Motivic cohomology of mixed characteristic schemes

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Abstract

We introduce a theory of motivic cohomology for quasi-compact quasi-separated schemes, which generalises the construction of Elmanto–Morrow in the case of schemes over a field. Our construction is non-$\mathbb{A}^1$-invariant in general, but it uses the classical $\mathbb{A}^1$-invariant motivic cohomology of smooth $\mathbb{Z}$-schemes as an input. The main new input of our construction is a global filtration on topological cyclic homology, whose graded pieces provide an integral refinement of derived de Rham cohomology and Bhatt–Morrow–Scholze's syntomic cohomology. Our theory satisfies various expected properties of motivic cohomology, including a relation to non-connective algebraic K-theory via an Atiyah–Hirzebruch spectral sequence, the projective bundle formula, and pro cdh descent.

CONTENTS

1 INTRODUCTION

Motivic cohomology is an analogue in algebraic geometry of singular cohomology. It was first envisioned to exist for schemes X of finite type over $\mathbb{Z}$ by Beilinson and Lichtenbaum [Lic73, Lic84, Beĭ86, Beĭ87, BMS87], as a way to better understand the special values of their L-functions. Motivic cohomology, in the form of complexes of abelian groups $\mathbb{Z}(i)^{\mathrm{mot}}(X)$ indexed by integers $i \geq 0$, should be an integral interpolation between étale cohomology, and Adams eigenspaces on rationalised algebraic K-theory. That is, there should be a natural filtration $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X)$ on the non-connective algebraic K-theory $\mathrm{K}(X)$, which splits rationally, and whose shifted graded pieces

$$\mathbb{Z}(i)^{\mathrm{mot}}(X) \simeq \mathrm{gr}^i_{\mathrm{mot}}\mathrm{K}(X)[-2i]$$

are given mod p, when p is invertible in X, and in degrees less than or equal to i, by the étale cohomology $R\Gamma_{\text{ét}}(X, \mu_p^{\otimes i})$:

$$\tau^{\leq i}\,\mathbb{F}_p(i)^{\mathrm{mot}}(X) \simeq \tau^{\leq i}R\Gamma_{\text{ét}}(X, \mu_p^{\otimes i}).$$

Such a theory was first developed in the smooth case at the initiative of Bloch and Voevodsky [Blo86, VSF00], using algebraic cycles and $\mathbb{A}^1$-homotopy theory. In this generality, the use of $\mathbb{A}^1$-invariant techniques is permitted by Quillen's fundamental theorem of algebraic K-theory [Qui73], stating that algebraic K-theory is $\mathbb{A}^1$-invariant on regular schemes. On more general schemes, algebraic K-theory fails to be $\mathbb{A}^1$-invariant, so motivic cohomology itself needs to be non-$\mathbb{A}^1$-invariant in general. The first non-$\mathbb{A}^1$-invariant theory of motivic cohomology was recently introduced by Elmanto and Morrow [EM23], using recent advances in algebraic K-theory and topological cyclic homology. Their theory is

developed in the generality of quasi-compact quasi-separated (qcqs) schemes over an arbitrary field, and recovers on smooth varieties the classical $\mathbb{A}^1$-invariant theory.

In this article, we extend the work of Elmanto–Morrow to mixed characteristic, thus producing a theory of motivic cohomology in the originally expected generality of Beilinson and Lichtenbaum. Our theory relies on recent progress in integral p-adic Hodge theory [BMS19, BS22, BL22], and offers in return a complete description of mod p motivic cohomology, even when p is not invertible in the qcqs scheme X.

1.1   A non-$\mathbb{A}^1$-invariant theory of motivic cohomology

The starting point of our construction is the following result, due to Kerz–Strunk–Tamme [KST18] (who prove that homotopy K-theory is the cdh sheafification of algebraic K-theory) and Land–Tamme [LT19] (who prove that the fibre $\mathrm{K}^{\mathrm{inf}}$ of the cyclotomic trace map satisfies cdh descent).

Theorem 1.1 ([KST18, LT19]). *Let X be a qcqs scheme. Then the natural commutative diagram*

$$
\begin{array}{ccc}
\mathrm{K}(X) & \longrightarrow & \mathrm{TC}(X) \\
\downarrow & & \downarrow \\
\mathrm{KH}(X) & \longrightarrow & \big(L_{\mathrm{cdh}}\mathrm{TC}\big)(X)
\end{array}
$$

is a cartesian square of spectra, where $\mathrm{KH}(X)$ is the homotopy K-theory of X, $\mathrm{TC}(X)$ is the topological cyclic homology of X, L_{cdh} is the cdh sheafification functor, the top horizontal map is the cyclotomic trace map, and the bottom horizontal map is the cdh sheafified cyclotomic trace map.

Theorem 1.1 states that algebraic K-theory of schemes can be reconstructed purely in terms of homotopy K-theory (*i.e.*, information coming from $\mathbb{A}^1$-homotopy theory) and topological cyclic homology (*i.e.*, information coming from trace methods). The cdh topology is a Grothendieck topology introduced by Voevodsky [SV00, Voe10], as a way to apply topos theoretic techniques to the study of resolution of singularities. In particular, assuming resolution of singularities, any qcqs scheme would be locally regular in the cdh topology. While homotopy K-theory and topological cyclic homology were originally introduced as tools to approximate the existing algebraic K-theory, we construct the motivic cohomology of schemes using refinements of homotopy K-theory and topological cyclic homology. More precisely, our motivic filtration on algebraic K-theory is defined as a pullback of appropriate filtrations on homotopy K-theory, topological cyclic homology, and cdh sheafified topological cyclic homology.

On homotopy K-theory, we use the recent work of Bachmann–Elmanto–Morrow [BEM], who construct a functorial multiplicative $\mathbb{N}$-indexed filtration $\mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{KH}(X)$ on the homotopy K-theory of qcqs schemes X. The shifted graded pieces of this filtration, that we will denote by $\mathbb{Z}(i)^{\mathrm{cdh}}(X)$, provide a good theory of *cdh-local motivic cohomology* for qcqs schemes. Their construction, which we review in Section 3.2, relies on the classical $\mathbb{A}^1$-invariant motivic cohomology of smooth $\mathbb{Z}$-schemes, and extends most of its properties to general qcqs schemes.

Our first construction is that of a functorial multiplicative $\mathbb{Z}$-indexed filtration $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(X)$ for qcqs schemes X. This filtration recovers the HKR filtration on $\mathrm{HC}^-(X/\mathbb{Q})$ in characteristic zero [Ant19, MRT22, Rak20], and the motivic filtration on $\mathrm{TC}(X;\mathbb{Z}_p)$ after p-completion [BMS19, Mor21, BL22, HRW22]. We describe the shifted graded pieces

$$
\mathbb{Z}(i)^{\mathrm{TC}}(X) \simeq \mathrm{gr}^i_{\mathrm{mot}}\mathrm{TC}(X)[-2i]
$$

of this filtration in the following definition.

Definition 1.2 (See Definition 2.30). For every qcqs scheme X and every integer $i \in \mathbb{Z}$, the complex $\mathbb{Z}(i)^{\mathrm{TC}}(X) \in \mathcal{D}(\mathbb{Z})$ is defined by a natural cartesian square

$$
\begin{CD}
\mathbb{Z}(i)^{\mathrm{TC}}(X) @>>> R\Gamma_{\mathrm{Zar}}\big(X, \widehat{\mathbb{L}\Omega}^{\geq i}_{-/\mathbb{Z}}\big) \\
@VVV @VVV \\
\prod_{p \in \mathbb{P}} \mathbb{Z}_p(i)^{\mathrm{BMS}}(X) @>>> \prod_{p \in \mathbb{P}} R\Gamma_{\mathrm{Zar}}\big(X, (\widehat{\mathbb{L}\Omega}^{\geq i}_{-/\mathbb{Z}})^{\wedge}_p\big),
\end{CD}
$$

where $\mathbb{Z}_p(i)^{\mathrm{BMS}}(X)$ is the syntomic cohomology of the p-adic formal scheme associated to X.

It is straightforward from this definition that the presheaf $\mathbb{Z}(i)^{\mathrm{TC}}$ is naturally identified with Bhatt–Morrow–Scholze's syntomic complex $\mathbb{Z}_p(i)^{\mathrm{BMS}}$ in characteristic p, and with the Hodge-completed derived de Rham complex $R\Gamma_{\mathrm{Zar}}(-, \widehat{\mathbb{L}\Omega}^{\geq i}_{-/\mathbb{Q}})$ in characteristic zero. Following [EM23], the motivic complex $\mathbb{Z}(i)^{\mathrm{mot}}$ is defined in characteristic p and zero respectively by cartesian squares

$$
\begin{CD}
\mathbb{Z}(i)^{\mathrm{mot}}(X) @>>> \mathbb{Z}_p(i)^{\mathrm{BMS}}(X) \\
@VVV @VVV \\
\mathbb{Z}(i)^{\mathrm{cdh}}(X) @>>> \big(L_{\mathrm{cdh}}\, \mathbb{Z}_p(i)^{\mathrm{BMS}}\big)(X)
\end{CD}
\qquad
\begin{CD}
\mathbb{Z}(i)^{\mathrm{mot}}(X) @>>> R\Gamma_{\mathrm{Zar}}\big(X, \widehat{\mathbb{L}\Omega}^{\geq i}_{-/\mathbb{Q}}\big) \\
@VVV @VVV \\
\mathbb{Z}(i)^{\mathrm{cdh}}(X) @>>> R\Gamma_{\mathrm{cdh}}\big(X, \widehat{\mathbb{L}\Omega}^{\geq i}_{-/\mathbb{Q}}\big).
\end{CD}
$$

The following definition is then a natural mixed characteristic generalisation of Elmanto–Morrow's definition over a field.

Definition 1.3 (Motivic cohomology; see Section 3.3). For every qcqs scheme X and every integer $i \in \mathbb{Z}$, the *weight-i motivic complex*
$$\mathbb{Z}(i)^{\mathrm{mot}}(X) \in \mathcal{D}(\mathbb{Z})$$
of X is defined by a natural cartesian square

$$
\begin{CD}
\mathbb{Z}(i)^{\mathrm{mot}}(X) @>>> \mathbb{Z}(i)^{\mathrm{TC}}(X) \\
@VVV @VVV \\
\mathbb{Z}(i)^{\mathrm{cdh}}(X) @>>> \big(L_{\mathrm{cdh}}\, \mathbb{Z}(i)^{\mathrm{TC}}\big)(X),
\end{CD}
$$

where the bottom horizontal map is induced by a filtered refinement of the cdh sheafified cyclotomic trace map.

However, proving the expected relation between the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$ and algebraic K-theory (Theorem C below) requires more efforts in mixed characteristic than over a field. The main foundational obstacle is to prove that the presheaves $\mathbb{Z}(i)^{\mathrm{mot}}$ vanish in weights $i < 0$, as we explain now. Note first that, by construction, the presheaves $\mathbb{Z}(i)^{\mathrm{cdh}}$ do vanish on all qcqs schemes in weights $i < 0$ (Section 3.2).

In characteristic p, the presheaves $\mathbb{Z}_p(i)^{\mathrm{BMS}}$ vanish in weights $i < 0$, thus so do the presheaves $\mathbb{Z}(i)^{\mathrm{mot}}$, and the zeroth step of the associated motivic filtration $\mathrm{Fil}^{\star}_{\mathrm{mot}}K$ recovers algebraic K-theory. Ignoring for a moment the completeness issues for this motivic filtration, this means that the presheaves $\mathbb{Z}(i)^{\mathrm{mot}}$ provide a natural cohomological refinement of algebraic K-theory on arbitrary characteristic p schemes.

In characteristic zero, the presheaves $R\Gamma_{\mathrm{Zar}}(-, \widehat{\mathbb{L}\Omega}^{\geq i}_{-/\mathbb{Q}})$ do not vanish in weights $i < 0$. Instead, they are equal to the presheaf $R\Gamma_{\mathrm{Zar}}(-, \widehat{\mathbb{L}\Omega}_{-/\mathbb{Q}})$, which happens to be a cdh sheaf on qcqs $\mathbb{Q}$-schemes, by results of Cortiñas–Haesemeyer–Schlichting–Weibel [CHSW08], Antieau [Ant19], and Elmanto–Morrow [EM23]. That is, the right vertical map of the previous diagram is an equivalence in weights $i < 0$, so the presheaves $\mathbb{Z}(i)^{\mathrm{mot}}$ vanish in weights $i < 0$, and the zeroth step of the associated motivic

filtration $\mathrm{Fil}^\star_{\mathrm{mot}} K$ recovers algebraic K-theory. This means that the presheaves $\mathbb{Z}(i)^{\mathrm{mot}}$ provide a natural cohomological refinement of algebraic K-theory on arbitrary characteristic zero schemes.

In mixed characteristic, we prove similarly that in weights $i < 0$, the presheaves $\mathbb{Z}(i)^{\mathrm{TC}}$ are cdh sheaves on qcqs schemes, *i.e.*, that the presheaves $\mathbb{Z}(i)^{\mathrm{mot}}$ vanish. The result modulo a prime number p is a consequence, as in characteristic p, of the fact that the presheaves $\mathbb{Z}_p(i)^{\mathrm{BMS}}$ vanish in weights $i < 0$. The difficulty is to then prove that the presheaves $\mathbb{Q}(i)^{\mathrm{TC}}$ are cdh sheaves in weight $i < 0$.

The main cdh descent result used in characteristic zero does not hold in mixed characteristic. That is, the presheaf $R\Gamma_{\mathrm{Zar}}(-, \widehat{\mathbb{L}\Omega}_{-/\mathbb{Z}})$ (or its rationalisation) is not a cdh sheaf on qcqs schemes. We avoid this difficulty by proving a rigid-analytic analogue of this cdh descent result. To formulate this result, denote by

$$R\Gamma_{\mathrm{Zar}}\big(X, \widehat{\mathbb{L}\Omega}^{\geq i}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p}\big)$$

the *rigid-analytic derived de Rham cohomology* of a qcqs $\mathbb{Z}_{(p)}$-scheme X, which we define as the pushout of the diagram

$$R\Gamma_{\mathrm{Zar}}\big(X, (\widehat{\mathbb{L}\Omega}^{\geq i}_{-/\mathbb{Z}})^\wedge_p\big) \longleftarrow R\Gamma_{\mathrm{Zar}}\big(X, \widehat{\mathbb{L}\Omega}^{\geq i}_{-/\mathbb{Z}}\big) \longrightarrow R\Gamma_{\mathrm{Zar}}\big(X, \widehat{\mathbb{L}\Omega}^{\geq i}_{-_{\mathbb{Q}}/\mathbb{Q}}\big)$$

in the derived category $\mathcal{D}(\mathbb{Z})$. Here, we restrict our attention to qcqs $\mathbb{Z}_{(p)}$-schemes for simplicity, and refer to Section 4 for the relevant statements over $\mathbb{Z}$. As a consequence of Definition 1.2, there is a natural cartesian square

$$
\begin{array}{ccc}
\mathbb{Q}(i)^{\mathrm{TC}}(X) & \longrightarrow & R\Gamma_{\mathrm{Zar}}(X, \widehat{\mathbb{L}\Omega}^{\geq i}_{-_{\mathbb{Q}}/\mathbb{Q}}) \\
\downarrow & & \downarrow \\
\mathbb{Q}_p(i)^{\mathrm{BMS}}(X) & \longrightarrow & R\Gamma_{\mathrm{Zar}}(X, \widehat{\mathbb{L}\Omega}^{\geq i}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p})
\end{array}
$$

in the derived category $\mathcal{D}(\mathbb{Q})$. In weights $i < 0$, the presheaves $\mathbb{Q}_p(i)^{\mathrm{BMS}}$ vanish. As already mentioned, the presheaf $R\Gamma_{\mathrm{Zar}}(-, \widehat{\mathbb{L}\Omega}_{-_{\mathbb{Q}}/\mathbb{Q}})$ is moreover a cdh sheaf on qcqs schemes. So the fact that the presheaves $\mathbb{Q}(i)^{\mathrm{TC}}$ are cdh sheaves in weights $i < 0$ reduces to the following result, which can be seen as a rigid-analytic analogue of the latter cdh descent over $\mathbb{Q}$.

Theorem A (Cdh descent for rigid-analytic derived de Rham cohomology; see Corollary 4.45). *For every prime number p, the presheaf $R\Gamma_{\mathrm{Zar}}\big(-, \widehat{\mathbb{L}\Omega}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p}\big)$ satisfies cdh descent on qcqs $\mathbb{Z}_{(p)}$-schemes.*

The modern proof of the analogous result over $\mathbb{Q}$ relies on the theory of truncating invariants of Land–Tamme [LT19] and on a theorem of Goodwillie [Goo85], who prove respectively that every truncating invariant is a cdh sheaf on qcqs schemes and that periodic cyclic homology over $\mathbb{Q}$ is a truncating invariant. By definition, a truncating invariant is a localizing invariant E such that for every connective $\mathbb{E}_1$-ring R, the natural map $E(R) \to E(\pi_0(R))$ is an equivalence. To prove Theorem A, we then use the condensed mathematics of Clausen–Scholze [CS19], and prove that a suitable rigid-analytic variant of periodic cyclic homology is a truncating invariant. In particular, the proof of Theorem A relies on a result on associative rings (actually, on general solid connective $\mathbb{E}_1$-rings).

As a consequence of Theorem A, we obtain the following cohomological description of rational motivic cohomology.

Theorem B (*p*-adic and rational motivic cohomology; see Corollaries 3.25 and 4.67). *Let X be a qcqs scheme, and p be a prime number. Then for any integers $i \in \mathbb{Z}$ and $k \geq 1$, the natural commutative diagrams*

$$
\begin{array}{ccc}
\mathbb{Z}/p^k(i)^{\mathrm{mot}}(X) & \longrightarrow & \mathbb{Z}/p^k(i)^{\mathrm{BMS}}(X) \\
\downarrow & & \downarrow \\
\mathbb{Z}/p^k(i)^{\mathrm{cdh}}(X) & \longrightarrow & \big(L_{\mathrm{cdh}}\,\mathbb{Z}/p^k(i)^{\mathrm{BMS}}\big)(X)
\end{array}
\qquad
\begin{array}{ccc}
\mathbb{Q}(i)^{\mathrm{mot}}(X) & \longrightarrow & R\Gamma_{\mathrm{Zar}}\big(X, \widehat{\mathbb{L}\Omega}^{\geq i}_{-_{\mathbb{Q}}/\mathbb{Q}}\big) \\
\downarrow & & \downarrow \\
\mathbb{Q}(i)^{\mathrm{cdh}}(X) & \longrightarrow & R\Gamma_{\mathrm{cdh}}\big(X, \widehat{\mathbb{L}\Omega}^{\geq i}_{-_{\mathbb{Q}}/\mathbb{Q}}\big)
\end{array}
$$

are cartesian squares in the derived category $\mathcal{D}(\mathbb{Z})$.

Together, these two cartesian squares recover the cartesian squares of Elmanto–Morrow that define the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$ over a field, and are thus natural mixed characteristic analogues of these. The p-adic part of Theorem B is a formal consequence of Definitions 1.2 and 1.3. The rational part of Theorem B implies that the difference between $\mathbb{Q}(i)^{\mathrm{mot}}(X)$ and $\mathbb{Q}(i)^{\mathrm{cdh}}(X)$ depends only on the rationalisation $X_{\mathbb{Q}}$ of the scheme X. If X is regular, this difference should vanish, and this is then more interesting in the presence of singularities. More precisely, Theorem B can be used to capture interesting information about the singularities of an arbitrary commutative ring R: cdh sheaves are typically insensitive to singularities, so the singular information in the motivic complex $\mathbb{Z}(i)^{\mathrm{mot}}(R)$ is controlled by the complexes $\mathbb{Z}/p^k(i)^{\mathrm{BMS}}(R)$ and $\mathbb{L}\Omega^{<i}_{(R\otimes_{\mathbb{Z}}\mathbb{Q})/\mathbb{Q}}$, which are accessible to computation.

Theorem B also implies that the presheaves $\mathbb{Q}(i)^{\mathrm{mot}}$ vanish in weights $i < 0$, which was the essential missing part to establish the following fundamental properties of motivic cohomology.

Theorem C (Relation to algebraic K-theory). *There exists a finitary Nisnevich sheaf of filtered spectra*

$$\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{K}(-) : \mathrm{Sch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

with the following properties:

(1) (Atiyah–Hirzebruch spectral sequence; see Section 4.5) *For every qcqs scheme X, the filtration $\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{K}(X)$ is a multiplicative $\mathbb{N}$-indexed filtration on the non-connective algebraic K-theory $\mathrm{K}(X)$, whose graded pieces are naturally given by*

$$\mathrm{gr}^{i}_{\mathrm{mot}}\mathrm{K}(X) \simeq \mathbb{Z}(i)^{\mathrm{mot}}(X)[2i], \quad i \geq 0.$$

In particular, writing $\mathrm{H}^{j}_{\mathrm{mot}}(X, \mathbb{Z}(i)) := \mathrm{H}^{j}(\mathbb{Z}(i)^{\mathrm{mot}}(X))$ for the corresponding motivic cohomology groups, there exists an Atiyah–Hirzebruch spectral sequence

$$E_2^{i,j} = \mathrm{H}^{i-j}_{\mathrm{mot}}(X, \mathbb{Z}(-j)) \Longrightarrow \mathrm{K}_{-i-j}(X).$$

If X has finite valuative dimension,[1] then the filtration $\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{K}(X)$ is complete, and the Atiyah–Hirzebruch spectral sequence is convergent.

(2) (Adams decomposition; see Corollary 4.57) *For every qcqs scheme X, the Atiyah–Hirzebruch spectral sequence degenerates rationally and, for every integer $n \in \mathbb{Z}$, there is a natural isomorphism of abelian groups*

$$\mathrm{K}_n(X) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \bigoplus_{i \geq 0} \left(\mathrm{H}^{2i-n}_{\mathrm{mot}}(X, \mathbb{Z}(i)) \otimes_{\mathbb{Z}} \mathbb{Q} \right)$$

induced by the Adams operations on rationalised algebraic K-theory.

One of the main historical motivations for developing motivic cohomology was to apply cohomological techniques to the study of algebraic K-theory [BMS87]. The following theorem summarizes our results on the relations between motivic cohomology and previously studied cohomological invariants. When X is smooth over $\mathbb{Z}$, we denote by

$$\mathbb{Z}(i)^{\mathrm{cla}}(X) := z^i(X, \bullet)[-2i]$$

the *weight-i classical motivic complex*, where $z^i(X, \bullet)$ is Bloch's cycle complex (and $\bullet$ is the cohomological index).

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[1] The valuative dimension of a commutative ring, defined in terms of the ranks of certain valuation rings, was introduced by Jaffard in [Jaf60, Chapter IV], and generalised to schemes in [EHIK21, Section 2.3]. The valuative dimension of a scheme is always at least equal to its Krull dimension, and both notions agree on noetherian schemes. For our purposes, the valuative dimension of a qcqs scheme X will be used as an upper bound on the cohomological dimension of the cdh topos of X ([EHIK21, Theorem 2.4.15]).

Theorem D. *Let X be a qcqs scheme, and $i \geq 0$ be an integer.*

(1) (Weight zero; see Example 4.69) *There is a natural equivalence*

$$\mathbb{Z}(0)^{\mathrm{mot}}(X) \xrightarrow{\sim} R\Gamma_{\mathrm{cdh}}(X, \mathbb{Z})$$

in the derived category $\mathcal{D}(\mathbb{Z})$.

(2) (Weight one; see Example 7.15) *There is a natural map*

$$R\Gamma_{\mathrm{Nis}}(X, \mathbb{G}_m)[-1] \longrightarrow \mathbb{Z}(1)^{\mathrm{mot}}(X)$$

in the derived category $\mathcal{D}(\mathbb{Z})$ which is an isomorphism in degrees less than or equal to three.

(3) (Étale cohomology; see Corollary 5.6) *For every prime number ℓ which is invertible in X and every integer $k \geq 1$, there is a natural map*

$$\mathbb{Z}/\ell^k(i)^{\mathrm{mot}}(X) \longrightarrow R\Gamma_{\mathrm{\acute{e}t}}(X, \mu_{\ell^k}^{\otimes i})$$

in the derived category $\mathcal{D}(\mathbb{Z}/\ell^k)$ which is an isomorphism in degrees less than or equal to i.

(4) (Syntomic cohomology; see Corollary 5.11) *For every prime number p and every integer $k \geq 1$, there is a natural map*

$$\mathbb{Z}/p^k(i)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{syn}}(X)$$

in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$ which is an isomorphism in degrees less than or equal to i, where $\mathbb{Z}/p^k(i)^{\mathrm{syn}}(X)$ denotes the weight-i syntomic cohomology of X in the sense of [BL22].

(5) (Milnor K-theory; see Theorem 7.21) *If $X = \mathrm{Spec}(A)$ is the spectrum of a henselian local ring A, then for every integer $n \geq 1$, there is a natural isomorphism*

$$\widehat{\mathrm{K}}_i^{\mathrm{M}}(A)/n \xrightarrow{\cong} \mathrm{H}^i_{\mathrm{mot}}(A, \mathbb{Z}(i))/n$$

of abelian groups, where $\widehat{\mathrm{K}}_i^{\mathrm{M}}(A)$ denotes the ith improved Milnor K-group of A in the sense of [Ker10].

(6) (Classical motivic cohomology; see Section 6) *If X is smooth over $\mathbb{Z}$, then there is a natural map*

$$\mathbb{Z}(i)^{\mathrm{cla}}(X) \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(X)$$

in the derived category $\mathcal{D}(\mathbb{Z})$ which is an isomorphism in degrees less than or equal to $i + 1$ in general, and an isomorphism in all degrees if X has dimension less than or equal to one over $\mathbb{Z}$.

(7) (Lisse motivic cohomology; see Corollary 7.12) *If $X = \mathrm{Spec}(A)$ is the spectrum of a local ring A, then for every integer $i \geq 0$, there is a natural equivalence*

$$\mathbb{Z}(i)^{\mathrm{lisse}}(A) \xrightarrow{\sim} \tau^{\leq i}\,\mathbb{Z}(i)^{\mathrm{mot}}(A)$$

in the derived category $\mathcal{D}(\mathbb{Z})$, where $\mathbb{Z}(i)^{\mathrm{lisse}}$ denotes the weight-i lisse motivic cohomology of A, defined as the left Kan extension from smooth $\mathbb{Z}$-algebras of the classical motivic complex $\mathbb{Z}(i)^{\mathrm{cla}}$. In particular, the functor $\tau^{\leq i}\,\mathbb{Z}(i)^{\mathrm{mot}}$ is left Kan extended on local rings from local essentially smooth $\mathbb{Z}$-algebras.

