the smooth case. The compatibility between the two definitions in the $p$-quasisyntomic case is proved in [BL22, Section 4]. Note in particular that $p$-Cartier smooth algebras are not necessarily $p$-quasisyntomic, as they are not necessarily $p$-flat.

The prismatic complex $\Delta_{S/A}$ naturally bears an $A$-linear Frobenius endomorphism

$$\phi : \Delta^{(1)}_{S/A} \longrightarrow \Delta_{S/A},$$

where $\Delta^{(1)}_{S/A} := \Delta_{S/A} \widehat{\otimes}^{\mathbb{L}}_{A,\phi_A} A$ is the Frobenius-twisted prismatic complex. Our main result on the prismatic cohomology of $p$-Cartier smooth algebras describes the image of this Frobenius endomorphism $\phi$, and is formulated in terms of the functor $L\eta_I$. Following [BMS18, Section 6] and for $A$ a commutative ring, $I \subset A$ an ideal defining a Cartier divisor in $\operatorname{Spec}(A)$ and $C \in \mathcal{D}(A)$ a complex, the object $L\eta_I C \in \mathcal{D}(A)$ is defined as follows. The complex $C$ in the derived category $\mathcal{D}(A)$ is represented by a complex $(C^\bullet, d)$ of $A$-modules such that for all $i \in \mathbb{Z}$, $C^i$ is $I$-torsionfree (*i.e.*, $C^i[I] = 0$). Define the complex $\eta_I C^\bullet$ with terms

$$\eta_I C^i := \{x \in I^i C^i \mid dx \in I^{i+1} C^{i+1}\}$$

and differential induced by the differential of $C^\bullet$. As an object of the derived category $\mathcal{D}(A)$ this construction does not depend on the choice of $(C^\bullet, d)$ ([BMS18, Corollary 6.5]), and we call this object $L\eta_I C \in \mathcal{D}(A)$. Following [BMS19, Section 5], an object $\operatorname{Fil}^\star C$ of the filtered derived category $\mathcal{DF}(A)$ is called *connective for the Beilinson t-structure* if for every integer $i \in \mathbb{Z}$, the graded piece $\operatorname{gr}^i C \in \mathcal{D}(A)$ is in degrees at most $i$. The *connective cover for the Beilinson t-structure* of a filtered complex $\operatorname{Fil}^\star C$ is the universal connective filtered complex with a map to $\operatorname{Fil}^\star C$.

**Proposition 11.1.11.** *Let $A$ be a commutative ring, $I \subset A$ an ideal defining a Cartier divisor in* $\operatorname{Spec}(A)$ *and $C$ an object in the derived category $\mathcal{D}(A)$.*

*(1) ([BMS18, Lemma 6.4]) For each integer $i \in \mathbb{Z}$, there is a canonical isomorphism of $A$-modules*

$$\mathrm{H}^i(L\eta_I C) \cong (\mathrm{H}^i(C)/\mathrm{H}^i(C)[I]) \otimes_A I^i.$$

*(2) ([BMS19, Proposition 5.8]) $L\eta_I C \in \mathcal{D}(A)$ is the complex underlying the connective cover for the Beilinson t-structure of the $I$-adically filtered object $I^\star C \in \mathcal{DF}(A)$.*

*(3) ([BMS18, Proposition 6.12]) There is a natural equivalence*

$$(L\eta_I C) \otimes^{\mathbb{L}}_A A/I \xrightarrow{\sim} \mathrm{H}^*(C/I)$$

*in the derived category $\mathcal{D}(A/I)$, where the differential on $\mathrm{H}^*(C/I)$ is the Bockstein operator induced by the $I$-adic filtration on $C$.*

By the previous proposition, the complex $L\eta_I \Delta_{S/A}$ is characterised by a universal property in terms of the $I$-adic filtration on the prismatic complex $\Delta_{S/A}$. Following [BS22, Section 15], define the Nygaard filtration $\mathcal{N}^\star \Delta^{(1)}_{S/A}$ on the Frobenius-twisted prismatic complex $\Delta^{(1)}_{S/A}$ as the preimage[3] of the $I$-adic filtration $I^\star \Delta_{S/A}$ via the morphism $\phi$. The Frobenius

$$\phi : \Delta^{(1)}_{S/A} \longrightarrow \Delta_{S/A}$$

naturally upgrades to a map of filtered complexes

$$\phi : \mathcal{N}^{\geq \star} \Delta^{(1)}_{S/A} \longrightarrow I^\star \Delta_{S/A} \tag{11.1}$$

in the filtered derived category $\mathcal{DF}(A)$. Beware that the filtration $\mathcal{N}^{\geq \star}\Delta^{(1)}_{S/A}$ is in general not complete, and we denote by $\widehat{\Delta}^{(1)}_{S/A} \in \mathcal{D}(A)$ (resp. $\mathcal{N}^i \Delta^{(1)}_{S/A} \in \mathcal{D}(A/I)$, for $i \geq 0$) the Nygaard-completed prismatic complex (resp. the Nygaard graded pieces). On the Nygaard graded pieces, the Frobenius (11.1) can be rewritten as a map

$$\phi : \mathcal{N}^i \Delta^{(1)}_{S/A} \longrightarrow \overline{\Delta}_{S/A}\{i\}$$

for each $i \geq 0$, where

$$\overline{\Delta}_{S/A}\{i\} := \overline{\Delta}_{S/A} \otimes_{A/I} (I/I^2)^{\otimes i}.$$

To describe the image of this Frobenius map, define the conjugate filtration $\mathrm{Fil}^{\mathrm{conj}}_\star \overline{\Delta}_{S/A}\{i\}$ on the Hodge–Tate complex $\overline{\Delta}_{S/A}\{i\}$ as the left Kan extension from smooth $A/I$-algebras of the increasing Postnikov filtration $\tau^{\leq \star}\overline{\Delta}_{-/A}\{i\}$.

**Theorem 11.1.12** ([BS22, BL22])**.** *Let $(A, I)$ be a bounded prism, and $S$ be an $A/I$-algebra.*

*(1) For each $i \geq 0$, the Frobenius map $\phi : \mathcal{N}^i \Delta^{(1)}_{S/A} \to \overline{\Delta}_{S/A}\{i\}$ factors as*

$$\mathcal{N}^i \Delta^{(1)}_{S/A} \xrightarrow{\tilde{\phi}} \mathrm{Fil}^{\mathrm{conj}}_i \overline{\Delta}_{S/A}\{i\} \xrightarrow{\mathrm{can}} \overline{\Delta}_{S/A}\{i\},$$

*and the map $\tilde{\phi} : \mathcal{N}^i \Delta^{(1)}_{S/A} \to \mathrm{Fil}^{\mathrm{conj}}_i \overline{\Delta}_{S/A}\{i\}$ is an equivalence. In particular, the complex $\mathcal{N}^i \Delta^{(1)}_{S/A} \in \mathcal{D}(A/I)$ is in degrees at most $i$.*

*(2) The Frobenius map $\phi : \Delta^{(1)}_{S/A} \to \Delta_{S/A}$ factors as*

$$\Delta^{(1)}_{S/A} \xrightarrow{\tilde{\phi}} L\eta_I \Delta_{S/A} \xrightarrow{\mathrm{can}} \Delta_{S/A}.$$

[3]To make sense of this preimage, one needs to restrict to large $p$-quasisyntomic $A/I$-algebras –for which the prismatic complex $\Delta_{S/A}$ is concentrated in degree zero–, and then generalise the definition using descent on the $p$-quasisyntomic site and left Kan extension.

*Proof.* The first part of (1) is a special case of [BL22, Proposition 5.1.1 and Remark 5.1.3], where the result is formulated for a general animated commutative $A/I$-algebra $S$. All the objects are left Kan extended from the smooth case, where the result is also given by [BS22, Theorem 15.3]. The functor $\tau^{\leq i}\overline{\Delta}_{-/A}\{i\}$ is in degrees at most $i$, so its left Kan extension (defined as a sifted colimit) is also in degrees at most $i$.

(2) The filtered complex $\mathcal{N}^{\geq\star}\Delta^{(1)}_{S/A} \in \mathcal{DF}(A)$ is connective for the Beilinson $t$-structure by (1). So the map $\phi : \mathcal{N}^{\geq\star}\Delta^{(1)}_{S/A} \to I^\star\Delta_{S/A}$ factors through the connective cover for the Beilinson $t$-structure of the target. The result follows from Proposition 11.1.11 (2), by looking at the underlying complexes. □

We recall some important features of the relative prismatic complex in the smooth case, following [BS22].

**Theorem 11.1.13** (Prismatic cohomology in the smooth case, [BS22])**.** *Let $(A, I)$ be a bounded prism, and $S$ be a smooth $A/I$-algebra.*

*(1) ($L\eta_I$ comparison) The Frobenius map*

$$\tilde{\phi} : \Delta^{(1)}_{S/A} \xrightarrow{\sim} L\eta_I\Delta_{S/A}$$

*of Theorem 11.1.12 (2) is an equivalence in the derived category $\mathcal{D}(A)$.*

*(2) (Hodge–Tate comparison) There is a canonical isomorphism*

$$(\Omega^*_{S/(A/I)})^\wedge_p \xrightarrow{\cong} \mathrm{H}^*(\overline{\Delta}_{S/A}\{*\}) := \bigoplus_{i\geq 0} \mathrm{H}^i(\overline{\Delta}_{S/A}\{i\})$$

*of differential graded $A/I$-algebras, where the differential on $\mathrm{H}^*(\overline{\Delta}_{S/A}\{*\})$ is the Bockstein operator induced by the $I$-adic filtration on $\Delta_{S/A}$.*

*(3) (de Rham comparison) There is a canonical equivalence*

$$\Delta^{(1)}_{S/A} \otimes^{\mathbb{L}}_A A/I \xrightarrow{\sim} (\Omega_{S/(A/I)})^\wedge_p$$

*in the derived category $\mathcal{D}(A/I)$.*

*Proof.* By [BL22, Corollary 4.1.14], the prismatic cohomology of $S$ is the same as the prismatic cohomology of the $p$-completion of $S$. (1) and (2) then respectively follow from [BS22, Theorem 15.3] and [BS22, Theorem 4.11]. Using Proposition 11.1.11 (3), (3) is a consequence of (1) and (2). □

### 11.1.3 Prismatic cohomology of Cartier smooth algebras

In this subsection we prove Theorem J (Theorem 11.1.16), characterising $p$-Cartier smooth algebras in terms of their prismatic cohomology.

Any $p$-Cartier smooth algebra is $p$-cotangent smooth by definition. We first extend some properties of smooth algebras to general $p$-cotangent smooth algebras. Given a morphism $R \to S$ of commutative rings, denote by $\widehat{\mathbb{L}\Omega}_{S/R}$ the Hodge-completed derived de Rham complex, and by $(\widehat{\mathbb{L}\Omega}_{S/R})^\wedge_p$ its $p$-completion.

**Proposition 11.1.14.** *Let $(A, I)$ be a bounded prism, and $S$ be a $p$-cotangent smooth $A/I$-algebra.*

*(1) The canonical map* $(\widehat{\mathbb{L}\Omega}_{S/(A/I)})^{\wedge}_p \to (\Omega_{S/(A/I)})^{\wedge}_p$ *is an equivalence in the derived category* $\mathcal{D}(A/I)$*.*

*(2) The conjugate filtration* $\mathrm{Fil}^{\mathrm{conj}}_{\star}\,\overline{\Delta}_{S/A}$ *on the Hodge–Tate complex* $\overline{\Delta}_{S/A} \in \mathcal{D}(A/I)$ *coincides with the Postnikov filtration* $\tau^{\leq\star}\overline{\Delta}_{S/A}$*. In particular for each* $i \geq 0$*, the Frobenius map*

$$\phi : \mathcal{N}^i \Delta^{(1)}_{S/A} \longrightarrow \overline{\Delta}_{S/A}\{i\}$$

*induces an equivalence*

$$\tilde{\phi} : \mathcal{N}^i \Delta^{(1)}_{S/A} \xrightarrow{\sim} \tau^{\leq i}\overline{\Delta}_{S/A}\{i\}$$

*in the derived category* $\mathcal{D}(A/I)$*.*

*(3) There is a canonical isomorphism*

$$(\Omega^*_{S/(A/I)})^{\wedge}_p \xrightarrow{\cong} \mathrm{H}^*(\overline{\Delta}_{S/A}\{*\})$$

*of differential graded* $A/I$*-algebras, where the differential on* $\mathrm{H}^*(\overline{\Delta}_{S/A}\{*\})$ *is the Bockstein operator induced by the* $I$*-adic filtration on* $\Delta_{S/A}$*.*

*(4) The Frobenius map* $\tilde{\phi} : \Delta^{(1)}_{S/A} \to L\eta_I \Delta_{S/A}$ *of Theorem 11.1.12* (2) *factors as*

$$\Delta^{(1)}_{S/A} \xrightarrow{\mathrm{can}} \widehat{\Delta}^{(1)}_{S/A} \xrightarrow{\tilde{\phi}} L\eta_I \Delta_{S/A},$$

*and the map* $\tilde{\phi} : \widehat{\Delta}^{(1)}_{S/A} \to L\eta_I \Delta_{S/A}$ *is an equivalence in the derived category* $\mathcal{D}(A)$*.*

*Proof.* (1) By the derived Nakayama lemma ([Sta19, 091N]), it suffices to prove the result after derived reduction modulo $p$. By base change for the Hodge-completed derived de Rham complex and $p$-discreteness, this is equivalent to proving that the canonical map

$$\widehat{\mathbb{L}\Omega}_{(S/p)/(A/(p,I))} \longrightarrow \Omega_{(S/p)/(A/(p,I))}$$

is an equivalence in the derived category $\mathcal{D}(A/(p,I))$. Both sides are complete for the Hodge filtration, so it suffices to prove that this canonical map is an equivalence on the Hodge graded pieces. The corresponding map

$$\mathbb{L}^n_{(S/p)/(A/(p,I))}[-n] \longrightarrow \Omega^n_{(S/p)/(A/(p,I))}[-n]$$

is the shift of a wedge power of the counit map

$$\mathbb{L}_{(S/p)/(A/(p,I))} \longrightarrow \Omega^1_{(S/p)/(A/(p,I))}[0]$$

for all $n \geq 0$, which is an equivalence by $p$-cotangent smoothness of the morphism $A/I \to S$.

(2) By left Kan extension of the Hodge–Tate comparison Theorem 11.1.13 (2) (or [BL22, Remark 4.1.7]), there is a canonical equivalence

$$(\mathbb{L}^n_{S/(A/I)})^{\wedge}_p[-n]\{-n\} \xrightarrow{\sim} \mathrm{gr}^{\mathrm{conj}}_n\,\overline{\Delta}_{S/A}$$

in the derived category $\mathcal{D}(A/I)$, for each $n \geq 0$. By the derived Nakayama lemma ([Sta19, 091N]) and $p$-cotangent smoothness of the morphism $A/I \to S$, the complex

$$(\mathbb{L}^n_{S/(A/I)})^{\wedge}_p \in \mathcal{D}(A/I)$$

is in degree zero and the conjugate graded piece $\mathrm{gr}_n^{\mathrm{conj}}\,\overline{\Delta}_{S/A}$ is thus in degree $n$. By induction on $n \geq 0$ and using the long exact sequence in cohomology groups associated to the homotopy cofibre sequence

$$\mathrm{Fil}_{n-1}^{\mathrm{conj}}\,\overline{\Delta}_{S/A} \longrightarrow \mathrm{Fil}_n^{\mathrm{conj}}\,\overline{\Delta}_{S/A} \longrightarrow \mathrm{gr}_n^{\mathrm{conj}}\,\overline{\Delta}_{S/A},$$

we deduce that $\mathrm{Fil}_n^{\mathrm{conj}}\,\overline{\Delta}_{S/A}$ belongs to $\mathcal{D}^{[0;n]}(A/I)$, that the induced morphism

$$\mathrm{Fil}_{n-1}^{\mathrm{conj}}\,\overline{\Delta}_{S/A} \longrightarrow \tau^{\leq n-1}\mathrm{Fil}_n^{\mathrm{conj}}\,\overline{\Delta}_{S/A}$$

is an equivalence and that the morphism

$$\mathrm{H}^n(\mathrm{Fil}_n^{\mathrm{conj}}\,\overline{\Delta}_{S/A}) \longrightarrow \mathrm{H}^n(\mathrm{gr}_n^{\mathrm{conj}}\,\overline{\Delta}_{S/A})$$

is an isomorphism of $A/I$-modules. The conjugate filtration $\mathrm{Fil}_\star^{\mathrm{conj}}\,\overline{\Delta}_{S/A}$ is moreover exhaustive, as a left Kan extension of the exhaustive Postnikov filtration $\tau^{\leq\star}\overline{\Delta}_{-/A}$. So the canonical map

$$\mathrm{Fil}_n^{\mathrm{conj}}\,\overline{\Delta}_{S/A} \longrightarrow \tau^{\leq n}\lim_k(\mathrm{Fil}_{n+k}^{\mathrm{conj}}\,\overline{\Delta}_{S/A}) \simeq \tau^{\leq n}\overline{\Delta}_{S/A}$$

is an equivalence for each $n \geq 0$, and the conjugate filtration $\mathrm{Fil}_\star^{\mathrm{conj}}\,\overline{\Delta}_{S/A}$ coincides with the Postnikov filtration $\tau^{\leq\star}\overline{\Delta}_{S/A}$. The last statement is a consequence of Theorem 11.1.12 (1).

(3) By left Kan extension of the Hodge–Tate comparison Theorem 11.1.13 (2) (or [BL22, Remark 4.1.7]) and (2), there is for each $n \geq 0$ a canonical isomorphism

$$(\Omega^n_{S/(A/I)})^\wedge_p \xrightarrow{\cong} \mathrm{H}^n(\overline{\Delta}_{S/A}\{n\})$$

of $A/I$-modules. To prove that the de Rham differential $d$ coincides via these isomorphisms with the Bockstein operator $\beta$ on $\mathrm{H}^*(\overline{\Delta}_{S/A}\{*\})$, it suffices to prove it when $S$ is a polynomial $A/I$-algebra, where this is Theorem 11.1.13 (2).

(4) For each integer $k \geq 0$, the map of filtered complexes $\phi : \mathcal{N}^{\geq\star}\Delta^{(1)}_{S/A} \to I^\star\Delta_{S/A}$ induces a map of filtered complexes

$$\phi : \mathcal{N}^{\geq\star}\Delta^{(1)}_{S/A}/\mathcal{N}^{\geq\star+k}\Delta^{(1)}_{S/A} \longrightarrow I^\star\Delta_{S/A}/I^{\star+k}\Delta_{S/A} \simeq \Delta_{S/A}\otimes^{\mathbb{L}}_A I^\star/I^{\star+k}.$$

Taking the inverse limit over $k \geq 0$ defines a map of filtered complexes

$$\phi : \mathcal{N}^{\geq\star}\widehat{\Delta}^{(1)}_{S/A} \longrightarrow I^\star\Delta_{S/A}.$$

The canonical map $\mathcal{N}^{\geq\star}\Delta^{(1)}_{S/A} \to \mathcal{N}^{\geq\star}\widehat{\Delta}^{(1)}_{S/A}$ is an equivalence on graded pieces, so the same argument as in Theorem 11.1.12 (2) proves that

$$\Delta^{(1)}_{S/A} \xrightarrow{\mathrm{can}} \widehat{\Delta}^{(1)}_{S/A} \xrightarrow{\phi} \Delta_{S/A}$$

factors as

$$\Delta^{(1)}_{S/A} \xrightarrow{\mathrm{can}} \widehat{\Delta}^{(1)}_{S/A} \xrightarrow{\tilde{\phi}} L\eta_I\Delta_{S/A} \xrightarrow{\mathrm{can}} \Delta_{S/A},$$

where the composite $\tilde{\phi} : \Delta^{(1)}_{S/A} \to L\eta_I\Delta_{S/A}$ of the first two maps is the map defined in Theorem 11.1.12 (2). Remark that the $I$-adic filtration on $L\eta_I\Delta_{S/A}$ is complete by [BL22, Remark D.10]. To prove that $\tilde{\phi} : \widehat{\Delta}^{(1)}_{S/A} \to L\eta_I\Delta_{S/A}$ is an equivalence, it suffices by completeness to prove that it is an equivalence on graded pieces. Proposition 11.1.11 (2) (more precisely, [BMS19, Theorem 5.4 (2)]) identifies the $i^{\text{th}}$ graded piece of $L\eta_I \Delta_{S/A}$ with the truncation $\tau^{\leq i}$ of the $I$-adic graded piece $\overline{\Delta}_{S/A}\{i\}$. Together with the identification $\mathcal{N}^i \Delta^{(1)}_{S/A} \xrightarrow{\sim} \mathcal{N}^i \widehat{\Delta}^{(1)}_{S/A}$, it then suffices to prove that

$$\tilde{\phi} : \mathcal{N}^i \Delta^{(1)}_{S/A} \longrightarrow \tau^{\leq i} \overline{\Delta}_{S/A}\{i\}$$

is an equivalence for all $i \geq 0$, which is (2). □

**Remark 11.1.15.** The factorisation part of Proposition 11.1.14 (4) and its proof hold more generally for $(A, I)$ a bounded prism and $S$ any $A/I$-algebra.

**Theorem 11.1.16.** *Let $(A, I)$ be a bounded prism, and $S$ be a $p$-cotangent smooth $A/I$-algebra. The following are equivalent:*

($CSm$) *The inverse Cartier map*

$$\Omega^n_{(S/p)/(A/(p,I))} \otimes_{A/(p,I),\phi_{A/(p,I)}} A/(p, I) \xrightarrow{C^{-1}} \mathrm{H}^n(\Omega^\bullet_{(S/p)/(A/(p,I))})$$

*is an isomorphism of $A/(p, I)$-modules for all $n \geq 0$, i.e., $S$ is $p$-Cartier smooth over $A/I$.*

($\mathbb{L}\Omega = \widehat{\mathbb{L}\Omega}$) *The Hodge-completion map $(\mathbb{L}\Omega_{S/(A/I)})^\wedge_p \to (\widehat{\mathbb{L}\Omega}_{S/(A/I)})^\wedge_p$ is an equivalence in the derived category $\mathcal{D}(A/I)$.*

($\mathbb{L}\Omega = \Omega$) *The counit map $(\mathbb{L}\Omega_{S/(A/I)})^\wedge_p \to (\Omega_{S/(A/I)})^\wedge_p$ is an equivalence in the derived category $\mathcal{D}(A/I)$.*

($dR$) *The de Rham comparison map $\Delta^{(1)}_{S/A} \otimes^{\mathbb{L}}_A A/I \to (\Omega_{S/(A/I)})^\wedge_p$ is an equivalence in the derived category $\mathcal{D}(A/I)$.*

($\overline{\Delta}$) *The canonical map $\Delta^{(1)}_{S/A} \otimes^{\mathbb{L}}_A A/I \to \mathrm{H}^*(\overline{\Delta}_{S/A}\{*\})$ is an equivalence in the derived category $\mathcal{D}(A/I)$.*

($\Delta = \widehat{\Delta}$) *The Nygaard-completion map $\Delta^{(1)}_{S/A} \to \widehat{\Delta}^{(1)}_{S/A}$ is an equivalence in the derived category $\mathcal{D}(A)$.*

($L\eta$) *The Frobenius map $\tilde{\phi} : \Delta^{(1)}_{S/A} \to L\eta_I \Delta_{S/A}$ is an equivalence in the derived category $\mathcal{D}(A)$.*

($\mathcal{N}$) *The canonical map $\mathrm{H}^0_{\mathrm{B}} : \Delta^{(1)}_{S/A} \otimes^{\mathbb{L}}_A A/I \to \mathrm{H}^*(\mathcal{N}^* \Delta^{(1)}_{S/A})$ is an equivalence in the derived category $\mathcal{D}(A/I)$, where the differential on $\mathrm{H}^*(\mathcal{N}^* \Delta^{(1)}_{S/A})$ is the Bockstein operator induced by the Nygaard filtration on $\Delta^{(1)}_{S/A}$.*

($\mathcal{N}^{\geq}$) *The Frobenius map $\tau^{\leq i}\phi : \tau^{\leq i} \mathcal{N}^{\geq i} \Delta^{(1)}_{S/A} \to \tau^{\leq i} I^i \Delta_{S/A}$ is an equivalence in the derived category $\mathcal{D}(A)$ for all $i \geq 0$.*

*Proof.* ($CSm$) ⇔ ($\mathbb{L}\Omega = \Omega$) By derived Nakayama ([Sta19, 091N]), the counit map

$$(\mathbb{L}\Omega_{S/(A/I)})^\wedge_p \longrightarrow (\Omega_{S/(A/I)})^\wedge_p$$

is an equivalence in the derived category $\mathcal{D}(A/I)$ if and only if the counit map

$$\mathbb{L}\Omega_{(S/p)/(A/(p,I))} \longrightarrow \Omega_{(S/p)/(A/(p,I))}$$

is an equivalence in the derived category $\mathcal{D}(A/(p,I))$, *i.e.*, if the induced map

$$\mathrm{H}^n(\mathbb{L}\Omega_{(S/p)/(A/(p,I))}) \longrightarrow \mathrm{H}^n(\Omega_{(S/p)/(A/(p,I))})$$

is an isomorphism of $A/(p,I)$-modules for each $n \geq 0$. By [Bha12, Proposition 3.5], the derived de Rham complex $\mathbb{L}\Omega_{(S/p)/(A/(p,I))}$ is equipped with an exhaustive $\mathbb{N}$-indexed increasing filtration $\mathrm{Fil}^{\mathrm{conj}}_\star \mathbb{L}\Omega_{(S/p)/(A/(p,I))}$, whose graded pieces are given by

$$C^{-1} : \wedge^n(\mathbb{L}_{(S/p)/(A/(p,I))} \otimes^{\mathbb{L}}_{A/(p,I),\phi_{A/(p,I)}} A/(p,I))[-n] \xrightarrow{\sim} \mathrm{gr}^{\mathrm{conj}}_n \mathbb{L}\Omega_{(S/p)/(A/(p,I))},$$

where $C^{-1}$ is the left Kan extension of the inverse Cartier map. By cotangent smoothness of the morphism $A/(p,I) \to S/p$, the graded piece $\mathrm{gr}^{\mathrm{conj}}_n \mathbb{L}\Omega_{(S/p)/(A/(p,I))}$ is thus concentrated in degree $n$ for each $n \geq 0$. Arguing by induction on $n \geq 0$ and by exhaustiveness of the conjugate filtration $\mathrm{Fil}^{\mathrm{conj}}_\star \mathbb{L}\Omega_{(S/p)/(A/(p,I))}$, there is a canonical isomorphism

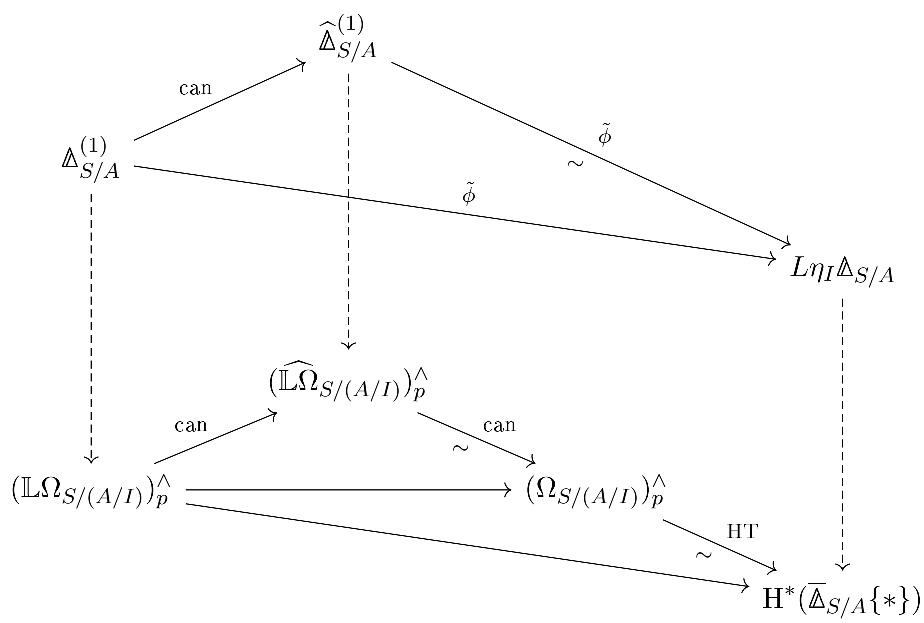
$$\mathrm{H}^n(\mathrm{gr}^{\mathrm{conj}}_n \mathbb{L}\Omega_{(S/p)/(A/(p,I))}) \xrightarrow{\cong} \mathrm{H}^n(\mathbb{L}\Omega_{(S/p)/(A/(p,I))}).$$

In particular the counit map $\mathbb{L}\Omega_{(S/p)/(A/(p,I))} \to \Omega_{(S/p)/(A/(p,I))}$ is an equivalence in the derived category $\mathcal{D}(A/(p,I))$ if and only if the inverse Cartier map

$$C^{-1} : \Omega^n_{(S/p)/(A/(p,I))} \otimes_{A/(p,I),\phi_{A/(p,I)}} A/(p,I) \longrightarrow \mathrm{H}^n(\Omega^\bullet_{(S/p)/(A/(p,I))})$$

is an isomorphism of $A/(p,I)$-modules for all $n \geq 0$.

$(\mathbb{L}\Omega = \widehat{\mathbb{L}\Omega}) \Leftrightarrow (\mathbb{L}\Omega = \Omega) \Leftrightarrow (dR) \Leftrightarrow (\overline{\mathbb{\Delta}}) \Leftrightarrow (\mathbb{\Delta} = \widehat{\mathbb{\Delta}}) \Leftrightarrow (L\eta)$ By [BL22, Proposition 5.2.3] there is a commutative diagram




in the derived category $\mathcal{D}(A)$, where the dashed arrows are derived reduction modulo $I$ ([BL22, Proposition 5.2.5 and Corollary 5.2.8] and Proposition 11.1.11 (3)). The map

$$(\mathbb{L}\Omega_{S/(A/I)})^\wedge_p \to \mathrm{H}^*(\overline{\mathbb{\Delta}}_{S/A}\{*\})$$

is defined as the composite

$$(\mathbb{L}\Omega_{S/(A/I)})_p^\wedge \longrightarrow (\Omega_{S/(A/I)})_p^\wedge \longrightarrow \mathrm{H}^*(\overline{\mathbb{\Delta}}_{S/A}\{*\}).$$

The three equivalences in this diagram hold for any $p$-cotangent smooth $A/I$-algebra $S$ by Proposition 11.1.14. By derived Nakayama ([Sta19, 091N]) and commutativity of this diagram, the conditions $(\mathbb{L}\Omega = \widehat{\mathbb{L}\Omega})$, $(\mathbb{L}\Omega = \Omega)$, $(dR)$, $(\overline{\mathbb{\Delta}})$, $(L\eta)$ and $(\mathbb{\Delta} = \widehat{\mathbb{\Delta}})$ are then equivalent.