(8) ($\mathbb{A}^1$-invariant motivic cohomology; see Theorem 10.5) *There is a natural equivalence*

$$\left(L_{\mathbb{A}^1}\,\mathbb{Z}(i)^{\mathrm{mot}}\right)(X) \xrightarrow{\sim} \mathbb{Z}(i)^{\mathbb{A}^1}(X)$$

in the derived category $\mathcal{D}(\mathbb{Z})$, where $\mathbb{Z}(i)^{\mathbb{A}^1}(X)$ denotes the cohomology represented by the slices of the K-theory motivic spectrum $\mathrm{KGL}_X \in \mathrm{SH}(X)$.

7

The following result is a consequence of Theorem D (8) and the fact that the $\mathbb{A}^1$-invariant motivic complexes $\mathbb{Z}(i)^{\mathbb{A}^1}$ recover the classical motivic complexes $\mathbb{Z}(i)^{\mathrm{cla}}$ on smooth $\mathbb{Z}$-schemes. In particular, although we expect our motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$ to actually coincide with the classical motivic complexes $\mathbb{Z}(i)^{\mathrm{cla}}$ on smooth $\mathbb{Z}$-schemes, this means that the former at least recover the latter after enforcing $\mathbb{A}^1$-invariance.

Corollary E (See Corollary 10.9)**.** *Let X be a smooth scheme over $\mathbb{Z}$. Then for every integer $i \geq 0$, there is a natural equivalence*
$$\mathbb{Z}(i)^{\mathrm{cla}}(X) \simeq \big(L_{\mathbb{A}^1} \mathbb{Z}(i)^{\mathrm{mot}}\big)(X)$$
in the derived category $\mathcal{D}(\mathbb{Z})$.

Assuming the existence of a well-behaved derived category of motives DM, the motivic cohomology groups of a scheme X should be given by
$$\mathrm{H}^j_{\mathrm{mot}}(X, \mathbb{Z}(i)) \cong \mathrm{Hom}_{\mathrm{DM}}(\mathrm{M}(X), \mathbb{Z}(i)[j]),$$
where $\mathrm{M}(X) \in \mathrm{DM}$ is the motive associated to X, and $\mathbb{Z}(i) \in \mathrm{DM}$ are the Tate motives, fitting for every integer $r \geq 0$ in a natural decomposition in DM:
$$\mathrm{M}(\mathbb{P}^r_{\mathbb{Z}}) \cong \bigoplus_{j=0}^r \mathbb{Z}(j)[2j].$$

In the $\mathbb{A}^1$-invariant framework, Voevodsky constructed such a derived category of motives, in which the classical motivic complexes $\mathbb{Z}(i)^{\mathrm{cla}}$ can be interpreted in terms of these Tate motives. Without assuming $\mathbb{A}^1$-invariance, Annala–Iwasa [AI23] and Annala–Hoyois–Iwasa [AHI23, AHI24] recently introduced a more general derived category of motives, where the decomposition of the motive $\mathrm{M}(\mathbb{P}^r_{\mathbb{Z}})$, *i.e.*, the *projective bundle formula*, is isolated as the key defining property. The following result states that the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$ fit within this theory of non-$\mathbb{A}^1$-invariant motives.

Theorem F (Projective bundle formula; see Theorem 8.16)**.** *Let X be a qcqs scheme, $i \geq 0$ be an integer, and $\mathcal{E}$ be a vector bundle of rank r on X. Then for every integer $i \geq 0$, the powers of the motivic first Chern class $c_1^{\mathrm{mot}}(\mathcal{O}(1)) \in \mathrm{H}^2_{\mathrm{mot}}(\mathbb{P}_X(\mathcal{E}), \mathbb{Z}(1))$ induce a natural equivalence*
$$\bigoplus_{i=0}^{r-1} \mathbb{Z}(i-j)^{\mathrm{mot}}(X)[-2j] \xrightarrow{\sim} \mathbb{Z}(i)^{\mathrm{mot}}(\mathbb{P}_X(\mathcal{E}))$$
in the derived category $\mathcal{D}(\mathbb{Z})$.

Theorem F is proved by Elmanto–Morrow in the equicharacteristic case [EM23], where the proof relies on the projective bundle formula for the complexes $\mathbb{Z}(i)^{\mathrm{cdh}}$ [BEM]. In mixed characteristic, however, the cdh-local motivic complexes $\mathbb{Z}(i)^{\mathrm{cdh}}$ are known to satisfy the projective bundle formula only conditionally on a certain property of valuation rings, called F-smoothness [BM23, BEM]. This condition can be proved in mixed characteristic for valuation rings over a perfectoid base: this is the main result of [Bou23]. The case of general valuation rings remaining open, our proof of Theorem F is different from that of Elmanto–Morrow, and uses in particular our description of motivic cohomology with finite coefficients in terms of syntomic cohomology (Theorem 5.10).

1.2 Motivic cohomology of regular schemes

Recall that on smooth $\mathbb{F}_p$-schemes, the Beilinson–Lichtenbaum conjecture, proved by Suslin and Voevodsky as a consequence of the Bloch–Kato conjecture [SV00], computes the ℓ-adic part of motivic cohomology in terms of the cohomology of the étale sheaf μ_{ℓ^k} of ℓ^k-roots of unity. To describe the p-adic part of motivic cohomology, one needs to replace $\mu_{\ell^k}^{\otimes i}$ (which is zero on smooth varieties when $\ell = p$ and $i > 0$) by the logarithmic de Rham–Witt sheaves $W_k\Omega^i_{-,\log}$ [GL00]. The corresponding description of p-adic algebraic K-theory, in terms of the logarithmic de Rham–Witt sheaves, is generalised in [KM21] to all Cartier smooth $\mathbb{F}_p$-algebras, and in particular to all characteristic p valuation rings.

On smooth schemes over a mixed characteristic Dedekind domain, the p-adic part of classical motivic cohomology is similarly described in low degrees by the étale cohomology of the generic fibre [Gei04]. This result is a consequence of the Gersten conjecture proved by Geisser [Gei04], and is unknown for general regular schemes. Combined with Theorem D (6), the following result extends this description of classical motivic cohomology to the regular case. More precisely, the notion of F-smoothness was introduced by Bhatt–Mathew [BM23] as a non-noetherian generalisation of regular schemes, and our result naturally applies to general p-torsionfree F-smooth schemes.

Theorem G (Beilinson–Lichtenbaum conjecture for F-smooth schemes; see Corollary 5.12). *Let X be a p-torsionfree F-smooth scheme (e.g., a regular scheme flat over $\mathbb{Z}$). Then for any integers $i \geq 0$ and $k \geq 1$, the Beilinson–Lichtenbaum comparison map*

$$\mathbb{Z}/p^k(i)^{\mathrm{mot}}(X) \longrightarrow R\Gamma_{\text{ét}}(X[\tfrac{1}{p}], \mu_{p^k}^{\otimes i})$$

is an isomorphism in degrees less than or equal to $i - 1$, and is injective in degree i.

Valuation ring extensions of a perfectoid valuation ring being F-smooth by [Bou23], we then obtain the following complete description of the p-adic motivic cohomology of valuation rings over a perfectoid base.

Corollary H (Motivic cohomology of valuation rings; see Theorem 11.13). *Let V_0 be a p-torsionfree valuation ring whose p-completion is a perfectoid ring, and V be a henselian valuation ring extension of V_0. Then for any integers $i \geq 0$ and $k \geq 1$, the motivic complex $\mathbb{Z}/p^k(i)^{\mathrm{mot}}(V)$ is in degrees at most i, and the Beilinson–Lichtenbaum comparison map*

$$\mathbb{Z}/p^k(i)^{\mathrm{mot}}(V) \longrightarrow R\Gamma_{\text{ét}}(\mathrm{Spec}(V[\tfrac{1}{p}]), \mu_{p^k}^{\otimes i})$$

is an isomorphism in degrees less than or equal to $i - 1$. On H^i, this map is injective, with image generated by symbols, via the symbol map

$$(V^{\times})^{\otimes i} \to \mathrm{H}^i_{\text{ét}}(\mathrm{Spec}(V[\tfrac{1}{p}]), \mu_{p^k}^{\otimes i}).$$

The proof of Theorem G relies on a syntomic-étale comparison theorem of Bhatt–Mathew [BM23] (see also [Bou23, Theorem B] for a proof over perfectoid bases using relative prismatic cohomology).

1.3 Motivic cohomology of singular schemes

For the rest of this introduction, we focus on the properties of motivic cohomology that are specific to singular schemes. One of the most interesting, yet mysterious features of the algebraic K-theory of singular schemes is the presence of nonzero negative K-groups. Most of the current understanding of negative K-groups relies on results on the behaviour of algebraic K-theory with respect to blowups [CHSW08, KST18]. It was proved in particular by Thomason [Tho93] that algebraic K-theory sends the blowup square associated to a regular closed immersion to a long exact sequence of K-groups. The following result is a cohomological refinement of Thomason's result.

Theorem I (Regular blowup formula; see Theorem 8.15). *For every regular closed immersion $Y \to X$ of qcqs schemes (i.e., the closed subscheme Y is Zariski-locally on X defined by a regular sequence) and every integer $i \geq 0$, the commutative diagram*

$$
\begin{array}{ccc}
\mathbb{Z}(i)^{\mathrm{mot}}(X) & \longrightarrow & \mathbb{Z}(i)^{\mathrm{mot}}(Y) \\
\downarrow & & \downarrow \\
\mathbb{Z}(i)^{\mathrm{mot}}(\mathrm{Bl}_Y(X)) & \longrightarrow & \mathbb{Z}(i)^{\mathrm{mot}}(\mathrm{Bl}_Y(X) \times_X Y)
\end{array}
$$

is a cartesian square in the derived category $\mathcal{D}(\mathbb{Z})$.

However, algebraic K-theory fails to associate long exact sequences to general blowups. Motivated by Grothendieck's theorem on formal functions for quasi-coherent cohomology [Gro61, Theorem 4.1.5], many people hoped for a formal analogue of these long exact sequences in algebraic K-theory, that would hold for general blowups. This was finally proved by Kerz–Strunk–Tamme [KST18], in the form of a pro cdh excision property for the algebraic K-theory of arbitrary noetherian schemes. Note that the pro cdh topology was recently introduced by Kelly–Saito [KS24], as a way to encode this pro cdh excision property in a descent property for this Grothendieck topology. The following result relies in particular on Theorem B and on Grothendieck's theorem on formal functions, thus shedding some light on the analogy between quasi-coherent and K-theoretic techniques.

Theorem J (Pro cdh descent; see Theorem 9.23). *For every integer $i \geq 0$, the presheaf $\mathbb{Z}(i)^{\mathrm{mot}}$ satisfies pro cdh descent on noetherian schemes. That is, for every abstract blowup square*

$$\begin{array}{ccc} Y' & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

of noetherian schemes, the associated commutative diagram

$$\begin{array}{ccc} \mathbb{Z}(i)^{\mathrm{mot}}(X) & \longrightarrow & \mathbb{Z}(i)^{\mathrm{mot}}(X') \\ \downarrow & & \downarrow \\ \{\mathbb{Z}(i)^{\mathrm{mot}}(rY)\}_r & \longrightarrow & \{\mathbb{Z}(i)^{\mathrm{mot}}(rY')\}_r \end{array}$$

is a weakly cartesian square of pro objects in the derived category $\mathcal{D}(\mathbb{Z})$.

Kelly–Saito moreover proved that non-connective algebraic K-theory not only satisfies pro cdh descent, but is the pro cdh sheafification of connective algebraic K-theory [KS24]. Combined with the observation of Bhatt–Lurie that connective algebraic K-theory is left Kan extended on commutative rings from smooth $\mathbb{Z}$-algebras [EHK$^+$20], this motivated the definition of the *pro cdh motivic complexes* $\mathbb{Z}(i)^{\mathrm{procdh}}$, as the pro cdh sheafification of the left Kan extension of the classical motivic complexes $\mathbb{Z}(i)^{\mathrm{cla}}$. The following result relies on the comparison to lisse motivic cohomology (Theorem D (7)) and pro cdh descent (Theorem J).

Corollary K (Comparison to pro cdh motivic cohomology; see Theorem 9.32). *Let X be a noetherian scheme. Then for every integer $i \geq 0$, there is a natural equivalence*

$$\mathbb{Z}(i)^{\mathrm{procdh}}(X) \overset{\sim}{\longrightarrow} \mathbb{Z}(i)^{\mathrm{mot}}(X)$$

in the derived category $\mathcal{D}(\mathbb{Z})$.

Note that the pro cdh motivic complexes $\mathbb{Z}(i)^{\mathrm{procdh}}$ are not finitary, so they cannot coincide with the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$ on general qcqs schemes.

An important conjecture of Weibel [Wei80] states that for every noetherian scheme X of dimension at most d, the negative K-groups $\mathrm{K}_{-n}(X)$ vanish for integers $n > d$. This conjecture was settled by Kerz–Strunk–Tamme [KST18], as a consequence of pro cdh descent for algebraic K-theory. The proof of the following result uses the techniques of Kerz–Strunk–Tamme [KST18] as reformulated by Elmanto–Morrow [EM23], who proved the same result over a field. In particular, Theorem L relies on pro cdh descent for motivic cohomology (Theorem J).

Theorem L (Motivic Weibel vanishing; see Theorem 9.27). *Let X be a noetherian scheme of finite dimension d, and $i \geq 0$ be an integer. Then for every integer $j > i + d$, the motivic cohomology group $\mathrm{H}^j_{\mathrm{mot}}(X, \mathbb{Z}(i))$ is zero.*

Via the Atiyah–Hirzebruch spectral sequence of Theorem C, Theorem L is a motivic refinement of Weibel's vanishing conjecture in K-theory.

1.4 Notation

Algebraic K-theory. By default, algebraic K-theory means non-connective algebraic K-theory, as introduced by Thomason–Trobaugh [TT90]. By a theorem of Blumberg–Gepner–Tabuada [BGT13], non-connective algebraic K-theory is the universal localizing invariant.

$\mathbb{A}^1$-invariance. A presheaf $F(-)$ on schemes is called $\mathbb{A}^1$-*invariant* if for every scheme X and every integer $m \geq 0$, the natural map $F(X) \to F(\mathbb{A}^m_X)$ is an equivalence. Given a presheaf $F(-)$ on schemes, the $\mathbb{A}^1$-*localisation* $L_{\mathbb{A}^1} F(-)$ of $F(-)$ is the initial $\mathbb{A}^1$-invariant presheaf with a map from $F(-)$. The $\mathbb{A}^1$-localisation functor $L_{\mathbb{A}^1}$ commutes with colimits.

Base change. Given a commutative ring R, an R-algebra S, and a scheme X over $\mathrm{Spec}(R)$, denote by X_S the base change $X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(S)$ of X from R to S. If X is a derived scheme, this base change is implicitly the derived base change from R to S. We sometimes use the derived base even on classical schemes, and say explicitly when we do so.

Bounded torsion. An abelian group A is said to have *bounded torsion* if there exists an integer $N \geq 1$ such that the multiplication by N of every element of A is zero.

Given a commutative ring R and an element d of R, an R-module M is said to have *bounded d-power torsion* if there exists an integer $n \geq 1$ such that $M[d^m] = M[d^n]$ for all $m \geq n$; this assumption guarantees that the derived d-completion of M is in degree zero, given by the classical d-completion of M.

Cdh topology. The cdh topology is a Grothendieck topology introduced by Voevodsky [SV00, Voe10]; see [EHIK21] for the definition and properties of the cdh topology in the generality of qcqs schemes. It is a completely decomposed version of the topology generated by Deligne's hypercoverings. The cdh sheafification functor L_{cdh} preserves multiplicative structures.

Coefficients. Given a functor $F(-)$ (resp. $\mathrm{Fil}^\star F(-)$) taking values in spectra (resp. filtered spectra) and a prime number p, we denote the rationalisation of $F(-)$ by $F(-;\mathbb{Q})$, its reduction modulo p by $F(-;\mathbb{F}_p)$, its derived reduction modulo powers of p by $F(-;\mathbb{Z}/p^k)$, its p-completion by $F(-;\mathbb{Z}_p)$, and the rationalisation of its p-completion by $F(-;\mathbb{Q}_p)$. We adopt a similar notation for a functor $\mathrm{Fil}^\star F(-)$ taking values in filtered spectra, *e.g.*, we denote its rationalisation by $\mathrm{Fil}^\star F(-;\mathbb{Q})$. Similarly, if $\mathbb{Z}(i)^F(-)$ is a functor taking values in the derived category $\mathcal{D}(\mathbb{Z})$, we denote the rationalisation of $\mathbb{Z}(i)^F(-)$ by $\mathbb{Q}(i)^F(-)$, its derived reduction modulo p by $\mathbb{F}_p(i)^F(-)$, etc. Following the same notation, we also write

$$\prod_{p\in\mathbb{P}}{}' F(-;\mathbb{Q}_p) := \Big(\prod_{p\in\mathbb{P}} F(-;\mathbb{Z}_p)\Big)_{\mathbb{Q}} \quad \Big(\text{resp.} \prod_{p\in\mathbb{P}}{}' \mathrm{Fil}^\star F(-;\mathbb{Q}_p) := \Big(\prod_{p\in\mathbb{P}} \mathrm{Fil}^\star F(-;\mathbb{Z}_p)\Big)_{\mathbb{Q}}\Big).$$

Derived categories and spectra. Denote by Sp the category of spectra. Given a commutative ring R, denote by $\mathcal{D}(R)$ the derived category of R-modules; it is implicitly the derived ∞-category of R-modules, and is in particular naturally identified with the category of R-linear spectra. Our convention for degrees is by default cohomological. In this context, the notions of fibre and cofibre sequences agree, and the fibre and cofibre of a given map satisfy the relation $\mathrm{fib} \simeq \mathrm{cofib}[-1]$. Given an element d of R, also denote by $(-)^\wedge_d$ the d-adic completion functor in the derived category $\mathcal{D}(R)$.

Extension by zero $j_!$. Given a prime number p (which is typically clear from context) and a scheme X, denote by $j : X[\frac{1}{p}] \to X$ the open immersion of the p-adic generic fibre of X, and by $j_! : (X[\frac{1}{p}])_{\text{ét}} \to X_{\text{ét}}$ the associated extension by zero functor.

Filtrations. By default, a filtration with values in a category $\mathcal{C}$ is a $\mathbb{Z}$-indexed decreasing filtered object in the category $\mathcal{C}$, *i.e.*, a functor from the category $(\mathbb{Z}, \geq)^{\mathrm{op}}$ to the category $\mathcal{C}$. A filtration is called $\mathbb{N}$-*indexed* if it is constant in non-positive degrees. Given a filtered object $\mathrm{Fil}^\star C$ and for each

integer $n \in \mathbb{Z}$, let $\mathrm{gr}^n C \in \mathcal{D}(R)$ denote the cofibre of the transition map $\mathrm{Fil}^{n+1} C \to \mathrm{Fil}^n C$. A filtered object $\mathrm{Fil}^\star C$ is said to be *complete* if the limit $\lim_n \mathrm{Fil}^n C$ vanishes. For instance, The Hodge filtration on the de Rham complex is given for each $n \in \mathbb{Z}$ by $\mathrm{Fil}^n_{\mathrm{Hod}} \Omega_{-/R} := \Omega^{\geq n}_{-/R}$; the Hodge filtration $\mathbb{L}\Omega^{\geq \star}_{-/R}$ on the derived de Rham complex $\mathbb{L}\Omega_{-/R}$ is defined as the left Kan extension of this filtration. It is $\mathbb{N}$-indexed, but not always complete. Its completion, the Hodge-completed derived de Rham complex, is denoted by $\widehat{\mathbb{L}\Omega}^{\geq \star}_{-/R}$.

Given a commutative ring R, denote by $\mathcal{DF}(R) := \mathrm{Fun}((\mathbb{Z}, \geq)^{\mathrm{op}}, \mathcal{D}(R))$ the filtered derived category of R-modules. Also denote by FilSp the category of filtered spectra, and by biFilSp the category of bifiltered spectra (*i.e.*, the category of filtered objects in the category of filtered spectra).

Henselian rings. Given a commutative ring R and an ideal I of R, the pair (R, I) is called *henselian* if it satisfies Hensel's lemma. A local ring R is called *henselian* if the pair $(R, \mathfrak{m})$ is henselian, where $\mathfrak{m}$ is the maximal ideal of R. Henselian local rings are the local rings for the Nisnevich topology. A commutative ring R is called *d-henselian*, for d an element of R, if the pair $(R, (d))$ is henselian.

Ind-smooth schemes. Given a scheme S, a scheme X is called ind-smooth (resp. ind-regular, ind-étale) over S if it is a cofiltered limit of smooth (resp. regular, étale) S-schemes.

Left Kan extensions. Given a commutative ring R, an ∞-category $\mathcal{D}$ which admits sifted colimits (*e.g.*, $\mathcal{D}(R)$ or $\mathcal{DF}(R)$), and a functor

$$F : \mathrm{Sm}_R := \{\text{smooth } R\text{-algebras}\} \longrightarrow \mathcal{D},$$

define

$$\mathbb{L}F : \quad R\text{-Alg} \longrightarrow \mathcal{D}$$
$$S \longmapsto \underset{P \to S}{\mathrm{colim}}\, F(P),$$

where the colimit is taken over all free R-algebras P mapping to S. The functor $\mathbb{L}F$ is called the left Kan extension from polynomial R-algebras of F. For instance, the cotangent complex $\mathbb{L}_{-/R} := \mathbb{L}\Omega^1_{-/R}$ is the left Kan extension from polynomial R-algebras of the module of Kähler differentials $\Omega^1_{-/R}$, and the derived de Rham complex $\mathbb{L}\Omega_{-/R}$ is the left Kan extension from polynomial R-algebras of the de Rham complex $\Omega_{-/R}$. We also consider more general left Kan extensions (*e.g.*, from smooth R-algebras), which are defined similarly –see [EM23, Section 2.3 and Remark 3.4] for a quick review of this formalism. The left Kan extension from a category $\mathcal{C}$ to a category $\mathcal{C}'$, when this makes sense, is denoted by $L_{\mathcal{C}'/\mathcal{C}}$.

Quasisyntomic rings. A morphism $R \to S$ of commutative rings is called *p-discrete*, for p a prime number, if the derived tensor product $S \otimes^{\mathbb{L}}_R R/p \in \mathcal{D}(R/p)$ is concentrated in degree zero, where it is given by S/p. It is called *p-flat* if it is p-discrete and if its reduction $R/p \to S/p$ modulo p is flat. It is called *p-quasisyntomic* if it is p-flat and if the cotangent complex $\mathbb{L}_{(S/p)/(R/p)} \in \mathcal{D}(S/p)$ has Tor-amplitude in $[-1; 0]$.

A commutative ring R is called *p-quasisyntomic* if it has bounded p-power torsion and if the complex $\mathbb{L}_{R/\mathbb{Z}} \otimes^{\mathbb{L}}_R R/p \in \mathcal{D}(R/p)$ has Tor-amplitude in $[-1; 0]$. Beware that p-quasisyntomic $\mathbb{Z}$-algebras are p-quasisyntomic rings, but the converse is not true: for instance, $\mathbb{F}_p$ is a p-quasisyntomic ring, but the morphism $\mathbb{Z} \to \mathbb{F}_p$ is not p-discrete. We refer to [BMS19] for the definition of the associated p-quasisyntomic topology on p-quasisyntomic rings.

Rigid functor. A functor $F(-)$ on commutative rings is called rigid if for every henselian pair (R, I), the natural map $F(R) \to F(R/I)$ is an equivalence.

Rings and schemes. Quasi-compact quasi-separated (derived) schemes are called qcqs (derived) schemes. These include all affine (derived) schemes, *i.e.*, (animated) commutative rings. Denote by $\mathrm{Sch}^{\mathrm{qcqs}}$ the category of qcqs schemes, $\mathrm{dSch}^{\mathrm{qcqs}}$ the category of qcqs derived schemes, Rings the category of commutative rings, AniRings the category of animated commutative rings. Schemes, resp. commutative rings, are sometimes called *classical* to emphasize that we are not working in the generality

of derived schemes, resp. animated commutative rings. Given a commutative base ring R, also denote by Sm_R the category of smooth schemes over $\mathrm{Spec}(R)$, $\mathrm{Sch}^{\mathrm{fp}}$ the category of finitely presented schemes over $\mathrm{Spec}(R)$, Poly_R the category of polynomial R-algebras, and $\mathbb{E}_1\text{-}\mathrm{Rings}_R$ the category of associative R-linear ring spectra.

Sheafification. We use several Grothendieck topologies, including the Zariski, Nisnevich, étale, and cdh topologies. Denote by L_{Zar}, L_{Nis}, $L_{\mathrm{\acute{e}t}}$, and L_{cdh} the sheafification functors for these topologies.

1.5 Acknowledgements

2   The motivic filtration on topological cyclic homology

In this section, we introduce a motivic filtration on the topological cyclic homology of qcqs derived schemes (Definition 2.30), whose shifted graded pieces $\mathbb{Z}(i)^{\mathrm{TC}}$ will serve as a building block for the definition of the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$ (Remark 3.20).

We first explain how to express topological cyclic homology in terms of its profinite completion and of negative cyclic homology. Following [NS18], and given a qcqs derived scheme X and a prime number p, the p-completed topological cyclic homology $\mathrm{TC}(X; \mathbb{Z}_p)$ of X is constructed from its p-completed topological negative cyclic homology $\mathrm{TC}^-(X; \mathbb{Z}_p)$ and its p-completed topological periodic cyclic homology $\mathrm{TP}(X; \mathbb{Z}_p)$ (see Section 2.2). Following [DGM13, Lemma 6.4.3.2] and [NS18, Section II.4], the topological cyclic homology $\mathrm{TC}(X)$ of X is then defined by a natural cartesian square of spectra

$$
\begin{array}{ccc}
\mathrm{TC}(X) & \longrightarrow & \mathrm{TC}^-(X) \\
\downarrow & & \downarrow \\
\prod_{p \in \mathbb{P}} \mathrm{TC}(X; \mathbb{Z}_p) & \longrightarrow & \prod_{p \in \mathbb{P}} \mathrm{TC}^-(X; \mathbb{Z}_p).
\end{array}
$$

The comparison map $\mathrm{THH}(X) \to \mathrm{HH}(X)$, induced by extension of scalars along the map of $\mathbb{E}_\infty$-rings $\mathrm{THH}(\mathbb{Z}) \to \mathbb{Z}$, is S^1-equivariant, and for every integer $n \in \mathbb{Z}$, the kernel and cokernel of the induced map on homotopy groups $\mathrm{THH}_n(X) \to \mathrm{HH}_n(X)$ are killed by an integer depending only on n. In particular, the natural commutative diagram

$$
\begin{array}{ccc}
\mathrm{THH}(X) & \longrightarrow & \mathrm{HH}(X) \\
\downarrow & & \downarrow \\
\prod_{p \in \mathbb{P}} \mathrm{THH}(X; \mathbb{Z}_p) & \longrightarrow & \prod_{p \in \mathbb{P}} \mathrm{HH}(X; \mathbb{Z}_p),
\end{array}
$$

is a cartesian square of spectra, which in turn defines a natural cartesian square of spectra

$$
\begin{array}{ccc}
\mathrm{TC}^-(X) & \longrightarrow & \mathrm{HC}^-(X) \\
\downarrow & & \downarrow \\
\prod_{p \in \mathbb{P}} \mathrm{TC}^-(X; \mathbb{Z}_p) & \longrightarrow & \prod_{p \in \mathbb{P}} \mathrm{HC}^-(X; \mathbb{Z}_p)
\end{array}
$$

by taking homotopy fixed points $(-)^{h\mathrm{S}^1}$. Composing this cartesian square with the cartesian square defining topological cyclic homology then induces a natural cartesian square

$$
\begin{array}{ccc}
\mathrm{TC}(X) & \longrightarrow & \mathrm{HC}^-(X) \\
\downarrow & & \downarrow \\
\prod_{p \in \mathbb{P}} \mathrm{TC}(X; \mathbb{Z}_p) & \longrightarrow & \prod_{p \in \mathbb{P}} \mathrm{HC}^-(X; \mathbb{Z}_p).
\end{array}
$$

We will use this cartesian square to define the motivic filtration on $\mathrm{TC}(X)$ (Definition 2.30), by glueing existing filtrations on the three other terms; namely, the HKR and BMS filtrations.

2.1   The HKR filtrations

In this subsection, we review the HKR filtrations on Hochschild homology and its variants, as defined, in the generality of qcqs derived schemes, by [Ant19] and [BL22, Section 6.3]. Only the HKR filtration on negative cyclic homology $\mathrm{HC}^-(-)$ (Definition 2.4) will be used to define the motivic filtration on topological cyclic homology $\mathrm{TC}(-)$ (Definition 2.30). We will use the other HKR filtrations of this section in Section 4.

The following result is [BL22, Example 6.1.3 and Remarks 6.1.4 and 6.1.5].

Proposition 2.1 (Tate filtration). *Let X be a spectrum equipped with an S^1-action. Then the Tate construction $X^{t\mathrm{S}^1} \in \mathrm{Sp}$ is naturally equipped with a $\mathbb{Z}$-indexed filtration*

$$\mathrm{Fil}_{\mathrm{T}}^{\star} X^{t\mathrm{S}^1} \in \mathrm{FilSp}.$$

This filtration is called the Tate filtration *on $X^{t\mathrm{S}^1}$, and satisfies the following properties:*

(1) *The filtration $\mathrm{Fil}_{\mathrm{T}}^{\star} X^{t\mathrm{S}^1} \in \mathrm{FilSp}$ is complete.*

(2) *$\mathrm{Fil}_{\mathrm{T}}^{0} X^{t\mathrm{S}^1}$ is the homotopy fixed point spectrum $X^{h\mathrm{S}^1}$, which is thus also equipped with an $\mathbb{N}$-indexed complete filtration $\mathrm{Fil}_{\mathrm{T}}^{\star} X^{h\mathrm{S}^1}$, which we call the Tate filtration on $X^{h\mathrm{S}^1}$.*

(3) *For every integer $n \in \mathbb{Z}$, the graded piece $\mathrm{gr}_{\mathrm{T}}^{n} X^{t\mathrm{S}^1}$ is naturally identified with the spectrum $X[-2n]$.*

Following [BMS19, Section 5], a filtered spectrum $\mathrm{Fil}^{\star} X$ is called *connective for the Beilinson t-structure* if for every integer $i \in \mathbb{Z}$, the graded piece $\mathrm{gr}^{i} X \in \mathrm{Sp}$ is in cohomological degrees at most i. For every integer $i \in \mathbb{Z}$, also denote by $\tau_{\geq i}^{\mathrm{B}}$ the truncation functor for the Beilinson t-structure on filtered spectra.

Definition 2.2 (Décalage filtration). Let $\mathrm{Fil}^{\star} X \in \mathrm{FilSp}$ be a filtered spectrum. The *décalage filtration* on $\mathrm{Fil}^{\star} X$ is the bifiltered spectrum

$$\mathrm{Fil}_{\mathrm{B}}^{\star} \mathrm{Fil}^{\star} X \in \mathrm{biFilSp}$$

where, for every integer $i \in \mathbb{Z}$, $\mathrm{Fil}_{\mathrm{B}}^{i} \mathrm{Fil}^{\star} X$ is the i-connective cover of $\mathrm{Fil}^{\star} X \in \mathrm{FilSp}$ with respect to the Beilinson t-structure on the category of filtered spectra:

$$\mathrm{Fil}_{\mathrm{B}}^{i} \mathrm{Fil}^{\star} X := \tau_{\geq i}^{\mathrm{B}} \mathrm{Fil}^{\star} X.$$

Construction 2.3 (HKR filtration on HP). For every integer $i \in \mathbb{Z}$, let

$$\mathrm{Fil}_{\mathrm{HKR}}^{i} \mathrm{Fil}_{\mathrm{T}}^{\star} \mathrm{HP}(-) := L_{\mathrm{Zar}} L_{\mathrm{dSch}^{\mathrm{qcqs,op}} / \mathrm{Poly}_{\mathbb{Z}}^{\mathrm{op}}} \mathrm{Fil}_{\mathrm{B}}^{i} \mathrm{Fil}_{\mathrm{T}}^{\star} \mathrm{HP}(-),$$

where $\mathrm{Fil}_{\mathrm{T}}^{\star} \mathrm{HP}(-)$ is the Tate filtration on periodic cyclic homology of qcqs derived schemes, $\mathrm{Fil}_{\mathrm{B}}^{\star}$ is the décalage filtration of Definition 2.2, and the left Kan extension $L_{\mathrm{dSch}^{\mathrm{qcqs,op}} / \mathrm{Poly}_{\mathbb{Z}}^{\mathrm{op}}}$ is taken in the category of filtration-complete filtered spectra. The *HKR filtration on periodic cyclic homology* of qcqs derived schemes is the functor

$$\mathrm{Fil}_{\mathrm{HKR}}^{\star} \mathrm{HP}(-) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

defined as the underlying filtered object of the bifiltered functor $\mathrm{Fil}_{\mathrm{HKR}}^{\star} \mathrm{Fil}_{\mathrm{T}}^{\star} \mathrm{HP}(-)$:

$$\mathrm{Fil}_{\mathrm{HKR}}^{\star} \mathrm{HP}(-) := \varinjlim_{n} \mathrm{Fil}_{\mathrm{HKR}}^{\star} \mathrm{Fil}_{\mathrm{T}}^{n} \mathrm{HP}(-).$$

The following definition is the one which will appear explicitly in the definition of the motivic filtration on topological cyclic homology (Definition 2.30).

Definition 2.4 (HKR filtration on HC$^-$). The *HKR filtration on negative cyclic homology* of qcqs derived schemes is the functor

$$\mathrm{Fil}_{\mathrm{HKR}}^{\star} \mathrm{HC}^{-}(-) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

defined as

$$\mathrm{Fil}_{\mathrm{HKR}}^{\star} \mathrm{HC}^{-}(-) := \mathrm{Fil}_{\mathrm{HKR}}^{\star} \mathrm{Fil}_{\mathrm{T}}^{0} \mathrm{HP}(-).$$

The following definition is motivated by Proposition 2.1 (3).

Definition 2.5 (HKR filtration on HH). The *HKR filtration on Hochschild homology* of qcqs derived schemes is the functor

$$\mathrm{Fil}_{\mathrm{HKR}}^{\star} \mathrm{HH}(-) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

defined as

$$\mathrm{Fil}_{\mathrm{HKR}}^{\star} \mathrm{HH}(-) := \mathrm{Fil}_{\mathrm{HKR}}^{\star} \mathrm{gr}_{\mathrm{T}}^{0} \mathrm{HP}(-).$$

Cyclic homology $\mathrm{HC}(-)$ is defined as the homotopy orbits $\mathrm{HH}(-)_{h\mathrm{S}^1}$ of the S^1-action on Hochschild homology $\mathrm{HH}(-)$, and is related to negative cyclic homology $\mathrm{HC}^-(-)$ and periodic cyclic homology $\mathrm{HP}(-)$ by a natural fibre sequence

$$\mathrm{HC}^-(-) \longrightarrow \mathrm{HP}(-) \longrightarrow \mathrm{HC}(-)[2].$$

Definition 2.6 (HKR filtration on HC)**.** The *HKR filtration on cyclic homology* of qcqs derived schemes is the functor

$$\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}(-) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

defined, for every integer $i \in \mathbb{Z}$, by

$$\mathrm{Fil}^i_{\mathrm{HKR}}\mathrm{HC}(-) := \mathrm{cofib}\Big(\mathrm{Fil}^{i+1}_{\mathrm{HKR}}\mathrm{HC}^-(-) \longrightarrow \mathrm{Fil}^{i+1}_{\mathrm{HKR}}\mathrm{HP}(-)\Big)[-2],$$

where the map on the right hand side is induced by Construction 2.3 and Definition 2.4.

Remark 2.7 (Graded pieces of the HKR filtrations)**.** Let X be a qcqs derived scheme. The main result of [Ant19] describes the graded pieces of the HKR filtrations on $\mathrm{HC}^-(X)$, $\mathrm{HP}(X)$, and $\mathrm{HC}(X)$ in terms of the Hodge-completed derived de Rham cohomology of X. In particular, Definition 2.6 provides a filtered refinement of the fibre sequence

$$\mathrm{HC}^-(-) \longrightarrow \mathrm{HP}(-) \longrightarrow \mathrm{HC}(-)[2],$$

which induces on graded pieces, for every integer $i \in \mathbb{Z}$, a natural fibre sequence

$$R\Gamma_{\mathrm{Zar}}\big(X, \widehat{\mathbb{L}\Omega}^{\geq i}_{-/\mathbb{Z}}\big)[2i] \longrightarrow R\Gamma_{\mathrm{Zar}}\big(X, \widehat{\mathbb{L}\Omega}_{-/\mathbb{Z}}\big)[2i] \longrightarrow R\Gamma_{\mathrm{Zar}}\big(X, \mathbb{L}\Omega^{\leq i}_{-/\mathbb{Z}}\big)[2i]$$

in the derived category $\mathcal{D}(\mathbb{Z})$.

Proposition 2.8 ([Ant19, BL22])**.** *For every integer $i \in \mathbb{Z}$, the functor $\mathrm{Fil}^i_{\mathrm{HKR}}\mathrm{HC}(-)$, from animated commutative rings to spectra, is left Kan extended from polynomial $\mathbb{Z}$-algebras, commutes with filtered colimits, and its values are in cohomological degrees at most $-i$.*

Proof. On animated commutative rings, the Tate filtrations

$$\mathrm{Fil}^{i+1}_{\mathrm{HKR}}\mathrm{Fil}^\star_{\mathrm{T}}\mathrm{HC}^-(-) \quad \text{and} \quad \mathrm{Fil}^{i+1}_{\mathrm{HKR}}\mathrm{Fil}^\star_{\mathrm{T}}\mathrm{HP}(-)$$

are by definition left Kan extended, as complete filtered objects, from polynomial $\mathbb{Z}$-algebras, thus so is the Tate filtration $\mathrm{Fil}^i_{\mathrm{HKR}}\mathrm{Fil}^\star_{\mathrm{T}}\mathrm{HC}(-)$. The Tate filtration $\mathrm{Fil}^i_{\mathrm{HKR}}\mathrm{Fil}^\star_{\mathrm{T}}\mathrm{HC}(-)$ is also finite by construction, so the functor $\mathrm{Fil}^i_{\mathrm{HKR}}\mathrm{HC}(-)$ is left Kan extended on animated commutative rings from polynomial $\mathbb{Z}$-algebras. In particular, the functor

$$\mathrm{Fil}^i_{\mathrm{HKR}}\mathrm{HC}(-)$$

commutes with filtered colimits of animated commutative rings. For every integer $j \in \mathbb{Z}$, the jth graded piece of the filtration $\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}(-)$ is naturally identified with the functor $R\Gamma_{\mathrm{Zar}}\big(-, \mathbb{L}\Omega^{\leq j}_{-/\mathbb{Z}}\big)[2j]$ ([Ant19]), whose values are in degrees at most $-j$ on animated commutative rings. The filtration $\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}(-)$ is moreover complete on animated commutative rings ([BL22, Remark 6.3.5]), hence the desired connectivity result. $\square$

Lemma 2.9 (Completeness of the HKR filtrations, after [BL22])**.** *Let X be a qcqs derived scheme. Then the HKR filtrations*

$$\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HP}(X), \ \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X), \ \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HH}(X), \ and \ \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}(X)$$

are complete.

Proof. The result for HC^- and HH is a direct consequence of [BL22, Remark 6.3.5]. The result for HC is a consequence of the connectivity result of Proposition 2.8. By Definition 2.6, the result for HP is then a consequence of the result for HC^- and HC. $\square$

Remark 2.10 (Variant over $\mathbb{Q}$)**.** Let X be a qcqs derived scheme. By the base change property for Hochschild homology, the natural map

$$\mathrm{HH}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \mathrm{HH}(X_{\mathbb{Q}}/\mathbb{Q})$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Q})$, where $\mathrm{HH}(-/\mathbb{Q})$ is Hochschild homology relative to $\mathbb{Q}$. Applying the functors $(-)^{h\mathrm{S}^1}$, $(-)^{t\mathrm{S}^1}$, and $(-)_{h\mathrm{S}^1}$ to this Hochschild homology relative to $\mathbb{Q}$ induces relative variants $\mathrm{HC}^-(-_{\mathbb{Q}}/\mathbb{Q})$ of negative cyclic homology, $\mathrm{HP}(-_{\mathbb{Q}}/\mathbb{Q})$ of periodic cyclic homology, and $\mathrm{HC}(-_{\mathbb{Q}}/\mathbb{Q})$ of cylic homology. One can then define similar HKR filtrations

$$\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HH}(X_{\mathbb{Q}}/\mathbb{Q}), \ \mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}^-(X_{\mathbb{Q}}/\mathbb{Q}), \ \mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HP}(-_{\mathbb{Q}}/\mathbb{Q}), \ \text{and} \ \mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}(-_{\mathbb{Q}}/\mathbb{Q}),$$

on these functors, whose graded pieces are versions of derived de Rham cohomology relative to $\mathbb{Q}$.

To introduce and study the motivic filtration on topological cyclic homology (Definition 2.30), we will need some p-complete variants of the previous HKR filtrations.

Definition 2.11 (HKR filtration on $\mathrm{HC}^-(-;\mathbb{Z}_p)$)**.** Let p be a prime number. The *HKR filtration on p-completed negative cyclic homology* of qcqs derived schemes is the functor

$$\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}^-(-;\mathbb{Z}_p) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

defined as

$$\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}^-(-;\mathbb{Z}_p) := \left(\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}^-(-)\right)^{\wedge}_p.$$

Remark 2.12. The HKR filtrations on $\mathrm{HP}(-;\mathbb{Z}_p)$, $\mathrm{HC}(-;\mathbb{Z}_p)$, and $\mathrm{HH}(-;\mathbb{Z}_p)$ of qcqs derived schemes are defined as in Definition 2.11, where $\mathrm{HC}^-(-;\mathbb{Z}_p)$ is replaced by $\mathrm{HP}(-)$, $\mathrm{HC}(-)$, or $\mathrm{HH}(-)$. In particular, for every qcqs derived scheme X, Definition 2.6 induces a fibre sequence of filtered spectra

$$\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}^-(X;\mathbb{Z}_p) \longrightarrow \mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HP}(X;\mathbb{Z}_p) \longrightarrow \mathrm{Fil}^{\star-1}_{\mathrm{HKR}}\mathrm{HC}(X;\mathbb{Z}_p)[2].$$

Lemma 2.13. *Let X be a qcqs derived scheme. Then the filtrations*

$$\prod_{p\in\mathbb{P}}\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HP}(X;\mathbb{Z}_p), \quad \prod_{p\in\mathbb{P}}\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}^-(X;\mathbb{Z}_p), \quad \prod_{p\in\mathbb{P}}\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HH}(X;\mathbb{Z}_p),$$

$$\text{and} \ \prod_{p\in\mathbb{P}}\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}(X;\mathbb{Z}_p)$$

are complete.

Proof. The collection of complete filtered spectra is closed under limits in the category of filtered spectra, so this is a consequence of Lemma 2.9. $\qquad\square$

Remark 2.14 (Exhaustivity of the HKR filtrations)**.** The HKR filtrations $\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}^-$ and $\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HP}$ are not exhaustive on general qcqs derived schemes ([BL22, Remark 6.3.6]). For the purpose of the motivic filtration on algebraic K-theory (Definition 3.17), we will however only need the fact that the HKR filtration $\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}$ is always $\mathbb{N}$-indexed, and in particular exhaustive.

2.2 The BMS filtrations

Let p be a prime number. In this subsection, we review the BMS filtrations on p-completed topological Hochschild homology $\mathrm{THH}(-;\mathbb{Z}_p)$ and its variants, as defined in [BMS19] for p-complete p-quasisyntomic rings, and generalised in [AMMN22] to p-complete rings and in [BL22, Section 6.2] to animated commutative rings. Only the BMS filtration on p-completed topological cyclic homology $\mathrm{TC}(-;\mathbb{Z}_p)$ (Definition 2.20) will appear in the definition of the motivic filtration on topological cyclic homology $\mathrm{TC}(-)$ (Definition 2.30). The other BMS filtrations are necessary to construct the BMS filtration on p-completed topological cyclic homology $\mathrm{TC}(-;\mathbb{Z}_p)$.

Construction 2.15 (BMS filtration on $\mathrm{Fil}_{\mathrm{T}}^{\star}\mathrm{TP}(-;\mathbb{Z}_p)$)**.** Topological Hochschild homology $\mathrm{THH}(-)$ of qcqs derived schemes admits a natural S^1-action, inducing a natural Tate filtration $\mathrm{Fil}_{\mathrm{T}}^{\star}\mathrm{TP}(-)$ on topological periodic cyclic homology $\mathrm{TP}(-) := \mathrm{THH}(-)^{tS^1}$ (Proposition 2.1). The Tate filtration

$$\mathrm{Fil}_{\mathrm{T}}^{\star}\mathrm{TP}(-;\mathbb{Z}_p) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

is then defined as the p-completion of the Tate filtration $\mathrm{Fil}_{\mathrm{T}}^{\star}\mathrm{TP}(-)$. For every quasiregular semiperfectoid ring R and every integer $i \in \mathbb{Z}$, define the filtered spectrum

$$\mathrm{Fil}_{\mathrm{BMS}}^{i}\mathrm{Fil}_{\mathrm{T}}^{\star}\mathrm{TP}(R;\mathbb{Z}_p) := \tau_{\geq 2i}\mathrm{Fil}_{\mathrm{T}}^{\star}\mathrm{TP}(R;\mathbb{Z}_p).$$

The filtered object

$$\mathrm{Fil}_{\mathrm{BMS}}^{i}\mathrm{Fil}_{\mathrm{T}}^{\star}\mathrm{TP}(-;\mathbb{Z}_p) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

is then first defined on p-quasisyntomic rings as the unique such functor satisfying p-complete faithfully flat descent (the existence and unicity of such a functor is [BMS19, Proposition 4.31]). In general, polynomial $\mathbb{Z}$-algebras are p-quasisyntomic rings, and this filtered object is defined as the Zariski sheafification of its left Kan extension from polynomial $\mathbb{Z}$-algebras

$$\mathrm{Fil}_{\mathrm{BMS}}^{i}\mathrm{Fil}_{\mathrm{T}}^{\star}\mathrm{TP}(-;\mathbb{Z}_p) := L_{\mathrm{Zar}}L_{\mathrm{AniRings/Poly}_{\mathbb{Z}}}\mathrm{Fil}_{\mathrm{BMS}}^{i}\mathrm{Fil}_{\mathrm{T}}^{\star}\mathrm{TP}(-;\mathbb{Z}_p),$$

where the left Kan extension is taken in the category of p-complete filtration-complete spectra. By [BL22, Theorem 6.2.4], the resulting functor is still given by the double-speed Postnikov filtration on quasiregular semiperfectoid rings and, as a functor from animated commutative rings to p-complete filtration-complete spectra, commutes with sifted colimits and satisfies p-complete faithfully flat descent.

Remark 2.16. The BMS filtrations were first defined in [BMS19] in the generality of p-complete p-quasisyntomic rings. On general animated commutative rings R, the BMS filtrations, by construction, depend only on the p-completion of R –and in particular vanish on animated commutative $\mathbb{Z}[\frac{1}{p}]$-algebras. Here the p-completion is the derived p-completion, even on classical commutative rings. On commutative rings with bounded p-power torsion (*e.g.*, on p-quasisyntomic rings), the derived and classical p-completions naturally coincide, and there is no conflict between the two definitions.

Definition 2.17 (BMS filtration on $\mathrm{TP}(-;\mathbb{Z}_p)$)**.** The *BMS filtration on p-completed topological periodic cyclic homology* of qcqs derived schemes is the functor

$$\mathrm{Fil}_{\mathrm{BMS}}^{\star}\mathrm{TP}(-;\mathbb{Z}_p) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

defined as the underlying filtered object of the bifiltered functor $\mathrm{Fil}_{\mathrm{BMS}}^{\star}\mathrm{Fil}_{\mathrm{T}}^{\star}\mathrm{TP}(-;\mathbb{Z}_p)$ of Construction 2.15:

$$\mathrm{Fil}_{\mathrm{BMS}}^{\star}\mathrm{TP}(-;\mathbb{Z}_p) := \varinjlim_{n} \mathrm{Fil}_{\mathrm{BMS}}^{\star}\mathrm{Fil}_{\mathrm{T}}^{n}\mathrm{TP}(-;\mathbb{Z}_p).$$

Topological negative cyclic homology is to topological periodic cyclic homology what negative cyclic homology is to periodic cyclic homology. Given Definition 2.17, the following definition then mimics Definition 2.4.

Definition 2.18 (BMS filtration on $\mathrm{TC}^{-}(-;\mathbb{Z}_p)$)**.** The *BMS filtration on p-completed topological negative cyclic homology* of qcqs derived schemes is the functor

$$\mathrm{Fil}_{\mathrm{BMS}}^{\star}\mathrm{TC}^{-}(-;\mathbb{Z}_p) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

defined as

$$\mathrm{Fil}_{\mathrm{BMS}}^{\star}\mathrm{TC}^{-}(-;\mathbb{Z}_p) := \mathrm{Fil}_{\mathrm{BMS}}^{\star}\mathrm{Fil}_{\mathrm{T}}^{0}\mathrm{TP}(-;\mathbb{Z}_p).$$

Similarly, topological Hochschild homology is to topological periodic and topological negative cyclic homologies what Hochschild homology is to periodic and negative cyclic homologies, and the following definition mimics Definition 2.5.

Definition 2.19 (BMS filtration on $\mathrm{THH}(-;\mathbb{Z}_p)$). The *BMS filtration on p-completed topological Hochschild homology* of qcqs derived schemes is the functor

$$\mathrm{Fil}^{\star}_{\mathrm{BMS}}\mathrm{THH}(-;\mathbb{Z}_p) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

defined as

$$\mathrm{Fil}^{\star}_{\mathrm{BMS}}\mathrm{THH}(-;\mathbb{Z}_p) := \mathrm{Fil}^{\star}_{\mathrm{BMS}}\mathrm{gr}^0_{\mathrm{T}}\mathrm{TP}(-;\mathbb{Z}_p).$$

Topological cyclic homology is however not to topological periodic cyclic homology what cyclic homology is to periodic cyclic homology. Following [NS18], it is rather defined, after p-completion, by a fibre sequence

$$\mathrm{TC}(-;\mathbb{Z}_p) \longrightarrow \mathrm{TC}^-(-;\mathbb{Z}_p) \xrightarrow{\phi_p - \mathrm{can}} \mathrm{TP}(-;\mathbb{Z}_p).$$

Unwinding the previous definitions, the map $\phi_p - \mathrm{can} : \mathrm{TC}^-(-;\mathbb{Z}_p) \to \mathrm{TP}(-;\mathbb{Z}_p)$ admits a unique refinement as a filtered map

$$\phi_p - \mathrm{can} : \mathrm{Fil}^{\star}_{\mathrm{BMS}}\mathrm{TC}^-(-;\mathbb{Z}_p) \longrightarrow \mathrm{Fil}^{\star}_{\mathrm{BMS}}\mathrm{TP}(-;\mathbb{Z}_p).$$

Definition 2.20 (BMS filtration on $\mathrm{TC}(-;\mathbb{Z}_p)$). The *BMS filtration on p-completed topological cyclic homology* of qcqs derived schemes is the functor

$$\mathrm{Fil}^{\star}_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

defined as

$$\mathrm{Fil}^{\star}_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p) := \mathrm{fib}\Big(\phi_p - \mathrm{can} : \mathrm{Fil}^{\star}_{\mathrm{BMS}}\mathrm{TC}^-(-;\mathbb{Z}_p) \longrightarrow \mathrm{Fil}^{\star}_{\mathrm{BMS}}\mathrm{TP}(-;\mathbb{Z}_p)\Big).$$

The BMS filtration on p-completed topological cyclic homology is always complete, as a consequence of a connectivity result of [AMMN22]. We will need the following slightly more precise result when studying the completeness of the motivic filtration on algebraic K-theory.

Lemma 2.21. *Let X be a qcqs derived scheme. Then the filtrations $\prod_{p\in\mathbb{P}} \mathrm{Fil}^{\star}_{\mathrm{BMS}}\mathrm{TC}(X;\mathbb{Z}_p)$ and $\left(\prod_{p\in\mathbb{P}} \mathrm{Fil}^{\star}_{\mathrm{BMS}}\mathrm{TC}(X;\mathbb{Z}_p)\right)_{\mathbb{Q}}$ are complete. More precisely, for every integer $i \in \mathbb{Z}$, the values of the presheaves $\prod_{p\in\mathbb{P}} \mathrm{Fil}^{i}_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)$ and $\left(\prod_{p\in\mathbb{P}} \mathrm{Fil}^{i}_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)\right)_{\mathbb{Q}}$ are in cohomological degrees at most $-i + 1$ on affine derived schemes.*

Proof. The presheaves

$$\prod_{p\in\mathbb{P}} \mathrm{Fil}^{\star}_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p) \quad \text{and} \quad \left(\prod_{p\in\mathbb{P}} \mathrm{Fil}^{\star}_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)\right)_{\mathbb{Q}}$$

are Zariski sheaves by construction, so it suffices to prove the result for affine derived schemes X. Let R be an animated commutative ring, and $i \in \mathbb{Z}$ be an integer. The spectrum $\mathrm{Fil}^{i}_{\mathrm{BMS}}\mathrm{TC}(R;\mathbb{Z}_p)$ is in cohomological degrees at most $-i + 1$ for every prime number p ([AMMN22, Theorem 5.1]). Taking the product over all primes p and rationalisation, this implies that the spectra $\prod_{p\in\mathbb{P}} \mathrm{Fil}^{i}_{\mathrm{BMS}}\mathrm{TC}(R;\mathbb{Z}_p)$ and $\left(\prod_{p\in\mathbb{P}} \mathrm{Fil}^{i}_{\mathrm{BMS}}\mathrm{TC}(R;\mathbb{Z}_p)\right)_{\mathbb{Q}}$ are also in cohomological degrees at most $-i + 1$, which implies that the associated filtrations are complete. $\square$

Remark 2.22 (Exhaustivity of the BMS filtrations). The BMS filtrations $\mathrm{Fil}^{\star}_{\mathrm{BMS}}\mathrm{TP}(-;\mathbb{Z}_p)$ and $\mathrm{Fil}^{\star}_{\mathrm{BMS}}\mathrm{TC}^-(-;\mathbb{Z}_p)$ are not exhaustive on general qcqs derived schemes ([BL22, Warning 6.2.7]). For the purpose of the motivic filtration on algebraic K-theory (Definition 3.17), we will however only need the fact that the BMS filtration $\mathrm{Fil}^{\star}_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)$ is always $\mathbb{N}$-indexed ([BMS19, proof of Proposition 7.16]), and in particular exhaustive.