$(L\eta) \Leftrightarrow (\mathcal{N})$ There is a commutative diagram

$$\begin{array}{ccc}
\mathbb{\Delta}^{(1)}_{S/A} \otimes^{\mathbb{L}}_A A/I & \xrightarrow{\mathrm{H}^0_{\mathrm{B}}} & \mathrm{H}^*(\mathcal{N}^*\mathbb{\Delta}^{(1)}_{S/A}) \\
\downarrow{\scriptstyle \tilde{\phi}} & & \sim\downarrow{\scriptstyle \tilde{\phi}} \\
(L\eta_I\mathbb{\Delta}_{S/A}) \otimes^{\mathbb{L}}_A A/I & \xrightarrow[\sim]{\mathrm{H}^0_{\mathrm{B}}} & \mathrm{H}^*(\overline{\mathbb{\Delta}}_{S/A}\{*\})
\end{array}$$

in the derived category $\mathcal{D}(A/I)$, where the maps $\tilde{\phi}$ are defined in Theorem 11.1.12, and the functor $\mathrm{H}^0_{\mathrm{B}}$ (where B refers to the Beilinson $t$-structure) is defined in [BMS19, Theorem 5.4]. More precisely, the functor $\mathrm{H}^0_{\mathrm{B}}$ sends the filtered complex

$$\mathcal{N}^{\geq \star}\mathbb{\Delta}^{(1)}_{S/A} \in \mathcal{DF}(A)$$

to

$$\mathrm{H}^*(\mathcal{N}^*\mathbb{\Delta}^{(1)}_{S/A}) \in \mathrm{Ch}(A) \simeq \mathcal{DF}(A)^{\heartsuit},$$

where $\mathcal{DF}(A)^{\heartsuit}$ is the heart of the filtered derived category $\mathcal{DF}(A)$ for its Beilinson $t$-structure. Because $\mathcal{N}^{\star}\mathbb{\Delta}^{(1)}_{S/A}$ is naturally an object of the derived category $\mathcal{D}(A/I)$, this induces a map

$$\mathbb{\Delta}^{(1)}_{S/A} \otimes^{\mathbb{L}}_A A/I \xrightarrow{\mathrm{H}^0_{\mathrm{B}}} \mathrm{H}^*(\mathcal{N}^*\mathbb{\Delta}^{(1)}_{S/A})$$

in the derived category $\mathcal{D}(A/I)$ on the underlying complexes. For any $p$-cotangent smooth $A/I$-algebra $S$, the right and bottom maps of the previous diagram are equivalences (Propositions 11.1.14 (2) and 11.1.11 (3)). By derived Nakayama and commutativity of the diagram, the Frobenius map

$$\tilde{\phi} : \mathbb{\Delta}^{(1)}_{S/A} \longrightarrow L\eta_I\mathbb{\Delta}_{S/A}$$

is an equivalence in the derived category $\mathcal{D}(A)$ if and only if the canonical map

$$\mathrm{H}^0_{\mathrm{B}} : \mathbb{\Delta}^{(1)}_{S/A} \otimes^{\mathbb{L}}_A A/I \longrightarrow \mathrm{H}^*(\mathcal{N}^*\mathbb{\Delta}^{(1)}_{S/A})$$

is an equivalence in the derived category $\mathcal{D}(A/I)$.

$(\mathcal{N}^{\geq}) \Rightarrow (L\eta)$ Assume that the Frobenius map

$$\tau^{\leq i}\phi : \tau^{\leq i}\mathcal{N}^{\geq i}\mathbb{\Delta}^{(1)}_{S/A} \longrightarrow \tau^{\leq i}I^i\mathbb{\Delta}_{S/A}$$

is an equivalence for each $i \geq 0$, and in particular that the Frobenius map

$$\phi : \mathrm{H}^i(\mathcal{N}^{\geq i}\mathbb{\Delta}^{(1)}_{S/A}) \longrightarrow \mathrm{H}^i(I^i\mathbb{\Delta}_{S/A})$$

is an isomorphism of $A$-modules for each $i \geq 0$. The homotopy cofibre sequence

$$\mathcal{N}^{\geq i}\mathbb{\Delta}^{(1)}_{S/A} \longrightarrow \mathcal{N}^{\geq i-1}\mathbb{\Delta}^{(1)}_{S/A} \longrightarrow \mathcal{N}^{i-1}\mathbb{\Delta}^{(1)}_{S/A}$$

induces an exact sequence

$$\mathrm{H}^{n-1}(\mathcal{N}^{i-1}\Delta^{(1)}_{S/A}) \longrightarrow \mathrm{H}^n(\mathcal{N}^{\geq i}\Delta^{(1)}_{S/A}) \longrightarrow \mathrm{H}^n(\mathcal{N}^{\geq i-1}\Delta^{(1)}_{S/A}) \longrightarrow \mathrm{H}^n(\mathcal{N}^{i-1}\Delta^{(1)}_{S/A})$$

for each integer $n$. By Proposition 11.1.14 (2), the graded piece $\mathcal{N}^{i-1}\Delta^{(1)}_{S/A}$ is in degrees $[0; i-1]$ and the Frobenius map $\phi : \mathcal{N}^{i-1}\Delta^{(1)}_{S/A} \to \overline{\Delta}_{S/A}\{i-1\}$ is an isomorphism in degrees at most $i-1$. In particular the morphism $\mathrm{H}^n(\mathcal{N}^{\geq i}\Delta^{(1)}_{S/A}) \to \mathrm{H}^n(\mathcal{N}^{\geq i-1}\Delta^{(1)}_{S/A})$ is an isomorphism for all $i, n \geq 0$ satisfying $i < n$, and the canonical morphism

$$\mathrm{H}^i(\mathcal{N}^{\geq i-1}\Delta^{(1)}_{S/A}) \longrightarrow \mathrm{H}^i(\mathcal{N}^{\geq 0}\Delta^{(1)}_{S/A}) = \mathrm{H}^i(\Delta^{(1)}_{S/A})$$

is an isomorphism for all $i \geq 0$. The $A$-module $\mathrm{H}^i(L\eta_I\Delta_{S/A})$ is canonically identified with the image of the morphism $\mathrm{H}^i(I^i\Delta_{S/A}) \to \mathrm{H}^i(I^{i-1}\Delta_{S/A})$ by Proposition 11.1.11 (1). The Frobenius map

$$\phi : \mathcal{N}^{\geq\star}\Delta^{(1)}_{S/A} \longrightarrow I^\star\Delta_{S/A}$$

thus induces a map of short exact sequences

$$\begin{array}{ccccccccc}
\mathrm{H}^{i-1}(\mathcal{N}^{\geq i-1}\Delta^{(1)}_{S/A}) & \longrightarrow & \mathrm{H}^{i-1}(\mathcal{N}^{i-1}\Delta^{(1)}_{S/A}) & \longrightarrow & \mathrm{H}^i(\mathcal{N}^{\geq i}\Delta^{(1)}_{S/A}) & \longrightarrow & \mathrm{H}^i(\Delta^{(1)}_{S/A}) & \longrightarrow & 0 \\
\downarrow\phi & & \downarrow\phi & & \downarrow\phi & & \downarrow\tilde{\phi} & & \\
\mathrm{H}^{i-1}(I^{i-1}\Delta_{S/A}) & \longrightarrow & \mathrm{H}^{i-1}(\overline{\Delta}_{S/A}\{i-1\}) & \longrightarrow & \mathrm{H}^i(I^i\Delta_{S/A}) & \longrightarrow & \mathrm{H}^i(L\eta_I\Delta_{S/A}) & \longrightarrow & 0.
\end{array}$$

The first three vertical maps are isomorphisms by assumption, thus so is the right vertical map, for each $i \geq 0$. So the Frobenius map $\tilde{\phi} : \Delta^{(1)}_{S/A} \to L\eta_I\Delta_{S/A}$ is an equivalence.

$(L\eta) \Rightarrow (\mathcal{N}^{\geq})$ Assume the Frobenius map $\tilde{\phi} : \Delta^{(1)}_{S/A} \to L\eta_I\Delta_{S/A}$ is an equivalence. We prove by induction on $i \geq 0$ that for every integer $n \leq i$, the Frobenius morphism

$$\phi : \mathrm{H}^n(\mathcal{N}^{\geq i}\Delta^{(1)}_{S/A}) \longrightarrow \mathrm{H}^n(I^i\Delta_{S/A})$$

is an isomorphism of $A$-modules. For $i = 0$, it suffices to prove that the Frobenius morphism

$$\phi : \mathrm{H}^0(\Delta^{(1)}_{S/A}) \longrightarrow \mathrm{H}^0(\Delta_{S/A})$$

is an isomorphism, or equivalently that the canonical map

$$\mathrm{H}^0(L\eta_I\Delta_{S/A}) \cong \mathrm{H}^0(\Delta_{S/A})/\mathrm{H}^0(\Delta_{S/A})[I] \longrightarrow \mathrm{H}^0(\Delta_{S/A})$$

is an isomorphism. For any prism $(B, IB)$ over $(A, I)$, the $A$-algebra $B$ is $I$-torsionfree [BS22, Lemma 3.5], so

$$\mathrm{H}^0(\Delta_{S/A}) = \lim_{(B,IB)\in(S/A)_\Delta} B$$

is also $I$-torsionfree ([BL22, Theorem 4.3.6]) and the morphism $\mathrm{H}^0(L\eta_I\Delta_{S/A}) \to \mathrm{H}^0(\Delta_{S/A})$ is the identity. Assume $i$ is now a non-negative integer for which the result holds. Using the equivalence

$$\tilde{\phi} : \Delta^{(1)}_{S/A} \xrightarrow{\sim} L\eta_I\Delta_{S/A},$$

the first, second and fourth vertical maps of the previous diagram are isomorphisms, so the morphism

$$\phi : \mathrm{H}^i(\mathcal{N}^{\geq i}\Delta^{(1)}_{S/A}) \longrightarrow \mathrm{H}^i(I^i\Delta_{S/A})$$

is also an isomorphism. For $n < i$, the Frobenius $\phi : \mathcal{N}^{\geq \star}\Delta^{(1)}_{S/A} \to I^\star\Delta_{S/A}$ induces a map of exact sequences

$$\begin{array}{ccccccccc}
\mathrm{H}^{n-1}(\mathcal{N}^{\geq i}\Delta^{(1)}_{S/A}) & \to & \mathrm{H}^{n-1}(\mathcal{N}^{i}\Delta^{(1)}_{S/A}) & \to & \mathrm{H}^{n}(\mathcal{N}^{\geq i+1}\Delta^{(1)}_{S/A}) & \to & \mathrm{H}^{n}(\mathcal{N}^{\geq i}\Delta^{(1)}_{S/A}) & \to & \mathrm{H}^{n}(\mathcal{N}^{i}\Delta^{(1)}_{S/A}) \\
\cong\downarrow\phi & & \cong\downarrow\phi & & \downarrow\phi & & \cong\downarrow\phi & & \cong\downarrow\phi \\
\mathrm{H}^{n-1}(I^{i-1}\Delta_{S/A}) & \to & \mathrm{H}^{n-1}(\overline{\Delta}_{S/A}\{i\}) & \longrightarrow & \mathrm{H}^{n}(I^{i+1}\Delta_{S/A}) & \longrightarrow & \mathrm{H}^{n}(I^{i}\Delta_{S/A}) & \longrightarrow & \mathrm{H}^{n}(\overline{\Delta}_{S/A}\{i\})
\end{array}$$

where the isomorphisms are given by Proposition 11.1.14 (2) and the induction hypothesis. It follows that the morphism $\phi : \mathrm{H}^n(\mathcal{N}^{\geq i}\Delta^{(1)}_{S/A}) \to \mathrm{H}^n(I^i\Delta_{S/A})$ is also an isomorphism, which concludes the proof. □

### 11.1.4 Comparison with $F$-smoothness

In this subsection, we compare the absolute notion of $F$-smoothness, introduced by Bhatt–Mathew [BM23], with our relative notion of $p$-Cartier smoothness, in the case when the base is perfectoid (Theorem 11.1.19). $F$-smoothness is a variant of ($p$-adic) smoothness designed to capture smoothness in an absolute sense. For instance, regular rings are $F$-smooth ([BM23, Theorem 4.15]).

Absolute prismatic cohomology can be defined using an absolute version of the relative prismatic site introduced in Section 11.1.2. We recall only the necessary notation, and refer the reader to [BL22] for the general theory. Following [BMS19, Section 4] or [BL22, Appendix C], a commutative ring $S$ is *$p$-quasisyntomic* if $S$ has bounded $p$-power torsion and

$$\mathbb{L}_{S/\mathbb{Z}} \otimes^{\mathbb{L}}_S S/p \in \mathcal{D}(S/p)$$

has Tor-amplitude in $[-1; 0]$. Following [BL22, Sections 4 and 5] and for any $p$-quasisyntomic ring $S$, the absolute prismatic site $(S)_\Delta$ is the site having as objects the prisms $(B, J)$ with a map $S \to B/J$ and covers given by flat covers. The absolute prismatic complex $\Delta_S \in \mathcal{D}(\mathbb{Z}_p)$ is the cohomology of the sheaf

$$\mathcal{O}_\Delta : (S)_\Delta \longrightarrow A\text{-Alg},\ (B, J) \longmapsto B.$$

It is equipped with a Nygaard filtration $\mathcal{N}^{\geq\star}\Delta_S$ and a map of filtered complexes

$$\phi : \mathcal{N}^{\geq\star}\Delta_S \longrightarrow \Delta_S^{[\star]},$$

where the filtered complex $\Delta_S^{[\star]}$ is an absolute version of the $I$-adic filtration on relative prismatic cohomology. This Frobenius map is compatible with the Frobenius map on relative prismatic cohomology when $S$ is defined over a base prism.

**Definition 11.1.17** ($F$-smoothness, [BM23])**.** A $p$-quasisyntomic ring $S$ is *$F$-smooth* if for each integer $i \geq 0$, the Nygaard filtration on $\Delta_S\{i\}$ is complete and the homotopy cofibre hocofib($\phi$) of the Frobenius map

$$\phi : \mathcal{N}^i\Delta_S \longrightarrow \overline{\Delta}_S\{i\},$$

where $\overline{\Delta}_S\{i\} := gr^i\,\Delta_S^{[\star]}$, has $p$-complete Tor-amplitude in degrees at least $i+1$.

Following [BS22, Section 3], a bounded prism $(A, I)$ is *perfect* if its Frobenius $\phi_A$ is an isomorphism. The functor $(A, I) \mapsto A/I$ induces an equivalence between the category of perfect prisms and the category of perfectoid rings ([BS22, Theorem 3.10]). Given a perfect prism $(A, I)$, a $p$-quasisyntomic $A/I$-algebra $S$ and an integer $i \geq 0$, the canonical maps

$$\Delta_S\{i\} \longrightarrow \Delta_{S/A}\{i\} \longrightarrow \Delta^{(1)}_{S/A}\{i\}$$

are equivalences and their composite can be refined into an equivalence of filtered complexes ([BL22, Construction 5.6.1 and Theorem 5.6.2])

$$\mathcal{N}^{\geq \star}\Delta_S\{i\} \xrightarrow{\sim} \mathcal{N}^{\geq \star}\Delta^{(1)}_{S/A}\{i\}.$$

**Theorem 11.1.18.** *Let $(A, I)$ be a bounded prism, and $S$ be a $p$-quasisyntomic $A/I$-algebra. Then $S$ is $p$-Cartier smooth over $A/I$ if and only if it satisfies the following relative version of $F$-smoothness: for each integer $i \geq 0$ (or equivalently $i \in \{0, 1\}$), the homotopy cofibre of the Frobenius map $\phi : \mathcal{N}^i\Delta^{(1)}_{S/A} \to \overline{\Delta}_{S/A}\{i\}$ has $p$-complete Tor-amplitude in degrees at least $i + 1$, and the Nygaard filtration on $\Delta^{(1)}_{S/A}\{i\}$ is complete.*

*Proof.* Assume first that $S$ is $p$-Cartier smooth over $A/I$. Let $i \geq 0$ be an integer. The cofibre of the Frobenius map

$$\phi : \mathcal{N}^i\Delta^{(1)}_{S/A} \longrightarrow \overline{\Delta}_{S/A}\{i\},$$

cofibre is naturally identified with $\tau^{\geq i+1}\overline{\Delta}_{S/A}\{i\}$ by Theorem 11.1.14 (2). For all integers $n \geq i+1$, the cohomology groups $\mathrm{H}^n(\tau^{\geq i+1}\overline{\Delta}_{S/A}\{i\})$ are canonically isomorphic to $(\Omega^n_{S/(A/I)})^\wedge_p$ by the Hodge–Tate comparison (Proposition 11.1.14 (3)). By $p$-cotangent smoothness of the $A/I$-algebra $S$, the $S$-modules $(\Omega^n_{S/(A/I)})^\wedge_p$ are $p$-completely flat for all $n$, hence the complex $\tau^{\geq i+1}\overline{\Delta}_{S/A}$ has $p$-complete Tor-amplitude in degrees at least $i + 1$. The Nygaard filtration on $\Delta^{(1)}_{S/A}\{i\}$ is defined as the tensor product of the Nygaard filtration on $\Delta^{(1)}_{S/A}$ with the invertible $A$-module $A\{i\}$ ([BL22, Construction 5.6.1]), and is thus complete by Theorem 11.1.16 ($\Delta = \widehat{\Delta}$).

Assume now that $S$ is $F$-smooth. In particular for every integer $i \geq 0$ (we only use $i \in \{0, 1\}$ in this argument), the Frobenius map

$$\phi : \mathcal{N}^i\Delta^{(1)}_{S/A} \longrightarrow \overline{\Delta}_{S/A}\{i\}$$

has cofibre in degrees at least $i + 1$, and $\mathcal{N}^i\Delta^{(1)}_{S/A}$ is in degrees at most $i$ (Theorem 11.1.12 (1)), so there is an equivalence

$$\tilde{\phi} : \mathcal{N}^i\Delta^{(1)}_{S/A} \longrightarrow \tau^{\leq i}\overline{\Delta}_{S/A}\{i\}.$$

The Frobenius factors through the equivalence

$$\tilde{\phi} : \mathcal{N}^i\Delta^{(1)}_{S/A} \xrightarrow{\sim} \mathrm{Fil}^{\mathrm{conj}}_i\,\overline{\Delta}_{S/A}\{i\}$$

of Theorem 11.1.12 (1), so the canonical map

$$\mathrm{Fil}^{\mathrm{conj}}_i\,\overline{\Delta}_{S/A}\{i\} \xrightarrow{\sim} \tau^{\leq i}\overline{\Delta}_{S/A}\{i\}$$

is an equivalence. The $F$-smoothness condition for $i = 0$ and $i = 1$ implies that the total cofibre of the commutative diagram

$$\begin{array}{ccc} \mathrm{Fil}_0^{\mathrm{conj}}\overline{\Delta}_{S/A} & \longrightarrow & \overline{\Delta}_{S/A} \\ \downarrow & & \downarrow \\ \mathrm{Fil}_1^{\mathrm{conj}}\overline{\Delta}_{S/A}\{1\} & \longrightarrow & \overline{\Delta}_{S/A}\{1\}, \end{array}$$

which is naturally identified with $\mathrm{gr}_1^{\mathrm{conj}}\overline{\Delta}_{S/A}\{1\}[-1] \in \mathcal{D}(S)$, has Tor-amplitude in degrees at least zero. By the Hodge-Tate comparison, this means that the cotangent complex

$$(\mathbb{L}_{S/(A/I)})_p^{\wedge} \in \mathcal{D}(S)$$

has Tor-amplitude in degrees at least zero, hence is concentrated in degree zero, where it is given by a flat $S$-module. So the morphism $A/I \to S$ is $p$-cotangent smooth. By Theorem 11.1.16 ($\Delta = \widehat{\Delta}$) $\Rightarrow$ $(CSm)$, $S$ is then $p$-Cartier smooth over $A/I$. □

**Corollary 11.1.19.** *Let $(A, I)$ be a perfect prism, and $S$ be a $p$-discrete $A/I$-algebra. Assume that the ring $S$ is $p$-quasisyntomic. Then $S$ is $F$-smooth if and only if $S$ is $p$-Cartier smooth over $A/I$.*

*Proof.* Because the base prism $(A, I)$ is perfect, the Nygaard filtration on $\Delta_{S/A}^{(1)}$ is naturally identified with the Nygaard filtration on $\Delta_S$ ([BL22, Theorem 5.6.2]), and this identification is compatible with Frobenius maps. The result is then a consequence of Theorem 11.1.18. □

**Remark 11.1.20.** In general, given a $p$-discrete morphism $R \to S$ of $p$-quasisyntomic rings, the notions of $F$-smoothness for $S$ and $p$-Cartier smoothness for $R \to S$ do not agree. For instance, the ring $\mathbb{Z}_p[p^{1/p}]$ is regular noetherian and is thus $F$-smooth ([BM23, Theorem 4.15]), but the morphism $\mathbb{Z}_p \to \mathbb{Z}_p[p^{1/p}]$ is not $p$-Cartier smooth (it is not $p$-cotangent smooth). On the other hand, the identity endomorphism of $R$ is always $p$-Cartier smooth, but not all $p$-quasisyntomic rings $R$ are $F$-smooth: any $F$-smooth $p$-complete noetherian ring is regular ([BM23, Theorem 4.15]). If $S$ is a $p$-Cartier smooth $\mathbb{Z}_p$-algebra, then $S$ is $F$-smooth ([BM23, Corollary 4.17]).

## 11.2 Valuation rings are Cartier smooth

Recall that a valuation ring is an integral domain $V$ such that for any elements $f$ and $g$ in $V$, either $f \in gV$ or $g \in fV$. The valuation on the fraction field $F$ of $V$ is defined as the canonical group homomorphism $v : F^\times \twoheadrightarrow F^\times/V^\times =: \Gamma_F$, where $\Gamma_F$ is naturally equipped with a structure of totally ordered abelian group, and is called the value group of $F$. The value group $\Gamma_F$ (resp. the valuation ring $V$) is said to be discrete if it is isomorphic to the ring of integers $\mathbb{Z}$. The rank of a valuation ring is defined as its number of nonzero prime ideals. In particular, a valuation ring $V$ has rank at most one if and only if its value group $\Gamma_F$ can be embedded in the ordered group of real numbers $\mathbb{R}$. A valuation ring extension $V'$ of $V$ is a valuation ring $V'$ equipped with a flat ring morphism $V \to V'$. The flat modules over a valuation ring $V$ are exactly the torsionfree $V$-modules, so a morphism $V \to V'$ of valuation rings is flat if and only if it is injective.

Valuation rings arise in various contexts in $p$-adic geometry, *e.g.*, in [BS17, Hub96, Sch12, BM21]. The goal of this section is to prove the following result.

**Theorem 11.2.1.** *Let $V$ be a valuation ring whose $p$-completion is a perfectoid ring. Let $V'$ be a valuation ring extension of $V$. Then the morphism $V \to V'$ is $p$-Cartier smooth.*[4]

The main consequence of Theorem 11.2.1 that we will use is the following result.

**Corollary 11.2.2.** *Let $V$ be a $p$-torsionfree valuation ring whose $p$-completion is a perfectoid ring, and $V'$ be a valuation ring extension of $V$. Then for any integers $i \geq 0$ and $k \geq 1$, the map*

$$\mathbb{Z}/p^k(i)^{\mathrm{syn}}(V') \longrightarrow R\Gamma_{\text{ét}}(\mathrm{Spec}(V'[\tfrac{1}{p}]), \mu_{p^k}^{\otimes i})$$

*is an isomorphism on cohomology in degrees $< i$, and is injective on $\mathrm{H}^i$.*

*Proof.* This is a consequence of Theorems 11.2.1 and 11.3.12. □

Examples of valuation rings $V$ satisfying the hypothesis of Theorem 11.2.1 include the ring of integers $\overline{\mathbb{Z}}_p$ of an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, the ring of integers $\mathcal{O}_{\mathbb{C}_p}$ of the $p$-adic complex numbers $\mathbb{C}_p$ and the ring $\mathbb{Z}_p[p^{1/p^\infty}]$ (Example 11.2.6 below).

**Remark 11.2.3.** By Example 11.1.2 (2), an extension of valuation rings is $p$-discrete. In particular, an extension of valuation rings $V \to V'$ is $p$-Cartier smooth if and only if the morphism $V/p \to V'/p$ is Cartier smooth, *i.e.*, if $V/p \to V'/p$ satisfies the Cartier isomorphism and the cotangent complex $\mathbb{L}_{(V'/p)/(V/p)}$ is a flat $(V'/p)$-module in degree zero.

We will distinguish three cases. If $p$ is invertible in the valuation ring $V$, then the rings $V/p$ and $V'/p$ are zero[5] and the result is trivial. If $p$ is zero in the valuation ring $V$, the result is essentially due to Gabber–Ramero and Gabber, as we recall now. We will then focus on the remaining case, *i.e.*, that of mixed characteristic.

Assume that $p = 0$ in the valuation ring $V$. By [BMS18, Example 3.15] an $\mathbb{F}_p$-algebra, or equivalently its $p$-completion, is a perfectoid ring if and only if it is perfect (*i.e.*, its Frobenius endomorphism is an isomorphism). Theorem 11.2.1 is then due to Gabber–Ramero and Gabber. Remark that Cartier smoothness in characteristic $p$, including its relation to valuation rings and algebraic $K$-theory, was previously studied in [KM21].

**Theorem 11.2.4** (Cartier smoothness of characteristic $p$ valuation rings, [GR03, KST21])**.** *Let $V$ be a perfect valuation ring of characteristic $p$, and $V'$ be a valuation ring extension of $V$. Then the morphism $V \to V'$ is Cartier smooth. Equivalently:*

*(1) The cotangent complex $\mathbb{L}_{V'/V}$ is concentrated in degree zero, and $\Omega^1_{V'/V}$ is a flat $V'$-module.*

*(2) The inverse Cartier map*

$$C^{-1} : \Omega^n_{V'/V} \otimes_{V,\phi_V} V \longrightarrow \mathrm{H}^n(\Omega^\bullet_{V'/V})$$

*is an isomorphism of $V$-modules for each $n \geq 0$.*

*Proof.* (1) The cotangent complexes $\mathbb{L}_{V'/V}$ and $\mathbb{L}_{V'/\mathbb{F}_p}$ are concentrated in degree zero ([GR03, Theorem 6.5.8 (*ii*)]) where they are given by $\Omega^1_{V'/V}$ and $\Omega^1_{V'/\mathbb{F}_p}$, and $\Omega^1_{V'/\mathbb{F}_p}$ is a torsionfree $V'$-module ([GR03, Corollary 6.5.21]). There is a homotopy transitivity fibre sequence

$$\mathbb{L}_{V/\mathbb{F}_p} \otimes^{\mathbb{L}}_V V' \longrightarrow \mathbb{L}_{V'/\mathbb{F}_p} \longrightarrow \mathbb{L}_{V'/V},$$

[4] By Theorem 11.1.19, note that this is equivalent to the fact that the valuation ring $V'$ is $F$-smooth.

[5] There is a slight ambiguity whether the zero ring is perfectoid or not. We include this case to avoid any ambiguity.

and $\mathbb{L}_{V/\mathbb{F}_p} \simeq 0$ because $V$ is a perfect $\mathbb{F}_p$-algebra. So the natural morphism

$$\Omega^1_{V'/\mathbb{F}_p} \longrightarrow \Omega^1_{V'/V}$$

is an isomorphism of $V'$-modules. Torsionfree modules over a valuation ring are flat, so $\Omega^1_{V'/V}$ is a flat $V'$-module.

(2) The morphism $\mathbb{F}_p \to V'$ satisfies the Cartier isomorphism ([KST21, Corollary A.4]). By the previous paragraph, there is a canonical isomorphism

$$\Omega^1_{V'/\mathbb{F}_p} \xrightarrow{\cong} \Omega^1_{V'/V}$$

of $V'$-modules, so the morphism $V \to V'$ also satisfies the Cartier isomorphism. □

Assume now, for the rest of this section, that $p$ is neither invertible nor zero in the valuation ring $V$; we say in this case that $V$ is a *mixed characteristic* valuation ring. The hypothesis on $V$ in Theorem 11.2.1 can be reformulated as follows.

**Lemma 11.2.5.** *Let $V$ be a mixed characteristic valuation ring. The following are equivalent:*

1. *(1) The $p$-completion of the valuation ring $V$ is a perfectoid ring.*
2. *(2) The ring $V/p$ has a nonzero nilpotent element, and its Frobenius endomorphism is surjective.*

*Proof.* The second condition depends only on the ring $V/p$, so we can assume the valuation ring $V$ is $p$-complete. Because $p$ is nonzero in the valuation ring $V$, the ring $V$ is in particular $p$-torsionfree.

By [BMS18, Lemmas 3.9 and 3.10], a $p$-torsionfree ring $R$ is perfectoid if and only if $R$ is $p$-complete, the Frobenius on $R/p$ is surjective and $up \in R$ admits a compatible system of $p$-power roots for some unit $u \in R^\times$. From now on, let $V$ be a nonzero $p$-complete and $p$-torsionfree valuation ring, such that the Frobenius on $V/p$ is surjective.

(1) ⇒ (2) Let $u \in V^\times$ be a unit such that $up$ admits a compatible system $((up)^{1/p^n})_{n\in\mathbb{N}}$ of $p$-power roots in the valuation ring $V$. Because $V$ is $p$-torsionfree and not the zero ring, $p$ is nonzero in $V$ and neither is $(up)^{1/p}$. The element $(up)^{1/p} \in V$ thus defines a nonzero nilpotent element in the ring $V/p$.