We refer to [BMS19, BS22, BL22] for the relation between prismatic cohomology and the graded pieces of the BMS filtrations on $\mathrm{TP}(-;\mathbb{Z}_p)$, $\mathrm{TC}^-(-;\mathbb{Z}_p)$, and $\mathrm{THH}(-;\mathbb{Z}_p)$. We only define here the shifted graded pieces of the BMS filtration on $\mathrm{TC}(-;\mathbb{Z}_p)$, which are a version of syntomic cohomology (see Remark 2.24), and which will serve as a building block for the p-adic motivic complexes (Corollary 3.25).

Definition 2.23 (BMS syntomic cohomology). For every integer $i \in \mathbb{Z}$, the *syntomic complex*

$$\mathbb{Z}_p(i)^{\mathrm{BMS}}(-) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathcal{D}(\mathbb{Z})$$

is the functor defined as the shifted graded piece of the BMS filtration on $\mathrm{TC}(-;\mathbb{Z}_p)$:

$$\mathbb{Z}_p(i)^{\mathrm{BMS}}(-) := \mathrm{gr}^i_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)[-2i].$$

Remark 2.24. Syntomic cohomology $\mathbb{Z}_p(i)^{\mathrm{syn}}(X)$ of qcqs derived (formal) schemes X is defined in [BL22, Section 8.4] (see also Notation 5.7), in terms of the syntomic complexes of Definition 2.23 and of étale cohomology. From this perspective, the syntomic complexes $\mathbb{Z}_p(i)^{\mathrm{BMS}}(X)$ of Definition 2.23 correspond to the syntomic cohomology $\mathbb{Z}_p(i)^{\mathrm{syn}}(\mathfrak{X})$ of the derived p-adic formal scheme $\mathfrak{X}$ associated to X.

Theorem 2.25. *(1) ([BMS19, BL22]) The functor $\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)$, viewed as a functor from p-quasisyntomic rings to p-complete filtered spectra, satisfies descent for the p-quasisyntomic topology.*

(2) ([AMMN22, BL22]) The functor $\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)$, viewed as a functor from animated commutative rings to p-complete filtered spectra, is left Kan extended from polynomial $\mathbb{Z}$-algebras.

Proof. (1) The filtration $\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)$ is complete on p-quasisyntomic rings (Lemma 2.21), so it suffices to prove the result on graded pieces. The result on graded pieces is a special case of [BL22, Proposition 7.4.7].

(2) By [AMMN22, Theorem 5.1 (2)], the functor $\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)$, viewed as a functor from p-complete animated commutative rings to p-complete filtered spectra, is left Kan extended from p-complete polynomial $\mathbb{Z}$-algebras.[2] Let R be an animated commutative ring, and $R_p^\wedge$ be its (derived) p-completion. The natural map

$$\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(R;\mathbb{Z}_p) \longrightarrow \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(R_p^\wedge;\mathbb{Z}_p)$$

is an equivalence of filtered spectra. Indeed, the filtrations

$$\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(R;\mathbb{Z}_p) \quad \text{and} \quad \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(R_p^\wedge;\mathbb{Z}_p)$$

are $\mathbb{N}$-indexed and complete (Lemma 2.21 and Remark 2.22), so it suffices to prove the result on graded pieces, where this is a direct consequence of [BL22, Corollary 7.4.11]. This implies the desired left Kan extension property. $\square$

Corollary 2.26. *Let $i \in \mathbb{Z}$ be an integer.*

(1) The functor $\mathbb{Z}_p(i)^{\mathrm{BMS}}(-)$, viewed as a functor from p-quasisyntomic rings to p-complete objects in the derived category $\mathcal{D}(\mathbb{Z})$, satisfies descent for the p-quasisyntomic topology.

(2) The functor $\mathbb{Z}_p(i)^{\mathrm{BMS}}(-)$, viewed as a functor from animated commutative rings to p-complete objects in the derived category $\mathcal{D}(\mathbb{Z})$, is left Kan extended from polynomial $\mathbb{Z}$-algebras.

Proof. (1) was already part of the proof of Theorem 2.25 (1).

(2) is a direct consequence of Theorem 2.25 (2). $\square$

Theorem 2.27 ([AMMN22]). *Let (A, I) be a henselian pair of commutative rings. Then for any integers $i \geq 0$ and $k \geq 1$, the fibre of the natural map*

$$\mathbb{Z}/p^k(i)^{\mathrm{BMS}}(A) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{BMS}}(A/I)$$

in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$ is in degrees at most i.

---

[2]More precisely, it is proved to be left Kan extended from p-complete polynomial $\mathbb{Z}$-algebras to p-complete p-quasisyntomic rings. By definition, p-quasisyntomic rings have bounded p-power torsion. Hence, their derived and classical p-completions are naturally identified, and the left Kan extension to p-complete animated commutative rings agrees with the left Kan extension to p-complete classical rings on p-complete p-quasisyntomic rings.

Proof. By [AMMN22, Theorem 5.2], for every henselian pair (A, I) such that the commutative rings A and A/I are (classically) p-complete, the fibre of the natural map

$$\mathbb{Z}/p^k(i)^{\mathrm{BMS}}(A) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{BMS}}(A/I)$$

is in degrees at most i. The proof of [AMMN22, Theorem 5.2] proves more generally that for (A, I) a general henselian pair of commutative rings, the fibre of the natural map

$$\mathbb{Z}/p^k(i)^{\mathrm{BMS}}(A_p^\wedge) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{BMS}}((A/I)_p^\wedge)$$

where $(-)_p^\wedge$ is the derived p-completion, is in degrees at most i. By [BL22, Corollary 7.4.11] (see also the proof of Theorem 2.25 (2)), the natural map $\mathbb{Z}/p^k(i)^{\mathrm{BMS}}(-) \to \mathbb{Z}/p^k(i)^{\mathrm{BMS}}((-)_p^\wedge)$ is an equivalence on animated commutative rings, hence for every henselian pair (A, I) of commutative rings, the fibre of the natural map

$$\mathbb{Z}/p^k(i)^{\mathrm{BMS}}(A) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{BMS}}(A/I)$$

is in degrees at most i. $\qquad\square$

2.3 The motivic filtration on TC

In this subsection, we introduce the motivic filtration on topological cyclic homology $\mathrm{TC}(-)$ of general qcqs derived schemes (Definition 2.30).

The following proposition is [BL22, Proposition 6.4.1].

Proposition 2.28 ([BL22])**.** *Let p be a prime number. The map*

$$\mathrm{Fil}_{\mathrm{T}}^\star \mathrm{TP}(-; \mathbb{Z}_p) \longrightarrow \mathrm{Fil}_{\mathrm{T}}^\star \mathrm{HP}(-; \mathbb{Z}_p),$$

viewed as a map of filtered spectra-valued presheaves on the category of qcqs derived schemes, admits a unique, multiplicative extension to a map of bifiltered presheaves of spectra

$$\mathrm{Fil}_{\mathrm{BMS}}^\star \mathrm{Fil}_{\mathrm{T}}^\star \mathrm{TP}(-; \mathbb{Z}_p) \longrightarrow \mathrm{Fil}_{\mathrm{HKR}}^\star \mathrm{Fil}_{\mathrm{T}}^\star \mathrm{HP}(-; \mathbb{Z}_p).$$

Construction 2.29 (BMS-HKR comparison map)**.** Let p be a prime number. The *BMS-HKR comparison map* is the map

$$\mathrm{Fil}_{\mathrm{BMS}}^\star \mathrm{TC}(-; \mathbb{Z}_p) \longrightarrow \mathrm{Fil}_{\mathrm{HKR}}^\star \mathrm{HC}^-(-; \mathbb{Z}_p)$$

of functors from (the opposite category of) qcqs derived schemes to the category of filtered spectra defined as the composite

$$\mathrm{Fil}_{\mathrm{BMS}}^\star \mathrm{TC}(-; \mathbb{Z}_p) \longrightarrow \mathrm{Fil}_{\mathrm{BMS}}^\star \mathrm{TC}^-(-; \mathbb{Z}_p) \longrightarrow \mathrm{Fil}_{\mathrm{HKR}}^\star \mathrm{HC}^-(-; \mathbb{Z}_p)$$

of the maps given by Definition 2.20, and Proposition 2.28 after restricting to the zeroth step of the Tate filtration.

Definition 2.30 (Motivic filtration on TC)**.** The *motivic filtration on topological cyclic homology* of qcqs derived schemes

$$\mathrm{Fil}_{\mathrm{mot}}^\star \mathrm{TC}(-) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

is the functor defined by the cartesian square

$$\begin{array}{ccc}
\mathrm{Fil}_{\mathrm{mot}}^\star \mathrm{TC}(-) & \longrightarrow & \mathrm{Fil}_{\mathrm{HKR}}^\star \mathrm{HC}^-(-) \\
\downarrow & & \downarrow \\
\prod_{p \in \mathbb{P}} \mathrm{Fil}_{\mathrm{BMS}}^\star \mathrm{TC}(-; \mathbb{Z}_p) & \longrightarrow & \prod_{p \in \mathbb{P}} \mathrm{Fil}_{\mathrm{HKR}}^\star \mathrm{HC}^-(-; \mathbb{Z}_p),
\end{array}$$

where the bottom horizontal map is the map of Construction 2.29, and the right vertical map is profinite completion. For every integer $i \in \mathbb{Z}$, also define the functor

$$\mathbb{Z}(i)^{\mathrm{TC}}(-) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathcal{D}(\mathbb{Z})^3$$

as the shifted graded piece of this motivic filtration:

$$\mathbb{Z}(i)^{\mathrm{TC}}(-) := \mathrm{gr}^i_{\mathrm{mot}} \mathrm{TC}(-)[-2i].$$

Remark 2.31 (Comparison to [EM23]). For every qcqs derived scheme X over $\mathbb{Q}$, the filtered spectrum $\mathrm{Fil}^\star_{\mathrm{HKR}} \mathrm{HC}^-(X)$ is $\mathbb{Q}$-linear by construction, so its profinite completion vanishes. The filtration $\prod_{p \in \mathbb{P}} \mathrm{Fil}^\star_{\mathrm{BMS}} \mathrm{TC}(X; \mathbb{Z}_p)$ also vanishes (Remark 2.16), and the natural map

$$\mathrm{Fil}^\star_{\mathrm{mot}} \mathrm{TC}(X) \longrightarrow \mathrm{Fil}^\star_{\mathrm{HKR}} \mathrm{HC}^-(X/\mathbb{Q})$$

is then an equivalence of filtered spectra.

Similarly, for every prime number p and every qcqs derived scheme X over $\mathbb{F}_p$, the filtered spectrum $\mathrm{Fil}^\star_{\mathrm{HKR}} \mathrm{HC}^-(X)$ is $\mathbb{Z}$-linear and p-complete, so it is naturally identified with its profinite completion. Again using Remark 2.16, the natural map

$$\mathrm{Fil}^\star_{\mathrm{mot}} \mathrm{TC}(X) \longrightarrow \mathrm{Fil}^\star_{\mathrm{BMS}} \mathrm{TC}(X; \mathbb{Z}_p)$$

is then an equivalence of filtered spectra.

Remark 2.32 (Comparison to [BL22]). In [BL22, Section 6.4], Bhatt–Lurie define filtered spectra $\mathrm{Fil}^\star_{\mathrm{mot}} \mathrm{TP}(X)$ and $\mathrm{Fil}^\star_{\mathrm{mot}} \mathrm{TC}^-(X)$ for qcqs derived schemes X, with shifted graded pieces called the global prismatic complexes $\widehat{\mathbb{\Delta}}^{\mathrm{gl}}_X\{i\}$ and $\mathcal{N}^{\geq i} \widehat{\mathbb{\Delta}}^{\mathrm{gl}}_X\{i\}$ respectively. These filtrations can be used to obtain an alternative definition of the $\mathrm{Fil}^\star_{\mathrm{mot}} \mathrm{TC}(X)$ of Definition 2.30. More precisely, for every prime number p the p-completion of Bhatt–Lurie's filtration $\mathrm{Fil}^\star_{\mathrm{mot}} \mathrm{TP}(X)$ is the filtration $\mathrm{Fil}^\star_{\mathrm{BMS}} \mathrm{TP}(X; \mathbb{Z}_p)$ of Definition 2.17, and there is a natural fibre sequence

$$\mathrm{Fil}^\star_{\mathrm{mot}} \mathrm{TC}(X) \longrightarrow \mathrm{Fil}^\star_{\mathrm{mot}} \mathrm{TC}^-(X) \longrightarrow \prod_{p \in \mathbb{P}} \mathrm{Fil}^\star_{\mathrm{BMS}} \mathrm{TP}(X; \mathbb{Z}_p)$$

of filtered spectra. In particular, for every integer $i \in \mathbb{Z}$, this induces a natural fibre sequence

$$\mathbb{Z}(i)^{\mathrm{TC}}(X) \longrightarrow \mathcal{N}^{\geq i} \widehat{\mathbb{\Delta}}^{\mathrm{gl}}_X\{i\} \longrightarrow \prod_{p \in \mathbb{P}} \widehat{\mathbb{\Delta}}_{X,p}\{i\}$$

in the derived category $\mathcal{D}(\mathbb{Z})$, where $\widehat{\mathbb{\Delta}}_{X,p}$ denotes the p-adic absolute prismatic cohomology of X.

Proposition 2.33. *Let X be a qcqs derived scheme, and p be a prime number. Then the natural map*

$$\mathrm{Fil}^\star_{\mathrm{mot}} \mathrm{TC}(X; \mathbb{Z}_p) \longrightarrow \mathrm{Fil}^\star_{\mathrm{BMS}} \mathrm{TC}(X; \mathbb{Z}_p)$$

is an equivalence of filtered spectra.

Proof. By definition, the natural map

$$\mathrm{Fil}^\star_{\mathrm{HKR}} \mathrm{HC}^-(X) \longrightarrow \prod_{\ell \in \mathbb{P}} \mathrm{Fil}^\star_{\mathrm{HKR}} \mathrm{HC}^-(X; \mathbb{Z}_\ell)$$

in Definition 2.30 is profinite completion, so its fibre becomes zero after p-completion. $\square$

---

[3] Every value of the functor $\mathbb{Z}(i)^{\mathrm{TC}}(-)$ has a natural module structure over the $\mathbb{E}_\infty$-ring $\mathbb{Z}(0)^{\mathrm{TC}}(\mathbb{Z})$, which, by unwinding the definition, is naturally identified with the $\mathbb{E}_\infty$-ring $H\mathbb{Z}$. This implies that the spectra-valued functor $\mathbb{Z}(i)^{\mathrm{TC}}(-)$ takes values in $H\mathbb{Z}$-linear spectra, *i.e.*, in the derived category $\mathcal{D}(\mathbb{Z})$.

Corollary 2.34. *Let X be a qcqs derived scheme, and p be a prime number. Then the natural map*

$$\mathbb{Z}_p(i)^{\mathrm{TC}}(X) \longrightarrow \mathbb{Z}_p(i)^{\mathrm{BMS}}(X)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z}_p)$.

Proof. This is a direct consequence of Proposition 2.33. □

Proposition 2.35. *Let X be a qcqs derived scheme. Then the filtrations*

$$\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(X) \text{ and } \mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(X;\mathbb{Q})$$

are complete.

Proof. The filtrations $\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X)$, $\prod_{p\in\mathbb{P}}\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X;\mathbb{Z}_p)$, and $\prod_{p\in\mathbb{P}}\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(X;\mathbb{Z}_p)$ are complete by Lemmas 2.9, 2.13, and 2.21 respectively. By Definition 2.30, the filtration $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(X)$ is then complete, as a pullback of three complete filtrations. To prove that the filtration $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(X;\mathbb{Q})$ is complete, consider the cartesian square of filtered spectra

$$\begin{array}{ccc}
\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(X;\mathbb{Q}) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X;\mathbb{Q}) \\
\downarrow & & \downarrow \\
\left(\prod_{p\in\mathbb{P}}\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(X;\mathbb{Z}_p)\right)_{\mathbb{Q}} & \longrightarrow & \left(\prod_{p\in\mathbb{P}}\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X;\mathbb{Z}_p)\right)_{\mathbb{Q}}
\end{array}$$

induced by taking the rationalisation of Definition 2.30. The filtration

$$\big(\prod_{p\in\mathbb{P}}\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(X;\mathbb{Z}_p)\big)_{\mathbb{Q}}$$

is complete by Lemma 2.21. The fibre of the natural map

$$\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X) \longrightarrow \prod_{p\in\mathbb{P}}\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X;\mathbb{Z}_p)$$

is complete as an object of the filtered derived category $\mathcal{D}\mathcal{F}(\mathbb{Z})$ (Lemmas 2.9 and 4.25), and is zero modulo p for every prime number p by construction. In particular, it is naturally identified with the fibre of the natural map

$$\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X;\mathbb{Q}) \longrightarrow \big(\prod_{p\in\mathbb{P}}\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X;\mathbb{Z}_p)\big)_{\mathbb{Q}},$$

which is thus complete as an object of the filtered derived category $\mathcal{D}\mathcal{F}(\mathbb{Q})$. This implies, by the previous cartesian square, that the filtration $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(X;\mathbb{Q})$ is also complete. □

Remark 2.36 (Exhaustivity of the motivic filtration on TC). The motivic filtration $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}$ is not exhaustive on general qcqs derived scheme. Although this will not be necessary to prove that the motivic filtration on algebraic K-theory is exhaustive (Proposition 4.47), one can prove, using [Ant19, Lemma 4.10] and its proof, that if X is a quasi-lci $\mathbb{Z}$-scheme,[4] then the motivic filtration $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(X)$ is exhaustive.

Proposition 2.37. *For every integer $i \in \mathbb{Z}$, the presheaf*

$$\mathrm{Fil}^i_{\mathrm{mot}}\mathrm{TC}(-) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{Sp}$$

is an étale sheaf.

---

[4]By this, we mean that Zariski-locally on the qcqs scheme X, the cotangent complex $\mathbb{L}_{-/\mathbb{Z}}$ has Tor-amplitude in $[-1;0]$.

Proof. By Definition 2.30, it suffices to prove that the presheaves

$$\prod_{p\in\mathbb{P}}\mathrm{Fil}_{\mathrm{BMS}}^i\mathrm{TC}(-;\mathbb{Z}_p),\ \mathrm{Fil}_{\mathrm{HKR}}^i\mathrm{HC}^-(-),\ \text{and}\ \prod_{p\in\mathbb{P}}\mathrm{Fil}_{\mathrm{HKR}}^i\mathrm{HC}^-(-;\mathbb{Z}_p)$$

are étale sheaves. A product of sheaves is a sheaf, and these BMS and HKR filtrations are complete (Lemmas 2.21, 2.9, and 2.13). It then suffices to prove that for every prime number p, the presheaves

$$\mathbb{Z}_p(i)^{\mathrm{BMS}}(-),\ R\Gamma_{\mathrm{Zar}}\big(-,\widehat{\mathbb{L}\Omega}_{-/\mathbb{Z}}^{\geq i}\big),\ \text{and}\ R\Gamma_{\mathrm{Zar}}\big(-,(\widehat{\mathbb{L}\Omega}_{-/\mathbb{Z}}^{\geq i})_p^{\wedge}\big)$$

are étale sheaves. The statement for $\mathbb{Z}_p(i)^{\mathrm{BMS}}(-)$ is a consequence of p-fpqc descent ([BL22, Proposition 7.4.7]). The statement for the other two presheaves reduces to the fpqc descent for the powers of the cotangent complex ([BMS19, Theorem 3.1]). □

Corollary 2.38. *For every integer $i \in \mathbb{Z}$, the presheaf*

$$\mathbb{Z}(i)^{\mathrm{TC}}(-) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathcal{D}(\mathbb{Z})$$

is an étale sheaf.

Proof. This is a direct consequence of Proposition 2.37. □

2.4 The motivic filtration on $L_{\mathbf{cdh}}\mathrm{TC}$

In this subsection, we introduce the motivic filtration on the cdh sheafification of topological cyclic homology of general qcqs schemes (Definition 2.39).

Definition 2.39 (Motivic filtration on $L_{\mathrm{cdh}}\mathrm{TC}$)**.** The *motivic filtration on cdh sheafified topological cyclic homology* of qcqs schemes

$$\mathrm{Fil}_{\mathrm{mot}}^{\star}L_{\mathrm{cdh}}\mathrm{TC}(-) : \mathrm{Sch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

is the functor defined as the cdh sheafification of the motivic filtration on topological cyclic homology (Definition 2.30)

$$\mathrm{Fil}_{\mathrm{mot}}^{\star}L_{\mathrm{cdh}}\mathrm{TC}(-) := \big(L_{\mathrm{cdh}}\mathrm{Fil}_{\mathrm{mot}}^{\star}\mathrm{TC}\big)(-).$$

Remark 2.40 (Graded pieces of $\mathrm{Fil}_{\mathrm{mot}}^{\star}L_{\mathrm{cdh}}\mathrm{TC}$)**.** Let X be a qcqs scheme. For every integer $i \in \mathbb{Z}$, the canonical map

$$\big(L_{\mathrm{cdh}}\mathbb{Z}(i)^{\mathrm{TC}}\big)(X) \longrightarrow \mathrm{gr}_{\mathrm{mot}}^i L_{\mathrm{cdh}}\mathrm{TC}(R)[-2i]$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$. We will usually refer to these shifted graded pieces by the complexes $\big(L_{\mathrm{cdh}}\mathbb{Z}(i)^{\mathrm{TC}}\big)(X)$.

Remark 2.41 (Completeness of $\mathrm{Fil}_{\mathrm{mot}}^{\star}L_{\mathrm{cdh}}\mathrm{TC}$)**.** It is not clear *a priori* that the filtered spectrum $\mathrm{Fil}_{\mathrm{mot}}^{\star}L_{\mathrm{cdh}}\mathrm{TC}(X)$ is complete, even for qcqs schemes of finite valuative dimension. Modulo any prime number p, this is a consequence of the connectivity bound [AMMN22, Theorem 5.1 (1)] and [EHIK21, Theorem 2.4.15]. The integral statement will be a consequence of certain cdh descent results in Section 4.

Remark 2.42 (Exhaustivity of $\mathrm{Fil}_{\mathrm{mot}}^{\star}L_{\mathrm{cdh}}\mathrm{TC}$)**.** The filtration $\mathrm{Fil}_{\mathrm{mot}}^{\star}L_{\mathrm{cdh}}\mathrm{TC}$ is not exhaustive on general qcqs derived schemes. We will prove however, in Section 4, that the fibre of the natural map $\mathrm{Fil}_{\mathrm{mot}}^{\star}\mathrm{TC} \to \mathrm{Fil}_{\mathrm{mot}}^{\star}L_{\mathrm{cdh}}\mathrm{TC}$ is $\mathbb{N}$-indexed, and in particular exhaustive.

3   Definition of motivic cohomology

In this section, we introduce motivic cohomology of general quasi-compact quasi-separated derived schemes (Definition 3.19) and establish some of the fundamental properties of the associated motivic filtration.

3.1   Classical motivic cohomology

In this subsection, we review the classical definition of motivic cohomology of smooth schemes in mixed characteristic. Following [Blo86, Lev01, Gei04], the motivic cohomology of smooth $\mathbb{Z}$-schemes X is classically defined in terms of Bloch's cycle complexes $z^i(X, \bullet)$. Recall that Bloch's cycle complex is a simplicial abelian group defined in terms of algebraic cycles. The homotopy groups of Bloch's cycle complexes, called Bloch's higher Chow groups, are a generalisation of Chow groups that are designed to generalise the relation between the K_0 and the Chow groups of a quasi-projective variety to higher K-groups. Via the Dold–Kan correspondence, we view Bloch's cycle complexes as objects of the derived category $\mathcal{D}(\mathbb{Z})$.

Definition 3.1 (Classical motivic cohomology of smooth schemes)**.** Let B be a field or a mixed characteristic Dedekind domain (*e.g.*, $B = \mathbb{Z}$), and X be a smooth B-scheme. For any integer $i \in \mathbb{Z}$, the *classical motivic complex*
$$\mathbb{Z}(i)^{\mathrm{cla}}(X) \in \mathcal{D}(\mathbb{Z})$$
is the shift of Bloch's cycle complex $z^i(X, \bullet)$:
$$\mathbb{Z}(i)^{\mathrm{cla}}(X) := z^i(X, \bullet)[-2i],$$
where $\bullet$ is the cohomological index.

Note that, by construction, the classical motivic complexes $\mathbb{Z}(i)^{\mathrm{cla}}$ vanish in degrees more than $2i$, and in all degrees for weights $i < 0$.

In the following definition, we use the slice filtration in stable homotopy theory, as introduced by Voevodsky [Voe02a, Voe02b, BH21].

Definition 3.2 (Motivic filtration on K-theory of smooth schemes)**.** Let B be a field or a mixed characteristic Dedekind domain. The *classical motivic filtration* on algebraic K-theory of smooth B-schemes is the functor
$$\mathrm{Fil}^\star_{\mathrm{cla}} \mathrm{K}(-) : \mathrm{Sm}_B^{\mathrm{op}} \longrightarrow \mathrm{FilSp}$$
defined as the image, via the mapping spectrum construction $\omega^\infty : \mathrm{SH}(B) \to \mathrm{PSh}(\mathrm{Sm}_B, \mathrm{Sp})$, of the slice filtration $f^\star \mathrm{KGL}$ on the K-theory motivic spectrum $\mathrm{KGL} \in \mathrm{SH}(B)$.