(2) ⇒ (1) Assume the valuation ring $V$ has an element $\pi \in V$ defining a nonzero nilpotent element in $V/p$. Because the Frobenius on $V/p$ is surjective, we can assume that $\pi^p$ is nonzero in $V/p$. Because $V$ is a valuation ring, this implies that $\pi^p$ divides $p$ and that $p$ divides some power of $\pi$ in $V$. In particular the valuation ring $V$ is $\pi$-complete for this element $\pi \in V$. By [BMS18, Lemma 3.9], there exists a unit $u \in V^\times$ such that $up \in V$ admits a compatible system of $p$-power roots in $V$. □

**Example 11.2.6.** The $p$-completion of the ring $\mathbb{Z}_p[p^{1/p^\infty}]$ is a perfectoid valuation ring, because it satisfies the hypotheses of Lemma 11.2.5 (2). In particular, any valuation ring extension $V'$ of $\mathbb{Z}_p$ containing a compatible system of $p$-power roots of $p$ is $p$-Cartier smooth over $\mathbb{Z}_p[p^{1/p^\infty}]$ by Theorem 11.2.1. Beware that $p$-completion (or even taking the $p$-adically separated quotient) does not preserve the value group of a valuation ring. For instance, the localisation of the ring $\mathbb{Z}_p[X/p^n, n \geq 0]$ at the ideal $(X/p^n, n \geq 0) \subset \mathbb{Z}_p[X/p^n, n \geq 0]$ is a valuation ring, with fraction field $\mathbb{Q}_p(X)$ and with $p$-completion $\mathbb{Z}_p$; an alternative description of this valuation ring is the fiber product $\mathbb{Z}_p \times_{\mathbb{Q}_p} \mathbb{Q}_p[X]_{(X)}$. Similarly, the localisation of the ring $\mathbb{Z}_p[p^{1/p^\infty}, X/p^n, n \geq 0]$ at the ideal $(X/p^n, n \geq 0) \subset \mathbb{Z}_p[p^{1/p^\infty}, X/p^n, n \geq 0]$ is a valuation ring, with fraction field $\mathbb{Q}_p(p^{1/p^\infty}, X)$ and $p$-adically separated quotient $\mathbb{Z}_p[p^{1/p^\infty}]$.

To prove Theorem 11.2.1 for the morphism $V \to V'$, it suffices to prove that the morphism $V/p \to V'/p$ is cotangent smooth (*i.e.*, its cotangent complex $\mathbb{L}_{(V'/p)/(V/p)}$ is given by a flat $V'/p$-module in degree zero) and satisfies the Cartier isomorphism (Remark 11.2.3). We first prove the cotangent smoothness.

**Lemma 11.2.7.** *Let $V'$ be a valuation ring in which $p$ is nonzero, and $M$ be a $V'$-module. If the $V'$-module $M$ is $p$-torsionfree, then its derived reduction $M \otimes^{\mathbb{L}}_{V'} V'/p$ modulo $p$ is a flat $V'/p$-module in degree zero.*

*Proof.* Because $p$ is a nonzerodivisor in the valuation ring $V'$, the derived reduction $M \otimes^{\mathbb{L}}_{V'} V'/p$ modulo $p$ of any $V'$-module $M$ is concentrated in degree zero if and only if $M$ is $p$-torsionfree.

Assuming $M$ is a $p$-torsionfree $V'$-module, we prove now that $M/p$ is a flat $V'/p$-module. Consider the $p$-adically separated quotient $\widetilde{V'}$ of $V'$, *i.e.*, the quotient of $V'$ by its ideal $\bigcap_{n\geq 0} p^n V'$. The ring $\widetilde{V'}$ is a valuation ring, because it is an integral domain and satisfies the divisibility condition of valuation rings. Define $M'$ as the $\widetilde{V'}$-module $M \otimes_{V'} \widetilde{V'}$. The morphism $M \to M'$ of $V'$-modules is surjective, with kernel given by a quotient of the $V'$-module $M \otimes_{V'} \bigcap_{n\geq 0} p^n V'$. The $V'$-module $M \otimes_{V'} \bigcap_{n\geq 0} p^n V'$ is $p$-divisible, so the morphism $M \to M'$ becomes an isomorphism after reduction modulo $p$. Because $\widetilde{V'}$ is a $p$-adically separated valuation ring, any $p$-torsionfree $\widetilde{V'}$-module is torsionfree and thus flat. In particular the $\widetilde{V'}$-module $M'$ is flat, and the $\widetilde{V'}/p$-module $M'/p$ is also flat. So the $V'/p$-module $M/p$ is flat. □

**Proposition 11.2.8** (Cotangent smoothness of the morphism $V/p \to V'/p$)**.** *Let $V$ be a mixed characteristic valuation ring whose $p$-completion is a perfectoid ring. Let $V'$ be a valuation ring extension of $V$. Then the $V'$-module $\Omega^1_{V'/V}$ is $p$-torsionfree and the cotangent complex $\mathbb{L}_{(V'/p)/(V/p)}$ is a flat $V'/p$-module in degree zero.*

*Proof.* Note the equivalence $\mathbb{L}_{(V'/p)/(V/p)} \simeq \mathbb{L}_{V'/V} \otimes^{\mathbb{L}}_{V'} V'/p$. The cotangent complex of any extension of valuation rings with characteristic zero fraction fields is concentrated in degree zero ([GR03, Theorem 6.5.8 (*ii*)][6]). In particular the cotangent complex $\mathbb{L}_{V'/V}$ is concentrated in degree zero, given by the $V'$-module $\Omega^1_{V'/V}$. If the $V'$-module $\Omega^1_{V'/V}$ is $p$-torsionfree, then the cotangent complex $\mathbb{L}_{(V'/p)/(V/p)}$ is a flat $V'/p$-module in degree zero (Lemma 11.2.7).

We prove now that the $V'$-module $\Omega^1_{V'/V}$ is $p$-torsionfree. Let $F'$ be the fraction field of the valuation ring $V'$, and $\overline{F'}$ an algebraic closure of $F'$. We fix a valuation on $\overline{F'}$ extending the valuation of the valued field $F'$, and denote by $\overline{V'}$ the corresponding ring of integers. Applying again [GR03, Theorem 6.5.8 (*ii*)] to $V$, $V'$ and $\overline{V'}$ and because the morphism $V' \to \overline{V'}$ is flat, the transivity sequence for the cotangent complex can be rewritten as a short exact sequence of $\overline{V'}$-modules:

$$0 \longrightarrow \Omega^1_{V'/V} \otimes_{V'} \overline{V'} \longrightarrow \Omega^1_{\overline{V'}/V} \longrightarrow \Omega^1_{\overline{V'}/V'} \longrightarrow 0.$$

An $V'$-module (resp. $\overline{V'}$-module) $M$ is $p$-torsionfree if and only if the multiplication by $p$ morphism $p : M \to M$ is injective. The morphism $V' \to \overline{V'}$ being faithfully flat, the $V'$-module $\Omega^1_{V'/V}$ is thus $p$-torsionfree if and only if the $\overline{V'}$-module $\Omega^1_{\overline{V'}/V} \otimes_{V'} \overline{V'}$ is $p$-torsionfree. By the previous short exact sequence, it then suffices to prove the $\overline{V'}$-module $\Omega^1_{\overline{V'}/V}$ is $p$-torsionfree. The $p$-completions $V^\wedge_p$ and $\overline{V'}^\wedge_p$ of $V$ and $\overline{V'}$ are perfectoid rings, respectively

[6] More precisely, [GR03, Theorem 6.5.8 (*ii*)] is formulated for extensions of valued fields. An extension of valuation rings $V \hookrightarrow V'$ is the composition of the localisation $V \hookrightarrow V_{\mathfrak{p}}$ at the prime ideal $\mathfrak{p} := V \cap \mathfrak{m}_{V'}$ and the morphism $V_{\mathfrak{p}} \hookrightarrow V'$. The cotangent complex of the first morphism is trivial and the second morphism is induced by an extension of valued fields.

by assumption and because the field $\overline{F'}$ is algebraically closed. In particular the cotangent complex $\mathbb{L}_{(\overline{V'}^{\wedge}_p)/p/(V_p^{\wedge})/p}$ is zero ([BMS18, Lemma 3.14]). Equivalently, the cotangent complex $\mathbb{L}_{(\overline{V'}/p)/(V/p)}$ is zero. In particular, the derived reduction $\Omega^1_{\overline{V'}/V} \otimes^{\mathbb{L}}_{\overline{V'}} \overline{V'}/p$ modulo $p$ of the $\overline{V'}$-module $\Omega^1_{\overline{V'}/V}$ is in degree zero, or equivalently the $\overline{V'}$-module $\Omega^1_{\overline{V'}/V}$ is $p$-torsionfree. □

It remains to prove the Cartier isomorphism for the reduction modulo $p$ of valuation ring extensions $V \to V'$, where $V$ is a mixed characteristic valuation ring whose $p$-completion is a perfectoid ring. The proof of the Cartier isomorphism for positive charateristic valuation rings [KST21, Corollary A.4] relies on subtle approximation results of Gabber, which do not immediately pass to mixed characteristic. Our strategy of proof in mixed characteristic is to reduce to this result in positive characteristic. We are immediately faced with the following issue: if $V$ is a mixed characteristic valuation ring, the ring $V/p$ is in general not an integral domain, and in particular not a valuation ring.

Here we use the perfectoid assumption on the base $V$ to remark that there is a perfect valuation ring $V^\flat$ of characteristic $p$ (the tilt of $V$) whose reduction modulo some element $d \in V^\flat$ is naturally isomorphic to $V/p$. The Cartier isomorphism is preserved by base change (Lemma 11.1.9). So it would suffice to find a valuation ring extension $V'^\flat$ of $V^\flat$ whose reduction modulo $d$ is $V'/p$ to prove the Cartier isomorphism for $V/p \to V'/p$. To construct such a valuation ring $V'^\flat$ over the $d$-complete lift $V^\flat$ of $V/p$, we use the deformation theory of Illusie [Ill71, III.2.1.2.3] (see also [Sch12, Theorem 5.11]). Namely, we will need the following result, where $R$ is a ring, $I \subset R$ is an ideal such that $I^2 = 0$, and $S_0$ is a flat $R_0 := R/I$-algebra.

**Theorem 11.2.9** (Deformation theory, [Ill71])**.** *There is an obstruction class $\omega(S_0)$ in the abelian group $\mathrm{Ext}^2_{S_0}(\mathbb{L}_{S_0/R_0}, S_0 \otimes_{R_0} I)$ which vanishes precisely when there exists a flat $R$-algebra $S$ such that $S \otimes_R R_0 = S_0$. If there exists such a deformation, then the set of all isomorphism classes of such deformations forms a torsor under $\mathrm{Ext}^1_{S_0}(\mathbb{L}_{S_0/R_0}, S_0 \otimes_{R_0} I)$, and every deformation has automorphism group $\mathrm{Hom}_{S_0}(\mathbb{L}_{S_0/R_0}, S_0 \otimes_{R_0} I)$.*

To apply this deformation result recursively up to a $d$-adic deformation $V'^\flat$ of the flat $V/p$-algebra $V'/p$, we need to have control on the cotangent complex $\mathbb{L}_{(V'/p)/(V/p)}$. More precisely, if the cotangent complex $\mathbb{L}_{(V'/p)/(V/p)}$ was a projective $V'/p$-module in degree zero then the higher Ext-groups in this deformation result would vanish and we could construct the lift $V'^\flat$ in a unique way. By Proposition 11.2.8, the cotangent complex $\mathbb{L}_{(V'/p)/(V/p)}$ is concentrated in degree zero, given by the flat $V'/p$-module $\Omega^1_{(V'/p)/(V/p)}$. The inverse Cartier map commutes with filtered colimits, so we can assume the field extension $F \to F'$ induced by the valuation ring extension $V \to V'$ is of finite type to prove the Cartier isomorphism. But even for such valuation ring extensions, the $V'/p$-module $\Omega^1_{(V'/p)/(V/p)}$ will not be projective in general. For instance, when $F' \cong F(X)$, it follows from the proof of [GR03, Proposition 6.5.6] that the $V'$-module $\Omega^1_{V'/V}$ can be isomorphic to $\mathfrak{m}_V V'$, where $\mathfrak{m}_V$ is the maximal ideal of $V$. The next result will ensure the vanishing of the relevant $\mathrm{Ext}^2$-groups in the deformation theory, and thus the existence of a lift $V'^\flat$ (see also Remark 11.2.15 about the uniqueness of such a lift).

**Lemma 11.2.10.** *Let $V$ be a finite rank valuation ring, with fraction field $F$. If $M$ is a torsion-free $V$-module such that $M \otimes_V F$ is a finite dimensional $F$-vector space, then the $V/p$-module $M/p$ is a countable filtered colimit of free $V/p$-modules of rank at most $\dim_F(M \otimes_V F)$. In particular, the $V'/p$-module $M/p$ has projective dimension at most one.*

*Proof.* The morphism of $V$-modules $M \to M \otimes_V F$ is injective because the $V$-module $M$ is torsionfree. We identify the $V$-module $M$ with a submodule of the $V$-module $M \otimes_V F$ via this morphism. We prove by induction on the dimension $d$ of the $F$-vector space $M \otimes_V F$ that $M$ is a countable filtered union of free $V$-modules.

For $d = 0$, the $V$-module $M$ is free of rank zero. For $d = 1$, the $V$-module $M$ is isomorphic to a submodule of $F$, and we consider two cases. If $M$ is equal to $F$, then $M$ is the filtered union of the free $V$-submodules of rank one of $F$. If $M$ is not equal to $F$, then up to shifting $M$ multiplicatively by an element of $V$, we can assume that $M$ is contained in $V$, *i.e.*, that $M$ is an ideal of $V$. Every ideal of a ring is the filtered union of its finitely generated subideals, and every such ideal is principal in a valuation ring. So the $V$-module $M$ can be written as a filtered union of free $V$-submodules of rank at most one. In both cases, and because the valuation ring $V$ has finite rank, one can assume that the filtered colimit is countable. So the $V$-module $M$ is a countable filtered union of free $V$-modules of rank at most one.

Fix an integer $d \geq 1$, assume the result is proved for all integers less than or equal to $d$, and let $M$ be a $V$-submodule of the $(d+1)$-dimensional $F$-vector space $\oplus_{i=1}^{d+1} F e_i$. Let $M_{e_{d+1}}$ be the image of $M$ under the projection $p_{d+1} : \oplus_{i=1}^{d+1} F e_i \to F e_{d+1}$. By the previous paragraph, the $V$-module $M_{e_{d+1}}$ is the countable filtered union of a system $(M_{e_{d+1}}^{(n)})_{n \in \mathbb{N}}$ of free $V$-modules of rank at most one. Let $M^{(n)} := p_{d+1}^{-1}(M_{e_{d+1}}^{(n)})$, so that $M$ is the filtered union of the $V$-modules $M^{(n)}$. For each integer $n \geq 0$, there is a short exact sequence of $V'$-modules

$$0 \longrightarrow M^{(n)} \cap \oplus_{i=1}^{d} F e_i \longrightarrow M^{(n)} \longrightarrow M_{e_{d+1}}^{(n)} \longrightarrow 0,$$

which is split since the $V$-module $M_{e_{d+1}}^{(n)}$ is free. The induction hypothesis implies that, for each integer $n \geq 0$, the $V$-module $M^{(n)} \cap \oplus_{i=1}^{d} F e_i$ is a countable filtered union of free $V$-modules. Taking the direct sum with the $V$-module $M_{e_{d+1}}^{(n)}$, this implies that $M^{(n)}$ is the countable filtered union of a system $(M^{(n,m)})_{m \in \mathbb{N}}$ of free $V'$-modules. The $V$-module $M$ is thus the countable union of the free $V$-modules $M^{(n,m)}$, $n, m \in \mathbb{N}$, and this union is filtered by construction. This concludes the induction.

In particular, the $V/p$-module $M/p$ is a countable filtered colimit $(P^{(n)})_{n \in \mathbb{N}}$ of free $V/p$-modules of rank at most $\dim_F(M \otimes_V F)$. To prove the last claim, consider the Milnor exact sequence

$$0 \longrightarrow \oplus_{n \in \mathbb{N}} P^{(n)} \xrightarrow{\partial} \oplus_{n \in \mathbb{N}} P^{(n)} \longrightarrow M/p \longrightarrow 0$$

of $V$-modules, where $f_n : P^{(n)} \to P^{(n+1)}$ is the transition map and

$$\partial : (x_n)_{n \geq 0} \longmapsto (x_n - f_{n-1}(x_{n-1})),$$

where $f_{-1}(x_{-1}) := 0$. For every $V/p$-module $Q$, the Ext-groups $\mathrm{Ext}^i_{V/p}(P^{(n)}, Q)$ vanish for all integers $n \geq 0$ and $i \geq 1$. Applying the long exact sequence of Ext-groups to the previous short exact sequence of $V/p$-modules implies that

$$\mathrm{Ext}^i_{V/p}(M/p, Q) \cong 0$$

for every integer $i \geq 2$ and every $V/p$-module $Q$, *i.e.*, that the $V/p$-module $M/p$ has projective dimension at most one. □

**Corollary 11.2.11.** *Let $V$ be a mixed characteristic valuation ring whose $p$-completion is a perfectoid ring. Let $V'$ be a finite rank valuation ring extension of $V$. Assume that $V$ and $V'$ are $p$-adically separated, and that the induced field extension $F \to F'$ is of finite type. Then the $V'/p$-module $\Omega^1_{(V'/p)/(V/p)}$ has projective dimension at most one.*

*Proof.* We apply Lemma 11.2.10 to the $V'$-module $\Omega^1_{V'/V}$. The $V'$-module $\Omega^1_{V'/V}$ is $p$-torsionfree (Proposition 11.2.8), and thus torsionfree since $V'$ is $p$-adically separated. The localisation at a prime ideal has trivial cotangent complex, so the canonical map $\Omega^1_{V'/V} \otimes_{V'} F' \to \Omega^1_{F'/F}$ is an isomorphism. Moreover, the $F'$-vector space $\Omega^1_{F'/F}$ is finite dimensional since $F \to F'$ is a field extension of finite type. The reduction modulo $p$ of the $V'$-module $\Omega^1_{V'/V}$ is the $V'/p$-module $\Omega^1_{(V'/p)/(V/p)}$, hence the conclusion. □

We now use this result on the projective dimension of $\Omega^1_{(V'/p)/(V/p)}$ to lift the $V/p$-algebra $V'/p$ to a valuation ring extension $V'^\flat$ of $V^\flat$.

**Notation 11.2.12.** Let $V$ be a mixed characteristic valuation ring whose $p$-completion is a perfectoid ring. Denote by

$$V^\flat := \lim_{x \mapsto x^p} V/p$$

the tilt of this perfectoid ring. The ring $V^\flat$ is a perfect valuation ring of characteristic $p$, which is $d$-complete for some element $d \in V^\flat$ such that there is a natural ring isomorphism

$$V/p \cong V^\flat/d.$$

We will need the following to ensure that the flat lift $V'^\flat$ produced by deformation theory is actually a valuation ring.

**Lemma 11.2.13.** *Let $V$ be a mixed characteristic valuation ring whose $p$-completion is a perfectoid ring. Let $V'$ be a valuation ring extension of $V$ and $V'^\flat$ a $d$-complete flat $V^\flat$-algebra. Assume there is an isomorphism of $V^\flat/d = V/p$-algebras $V'^\flat/d \cong V'/p$. Then $V'^\flat$ is a valuation ring extension of $V^\flat$.*

*Proof.* The commutative ring $V'^\flat$ is a flat $V^\flat$-algebra, so it suffices to prove that it is an integral domain such that for any elements $f$ and $g$ in $V'^\flat$, either $f$ divides $g$ or $g$ divides $f$. We first prove the latter property.

The valuation ring $V^\flat$ is perfect by definition. Denote by $(d^{1/p^n})_{n\in\mathbb{N}} \in (V^\flat)^{\mathbb{N}}$ a compatible system of $p$-power roots of $d \in V^\flat$. A module over the valuation ring $V^\flat$ is flat if and only if it is torsionfree, so the $V^\flat$-module $V'^\flat$ is $d^{1/p^n}$-torsionfree for each $n \geq 0$. In the following we identify the rings $V/p$ and $V^\flat/d$; in particular any (rational) power $d^\alpha$ of $d \in V^\flat$ defines an element of $V/p$, which we also denote by $d^\alpha$.

Let $f$ and $g$ be elements of the ring $V'^\flat$. Let us prove that either $f$ divides $g$ or $g$ divides $f$. First assume in this paragraph that $g = d$. We assume that $d$ does not divide $f$ in $V'^\flat$, and prove that $f$ divides $d$. Because $V'$ is a valuation ring, and for every $\alpha \in \mathbb{Z}[\frac{1}{p}]$, $\alpha < 1$, either $f$ divides $d^\alpha$, or $d^\alpha$ divides $f$ in the ring $V'^\flat/d$. If $f$ divides $d^\alpha$ for such an $\alpha$ in $V'^\flat/d$, then we can write $fh = d^\alpha + dt$ in $V'^\flat$ for some elements $h, t \in V'^\flat$. Because $V'^\flat$ is $d$-complete it is also $d^{1-\alpha}$-complete. In particular $1 + d^{1-\alpha}t$ is a unit in $V'^\flat$, so $f$ divides $d^\alpha$, and thus $f$ divides $d$. If $f$ does not divide $d^\alpha$ for any $\alpha \in \mathbb{Z}[\frac{1}{p}]$, $\alpha < 1$, then in particular $d^{1/p}$ divides $f$ in $V'^\flat/d$. Because $d^{1/p}$ divides $d$ in $V'^\flat$, it then also divides $f$ in $V'^\flat$. Because $V'^\flat$ is $d^{1/p}$-torsionfree, we can consider the element $f/d^{1/p} \in V'^\flat$. By construction, any lifts $\tilde{f}, \widetilde{d}^{1/p} \in V'$ of $f, d^{1/p} \in V'^\flat/d$ satisfy that $\tilde{f}$ divides $p$, $\tilde{f}$ is divisible by $p^{1/p}$, and the quotient $\tilde{f}/\widetilde{d}^{1/p}$ is sent to $f/d^{1/p}$ by the map $V' \to V'/p$. This implies that $f/d^{1/p}$ divides $d^{1-1/p}$ in $V'^\flat/d$. The same argument as in the previous case implies that $f$ divides $d$ in $V'^\flat$.

Going back to the elements $f$ and $g$ of $V'^\flat$, we deduce the following: if $f$ is not divisible by $d$ and $g$ is divisible by $d$, then $f$ divides $g$. The ring $V'^\flat$ is $d$-torsionfree, so we can also assume that $d$ does not divide one of $f$ and $g$. We now assume that $d$ does not divide $f$ or $g$. Without loss of generality and because $V'$ is a valuation ring, there is an element $h \in V'/p$ such that $f = gh$ in $V'^\flat/d$. We can then write $f = g\tilde{h} + dt$ for some elements $\tilde{h}, t \in V'^\flat$. By the previous paragraph, and because $d$ does not divide $g$, we know that $g$ divides $d$. So $g$ divides $f$ in $V'^\flat$.

We now prove that the ring $V'^\flat$ is an integral domain. By the previous divisibility property, and because the ring $V'^\flat$ is $d$-adically separated, any nonzero element in $V'^\flat$ divides $d^n$ for some integer $n \geq 1$. The product of two nonzero elements of $V'^\flat$ would thus imply that a certain power of $d$ vanishes in $V'^\flat$, which is impossible since the ring $V'^\flat$ is $d$-torsionfree. The ring $V'^\flat$ is then an integral domain, as desired. □

**Theorem 11.2.14.** *Let $V$ be a mixed characteristic valuation ring whose $p$-completion is a perfectoid ring, and $V'$ be a valuation ring extension of $V$. If the $V'/p$-module $\Omega^1_{(V'/p)/(V/p)}$ has projective dimension at most one, then there exists a $d$-complete valuation ring extension $V'^\flat$ of $V^\flat$ such that there is a ring isomorphism $V'^\flat/d \cong V'/p$.*

The projective dimension hypothesis is satisfied in particular if the induced field extension $F \to F'$ is of finite type and $V'$ is of finite rank (Corollary 11.2.11). It is also satisfied for the $p$-completion of such an extension, as it depends only on its reduction modulo $p$.

*Proof.* We prove by induction on $n \geq 1$ that there are, for all $1 \leq m \leq n$, flat $V^\flat/d^m$-algebras $V'^\flat_m$ with isomorhisms $V'^\flat_m/d \xrightarrow{\cong} V'/p$ and with discrete cotangent complexes $\mathbb{L}_{V'^\flat_m/(V^\flat/d^m)}$ having projective dimension at most one. By Theorem 11.2.9, the vanishing of the obstructions in this process can be expressed in terms of the cotangent complexes $\mathbb{L}_{(V'^\flat_n)/(V^\flat/d^n)}$. For $n = 1$, define $V'^\flat_1$ as the flat $V^\flat/d$-algebra $V'/p$.

Now let $n \geq 1$ be an integer, and assume we are given for each $1 \leq m \leq n$ a flat $V^\flat/d^m$-algebra $V'^\flat_m$, with isomorphisms $V'^\flat_m/d^{m-1} \cong V'^\flat_{m-1}$ for all $2 \leq m \leq n$. We claim that the cotangent complex $\mathbb{L}_{V'^\flat_n/(V^\flat/d^n)}$ is in degree zero, given by a $V'^\flat_n$-module of projective dimension at most one. Tensoring the short exact sequence

$$0 \longrightarrow V'^\flat_1 \xrightarrow{d^{n-1}} V'^\flat_n \longrightarrow V'^\flat_{n-1} \longrightarrow 0$$

of $V^\flat/d^n$-modules by the cotangent complex $\mathbb{L}_{V'^\flat_n/(V^\flat/d^n)}$ induces a distinguished triangle

$$\mathbb{L}_{(V'/p)/(V/p)} \longrightarrow \mathbb{L}_{V'^\flat_n/(V^\flat/d^n)} \longrightarrow \mathbb{L}_{V'^\flat_{n-1}/(V^\flat/d^{n-1})} \longrightarrow$$

in the derived category $\mathcal{D}(V'^\flat_n)$. The discreteness and the projective dimension of $\mathbb{L}_{V'^\flat_n/(V^\flat/d^n)}$ being at most one thus reduce inductively to the case $n = 1$. For $n = 1$, $\mathbb{L}_{(V'/p)/(V/p)}$ is in degree zero by Proposition 11.2.8, and the $V'/p$-module $\Omega^1_{(V'/p)/(V/p)}$ has projective dimension at most one.

In particular, the $\mathrm{Ext}^2$-group in the deformation theory Theorem 11.2.9 vanishes, and so does the deformation class. This implies that there is a flat $V^\flat/d^{n+1}$-algebra $V'^\flat_{n+1}$ with an isomorphism $V'^\flat_{n+1}/d^n \xrightarrow{\cong} V'^\flat_n$. This concludes the induction.

Let $V'^\flat$ be the inverse limit of the system $(V'^\flat_n)_{n\geq 1}$. In particular, the $V^\flat$-algebra $V'^\flat$ is $d$-complete. For each integer $n \geq 2$, there is a natural exact sequence of $V^\flat$-modules:

$$0 \longrightarrow V'^\flat_{n-1} \xrightarrow{d} V'^\flat_n \longrightarrow V'^\flat_1.$$

Passing to the inverse limit over $n \geq 2$ implies that the $V^\flat$-modules $V'^\flat$ has no $d$-torsion, and is thus flat. By Lemma 11.2.13, $V'^\flat$ is thus a valuation ring extension of $V^\flat$. □

*Proof of Theorem 11.2.1.* As a flat morphism of integral domains, the valuation ring extension $V \to V'$ is $p$-Cartier smooth if and only if the morphism $V/p \to V'/p$ is Cartier smooth (Remark 11.2.3). The result in characteristic $p$ is Theorem 11.2.4, and the result when $p$ is invertible in $V'$ is trivial. In mixed characteristic, the cotangent smoothness of the morphism $V/p \to V'/p$ is Proposition 11.2.8. The inverse Cartier map depends only on the morphism $V/p \to V'/p$, so we can replace $V$ and $V'$ by their $p$-adically separated quotients to prove the Cartier isomorphism.

We now reduce to the case where $V$ is a rank one valuation ring. The ideal $\mathfrak{p} := \sqrt{(p)}$ of the valuation ring $V$ is a prime ideal and, as such, induces a natural short exact sequence

$$0 \longrightarrow V \longrightarrow V_{\mathfrak{p}} \oplus V/\mathfrak{p} \longrightarrow V_{\mathfrak{p}}/\mathfrak{p}V_{\mathfrak{p}} \longrightarrow 0$$

of $V$-modules ([HKK18, Lemma 3.12]). Note that the commutative ring $V_{\mathfrak{p}}$ is a mixed characteristic rank one valuation ring whose $p$-completion is a perfectoid ring. For every morphism $A \to B$ of commutative rings, let

$$F(B/A) := \mathrm{fib}\big((\mathbb{L}\Omega_{B/A})^{\wedge}_p \longrightarrow (\Omega_{B/A})^{\wedge}_p\big).$$

By Theorem 11.1.16 ($\mathbb{L}\Omega = \widehat{\mathbb{L}\Omega}$), a $p$-cotangent smooth morphism of commutative rings $A \to B$ (*e.g.*, $V \to V'$) is $p$-Cartier smooth if and only if $F(B/A) \in \mathcal{D}(A)$ vanishes. Tensoring and $p$-completing the previous short exact sequence with $F(V'/V)$ in the derived category $\mathcal{D}(V)$ induces, by base change for de Rham cohomology, a natural fibre sequence

$$F(V'/V) \longrightarrow F(V'_{\mathfrak{p}V'}/V_{\mathfrak{p}}) \oplus F\big((V'/\mathfrak{p}V')/(V/\mathfrak{p})\big) \longrightarrow F\big((V'_{\mathfrak{p}V'}/\mathfrak{p}V_{\mathfrak{p}V'})/(V_{\mathfrak{p}}/\mathfrak{p}V_{\mathfrak{p}})\big)$$

in the derived category $\mathcal{D}(V)$. Here we use that the morphism $V \to V'$ is flat, that $\mathfrak{p}V'$ is a prime ideal of the valuation ring $V'$, and that there is a natural isomorphism of commutative rings $V'_{\mathfrak{p}V'} \cong V' \otimes_V V_{\mathfrak{p}}$. The morphisms $V/\mathfrak{p} \to V'/\mathfrak{p}V'$ and $V_{\mathfrak{p}}/\mathfrak{p}V_{\mathfrak{p}} \to V'_{\mathfrak{p}V'}/\mathfrak{p}V'_{\mathfrak{p}V'}$ are extensions of characteristic $p$ valuation rings, so $F\big((V'/\mathfrak{p}V')/(V/\mathfrak{p})\big)$ and $F\big((V'_{\mathfrak{p}V'}/\mathfrak{p}V_{\mathfrak{p}V'})/(V_{\mathfrak{p}}/\mathfrak{p}V_{\mathfrak{p}})\big)$ are zero in the derived category $\mathcal{D}(V)$ (Theorem 11.2.4). The morphism $V \to V'$ is then $p$-Cartier smooth if and only if the morphism $V_{\mathfrak{p}} \to V'_{\mathfrak{p}V'}$ is $p$-Cartier smooth, *i.e.*, we can assume that the valuation ring $V$ has rank one.