Remark 3.3. The pullback of algebraic cycles being well-defined only along flat maps, it is not straightforward to prove that the classical motivic complexes $\mathbb{Z}(i)^{\mathrm{cla}}$ of Definition 3.1 are functorial. Over a field, Voevodsky overcomes this technicality by proving that Bloch's cycle complexes are represented in SH by the zeroth slice of the K-theory motivic spectrum KGL. Over a mixed characteristic Dedekind domain, this identification is proved by Bachmann [Bac22]. In particular, this means that Bloch's cycle complexes $z^i(-, \bullet)$, when seen as a construction taking values in the derived category $\mathcal{D}(\mathbb{Z})$, is indeed functorial, and multiplicative. In terms of Definitions 3.1 and 3.2, this implies that for every integer $i \in \mathbb{Z}$, there is an equivalence of $\mathcal{D}(\mathbb{Z})$-valued[5] functors
$$\mathbb{Z}(i)^{\mathrm{cla}}(-) := \mathrm{gr}^i_{\mathrm{cla}} \mathrm{K}(-)[-2i].$$

Example 3.4 (Weight zero classical motivic cohomology)**.** For every smooth scheme X over a field or a mixed characteristic Dedekind domain, there is a natural equivalence
$$\mathbb{Z}(0)^{\mathrm{cla}}(X) \simeq R\Gamma_{\mathrm{Zar}}(X, \mathbb{Z})$$
in the derived category $\mathcal{D}(\mathbb{Z})$ ([Spi18, Proposition 6.1]).

---

[5] Every value of the functor $\mathrm{gr}^i_{\mathrm{cla}} \mathrm{K}(-)[-2i]$ has a natural module structure over the $\mathbb{E}_\infty$-ring $\mathrm{gr}^0_{\mathrm{cla}} \mathrm{K}(\mathbb{Z})$, which, by [Spi18, Proposition 6.1] and [Bac22], is naturally identified with the $\mathbb{E}_\infty$-ring $H\mathbb{Z}$. This implies that the spectra-valued functor $\mathrm{gr}^i_{\mathrm{cla}} \mathrm{K}(-)[-2i]$ takes values in $H\mathbb{Z}$-linear spectra, *i.e.*, in the derived category $\mathcal{D}(\mathbb{Z})$.

Example 3.5 (Weight one classical motivic cohomology). For every smooth scheme X over a field or a mixed characteristic Dedekind domain, there is a natural equivalence

$$\mathbb{Z}(1)^{\mathrm{cla}}(X) \simeq R\Gamma_{\mathrm{Zar}}(X, \mathbb{G}_m)[-1]$$

in the derived category $\mathcal{D}(\mathbb{Z})$ ([Spi18, Theorem 7.10]). In particular, the complex $\mathbb{Z}(1)^{\mathrm{cla}}(X)$ is concentrated in degrees one and two, where it is given by

$$\mathrm{H}^1(\mathbb{Z}(1)^{\mathrm{cla}}(X)) \cong \mathcal{O}(X)^{\times} \quad \text{and} \quad \mathrm{H}^2(\mathbb{Z}(1)^{\mathrm{cla}}(X)) \cong \mathrm{Pic}(X).$$

3.2 The cdh-local motivic filtration

Following [BEM], we review in this subsection the cdh-local motivic filtration on homotopy K-theory $\mathrm{KH}(-)$ of qcqs schemes (Definition 3.10), whose shifted graded pieces $\mathbb{Z}(i)^{\mathrm{cdh}}$ will serve as a building block for the definition of the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$ (Remark 3.20). We first define the lisse motivic complexes $\mathbb{Z}(i)^{\mathrm{lisse}}$ as an intermediate construction, and as a practical tool for later sections.

The following definition is motivated by the observation of Bhatt–Lurie that connective algebraic K-theory $\mathrm{K}^{\mathrm{conn}}(-)$ is left Kan extended on animated commutative rings from smooth $\mathbb{Z}$-algebras [EHK$^+$20, Proposition A.0.4].

Definition 3.6 (Motivic filtration on connective K-theory of animated rings). The *motivic filtration on connective algebraic K-theory* of animated commutative rings is the functor

$$\mathrm{Fil}^{\star}_{\mathrm{lisse}}\mathrm{K}^{\mathrm{conn}}(-) : \mathrm{AniRings} \longrightarrow \mathrm{FilSp}$$

defined as the left Kan extension of the classical motivic filtration on algebraic K-theory of smooth $\mathbb{Z}$-algebras

$$\mathrm{Fil}^{\star}_{\mathrm{lisse}}\mathrm{K}^{\mathrm{conn}}(-) := \left(L_{\mathrm{AniRings}/\mathrm{Sm}_{\mathbb{Z}}}\mathrm{Fil}^{\star}_{\mathrm{cla}}\mathrm{K}\right)(-).$$

Note that connective algebraic K-theory is not a Zariski sheaf on commutative rings. Most of our results on the following lisse motivic complexes $\mathbb{Z}(i)^{\mathrm{lisse}}$ will be formulated in the generality of local rings.

Definition 3.7 (Lisse motivic cohomology of animated rings). For any integer $i \in \mathbb{Z}$, the *lisse motivic complex*

$$\mathbb{Z}(i)^{\mathrm{lisse}}(-) : \mathrm{AniRings} \longrightarrow \mathcal{D}(\mathbb{Z})$$

is the shifted graded piece of the motivic filtration of Definition 3.6:

$$\mathbb{Z}(i)^{\mathrm{lisse}}(-) := \mathrm{gr}^i_{\mathrm{lisse}}\mathrm{K}^{\mathrm{conn}}(-)[-2i].$$

Note that the lisse motivic complexes $\mathbb{Z}(i)^{\mathrm{lisse}}$ are the left Kan extension of the classical motivic complexes $\mathbb{Z}(i)^{\mathrm{cla}}$, and in particular vanish in weights $i < 0$.

Example 3.8 (Weight zero lisse motivic cohomology). For every local ring R, there is a natural equivalence

$$\mathbb{Z}(0)^{\mathrm{lisse}}(R) \simeq \mathbb{Z}[0]$$

in the derived category $\mathcal{D}(\mathbb{Z})$. This is a consequence of Example 3.4, by using that the functor $\mathbb{Z}(0)^{\mathrm{lisse}}(-)$ is left Kan extended on local rings from its restriction to local essentially smooth $\mathbb{Z}$-algebras.

Example 3.9 (Weight one lisse motivic cohomology). For every commutative ring R, there is a natural equivalence

$$\mathbb{Z}(1)^{\mathrm{lisse}}(R) \simeq \left(\tau^{\leq 1}R\Gamma_{\mathrm{Zar}}(R, \mathbb{G}_m)\right)[-1]$$

in the derived category $\mathcal{D}(\mathbb{Z})$. In particular, the complex $\mathbb{Z}(1)^{\mathrm{lisse}}(R) \in \mathcal{D}(\mathbb{Z})$ is concentrated in degrees one and two, where it is given by

$$\mathrm{H}^1(\mathbb{Z}(1)^{\mathrm{lisse}}(R)) \cong \mathcal{O}(R)^{\times} \quad \text{and} \quad \mathrm{H}^2(\mathbb{Z}(1)^{\mathrm{lisse}}(R)) \cong \mathrm{Pic}(R).$$

This is a consequence of Example 3.5, by using that the left Kan extension of a functor taking values in degrees at most two takes values in degrees at most two, and that the functor $\tau^{\leq 1} R\Gamma_{\mathrm{Zar}}(-, \mathbb{G}_m)$ on commutative rings is left Kan extended from its restriction to smooth $\mathbb{Z}$-algebras. Here the latter left Kan extension property is a consequence of the same left Kan extension property for the functors $\mathbb{G}_m(-)$ (which is a special case of Mathew's criterion [EHK+20, Proposition A.0.4]) and $\mathrm{Pic}(-)$ (which is a consequence of rigidity, see [EM23, Lemma 7.6]).

Definition 3.10 (Cdh-local motivic filtration on KH-theory of schemes)**.** The *cdh-local motivic filtration on homotopy K-theory* of qcqs schemes is the functor

$$\mathrm{Fil}^\star_{\mathrm{cdh}} \mathrm{KH}(-) : \mathrm{Sch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

defined as

$$\mathrm{Fil}^\star_{\mathrm{cdh}} \mathrm{KH}(-) := \big(L_{\mathrm{cdh}} \mathrm{Fil}^\star_{\mathrm{lisse}} \mathrm{K}^{\mathrm{conn}}\big)(-) = \big(L_{\mathrm{cdh}} L_{\mathrm{Sch}^{\mathrm{qcqs,op}}/\mathrm{Sm}_{\mathbb{Z}}^{\mathrm{op}}} \mathrm{Fil}^\star_{\mathrm{cla}} \mathrm{K}\big)(-).$$

Remark 3.11. By construction of the cdh-local motivic filtration (Definition 3.10), there is a natural comparison map of presheaves

$$\mathrm{Fil}^\star_{\mathrm{cla}} \mathrm{K}(-) \longrightarrow \mathrm{Fil}^\star_{\mathrm{cdh}} \mathrm{KH}(-)$$

on smooth $\mathbb{Z}$-schemes.

Definition 3.12 (Cdh-local motivic cohomology of schemes)**.** For any integer $i \in \mathbb{Z}$, the *cdh-local motivic complex*

$$\mathbb{Z}(i)^{\mathrm{cdh}}(-) : \mathrm{Sch}^{\mathrm{qcqs,op}} \longrightarrow \mathcal{D}(\mathbb{Z})$$

is the shifted graded piece of the motivic filtration of Definition 3.10:

$$\mathbb{Z}(i)^{\mathrm{cdh}}(-) := \mathrm{gr}^i_{\mathrm{cdh}} \mathrm{KH}(-)[-2i].$$

Although we will refer throughout the text to [BEM] for the properties of these cdh-local motivic complexes $\mathbb{Z}(i)^{\mathrm{cdh}}$, we already mention the following result, as it will play an important role to establish the completeness of the motivic filtration on algebraic K-theory.

Proposition 3.13 (Completeness of the cdh-local motivic filtration, after [BEM])**.** *Let $d \geq 0$ be an integer, and X be a qcqs scheme of valuative dimension at most d. Then for every integer $i \in \mathbb{Z}$, the spectrum $\mathrm{Fil}^i_{\mathrm{cdh}} \mathrm{KH}(X)$ is in cohomological degrees at most $-i + d$. In particular, the filtration $\mathrm{Fil}^\star_{\mathrm{cdh}} \mathrm{KH}(X)$, and its rationalisation $\mathrm{Fil}^\star_{\mathrm{cdh}} \mathrm{KH}(X; \mathbb{Q})$, are complete.*

3.3 Definition of motivic cohomology

In this subsection, we introduce the motivic filtration on algebraic K-theory of qcqs derived schemes (Definitions 3.17 and 3.18), by constructing a filtered extension of the cartesian square of Theorem 1.1. To do so, we first define a filtered cdh-local cyclotomic trace map $\mathrm{Fil}^\star_{\mathrm{cdh}} \mathrm{KH}(-) \to \mathrm{Fil}^\star_{\mathrm{mot}} L_{\mathrm{cdh}} \mathrm{TC}(-)$ (Construction 3.16).

Theorem 3.14 (Filtered cyclotomic trace in the smooth case)**.** *The cyclotomic trace map*

$$\mathrm{K}(-) \longrightarrow \mathrm{TC}(-),$$

viewed as a map of spectra-valued presheaves on the category of smooth $\mathbb{Z}$-schemes, admits a unique, multiplicative extension to a map

$$\mathrm{Fil}^\star_{\mathrm{cla}} \mathrm{K}(-) \longrightarrow \mathrm{Fil}^\star_{\mathrm{mot}} \mathrm{TC}(-)$$

of filtered presheaves of spectra.

Proof. By Definition 2.30, the natural cartesian square

$$
\begin{array}{ccc}
\mathrm{TC}(-) & \longrightarrow & \mathrm{HC}^-(-) \\
\downarrow & & \downarrow \\
\prod_{p \in \mathbb{P}} \mathrm{TC}(-;\mathbb{Z}_p) & \longrightarrow & \prod_{p \in \mathbb{P}} \mathrm{HC}^-(-;\mathbb{Z}_p)
\end{array}
$$

admits a multiplicative filtered extension

$$
\begin{array}{ccc}
\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(-) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(-) \\
\downarrow & & \downarrow \\
\prod_{p \in \mathbb{P}} \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p) & \longrightarrow & \prod_{p \in \mathbb{P}} \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(-;\mathbb{Z}_p)
\end{array}
$$

on qcqs derived schemes. Let p be a prime number. It then suffices to prove that the natural maps

$$
\mathrm{K}(-) \longrightarrow \mathrm{HC}^-(-), \ \mathrm{K}(-) \longrightarrow \mathrm{HC}^-(-;\mathbb{Z}_p), \ \text{and } \mathrm{K}(-) \longrightarrow \mathrm{TC}(-;\mathbb{Z}_p),
$$

viewed as maps of spectra-valued presheaves on the category of smooth $\mathbb{Z}$-schemes, admit unique multiplicative filtered extensions to maps of filtered presheaves of spectra

$$
\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(-) \longrightarrow \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(-), \quad \mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(-) \longrightarrow \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(-;\mathbb{Z}_p), \ \text{and}
$$

$$
\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(-) \longrightarrow \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)
$$

respectively.

The proof of [EM23, Proposition 4.6], which is stated over a quasismooth map of commutative rings $k_0 \to k$ such that k is a field, applies readily to the case where $k_0 = k = \mathbb{Z}$. More precisely, we use in this proof the Gersten conjecture for classical motivic cohomology on smooth $\mathbb{Z}$-schemes, which is [Gei04, Theorem 1.1]. In particular, the natural map $\mathrm{K}(-) \to \mathrm{HC}^-(-)$, viewed as a map of spectra-valued presheaves on the category of smooth $\mathbb{Z}$-schemes, admits a unique multiplicative filtered extension to a map of filtered presheaves of spectra $\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(-) \to \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(-)$.

Similarly, the p-completed cotangent complex $(\mathbb{L}_{R/\mathbb{Z}})^\wedge_p$ of a smooth $\mathbb{Z}$-algebra R is concentrated in degree zero, given by the p-flat R-module $(\Omega^1_{R/\mathbb{Z}})^\wedge_p$. So for every integer $i \in \mathbb{Z}$, this implies that the object $\mathrm{gr}^i_{\mathrm{HKR}}\mathrm{HC}^-(R;\mathbb{Z}_p) \in \mathcal{D}(\mathbb{Z})$ is in cohomological degrees less than $-i$. The proof of [EM23, Proposition 4.6] then adapts readily to prove that the natural map $\mathrm{K}(-) \to \mathrm{HC}^-(-;\mathbb{Z}_p)$, viewed as a map of spectra-valued presheaves on the category of smooth $\mathbb{Z}$-schemes, admits a unique multiplicative extension to a map of filtered presheaves of spectra $\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(-) \to \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(-;\mathbb{Z}_p)$.

Finally, the natural map $\mathrm{K}(-) \to \mathrm{TC}(-;\mathbb{Z}_p)$, viewed as a map of spectra-valued presheaves on the category of smooth $\mathbb{Z}$-schemes, admits a unique multiplicative extension to a map of filtered presheaves of spectra $\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(-) \to \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)$, by [AHI24, Proposition 6.12]. $\qquad\square$

Remark 3.15. For every prime number p, the BMS filtration $\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)$ is determined by its p-quasisyntomic-local values (Theorem 2.25). By the proof of [AHI24, Lemma 6.10], the p-completed left Kan extension of the functor $\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(-)$, from smooth $\mathbb{Z}$-algebras to p-quasisyntomic rings, is p-quasisyntomic-locally identified, via the map induced by [AHI24, Proposition 6.12], with the functor $\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)$. In particular, one can reconstruct the BMS filtration $\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)$ on qcqs derived schemes (Definition 2.20) from the classical motivic filtration $\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(-)$ on smooth $\mathbb{Z}$-schemes (Definition 3.2). This will be used in Section 4 to construct Adams operations on the BMS filtration $\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)$.

Construction 3.16 (Filtered cdh-local cyclotomic trace). The *filtered cdh-local cyclotomic trace map* is the map

$$
\mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{KH}(-) \longrightarrow \mathrm{Fil}^\star_{\mathrm{mot}}L_{\mathrm{cdh}}\mathrm{TC}(-)
$$

of functors from (the opposite category of) qcqs schemes to the category of multiplicative filtered spectra defined as the cdh sheafification of the composite

$$\left(L_{\mathrm{Sch}^{\mathrm{qcqs,op}}/\mathrm{Sm}_{\mathbb{Z}}^{\mathrm{op}}}\mathrm{Fil}_{\mathrm{cla}}^{\star}\mathrm{K}\right)(-) \longrightarrow \left(L_{\mathrm{Sch}^{\mathrm{qcqs,op}}/\mathrm{Sm}_{\mathbb{Z}}^{\mathrm{op}}}\mathrm{Fil}_{\mathrm{mot}}^{\star}\mathrm{TC}\right)(-) \longrightarrow \mathrm{Fil}_{\mathrm{mot}}^{\star}\mathrm{TC}(-),$$

where the first map is the map induced by Theorem 3.14 and the second map is the canonical map. Note here that sheafification is a multiplicative operation, and that the compatibility between left Kan extension and multiplicative structures is ensured by [Lur17, Corollary 3.2.3.2] (see also [EM23, 2.3.2]).

Definition 3.17 (Motivic filtration on K-theory of schemes). The *motivic filtration on non-connective algebraic K-theory* of qcqs schemes is the functor

$$\mathrm{Fil}_{\mathrm{mot}}^{\star}\mathrm{K}(-) : \mathrm{Sch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

defined by the cartesian square of functors of multiplicative filtered spectra

$$\begin{array}{ccc} \mathrm{Fil}_{\mathrm{mot}}^{\star}\mathrm{K}(-) & \longrightarrow & \mathrm{Fil}_{\mathrm{mot}}^{\star}\mathrm{TC}(-) \\ \downarrow & & \downarrow \\ \mathrm{Fil}_{\mathrm{cdh}}^{\star}\mathrm{KH}(-) & \longrightarrow & \mathrm{Fil}_{\mathrm{mot}}^{\star}L_{\mathrm{cdh}}\mathrm{TC}(-), \end{array}$$

where the bottom horizontal map is the map of Construction 3.16, and the right vertical map is cdh sheafification.

Definition 3.18 (Motivic filtration on K-theory of derived schemes). The *motivic filtration on non-connective algebraic K-theory* of qcqs derived schemes is the functor

$$\mathrm{Fil}_{\mathrm{mot}}^{\star}\mathrm{K}(-) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

defined by the cartesian square of functors of multiplicative filtered spectra

$$\begin{array}{ccc} \mathrm{Fil}_{\mathrm{mot}}^{\star}\mathrm{K}(-) & \longrightarrow & \mathrm{Fil}_{\mathrm{mot}}^{\star}\mathrm{TC}(-) \\ \downarrow & & \downarrow \\ \mathrm{Fil}_{\mathrm{mot}}^{\star}\mathrm{K}(\pi_0(-)) & \longrightarrow & \mathrm{Fil}_{\mathrm{mot}}^{\star}\mathrm{TC}(\pi_0(-)) \end{array}$$

where $\pi_0(-) : \mathrm{dSch} \to \mathrm{Sch}$ is restriction to the classical locus, the filtration on $\mathrm{K}(\pi_0(-))$ is given by Definition 3.17, and the filtrations on $\mathrm{TC}(-)$ and $\mathrm{TC}(\pi_0(-))$ are given by Definition 2.30.

Definition 3.19 (Motivic cohomology of derived schemes). For any integer $i \in \mathbb{Z}$, the *motivic complex*

$$\mathbb{Z}(i)^{\mathrm{mot}} : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathcal{D}(\mathbb{Z})$$

is the shifted graded piece of the motivic filtration of Definition 3.18:

$$\mathbb{Z}(i)^{\mathrm{mot}}(-) := \mathrm{gr}_{\mathrm{mot}}^{i}\mathrm{K}(-)[-2i].$$

For every qcqs derived scheme X, also denote by

$$\mathrm{H}_{\mathrm{mot}}^{n}(X, \mathbb{Z}(i)) := \mathrm{H}^{n}(\mathbb{Z}(i)^{\mathrm{mot}}(X)) \quad n \in \mathbb{Z}$$

the *motivic cohomology groups* of X.

Remark 3.20 (Motivic cohomology of schemes). Let X be a qcqs scheme, and $i \in \mathbb{Z}$ be an integer. By Definition 3.18, there is a natural cartesian square

$$\begin{array}{ccc} \mathbb{Z}(i)^{\mathrm{mot}}(X) & \longrightarrow & \mathbb{Z}(i)^{\mathrm{TC}}(X) \\ \downarrow & & \downarrow \\ \mathbb{Z}(i)^{\mathrm{cdh}}(X) & \longrightarrow & \left(L_{\mathrm{cdh}}\,\mathbb{Z}(i)^{\mathrm{TC}}\right)(X) \end{array}$$

in the derived category $\mathcal{D}(\mathbb{Z})$, where the bottom horizontal map is induced by Construction 3.16 and the right vertical map is cdh sheafification. This cartesian square can serve as a definition for the motivic cohomology of the scheme X.

We now construct, for later use, a comparison map from classical motivic cohomology to the motivic cohomology of Definition 3.19.

Definition 3.21 (Filtered classical-motivic comparison map). The *filtered classical-motivic comparison map* is the map of presheaves

$$\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(-) \longrightarrow \mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(-)$$

on smooth $\mathbb{Z}$-schemes induced by the maps Remark 3.11 and Theorem 3.14. Note here that the compatibility between these two maps and Definition 3.17 is automatic by Construction 3.16.

Definition 3.22 (Classical-motivic comparison map). For any integer $i \in \mathbb{Z}$, the *classical-motivic comparison map* is the map of $\mathcal{D}(\mathbb{Z})$-valued presheaves

$$\mathbb{Z}(i)^{\mathrm{cla}}(-) \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(-)$$

on smooth $\mathbb{Z}$-schemes induced by taking the ith shifted graded piece of the filtered map of Definition 3.21.

In the rest of this section, we discuss some of the first properties of the motivic filtration.

Remark 3.23 (Comparison to cdh-local motivic cohomology). By construction (Definition 3.10), the cdh-local motivic complex

$$\mathbb{Z}(i)^{\mathrm{cdh}} : \mathrm{Sch}^{\mathrm{qcqs,op}} \longrightarrow \mathcal{D}(\mathbb{Z})$$

is a cdh sheaf, so the common fibre of the horizontal maps in the cartesian square of Remark 3.20 is also a cdh sheaf. In particular, for every qcqs scheme X, the left vertical map of this cartesian square exhibits cdh-local motivic cohomology as the cdh sheafification of motivic cohomology:

$$\mathbb{Z}(i)^{\mathrm{cdh}}(X) \simeq \left(L_{\mathrm{cdh}}\,\mathbb{Z}(i)^{\mathrm{mot}}\right)(X).$$

Proposition 3.24. *Let X be a qcqs scheme, and p be a prime number. Then for every integer $k \geq 1$, there is a natural cartesian square*

$$\begin{array}{ccc}
\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X;\mathbb{Z}/p^k) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(X;\mathbb{Z}/p^k) \\
\downarrow & & \downarrow \\
\mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{KH}(X;\mathbb{Z}/p^k) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{BMS}}L_{\mathrm{cdh}}\mathrm{TC}(X;\mathbb{Z}/p^k)
\end{array}$$

of filtered spectra.

Proof. This is a consequence of Proposition 2.33 and Definition 3.18. $\square$

Corollary 3.25. *Let X be a qcqs scheme, and p be a prime number. Then for any integers $i \in \mathbb{Z}$ and $k \geq 1$, there is natural cartesian square*

$$\begin{array}{ccc}
\mathbb{Z}/p^k(i)^{\mathrm{mot}}(X) & \longrightarrow & \mathbb{Z}/p^k(i)^{\mathrm{BMS}}(X) \\
\downarrow & & \downarrow \\
\mathbb{Z}/p^k(i)^{\mathrm{cdh}}(X) & \longrightarrow & \left(L_{\mathrm{cdh}}\,\mathbb{Z}/p^k(i)^{\mathrm{BMS}}\right)(X)
\end{array}$$

in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$.

Proof. This is a direct consequence of Proposition 3.24. $\square$

Remark 3.26 (ℓ-adic motivic cohomology). For any prime number p and integer $k \geq 1$, the filtered presheaf $\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}/p^k)$ and its cdh sheafification vanish on qcqs $\mathbb{Z}[\frac{1}{p}]$-schemes. In particular, Proposition 3.24 implies that for every qcqs $\mathbb{Z}[\frac{1}{p}]$-scheme X, the natural map

$$\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X;\mathbb{Z}/p^k) \longrightarrow \mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{KH}(X;\mathbb{Z}/p^k)$$

is an equivalence of filtered spectra.

Remark 3.27 (Completeness and exhaustivity of $\mathrm{Fil}^\star_{\mathrm{mot}} \mathrm{K}$). The filtration $\mathrm{Fil}^\star_{\mathrm{mot}} \mathrm{K}(X)$ of Definition 3.18 will be proved to be $\mathbb{N}$-indexed, hence exhaustive, on general qcqs derived schemes X (Proposition 4.47), and complete on qcqs schemes of finite valuative dimension (Proposition 4.50). Note that these results can already be proved modulo any prime number p, as a formal consequence of Proposition 3.24 and Section 2.

The following result is a filtered version of the classical Dundas–Goodwillie–McCarthy theorem ([DGM13, Theorem 7.0.0.2]).

Proposition 3.28. *Let $A \to B$ be a map of animated commutative rings such that the induced map $\pi_0(A) \to \pi_0(B)$ of commutative rings is surjective with finitely generated nilpotent kernel. Then the natural commutative diagram*

$$\begin{array}{ccc} \mathrm{Fil}^\star_{\mathrm{mot}} \mathrm{K}(A) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{mot}} \mathrm{TC}(A) \\ \downarrow & & \downarrow \\ \mathrm{Fil}^\star_{\mathrm{mot}} \mathrm{K}(B) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{mot}} \mathrm{TC}(B) \end{array}$$

is a cartesian square of filtered spectra.

Proof. By Definition 3.18, it suffices to prove that the natural commutative diagram

$$\begin{array}{ccc} \mathrm{Fil}^\star_{\mathrm{mot}} \mathrm{K}(\pi_0(A)) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{mot}} \mathrm{TC}(\pi_0(A)) \\ \downarrow & & \downarrow \\ \mathrm{Fil}^\star_{\mathrm{mot}} \mathrm{K}(\pi_0(B)) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{mot}} \mathrm{TC}(\pi_0(B)) \end{array}$$

is a cartesian square of filtered spectra. For every cdh sheaf F defined on qcqs schemes, the natural map $F(\pi_0(A)) \to F(\pi_0(B))$ is an equivalence. The result is then a consequence of Definition 3.17. $\square$

4 Rational structure of motivic cohomology

In this section, we prove some fundamental properties of the motivic filtration that we introduced in the previous section: namely, that it is always exhaustive and finitary, and complete on schemes of finite valuative dimension. These properties modulo a prime number p are a formal consequence of the analogous properties for the BMS filtration on p-completed topological cyclic homology. Proving these results integrally will however require more understanding of the rational part of this motivic filtration. Our two main results on rational motivic cohomology are as follows.

The first main result is the following generalisation of the rational splitting of the classical motivic filtration.