The fraction field $F'$ of the valuation ring $V'$ is the filtered union of the finite type field extensions $F'_i$ of $F$ contained in $F'$. The valuation ring $V'$ is the filtered union of the associated valuation rings $V'_i$ and the inverse Cartier map commutes with filtered colimits, so we can assume that the field extension $F \to F'$ is of finite type. We assume now that $V$ is a mixed characteristic rank one valuation ring (in particular $p$-adically separated) whose $p$-completion is a perfectoid ring, that $V'$ is a $p$-adically separated valuation ring extension of $V$ such that the field extension $F \to F'$ is of finite type, and prove the Cartier isomorphism for the morphism $V/p \to V'/p$. In this case the valuation ring $V'$ has finite rank ([GR03, 6.1.24]), so there exists a valuation ring extension $V^\flat \to V'^\flat$ such that $V'^\flat/d$ is isomorphic to $V'/p$ (Theorem 11.2.14). The Cartier isomorphism for $V^\flat \to V'^\flat$ (Theorem 11.2.4) then implies the Cartier isomorphism for $V/p \to V'/p$ by base change along the morphism $V^\flat \to V^\flat/d$ (Lemma 11.1.9). □

**Remark 11.2.15.** Let $V$ be a mixed characteristic rank one perfectoid valuation ring, and $V'$ a rank one valuation ring extension of $V$, such that the induced field extension $F \to F'$ is of finite type. In this remark, we sketch the proof of the fact that the $d$-complete valuation ring $V'^\flat$, introduced in Theorem 11.2.1, is unique up to isomorphism. To do so, we consider almost mathematics ([GR03, AGT16]) with respect to the pair $(V^\flat, \sqrt{(d)})$ (Notation 11.2.12). The proof of Lemma 11.2.10, where we use [GR03, Lemma 2.4.15], can be adapted to prove that the $V'/p$-module $\Omega^1_{(V'/p)/(V/p)}$ is almost free, and in particular almost projective. The almost deformation theory [GR03, Corollary 3.2.11] then implies that the $d$-complete flat lift $V'^\flat$ of Theorem 11.2.14 must be unique when seen in the category of almost $V^\flat$-algebras. One can check that the functor

$$(-)_* : V\text{-Alg} \xrightarrow{(-)^a} V^a\text{-Alg} \xrightarrow{(-)_*} V\text{-Alg},$$

where the second map is the right adjoint to the localisation functor from $V$-algebras to almost $V$-algebras ([GR03, Proposition 2.2.13 (*ii*)]), is the identity on rank one valuation rings extensions of $V^\flat$. By construction, the valuation ring $V'^\flat$ is of rank one because the valuation ring $V'$ is of rank one, hence the result.

## 11.3 The syntomic-étale comparison theorem

In this section, we prove Theorems 11.3.10 and 11.3.12, comparing the syntomic cohomology of a $p$-Cartier smooth algebra over a perfectoid ring to the étale cohomology of its generic fibre. This comparison theorem was also proved in [KM21, LM23] in characteristic $p$, and in [BM23] for $p$-torsionfree $F$-smooth schemes, using different methods.

### 11.3.1 Syntomic cohomology

The syntomic complexes $\mathbb{Z}_p(i)^{\mathrm{syn}}$ were first defined in [BMS19, Section 7.4] for $p$-complete $p$-quasisyntomic rings in terms of $p$-completed topological cyclic homology. Another equivalent definition was given in [BS22] in terms of absolute prismatic cohomology, and a generalisation for general schemes was developed in [BL22, Section 8.4]. In this section, we will be interested mainly in $p$-complete[7] algebras over $\mathbb{Z}_p^{\mathrm{cyc}} := \mathbb{Z}[\zeta_{p^\infty}]^\wedge_p$, whose syntomic complexes can be defined in terms of relative prismatic cohomology (Definition 11.3.1).

The ideal $I \subset A$ of a perfect prism (Section 11.1.4 or [BS22, Definition 3.2 (2)]) is necessarily principal, generated by a nonzerodivisor $d \in A$. When defining a perfect prism $(A, (d))$ in this section, we implicitly fix a choice of generator $d \in A$. If the perfect prism $(A, (d))$ is a prism over the perfected $q$-de Rham prism $(\mathbb{Z}[q^{1/p^\infty}]^\wedge_{(p,q-1)}, ([p]_q))$, we implicitly assume that this element $d$ is $[p]_q \in A$. When the base prism $(A, I)$ is perfect, one can define a Nygaard filtration $\mathcal{N}^{\geq \star}\Delta_{S/A}$ on the prismatic complex $\Delta_{S/A}$ (without Frobenius twist), and we denote by

$$\phi : \mathcal{N}^{\geq \star}\Delta_{S/A} \xrightarrow{\sim} \mathcal{N}^{\geq \star}\Delta^{(1)}_{S/A} \xrightarrow{\phi} I^\star \Delta_{S/A}$$

the Frobenius on the relative prismatic complex (see Section 11.1.2 for the definition of the second map). Following [BS22, Section 12] and for each $i \geq 0$, a divided Frobenius map

$$\phi_i = \text{“}\frac{\phi}{d^i}\text{”} : \mathcal{N}^{\geq i}\Delta_{S/A} \longrightarrow \Delta_{S/A}$$

[7] As in Remark 3.2.2, one can consider the derived or the classical $p$-completion of a given commutative ring. The two notions coincide on commutative rings with bounded $p$-power torsion. In this section, we assume that $p$-complete commutative ring means $p$-complete commutative ring with bounded $p$-power torsion. Note that, by definition, $p$-quasisyntomic rings have bounded $p$-power torsion.

is defined and sits in the commutative diagram

$$\begin{array}{ccc} \mathcal{N}^{\geq i}\mathbb{\Delta}_{S/A} & \xrightarrow{\phi_i} & \mathbb{\Delta}_{S/A} \\ & \searrow^{\phi} & \downarrow{\scriptstyle d^i} \\ & & d^i\mathbb{\Delta}_{S/A}. \end{array}$$

**Definition 11.3.1** (Syntomic complexes)**.** Let $(A,(d))$ be a perfect prism over the prism $(\mathbb{Z}[q^{1/p^\infty}]^\wedge_{(p,q-1)}, ([p]_q))$, and $S$ be a $p$-complete $A/d$-algebra. For each integer $i \geq 0$, the syntomic complex $\mathbb{Z}_p(i)^{\mathrm{syn}}(S) \in \mathcal{D}^{\geq 0}(\mathbb{Z}_p)$ is

$$\mathbb{Z}_p(i)^{\mathrm{syn}}(S) := \mathrm{hofib}(\phi_i - 1 : \mathcal{N}^{\geq i}\mathbb{\Delta}_{S/A} \longrightarrow \mathbb{\Delta}_{S/A}).$$

For each integer $k \geq 1$, also define $\mathbb{Z}/p^k(i)^{\mathrm{syn}}(S) := \mathbb{Z}_p(i)^{\mathrm{syn}}(S) \otimes^{\mathbb{L}}_{\mathbb{Z}_p} \mathbb{Z}/p^k$.

Over a scheme in which $p$ is invertible, the object $\mathbb{Z}_p(i)$, called *$p$-adic étale Tate twist*, will denote the (pro-)étale sheaf defined as the inverse limit over $k \geq 1$ of the étale sheaves $\mu_{p^k}^{\otimes i}$. Following [BL22, Section 8.3], there is a map comparing the syntomic complexes and the $p$-adic étale Tate twists.

**Construction 11.3.2** (The syntomic-étale comparison map, [BL22])**.** Let $S$ be a $p$-complete ring (*e.g.*, a $p$-complete $\mathbb{Z}_p^{\mathrm{cyc}}$-algebra). For every integer $i \geq 0$, there is a canonical map

$$\mathbb{Z}_p(i)^{\mathrm{syn}}(S) \longrightarrow R\Gamma_{\mathrm{pro\acute{e}t}}(\mathrm{Spec}(S[\tfrac{1}{p}]), \mathbb{Z}_p(i)),$$

called the *syntomic-étale comparison map*. For every integer $k \geq 1$, one can also consider the derived reduction modulo $p^k$ of this canonical map

$$\mathbb{Z}/p^k(i)^{\mathrm{syn}}(S) \longrightarrow R\Gamma_{\mathrm{\acute{e}t}}(\mathrm{Spec}(S[\tfrac{1}{p}]), \mu_{p^k}^{\otimes i}).$$

**Remark 11.3.3.** Let $(A,(d))$ be a perfect prism, and $S$ be an $A/d$-algebra. For each $i \geq 0$, the canonical map

$$(\mathcal{N}^{\geq i}\mathbb{\Delta}_{S/A})[\tfrac{1}{d}]/p \longrightarrow \mathbb{\Delta}_{S/A}[\tfrac{1}{d}]/p$$

is an equivalence in the derived category $\mathcal{D}(A)$. By left Kan extension and $p$-quasisyntomic descent, it suffices to prove it for quasiregular semiperfectoid $A/d$-algebras $S$, for which there are natural inclusions of $(p,d)$-completely flat $A$-modules $\phi_A^{-1}(d)^i\mathbb{\Delta}_{S/A} \subseteq \mathcal{N}^{\geq i}\mathbb{\Delta}_{S/A} \subseteq \mathbb{\Delta}_{S/A}$. The result follows by using the equality $(\phi_A^{-1}(d))^p = d$ in $A/p$.

**Theorem 11.3.4** (Prismatic-étale comparison, [BS22, BL22])**.** *Let $(A,(d))$ be a perfect prism over the prism $(\mathbb{Z}[q^{1/p^\infty}]^\wedge_{(p,q-1)}, ([p]_q))$, and $S$ be a $p$-complete $A/d$-algebra. Then for any integers $i \geq 0$ and $k \geq 1$, there are canonical identifications*

$$R\Gamma_{\mathrm{\acute{e}t}}(\mathrm{Spec}(S[\tfrac{1}{p}]), \mu_{p^k}^{\otimes i}) \xrightarrow{\sim} \mathrm{hofib}(\phi_i - 1 : \mathbb{\Delta}_{S/A}[\tfrac{1}{d}]/p^k \longrightarrow \mathbb{\Delta}_{S/A}[\tfrac{1}{d}]/p^k)$$

*and*

$$R\Gamma_{\mathrm{pro\acute{e}t}}(\mathrm{Spec}(S[\tfrac{1}{p}]), \mathbb{Z}_p(i)) \xrightarrow{\sim} \mathrm{hofib}(\phi_i - 1 : \mathbb{\Delta}_{S/A}[\tfrac{1}{d}]^\wedge_p \longrightarrow \mathbb{\Delta}_{S/A}[\tfrac{1}{d}]^\wedge_p).$$

*Proof.* For $i = 0$ this is [BS22, Theorem 9.1]. When $(A,(d))$ is the perfected $q$-de Rham prism $(\mathbb{Z}[q^{1/p^\infty}]^\wedge_{(p,q-1)}, ([p]_q))$, [BL22, Theorem 8.5.1] and its proof extend [BS22, Theorem 9.1] to the general case $i \geq 0$. By independence of the perfect base prism for prismatic cohomology [BL22, Theorem 5.6.2], the same result also holds for any perfect base prism $(A,(d))$ admits a map from the perfected $q$-de Rham prism $(\mathbb{Z}[q^{1/p^\infty}]^\wedge_{(p,q-1)}, ([p]_q))$. ☐

### 11.3.2 The syntomic-étale comparison theorem in characteristic $p$

In this subsection, we review the description of the syntomic complexes in characteristic $p$ in terms of logarithmic de Rham–Witt forms (following [KM21, LM23]), and its consequence on the syntomic-étale comparison map (Construction 11.3.2).

**Proposition 11.3.5** (Syntomic-étale comparison theorem in characteristic $p$)**.** *Let $R$ be perfect $\mathbb{F}_p$-algebra, and $S$ be a Cartier smooth $R$-algebra. Then for any integers $i \geq 0$ and $k \geq 1$, there are canonical identifications*

$$\mathbb{Z}/p^k(i)^{\mathrm{syn}}(S) \simeq R\Gamma_{\text{ét}}(\mathrm{Spec}(S), W_k\Omega^i_{\log})[-i]$$

*and*

$$\mathbb{Z}_p(i)^{\mathrm{syn}}(S) \simeq R\Gamma_{\text{proét}}(\mathrm{Spec}(S), W\Omega^i_{\log})[-i].$$

*Proof.* The first equivalence follows from [LM23, Proposition 5.1 (*ii*)], the second by taking limits over $k \geq 1$. □

**Corollary 11.3.6.** *Let $R$ be a perfect $\mathbb{F}_p$-algebra, and $S$ be a Cartier smooth $R$-algebra. Then for any integers $i \geq 0$ and $k \geq 1$, the syntomic-étale comparison maps*

$$\mathbb{Z}/p^k(i)^{\mathrm{syn}}(S) \longrightarrow R\Gamma_{\text{ét}}(\mathrm{Spec}(S[\tfrac{1}{p}]), \mu_{p^k}^{\otimes i})$$

*and*

$$\mathbb{Z}_p(i)^{\mathrm{syn}}(S) \longrightarrow R\Gamma_{\text{proét}}(\mathrm{Spec}(S[\tfrac{1}{p}]), \mathbb{Z}_p(i))$$

*have homotopy cofibres in degrees at least $i-1$.*

*Proof.* The syntomic complexes $\mathbb{Z}/p^k(i)^{\mathrm{syn}}(S)$ and $\mathbb{Z}_p(i)^{\mathrm{syn}}(S)$ are in degrees at least $i$ by Proposition 11.3.5, and the generic fibre $S[\frac{1}{p}]$ of $S$ is zero. □

### 11.3.3 The syntomic-étale comparison theorem

In this subsection, we prove the syntomic-étale comparison theorem over a general perfectoid $\mathbb{Z}_p^{\mathrm{cyc}}$-algebra (Theorem 11.3.10), generalising Corollary 11.3.6.

The following result is a direct consequence of Theorem 11.1.16 ($\mathcal{N}^{\geq}$) when the base prism is perfect.

**Lemma 11.3.7.** *Let $(A, (d))$ be a perfect prism, and $S$ be a $p$-Cartier smooth $A/d$-algebra. Then for every integer $i \geq 0$, the map*

$$\tau^{\leq i}\mathcal{N}^{\geq i}\mathbb{\Delta}_{S/A} \xrightarrow{\phi_i} \tau^{\leq i}\mathbb{\Delta}_{S/A}$$

*is an equivalence.*

*Proof.* The diagram

$$\begin{array}{ccc} \mathcal{N}^{\geq i}\mathbb{\Delta}_{S/A} & \xrightarrow{\phi_i} & \mathbb{\Delta}_{S/A} \\ & \searrow^{\phi} & \downarrow{\scriptstyle d^i} \\ & & d^i\mathbb{\Delta}_{S/A} \end{array}$$

is commutative for any $A/d$-algebra $S$ and the map $d^i : \mathbb{\Delta}_{S/A} \to d^i\mathbb{\Delta}_{S/A}$ is an equivalence. Locally on the $p$-quasisyntomic site $\mathbb{\Delta}_{S/A}$ is a $d$-torsionfree $A$-module in degree zero and this is by definition of the divided Frobenius map $\phi_i$, and in general this is true by descent on the $p$-quasisyntomic site and left Kan extension. When $S$ is $p$-Cartier smooth over $A/d$, the Frobenius map $\phi : \mathcal{N}^{\geq i}\Delta_{S/A} \to d^i\Delta_{S/A}$ is an isomorphism in degrees at most $i$, hence the result. $\square$

**Lemma 11.3.8.** *Let $(A,(d))$ be a perfect prism, and $S$ be a $p$-Cartier smooth $A/d$-algebra. For all integers $i \geq 0$ and $k \leq i-1$, define the map*

$$\phi_i^{-1} : \mathrm{H}^k(\Delta_{S/A}/p) \longrightarrow \mathrm{H}^k(\Delta_{S/A}/p)$$

*as the composite*

$$\mathrm{H}^k(\Delta_{S/A}/p) \xrightarrow{\phi_i^{-1}} \mathrm{H}^k(\mathcal{N}^{\geq i}\Delta_{S/A}/p) \xrightarrow{\mathrm{can}} \mathrm{H}^k(\Delta_{S/A}/p)$$

*where the first map is given by the derived reduction modulo $p$ of Lemma 11.3.7. Then for every $k \leq i-2$, the map $\phi_i^{-1}$ is zero on the $d$-torsion subgroup $\mathrm{H}^k(\Delta_{S/A}/p)[d]$.*[8] *For $k = i-1$, the map $\phi_i^{-1} \circ \phi_i^{-1}$ is zero on the $d$-torsion subgroup $\mathrm{H}^{i-1}(\Delta_{S/A}/p)[d]$.*

*Proof.* First assume that $k \leq i-2$. In this case, the map

$$\phi_i^{-1} : \mathrm{H}^k(\Delta_{S/A}/p) \longrightarrow \mathrm{H}^k(\Delta_{S/A}/p)$$

can be rewritten as the map $d^{1/p}\phi_{i-1}^{-1}$, where $\phi_{i-1}^{-1}$ is defined as $\phi_i^{-1}$. Let $x$ be a $d$-torsion element of the $A/p$-module $\mathrm{H}^k(\Delta_{S/A}/p)$. Then

$$\phi_i^{-1}(x) = d^{1/p}\phi_{i-1}^{-1}(x) = \phi_{i-1}^{-1}(dx) = 0,$$

hence the result.

Now assume that $k = i-1$, and let us prove that $(\phi_i^{-1})^2 := \phi_i^{-1} \circ \phi_i^{-1}$ is the zero map on the $A/p$-module $\mathrm{H}^{i-1}(\Delta_{S/A}/p)[d]$. There is a natural short exact sequence of $A/p$-modules

$$0 \longrightarrow \mathrm{H}^{i-1}(\Delta_{S/A})/p \longrightarrow \mathrm{H}^{i-1}(\Delta_{S/A}/p) \longrightarrow \mathrm{H}^i(\Delta_{S/A})[p] \longrightarrow 0,$$

and compatible maps $\phi_i^{-1}$ on each of its terms. This short exact sequence induces, by the snake lemma, an exact sequence of $A/p$-modules

$$0 \longrightarrow (\mathrm{H}^{i-1}(\Delta_{S/A})/p)[d] \longrightarrow \mathrm{H}^{i-1}(\Delta_{S/A}/p)[d] \longrightarrow (\mathrm{H}^i(\Delta_{S/A})[p])[d],$$

and the maps $\phi_i^{-1}$ restrict to these $A/p$-modules. It suffices to prove that the map $\phi_i^{-1}$ is zero on both $(\mathrm{H}^{i-1}(\Delta_{S/A})/p)[d]$ and $(\mathrm{H}^i(\Delta_{S/A})[p])[d]$. Indeed, if $x$ is an element of $\mathrm{H}^{i-1}(\Delta_{S/A}/p)[d]$, then $\phi_i^{-1}(x)$ is naturally in the kernel of the canonical map

$$\mathrm{H}^{i-1}(\Delta_{S/A}/p)[d] \longrightarrow (\mathrm{H}^i(\Delta_{S/A})[p])[d].$$

So $\phi_i^{-1}(x)$ is in the subgroup $(\mathrm{H}^{i-1}(\Delta_{S/A})/p)[d] \subseteq \mathrm{H}^{i-1}(\Delta_{S/A}/p)[d]$, and thus $\phi_i^{-1}(\phi_i^{-1}(x)) = 0$.

[8] Equivalently, by Lemma 11.3.7, the canonical map

$$\mathrm{can} : \mathrm{H}^k(\mathcal{N}^{\geq i}\Delta_{S/A}/p) \longrightarrow \mathrm{H}^k(\Delta_{S/A}/p)$$

is zero on the $d^{1/p}$-torsion subgroup $\mathrm{H}^k(\mathcal{N}^{\geq i}\Delta_{S/A}/p)[d^{1/p}]$.

On the $A/p$-module $\mathrm{H}^{i-1}(\Delta_{S/A})/p$, the map $\phi_i^{-1}$ can be rewritten as the map $d^{1/p}\phi_{i-1}^{-1}$, and is thus zero on $(\mathrm{H}^{i-1}(\Delta_{S/A})/p)[d]$, arguing as in the first paragraph of this proof. We now prove that the map $\phi_i^{-1}$ is zero on the $A/p$-module $(\mathrm{H}^i(\Delta_{S/A})[p])[d]$. By definition of the map $\phi_i^{-1}$, it is equivalent to prove that the canonical map

$$\mathrm{can} : \mathrm{H}^i(\mathcal{N}^{\geq i}\Delta_{S/A})[p] \longrightarrow \mathrm{H}^i(\Delta_{S/A})[p]$$

is zero on the $d^{1/p}$-torsion subgroup $(\mathrm{H}^i(\mathcal{N}^{\geq i}\Delta_{S/A})[p])[d^{1/p}]$. By Theorem 11.1.16, there is a commutative diagram

$$\begin{array}{ccc} \mathrm{H}^i(\mathcal{N}^{\geq i}\Delta_{S/A})[p] & \xrightarrow[\cong]{\phi} & \mathrm{H}^i(d^i\Delta_{S/A})[p] \\ \downarrow{\scriptstyle \mathrm{can}} & & \downarrow{\scriptstyle \mathrm{can}} \\ \mathrm{H}^i(\Delta_{S/A})[p] & \xrightarrow[\cong]{\phi} & \mathrm{H}^i(L\eta_d\Delta_{S/A})[p], \end{array}$$

where the right vertical map is defined in [BMS18, Lemma 6.9]. By [BMS18, Lemma 6.4] and its proof, the canonical map

$$\mathrm{can} : \mathrm{H}^i(d^i\Delta_{S/A}) \longrightarrow \mathrm{H}^i(L\eta_d\Delta_{S/A})$$

is surjective, with kernel given by the $d$-torsion subgroup $\mathrm{H}^i(d^i\Delta_{S/A})[d]$. In particular, the right vertical map in the previous diagram is surjective, with kernel given by the $d$-torsion subgroup $(\mathrm{H}^i(d^i\Delta_{S/A})[p])[d]$. The left vertical map of this diagram is thus also surjective, with kernel given by the $d^{1/p}$-torsion subgroup $(\mathrm{H}^i(\mathcal{N}^{\geq i}\Delta_{S/A})[p])[d^{1/p}]$. Hence the result. □

**Remark 11.3.9.** In the previous lemma, the result can be slightly improved if the perfectoid base ring $A/d$ is $p$-torsionfree. In this case, the map $\phi_i^{-1}$ is zero on $\mathrm{H}^k(\Delta_{S/A}/p)[d]$, for every integer $k \leq i-1$; see the proof of Theorem 11.3.12 below.

**Theorem 11.3.10** (Syntomic-étale comparison theorem)**.** *Let $R$ be a perfectoid $\mathbb{Z}_p^{cyc}$-algebra, and $S$ be a $p$-Cartier smooth $R$-algebra. Then for any integers $i \geq 0$ and $k \geq 1$, the homotopy cofibres of the syntomic-étale comparison maps*

$$\mathbb{Z}/p^k(i)^{\mathrm{syn}}(S) \longrightarrow R\Gamma_{\text{ét}}(\mathrm{Spec}(S[\tfrac{1}{p}]), \mu_{p^k}^{\otimes i})$$

*and*

$$\mathbb{Z}_p(i)^{\mathrm{syn}}(S) \longrightarrow R\Gamma_{\text{proét}}(\mathrm{Spec}(S[\tfrac{1}{p}]), \mathbb{Z}_p(i))$$

*are in degrees at least $i-1$.*

*Proof.* We first prove the result modulo $p$. Let $i \geq 0$ be an integer, $(A,(d))$ the perfect prism corresponding to the perfectoid ring $R$ and $R\Phi(S, \mathbb{F}_p(i)) \in \mathcal{D}(\mathbb{F}_p)$ the homotopy cofibre of the syntomic-étale comparison map

$$\mathbb{F}_p(i)^{\mathrm{syn}}(S) \longrightarrow R\Gamma_{\text{ét}}(\mathrm{Spec}(S[\tfrac{1}{p}]), \mu_p^{\otimes i}).$$

Following [BL22, Section 8.4] (see also Notation 6.2.1), the syntomic complex $\mathbb{F}_p(i)^{\mathrm{syn}}(S)$, where $S$ is a not necessarily $p$-complete ring, is defined by the cartesian square

$$\begin{array}{ccc} \mathbb{F}_p(i)^{\mathrm{syn}}(S) & \longrightarrow & R\Gamma_{\text{ét}}(\mathrm{Spec}(S[\tfrac{1}{p}]), \mu_p^{\otimes i}) \\ \downarrow & & \downarrow \\ \mathbb{F}_p(i)^{\mathrm{syn}}(S^\wedge_p) & \longrightarrow & R\Gamma_{\text{ét}}(\mathrm{Spec}(S^\wedge_p[\tfrac{1}{p}]), \mu_p^{\otimes i}), \end{array}$$

where the bottom horizontal map is the syntomic-étale comparison map (Construction 11.3.2). In particular, the homotopy cofibre of the top horizontal map is naturally identified with that of the bottom horizontal map. The desired statement depending only on this homotopy cofibre, and in light of Remark 11.1.6, we assume for the rest of the proof that the ring $S$ is $p$-complete.

By Theorem 11.3.4 and Remark 11.3.3, there is a commutative diagram

$$\begin{array}{ccccc}
\mathbb{F}_p(i)^{\mathrm{syn}}(S) & \longrightarrow & \mathcal{N}^{\geq i}\Delta_{S/A}/p & \xrightarrow{\phi_i-1} & \Delta_{S/A}/p \\
\downarrow & & \downarrow{\scriptstyle \lambda_i} & & \downarrow{\scriptstyle \lambda} \\
R\Gamma_{\text{ét}}(\mathrm{Spec}(S[\tfrac{1}{p}]),\mu_p^{\otimes i}) & \longrightarrow & \Delta_{S/A}[\tfrac{1}{d}]/p & \xrightarrow{\phi_i-1} & \Delta_{S/A}[\tfrac{1}{d}]/p \\
\downarrow & & \downarrow & & \downarrow \\
R\Phi(S,\mathbb{F}_p(i)) & \longrightarrow & \mathrm{hocofib}(\lambda_i) & \xrightarrow{\phi_i-1} & \mathrm{hocofib}(\lambda)
\end{array}$$

where all the horizontal maps are homotopy fibre sequences. The maps $\lambda_i$ and $\lambda$ correspond to inverting $d$.

We want to prove that $R\Phi(S,\mathbb{F}_p(i)) \in \mathcal{D}^{\geq i-1}(\mathbb{F}_p)$, *i.e.*, that $R\Phi(S,\mathbb{F}_p(i))$ is zero in degrees at most $i-2$. This statement depends only on

$$\tau^{\leq i-2}\mathrm{hocofib}(\lambda_i) \xrightarrow{\phi_i-1} \tau^{\leq i-2}\mathrm{hocofib}(\lambda),$$

and thus only on the commutative diagram

$$\begin{array}{ccc}
\tau^{\leq i-1}(\mathcal{N}^{\geq i}\Delta_{S/A}/p) & \xrightarrow{\phi_i-1} & \tau^{\leq i-1}(\Delta_{S/A}/p) \\
\downarrow{\scriptstyle \tau^{\leq i-1}\lambda_i} & & \downarrow{\scriptstyle \tau^{\leq i-1}\lambda} \\
\tau^{\leq i-1}(\Delta_{S/A}[\tfrac{1}{d}]/p) & \xrightarrow{\phi_i-1} & \tau^{\leq i-1}(\Delta_{S/A}[\tfrac{1}{d}]/p).
\end{array}$$

In terms of this diagram, we want to prove that the map

$$\mathrm{hocofib}(\tau^{\leq i-1}\lambda_i) \xrightarrow{\phi_i-1} \mathrm{hocofib}(\tau^{\leq i-1}\lambda)$$

is an isomorphism in degrees at most $i-3$ and is injective in degree $i-2$.

By Lemma 11.3.7, the divided Frobenius map $\phi_i : \mathcal{N}^{\geq i}\Delta_{S/A}/p \to \Delta_{S/A}/p$ is an isomorphism in degrees at most $i-1$. Define

$$1-\phi_i^{-1} : \tau^{\leq i-1}(\Delta_{S/A}/p) \longrightarrow \tau^{\leq i-1}(\Delta_{S/A}/p)$$

as the map $\phi_i - 1 : \tau^{\leq i-1}(\mathcal{N}^{\geq i}\Delta_{S/A}/p) \to \tau^{\leq i-1}(\Delta_{S/A}/p)$ precomposed with the inverse of the divided Frobenius map $\phi_i : \tau^{\leq i-1}(\mathcal{N}^{\geq i}\Delta_{S/A}/p) \xrightarrow{\sim} \tau^{\leq i-1}(\Delta_{S/A}/p)$. Similarly the divided Frobenius map $\phi_i : \Delta_{S/A}[\frac{1}{d}]/p \to \Delta_{S/A}[\frac{1}{d}]/p$ is an equivalence by Theorem 11.1.16 ($L\eta$), and we define

$$1-\phi_i^{-1} : \tau^{\leq i-1}(\Delta_{S/A}[\tfrac{1}{d}]/p) \longrightarrow \tau^{\leq i-1}(\Delta_{S/A}[\tfrac{1}{d}]/p)$$

as the map $\phi_i - 1 : \tau^{\leq i-1}(\Delta_{S/A}[\frac{1}{d}]/p) \to \tau^{\leq i-1}(\Delta_{S/A}[\frac{1}{d}]/p)$ precomposed with the inverse of the divided Frobenius map $\phi_i : \tau^{\leq i-1}(\Delta_{S/A}[\frac{1}{d}]/p) \xrightarrow{\sim} \tau^{\leq i-1}(\Delta_{S/A}[\frac{1}{d}]/p)$. We thus want to prove that the induced map

$$\mathrm{hocofib}(\tau^{\leq i-1}\lambda) \xrightarrow{1-\phi_i^{-1}} \mathrm{hocofib}(\tau^{\leq i-1}\lambda)$$

is an isomorphism in degrees at most $i-3$ and is injective in degree $i-2$. We prove that it is an isomorphism in degrees at most $i-2$.

Let $k$ be an integer such that $k \leq i-2$. There is a short exact sequence of $A/p$-modules:

$$0 \longrightarrow \mathrm{Coker}(\mathrm{H}^k(\lambda)) \longrightarrow \mathrm{H}^k(\mathrm{hocofib}(\lambda)) \longrightarrow \mathrm{Ker}(\mathrm{H}^{k+1}(\lambda)) \longrightarrow 0.$$

To prove that $1-\phi_i^{-1}$ is an isomorphism on $\mathrm{H}^k(\mathrm{hocofib}(\lambda))$, let us prove that it is an isomorphism on both $\mathrm{Ker}(\mathrm{H}^{k+1}(\lambda))$ and $\mathrm{Coker}(\mathrm{H}^k(\lambda))$.

First consider the map

$$\mathrm{H}^{k+1}(\lambda) : \mathrm{H}^{k+1}(\Delta_{S/A}/p) \longrightarrow \mathrm{H}^{k+1}(\Delta_{S/A}/p)[\tfrac{1}{d}].$$

Its kernel is given by the $d$-power torsion subgroup of $\mathrm{H}^{k+1}(\Delta_{S/A}/p)$, so we want to prove that the map $1-\phi_i^{-1}$ is an isomorphism on the $A/p$-module $\mathrm{H}^{k+1}(\Delta_{S/A}/p)[d^\infty]$. The relation $\phi_A(d) = d^p$ holds in the ring $A/p$. So for every integer $j \geq 1$ and every element $x \in \mathrm{H}^{k+1}(\Delta_{S/A}/p)[d^{p^j}]$:

$$d^{p^{j-1}}\phi_i^{-1}(x) = \phi_i^{-1}(d^{p^j}x) = 0,$$

and thus $\phi_i^{-1}(x) \in \mathrm{H}^{k+1}(\Delta_{S/A}/p)[d^{p^{j-1}}]$. By Lemma 11.3.8, the map $(\phi_i^{-1})^2$ is zero on the $A/p$-module $\mathrm{H}^{k+1}(\Delta_{S/A}/p)[d]$. So the map $1-\phi_i^{-1}$ is an isomorphism on the $A/p$-module $\mathrm{H}^{k+1}(\Delta_{S/A}/p)[d^{p^j}]$, for each integer $j \geq 1$, with inverse given by the map

$$1+\phi_i^{-1}+\cdots+(\phi_i^{-1})^{j+1}.$$

So the map $1-\phi_i^{-1}$ is an isomorphism on the $A/p$-module $\mathrm{H}^{k+1}(\Delta_{S/A}/p)[d^\infty]$.

Now consider the map

$$\mathrm{H}^k(\lambda) : \mathrm{H}^k(\Delta_{S/A}/p) \longrightarrow \mathrm{H}^k(\Delta_{S/A}/p)[\tfrac{1}{d}].$$

It is the filtered colimit over $m \geq 0$ of the maps

$$\lambda^{(m)} : \Delta_{S/A}/p \longrightarrow \Delta_{S/A}/p$$

given by multiplication by $d^m$. The $A/p$-module $\mathrm{Coker}(\mathrm{H}^k(\lambda))$ and the map $1-\phi_i^{-1}$ acting on it can be rewritten as the colimit over $m \geq 0$ of the $A/p$-modules $\mathrm{H}^k(\Delta_{S/A}/p)/d^m$ with maps $1-d^{\frac{(p-1)m}{p}}\phi_i^{-1}$. Let $m \geq 0$ be an integer. We claim that the map

$$d^{\frac{(p-1)m}{p}}\phi_i^{-1} : \mathrm{H}^k(\Delta_{S/A}/p)/d^m \longrightarrow \mathrm{H}^k(\Delta_{S/A}/p)/d^m$$

is nilpotent. Because $k \leq i-2$, this map $d^{\frac{(p-1)m}{p}}\phi_i^{-1}$ is naturally identified with the map

$$d^{\frac{(p-1)m}{p}+\frac{1}{p}}\phi_{i-1}^{-1} : \mathrm{H}^k(\Delta_{S/A}/p)/d^m \longrightarrow \mathrm{H}^k(\Delta_{S/A}/p)/d^m,$$

where

$$\phi_{i-1} : \tau^{\leq i-1}\mathcal{N}^{\geq i-1}\Delta_{S/A} \longrightarrow \tau^{\leq i-1}\Delta_{S/A}$$

is the equivalence of Lemma 11.3.7. Composing with itself $k$ times for some integer $k \geq 1$ and using the $\phi_A^{-1}$-linearity of $\phi_{i-1}^{-1}$ gives the map $d^{\frac{(p^k-1)m}{p^k}+\frac{p^k-1}{p^k(p-1)}}(\phi_{i-1}^{-1})^k$. This map is zero for every integer $k \geq 1$ satisfying $\frac{(p^k-1)m}{p^k} + \frac{p^k-1}{p^k(p-1)} \geq m$, that is $p^k \geq m(p-1)+1$. The map $1 - d^{\frac{(p-1)m}{p}} \phi_i^{-1}$ is thus a sum of an isomorphism and a nilpotent map, so it is an equivalence. Taking the colimit over $m \geq 0$, the map $1 - \phi_i^{-1}$ is an isomorphism on $\mathrm{Coker}(\mathrm{H}^k(\lambda))$. This concludes the proof of the result modulo $p$.

We now prove the result for integral coefficients. Let $R\Phi(S, \mathbb{Z}_p(i)) \in \mathcal{D}(\mathbb{Z}_p)$ be the homotopy cofibre of the syntomic-étale comparison map

$$\mathbb{Z}_p(i)^{\mathrm{syn}}(S) \longrightarrow R\Gamma_{\mathrm{proét}}(\mathrm{Spec}(S[\tfrac{1}{p}]), \mathbb{Z}_p(i)).$$

We want to prove that $R\Phi(S, \mathbb{Z}_p(i)) \in \mathcal{D}^{\geq i-1}(\mathbb{Z}_p)$, *i.e.*, that $\tau^{\leq i-2} R\Phi(S, \mathbb{Z}_p(i)) \simeq 0$. The truncation of a derived $p$-complete object is derived $p$-complete ([Sta19, 091N]). By derived Nakayama, it thus suffices to prove that

$$(\tau^{\leq i-2} R\Phi(S, \mathbb{Z}_p(i)))/p \simeq 0.$$

For every integer $k \leq i-3$, the natural map

$$\mathrm{H}^k((\tau^{\leq i-2} R\Phi(S, \mathbb{Z}_p(i)))/p) \longrightarrow \mathrm{H}^k(R\Phi(S, \mathbb{Z}_p(i))/p) \cong \mathrm{H}^k(R\Phi(S, \mathbb{F}_p(i)))$$

is an isomorphism, and its target is zero by the first part of the proof. In degree $i-2$, the cohomology group $\mathrm{H}^{i-2}((\tau^{\leq i-2} R\Phi(S, \mathbb{Z}_p(i)))/p)$ is naturally identified with the (classical) reduction modulo $p$ of the cohomology group $\mathrm{H}^{i-2}(R\Phi(S, \mathbb{Z}_p(i)))$, and there is a short exact sequence of abelian groups:

$$0 \longrightarrow \mathrm{H}^{i-2}(R\Phi(S, \mathbb{Z}_p(i)))/p \longrightarrow \mathrm{H}^{i-2}(R\Phi(S, \mathbb{F}_p(i))) \longrightarrow \mathrm{H}^{i-1}(R\Phi(S, \mathbb{Z}_p(i)))[p] \longrightarrow 0.$$

The middle term of this short exact sequence is zero by the first part of the proof; so the left one also is, which concludes the proof for integral coefficients. The result modulo $p^k$ can be proved like the result for integral coefficients, or deduced from it by reduction modulo $p^k$. □

### 11.3.4 The syntomic-étale comparison theorem over a $p$-torsionfree base

In this subsection, we prove a refined version of the syntomic-étale comparison theorem (Theorem 11.3.12), assuming the perfectoid base ring is $p$-torsionfree.

**Lemma 11.3.11.** *Let $(A, (d))$ be a perfect prism such that $A/d$ is $p$-torsionfree, and $S$ be a $p$-Cartier smooth $A/d$-algebra. Then for every integer $i \geq 0$, the Frobenius maps and divided Frobenius map*

$$\phi : \Delta_{S/A}/p \longrightarrow L\eta_d(\Delta_{S/A}/p)$$

$$\phi : \tau^{\leq i}(\mathcal{N}^{\geq i}\Delta_{S/A}/p) \longrightarrow \tau^{\leq i}(d^i \Delta_{S/A}/p)$$

$$\phi_i : \tau^{\leq i}(\mathcal{N}^{\geq i}\Delta_{S/A}/p) \longrightarrow \tau^{\leq i}(\Delta_{S/A}/p)$$

*are equivalences.*

*Proof.* By Theorem 11.1.16 ($L\eta$), the Frobenius map

$$\phi : \Delta_{S/A} \longrightarrow L\eta_d \Delta_{S/A}$$

is an equivalence, thus so is its derived reduction modulo $p$. Note that $p$ is a nonzerodivisor in the ring $A$ ([BS22, Lemma 2.28 (1)]). Moreover, $S$ is $p$-cotangent smooth over $A/d$, so the groups $(\Omega^n_{S/(A/d)})^\wedge_p$ are $p$-flat modules over the $p$-torsionfree ring $A/d$ and are in particular $p$-torsionfree. The groups $\mathrm{H}^n(\overline{\Delta}_{S/A})$ are thus also $p$-torsionfree (Proposition 11.1.14 (3)), and the natural map

$$(L\eta_d \Delta_{S/A})/p \longrightarrow L\eta_d(\Delta_{S/A}/p)$$

is an equivalence in the derived category $\mathcal{D}(A/p)$ ([Bha18, Lemma 5.16]), which proves the first statement.

The proof of Theorem 11.1.16 $(L\eta) \Rightarrow (\mathcal{N}^{\geq})$, where we use that the short exact sequence

$$0 \longrightarrow \mathrm{H}^{-1}(\Delta_{S/A}/p)/d \longrightarrow \mathrm{H}^{-1}(\Delta_{S/A}/(p,d)) \longrightarrow \mathrm{H}^0(\Delta_{S/A}/p)[d] \longrightarrow 0$$

to prove that $\mathrm{H}^0(\Delta_{S/A}/p)$ is $d$-torsionfree, then adapts readily to prove that the Frobenius map

$$\phi : \tau^{\leq i}(\mathcal{N}^{\geq i}\Delta_{S/A}/p) \longrightarrow \tau^{\leq i}(d^i\Delta_{S/A}/p)$$

is an equivalence in the derived category $\mathcal{D}(A/p)$. The proof of the third statement is the same as in Lemma 11.3.7. □

**Theorem 11.3.12** (Syntomic-étale comparison theorem over a $p$-torsionfree base)**.** *Let $R$ be a $p$-torsionfree perfectoid $\mathbb{Z}_p^{cyc}$-algebra, and $S$ be a $p$-Cartier smooth $R$-algebra. Then for any integers $i \geq 0$ and $k \geq 1$, the homotopy cofibres of the syntomic-étale comparison maps*

$$\mathbb{Z}/p^k(i)^{\mathrm{syn}}(S) \longrightarrow R\Gamma_{\text{ét}}(\mathrm{Spec}(S[\tfrac{1}{p}]), \mu_{p^k}^{\otimes i})$$

*and*

$$\mathbb{Z}_p(i)^{\mathrm{syn}}(S) \longrightarrow R\Gamma_{\text{proét}}(\mathrm{Spec}(S[\tfrac{1}{p}]), \mathbb{Z}_p(i))$$

*are in degrees at least $i$.*

*Proof.* As in the proof of Theorem 11.3.10, we first reduce to the case where $S$ is a $p$-complete ring and, by derived Nakayama, it suffices to prove the result modulo $p$. We keep the same notation as in the proof of Theorem 11.3.10. We want to prove that

$$R\Phi(S, \mathbb{F}_p(i)) \in \mathcal{D}^{\geq i}(\mathbb{F}_p),$$

*i.e.*, that $R\Phi(S, \mathbb{F}_p(i))$ is zero in degrees at most $i-1$. This statement depends only on

$$\tau^{\leq i-1}\mathrm{hocofib}(\lambda_i) \xrightarrow{\phi_i - 1} \tau^{\leq i-1}\mathrm{hocofib}(\lambda),$$

and thus only on the commutative diagram

$$\begin{array}{ccc}
\tau^{\leq i}(\mathcal{N}^{\geq i}\Delta_{S/A}/p) & \xrightarrow{\phi_i - 1} & \tau^{\leq i}(\Delta_{S/A}/p) \\
\downarrow{\scriptstyle \tau^{\leq i}\lambda_i} & & \downarrow{\scriptstyle \tau^{\leq i}\lambda} \\
\tau^{\leq i}(\Delta_{S/A}[\tfrac{1}{d}]/p) & \xrightarrow{\phi_i - 1} & \tau^{\leq i}(\Delta_{S/A}[\tfrac{1}{d}]/p).
\end{array}$$

In terms of this diagram, we want to prove that the map

$$\mathrm{hocofib}(\tau^{\leq i}\lambda_i) \xrightarrow{\phi_i - 1} \mathrm{hocofib}(\tau^{\leq i}\lambda)$$

is an isomorphism in degrees at most $i-2$ and is injective in degree $i-1$.

The divided Frobenius map $\phi_i : \mathcal{N}^{\geq i}\Delta_{S/A}/p \to \Delta_{S/A}/p$ is an isomorphism in degrees at most $i$ by Lemma 11.3.11. Define

$$1-\phi_i^{-1} : \tau^{\leq i}(\Delta_{S/A}/p) \longrightarrow \tau^{\leq i}(\Delta_{S/A}/p)$$

as the map $\phi_i - 1 : \tau^{\leq i}(\mathcal{N}^{\geq i}\Delta_{S/A}/p) \to \tau^{\leq i}(\Delta_{S/A}/p)$ precomposed with the inverse of the divided Frobenius map $\phi_i : \tau^{\leq i}(\mathcal{N}^{\geq i}\Delta_{S/A}/p) \xrightarrow{\sim} \tau^{\leq i}(\Delta_{S/A}/p)$. Similarly the divided Frobenius map $\phi_i : \Delta_{S/A}[\frac{1}{d}]/p \to \Delta_{S/A}[\frac{1}{d}]/p$ is an equivalence (Theorem 11.1.16 ($L\eta$)), and we define

$$1-\phi_i^{-1} : \tau^{\leq i}(\Delta_{S/A}[\tfrac{1}{d}]/p) \longrightarrow \tau^{\leq i}(\Delta_{S/A}[\tfrac{1}{d}]/p)$$

as the map $\phi_i - 1 : \tau^{\leq i}(\Delta_{S/A}[\frac{1}{d}]/p) \to \tau^{\leq i}(\Delta_{S/A}[\frac{1}{d}]/p)$ precomposed with the inverse of the divided Frobenius map $\phi_i : \tau^{\leq i}(\Delta_{S/A}[\frac{1}{d}]/p) \xrightarrow{\sim} \tau^{\leq i}(\Delta_{S/A}[\frac{1}{d}]/p)$. We thus want to prove that the induced map

$$\mathrm{hocofib}(\tau^{\leq i}\lambda) \xrightarrow{1-\phi_i^{-1}} \mathrm{hocofib}(\tau^{\leq i}\lambda)$$

is an isomorphism in degrees at most $i-2$ and is injective in degree $i-1$. We prove that it is an isomorphism in degrees at most $i-1$.

Let $k$ be an integer such that $k \leq i-1$. There is a short exact sequence of $A/p$-modules:

$$0 \longrightarrow \mathrm{Coker}(\mathrm{H}^k(\lambda/p)) \longrightarrow \mathrm{H}^k(\mathrm{hocofib}(\lambda/p)) \longrightarrow \mathrm{Ker}(\mathrm{H}^{k+1}(\lambda/p)) \longrightarrow 0.$$

To prove that $1-\phi_i^{-1}$ is an isomorphism on $\mathrm{H}^k(\mathrm{hocofib}(\lambda/p))$, let us prove that it is an isomorphism on both $\mathrm{Coker}(\mathrm{H}^k(\lambda/p))$ and $\mathrm{Ker}(\mathrm{H}^{k+1}(\lambda/p))$.

For $\mathrm{Coker}(\mathrm{H}^k(\lambda/p))$, the argument is the same as in the proof of Theorem 11.3.10, where we need Lemma 11.3.11 for the case of $k = i-1$.

For $\mathrm{Ker}(\mathrm{H}^{k+1}(\lambda/p))$, we follow the lines of the proof of Theorem 11.3.10. It suffices to prove the case of $k = i-1$, *i.e.*, that $1-\phi_i^{-1}$ is an isomorphism on the $A/p$-module $\mathrm{H}^i(\Delta_{S/A}/p)[d^\infty]$. It then suffices to prove that $\phi_i^{-1}$ is nilpotent on the $A/p$-module $\mathrm{H}^i(\Delta_{S/A}/p)[d]$; we prove that it is zero. By definition of the map $\phi_i^{-1}$, it is equivalent to proving that the canonical map

$$\mathrm{can} : \mathrm{H}^i(\mathcal{N}^{\geq i}\Delta_{S/A}/p) \longrightarrow \mathrm{H}^i(\Delta_{S/A}/p)$$

is zero on the $d^{1/p}$-torsion subgroup $\mathrm{H}^i(\mathcal{N}^{\geq i}\Delta_{S/A}/p)[d^{1/p}]$. By Lemma 11.3.11, there is a commutative diagram

$$\begin{array}{ccc}
\mathrm{H}^i((\mathcal{N}^{\geq i}\Delta_{S/A})/p) & \xrightarrow[\cong]{\phi} & \mathrm{H}^i(d^i\Delta_{S/A}/p) \\
\downarrow{\scriptstyle \mathrm{can}} & & \downarrow{\scriptstyle \mathrm{can}} \\
\mathrm{H}^i(\Delta_{S/A}/p) & \xrightarrow[\cong]{\phi} & \mathrm{H}^i(L\eta_d(\Delta_{S/A}/p)),
\end{array}$$

where the right vertical map is defined in [BMS18, Lemma 6.9]. As in the proof of 11.3.8, the right vertical map of this diagram is surjective, with kernel given by the $d$-torsion subgroup of $\mathrm{H}^i(d^i\Delta_{S/A}/p)$. So the left vertical map is also surjective, with kernel given by the $d^{1/p}$-torsion subgroup $\mathrm{H}^i((\mathcal{N}^{\geq i}\Delta_{S/A})/p)[d^{1/p}]$. Hence the result. $\square$

**Remark 11.3.13** (Comparison with [BMS19])**.** Let $C$ be a complete and algebraically closed extension of $\mathbb{Q}_p$, and $\mathcal{O}_C$ be its ring of integers. In particular, $\mathcal{O}_C$ is a $p$-torsionfree perfectoid ring. When $S$ is a smooth $\mathcal{O}_C$-algebra, the previous result was already proved by Bhatt–Morrow–Scholze ([BMS19, Theorem 10.1]). In this situation, their result is slighlty stronger, as they prove that the syntomic-étale comparison map is an isomorphism in degree $i$ (and is thus not only injective). Note that this fact does not hold for general $p$-Cartier smooth $\mathcal{O}_C$-algebras, *e.g.*, for general valuation ring extensions of $\mathcal{O}_C$.

**Remark 11.3.14.** Without assuming that the perfectoid base is $p$-torsionfree, the previous result would be false: for instance, in characteristic $p$, the homotopy cofibre of the syntomic-étale comparison map is typically nonzero in degree $i-1$ (Proposition 11.3.5).

## 11.4 Motivic cohomology of valuation rings

In this section, we describe the motivic cohomology of valuation rings (Theorems 11.4.1 and 11.4.6). We start with the following result, stating that the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$, on henselian valuation rings, have a description purely in terms of algebraic cycles. See [EM23, Section 9] for related results over a field.

**Theorem 11.4.1.** *Let $V$ be a henselian valuation ring. Then for every integer $i \geq 0$, the motivic complex $\mathbb{Z}(i)^{\mathrm{mot}}(V) \in \mathcal{D}(\mathbb{Z})$ is in degrees at most $i$, and the lisse-motivic comparison map (Definition 8.1.1)*

$$\mathbb{Z}(i)^{\mathrm{lisse}}(V) \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(V)$$

*is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$.*

*Proof.* The second statement already appears in the proof of Lemma 10.3.1. As in Lemma 10.3.1 or Corollary 8.1.12, the first statement is then a consequence of [Gei04, Corollary 4.4]. □

**Example 11.4.2.** Let $V$ be a henselian valuation ring. By Example 5.6.8, there is a natural equivalence

$$\mathbb{Z}(0)^{\mathrm{mot}}(V) \simeq \mathbb{Z}[0]$$

in the derived category $\mathcal{D}(\mathbb{Z})$. Similarly, Theorem 11.4.1, Example 4.2.4, and the fact that the Picard group of a local ring is zero, imply that the motivic complex $\mathbb{Z}(1)^{\mathrm{mot}}(V) \in \mathcal{D}(\mathbb{Z})$ is concentrated in degree one, where it is given by

$$\mathrm{H}^1_{\mathrm{mot}}(V, \mathbb{Z}(1)) \cong V^{\times}.$$

We now apply the results of the previous sections to give an alternative description of the motivic cohomology of valuation rings with finite coefficients. The following proposition will be used to reformulate the results of the previous sections on syntomic cohomology in terms of motivic cohomology.

**Proposition 11.4.3.** *Let $p$ be a prime number, and $V$ be a henselian valuation ring. Then for any integers $i \geq 0$ and $k \geq 1$, there is a natural equivalence*

$$\mathbb{Z}/p^k(i)^{\mathrm{mot}}(V) \xrightarrow{\sim} \tau^{\leq i} \mathbb{Z}/p^k(i)^{\mathrm{syn}}(V)$$

*in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$.*

*Proof.* Henselian valuation rings are local rings for the cdh topology, so this is a consequence of Theorem 6.2.4. □

The following result is an analogue for valuation rings of Geisser–Levine's description of motivic cohomology of smooth $\mathbb{F}_p$-algebras [GL00]. It can be deduced from the results of Kelly–Morrow [KM21] and Elmanto–Morrow [EM23].

**Theorem 11.4.4.** *Let $p$ be a prime number, and $V$ be a henselian valuation ring of characteristic $p$. Then for any integers $i \geq 0$ and $k \geq 1$, there is a natural equivalence*

$$\mathbb{Z}/p^k(i)^{\mathrm{mot}}(V) \xrightarrow{\sim} W_k\Omega^i_{V,\log}[-i]$$

*in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$.*

*Proof.* Valuation rings of characteristic $p$ are Cartier smooth over $\mathbb{F}_p$ (Theorem 11.2.4), so this is a consequence of Propositions 11.3.5 and 11.4.3. □

We then prove a mixed characteristic version of Theorem 11.4.4, starting with the following $\ell$-adic general result.

**Proposition 11.4.5.** *Let $p$ be a prime number, and $V$ be a henselian valuation ring such that $p$ is invertible in $V$. Then for any integers $i \geq 0$ and $k \geq 1$, the Beilinson–Lichtenbaum comparison map (Definition 6.1.3) naturally factors through an equivalence*

$$\mathbb{Z}/p^k(i)^{\mathrm{mot}}(V) \xrightarrow{\sim} \tau^{\leq i} R\Gamma_{\acute{e}t}(\mathrm{Spec}(V), \mu_{p^k}^{\otimes i})$$

*in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$.*

*Proof.* By Proposition 11.4.3, the motivic complex $\mathbb{Z}/p^k(i)^{\mathrm{mot}}(V) \in \mathcal{D}(\mathbb{Z}/p^k)$ is in degrees at most $i$, so the result is a consequence of Corollary 6.1.6. □

The following result generalises Proposition 11.4.5 when $p$ is not necessarily invertible in the valuation ring $V$, at least over a perfectoid base.

**Theorem 11.4.6** (Motivic cohomology of valuation rings with finite coefficients)**.** *Let $p$ be a prime number, $V_0$ be a $p$-torsionfree valuation ring whose $p$-completion is a perfectoid ring, and $V$ be a henselian valuation ring extension of $V_0$. Then for any integers $i \geq 0$ and $k \geq 1$, the Beilinson–Lichtenbaum comparison map (Definition 6.1.3) induces a natural map*

$$\mathbb{Z}/p^k(i)^{\mathrm{mot}}(V) \longrightarrow \tau^{\leq i} R\Gamma_{\acute{e}t}(\mathrm{Spec}(V[\tfrac{1}{p}]), \mu_{p^k}^{\otimes i})$$

*in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$, which is an isomorphism in degrees at most $i-1$. On $\mathrm{H}^i$, this map is injective, with image generated by symbols, via the symbol map*

$$(V^\times)^{\otimes i} \to \mathrm{H}^i_{\acute{e}t}(\mathrm{Spec}(V[\tfrac{1}{p}]), \mu_{p^k}^{\otimes i}).$$

*Proof.* The fact that the Beilinson–Lichtenbaum comparison map factors through the complex

$$\tau^{\leq i} R\Gamma_{\acute{e}t}(\mathrm{Spec}(V[\tfrac{1}{p}]), \mu_{p^k}^{\otimes i}) \in \mathcal{D}(\mathbb{Z}/p^k)$$

is a consequence of Proposition 11.4.3. The isomorphism in degrees at most $i-1$ and the injectivity in degree $i$ of this map are then a consequence of Theorems 11.2.1 and 11.3.12. The last statement is a consequence of the isomorphism

$$\widehat{\mathrm{K}}{}^{\mathrm{M}}_i(V)/p^k \xrightarrow{\cong} \mathrm{H}^i_{\mathrm{mot}}(V, \mathbb{Z}/p^k(i))$$

of abelian groups (Theorem 8.2.6 and Corollary 8.1.10). □

**Remark 11.4.7.** The generation by symbols appearing in Theorem 11.4.6 was also studied in the context of syntomic cohomology of general $p$-torsionfree $F$-smooth schemes by Bhatt–Mathew [BM23]. Note that all valuation rings are conjecturally $F$-smooth (see Conjecture 12.2.1), and that the proof of Theorem 11.4.6 adapts more generally to any henselian $F$-smooth valuation ring.

# Chapter 12

# $\mathbb{A}^1$-invariant motivic cohomology

The theory of classical motivic cohomology of smooth schemes over a mixed characteristic Dedekind domain [Blo86, Lev01, Gei04], as a theory of $\mathbb{A}^1$-invariant motivic cohomology, admits a natural generalisation to general qcqs schemes. More precisely, Spitzweck constructed in [Spi18], for every qcqs scheme $X$, an $\mathbb{A}^1$-motivic spectrum $H\mathbb{Z}^{\text{Spi}} \in \text{SH}(\mathbb{Z})$, which represents Bloch's cycle complexes on smooth $\mathbb{Z}$-schemes, and whose pullback to $\text{SH}(B)$, for $B$ a field or a mixed characteristic Dedekind domain, still represents Bloch's cycle complexes on smooth $B$-schemes. Bachmann then proved in [Bac22] that Spitzweck's construction coincides with the zeroth slice of the homotopy $K$-theory motivic spectrum $\text{KGL} \in \text{SH}(\mathbb{Z})$. Finally, Bachmann–Elmanto–Morrow recently proved in [BEM24] that the slice filtration is compatible with arbitrary pullbacks, thus defining a well-behaved $\mathbb{A}^1$-motivic spectrum $H\mathbb{Z}_X \in \text{SH}(X)$ for arbitrary qcqs schemes $X$. The associated $\mathbb{A}^1$-invariant motivic complexes

$$\mathbb{Z}(i)^{\mathbb{A}^1}(X) \in \mathcal{D}(\mathbb{Z})$$

are related to the homotopy $K$-theory $\text{KH}(X)$ by an $\mathbb{A}^1$-invariant Atiyah–Hirzebruch spectral sequence. For our purposes, we will only use that there is a natural map

$$\mathbb{Z}(i)^{\text{cdh}}(X) \longrightarrow \mathbb{Z}(i)^{\mathbb{A}^1}(X)$$

which exhibits the target as the $\mathbb{A}^1$-localisation of the source, and which is an equivalence if the qcqs scheme $X$ satisfies the condition $\text{Val}(X)$ (see Theorem 12.3.1).

## 12.1 Comparison to $\mathbb{A}^1$-invariant motivic cohomology

In this section, we prove that the $\mathbb{A}^1$-localisation of the motivic complexes $\mathbb{Z}(i)^{\text{mot}}$ recover Bachmann–Elmanto–Morrow's $\mathbb{A}^1$-invariant motivic complexes $\mathbb{Z}(i)^{\mathbb{A}^1}$ (Theorem 12.1.5). This is a motivic refinement of [Elm21, Theorem 1.0.1], and implies that the motivic complexes $\mathbb{Z}(i)^{\text{mot}}$ recover the classical motivic complexes $\mathbb{Z}(i)^{\text{cla}}$ on smooth schemes over a mixed characteristic Dedekind domain after $\mathbb{A}^1$-localisation (Corollary 12.1.9).