Theorem 4.1 (The motivic filtration is rationally split). *Let X be a qcqs derived scheme. Then there exists a natural multiplicative equivalence of filtered spectra*

$$\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{K}(X;\mathbb{Q}) \simeq \bigoplus_{j \geq \star} \mathbb{Q}(j)^{\mathrm{mot}}(X)[2j].$$

As for the classical motivic filtration, this splitting is induced by suitable Adams operations, which we construct in the generality of qcqs derived schemes in the first section.

This result is however not enough to prove the exhaustivity, completeness, or finitariness of the motivic filtration. Instead, these will be proved along the way to the following second main result on rational motivic cohomology.

Theorem 4.2. *Let X be a qcqs scheme. Then there is a natural fibre sequence of filtered spectra*

$$\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{K}(X;\mathbb{Q}) \to \mathrm{Fil}^{\star}_{\mathrm{cdh}}\mathrm{KH}(X;\mathbb{Q}) \to \mathrm{cofib}\big(\mathrm{Fil}^{\star-1}_{\mathrm{HKR}}\mathrm{HC}(X_{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Fil}^{\star-1}_{\mathrm{HKR}}L_{\mathrm{cdh}}\mathrm{HC}(X_{\mathbb{Q}}/\mathbb{Q})\big)[1].$$

To prove this result, we use a classical argument of Weibel at the level of K-theory (Lemma 4.62) and the rational splitting Theorem 4.1 to reduce the statement to the case of characteristic zero, where the result is essentially [EM23, Theorem 4.10 (3)].

The key step in this argument, in order to pass from a statement at the level of K-theory to a filtered statement, is to prove beforehand that the motivic filtration $\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{K}$ is $\mathbb{N}$-indexed (Proposition 4.47). The strategy to prove this is as follows. We first introduce a new rigid-analytic variant of the HKR filtrations in the generality of qcqs derived schemes, whose graded pieces are rigid-analytic variants of derived de Rham cohomology. We then adapt a theorem of Goodwillie –stating in modern language that periodic cyclic homology is truncating in characteristic zero– to this rigid-analytic variant of periodic cyclic homology. This rigid-analytic Goodwillie theorem implies, by the work of Land–Tamme on truncating invariants, that the rigid-analytic variant of periodic cyclic homology is a cdh sheaf on qcqs schemes. A filtered consequence of this cdh descent result then formally implies the desired result, *i.e.*, that the motivic filtration $\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{K}$ is $\mathbb{N}$-indexed.

4.1 Adams operations

In this subsection we construct Adams operations on filtered algebraic K-theory of qcqs derived schemes (Construction 4.10).

In [EM23, Appendix B], Elmanto–Morrow construct Adams operations ψ^m on the m-periodic filtered K-theory $\mathrm{Fil}^{\star}_{\mathrm{cla}}\mathrm{K}(X)[\frac{1}{m}]$ of smooth $\mathbb{Z}$-schemes X, acting on the ith graded piece as multiplication by m^i. Using this construction, Bachmann–Elmanto–Morrow construct the following Adams operations ψ^m on the filtered KH-theory of arbitrary qcqs schemes X.

Proposition 4.3 (Adams operations on filtered KH-theory, [BEM]). *Let $m \geq 2$ be an integer, and X be a qcqs scheme. Then there exists a natural automorphism ψ^m of the filtered spectrum $\mathrm{Fil}^{\star}_{\mathrm{cdh}}\mathrm{KH}(X)[\frac{1}{m}]$ such that for every integer $i \in \mathbb{Z}$, the induced automorphism on the ith graded piece $\mathbb{Z}[\frac{1}{m}](i)^{\mathrm{cdh}}(X)[2i]$ is multiplication by m^i.*

We now use the Adams operations ψ^m on the m-periodic filtered K-theory $\mathrm{Fil}^{\star}_{\mathrm{cla}}\mathrm{K}(X)[\frac{1}{m}]$ of smooth $\mathbb{Z}$-schemes X ([EM23, Appendix B]) to construct Adams operations ψ^m on the filtered p-completed topological cyclic homology $\mathrm{Fil}^{\star}_{\mathrm{BMS}}\mathrm{TC}(X;\mathbb{Z}_p)$ of qcqs $\mathbb{Z}[\frac{1}{m}]$-schemes X.

Proposition 4.4 (Adams operations on filtered $\mathrm{TC}(-;\mathbb{Z}_p)$)**.** *Let $m \geq 2$ be an integer, X be a qcqs derived $\mathbb{Z}[\frac{1}{m}]$-scheme, and p be a prime number. Then there exists a natural multiplicative automorphism ψ^m of the filtered spectrum $\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(X;\mathbb{Z}_p)$ such that for every integer $i \in \mathbb{Z}$, the induced automorphism on the ith graded piece $\mathbb{Z}_p(i)^{\mathrm{BMS}}(X)[2i]$ is multiplication by m^i. Moreover, this automorphism ψ^m is uniquely determined by its naturality and the fact that on smooth $\mathbb{Z}[\frac{1}{m}]$-schemes X, the diagram of filtered spectra*

$$
\begin{array}{ccc}
\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(X)[\tfrac{1}{m}] & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(X;\mathbb{Z}_p) \\
\downarrow{\psi^m} & & \downarrow{\psi^m} \\
\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(X)[\tfrac{1}{m}] & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(X;\mathbb{Z}_p)
\end{array}
$$

where the horizontal maps are induced by Remark 3.15, and the left vertical map is defined in [EM23, Construction B.4], is commutative.

Proof. The functor $\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)$ on p-quasisyntomic rings is p-quasisyntomic-locally identified with the p-completed left Kan extension of the functor $\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(-)$ from smooth $\mathbb{Z}$-algebras to p-quasisyntomic rings (Remark 3.15). The functor $\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)$ on p-quasisyntomic rings moreover satisfies p-quasisyntomic descent (Theorem 2.25 (1)), so the Adams operation ψ^m on the functor $\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(-)[\frac{1}{m}]$ ([EM23, Appendix B]) induces a natural automorphism ψ^m of the presheaf $\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)$, acting as multiplication by m^i on the ith graded piece $\mathbb{Z}_p(i)^{\mathrm{BMS}}(-)[2i]$. The same result then applies to animated commutative $\mathbb{Z}[\frac{1}{m}]$-algebras by left Kan extending the result on polynomial $\mathbb{Z}[\frac{1}{m}]$-algebras (Theorem 2.25 (2)), and to general qcqs derived $\mathbb{Z}[\frac{1}{m}]$-schemes by Zariski sheafifying the result on animated commutative $\mathbb{Z}[\frac{1}{m}]$-algebras. $\qquad\square$

The following result is [Rak20, Construction 6.4.8 and Proposition 6.4.12]. Note that if X is a qcqs derived $\mathbb{Z}[\frac{1}{m}]$-scheme, then the filtered spectrum $\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X)$ is naturally $\mathbb{Z}[\frac{1}{m}]$-linear.

Proposition 4.5 (Adams operations on filtered HC^-, [Rak20])**.** *Let $m \geq 2$ be an integer, and X be qcqs derived $\mathbb{Z}[\frac{1}{m}]$-scheme. Then there exists a natural multiplicative automorphism ψ^m of the filtered spectrum $\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X)$ such that for every integer $i \in \mathbb{Z}$, the induced automorphism on the ith graded piece $\widehat{\mathbb{L}\Omega}^{\geq i}_{X/\mathbb{Z}[\frac{1}{m}]}[2i]$ is multiplication by m^i.*

Lemma 4.6. *Let $m \geq 2$ be an integer, X be a qcqs derived $\mathbb{Z}[\frac{1}{m}]$-scheme, and p be a prime number. Then the natural diagram of filtered spectra*

$$
\begin{array}{ccc}
\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(X;\mathbb{Z}_p) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X;\mathbb{Z}_p) \\
\downarrow{\psi^m} & & \downarrow{\psi^m} \\
\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(X;\mathbb{Z}_p) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X;\mathbb{Z}_p)
\end{array}
$$

where the horizontal maps are defined in Construction 2.29, the left map is the map defined in Proposition 4.4, and the right map is the map induced by Proposition 4.5, is commutative.

Proof. By [EM23, Lemma B.8], the natural diagram of filtered spectra

$$
\begin{array}{ccc}
\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(X)[\tfrac{1}{m}] & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X) \\
\downarrow{\psi^m} & & \downarrow{\psi^m} \\
\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(X)[\tfrac{1}{m}] & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X)
\end{array}
$$

is commutative for every smooth $\mathbb{Z}[\frac{1}{m}]$-scheme X. The result is then a consequence of Proposition 4.4, where the compatibility between the filtered maps is a consequence of the proof of Theorem 3.14. $\quad\square$

Construction 4.7 (Adams operations on filtered TC)**.** Let $m \geq 2$ be an integer, and X be a qcqs derived $\mathbb{Z}[\frac{1}{m}]$-scheme. The automorphism ψ^m of the filtered spectrum $\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{TC}(X)$ is the automorphism defined by pullback along the natural cartesian square of filtered spectra

$$\begin{array}{ccc} \mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{TC}(X) & \longrightarrow & \mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}^-(X) \\ \downarrow & & \downarrow \\ \prod_{p\in\mathbb{P}}\mathrm{Fil}^{\star}_{\mathrm{BMS}}\mathrm{TC}(X;\mathbb{Z}_p) & \longrightarrow & \prod_{p\in\mathbb{P}}\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}^-(X;\mathbb{Z}_p), \end{array}$$

where the automorphism ψ^m of $\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}^-(X)$ is the automorphism of Proposition 4.5, the automorphism ψ^m of $\prod_{p\in\mathbb{P}}\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}^-(X;\mathbb{Z}_p)$ is induced by the automorphism ψ^m of $\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}^-(X)$, and the automorphism ψ^m of $\prod_{p\in\mathbb{P}}\mathrm{Fil}^{\star}_{\mathrm{BMS}}\mathrm{TC}(X;\mathbb{Z}_p)$ is the automorphism of Proposition 4.4. Note here that the compatibility between the automorphisms ψ^m and the bottom map is given by Lemma 4.6.

Note in the following result that if X is a qcqs derived $\mathbb{Z}[\frac{1}{m}]$-scheme, then the filtered spectrum $\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{TC}(X)$ is naturally $\mathbb{Z}[\frac{1}{m}]$-linear.

Corollary 4.8. *Let $m \geq 2$ be an integer, and X be a qcqs derived $\mathbb{Z}[\frac{1}{m}]$-scheme. Then for every integer $i \in \mathbb{Z}$, the automorphism ψ^m induced on the ith graded piece $\mathbb{Z}(i)^{\mathrm{TC}}(X)[2i]$ of the filtered spectrum $\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{TC}(X)$ is multiplication by m^i. Moreover, if X is smooth over $\mathbb{Z}[\frac{1}{m}]$, then the natural diagram of filtered spectra*

$$\begin{array}{ccc} \mathrm{Fil}^{\star}_{\mathrm{cla}}\mathrm{K}(X)[\frac{1}{m}] & \longrightarrow & \mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{TC}(X) \\ \downarrow{\scriptstyle\psi^m} & & \downarrow{\scriptstyle\psi^m} \\ \mathrm{Fil}^{\star}_{\mathrm{cla}}\mathrm{K}(X)[\frac{1}{m}] & \longrightarrow & \mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{TC}(X) \end{array}$$

where the horizontal maps are defined in Theorem 3.14, and the left vertical map is defined in [EM23, Construction B.4], is commutative.

Proof. The identification of the automorphism ψ^m on the graded pieces is a consequence of Propositions 4.4 and 4.5. The second statement is a consequence of the analogous compatibilities for $\prod_{p\in\mathbb{P}}\mathrm{Fil}^{\star}_{\mathrm{BMS}}\mathrm{TC}(X;\mathbb{Z}_p)$ (Proposition 4.4) and for $\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}^-(X)$ ([EM23, Lemma B.8]), and of Lemma 4.6. $\square$

Lemma 4.9. *Let $m \geq 2$ be an integer, and X be a qcqs $\mathbb{Z}[\frac{1}{m}]$-scheme. Then the natural diagram of filtered spectra*

$$\begin{array}{ccc} \mathrm{Fil}^{\star}_{\mathrm{cdh}}\mathrm{KH}(X)[\frac{1}{m}] & \longrightarrow & \mathrm{Fil}^{\star}_{\mathrm{mot}}L_{\mathrm{cdh}}\mathrm{TC}(X) \\ \downarrow{\scriptstyle\psi^m} & & \downarrow{\scriptstyle\psi^m} \\ \mathrm{Fil}^{\star}_{\mathrm{cdh}}\mathrm{KH}(X)[\frac{1}{m}] & \longrightarrow & \mathrm{Fil}^{\star}_{\mathrm{mot}}L_{\mathrm{cdh}}\mathrm{TC}(X) \end{array}$$

where the horizontal maps are defined in Construction 3.16, the left map is the map of Proposition 4.3, and the right map is the map induced by Construction 4.7, is commutative.

Proof. The left map ψ^m of this diagram is defined by cdh sheafifying the left Kan extension from smooth $\mathbb{Z}[\frac{1}{m}]$-schemes to qcqs $\mathbb{Z}[\frac{1}{m}]$-schemes of the automorphism ψ^m on $\mathrm{Fil}^{\star}_{\mathrm{cla}}\mathrm{K}(-)[\frac{1}{m}]$ ([EM23, Appendix B]). The result then follows from Construction 3.16 and Corollary 4.8. $\square$

Construction 4.10 (Adams operations on filtered K-theory)**.** Let $m \geq 2$ be an integer. Following Definition 3.17, if X is a qcqs $\mathbb{Z}[\frac{1}{m}]$-scheme, the automorphism ψ^m of the filtered spectrum $\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{K}(X)[\frac{1}{m}]$ is the automorphism defined by pullback along the natural cartesian square of filtered spectra

$$\begin{array}{ccc} \mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{K}(X)[\frac{1}{m}] & \longrightarrow & \mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{TC}(X) \\ \downarrow & & \downarrow \\ \mathrm{Fil}^{\star}_{\mathrm{cdh}}\mathrm{KH}(X)[\frac{1}{m}] & \longrightarrow & \mathrm{Fil}^{\star}_{\mathrm{mot}}L_{\mathrm{cdh}}\mathrm{TC}(X), \end{array}$$

where the automorphism ψ^m of $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(X)$ is the automorphism of Construction 4.7, the automorphism ψ^m of $\mathrm{Fil}^\star_{\mathrm{mot}}L_{\mathrm{cdh}}\mathrm{TC}(X)$ is defined by cdh sheafifying the automorphism ψ^m of $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(-)$, and the automorphism ψ^m of $\mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{KH}(X)[\frac{1}{m}]$ is the automorphism of Proposition 4.3. Note here that the compatibility between the automorphisms ψ^m and the bottom map is given by Lemma 4.9.

Following Definition 3.18, if X is a qcqs derived $\mathbb{Z}[\frac{1}{m}]$-scheme, the automorphism ψ^m of the filtered spectrum $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X)[\frac{1}{m}]$ is the automorphism defined by pullback along the natural cartesian square of filtered spectra

$$
\begin{array}{ccc}
\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X)[\frac{1}{m}] & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(X) \\
\downarrow & & \downarrow \\
\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(\pi_0(X))[\frac{1}{m}] & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(\pi_0(X)),
\end{array}
$$

where the automorphisms ψ^m of $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(X)$ and $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(\pi_0(X))$ are defined in Construction 4.7, and the automorphism ψ^m of $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(\pi_0(X))[\frac{1}{m}]$ is the automorphism of the previous paragraph. Note here that the compatibility between the automorphisms ψ^m is automatic by construction.

Corollary 4.11. *Let $m \geq 2$ be an integer, and X be a qcqs derived $\mathbb{Z}[\frac{1}{m}]$-scheme. Then for every integer $i \in \mathbb{Z}$, the automorphism ψ^m induced on the ith graded piece $\mathbb{Z}[\frac{1}{m}](i)^{\mathrm{mot}}(X)[2i]$ of the filtered spectrum $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X)[\frac{1}{m}]$ is multiplication by m^i.*

Proof. This is a consequence of Proposition 4.3 and Corollary 4.8. $\square$

The following lemma explains how to use Adams operations to deduce splitting results on the rationalisation of certain filtrations.

Lemma 4.12. *Let*

$$\mathrm{Fil}^\star F(-) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

be a Zariski sheaf of filtered spectra. For each integer $m \geq 2$, let ψ^m be a natural multiplicative automorphism of the filtered spectrum $\mathrm{Fil}^\star F(X)$ on qcqs derived $\mathbb{Z}[\frac{1}{m}]$-schemes X, satisfying the following properties:

(i) *for every qcqs derived scheme X, the rationalised filtration $\mathrm{Fil}^\star F(X; \mathbb{Q})$ is complete;*

(ii) *for any integers $i \in \mathbb{Z}$ and $m \geq 2$, and every qcqs derived $\mathbb{Z}[\frac{1}{m}]$-scheme X, the induced automorphism on the ith graded piece $\mathrm{gr}^i F(X)$ is multiplication by m^i;*

(iii) *for any integers $m, m' \geq 2$, and every qcqs derived $\mathbb{Z}[\frac{1}{mm'}]$-scheme X, the space of natural transformations from $\psi^m \circ \psi^{m'}$ to $\psi^{mm'}$, as endomorphisms of the filtered spectrum $\mathrm{Fil}^\star F(X)$, is contractible.*

Then for every qcqs derived scheme X, and any integers $i, k \in \mathbb{Z}$ such that $k \geq i$, there exists a natural equivalence of spectra

$$\mathrm{Fil}^i F(X; \mathbb{Q}) \simeq \Big(\bigoplus_{i \leq j < k} \mathrm{gr}^j F(X; \mathbb{Q}) \Big) \oplus \mathrm{Fil}^k F(X; \mathbb{Q}).$$

Proof. For every spectrum C equipped with a map $F : C \to C$, denote by C^F the homotopy fibre of the map F. Let $m \geq 2$ be an integer, $i, k \in \mathbb{Z}$ be integers such that $k \geq i$, and X be a qcqs derived $\mathbb{Z}[\frac{1}{m}]$-scheme, for which we first construct the desired equivalence of spectra

$$\mathrm{Fil}^i F(X; \mathbb{Q}) \simeq \Big(\bigoplus_{i \leq j < k} \mathrm{gr}^j F(X; \mathbb{Q}) \Big) \oplus \mathrm{Fil}^k F(X; \mathbb{Q}).$$

We first prove that the spectrum $\big(\mathrm{Fil}^{i+1} F(X; \mathbb{Q}) \big)^{\psi^m - m^i}$ is zero. The filtration $\mathrm{Fil}^{i+1+\star} F(X; \mathbb{Q})$ induced on the spectrum $\mathrm{Fil}^{i+1} F(X; \mathbb{Q})$ is complete (hypothesis (i)), so it suffices to prove that the natural map

$$\psi^m - m^i : \mathrm{gr}^j F(X; \mathbb{Q}) \longrightarrow \mathrm{gr}^j F(X; \mathbb{Q})$$

is an equivalence of spectra for every integer $j \geq i + 1$. For every integer $j \geq i + 1$, this map can be identified with multiplication by the nonzero integer $m^i(m^{j-i} - 1)$ on the $\mathbb{Q}$-linear spectrum $\mathrm{gr}^j F(X; \mathbb{Q})$ (hypothesis (ii)), and is thus an equivalence. Taking the fibre of the natural map $\psi^m - m^i$ on the fibre sequence of spectra

$$\mathrm{Fil}^{i+1} F(X; \mathbb{Q}) \longrightarrow \mathrm{Fil}^i F(X; \mathbb{Q}) \longrightarrow \mathrm{gr}^i F(X; \mathbb{Q}),$$

this implies that the natural map

$$\left(\mathrm{Fil}^i F(X; \mathbb{Q}) \right)^{\psi^m - m^i} \longrightarrow \left(\mathrm{gr}^i F(X; \mathbb{Q}) \right)^{\psi^m - m^i}$$

is an equivalence of spectra. The spectrum $\left(\mathrm{gr}^i F(X; \mathbb{Q}) \right)^{\psi^m - m^i}$ can be naturally identified with the spectrum $\mathrm{gr}^i F(X; \mathbb{Q}) \oplus \mathrm{gr}^i F(X; \mathbb{Q})[-1]$ (hypothesis (ii)), and the induced composite map

$$\mathrm{gr}^i F(X; \mathbb{Q}) \longrightarrow \left(\mathrm{gr}^i F(X; \mathbb{Q}) \right)^{\psi^m - m^i} \xrightarrow{\sim} \left(\mathrm{Fil}^i F(X; \mathbb{Q}) \right)^{\psi^m - m^i} \xrightarrow{\mathrm{can}} \mathrm{Fil}^i F(X; \mathbb{Q})$$

induces a natural splitting of spectra

$$\mathrm{Fil}^i F(X; \mathbb{Q}) \simeq \mathrm{gr}^i F(X; \mathbb{Q}) \oplus \mathrm{Fil}^{i+1} F(X; \mathbb{Q}).$$

By induction, this implies that for every integer $k \geq i$, there is a natural equivalence of spectra

$$\mathrm{Fil}^i F(X; \mathbb{Q}) \simeq \big(\bigoplus_{i \leq j < k} \mathrm{gr}^j F(X; \mathbb{Q}) \big) \oplus \mathrm{Fil}^k F(X; \mathbb{Q}).$$

We now prove the desired equivalence of spectra for a general qcqs derived scheme X. The presheaf $\mathrm{Fil}^\star F(-)$ being a Zariski sheaf of filtered spectra, the presheaves

$$\mathrm{Fil}^i F(-; \mathbb{Q}) \text{ and } \big(\bigoplus_{i \leq j < k} \mathrm{gr}^j F(-; \mathbb{Q}) \big) \oplus \mathrm{Fil}^k F(-; \mathbb{Q})$$

are Zariski sheaves of spectra. It thus suffices to construct compatible equivalences of spectra

$$\mathrm{Fil}^i F(X_{\mathbb{Z}[\frac{1}{m}]}; \mathbb{Q}) \simeq \big(\bigoplus_{i \leq j < k} \mathrm{gr}^j F(X_{\mathbb{Z}[\frac{1}{m}]}; \mathbb{Q}) \big) \oplus \mathrm{Fil}^k F(X_{\mathbb{Z}[\frac{1}{m}]}; \mathbb{Q})$$

for all integers $m \geq 2$. The construction of this equivalence for each integer $m \geq 2$, which depends on the map $\psi^m : \mathrm{Fil}^\star F(X_{\mathbb{Z}[\frac{1}{m}]}) \to \mathrm{Fil}^\star F(X_{\mathbb{Z}[\frac{1}{m}]})$, is the first part of this proof. Let $m, m' \geq 2$ be integers. The compatibility between the constructions of the equivalences for m, m', and mm' only depend on the choices, for every integer $i \in \mathbb{Z}$, of the identification of the maps ψ^m, $\psi^{m'}$, and $\psi^{mm'}$ on the ith graded piece with multiplication by m^i, $(m')^i$, and $(mm')^i$ respectively. These choices are compatible up to homotopy (hypothesis (iii)), which concludes the proof. $\qquad\square$

Remark 4.13. Lemma 4.12 can also be proved, with the same proof, for Zariski sheaves of filtered spectra $\mathrm{Fil}^\star F(-)$ that are defined on qcqs schemes, on qcqs schemes of finite valuative dimension, on noetherian schemes of finite dimension, or on smooth schemes over a given commutative ring.

Remark 4.14. Hypothesis (iii) in Lemma 4.12 follows from the construction of the Adams operations on the filtrations $\mathrm{Fil}^\star_{\mathrm{cla}} K(-)$ and $\mathrm{Fil}^\star_{\mathrm{HKR}} \mathrm{HC}^-(-)$, when these are defined. This hypothesis (iii) then follows formally for all the other filtrations considered in this subsection.

Proposition 4.15. *Let*

$$\mathrm{Fil}^\star F(-) : \mathrm{Sch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

be a finitary Zariski sheaf of filtered spectra. For each integer $m \geq 2$, let ψ^m be a natural multiplicative automorphism of the filtered spectrum $\mathrm{Fil}^\star F(X)$ on qcqs $\mathbb{Z}[\frac{1}{m}]$-schemes X, satisfying the following properties:

(i) *for every noetherian scheme X of finite dimension, there exists an integer $d \in \mathbb{Z}$ such that for every integer $i \in \mathbb{Z}$, the spectrum $\mathrm{Fil}^i F(X; \mathbb{Q})$ is in cohomological degrees at most $-i + d$;*

(ii) *for any integers $i \in \mathbb{Z}$ and $m \geq 2$, and every qcqs $\mathbb{Z}[\frac{1}{m}]$-scheme X, the induced automorphism on the ith graded piece $\mathrm{gr}^i F(X)$ is multiplication by m^i;*

(iii) *for any integers $m, m' \geq 2$, and every qcqs $\mathbb{Z}[\frac{1}{mm'}]$-scheme X, the space of natural transformations from $\psi^m \circ \psi^{m'}$ to $\psi^{mm'}$, as endomorphisms of the filtered spectrum $\mathrm{Fil}^\star F(X)$, is contractible.*

Then for every qcqs scheme X, there exists a natural multiplicative equivalence of filtered spectra

$$\mathrm{Fil}^\star F(X; \mathbb{Q}) \simeq \bigoplus_{j \geq \star} \mathrm{gr}^j F(X; \mathbb{Q}).$$

Proof. By finitariness, it suffices to prove the result on noetherian schemes of finite dimension. Hypothesis (i) implies that the filtration $\mathrm{Fil}^\star F(X; \mathbb{Q})$ is complete on such schemes X. Lemma 4.12 and Remark 4.13 then imply that there exist natural equivalences of spectra

$$\mathrm{Fil}^i F(X; \mathbb{Q}) \simeq \left(\oplus_{i \leq j < k} \mathrm{gr}^j F(X; \mathbb{Q}) \right) \oplus \mathrm{Fil}^k F(X; \mathbb{Q})$$

for all integers $i, k \in \mathbb{Z}$ such that $k \geq i$. Again using completeness, taking the limit over $k \to +\infty$ induces a natural equivalence of spectra

$$\mathrm{Fil}^i F(X; \mathbb{Q}) \simeq \prod_{j \geq i} \mathrm{gr}^j F(X; \mathbb{Q}).$$

Hypothesis (i) then implies that, at each cohomological degree, only a finite number of terms in the previous product are nonzero. In particular, the previous equivalence can be rewritten as a natural equivalence of spectra

$$\mathrm{Fil}^i F(X; \mathbb{Q}) \simeq \bigoplus_{j \geq i} \mathrm{gr}^j F(X; \mathbb{Q}),$$

which induces the desired multiplicative equivalence of filtered spectra. $\qquad\square$

Remark 4.16. Let X be a qcqs derived scheme. The argument of Proposition 4.15, where the necessary hypotheses are satisfied by Corollary 4.8 and Proposition 2.35, implies that there is a natural multiplicative equivalence of filtered spectra

$$\mathrm{Fil}^\star_{\mathrm{mot}} \mathrm{TC}(X; \mathbb{Q}) \simeq \prod_{j \geq \star} \mathbb{Q}(j)^{\mathrm{TC}}(X)[2i].$$

4.2 Rigid-analytic HKR filtrations

In this subsection, we define variants, which we call rigid-analytic, of the HKR filtrations. We start by explaining the relevant objects at the level of Hochschild homology.