**Lemma 12.1.1.** *Let $\mathcal{C}$ be a presentable stable $\infty$-category, and $F : \text{Sch}^{\text{qcqs,op}} \to \mathcal{C}$ be a presheaf. Then the natural map*

$$L_{\mathbb{A}^1} L_{\text{cdh}} F \longrightarrow L_{\mathbb{A}^1} L_{\text{cdh}} L_{\mathbb{A}^1} F$$

*induced by $\mathbb{A}^1$-localisation is an equivalence of $\mathcal{C}$-valued presheaves.*

*Proof.* A filtered colimit of cdh sheaves is a cdh sheaf (Lemma 5.5.12). In particular, the $\mathbb{A}^1$-localisation of a cdh sheaf is a cdh sheaf, so the natural composites

$$L_{\mathbb{A}^1} L_{\mathrm{cdh}} F \longrightarrow L_{\mathbb{A}^1} L_{\mathrm{cdh}} L_{\mathbb{A}^1} F \longrightarrow L_{\mathbb{A}^1} L_{\mathrm{cdh}} L_{\mathbb{A}^1} L_{\mathrm{cdh}} F$$

and

$$L_{\mathbb{A}^1} L_{\mathrm{cdh}} L_{\mathbb{A}^1} F \longrightarrow L_{\mathbb{A}^1} L_{\mathrm{cdh}} L_{\mathbb{A}^1} L_{\mathrm{cdh}} F \longrightarrow L_{\mathbb{A}^1} L_{\mathrm{cdh}} L_{\mathbb{A}^1} L_{\mathrm{cdh}} L_{\mathbb{A}^1} F$$

are equivalences in the $\infty$-category $\mathcal{C}$. This implies the desired result. □

**Lemma 12.1.2.** *For every integer $i \geq 0$, the $\mathbb{A}^1$-localisation of the presheaf*

$$R\Gamma_{\mathrm{Zar}}\big(-, \mathbb{L}\Omega^{<i}_{-_{\mathbb{Q}}/\mathbb{Q}}\big) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathcal{D}(\mathbb{Q})$$

*is zero.*

*Proof.* By Zariski descent, it suffices to prove the result on animated commutative rings. The functor $\mathbb{L}\Omega^{<i}_{-_{\mathbb{Q}}/\mathbb{Q}}$, from animated commutative rings to the derived category $\mathcal{D}(\mathbb{Q})$, is left Kan extended from polynomial $\mathbb{Z}$-algebras. Equivalently, it commutes with sifted colimits. This property being preserved by $\mathbb{A}^1$-localisation, the functor $L_{\mathbb{A}^1}\mathbb{L}\Omega^{<i}_{-_{\mathbb{Q}}/\mathbb{Q}}$ is also left Kan extended from polynomial $\mathbb{Z}$-algebras. As it is also constant on polynomial $\mathbb{Z}$-algebras, and zero on the zero ring, it is the zero functor. □

**Corollary 12.1.3.** *Let $X$ be a qcqs derived scheme. Then for every integer $i \geq 0$, the natural map*

$$\big(L_{\mathbb{A}^1}\widehat{\mathbb{L}\Omega}^{\geq i}_{-_{\mathbb{Q}}/\mathbb{Q}}\big)(X) \longrightarrow \big(L_{\mathbb{A}^1}\widehat{\mathbb{L}\Omega}_{-_{\mathbb{Q}}/\mathbb{Q}}\big)(X)$$

*is an equivalence in the derived category $\mathcal{D}(\mathbb{Q})$.*

*Proof.* There is a natural fibre sequence

$$\big(L_{\mathbb{A}^1}\widehat{\mathbb{L}\Omega}^{\geq i}_{-_{\mathbb{Q}}/\mathbb{Q}}\big)(X) \longrightarrow \big(L_{\mathbb{A}^1}\widehat{\mathbb{L}\Omega}_{-_{\mathbb{Q}}/\mathbb{Q}}\big)(X) \longrightarrow \big(L_{\mathbb{A}^1}\mathbb{L}\Omega^{<i}_{-_{\mathbb{Q}}/\mathbb{Q}}\big)$$

in the derived category $\mathcal{D}(\mathbb{Q})$. The result is then a consequence of Lemma 12.1.2. □

**Lemma 12.1.4.** *Let $p$ be a prime number. Then for every integer $i \geq 0$, the $\mathbb{A}^1$-localisation of the presheaf*

$$\mathbb{F}_p(i)^{\mathrm{BMS}}(-) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathcal{D}(\mathbb{F}_p)$$

*is zero.*

*Proof.* The presheaf $\mathbb{F}_p(i)^{\mathrm{BMS}}$ is a Zariski sheaf, and its restriction to animated commutative rings is left Kan extended from polynomial $\mathbb{Z}$-algebras (Corollary 3.2.12). The result then follows by the same argument as in Lemma 12.1.2. □

**Theorem 12.1.5.** *Let $X$ be a qcqs scheme. Then for every integer $i \geq 0$, the natural map*

$$\big(L_{\mathbb{A}^1}\,\mathbb{Z}(i)^{\mathrm{mot}}\big)(X) \longrightarrow \big(L_{\mathbb{A}^1}\,\mathbb{Z}(i)^{\mathrm{cdh}}\big)(X) \simeq \mathbb{Z}(i)^{\mathbb{A}^1}(X)$$

*is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$.*

*Proof.* It suffices to prove the result rationally, and modulo $p$ for every prime number $p$. By Lemma 12.1.2, the object

$$\big(L_{\mathbb{A}^1} R\Gamma_{\mathrm{Zar}}\big(X, \mathbb{L}\Omega^{<i}_{-_{\mathbb{Q}}/\mathbb{Q}}\big)\big)(X)$$

is zero in the derived category $\mathcal{D}(\mathbb{Q})$. Lemma 12.1.1 then implies that the object

$$\big(L_{\mathbb{A}^1} R\Gamma_{\mathrm{cdh}}(-, \mathbb{L}\Omega^{<i}_{-_{\mathbb{Q}}/\mathbb{Q}})\big)(X)$$

is zero in the derived category $\mathcal{D}(\mathbb{Q})$. In particular, the natural map

$$\big(L_{\mathbb{A}^1} R\Gamma_{\mathrm{Zar}}(-, \mathbb{L}\Omega^{<i}_{-_{\mathbb{Q}}/\mathbb{Q}})\big)(X) \longrightarrow \big(L_{\mathbb{A}^1} R\Gamma_{\mathrm{cdh}}(-, \mathbb{L}\Omega^{<i}_{-_{\mathbb{Q}}/\mathbb{Q}})\big)(X)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Q})$, which implies the desired result rationally by Corollary 5.6.7. Similarly, for every prime number $p$, the syntomic complex $\mathbb{F}_p(i)^{\mathrm{BMS}}$ vanishes after $\mathbb{A}^1$-localisation by Lemma 12.1.4, which implies the desired result modulo $p$ by Lemma 12.1.1 and Corollary 4.3.12. □

**Remark 12.1.6.** Let $X$ be a qcqs derived scheme. One can prove, using similar arguments and Corollary 5.2.16, that there is a natural fibre sequence

$$\big(L_{\mathbb{A}^1}\, \mathbb{Z}(i)^{\mathrm{TC}}\big)(X) \longrightarrow R\Gamma_{\mathrm{Zar}}\big(X, \widehat{\mathbb{L}\Omega}_{-_{\mathbb{Q}}/\mathbb{Q}}\big) \longrightarrow R\Gamma_{\mathrm{Zar}}\big(X, \prod_{p\in\mathbb{P}}{}' \underline{\widehat{\mathbb{L}\Omega}}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p}\big)$$

in the derived category $\mathcal{D}(\mathbb{Z})$. Note that this implies Theorem 12.1.5 on qcqs classical schemes, via the cdh descent results [EM23, Lemma 4.5] and Corollary 5.4.4.

**Theorem 12.1.7.** *Let $X$ be a smooth scheme over a mixed characteristic Dedekind domain. Then the natural map*

$$\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(X) \longrightarrow \big(L_{\mathbb{A}^1}\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}\big)(X)$$

*is an equivalence of filtered spectra.*

*Proof.* By Proposition 7.1.2 and its proof, this natural map is an equivalence rationally, so it suffices to prove that it is an equivalence modulo every prime number $p$. Let $p$ be a prime number. By [BEM24], the natural map

$$\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(X) \longrightarrow \big(L_{\mathbb{A}^1}\mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{KH}\big)(X)$$

is an equivalence of filtered spectra, so it suffices to prove that the natural map

$$\big(L_{\mathbb{A}^1}\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}\big)(X)/p \longrightarrow \big(L_{\mathbb{A}^1}\mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{KH}\big)(X)/p$$

is an equivalence of filtered spectra. By Proposition 4.3.11, this is equivalent to the fact that the natural map

$$\big(L_{\mathbb{A}^1}\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{F}_p)\big)(X) \longrightarrow \big(L_{\mathrm{A}^1}L_{\mathrm{cdh}}\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{F}_p)\big)(X)$$

is an equivalence of filtered spectra. The filtered spectrum $\big(L_{\mathbb{A}^1}\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{F}_p)\big)(X)$ is complete by the connectivity bound of Lemma 3.2.7, and its graded pieces are zero by Lemma 12.1.4, so it is zero. By Lemma 12.1.1, the target $\big(L_{\mathrm{A}^1}L_{\mathrm{cdh}}\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{F}_p)\big)(X)$ of the previous map is then also zero, and this map is in particular an equivalence. □

**Remark 12.1.8.** We expect that the $\mathbb{A}^1$-localisation in Theorem 12.1.7 is not necessary, *i.e.*, that the natural map

$$\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X) \longrightarrow \big(L_{\mathbb{A}^1}\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}\big)(X)$$

is an equivalence of filtered spectra for smooth schemes $X$ over a torsionfree Dedekind domain. By Theorem 12.1.5, this is equivalent to the fact that the composite

$$\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X) \xrightarrow{L_{\mathrm{cdh}}} \mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{KH}(X) \xrightarrow{L_{\mathbb{A}^1}} \big(L_{\mathbb{A}^1}\mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{KH}\big)(X)$$

is an equivalence of filtered spectra, where the second map is expected to be an equivalence for every qcqs scheme $X$ ([BEM24]).

**Corollary 12.1.9.** *Let $X$ be a smooth scheme over a mixed characteristic Dedekind domain. Then for every integer $i \geq 0$, there is a natural equivalence*

$$z^i(X,\bullet)[-2i] \simeq \big(L_{\mathbb{A}^1}\,\mathbb{Z}(i)^{\mathrm{mot}}\big)(X)$$

*in the derived category $\mathcal{D}(\mathbb{Z})$.*

*Proof.* This is a consequence of Theorem 12.1.7. □

## 12.2 $F$-smoothness of valuation rings

In this section, we formulate the key hypothesis used in the work of Bachmann–Elmanto–Morrow [BEM24] on $\mathbb{A}^1$-invariant motivic cohomology (Definition 12.2.3). This hypothesis relies on the following conjecture, implicit in the work of Bhatt–Mathew [BM23] on $F$-smoothness.

**Conjecture 12.2.1** (Bhatt–Mathew)**.** Every valuation ring is $F$-smooth.

The following theorem summarizes the known cases of Conjecture 12.2.1. Note that the notion of $F$-smoothness implicitly depends on a fixed prime number $p$, and that a commutative ring with bounded $p$-power torsion is $F$-smooth if and only if its $p$-completion is $F$-smooth.

**Theorem 12.2.2.** *Let $V$ be a valuation ring, and $p$ be a prime number.*

*(1) (Bhatt–Mathew [BM23]) If $V$ is a discrete valuation ring, then $V$ is $F$-smooth.*

*(2) (Gabber–Ramero [GR03], Gabber [KST21], Kelly–Morrow [KM21]) If $V$ is a valuation ring extension of $\mathbb{F}_p$, then $V$ is $F$-smooth.*

*(3) (Bouis [Bou23]) If $V$ is a valuation ring extension of a $p$-torsionfree perfectoid valuation ring of mixed characteristic $(0,p)$, then $V$ is $F$-smooth.*

**Definition 12.2.3.** (1) Let $p$ be a prime number (on which the notion of $F$-smoothness implicitly depends). A qcqs scheme $X$ *satisfies the condition* $\mathrm{Val}(X,p)$ if every valuation ring $V$ with a map $\mathrm{Spec}(V) \to X$ is $F$-smooth.

(2) A qcqs scheme $X$ *satisfies the condition* $\mathrm{Val}(X)$ if it satisfies the condition $\mathrm{Val}(X,p)$ for every prime number $p$.

**Examples 12.2.4.** Let $p$ be a prime number.

(1) Every qcqs $\mathbb{Z}[\frac{1}{p}]$-scheme satisfies the condition $\mathrm{Val}(X,p)$ because every valuation ring in which $p$ is invertible is vacuously $F$-smooth. Consequently, every qcqs $\mathbb{Q}$-scheme satisfies the condition $\mathrm{Val}(X)$.

(2) Every qcqs $\mathbb{F}_p$-scheme satisfies the condition $\mathrm{Val}(X)$ (Theorem 12.2.2 (2)). More generally, for every integer $N \geq 1$, every qcqs $\mathbb{Z}/N$-scheme satisfies the condition $\mathrm{Val}(X)$.

(3) Every qcqs $V$-scheme, where $V$ is a commutative ring with bounded $p$-power torsion and whose $p$-completion is a perfectoid valuation ring of mixed characteristic $(0,p)$, satisfies the condition $\mathrm{Val}(X)$ (Theorem 12.2.2 (3) for the condition $\mathrm{Val}(X,p)$, and (1) for the condition $\mathrm{Val}(X,\ell)$ at all other primes $\ell$).

## 12.3 Motivic regularity

By Quillen's fundamental theorem of algebraic $K$-theory, the algebraic $K$-theory of a regular noetherian ring is $\mathbb{A}^1$-invariant. Vorst conjectured a partial converse of this result, *i.e.*, that an essentially finite type algebra over a field whose $K$-groups are $\mathbb{A}^1$-invariant is regular ([Vor79]). The conjecture was proved in characteristic zero by Cortiñas–Haesemeyer–Weibel ([CHW08]). Kerz–Strunk–Tamme then proved a variant of the conjecture in positive characteristic ([KST21, Theorem A]), and asked if Vorst's conjecture holds for general excellent noetherian rings ([KST21, Question D]).

In this section, we study the extent to which these questions have a natural analogue in motivic cohomology. More precisely, given a qcqs scheme $X$ and an integer $i \geq 0$, and in light of Theorem 12.1.5, we will be interested in when the natural map

$$\mathbb{Z}(i)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Z}(i)^{\mathbb{A}^1}(X)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$. We will use repeatedly the following result of [BEM24].

**Theorem 12.3.1** ([BEM24])**.** *Let $i \geq 0$ be an integer.*

*(1) For every qcqs scheme $X$, the natural map*

$$\mathbb{Q}(i)^{\mathrm{cdh}}(X) \longrightarrow \mathbb{Q}(i)^{\mathbb{A}^1}(X)$$

*is an equivalence in the derived category $\mathcal{D}(\mathbb{Q})$.*

*(2) For every prime number $p$ and every qcqs $\mathbb{Z}[\frac{1}{p}]$-scheme $X$, the natural map*

$$\mathbb{F}_p(i)^{\mathrm{cdh}}(X) \longrightarrow \mathbb{F}_p(i)^{\mathbb{A}^1}(X)$$

*is an equivalence in the derived category $\mathcal{D}(\mathbb{F}_p)$.*

*(3) For every qcqs scheme $X$ satisfying the condition* $\mathrm{Val}(X)$ *(Definition 12.2.3), the natural map*

$$\mathbb{Z}(i)^{\mathrm{cdh}}(X) \longrightarrow \mathbb{Z}(i)^{\mathbb{A}^1}(X)$$

*is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$.*

**Theorem 12.3.2** (Rational motivic regularity)**.** *Let $X$ be a qcqs scheme. Then for every integer $i \geq 0$, the following are equivalent:*

*(1) the natural map* $\mathbb{Q}(i)^{\mathrm{mot}}(X) \to \mathbb{Q}(i)^{\mathbb{A}^1}(X)$ *is an equivalence in the derived category* $\mathcal{D}(\mathbb{Q})$*;*

*(2) the natural map* $\mathbb{Q}(i)^{\mathrm{mot}}(X_{\mathbb{Q}}) \to \mathbb{Q}(i)^{\mathbb{A}^1}(X_{\mathbb{Q}})$ *is an equivalence in the derived category* $\mathcal{D}(\mathbb{Q})$*.*

*Moreover,* (1) *or* (2) *for all integers* $i \geq 0$ *is equivalent to each of the following statements:*

*(3) for every integer* $j \geq 0$*, the natural map* $R\Gamma_{\mathrm{Zar}}(X, \mathbb{L}^j_{-_{\mathbb{Q}}/\mathbb{Q}}) \to R\Gamma_{\mathrm{cdh}}(X, \Omega^j_{-_{\mathbb{Q}}/\mathbb{Q}})$ *is an equivalence in the derived category* $\mathcal{D}(\mathbb{Q})$*;*

*(4) the natural map* $\mathrm{K}(X;\mathbb{Q}) \to \mathrm{KH}(X;\mathbb{Q})$ *is an equivalence of spectra;*

*(5) the natural map* $\mathrm{HC}(X_{\mathbb{Q}}/\mathbb{Q}) \to L_{\mathrm{cdh}}\mathrm{HC}(-_{\mathbb{Q}}/\mathbb{Q})(X)$ *is an equivalence of spectra.*

*Proof.* (4) and (5) are equivalent by Corollary 5.6.7. Theorem 12.3.1 (1) and the Adams decompositions Corollary 5.5.11 and

$$\mathrm{KH}(X;\mathbb{Q}) \simeq \bigoplus_{i\geq 0} \mathbb{Q}(i)^{\mathrm{cdh}}(X)$$

then imply that (1) for all integers $i \geq 0$ and (4) are equivalent. By Corollary 5.6.7, (1) is equivalent to the fact that the natural map

$$R\Gamma_{\mathrm{Zar}}\big(X, \mathbb{L}\Omega^{<i}_{-_{\mathbb{Q}}/\mathbb{Q}}\big) \longrightarrow R\Gamma_{\mathrm{cdh}}\big(X, \Omega^{<i}_{-_{\mathbb{Q}}/\mathbb{Q}}\big)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Q})$, which is in turn equivalent to (2), and implies that (1) for all integers $i \geq 0$ is equivalent to (3). □

**Corollary 12.3.3** (Motivic regularity in characteristic zero)**.** *Let* $X$ *be a qcqs* $\mathbb{Q}$*-scheme. Then the following are equivalent:*

*(1) for every integer* $i \geq 0$*, the natural map* $\mathbb{Z}(i)^{\mathrm{mot}}(X) \to \mathbb{Z}(i)^{\mathbb{A}^1}(X)$ *is an equivalence in the derived category* $\mathcal{D}(\mathbb{Z})$*;*

*(2) for every integer* $j \geq 0$*, the natural map* $R\Gamma_{\mathrm{Zar}}(X, \mathbb{L}^j_{-/\mathbb{Q}}) \to R\Gamma_{\mathrm{cdh}}(X, \Omega^j_{-/\mathbb{Q}})$ *is an equivalence in the derived category* $\mathcal{D}(\mathbb{Q})$*;*

*(3) the natural map* $\mathrm{K}(X) \to \mathrm{KH}(X)$ *is an equivalence of spectra;*

*(4) the natural map* $\mathrm{HC}(X/\mathbb{Q}) \to L_{\mathrm{cdh}}\mathrm{HC}(-/\mathbb{Q})(X)$ *is an equivalence of spectra.*

*Proof.* For every prime number $p$, the natural map

$$\mathrm{K}(X;\mathbb{F}_p) \longrightarrow \mathrm{KH}(X;\mathbb{F}_p)$$

is an equivalence of spectra on qcqs $\mathbb{Q}$-schemes ([Wei89, Proposition 1.6]). Similarly, for any prime number $p$ and integer $i \geq 0$, the natural map

$$\mathbb{F}_p(i)^{\mathrm{mot}}(X) \longrightarrow \mathbb{F}_p(i)^{\mathrm{cdh}}(X)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{F}_p)$ (Remark 4.3.13). In particular, the natural map

$$\mathrm{K}(X) \longrightarrow \mathrm{KH}(X) \quad (\text{resp. } \mathbb{Z}(i)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Z}(i)^{\mathrm{cdh}}(X))$$

is an equivalence of spectra (resp. in the derived category $\mathcal{D}(\mathbb{Z})$) if and only if the natural map

$$\mathrm{K}(X;\mathbb{Q}) \longrightarrow \mathrm{KH}(X;\mathbb{Q}) \quad (\text{resp. } \mathbb{Q}(i)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Q}(i)^{\mathrm{cdh}}(X))$$

is an equivalence of spectra (resp. in the derived category $\mathcal{D}(\mathbb{Q})$). The result then follows from Theorems 12.3.1 (1) and 12.3.2. ☐

**Proposition 12.3.4** ($\ell$-adic motivic regularity). *Let $\ell$ be a prime number, and $X$ be a qcqs $\mathbb{Z}[\frac{1}{\ell}]$-scheme. Then for any integers $i \geq 0$ and $k \geq 1$, the natural map*

$$\mathbb{Z}/\ell^k(i)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Z}/\ell^k(i)^{\mathbb{A}^1}(X)$$

*is an equivalence in the derived category $\mathcal{D}(\mathbb{Z}/\ell^k)$.*

*Proof.* This is a consequence of Remark 4.3.13 and Theorem 12.3.1 (2). ☐

Recall the following known result at the level of $K$-theory in positive characteristic.

**Corollary 12.3.5.** *Let $N \geq 1$ be an integer, and $X$ be a qcqs $\mathbb{Z}/N$-scheme. Then the natural map $\mathrm{K}(X) \to \mathrm{KH}(X)$ is an equivalence of spectra if and only if for every prime number $p$ dividing $N$, the natural map $\mathrm{TC}(X;\mathbb{F}_p) \to L_{\mathrm{cdh}}\mathrm{TC}(-;\mathbb{F}_p)(X)$ is an equivalence of spectra.*

*Proof.* For every prime number $p$, the functor $\mathrm{TC}(-;\mathbb{F}_p)$ is zero on qcqs $\mathbb{Z}[\frac{1}{p}]$-schemes. The natural map $\mathrm{K}(X) \to \mathrm{KH}(X)$ is an equivalence if and only if it is an equivalence rationally, and modulo $p$ for every prime number $p$. The result is then a consequence of Theorems 2.1.1 and 12.3.2 (4)-(5). ☐

The following result is a motivic analogue of Corollary 12.3.5.

**Corollary 12.3.6** (Motivic regularity in positive characteristic). *Let $N \geq 1$ and $i \geq 0$ be integers, and $X$ be a qcqs $\mathbb{Z}/N$-scheme. Then the natural map $\mathbb{Z}(i)^{\mathrm{mot}}(X) \to \mathbb{Z}(i)^{\mathbb{A}^1}(X)$ is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$ if and only if for every prime number $p$ dividing $N$, the natural map $\mathbb{F}_p(i)^{\mathrm{BMS}}(X) \to (L_{\mathrm{cdh}}\,\mathbb{F}_p(i)^{\mathrm{BMS}})(X)$ is an equivalence in the derived category $\mathcal{D}(\mathbb{F}_p)$.*

*Proof.* By Theorem 12.3.1 (3) and Example 12.2.4 (2), the natural map

$$\mathbb{Z}(i)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Z}(i)^{\mathbb{A}^1}(X)$$

is an equivalence if and only if the natural map $\mathbb{Z}(i)^{\mathrm{mot}}(X) \to \mathbb{Z}(i)^{\mathrm{cdh}}(X)$ is an equivalence. The natural map $\mathbb{Z}(i)^{\mathrm{mot}}(X) \to \mathbb{Z}(i)^{\mathrm{cdh}}(X)$ is an equivalence if and only if it is an equivalence rationally, and modulo $p$ for every prime number $p$. For every prime number $p$, the functor $\mathbb{F}_p(i)^{\mathrm{BMS}}$ is zero on qcqs $\mathbb{Z}[\frac{1}{p}]$-schemes. The result is then a consequence of Corollary 4.3.12 and Theorem 12.3.2 (1)-(2). ☐

The following result is [EM23, Theorem 6.1 (2)], where we use Theorem 12.3.1 and Example 12.2.4 (2) to identify cdh-local and $\mathbb{A}^1$-invariant motivic cohomologies in characteristic $p$.

**Theorem 12.3.7** ([EM23]). *Let $p$ be a prime number, and $X$ be an ind-regular $\mathbb{F}_p$-scheme. Then for every integer $i \geq 0$, the natural map*

$$\mathbb{Z}(i)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Z}(i)^{\mathbb{A}^1}(X)$$

*is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$.*

**Corollary 12.3.8.** *Let $p$ be a prime number, $V$ be a characteristic $p$ perfect valuation ring, and $X$ be an ind-smooth scheme over $V$. Then for every integer $i \geq 0$, the natural map*

$$\mathbb{Z}(i)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Z}(i)^{\mathbb{A}^1}(X)$$

*is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$.*

*Proof.* By [AD21, Proposition 4.1.1], the perfect valuation ring $V$ is the filtered colimit of its smooth $\mathbb{F}_p$-subalgebras. So the scheme $X$ is in particular ind-smooth over $\mathbb{F}_p$, and the result is a consequence of Theorem 12.3.7. □

We now study motivic regularity in mixed characteristic.

**Proposition 12.3.9** (Motivic regularity of valuation rings)**.** *Let $V$ be a henselian valuation ring. If $V$ satisfies the condition* Val$(V)$ *(e.g., if $V$ is an extension of a perfectoid valuation ring), then for every integer $i \geq 0$, the natural map*

$$\mathbb{Z}(i)^{\mathrm{mot}}(V) \longrightarrow \mathbb{Z}(i)^{\mathbb{A}^1}(V)$$

*is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$.*

*Proof.* If the ring $V$ satisfies the condition Val$(V)$, the natural map

$$\mathbb{Z}(i)^{\mathrm{mot}}(V) \longrightarrow \mathbb{Z}(i)^{\mathbb{A}^1}(V)$$

is naturally identified with the natural map

$$\mathbb{Z}(i)^{\mathrm{mot}}(V) \longrightarrow \mathbb{Z}(i)^{\mathrm{cdh}}(V)$$

in the derived category $\mathcal{D}(\mathbb{Z})$ (Theorem 12.3.1 (3)). This latter map is an equivalence by Remark 4.3.10, and because henselian valuation rings are local for the cdh topology. □

**Proposition 12.3.10** ([BEM24])**.** *Let $p$ be a prime number, $S$ be a qcqs scheme of finite valuative dimension and satisfying the condition* Val$(S,p)$*, and $X$ be a qcqs $S$-scheme. Then for any integers $i \geq 0$ and $k \geq 1$, the fibre of the cdh-local Beilinson–Lichtenbaum comparison map (Definition 6.1.2)*

$$\mathbb{Z}/p^k(i)^{\mathrm{cdh}}(X) \longrightarrow R\Gamma_{\text{ét}}(X[\tfrac{1}{p}], \mu_{p^k}^{\otimes i})$$

*in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$ is in degrees at least $i$.*

*Proof.* The presheaves $\mathbb{Z}/p^k(i)^{\mathrm{cdh}}(-)$ and $R\Gamma_{\text{ét}}(-[\frac{1}{p}], \mu_{p^k}^{\otimes i})$ are finitary cdh sheaves on qcqs $S$-schemes ([BEM24] and Theorem 6.1.1), so it suffices to prove the result on henselian valuation rings $V$ with a map $\mathrm{Spec}(V) \to S$ ([EHIK21, Corollary 2.4.19]). If the henselian valuation ring $V$ is $p$-torsionfree, the condition Val$(S,p)$ and Corollary 6.2.6 imply that the fibre of the Beilinson–Lichtenbaum comparison map

$$\mathbb{Z}/p^k(i)^{\mathrm{mot}}(V) \longrightarrow R\Gamma_{\text{ét}}(V[\tfrac{1}{p}], \mu_{p^k}^{\otimes i})$$

is in degrees at least $i+1$. If the henselian valuation ring $V$ is not $p$-torsionfree, then it is an $\mathbb{F}_p$-algebra, and it is $F$-smooth (Theorem 12.2.2 (2)). In particular, there is a natural equivalence

$$\mathbb{Z}/p^k(i)^{\mathrm{BMS}}(V) \xrightarrow{\sim} R\Gamma_{\text{ét}}(V, W_k\Omega^i_{\log})[-i]$$

in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$ ([LM23, Proposition 5.1 (*ii*)]), and Theorem 6.2.4 implies that the fibre of the Beilinson–Lichtenbaum comparison map is in degrees at least $i$. □

**Theorem 12.3.11.** *Let $p$ be a prime number, $V$ be a valuation ring whose $p$-completion is perfectoid, and $X$ be a $p$-torsionfree $F$-smooth scheme over* $\mathrm{Spec}(V)$*. Then for any integers $i \geq 0$ and $k \geq 1$, the fibre of the natural map*

$$\mathbb{Z}/p^k(i)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Z}/p^k(i)^{\mathbb{A}^1}(X)$$

*in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$ is in degrees at least $i+1$.*

*Proof.* By Theorem 12.3.1 (3) and Example 12.2.4 (3), this is equivalent to the fact that the fibre of the left vertical map in the natural commutative diagram

$$\begin{array}{ccc} \mathbb{Z}/p^k(i)^{\mathrm{mot}}(X) & \longrightarrow & R\Gamma_{\text{ét}}(X[\tfrac{1}{p}], \mu_{p^k}^{\otimes i}) \\ \downarrow & & \downarrow \\ \mathbb{Z}/p^k(i)^{\mathrm{cdh}}(X) & \longrightarrow & \big(L_{\mathrm{cdh}} R\Gamma_{\text{ét}}(-[\tfrac{1}{p}], \mu_{p^k}^{\otimes i})\big)(X) \end{array}$$

is in degrees at least $i+1$, *i.e.*, that it is an isomorphism on cohomology groups in degrees at most $i-1$, and injective in degree $i$. The right vertical map is an equivalence (Theorem 6.1.1). The fibre of the top horizontal map is in degrees at least $i+1$ (Corollary 6.2.6). The fibre of the bottom horizontal map is in degrees at least $i$ (Proposition 12.3.10 and Theorem 12.2.2 (3)). This implies the desired result, by the following elementary argument. On cohomology groups in degrees at most $i-2$, all but the left vertical map are isomorphisms, so the left vertical map is an isomorphism. In degree $i-1$, the top horizontal and right vertical maps are isomorphisms, so the left vertical map is injective; the bottom horizontal map is moreover injective, so the left vertical map is an isomorphism. In degree $i$, the top horizontal map is injective and the right vertical map is an isomorphism, so the left vertical map is injective. □

**Corollary 12.3.12.** *Let $V$ be a valuation ring whose $p$-completion is perfectoid for every prime number $p$,*[1] *and $X$ be an ind-smooth $V$-scheme. Then for every integer $i \geq 0$, the fibre of the natural map*

$$\mathbb{Z}(i)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Z}(i)^{\mathbb{A}^1}(X)$$

*in the derived category $\mathcal{D}(\mathbb{Z})$ is in degrees at least $i+2$.*

*Proof.* By [AMM22, Corollary 2.3 and Proposition 2.4.2], the natural map

$$\mathrm{K}(X) \longrightarrow \mathrm{KH}(X)$$

is an equivalence of spectra for every ind-smooth scheme over a general valuation ring $V$. By Theorem 10.1, this implies that the natural map

$$\mathbb{Q}(i)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Q}(i)^{\mathbb{A}^1}(X)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Q})$. That is, the cohomology groups of the fibre of the natural map

$$\mathbb{Z}(i)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Z}(i)^{\mathbb{A}^1}(X)$$

are torsion. In degrees at most $i+2$, and if the $p$-completion of the valuation ring $V$ is perfectoid for every prime number $p$, then these cohomology groups are also $p$-torsionfree for every prime number $p$ by Theorem 12.3.11, and are thus zero. □

[1] Note that this condition is vacuously true for every prime number $p$ different from the residue characteristic of the valuation ring $V$.

# Chapter 13

# Examples

In this chapter, we revisit certain known results on algebraic $K$-theory in terms of the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$.

## 13.1 Perfect and semiperfect rings

Let $p$ be a prime number. It was proved by Kratzer [Kra80, Corollary 5.5] that for every perfect $\mathbb{F}_p$-algebra $R$ and every integer $n \geq 1$, the $K$-group $\mathrm{K}_n(R)$ is uniquely $p$-divisible (see also [AMM22] for a mixed characteristic generalisation). It was also proved by Kelly–Morrow that for every $\mathbb{F}_p$-algebra $R$ with perfection $R_{\mathrm{perf}}$, the natural map $\mathrm{K}(R) \to \mathrm{K}(R_{\mathrm{perf}})$ is an equivalence after inverting $p$ ([KM21, Lemma 4.1], see also [EK20, Example 2.1.11] and [Cou23, Theorem 3.1.2 and Proposition 3.3.1] for different proofs). The following result is a motivic refinement of these two facts.

**Theorem 13.1.1** (Motivic cohomology of perfect $\mathbb{F}_p$-schemes, after [EM23])**.** *Let $X$ be a qcqs $\mathbb{F}_p$-scheme.*

*(1) For every integer $i \geq 0$, the natural map*

$$\mathbb{Z}(i)^{\mathrm{mot}}(X)[\tfrac{1}{p}] \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(X_{\mathrm{perf}})[\tfrac{1}{p}]$$

*is an equivalence in the derived category $\mathcal{D}(\mathbb{Z}[\frac{1}{p}])$.*

*(2) For every integer $i \geq 1$, the natural map*

$$\mathbb{Z}(i)^{\mathrm{mot}}(X_{\mathrm{perf}}) \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(X_{\mathrm{perf}})[\tfrac{1}{p}]$$

*is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$.*

*Proof.* By [EM23, Theorem 4.24 (5)],[1] for every integer $i \geq 0$, the natural map

$$\phi_X^* : \mathbb{Z}(i)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(X)$$

induced by the absolute Frobenius $\phi_X : X \to X$ of a qcqs $\mathbb{F}_p$-scheme $X$ is multiplication by $p^i$. In particular, this natural map is an equivalence after inverting $p$, and (1) is a consequence of

[1] This result is proved as a consequence of the same result in classical motivic cohomology [GL00] and in syntomic cohomology [AMMN22], and ultimately goes back to the fact that the Frobenius acts by multiplication by $p^i$ on the logarithmic de Rham–Witt sheaf $W\Omega^i_{\log}$.

this and the fact that the presheaf $\mathbb{Z}(i)^{\mathrm{mot}}$ is finitary ([EM23, Theorem 4.24 (4)]). Similarly, the same result applied to the perfect $\mathbb{F}_p$-scheme $X_{\mathrm{perf}}$ implies that multiplication by $p^i$ on the complex $\mathbb{Z}(i)^{\mathrm{mot}}(X_{\mathrm{perf}}) \in \mathcal{D}(\mathbb{Z})$ is an equivalence. If $i \geq 1$, this is equivalent to the fact that the natural map

$$\mathbb{Z}(i)^{\mathrm{mot}}(X_{\mathrm{perf}}) \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(X_{\mathrm{perf}})[\tfrac{1}{p}]$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$. □

**Remark 13.1.2** (Negative $K$-groups of perfect $\mathbb{F}_p$-algebras)**.** It is possible to construct examples of perfect $\mathbb{F}_p$-algebras whose negative $K$-groups are not $p$-divisible ([Cou23, Section 3.3]). Theorem 13.1.1 (2) states that the only non-$p$-divisible information in the negative $K$-groups of a perfect $\mathbb{F}_p$-algebra $R$ actually come from weight zero motivic cohomology, *i.e.*, from the complex $R\Gamma_{\mathrm{cdh}}(R, \mathbb{Z})$ (Example 5.6.8).

Recall that a $\mathbb{F}_p$-algebra is *semiperfect* if its Frobenius is surjective.

**Corollary 13.1.3** (Motivic cohomology of semiperfect $\mathbb{F}_p$-algebras)**.** *Let $S$ be a semiperfect $\mathbb{F}_p$-algebra. Then for every integer $i \geq 1$, the natural commutative diagram*

$$\begin{array}{ccc} \mathbb{Z}(i)^{\mathrm{mot}}(S) & \longrightarrow & \mathbb{Z}_p(i)^{\mathrm{syn}}(S) \\ \downarrow & & \downarrow \\ \mathbb{Z}(i)^{\mathrm{mot}}(S_{\mathrm{perf}}) & \longrightarrow & \mathbb{Z}_p(i)^{\mathrm{syn}}(S_{\mathrm{perf}}) \end{array}$$

*is a cartesian square in the derived category $\mathcal{D}(\mathbb{Z})$.*

*Proof.* It suffices to prove the result modulo $p$, and after inverting $p$. After inverting $p$, the vertical maps become equivalences by Theorem 13.1.1 (1) (and the same argument for syntomic cohomology). We prove now the result modulo $p$. By Theorem 13.1.1 (2) (and the same argument for syntomic cohomology), the bottom terms of the commutative diagram are zero modulo $p$, so it suffices to prove that the natural map

$$\mathbb{F}_p(i)^{\mathrm{mot}}(S) \longrightarrow \mathbb{F}_p(i)^{\mathrm{syn}}(S)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{F}_p)$. By [EM23, Corollary 4.32] (see also Theorem 6.2.4 for a mixed characteristic generalisation), this is equivalent to the fact that

$$R\Gamma_{\mathrm{cdh}}(S, \widetilde{\nu}(i))[-i-1] \simeq 0$$

in the derived category $\mathcal{D}(\mathbb{F}_p)$. By definition, the Frobenius map $\phi_S : S \to S$ is surjective, and has nilpotent kernel. The presheaf $R\Gamma_{\mathrm{cdh}}(-, \widetilde{\nu}(i))[-i-1]$ is a finitary cdh sheaf, so the natural map

$$R\Gamma_{\mathrm{cdh}}(S, \widetilde{\nu}(i))[-i-1] \longrightarrow R\Gamma_{\mathrm{cdh}}(S_{\mathrm{perf}}, \widetilde{\nu}(i))[-i-1]$$

is then an equivalence in the derived category $\mathcal{D}(\mathbb{F}_p)$. The target of this map is zero by Theorem 13.1.1 (2) (where we use that $i \geq 1$, and the same argument for syntomic cohomology), and applying [EM23, Corollary 4.32] to the perfect $\mathbb{F}_p$-algebra $S_{\mathrm{perf}}$. □

## 13.2 Finite chain rings

Finite chain rings are commutative rings $\mathcal{O}_K/\pi^n$, where $\mathcal{O}_K$ is a mixed characteristic discrete valuation ring with finite residue field, $\pi$ is a uniformizer of $\mathcal{O}_K$, and $n \geq 1$ is an integer. Examples of finite chain rings thus include finite fields, rings of the form $\mathbb{Z}/p^n$, and truncated polynomials over a finite field.

**Lemma 13.2.1.** *Let $\mathcal{O}_K$ be a discrete valuation ring of mixed characteristic $(0,p)$ and with finite residue field $\mathbb{F}_q$, $\pi$ be a uniformizer of $\mathcal{O}_K$, and $n \geq 1$ be an integer. Then for every integer $i \geq 0$, there is a natural equivalence*

$$\mathbb{Z}(i)^{\mathrm{mot}}(\mathcal{O}_K/\pi^n) \simeq \begin{cases} \mathbb{Z}[0] & \text{if } i = 0 \\ \mathbb{Z}_p(i)^{\mathrm{BMS}}(\mathcal{O}_K/\pi^n) \oplus \mathbb{Z}(i)^{\mathrm{mot}}(\mathbb{F}_q)[\frac{1}{p}] & \text{if } i \geq 1 \end{cases}$$

*in the derived category $\mathcal{D}(\mathbb{Z})$.*

*Proof.* The result for $i = 0$ follows from the equivalences

$$\mathbb{Z}(0)^{\mathrm{mot}}(\mathcal{O}_K/\pi^n) \simeq R\Gamma_{\mathrm{cdh}}(\mathcal{O}_K/\pi^n, \mathbb{Z}) \simeq R\Gamma_{\mathrm{cdh}}(\mathbb{F}_q, \mathbb{Z}) \simeq \mathbb{Z}[0]$$

in the derived category $\mathcal{D}(\mathbb{Z})$, the first equivalence being Example 5.6.8, the second equivalence being nilpotent invariance of cdh sheaves, and the last equivalence being a consequence of the fact that fields are local for the cdh topology.

For every integer $i \geq 0$, the commutative diagram

$$\begin{array}{ccc} \mathbb{Z}(i)^{\mathrm{mot}}(\mathcal{O}_K/\pi^n) & \longrightarrow & \mathbb{Z}_p(i)^{\mathrm{BMS}}(\mathcal{O}_K/\pi^n) \\ \downarrow & & \downarrow \\ \mathbb{Z}(i)^{\mathrm{mot}}(\mathbb{F}_q) & \longrightarrow & \mathbb{Z}_p(i)^{\mathrm{BMS}}(\mathbb{F}_q) \end{array}$$

is a cartesian square in the derived category $\mathcal{D}(\mathbb{Z})$ (Theorem 4.3.15). If $i \geq 1$, the bottom right term vanishes (use for instance the description of Bhatt–Morrow–Scholze's syntomic cohomology in characteristic $p$ in terms of logarithmic de Rham–Witt forms), and there is a natural equivalence

$$\mathbb{Z}(i)^{\mathrm{mot}}(\mathbb{F}_q) \xrightarrow{\sim} \mathbb{Z}(i)^{\mathrm{mot}}(\mathbb{F}_q)[\tfrac{1}{p}]$$

in the derived category $\mathcal{D}(\mathbb{Z})$ (by a classical result in motivic cohomology, see also Theorem 13.1.1 (2) for a more general statement), hence the desired result. ☐

**Proposition 13.2.2.** *Let $\mathcal{O}_K$ be a mixed characteristic discrete valuation ring with finite residue field, $\pi$ be a uniformizer of $\mathcal{O}_K$, and $n \geq 1$ be an integer. Then for every integer $m \in \mathbb{Z}$, there is a natural isomorphism*

$$\mathrm{K}_m(\mathcal{O}_K/\pi^n) \cong \begin{cases} \mathbb{Z} & \text{if } m = 0 \\ \mathrm{H}^1_{\mathrm{mot}}(\mathcal{O}_K/\pi^n, \mathbb{Z}(i)) & \text{if } m = 2i-1,\ i \geq 1 \\ \mathrm{H}^2_{\mathrm{mot}}(\mathcal{O}_K/\pi^n, \mathbb{Z}(i)) & \text{if } m = 2i-2,\ i \geq 2 \\ 0 & \text{if } m < 0 \end{cases}$$

*of abelian groups.*

*Proof.* Let $p$ be the residue characteristic of the discrete valuation ring $\mathcal{O}_K$. The result with $p$-adic coefficients is [AKN24, Corollary 2.16]. The result with $\mathbb{Z}[\frac{1}{p}]$-coefficients reduces to the case $n = 1$, where the result follows from the description of the (classical) motivic cohomology of finite fields. The integral result is then a consequence of Lemma 13.2.1. ☐

**Theorem 13.2.3** (Motivic cohomology of finite chain rings, after [AKN24]). *Let $\mathcal{O}_K$ be a discrete valuation ring of mixed characteristic $(0,p)$ and with finite residue field $\mathbb{F}_q$, $\pi$ be a uniformizer of $\mathcal{O}_K$, and $n \geq 1$ be an integer. Then for every integer $i \geq 4p^n$,*[2] *the motivic complex*

$$\mathbb{Z}(i)^{\mathrm{mot}}(\mathcal{O}_K/\pi^n) \in \mathcal{D}(\mathbb{Z})$$

*is concentrated in degree one, where it is given by a group of order $(q^i-1)q^{i(n-1)}$.*

*Proof.* This is a consequence of Lemma 13.2.1, the classical computation of the motivic cohomology of $\mathbb{F}_q$, and [AKN24, Theorem 1.4 and Proposition 1.5]. □

**Remark 13.2.4** (Nilpotence of $v_1$). Antieau–Krause–Nikolaus also determine the nilpotence degree of the element $v_1$ in the mod $p$ syntomic cohomology of $\mathbb{Z}/p^n$ ([AKN24, Theorem 1.8]). This is a refinement of the key result in the study of $K(1)$-local $K$-theory of Bhatt–Clausen–Mathew [BCM20]. Note that this result on the nilpotence degree of $v_1$ can be reformulated, via Lemma 13.2.1, as a statement on the mod $p$ motivic cohomology of $\mathbb{Z}/p^n$.

## 13.3 $\mathbb{C}^\star$-algebras

By Gelfand representation theorem, the commutative $\mathbb{C}^\star$-algebras are exactly the algebras of continuous complex-valued functions $\mathscr{C}(X;\mathbb{C})$ on a compact Hausdorff space $X$. An important theorem of Cortiñas–Thom states that commutative $\mathbb{C}^\star$-algebras are $K$-regular ([CT12, Theorem 1.5]). This result was further generalised recently by Aoki to all smooth algebras over commutative $\mathbb{C}^\star$-algebras, and over a general local field ([Aok24, Theorem 8.7]). The following result is a motivic analogue of the latter result.

**Theorem 13.3.1** ($\mathbb{C}^\star$-algebras are motivically regular, after [CT12, Aok24]). *Let $X$ be a compact Hausdorff space, $F$ be a characteristic zero local field, and $A$ be a smooth $\mathscr{C}(X;F)$-algebra. Then for any integers $i \geq 0$ and $n \geq 0$, the natural map*

$$\mathbb{Z}(i)^{\mathrm{mot}}(A) \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(A[T_1,\dots,T_n])$$

*is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$.*

*Proof.* By [Aok24, Theorem 8.7 (2)], the natural map

$$\mathrm{K}(A[T_1,\dots,T_n]) \longrightarrow \mathrm{KH}(A[T_1,\dots,T_n])$$

is an equivalence of spectra for every integer $n \geq 0$. By Corollary 12.3.3, this implies that the vertical maps in the commutative diagram

$$\begin{array}{ccc} \mathbb{Z}(i)^{\mathrm{mot}}(A) & \longrightarrow & \mathbb{Z}(i)^{\mathrm{mot}}(A[T_1,\dots,T_n]) \\ \downarrow & & \downarrow \\ \mathbb{Z}(i)^{\mathbb{A}^1}(A) & \longrightarrow & \mathbb{Z}(i)^{\mathbb{A}^1}(A[T_1,\dots,T_n]) \end{array}$$

are equivalences in the derived category $\mathcal{D}(\mathbb{Z})$. The bottom horizontal map is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$ by definition of the presheaf $\mathbb{Z}(i)^{\mathbb{A}^1}$ ([BEM24], see also Chapter 12). So the top horizontal map is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$. □

[2]Note that this is not an optimal lower bound on the integer $i$. See [AKN24, Theorem 1.4] for a more precise result, in terms of the ramification index of $\mathcal{O}_K$.

## 13.4 Truncated polynomials

In this section, we study the motivic cohomology of truncated polynomials, *i.e.*, the motivic cohomology of commutative rings of the form $R[x]/(x^e)$. Given a $\mathcal{D}(\mathbb{Z})$-valued functor $F(-)$, a commutative ring $R$, and an integer $e \geq 1$, we use the notation

$$F(R[x]/(x^e),(x)) := \mathrm{fib}(F(R[x]/(x^e)) \longrightarrow F(R)),$$

where the map is induced by the canonical projection $R[x]/(x^e) \to R$.

The relative $K$-theory $\mathrm{K}(k[x]/(x^e),(x))$ of truncated polynomials over a perfect field $k$ of positive characteristic was computed by Hesselholt–Madsen [HM97b, HM97a], using topological restriction homology. Their calculation was reproved by Speirs [Spe20] using Nikolaus–Scholze's approach to topological cyclic homology [NS18], and by Mathew [Mat22] and Sulyma [Sul23] using Bhatt–Morrow–Scholze's filtration on topological cyclic homology [BMS19]. This last approach was then extended to mixed characteristic by Riggenbach [Rig22]. More precisely, Riggenbach used computations in prismatic cohomology to extend the previous result to a computation of the $p$-adic relative $K$-theory $\mathrm{K}(R[x]/(x^e),(x);\mathbb{Z}_p)$ of perfectoid rings $R$, and also reproved the $p$-adic part of the known description of $\mathrm{K}(\mathbb{Z}[x]/(x^e),(x))$, originally due to Angeltveit–Gerhardt–Hesselholt [AGH09].

This recent progress would seem to indicate that $K$-theory calculations using equivariant stable homotopy may be pushed further by using cohomological techniques. Note however that the calculations in [Mat22, Sul23, Rig22] are purely $p$-adic ones, as they rely on (instances of) prismatic cohomology. In fact, all of the previous integral calculations in mixed characteristic (*i.e.*, for $R$ the ring of integers of a number field) rely on a rational result of Soulé [Sou81] and Staffeldt [Sta85], who compute the ranks of the associated relative $K$-groups using equivariant homotopy theory. In this section, we revisit and extend this rational computation, and discuss some natural motivic refinements of the previous results.

All of the above calculations use trace methods, via the Dundas–Goodwillie–McCarthy theorem. We first state the corresponding results at the level of cohomology theories.

**Lemma 13.4.1.** *Let $R$ be a commutative ring, and $e \geq 1$ be an integer. Then for every integer $i \geq 0$, the natural map*

$$\mathbb{Z}(i)^{\mathrm{mot}}(R[x]/(x^e),(x)) \longrightarrow \mathbb{Z}(i)^{\mathrm{TC}}(R[x]/(x^e),(x))$$

*is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$.*

*Proof.* This is a direct consequence of Remark 4.3.7, and the fact that cdh sheaves are invariant under nilpotent extensions. □

**Corollary 13.4.2.** *Let $R$ be a commutative ring, $e \geq 1$ be an integer, and $p$ be a prime number. Then for every integer $i \geq 0$, the natural map*

$$\mathbb{Z}_p(i)^{\mathrm{mot}}(R[x]/(x^e),(x)) \longrightarrow \mathbb{Z}_p(i)^{\mathrm{BMS}}(R[x]/(x^e),(x))$$

*is an equivalence in the derived category $\mathcal{D}(\mathbb{Z}_p)$.*

*Proof.* This is a consequence of Lemma 13.4.1. □

**Corollary 13.4.3.** *Let $R$ be a commutative ring, and $e \geq 1$ be an integer. Then for every integer $i \geq 0$, there is a natural equivalence*

$$\mathbb{Q}(i)^{\mathrm{mot}}(R[x]/(x^e),(x)) \simeq \mathbb{L}\Omega^{<i}_{(R[x]/(x^e),(x))_{\mathbb{Q}}/\mathbb{Q}}[-1]$$

*in the derived category $\mathcal{D}(\mathbb{Q})$.*

*Proof.* This is a consequence of Lemma 13.4.1 and cdh descent for the presheaf $\widehat{\mathbb{L}\Omega}_{-/\mathbb{Q}}$ on commutative $\mathbb{Q}$-algebras ([EM23, Lemma 4.5]). □

**Lemma 13.4.4.** *For every commutative ring $R$ and integer $e \geq 1$, the object*

$$\mathbb{Z}(0)^{\mathrm{mot}}\big(R[x]/(x^e),(x)\big)$$

*is zero in the derived category $\mathcal{D}(\mathbb{Z})$.*

*Proof.* This is a consequence of the fact that the motivic complex $\mathbb{Z}(0)^{\mathrm{mot}}$ is a cdh sheaf (Example 5.6.8). □

**Lemma 13.4.5.** *For any integers $e \geq 1$ and $i \geq 0$, the complex*

$$\mathbb{L}\Omega^{\leq i}_{(\mathbb{Q}[x]/(x^e),(x))/\mathbb{Q}} \in \mathcal{D}(\mathbb{Q})$$

*is concentrated in degree zero, given by a $\mathbb{Q}$-vector space of dimension $e-1$.*

*Proof.* This follows from a standard argument using the natural grading of the $\mathbb{Q}$-algebra $\mathbb{Q}[x]/(x^e)$ and the $\mathbb{Q}$-linear derivation $d : \mathbb{Q}[x]/(x^e) \to \mathbb{Q}[x]/(x^e)$ given by $d(x^j) = jx^j$; see for instance the proof of [Sta85, Proposition 5]. □

**Theorem 13.4.6.** *Let $R$ be a commutative ring such that the cotangent complex $\mathbb{L}_{(R\otimes_{\mathbb{Z}}\mathbb{Q})/\mathbb{Q}}$ vanishes (e.g., if $R \otimes_{\mathbb{Z}} \mathbb{Q}$ is ind-étale over $\mathbb{Q}$),*[3] *and $e \geq 1$ be an integer. Then for every integer $i \geq 1$, there is a natural equivalence*

$$\mathbb{Q}(i)^{\mathrm{mot}}(R[x]/(x^e),(x)) \simeq (R \otimes_{\mathbb{Z}} \mathbb{Q})^{e-1}[-1]$$

*in the derived category $\mathcal{D}(\mathbb{Q})$.*

*Proof.* By Corollary 13.4.3, there is a natural equivalence

$$\mathbb{Q}(i)^{\mathrm{mot}}(R[x]/(x^e),(x)) \simeq \mathbb{L}\Omega^{<i}_{(R[x]/(x^e),(x))_{\mathbb{Q}}/\mathbb{Q}}[-1]$$

in the derived category $\mathcal{D}(\mathbb{Q})$. By the Künneth formula for derived de Rham cohomology, and because all the positive powers of the cotangent complex $\mathbb{L}_{(R\otimes_{\mathbb{Z}}\mathbb{Q})/\mathbb{Q}}$ vanish, there is a natural equivalence

$$\mathbb{L}\Omega^{<i}_{(R[x]/(x^e),(x))_{\mathbb{Q}}/\mathbb{Q}} \simeq \mathbb{L}\Omega^{<i}_{(\mathbb{Q}[x]/(x^e),(x))/\mathbb{Q}} \otimes_{\mathbb{Q}} R$$

in the derived category $\mathcal{D}(\mathbb{Q})$. The result is then a consequence of Lemma 13.4.5. □

When $R$ is the ring of integers of a number field, the following result is due to Soulé [Sou81] when $e = 2$, and to Staffeldt [Sta85] for $e \geq 2$ a general integer. Their proof uses rational homotopy theory, and ultimately reduces to a computation in cyclic homology.

**Corollary 13.4.7.** *Let $R$ be a commutative ring such that the cotangent complex $\mathbb{L}_{(R\otimes_{\mathbb{Z}}\mathbb{Q})/\mathbb{Q}}$ vanishes, and $e \geq 1$ be an integer. Then for every integer $n \in \mathbb{Z}$, there is a natural isomorphism*

$$\mathrm{K}_n(R[x]/(x^e),(x);\mathbb{Q}) \cong \begin{cases} (R \otimes_{\mathbb{Z}} \mathbb{Q})^{e-1} & \text{if } n \text{ is odd and } n \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

*of abelian groups.*

[3] See [MM22] for more on this condition.

*Proof.* This is a consequence of Theorem 13.4.6 and Corollary 5.5.11. □

**Remark 13.4.8.** Let $K$ be a number field, $\mathcal{O}_K$ be its ring of integers, and $e \geq 1$ be an integer. The orders in the torsion part of the relative $K$-theory $\mathrm{K}(\mathcal{O}_K[x]/(x^e), (x))$ were completely determined in [Rig22, Remark 1.8]. It would be interesting to use this result and Theorem 13.4.6 to obtain an integral description of the relative motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}(\mathcal{O}_K[x]/(x^e), (x))$ for all $i \geq 0$. This would in particular reprove and generalise the result for $K = \mathbb{Q}$ of Angeltveit–Gerhardt–Hesselholt [AGH09].

We also deduce from the work of Riggenbach the following motivic interpretation of the analogous result in $K$-theory ([Rig22, Theorem 1.1]).

**Theorem 13.4.9** (Truncated polynomials over perfectoids, after [Rig22])**.** *Let $R$ be a perfectoid ring, and $e \geq 1$ be an integer. Then for every integer $i \geq 1$, there is a natural equivalence*

$$\mathbb{Z}_p(i)^{\mathrm{mot}}(R[x]/(x^e), (x)) \simeq \mathbb{W}_{ei}(R)/V_e\mathbb{W}_i(R)[-1]$$

*in the derived category $\mathcal{D}(\mathbb{Z}_p)$, where $\mathbb{W}(R)$ denotes the big Witt vectors of $R$, and $V$ the associated Verschiebung operator.*

*Proof.* This is a consequence of [Rig22, proof of Corollary 6.5] and Corollary 13.4.2. □

**Remark 13.4.10** (Cuspidal curves)**.** The algebraic $K$-theory of cuspidal curves (*i.e.*, curves that are defined by an equation of the form $y^a - x^b$, for $a, b \geq 2$ coprime integers) was completely determined over a perfect $\mathbb{F}_p$-algebra by Hesselholt–Nikolaus [HN20], using Nikolaus–Scholze's approach [NS18] to topological cyclic homology. This result was then generalised to mixed characteristic perfectoid rings by Riggenbach [Rig23], ultimately relying on computations in relative topological Hochschild homology. It would seem that the associated Atiyah–Hirzebruch spectral sequence should degenerate in this context, thus providing a similar computation of the motivic cohomology of cuspidal curves. An interesting question would be whether these results can be reproved, or even extended to more general base rings, using techniques from prismatic cohomology and derived de Rham cohomology.

# Notation

## Algebraic $K$-theory

By default, algebraic $K$-theory means non-connective algebraic $K$-theory, as introduced by Thomason–Trobaugh [TT90]. By a theorem of Blumberg–Gepner–Tabuada [BGT13], non-connective algebraic $K$-theory is the universal localizing invariant.

## $\mathbb{A}^1$-invariance

A presheaf $F(-)$ on schemes is called $\mathbb{A}^1$-*invariant* if for every scheme $X$ and every integer $m \geq 0$, the natural map $F(X) \to F(\mathbb{A}^m_X)$ is an equivalence. Given a presheaf $F(-)$ on schemes, the $\mathbb{A}^1$-*localisation* $L_{\mathbb{A}^1}F(-)$ of $F(-)$ is the initial $\mathbb{A}^1$-invariant presheaf with a map from $F(-)$. The $\mathbb{A}^1$-localisation functor $L_{\mathbb{A}^1}$ commutes with colimits.

## Base change

Given a commutative ring $R$, an $R$-algebra $S$, and a scheme $X$ over $\mathrm{Spec}(R)$, denote by $X_S$ the base change $X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(S)$ of $X$ from $R$ to $S$. If $X$ is a derived scheme, this base change is implicitly the derived base change from $R$ to $S$. We sometimes use the derived base even on classical schemes, and say explicitly when we do so.

## Bounded torsion

An abelian group $A$ is said to have *bounded torsion* if there exists an integer $N \geq 1$ such that the multiplication by $N$ of every element of $A$ is zero.

Given a commutative ring $R$ and an element $d$ of $R$, an $R$-module $M$ is said to have *bounded $d$-power torsion* if there exists an integer $n \geq 1$ such that $M[d^m] = M[d^n]$ for all $m \geq n$; this assumption guarantees that the derived $d$-completion of $M$ is in degree zero, given by the classical $d$-completion of $M$.

## Cdh topology

The cdh topology is a Grothendieck topology introduced by Voevodsky [SV00, Voe10]; see [EHIK21] for the definition and properties of the cdh topology in the generality of qcqs schemes. It is a completely decomposed version of the topology generated by Deligne's hypercoverings. The cdh sheafification functor $L_{\mathrm{cdh}}$ preserves multiplicative structures.

## Coefficients

Given a functor $F(-)$ (resp. $\mathrm{Fil}^\star F(-)$) taking values in spectra (resp. filtered spectra) and a prime number $p$, we denote the rationalisation of $F(-)$ by $F(-;\mathbb{Q})$, its reduction modulo $p$ by $F(-;\mathbb{F}_p)$, its derived reduction modulo powers of $p$ by $F(-;\mathbb{Z}/p^k)$, its $p$-completion by

$F(-;\mathbb{Z}_p)$, and the rationalisation of its $p$-completion by $F(-;\mathbb{Q}_p)$. We adopt a similar notation for a functor $\mathrm{Fil}^\star F(-)$ taking values in filtered spectra, *e.g.*, we denote its rationalisation by $\mathrm{Fil}^\star F(-;\mathbb{Q})$. Similarly, if $\mathbb{Z}(i)^F(-)$ is a functor taking values in the derived category $\mathcal{D}(\mathbb{Z})$, we denote the rationalisation of $\mathbb{Z}(i)^F(-)$ by $\mathbb{Q}(i)^F(-)$, its derived reduction modulo $p$ by $\mathbb{F}_p(i)^F(-)$, etc. Following the same notation, we also write

$$\prod_{p\in\mathbb{P}}{}' F(-;\mathbb{Q}_p) := \Big(\prod_{p\in\mathbb{P}} F(-;\mathbb{Z}_p)\Big)_{\mathbb{Q}} \quad \Big(\text{resp.} \prod_{p\in\mathbb{P}}{}' \mathrm{Fil}^\star F(-;\mathbb{Q}_p) := \Big(\prod_{p\in\mathbb{P}} \mathrm{Fil}^\star F(-;\mathbb{Z}_p)\Big)_{\mathbb{Q}}\Big).$$

### Derived categories and spectra

Denote by Sp the category of spectra. Given a commutative ring $R$, denote by $\mathcal{D}(R)$ the derived category of $R$-modules; it is implicitly the derived $\infty$-category of $R$-modules, and is in particular naturally identified with the category of $R$-linear spectra. Our convention for degrees is by default cohomological. In this context, the notions of fibre and cofibre sequences agree, and the fibre and cofibre of a given map satisfy the relation $\mathrm{fib} \simeq \mathrm{cofib}[-1]$. Given an element $d$ of $R$, also denote by $(-)^\wedge_d$ the $d$-adic completion functor in the derived category $\mathcal{D}(R)$.