For every qcqs derived scheme X, there is a natural S^1-equivariant arithmetic fracture square

$$\begin{array}{ccc} \mathrm{HH}(X) & \longrightarrow & \mathrm{HH}(X_{\mathbb{Q}}/\mathbb{Q}) \\ \downarrow & & \downarrow \\ \prod_{p \in \mathbb{P}} \mathrm{HH}(X; \mathbb{Z}_p) & \longrightarrow & \prod'_{p \in \mathbb{P}} \mathrm{HH}(X; \mathbb{Q}_p) \end{array} \qquad (1)$$

in the derived category $\mathcal{D}(\mathbb{Z})$, where we use base change for Hochschild homology for the top right corner, and the convention adopted in the Notation part for the bottom left and bottom right corners. Applying homotopy fixed points $(-)^{h\mathrm{S}^1}$ for the S^1-action induces a natural cartesian square

$$\begin{array}{ccc} \mathrm{HC}^-(X) & \longrightarrow & \mathrm{HC}^-(X_{\mathbb{Q}}/\mathbb{Q}) \\ \downarrow & & \downarrow \\ \prod_{p \in \mathbb{P}} \mathrm{HC}^-(X; \mathbb{Z}_p) & \longrightarrow & \left(\prod'_{p \in \mathbb{P}} \mathrm{HH}(X; \mathbb{Q}_p) \right)^{h\mathrm{S}^1} \end{array}$$

in the derived category $\mathcal{D}(\mathbb{Z})$, where we use that taking homotopy fixed points $(-)^{h\mathrm{S}^1}$ commutes with limits for the bottom left corner. We call *rigid-analytic variant of Hochschild homology and of negative cyclic homology* the bottom right corners of the previous two cartesian squares, respectively. This terminology should find some justification in Section 4.3. Following Section 2.1, we use the previous cartesian square to introduce a new HKR filtration on this rigid-analytic variant of negative cyclic homology $\big(\prod_{p\in\mathbb{P}}' \mathrm{HH}(X;\mathbb{Q}_p)\big)^{h\mathrm{S}^1}$.

Definition 4.17 (HKR filtration on rigid-analytic HC^-). The *HKR filtration on rigid-analytic negative cyclic homology* of qcqs derived schemes is the functor

$$\mathrm{Fil}^\star_{\mathrm{HKR}}\big(\prod_{p\in\mathbb{P}}{}' \mathrm{HH}(-;\mathbb{Q}_p)\big)^{h\mathrm{S}^1} : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

defined by the cocartesian square

$$\begin{array}{ccc}
\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(-) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(-_\mathbb{Q}/\mathbb{Q}) \\
\downarrow & & \downarrow \\
\prod_{p\in\mathbb{P}}\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(-;\mathbb{Z}_p) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{HKR}}\big(\prod_{p\in\mathbb{P}}' \mathrm{HH}(-;\mathbb{Q}_p)\big)^{h\mathrm{S}^1}.
\end{array}$$

Remark 4.18. Let p be a prime number, and X be a qcqs derived scheme over $\mathbb{Z}_{(p)}$. The natural map

$$\prod_{\ell\in\mathbb{P}}\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X;\mathbb{Z}_\ell) \longrightarrow \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X;\mathbb{Z}_p)$$

is then an equivalence of filtered spectra, and we then denote by $\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HH}(X;\mathbb{Q}_p)^{h\mathrm{S}^1}$ the filtered spectrum $\mathrm{Fil}^\star_{\mathrm{HKR}}\prod_{\ell\in\mathbb{P}}' \mathrm{HH}(X;\mathbb{Q}_\ell)^{h\mathrm{S}^1}$. In particular, the natural commutative diagram

$$\begin{array}{ccc}
\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X_\mathbb{Q}/\mathbb{Q}) \\
\downarrow & & \downarrow \\
\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X;\mathbb{Z}_p) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HH}(X;\mathbb{Q}_p)^{h\mathrm{S}^1}
\end{array}$$

is a cartesian square of filtered spectra.

Similarly, one can apply homotopy orbits $(-)_{h\mathrm{S}^1}$ for the S^1-action to the arithmetic fracture square for Hochschild homology (1). The functor $(-)_{h\mathrm{S}^1}$ does not commute with limits, but the S^1-action on the product $\prod_{p\in\mathbb{P}}\mathrm{HH}(X;\mathbb{Z}_p)$ is diagonal. Using the natural fibre sequence

$$\mathrm{HH}(X;\mathbb{Z}_p) \longrightarrow \mathrm{HH}(X;\mathbb{Z}_p)_{h\mathrm{S}^1} \longrightarrow \mathrm{HH}(X;\mathbb{Z}_p)_{h\mathrm{S}^1}[2]$$

in the derived category $\mathcal{D}(\mathbb{Z})$ and the fact that the functors $\mathrm{HH}(-;\mathbb{Z}_p)$ and $\mathrm{HH}(-;\mathbb{Z}_p)_{h\mathrm{S}^1}$ are in non-positive cohomological degrees on animated commutative rings, one can prove that the complex $\mathrm{HH}(X;\mathbb{Z}_p)_{h\mathrm{S}^1} \in \mathcal{D}(\mathbb{Z})$ is derived p-complete, hence the natural map

$$\mathrm{HC}(X;\mathbb{Z}_p) \longrightarrow \mathrm{HH}(X;\mathbb{Z}_p)_{h\mathrm{S}^1}$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$. In particular, applying homotopy orbits $(-)_{h\mathrm{S}^1}$ to the arithmetic fracture square for Hochschild homology (1) induces a natural cartesian square

$$\begin{array}{ccc}
\mathrm{HC}(X) & \longrightarrow & \mathrm{HC}(X_\mathbb{Q}/\mathbb{Q}) \\
\downarrow & & \downarrow \\
\prod_{p\in\mathbb{P}}\mathrm{HC}(X;\mathbb{Z}_p) & \longrightarrow & \big(\prod_{p\in\mathbb{P}}' \mathrm{HH}(X;\mathbb{Q}_p)\big)_{h\mathrm{S}^1}
\end{array}$$

in the derived category $\mathcal{D}(\mathbb{Z})$. We use this cartesian square to introduce the following HKR filtration on the bottom right corner.

Definition 4.19 (HKR filtration on rigid-analytic HC)**.** The *HKR filtration on rigid-analytic cyclic homology* of qcqs derived schemes is the functor

$$\mathrm{Fil}^\star_{\mathrm{HKR}}\big(\prod_{p\in\mathbb{P}}{}' \mathrm{HH}(-;\mathbb{Q}_p)\big)_{h\mathrm{S}^1} : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

defined by the cocartesian square

$$\begin{array}{ccc} \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}(-) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}(-_\mathbb{Q}/\mathbb{Q}) \\ \downarrow & & \downarrow \\ \prod_{p\in\mathbb{P}}\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}(-;\mathbb{Z}_p) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{HKR}}\big(\prod'_{p\in\mathbb{P}}\mathrm{HH}(-;\mathbb{Q}_p)\big)_{h\mathrm{S}^1}. \end{array}$$

Remark 4.20. Taking homotopy orbits $(-)_{h\mathrm{S}^1}$ commutes with colimits, so the natural map

$$\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}(X;\mathbb{Q}) \longrightarrow \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}(X_\mathbb{Q}/\mathbb{Q})$$

is an equivalence in the filtered derived category $\mathcal{DF}(\mathbb{Q})$. Upon applying rationalisation to the natural cartesian square

$$\begin{array}{ccc} \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}(X) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}(X_\mathbb{Q}/\mathbb{Q}) \\ \downarrow & & \downarrow \\ \prod_{p\in\mathbb{P}}\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}(X;\mathbb{Z}_p) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{HKR}}\big(\prod'_{p\in\mathbb{P}}\mathrm{HH}(X;\mathbb{Q}_p)\big)_{h\mathrm{S}^1}, \end{array}$$

this implies that the natural map

$$\prod_{p\in\mathbb{P}}{}' \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}(X;\mathbb{Q}_p) := \big(\prod_{p\in\mathbb{P}}\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}(X;\mathbb{Z}_p)\big)_\mathbb{Q} \longrightarrow \mathrm{Fil}^\star_{\mathrm{HKR}}\big(\prod_{p\in\mathbb{P}}{}' \mathrm{HH}(X;\mathbb{Q}_p)\big)_{h\mathrm{S}^1}$$

is an equivalence in the filtered derived category $\mathcal{DF}(\mathbb{Q})$.

Remark 4.21. Let p be a prime number, and X be a qcqs derived scheme over $\mathbb{Z}_{(p)}$. As in Remark 4.18, we denote the filtered complex $\mathrm{Fil}^\star_{\mathrm{HKR}}\big(\prod'_{\ell\in\mathbb{P}}\mathrm{HH}(X;\mathbb{Q}_\ell)\big)_{h\mathrm{S}^1}$ by

$$\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HH}(X;\mathbb{Q}_p)_{h\mathrm{S}^1} \in \mathcal{DF}(\mathbb{Q}).$$

In particular, the natural map

$$\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}(X;\mathbb{Q}_p) \longrightarrow \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HH}(X;\mathbb{Q}_p)_{h\mathrm{S}^1}$$

is an equivalence in the filtered derived category $\mathcal{DF}(\mathbb{Q})$ by Remark 4.20.

The Tate construction $(-)^{t\mathrm{S}^1}$ is by definition the cofibre of the norm map

$$(-)_{h\mathrm{S}^1}[1] \to (-)^{h\mathrm{S}^1}.$$

Applying the Tate construction $(-)^{t\mathrm{S}^1}$ to the arithmetic fracture square for Hochschild homology (1) then induces a natural cartesian square

$$\begin{array}{ccc} \mathrm{HP}(X) & \longrightarrow & \mathrm{HP}(X_\mathbb{Q}/\mathbb{Q}) \\ \downarrow & & \downarrow \\ \prod_{p\in\mathbb{P}}\mathrm{HP}(X;\mathbb{Z}_p) & \longrightarrow & \big(\prod'_{p\in\mathbb{P}}\mathrm{HH}(X;\mathbb{Q}_p)\big)^{t\mathrm{S}^1} \end{array}$$

in the derived category $\mathcal{D}(\mathbb{Z})$, where we use the analogous cartesian squares for HC^- and HC to identify the bottom left corner.

Definition 4.22 (HKR filtration on rigid-analytic HP). The *HKR filtration on rigid-analytic periodic cyclic homology* of qcqs derived schemes is the functor

$$\mathrm{Fil}^\star_{\mathrm{HKR}}\big(\prod_{p\in\mathbb{P}}{}'\,\mathrm{HH}(-;\mathbb{Q}_p)\big)^{t\mathrm{S}^1} : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

defined by the cocartesian square

$$
\begin{array}{ccc}
\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HP}(-) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HP}(-_\mathbb{Q}/\mathbb{Q}) \\
\downarrow & & \downarrow \\
\prod_{p\in\mathbb{P}}\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HP}(-;\mathbb{Z}_p) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{HKR}}\big(\prod'_{p\in\mathbb{P}}\mathrm{HH}(-;\mathbb{Q}_p)\big)^{t\mathrm{S}^1}.
\end{array}
$$

Remark 4.23. Let p be a prime number, and X be a qcqs derived scheme over $\mathbb{Z}_{(p)}$. As in Remarks 4.18 and 4.21, we denote the filtered complex $\mathrm{Fil}^\star_{\mathrm{HKR}}\big(\prod_{\ell\in\mathbb{P}}\mathrm{HH}(X;\mathbb{Q}_\ell)\big)^{t\mathrm{S}^1}$ by

$$\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HH}(X;\mathbb{Q}_p)^{t\mathrm{S}^1} \in \mathcal{DF}(\mathbb{Q}).$$

In particular, the natural commutative diagram

$$
\begin{array}{ccc}
\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HP}(X) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HP}(X_\mathbb{Q}/\mathbb{Q}) \\
\downarrow & & \downarrow \\
\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HP}(X;\mathbb{Z}_p) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HH}(X;\mathbb{Q}_p)^{t\mathrm{S}^1}
\end{array}
$$

is a cartesian square of filtered spectra.

Lemma 4.24. *Let X be a qcqs derived scheme. Then the natural commutative diagram*

$$
\begin{array}{ccc}
\prod'_{p\in\mathbb{P}}\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X;\mathbb{Q}_p) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{HKR}}\big(\prod'_{p\in\mathbb{P}}\mathrm{HH}(X;\mathbb{Q}_p)\big)^{h\mathrm{S}^1} \\
\downarrow & & \downarrow \\
\prod'_{p\in\mathbb{P}}\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HP}(X;\mathbb{Q}_p) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{HKR}}\big(\prod'_{p\in\mathbb{P}}\mathrm{HH}(X;\mathbb{Q}_p)\big)^{t\mathrm{S}^1}
\end{array}
$$

is a cartesian square of filtered spectra.

Proof. There is a natural commutative diagram of filtered spectra

$$
\begin{array}{ccccc}
\prod'_{p\in\mathbb{P}}\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X;\mathbb{Q}_p) & \longrightarrow & \prod'_{p\in\mathbb{P}}\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HP}(X;\mathbb{Q}_p) & \longrightarrow & \prod'_{p\in\mathbb{P}}\mathrm{Fil}^{\star-1}_{\mathrm{HKR}}\mathrm{HC}(X;\mathbb{Q}_p)[2] \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Fil}^\star_{\mathrm{HKR}}\big(\prod'_{p\in\mathbb{P}}\mathrm{HH}(X;\mathbb{Q}_p)\big)^{h\mathrm{S}^1} & \to & \mathrm{Fil}^\star_{\mathrm{HKR}}\big(\prod'_{p\in\mathbb{P}}\mathrm{HH}(X;\mathbb{Q}_p)\big)^{t\mathrm{S}^1} & \to & \mathrm{Fil}^{\star-1}_{\mathrm{HKR}}\big(\prod'_{p\in\mathbb{P}}\mathrm{HH}(X;\mathbb{Q}_p)\big)_{h\mathrm{S}^1}[2]
\end{array}
$$

where, by definition of the bottom terms (Definitions 4.17, 4.22, and 4.19), the horizontal lines are fibre sequences. In this diagram, the right vertical map is an equivalence (Remark 4.20), so the left square is a cartesian square. □

Lemma 4.25. *Let X be a qcqs derived scheme. Then the filtrations*

$$\prod_{p\in\mathbb{P}}{}'\big(\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HH}(X;\mathbb{Q}_p)\big)^{h\mathrm{S}^1}, \quad \prod_{p\in\mathbb{P}}{}'\big(\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HH}(X;\mathbb{Q}_p)\big)_{h\mathrm{S}^1}, \ \text{and} \ \prod_{p\in\mathbb{P}}{}'\big(\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HH}(X;\mathbb{Q}_p)\big)^{t\mathrm{S}^1}$$

are complete.

Proof. The HKR filtrations on $\mathrm{HC}^-(X)$, $\mathrm{HC}(X)$, and $\mathrm{HP}(X)$ are complete by Lemma 2.9. The product on prime numbers p of the p-completions of these filtrations are also complete, since p-completions and products commute with limits. Similarly, the HKR filtrations on $\mathrm{HC}^-(X_\mathbb{Q}/\mathbb{Q})$, $\mathrm{HC}(X_\mathbb{Q}/\mathbb{Q})$, and $\mathrm{HP}(X_\mathbb{Q}/\mathbb{Q})$ are complete by [Ant19, Theorem 1.1]. The rigid-analytic HKR filtrations on HC^-, HC, and HP are thus also complete, as pushouts of three complete filtrations. $\square$

We now describe the graded pieces of these rigid-analytic HKR filtrations, by analogy with the classical HKR filtrations.

Definition 4.26 (Rigid-analytic Hodge-complete derived de Rham cohomology). For every integer $i \in \mathbb{Z}$, the functor

$$R\Gamma_{\mathrm{Zar}}\Big(-, \prod_{p\in\mathbb{P}}{}' \, \widehat{\mathbb{L}\Omega}^{\geq i}_{-\mathbb{Q}_p/\mathbb{Q}_p}\Big) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathcal{D}(\mathbb{Q})$$

is defined as the shifted graded piece of the HKR filtration on $\prod'_{p\in\mathbb{P}} \mathrm{HH}(-;\mathbb{Q}_p)^{h\mathrm{S}^1}$:

$$R\Gamma_{\mathrm{Zar}}\Big(-, \prod_{p\in\mathbb{P}}{}' \, \widehat{\mathbb{L}\Omega}^{\geq i}_{-\mathbb{Q}_p/\mathbb{Q}_p}\Big) := \mathrm{gr}^i_{\mathrm{HKR}}\big(\prod_{p\in\mathbb{P}}{}' \, \mathrm{HH}(-;\mathbb{Q}_p)\big)^{h\mathrm{S}^1}[-2i].$$

Remark 4.27. Let X be a qcqs derived scheme, and $i \in \mathbb{Z}$ be an integer. By Definition 4.17, there is a natural cartesian square

$$\begin{array}{ccc}
R\Gamma_{\mathrm{Zar}}\Big(X, \widehat{\mathbb{L}\Omega}^{\geq i}_{-/\mathbb{Z}}\Big) & \longrightarrow & R\Gamma_{\mathrm{Zar}}\Big(X, \widehat{\mathbb{L}\Omega}^{\geq i}_{-\mathbb{Q}/\mathbb{Q}}\Big) \\
\downarrow & & \downarrow \\
R\Gamma_{\mathrm{Zar}}\Big(X, \prod_{p\in\mathbb{P}} \big(\widehat{\mathbb{L}\Omega}^{\geq i}_{-/\mathbb{Z}}\big)^\wedge_p\Big) & \longrightarrow & R\Gamma_{\mathrm{Zar}}\Big(X, \prod'_{p\in\mathbb{P}} \widehat{\mathbb{L}\Omega}^{\geq i}_{-\mathbb{Q}_p/\mathbb{Q}_p}\Big)
\end{array}$$

in the derived category $\mathcal{D}(\mathbb{Z})$, which can serve as an alternative definition for the bottom right term.

Remark 4.28. Let X be a qcqs derived scheme, and $i \in \mathbb{Z}$ be an integer. The complexes

$$R\Gamma_{\mathrm{Zar}}\Big(X, \prod_{p\in\mathbb{P}}{}' \, \widehat{\mathbb{L}\Omega}_{-\mathbb{Q}_p/\mathbb{Q}_p}\Big) \quad \text{and} \quad R\Gamma_{\mathrm{Zar}}\Big(X, \prod_{p\in\mathbb{P}}{}' \, \mathbb{L}\Omega^{\leq i}_{-\mathbb{Q}_p/\mathbb{Q}_p}\Big)$$

are defined as in Definition 4.26, where $(-)^{h\mathrm{S}^1}$ is replaced by $(-)^{t\mathrm{S}^1}$ and $(-)_{h\mathrm{S}^1}$ respectively. In particular, the natural fibre sequence

$$\mathrm{Fil}^\star_{\mathrm{HKR}}\big(\prod_{p\in\mathbb{P}}{}' \, \mathrm{HH}(X;\mathbb{Q}_p)\big)^{h\mathrm{S}^1} \to \mathrm{Fil}^\star_{\mathrm{HKR}}\big(\prod_{p\in\mathbb{P}}{}' \, \mathrm{HH}(X;\mathbb{Q}_p)\big)^{t\mathrm{S}^1} \to \mathrm{Fil}^{\star-1}_{\mathrm{HKR}}\big(\prod_{p\in\mathbb{P}}{}' \, \mathrm{HH}(X;\mathbb{Q}_p)\big)_{h\mathrm{S}^1}[2]$$

induces a natural fibre sequence

$$R\Gamma_{\mathrm{Zar}}\Big(X, \prod_{p\in\mathbb{P}}{}' \, \widehat{\mathbb{L}\Omega}^{\geq i}_{-\mathbb{Q}_p/\mathbb{Q}_p}\Big) \longrightarrow R\Gamma_{\mathrm{Zar}}\Big(X, \prod_{p\in\mathbb{P}}{}' \, \widehat{\mathbb{L}\Omega}_{-\mathbb{Q}_p/\mathbb{Q}_p}\Big) \longrightarrow R\Gamma_{\mathrm{Zar}}\Big(X, \prod_{p\in\mathbb{P}}{}' \, \mathbb{L}\Omega^{< i}_{-\mathbb{Q}_p/\mathbb{Q}_p}\Big),$$

where the right term is naturally identified with the complex

$$\Big(\prod_{p\in\mathbb{P}} R\Gamma_{\mathrm{Zar}}\big(X, \big(L\Omega^{< i}_{-/\mathbb{Z}}\big)^\wedge_p\big)\Big)_\mathbb{Q} \in \mathcal{D}(\mathbb{Q})$$

by Remark 4.20.

Remark 4.29. Let p be a prime number, X be a qcqs derived scheme over $\mathbb{Z}_{(p)}$, and $i \in \mathbb{Z}$ be an integer. Following Remarks 4.18, 4.21, and 4.23, we denote the complexes

$$R\Gamma_{\mathrm{Zar}}\Big(X, \prod_{\ell\in\mathbb{P}}{}' \, \widehat{\mathbb{L}\Omega}^{\geq i}_{-\mathbb{Q}_\ell/\mathbb{Q}_\ell}\Big), \ R\Gamma_{\mathrm{Zar}}\Big(X, \prod_{\ell\in\mathbb{P}}{}' \, \widehat{\mathbb{L}\Omega}_{-\mathbb{Q}_\ell/\mathbb{Q}_\ell}\Big), \ \text{and} \ R\Gamma_{\mathrm{Zar}}\Big(X, \prod_{\ell\in\mathbb{P}}{}' \, \mathbb{L}\Omega^{< i}_{-\mathbb{Q}_\ell/\mathbb{Q}_\ell}\Big)$$

by

$$R\Gamma_{\mathrm{Zar}}\big(X, \widehat{\underline{\mathbb{L}\Omega}}^{\geq i}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p}\big),\ R\Gamma_{\mathrm{Zar}}\big(X, \widehat{\underline{\mathbb{L}\Omega}}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p}\big),\ \text{and } R\Gamma_{\mathrm{Zar}}\big(X, \underline{\mathbb{L}\Omega}^{<i}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p}\big)$$

respectively. In particular, by Remark 4.28, there is a natural fibre sequence

$$R\Gamma_{\mathrm{Zar}}\big(X, \widehat{\underline{\mathbb{L}\Omega}}^{\geq i}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p}\big) \longrightarrow R\Gamma_{\mathrm{Zar}}\big(X, \widehat{\underline{\mathbb{L}\Omega}}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p}\big) \longrightarrow R\Gamma_{\mathrm{Zar}}\big(X, \underline{\mathbb{L}\Omega}^{<i}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p}\big)$$

in the derived category $\mathcal{D}(\mathbb{Q}_p)$, where the right term is naturally identified with the complex

$$R\Gamma_{\mathrm{Zar}}\big(X, \big(\mathbb{L}\Omega^{<i}_{-/\mathbb{Z}}\big)^{\wedge}_p[\tfrac{1}{p}]\big) \in \mathcal{D}(\mathbb{Q}_p).$$

In the following result, we reformulate the motivic filtration on topological cyclic homology of Definition 2.30 in terms of the rigid-analytic HKR filtration on negative cyclic homology. This can be interpreted as a filtered arithmetic fracture square for topological cyclic homology.

Proposition 4.30. *Let X be qcqs derived scheme. Then the natural commutative diagram*

$$\begin{array}{ccc}
\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{TC}(X) & \longrightarrow & \mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}^{-}(X_{\mathbb{Q}}/\mathbb{Q}) \\
\downarrow & & \downarrow \\
\prod_{p\in\mathbb{P}}\mathrm{Fil}^{\star}_{\mathrm{BMS}}\mathrm{TC}(X;\mathbb{Z}_p) & \longrightarrow & \mathrm{Fil}^{\star}_{\mathrm{HKR}}\big(\prod'_{p\in\mathbb{P}}\mathrm{HH}(X;\mathbb{Q}_p)\big)^{h\mathrm{S}^1}
\end{array}$$

is a cartesian square of filtered spectra.

Proof. This is a consequence of Definitions 2.30 and 4.17. $\qquad\square$

Corollary 4.31. *Let X be a qcqs scheme. Then the natural commutative diagram*

$$\begin{array}{ccc}
\big(L_{\mathrm{cdh}}\,\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{TC}(-)\big)(X) & \longrightarrow & \big(L_{\mathrm{cdh}}\,\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}^{-}(-_{\mathbb{Q}}/\mathbb{Q})\big)(X) \\
\downarrow & & \downarrow \\
\big(L_{\mathrm{cdh}}\prod_{p\in\mathbb{P}}\mathrm{Fil}^{\star}_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)\big)(X) & \longrightarrow & \big(L_{\mathrm{cdh}}\,\mathrm{Fil}^{\star}_{\mathrm{HKR}}\big(\prod'_{p\in\mathbb{P}}\mathrm{HH}(-;\mathbb{Q}_p)\big)^{h\mathrm{S}^1}\big)(X),
\end{array}$$

is a cartesian square of filtered spectra.

Proof. This is a consequence of Proposition 4.30, which we restrict to qcqs schemes and then sheafify for the cdh topology. $\qquad\square$

Corollary 4.32. *Let X be a qcqs derived scheme. Then for every integer $i \in \mathbb{Z}$, the natural commutative diagram*

$$\begin{array}{ccc}
\mathbb{Z}(i)^{\mathrm{TC}}(X) & \longrightarrow & R\Gamma_{\mathrm{Zar}}\big(X, \widehat{\underline{\mathbb{L}\Omega}}^{\geq i}_{-_{\mathbb{Q}}/\mathbb{Q}}\big) \\
\downarrow & & \downarrow \\
\prod_{p\in\mathbb{P}}\mathbb{Z}_p(i)^{\mathrm{BMS}}(X) & \longrightarrow & R\Gamma_{\mathrm{Zar}}\big(X, \prod'_{p\in\mathbb{P}}\widehat{\underline{\mathbb{L}\Omega}}^{\geq i}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p}\big),
\end{array}$$

is a cartesian square in the derived category $\mathcal{D}(\mathbb{Z})$.

Proof. This is a direct consequence of Proposition 4.30. $\qquad\square$

4.3 A rigid-analytic Goodwillie theorem

In this subsection, we prove that the rigid-analytic version of periodic cyclic homology is a truncating invariant (Theorem 4.41). We first recall the definition of Hochschild, cyclic, negative cyclic, and periodic cyclic homologies of a general cyclic object.