### Extension by zero $j_!$

Given a prime number $p$ (which is typically clear from context) and a scheme $X$, denote by $j : X[\tfrac{1}{p}] \to X$ the open immersion of the $p$-adic generic fibre of $X$, and by $j_! : (X[\tfrac{1}{p}])_{\mathrm{\acute{e}t}} \to X_{\mathrm{\acute{e}t}}$ the associated extension by zero functor.

### Filtrations

By default, a filtration with values in a category $\mathcal{C}$ is a $\mathbb{Z}$-indexed decreasing filtered object in the category $\mathcal{C}$, *i.e.*, a functor from the category $(\mathbb{Z},\geq)^{\mathrm{op}}$ to the category $\mathcal{C}$. A filtration is called $\mathbb{N}$-*indexed* if it is constant in non-positive degrees. Given a filtered object $\mathrm{Fil}^\star C$ and for each integer $n \in \mathbb{Z}$, let $\mathrm{gr}^n C \in \mathcal{D}(R)$ denote the cofibre of the transition map $\mathrm{Fil}^{n+1} C \to \mathrm{Fil}^n C$. A filtered object $\mathrm{Fil}^\star C$ is said to be *complete* if the limit $\lim_n \mathrm{Fil}^n C$ vanishes. For instance, The Hodge filtration on the de Rham complex is given for each $n \in \mathbb{Z}$ by $\mathrm{Fil}^n_{\mathrm{Hod}} \Omega_{-/R} := \Omega^{\geq n}_{-/R}$; the Hodge filtration $\mathbb{L}\Omega^{\geq\star}_{-/R}$ on the derived de Rham complex $\mathbb{L}\Omega_{-/R}$ is defined as the left Kan extension of this filtration. It is $\mathbb{N}$-indexed, but not always complete. Its completion, the Hodge-completed derived de Rham complex, is denoted by $\widehat{\mathbb{L}\Omega}^{\geq\star}_{-/R}$.

Given a commutative ring $R$, denote by $\mathcal{DF}(R) := \mathrm{Fun}((\mathbb{Z},\geq)^{\mathrm{op}}, \mathcal{D}(R))$ the filtered derived category of $R$-modules. Also denote by FilSp the category of filtered spectra, and by biFilSp the category of bifiltered spectra (*i.e.*, the category of filtered objects in the category of filtered spectra).

### Henselian rings

Given a commutative ring $R$ and an ideal $I$ of $R$, the pair $(R,I)$ is called *henselian* if it satisfies Hensel's lemma. A local ring $R$ is called *henselian* if the pair $(R,\mathfrak{m})$ is henselian, where $\mathfrak{m}$ is the maximal ideal of $R$. Henselian local rings are the local rings for the Nisnevich topology. A commutative ring $R$ is called $d$-*henselian*, for $d$ an element of $R$, if the pair $(R,(d))$ is henselian.

## Ind-smooth schemes

Given a scheme $S$, a scheme $X$ is called ind-smooth (resp. ind-regular, ind-étale) over $S$ if it is a cofiltered limit of smooth (resp. regular, étale) $S$-schemes.

## Left Kan extensions

Given a commutative ring $R$, an $\infty$-category $\mathcal{D}$ which admits sifted colimits (*e.g.*, $\mathcal{D}(R)$ or $\mathcal{DF}(R)$), and a functor

$$F : \mathrm{Sm}_R := \{\text{smooth } R\text{-algebras}\} \longrightarrow \mathcal{D},$$

define

$$\begin{aligned} \mathbb{L}F : \quad & R\text{-Alg} \longrightarrow \mathcal{D} \\ & S \longmapsto \operatorname*{colim}_{P \to S} F(P), \end{aligned}$$

where the colimit is taken over all free $R$-algebras $P$ mapping to $S$. The functor $\mathbb{L}F$ is called the left Kan extension from polynomial $R$-algebras of $F$. For instance, the cotangent complex $\mathbb{L}_{-/R} := \mathbb{L}\Omega^1_{-/R}$ is the left Kan extension from polynomial $R$-algebras of the module of Kähler differentials $\Omega^1_{-/R}$, and the derived de Rham complex $\mathbb{L}\Omega_{-/R}$ is the left Kan extension from polynomial $R$-algebras of the de Rham complex $\Omega_{-/R}$. We also consider more general left Kan extensions (*e.g.*, from smooth $R$-algebras), which are defined similarly –see [EM23, Section 2.3 and Remark 3.4] for a quick review of this formalism. The left Kan extension from a category $\mathcal{C}$ to a category $\mathcal{C}'$, when this makes sense, is denoted by $L_{\mathcal{C}'/\mathcal{C}}$.

## Quasisyntomic rings

A morphism $R \to S$ of commutative rings is called *$p$-discrete*, for $p$ a prime number, if the derived tensor product $S \otimes^{\mathbb{L}}_R R/p \in \mathcal{D}(R/p)$ is concentrated in degree zero, where it is given by $S/p$. It is called *$p$-flat* if it is $p$-discrete and if its reduction $R/p \to S/p$ modulo $p$ is flat. It is called *$p$-quasisyntomic* if it is $p$-flat and if the cotangent complex $\mathbb{L}_{(S/p)/(R/p)} \in \mathcal{D}(S/p)$ has Tor-amplitude in $[-1;0]$.

A commutative ring $R$ is called *$p$-quasisyntomic* if it has bounded $p$-power torsion and if the complex $\mathbb{L}_{R/\mathbb{Z}} \otimes^{\mathbb{L}}_R R/p \in \mathcal{D}(R/p)$ has Tor-amplitude in $[-1;0]$. Beware that $p$-quasisyntomic $\mathbb{Z}$-algebras are $p$-quasisyntomic rings, but the converse is not true: for instance, $\mathbb{F}_p$ is a $p$-quasisyntomic ring, but the morphism $\mathbb{Z} \to \mathbb{F}_p$ is not $p$-discrete. We refer to [BMS19] for the definition of the associated $p$-quasisyntomic topology on $p$-quasisyntomic rings.

## Rigid functor

A functor $F(-)$ on commutative rings is called rigid if for every henselian pair $(R, I)$, the natural map $F(R) \to F(R/I)$ is an equivalence.

## Rings and schemes

Quasi-compact quasi-separated (derived) schemes are called qcqs (derived) schemes. These include all affine (derived) schemes, *i.e.*, (animated) commutative rings. Denote by $\mathrm{Sch}^{\mathrm{qcqs}}$ the category of qcqs schemes, $\mathrm{dSch}^{\mathrm{qcqs}}$ the category of qcqs derived schemes, Rings the category of commutative rings, AniRings the category of animated commutative rings. Schemes, resp. commutative rings, are sometimes called *classical* to emphasize that we are not working in the generality of derived schemes, resp. animated commutative rings. Given a commutative base ring $R$, also denote by $\mathrm{Sm}_R$ the category of smooth schemes over $\mathrm{Spec}(R)$, $\mathrm{Sch}^{\mathrm{fp}}$ the category of finitely presented schemes over $\mathrm{Spec}(R)$, $\mathrm{Poly}_R$ the category of polynomial $R$-algebras, and $\mathbb{E}_1\text{-}\mathrm{Rings}_R$ the category of associative $R$-linear ring spectra.

### Sheafification

We use several Grothendieck topologies, including the Zariski, Nisnevich, étale, and cdh topologies. Denote by $L_{\mathrm{Zar}}$, $L_{\mathrm{Nis}}$, $L_{\text{ét}}$, and $L_{\mathrm{cdh}}$ the sheafification functors for these topologies.

# Bibliography


[AD21] Benjamin Antieau and Rankeya Datta. Valuation rings are derived splinters. *Math. Z.*, 299(1-2):827–851, 2021.

[AGH09] Vigleik Angeltveit, Teena Gerhardt, and Lars Hesselholt. On the $K$-theory of truncated polynomial algebras over the integers. *J. Topol.*, 2(2):277–294, 2009.

[AGT16] Ahmed Abbes, Michel Gros, and Takeshi Tsuji. *The p-adic Simpson correspondence*, volume 193 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2016.

[AHI23] Toni Annala, Marc Hoyois, and Ryomei Iwasa. Algebraic cobordism and a Conner–Floyd isomorphism for algebraic $K$-theory. *J. Amer. Math. Soc.*, 2023. To appear.

[AHI24] Toni Annala, Marc Hoyois, and Ryomei Iwasa. Atiyah duality for motivic spectra. `https://arxiv.org/abs/2403.01561`, 2024.

[AI23] Toni Annala and Ryomei Iwasa. Motivic spectra and universality of $K$-theory. `https://arxiv.org/abs/2204.03434`, 2023.

[AKN24] Benjamin Antieau, Achim Krause, and Thomas Nikolaus. On the $K$-theory of $\mathbb{Z}/p^n$. `https://arxiv.org/abs/2405.04329`, 2024.

[AMM22] Benjamin Antieau, Akhil Mathew, and Matthew Morrow. The $K$-theory of perfectoid rings. *Doc. Math.*, 27:1923–1952, 2022.

[AMMN22] Benjamin Antieau, Akhil Mathew, Matthew Morrow, and Thomas Nikolaus. On the Beilinson fiber square. *Duke Math. J.*, 2022. To appear.

[Ant19] Benjamin Antieau. Periodic cyclic homology and derived de Rham cohomology. *Ann. K-Theory*, 4(3):505–519, 2019.

[Aok24] Ko Aoki. $K$-theory of rings of continuous functions. `https://arxiv.org/abs/2402.05257`, 2024.

[Avr99] Luchezar L. Avramov. Locally complete intersection homomorphisms and a conjecture of Quillen on the vanishing of cotangent homology. *Ann. of Math.*, 150(2):455–487, 1999.

[Bac22] Tom Bachmann. The very effective covers over KO and KGL over Dedekind domains. *J. Eur. Math. Soc.*, 2022. To appear.

[Bal23] Konrad Bals. Periodic cyclic homology over $\mathbb{Q}$. https://arxiv.org/abs/2301.03112, 2023.

[BCM20] Bhargav Bhatt, Dustin Clausen, and Akhil Mathew. Remarks on $K(1)$-local $K$-theory. *Selecta Math. (N.S.)*, 26(3):Paper No. 39, 16, 2020.

[Beĭ86] Alexander A. Beĭlinson. Notes on absolute Hodge cohomology. In *Applications of algebraic $K$-theory to algebraic geometry and number theory, Part I, II (Boulder, Colo., 1983)*, volume 55 of *Contemp. Math.*, pages 35–68. Amer. Math. Soc., Providence, RI, 1986.

[Beĭ87] Alexander A. Beĭlinson. Height pairing between algebraic cycles. In *Current trends in arithmetical algebraic geometry (Arcata, Calif., 1985)*, volume 67 of *Contemp. Math.*, pages 1–24. Amer. Math. Soc., Providence, RI, 1987.

[BEM24] Tom Bachmann, Elden Elmanto, and Matthew Morrow. $\mathbb{A}^1$-invariant motivic cohomology of schemes. In preparation, 2024.

[BGT13] Andrew J. Blumberg, David Gepner, and Goncalo Tabuada. A universal characterization of higher algebraic $K$-theory. *Geom. Topol.*, 17(2):733–838, 2013.

[BH21] Tom Bachmann and Marc Hoyois. Norms in motivic homotopy theory. *Astérisque*, (425):ix+207, 2021.

[Bha12] Bhargav Bhatt. $p$-adic derived de Rham cohomology. https://arxiv.org/abs/1204.6560, 2012.

[Bha13] Bhargav Bhatt. An imperfect ring with trivial cotangent complex. https://www.math.ias.edu/~bhatt/math/trivial-cc.pdf, 2013.

[Bha18] Bhargav Bhatt. Specializing varieties and their cohomology from characteristic 0 to characteristic $p$. *Algebraic Geometry: Salt Lake City 2015*, 2018.

[Bha19] Bhargav Bhatt. Torsion completions are bounded. *J. Pure Appl. Algebra*, 223(5):1940–1945, 2019.

[BL22] Bhargav Bhatt and Jacob Lurie. Absolute prismatic cohomology. https://arxiv.org/abs/2201.06120, 2022.

[Blo86] Spencer Bloch. Algebraic cycles and higher $K$-theory. *Adv. in Math.*, 61(3):267–304, 1986.

[BM21] Bhargav Bhatt and Akhil Mathew. The arc topology. *Duke Math. J.*, 170(9):1899–1988, 2021.

[BM23] Bhargav Bhatt and Akhil Mathew. Syntomic complexes and $p$-adic étale Tate twists. *Forum Math. Pi*, 11:Paper No. e1, 26, 2023.

[BMS87] Alexander Beĭlinson, Robert MacPherson, and Vadim Schechtman. Notes on motivic cohomology. *Duke Math. J.*, 54(2):679–710, 1987.

[BMS18] Bhargav Bhatt, Matthew Morrow, and Peter Scholze. Integral $p$-adic Hodge theory. *Publ. Math. Inst. Hautes Études Sci.*, 128:219–397, 2018.

[BMS19] Bhargav Bhatt, Matthew Morrow, and Peter Scholze. Topological Hochschild homology and integral $p$-adic Hodge theory. *Publ. Math. Inst. Hautes Études Sci.*, 129:199–310, 2019.

[Bou23] Tess Bouis. Cartier smoothness in prismatic cohomology. *J. Reine Angew. Math.*, 805:241–282, 2023.

[BS17] Bhargav Bhatt and Peter Scholze. Projectivity of the Witt vector affine Grassmannian. *Invent. Math.*, 209(2):329–423, 2017.

[BS22] Bhargav Bhatt and Peter Scholze. Prisms and prismatic cohomology. *Ann. of Math. (2)*, 196(3):1135–1275, 2022.

[CHSW08] Guillermo Cortiñas, Christian Haesemeyer, Marco Schlichting, and Charles Weibel. Cyclic homology, cdh-cohomology and negative $K$-theory. *Ann. of Math.*, 167:549–573, 2008.

[CHW08] Guillermo Cortiñas, Christian Haesemeyer, and Charles Weibel. $K$-regularity, cdh-fibrant Hochschild homology, and a conjecture of Vorst. *J. Amer. Math. Soc.*, 21(2):547–561, 2008.

[CM21] Dustin Clausen and Akhil Mathew. Hyperdescent and étale $K$-theory. *Invent. Math.*, 225(3):981–1076, 2021.

[CMNN20] Dustin Clausen, Akhil Mathew, Niko Naumann, and Justin Noel. Descent in algebraic $K$-theory and a conjecture of Ausoni-Rognes. *J. Eur. Math. Soc. (JEMS)*, 22(4):1149–1200, 2020.

[Cou23] Kevin Coulembier. $K$-theory and perfection. `https://arxiv.org/abs/2304.01421`, 2023.

[CS19] Dustin Clausen and Peter Scholze. Lectures on condensed mathematics. `https://www.math.uni-bonn.de/people/scholze/Condensed.pdf`, 2019.

[CS21] Dustin Clausen and Peter Scholze. Witt vectors, the cotangent complex, and a solid construction of $B_{dR}^+$. `https://mathoverflow.net/questions/407484/witt-vectors-the-cotangent-complex-and-a-solid-construction-of-b-dr`, 2021.

[CT12] Guillermo Cortiñas and Andreas Thom. Algebraic geometry of topological spaces I. *Acta Math.*, 209(1):83–131, 2012.

[CTHK97] Jean-Louis Colliot-Thélène, Raymond T. Hoobler, and Bruno Kahn. The Bloch-Ogus-Gabber theorem. In *Algebraic $K$-theory (Toronto, ON, 1996)*, volume 16 of *Fields Inst. Commun.*, pages 31–94. Amer. Math. Soc., Providence, RI, 1997.

[DGM13] Bjørn Ian Dundas, Thomas G. Goodwillie, and Randy McCarthy. *The local structure of algebraic $K$-theory*, volume 18 of *Algebra and Applications*. Springer-Verlag London, Ltd., London, 2013.

[DI87] Pierre Deligne and Luc Illusie. Relèvements modulo $p^2$ et décomposition du complexe de de Rham. *Invent. Math.*, 89(2):247–270, 1987.

[Dri06] Vladimir Drinfeld. Infinite-dimensional vector bundles in algebraic geometry: an introduction. In *The unity of mathematics*, volume 244 of *Progr. Math.*, pages 263–304. Birkhäuser Boston, Boston, MA, 2006.

[DS75] R. Keith Dennis and Michael R. Stein. $K_2$ of discrete valuation rings. *Advances in Math.*, 18(2):182–238, 1975.

[EHIK21] Elden Elmanto, Marc Hoyois, Ryomei Iwasa, and Shane Kelly. Cdh descent, cdarc descent, and Milnor excision. *Math. Ann.*, 379(3-4):1011–1045, 2021.

[EHK$^+$20] Elden Elmanto, Marc Hoyois, Adeel A. Khan, Vladimir Sosnilo, and Maria Yakerson. Modules over algebraic cobordism. *Forum Math. Pi*, 8:e14, 44, 2020.

[EK20] Elden Elmanto and Adeel A. Khan. Perfection in motivic homotopy theory. *Proc. Lond. Math. Soc. (3)*, 120(1):28–38, 2020.

[Elm21] Elden Elmanto. THH and TC are (very) far from being homotopy functors. *J. Pure Appl. Algebra*, 225(8):Paper No. 106640, 12, 2021.

[EM23] Elden Elmanto and Matthew Morrow. Motivic cohomology of equicharacteristic schemes. `https://arxiv.org/abs/2309.08463`, 2023.

[Gab94] Ofer Gabber. Affine analog of the proper base change theorem. *Israel J. Math.*, 87(1-3):325–335, 1994.

[Gei04] Thomas Geisser. Motivic cohomology over Dedekind rings. *Math. Z.*, 248(4):773–794, 2004.

[GL87] Henri Gillet and Marc Levine. The relative form of Gersten's conjecture over a discrete valuation ring: the smooth case. *J. Pure Appl. Algebra*, 46(1):59–71, 1987.

[GL00] Thomas Geisser and Marc Levine. The $K$-theory of fields in characteristic $p$. *Invent. Math.*, 139(3):459–493, 2000.

[Goo85] Thomas G. Goodwillie. Cyclic homology, derivations, and the free loopspace. *Topology*, 24(2):187–215, 1985.

[GR03] Ofer Gabber and Lorenzo Ramero. *Almost ring theory*, volume 1800 of *Lecture Notes in Math.* Springer-Verlag, Berlin, 2003.

[Gro61] Alexandre Grothendieck. Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. I. *Inst. Hautes Études Sci. Publ. Math.*, 11:167, 1961.

[HKK18] Annette Huber-Klawitter and Shane Kelly. Differential forms in positive characteristic, II: cdh-descent via functorial Riemann–Zariski spaces. *Algebra Number Theory*, 12(3):649–692, 2018.

[HM97a] Lars Hesselholt and Ib Madsen. Cyclic polytopes and the $K$-theory of truncated polynomial algebras. *Invent. Math.*, 130(1):73–97, 1997.

[HM97b] Lars Hesselholt and Ib Madsen. On the $K$-theory of finite algebras over Witt vectors of perfect fields. *Topology*, 36(1):29–101, 1997.

[HN20] Lars Hesselholt and Thomas Nikolaus. Algebraic $K$-theory of planar cuspidal curves. In *$K$-theory in algebra, analysis and topology*, volume 749 of *Contemp. Math.*, pages 139–148. Amer. Math. Soc., Providence, RI, 2020.

[HRW22] Jeremy Hahn, Arpon Raksit, and Dylan Wilson. A motivic filtration on the topological cyclic homology of commutative ring spectra. https://arxiv.org/abs/2206.11208, 2022.

[HSS17] Marc Hoyois, Sarah Scherotzke, and Nicolò Sibilla. Higher traces, noncommutative motives, and the categorified Chern character. *Adv. Math.*, 309:97–154, 2017.

[Hub96] Roland Huber. Adic spaces. In *Étale cohomology of rigid analytic varieties and adic spaces*, volume E30 of *Aspects of Mathematics*, pages 36–107. Friedr. Vieweg & Sohn, Braunschweig, 1996.

[Ill71] Luc Illusie. *Complexe cotangent et déformations. I*, volume 239 of *Lecture Notes in Math.* Springer-Verlag, Berlin-New York, 1971.

[Jaf60] Paul Jaffard. *Théorie de la dimension dans les anneaux de polynomes*. Mémor. Sci. Math., Fasc. 146. Gauthier-Villars, Paris, 1960.

[Ker10] Moritz Kerz. Milnor $K$-theory of local rings with finite residue fields. *J. Algebraic Geom.*, 19(1):173–191, 2010.

[KM21] Shane Kelly and Matthew Morrow. $K$-theory of valuation rings. *Compos. Math.*, 157(6):1121–1142, 2021.

[Kra80] Charles Kratzer. $\lambda$-structure en $K$-théorie algébrique. *Comment. Math. Helv.*, 55(2):233–254, 1980.

[KS24] Shane Kelly and Shuji Saito. A procdh topology. https://arxiv.org/abs/2401.02699, 2024.

[KST18] Moritz Kerz, Florian Strunk, and Georg Tamme. Algebraic $K$-theory and descent for blow-ups. *Invent. Math.*, 211(2):523–577, 2018.

[KST21] Moritz Kerz, Florian Strunk, and Georg Tamme. Towards Vorst's conjecture in positive characteristic. *Compos. Math.*, 157(6):1143–1171, 2021.

[Lev01] Marc Levine. Techniques of localization in the theory of algebraic cycles. *J. Algebraic Geom.*, 10(2):299–363, 2001.

[Lic73] Stephen Lichtenbaum. Values of zeta-functions, étale cohomology, and algebraic $K$-theory. In *Algebraic $K$-theory, II: "Classical" algebraic $K$-theory and connections with arithmetic (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, Lecture Notes in Math., Vol. 342, pages 489–501. Springer, Berlin-New York, 1973.

[Lic84] Stephen Lichtenbaum. Values of zeta-functions at nonnegative integers. In *Number theory, Noordwijkerhout 1983 (Noordwijkerhout, 1983)*, volume 1068 of *Lecture Notes in Math.*, pages 127–138. Springer, Berlin, 1984.

[LM23] Morten Lüders and Matthew Morrow. Milnor $K$-theory of $p$-adic rings. *J. Reine Angew. Math.*, 796:69–116, 2023.

[LT19] Markus Land and Georg Tamme. On the $K$-theory of pullbacks. *Ann. of Math.*, 190(3):877–930, 2019.

[Lüd22] Morten Lüders. On the relative Gersten conjecture for Milnor $K$-theory in the smooth case. `https://arxiv.org/abs/2010.02622`, 2022.

[Lur17] Jacob Lurie. Higher algebra. `https://people.math.harvard.edu/~lurie/papers/HA.pdf`, 2017.

[Lur19] Jacob Lurie. Spectral algebraic geometry. `https://www.math.ias.edu/~lurie/papers/SAG-rootfile.pdf`, 2019.

[Mat22] Akhil Mathew. Some recent advances in topological Hochschild homology. *Bull. Lond. Math. Soc.*, 54(1):1–44, 2022.

[MM22] Shubhodip Mondal and Alapan Mukhopadhyay. Ind-étale vs formally étale. `https://arxiv.org/abs/2209.09392`, 2022.

[Mor16] Matthew Morrow. Pro cdh-descent for cyclic homology and $K$-theory. *J. Inst. Math. Jussieu*, 15(3):539–567, 2016.

[Mor18a] Matthew Morrow. Pro unitality and pro excision in algebraic $K$-theory and cyclic homology. *J. Reine Angew. Math.*, 736:95–139, 2018.

[Mor18b] Matthew Morrow. Topological cyclic homology in arithmetic geometry. In *AWS 2019: Topology and Arithmetic*, 2018.

[Mor21] Baptiste Morin. Topological Hochschild homology and Zeta-values. `https://arxiv.org/abs/2011.11549`, 2021.

[MRT22] Tasos Moulinos, Marco Robalo, and Bertrand Toën. A universal Hochschild-Kostant-Rosenberg theorem. *Geom. Topol.*, 26(2):777–874, 2022.

[NS89] Yuri P. Nesterenko and Andreï A. Suslin. Homology of the general linear group over a local ring, and Milnor's $K$-theory. *Izv. Akad. Nauk SSSR Ser. Mat.*, 53(1):121–146, 1989.

[NS18] Thomas Nikolaus and Peter Scholze. On topological cyclic homology. *Acta Math.*, 221(2):203–409, 2018.

[Qui73] Daniel Quillen. Higher algebraic $K$-theory. I. In *Algebraic $K$-theory, I: Higher $K$-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, Lecture Notes in Math., Vol. 341, pages 85–147. Springer, Berlin-New York, 1973.

[Rak20] Arpon Raksit. Hochschild homology and the derived de Rham complex revisited. `https://arxiv.org/abs/2007.02576`, 2020.

[Rig22] Noah Riggenbach. $K$-theory of truncated polynomials. `https://arxiv.org/abs/2211.11110`, 2022.

[Rig23] Noah Riggenbach. $K$-theory of cuspidal curves over a perfectoid base and formal analogues. *Adv. Math.*, 433:Paper No. 109289, 40, 2023.

[Sch12] Peter Scholze. Perfectoid spaces. *Publ. Math. Inst. Hautes Études Sci.*, 116:245–313, 2012.

[Sou81] Christophe Soulé. Rational $K$-theory of the dual numbers of a ring of algebraic integers. In *Algebraic $K$-theory, Evanston 1980 (Proc. Conf., Northwestern Univ., Evanston, Ill., 1980)*, volume 854 of *Lecture Notes in Math.*, pages 402–408. Springer, Berlin, 1981.

[Spe20] Martin Speirs. On the $K$-theory of truncated polynomial algebras, revisited. *Adv. Math.*, 366:107083, 18, 2020.

[Spi18] Markus Spitzweck. A commutative $\mathbb{P}^1$-spectrum representing motivic cohomology over Dedekind domains. *Mém. Soc. Math. Fr. (N.S.)*, 157:110, 2018.

[Sta85] Ross E. Staffeldt. Rational algebraic $K$-theory of certain truncated polynomial rings. *Proc. Amer. Math. Soc.*, 95(2):191–198, 1985.

[Sta19] The Stacks Project Authors. *Stacks Project*. `https://stacks.math.columbia.edu`, 2019.

[Sul23] Yuri J. F. Sulyma. Floor, ceiling, slopes, and $K$-theory. *Ann. $K$-Theory*, 8(3):331–354, 2023.

[SV00] Andrei Suslin and Vladimir Voevodsky. Bloch–Kato conjecture and motivic cohomology with finite coefficients. In *The arithmetic and geometry of algebraic cycles (Banff, AB, 1998)*, volume 548 of *NATO Sci. Ser. C Math. Phys. Sci.*, pages 117–189. Kluwer Acad. Publ., Dordrecht, 2000.

[Tho93] Robert W. Thomason. Les $K$-groupes d'un schéma éclaté et une formule d'intersection excédentaire. *Invent. Math.*, 112(1):195–215, 1993.

[Tot92] Burt Totaro. Milnor $K$-theory is the simplest part of algebraic $K$-theory. *$K$-Theory*, 6(2):177–189, 1992.

[TT90] Robert W. Thomason and Thomas Trobaugh. Higher algebraic $K$-theory of schemes and of derived categories. In *The Grothendieck Festschrift, Vol. III*, volume 88 of *Progr. Math.*, pages 247–435. Birkhäuser Boston, Boston, MA, 1990.

[Voe02a] Vladimir Voevodsky. Open problems in the motivic stable homotopy theory. I. In *Motives, polylogarithms and Hodge theory, Part I (Irvine, CA, 1998)*, volume 3 of *Int. Press Lect. Ser.*, pages 3–34. Int. Press, Somerville, MA, 2002.

[Voe02b] Vladimir Voevodsky. A possible new approach to the motivic spectral sequence for algebraic $K$-theory. In *Recent progress in homotopy theory (Baltimore, MD, 2000)*, volume 293 of *Contemp. Math.*, pages 371–379. Amer. Math. Soc., Providence, RI, 2002.

[Voe10] Vladimir Voevodsky. Unstable motivic homotopy categories in Nisnevich and cdh-topologies. *J. Pure Appl. Algebra*, 214(8):1399–1406, 2010.

[Vor79] Ton Vorst. Localization of the $K$-theory of polynomial extensions. *Math. Ann.*, 244(1):33–53, 1979. With an appendix by Wilberd van der Kallen.

[VSF00] Vladimir Voevodsky, Andrei Suslin, and Eric M. Friedlander. *Cycles, transfers, and motivic homology theories*, volume 143 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2000.

[Wei80] Charles A. Weibel. $K$-theory and analytic isomorphisms. *Invent. Math.*, 61(2):177–197, 1980.

[Wei81] Charles A. Weibel. Meyer-Vietoris sequences and module structures on $NK_*$. In *Algebraic $K$-theory, Evanston 1980 (Proc. Conf., Northwestern Univ., Evanston, Ill., 1980)*, volume 854 of *Lecture Notes in Math.*, pages 466–493. Springer, Berlin, 1981.

[Wei89] Charles A. Weibel. Homotopy algebraic $K$-theory. In *Algebraic $K$-theory and algebraic number theory (Honolulu, HI, 1987)*, volume 83 of *Contemp. Math.*, pages 461–488. Amer. Math. Soc., Providence, RI, 1989.