Definition 4.33 (Cyclic object). Let $\mathcal{C}$ be a category, or an ∞-category. The *cyclic category* Λ is the category with objects $[n]$ indexed by non-negative integers n, and morphisms $[m] \to [n]$ given by homotopy classes of degree one increasing maps from S^1 to itself that map the subgroup $\mathbb{Z}/(m+1)$ to $\mathbb{Z}/(n+1)$. A *cyclic object* in $\mathcal{C}$ is then a contravariant functor from the cyclic category Λ to $\mathcal{C}$.

Notation 4.34 (Hochschild homology of a cyclic object). Let $\mathcal{A}$ be an abelian category with exact infinite products, and $X_\bullet$ be a cyclic object in $\mathcal{A}$. Following [Goo85, Section II] (see also [Mor18b, Section 2.2]), one can define the Hochschild homology $\mathrm{HH}(X_\bullet)$ of $X_\bullet$, as an object of the derived category $\mathcal{D}(\mathcal{A})$ equipped with a natural S^1-action. The cyclic, negative cyclic, and periodic cyclic homologies of the cyclic object $X_\bullet$ are then defined by

$$\mathrm{HC}(X_\bullet) := \mathrm{HH}(X_\bullet)_{hS^1}, \ \mathrm{HC}^-(X_\bullet) := \mathrm{HH}(X_\bullet)^{hS^1}, \ \text{and } \mathrm{HP}(X_\bullet) := \mathrm{HH}(X_\bullet)^{tS^1}.$$

In this context, there is moreover a natural map

$$s : \mathrm{HC}(X_\bullet)[-2] \longrightarrow \mathrm{HC}(X_\bullet)$$

in the derived category $\mathcal{D}(\mathcal{A})$, from which periodic cyclic homology $\mathrm{HP}(X_\bullet) \in \mathcal{D}(\mathcal{A})$ can be recovered by the formula

$$\mathrm{HP}(X_\bullet) \simeq \varprojlim \left(\cdots \xrightarrow{s} \mathrm{HC}(X_\bullet)[-4] \xrightarrow{s} \mathrm{HC}(X_\bullet)[-2] \xrightarrow{s} \mathrm{HC}(X_\bullet) \right).$$

We now apply the previous general construction to define Hochschild homology and its variants on solid associative derived algebras over a solid commutative ring.

Definition 4.35 (Solid Hochschild homology). Let k be a solid commutative ring, and R be a connective solid k-$\mathbb{E}_1$-algebra. The simplicial object

$$\cdots \, \substack{\longrightarrow \\[-3pt] \longrightarrow \\[-3pt] \longrightarrow} \, R \otimes_k R \otimes_k R \, \substack{\longrightarrow \\[-3pt] \longrightarrow} \, R \otimes_k R \, \rightrightarrows \, R$$

has a natural structure of cyclic object in the derived ∞-category of solid k-modules (Definition 4.33), induced by permutation of the tensor factors. We write $\underline{\mathrm{HH}}(R/k)$, $\underline{\mathrm{HC}}(R/k)$, $\underline{\mathrm{HC}^-}(R/k)$ and $\underline{\mathrm{HP}}(R/k)$ for the Hochschild, cyclic, negative cyclic and periodic cyclic homologies of this cyclic object (Notation 4.34). If $k = \mathbb{Z}$, we simply denote these by $\underline{\mathrm{HH}}(R)$, $\underline{\mathrm{HC}}(R)$, $\underline{\mathrm{HC}^-}(R)$ and $\underline{\mathrm{HP}}(R)$.

For $f : R \to R'$ a map of $\mathbb{Z}$-$\mathbb{E}_1$-algebras and $F : \mathbb{E}_1\text{-Alg}_{\mathbb{Z}} \to \mathcal{D}(\mathbb{Z})$ a functor, we denote by $F(f) \in \mathcal{D}(\mathbb{Z})$ the cofibre of the map $F(R) \to F(R')$. More generally, for $f : R \to R'$ a map of solid $\mathbb{Z}$-$\mathbb{E}_1$-algebras and F a $\mathcal{D}(\text{Solid})$-valued functor on solid $\mathbb{Z}$-$\mathbb{E}_1$-algebras, denote by $F(f) \in \mathcal{D}(\text{Solid})$ the cofibre of the map $F(R) \to F(R')$.

The following result is [Goo85, Theorem IV.2.6]. More precisely, this result of Goodwillie is for maps of simplicial $\mathbb{Z}$-algebras, and the underlying ∞-category of simplicial $\mathbb{Z}$-algebras is naturally identified with the ∞-category of connective $\mathbb{Z}$-$\mathbb{E}_1$-algebras, by the monoidal Dold–Kan correspondence and [Lur17, Proposition 7.1.4.6].

Theorem 4.36 ([Goo85]). *Let $f : R \to R'$ be a 1-connected map of connective $\mathbb{Z}$-$\mathbb{E}_1$-algebras. For every integer $n \geq 0$, the natural map*

$$n!s^n : \mathrm{HC}_{*+2n}(f) \longrightarrow \mathrm{HC}_*(f)$$

is the zero map for $* \leq n - 1$.

Goodwillie's proof of Theorem 4.36, although stated with respect to the abelian category of $\mathbb{Z}$-modules (or k-modules, for k an arbitrary discrete commutative ring), is valid for any abelian symmetric monoidal category with exact infinite products (see Notation 4.34 and Definition 4.35). One can thus prove the following generalisation of the previous result.

Theorem 4.37. *Let $f : R \to R'$ be a 1-connected map of connective solid $\mathbb{Z}$-$\mathbb{E}_1$-algebras. Then for every integer $n \geq 0$, the natural map*

$$n! s^n : \underline{\mathrm{HC}}(f)[-2n] \longrightarrow \underline{\mathrm{HC}}(f),$$

in the derived category $\mathcal{D}(\mathrm{Solid})$, is the zero map on cohomology groups[6] in degrees at least $-n+1$.

Proof. To prove the result for simplicial solid $\mathbb{Z}$-algebras, it suffices to prove that the abelian category of solid abelian groups is symmetric monoidal, and has exact infinite products. The first claim is [CS19, Theorem 6.2 (i)]. The second claim is a consequence of the fact that the abelian category of condensed abelian groups has exact arbitrary products ([CS19, Theorem 1.10]), and that the category of solid abelian groups, as a subcategory of the abelian category of condensed abelian groups, is stable under all limits ([CS19, Theorem 5.8 (i)]). We omit the proof of the analogue of [Lur17, Proposition 7.1.4.6] to pass from simplicial solid $\mathbb{Z}$-algebras to connective solid $\mathbb{Z}$-$\mathbb{E}_1$-algebras. $\square$

For the rest of this section, and following the convention of condensed mathematics, a condensed object is called *discrete* if its condensed structure is trivial. Given a classical object X, we denote by $\underline{X}$ the associated discrete condensed object.

Lemma 4.38. *Let R be a connective $\mathbb{Z}$-$\mathbb{E}_1$-algebra, and p be a prime number. Then the natural map*

$$\underline{\mathrm{HH}}(R) \longrightarrow \underline{\mathrm{HH}}(R_p^\wedge),$$

in the derived category $\mathcal{D}(\mathrm{Solid})$, exhibits the target as the p-completion of the source. In particular, there is a natural equivalence

$$\underline{\mathrm{HH}}(R)_p^\wedge[\tfrac{1}{p}] \xrightarrow{\sim} \underline{\mathrm{HH}}(R_p^\wedge[\tfrac{1}{p}]/\mathbb{Q}_p)$$

in the derived category $\mathcal{D}(\mathrm{Solid})$.

Proof. The solid tensor product of p-complete solid connective $\mathbb{Z}$-$\mathbb{E}_1$-algebras is p-complete, and so is the totalisation of a complex of p-complete solid connective $\mathbb{Z}$-modules ([CS21]). In particular, the complex $\underline{\mathrm{HH}}(R_p^\wedge)$, as a totalisation of tensor powers of the p-complete solid connective $\mathbb{Z}$-$\mathbb{E}_1$-algebra $\underline{R_p^\wedge}$, is p-complete. By the derived Nakayama lemma ([Sta19, 091N], see also [CS21]), it thus suffices to prove the first statement modulo p. By base change for Hochschild homology (resp. solid Hochschild homology), this is equivalent to proving that the natural map

$$\underline{\mathrm{HH}}((R/p)/\mathbb{F}_p) \longrightarrow \underline{\mathrm{HH}}((\underline{R}/p)/\underline{\mathbb{F}_p})$$

is an equivalence in the derived category $\mathcal{D}(\mathrm{Solid})$. The desired equivalence is then a consequence of the fact that reduction modulo p and tensor products commute with the functor $\underline{(-)}$ from $\mathbb{Z}$-$\mathbb{E}_1$-algebras to solid $\mathbb{Z}$-$\mathbb{E}_1$-algebras. The second statement follows from the fact that rationalisation commutes with the functor

$$\underline{(-)} : \mathcal{D}(\mathbb{Z}) \longrightarrow \mathcal{D}(\mathrm{Solid}),$$

and base change for solid Hochschild homology. $\square$

Proposition 4.39. *Let R be a connective $\mathbb{Z}$-$\mathbb{E}_1$-algebra, and p be a prime number. Then the natural map*

$$\underline{\mathrm{HC}}(R) \longrightarrow \underline{\mathrm{HC}}(R_p^\wedge),$$

in the derived category $\mathcal{D}(\mathrm{Solid})$, exhibits the target as the p-completion of the source. In particular, there is a natural equivalence

$$\underline{\mathrm{HC}}(R)_p^\wedge[\tfrac{1}{p}] \xrightarrow{\sim} \underline{\mathrm{HC}}(R_p^\wedge[\tfrac{1}{p}]/\mathbb{Q}_p)$$

in the derived category $\mathcal{D}(\mathrm{Solid})$.

---

[6]By this, we mean that it is the zero map as a map of solid abelian groups, *i.e.*, that it factors through the zero object of the abelian category of solid abelian groups. Note that the underlying abelian group of a nonzero solid abelian group can be zero (*e.g.*, the quotient of $\mathbb{Z}_p$ with the p-adic topology by $\mathbb{Z}_p$ with the discrete topology). In particular, being zero for a map of solid abelian groups cannot be detected on the underlying map of abelian groups.

Proof. The first statement is a consequence of Lemma 4.38, and the description [Mor18b, Definition 2.19] of (solid) cyclic homology in terms of the cyclic object $\mathrm{HH}(R)$ (resp. $\underline{\mathrm{HH}}(\underline{R}_p^\wedge)$) in the stable ∞-category $\mathcal{D}(\mathbb{Z})$ (resp. $\mathcal{D}(\mathrm{Solid})$). The second statement follows from the fact that rationalisation commutes direct sums (or equivalently, with the functor $(-)_{h\mathrm{S}^1}$) and with the functor $\underline{(-)}$ from the derived category $\mathcal{D}(\mathbb{Z})$ to the derived category $\mathcal{D}(\mathrm{Solid})$. $\qquad\square$

Corollary 4.40. *Let $f : R \to R'$ a 1-connected map of connective $\mathbb{Z}$-$\mathbb{E}_1$-algebras. Then for every integer $n \geq 0$, the map*

$$n!s^n : \prod_{p\in\mathbb{P}}{}' \mathrm{HC}(f;\mathbb{Q}_p)[-2n] \longrightarrow \prod_{p\in\mathbb{P}}{}' \mathrm{HC}(f;\mathbb{Q}_p),$$

in the derived category $\mathcal{D}(\mathbb{Q})$, is the zero map on cohomology groups in degrees at least $-n+1$.

Proof. For every prime number p, the natural map

$$n!s^n : \underline{\mathrm{HC}}(\underline{f}_p^\wedge)[-2n] \longrightarrow \underline{\mathrm{HC}}(\underline{f}_p^\wedge),$$

in the derived category $\mathcal{D}(\mathrm{Solid})$, is zero in cohomological degrees at least $-n+1$ (Theorem 4.37). Forgetting the condensed structure and taking the product over all primes p, this implies that the natural map

$$n!s^n : \prod_{p\in\mathbb{P}} \mathrm{HC}(f;\mathbb{Z}_p)[-2n] \longrightarrow \prod_{p\in\mathbb{P}} \mathrm{HC}(f;\mathbb{Z}_p),$$

in the derived category $\mathcal{D}(\mathbb{Z})$, is zero in cohomological degrees at least $-n+1$ (Proposition 4.39). Taking rationalisation then implies the desired result. $\qquad\square$

Theorem 4.41 (Rigid-analytic HP is truncating)**.** *The construction*

$$R \longmapsto \Big(\prod_{p\in\mathbb{P}}{}' \mathrm{HH}(R;\mathbb{Q}_p) \Big)^{t\mathrm{S}^1} := \Big(\Big(\prod_{p\in\mathbb{P}} \mathrm{HH}(R;\mathbb{Z}_p) \Big)_{\mathbb{Q}} \Big)^{t\mathrm{S}^1},$$

from connective $\mathbb{Z}$-$\mathbb{E}_1$-algebras R to the derived category $\mathcal{D}(\mathbb{Q})$, is truncating. More precisely, there exists a truncating invariant $E : \mathrm{Cat}_\infty^{\mathrm{perf}} \to \mathcal{D}(\mathbb{Q})$ such that the previous construction is the composite $R \mapsto \mathrm{Perf}(R) \mapsto E(\mathrm{Perf}(R))$.

Proof. Let $f : R \to R'$ be a 1-connected map of connective $\mathbb{Z}$-$\mathbb{E}_1$-algebras. We want to prove that the natural map

$$\Big(\prod_{p\in\mathbb{P}}{}' \mathrm{HH}(R;\mathbb{Q}_p) \Big)^{t\mathrm{S}^1} \longrightarrow \Big(\prod_{p\in\mathbb{P}}{}' \mathrm{HH}(R';\mathbb{Q}_p) \Big)^{t\mathrm{S}^1}$$

is an equivalence, or equivalently that its homotopy cofibre $\big(\prod'_{p\in\mathbb{P}} \mathrm{HH}(f;\mathbb{Q}_p) \big)^{t\mathrm{S}^1}$ vanishes in the derived category $\mathcal{D}(\mathbb{Q})$. For every integer $n \geq 0$, the natural map

$$n!s^n : \prod_{p\in\mathbb{P}}{}' \mathrm{HC}(f;\mathbb{Q}_p)[-2n] \longrightarrow \prod_{p\in\mathbb{P}}{}' \mathrm{HC}(f;\mathbb{Q}_p)$$

is the zero map in cohomological degrees at least $-n+1$ (Corollary 4.40). This map is $\mathbb{Q}$-linear, so the map

$$s^n : \prod_{p\in\mathbb{P}}{}' \mathrm{HC}(f;\mathbb{Q}_p)[-2n] \longrightarrow \prod_{p\in\mathbb{P}}{}' \mathrm{HC}(f;\mathbb{Q}_p)$$

is also the zero map in cohomological degrees at least $-n+1$. Taking the inverse limit over integers $n \geq 0$ and using the equivalence at the end of Notation 4.34 then implies that the complex $\big(\prod'_{p\in\mathbb{P}} \mathrm{HH}(f;\mathbb{Q}_p) \big)^{t\mathrm{S}^1}$ is zero in the derived category $\mathcal{D}(\mathbb{Q})$. $\qquad\square$

4.4 Rigid-analytic derived de Rham cohomology is a cdh sheaf

The aim of this section is to prove the cdh descent result Corollary 4.46, which is a rigid-analytic analogue of the following result, used in [EM23] to prove Theorem 4.2 in characteristic zero.

Proposition 4.42 ([CHSW08, Bal23]). *For every integer $i \in \mathbb{Z}$, the presheaf*

$$\mathrm{Fil}^i_{\mathrm{HKR}}\mathrm{HP}(-_\mathbb{Q}/\mathbb{Q}) : \mathrm{Sch}^{\mathrm{qcqs,op}} \longrightarrow \mathcal{D}(\mathbb{Q})$$

is a cdh sheaf.

Proof. This is a consequence of [LT19, Corollary A.6] and [Bal23, Theorem 1.3]. $\qquad\square$

We first extract the following cdh descent result from Theorem 4.41. Note that this argument uses the theory of truncating invariants, as developed by Land–Tamme [LT19], in a crucial way.

Corollary 4.43. *The presheaf*

$$\Big(\prod_{p \in \mathbb{P}}{}' \mathrm{HH}(-; \mathbb{Q}_p) \Big)^{t\mathrm{S}^1} : \mathrm{Sch}^{\mathrm{qcqs,op}} \longrightarrow \mathcal{D}(\mathbb{Q})$$

is a cdh sheaf.

Proof. This is a consequence of Theorem 4.41 and [LT19, Theorem E]. $\qquad\square$

We then adapt the main splitting result of [Bal23] on periodic cyclic homology over $\mathbb{Q}$ to our rigid-analytic setting.

Proposition 4.44. *Let X be a qcqs derived scheme. Then there exists a natural equivalence*

$$\Big(\prod_{p \in \mathbb{P}}{}' \mathrm{HH}(X; \mathbb{Q}_p) \Big)^{t\mathrm{S}^1} \simeq \prod_{i \in \mathbb{Z}} R\Gamma_{\mathrm{Zar}}\Big(X, \prod_{p \in \mathbb{P}}{}' \widehat{\mathbb{L}\Omega}_{-\mathbb{Q}_p/\mathbb{Q}_p} \Big)[2i]$$

in the derived category $\mathcal{D}(\mathbb{Q})$.

Proof. We adapt the proof of [Bal23, Theorem 3.4], which proves a similar decomposition for periodic cyclic homology over $\mathbb{Q}$. The crucial point to adapt this proof is to note that there is a natural equivalence

$$\prod_{p \in \mathbb{P}}{}' \mathrm{HH}(X; \mathbb{Q}_p) \simeq \bigoplus_{i \in \mathbb{N}} R\Gamma_{\mathrm{Zar}}\Big(X, \big(\prod_{p \in \mathbb{P}}(\mathbb{L}^i_{-/\mathbb{Z}})^\wedge_p \big)_\mathbb{Q} \Big)[i]$$

in the derived category $\mathcal{D}(\mathbb{Q})$ (see [Bal23, Remark 2.8]). This is the rigid-analytic analogue of [Bal23, Proposition 2.7], and it is for instance a consequence of Lemma 4.12 applied to the $\mathbb{N}$-indexed filtration $\mathrm{Fil}^\star_{\mathrm{HKR}} \prod'_{p \in \mathbb{P}} \mathrm{HH}(-; \mathbb{Q}_p)$. By [Bal23, Theorem 3.2], this implies that there is a natural equivalence

$$\Big(\mathrm{Fil}^\star_{\mathrm{H}} R\Gamma_{\mathrm{Zar}}\big(X, \prod_{p \in \mathbb{P}}{}' \widehat{\mathbb{L}\Omega}_{-\mathbb{Q}_p/\mathbb{Q}_p} \big) \otimes \mathrm{Fil}^\star_{\mathrm{T}}\big(\prod_{p \in \mathbb{P}}{}' \mathbb{Q}_p \big)^{t\mathrm{S}^1} \Big)^\wedge \xrightarrow{\sim} \mathrm{Fil}^\star_{\mathrm{T}}\Big(\prod_{p \in \mathbb{P}}{}' \mathrm{HH}(X; \mathbb{Q}_p) \Big)^{t\mathrm{S}^1}$$

in the filtered derived category $\mathcal{DF}(\mathbb{Q})$, where the tensor product $\otimes$ is the Day convolution tensor product, $\mathrm{Fil}^\star_{\mathrm{H}}$ is the Hodge filtration on derived de Rham cohomology, and $(-)^\wedge$ is the completion with respect to the associated filtration. By a degree argument explained in [Bal23, proof of Theorem 3.4], the filtered object $\mathrm{Fil}^\star_{\mathrm{T}}\big(\prod'_{p \in \mathbb{P}} \mathbb{Q}_p \big)^{t\mathrm{S}^1}$ carries a canonical splitting, which induces an equivalence

$$\prod_{i \in \mathbb{Z}} \Big(\mathrm{Fil}^{\star+i}_{\mathrm{H}} R\Gamma_{\mathrm{Zar}}\big(X, \prod_{p \in \mathbb{P}}{}' \widehat{\mathbb{L}\Omega}_{-\mathbb{Q}_p/\mathbb{Q}_p} \big) \Big)[2i] \xrightarrow{\sim} \mathrm{Fil}^\star_{\mathrm{T}}\Big(\prod_{p \in \mathbb{P}}{}' \mathrm{HH}(X; \mathbb{Q}_p) \Big)^{t\mathrm{S}^1}$$

in the filtered derived category $\mathcal{DF}(\mathbb{Q})$.

It then suffices to prove that the desired result is indeed obtained by taking the colimit over $\star \to -\infty$ of the previous equivalence. Following [Bal23, proof of Theorem 3.4], the result for the source is a formal consequence of the connectivity estimate for the functor

$$R\Gamma_{\mathrm{Zar}}\Big(-, \prod_{p\in\mathbb{P}}{}' \underline{\mathbb{L}\Omega}^{\leq i}_{-\mathbb{Q}_p/\mathbb{Q}_p}\Big)$$

on animated commutative ring, which is itself a consequence of Remark 4.20, and of the classical connectivity estimate for the HKR filtration on cyclic homology (Proposition 2.8). The result for the target is a consequence of the fact that the Tate filtration is always exhaustive ([Bal23, Proposition B.6]). □

Corollary 4.45. *The presheaf*

$$R\Gamma_{\mathrm{Zar}}\Big(-, \prod_{p\in\mathbb{P}}{}' \widehat{\underline{\mathbb{L}\Omega}}_{-\mathbb{Q}_p/\mathbb{Q}_p}\Big) : \mathrm{Sch}^{\mathrm{qcqs,op}} \longrightarrow \mathcal{D}(\mathbb{Q})$$

is a cdh sheaf.

Proof. This is a consequence of the natural splitting Proposition 4.44, and of the cdh descent result Corollary 4.43. □

Corollary 4.46. *For every integer $i \in \mathbb{Z}$, the presheaf*

$$\mathrm{Fil}^i_{\mathrm{HKR}}\Big(\prod_{p\in\mathbb{P}}{}' \mathrm{HH}(-; \mathbb{Q}_p)\Big)^{t\mathrm{S}^1} : \mathrm{Sch}^{\mathrm{qcqs,op}} \longrightarrow \mathcal{D}(\mathbb{Q})$$

is a cdh sheaf.

Proof. The HKR filtration on $\big(\prod'_{p\in\mathbb{P}} \mathrm{HH}(-;\mathbb{Q}_p)\big)^{t\mathrm{S}^1}$ is complete by Lemma 4.25, so it suffices to prove the result on graded pieces. The result is then Corollary 4.45. □

4.5 The Atiyah–Hirzebruch spectral sequence

In this subsection, we use the results of the previous sections to prove Theorem C.

Proposition 4.47 (The motivic filtration is $\mathbb{N}$-indexed). *Let X be a qcqs derived scheme. Then for every integer $i \leq 0$, the natural map*

$$\mathrm{Fil}^i_{\mathrm{mot}}\mathrm{K}(X) \longrightarrow \mathrm{K}(X)$$

is an equivalence of spectra. In particular, the motivic filtration $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X)$ on $\mathrm{K}(X)$ is exhaustive.

Proof. First assume that X is a qcqs classical scheme. The filtration $\mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{KH}(X)$ is $\mathbb{N}$-indexed by [BEM], so it suffices to prove that the filtration

$$\mathrm{cofib}\Big(\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(X) \longrightarrow \big(L_{\mathrm{cdh}}\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}\big)(X)\Big)$$

is $\mathbb{N}$-indexed (Definition 3.17). To prove this, we use Proposition 4.30 and Corollary 4.31. For every prime number p, the filtration $\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)$ is $\mathbb{N}$-indexed on qcqs schemes ([BMS19, Theorem 7.2 (1)] and Theorem 2.25 (2)). In particular, the filtration

$$\mathrm{cofib}\Big(\prod_{p\in\mathbb{P}} \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(X; \mathbb{Z}_p) \longrightarrow \big(L_{\mathrm{cdh}} \prod_{p\in\mathbb{P}} \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)\big)(X)\Big)$$

is $\mathbb{N}$-indexed. The presheaf $\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HP}(-\mathbb{Q}/\mathbb{Q})$ is a cdh sheaf on qcqs schemes (Proposition 4.42), and the filtration $\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}(X_\mathbb{Q}/\mathbb{Q})$ is $\mathbb{N}$-indexed by construction. In particular, the filtration

$$\mathrm{cofib}\Big(\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X_\mathbb{Q}/\mathbb{Q}) \longrightarrow \big(L_{\mathrm{cdh}}\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(-\mathbb{Q}/\mathbb{Q})\big)(X)\Big)$$

is naturally identified with the filtration

$$\mathrm{cofib}\Big(\mathrm{Fil}_{\mathrm{HKR}}^{\star-1}\mathrm{HC}(X_{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \big(L_{\mathrm{cdh}}\mathrm{Fil}_{\mathrm{HKR}}^{\star-1}\mathrm{HC}(-_{\mathbb{Q}}/\mathbb{Q})\big)(X)\Big)[1],$$

which is $\mathbb{N}$-indexed. Similarly, the presheaf $\mathrm{Fil}_{\mathrm{HKR}}^{\star}\big(\prod'_{p\in\mathbb{P}}\mathrm{HH}(-;\mathbb{Q}_p)\big)^{t\mathrm{S}^1}$ is a cdh sheaf on qcqs schemes (Corollary 4.46), and the filtration $\mathrm{Fil}_{\mathrm{HKR}}^{\star}\big(\prod'_{p\in\mathbb{P}}\mathrm{HH}(-;\mathbb{Q}_p)\big)_{h\mathrm{S}^1}$ is $\mathbb{N}$-indexed (Remark 4.20). In particular, the filtration

$$\mathrm{cofib}\Big(\mathrm{Fil}_{\mathrm{HKR}}^{\star}\big(\prod_{p\in\mathbb{P}}{}'\mathrm{HH}(X;\mathbb{Q}_p)\big)^{h\mathrm{S}^1} \longrightarrow \big(L_{\mathrm{cdh}}\mathrm{Fil}_{\mathrm{HKR}}^{\star}\big(\prod_{p\in\mathbb{P}}{}'\mathrm{HH}(-;\mathbb{Q}_p)\big)^{t\mathrm{S}^1}\big)(X)\Big)$$

is naturally identified with the filtration

$$\mathrm{cofib}\Big(\mathrm{Fil}_{\mathrm{HKR}}^{\star-1}\big(\prod_{p\in\mathbb{P}}{}'\mathrm{HH}(X;\mathbb{Q}_p)\big)_{h\mathrm{S}^1} \longrightarrow \big(L_{\mathrm{cdh}}\mathrm{Fil}_{\mathrm{HKR}}^{\star-1}\big(\prod_{p\in\mathbb{P}}{}'\mathrm{HH}(-;\mathbb{Q}_p)\big)_{h\mathrm{S}^1}\big)(X)\Big)[1],$$

which is $\mathbb{N}$-indexed.

Assume now that X is a general qcqs derived scheme. By Definition 3.18 and the previous paragraph, it suffices to prove that the filtration

$$\mathrm{cofib}\Big(\mathrm{Fil}_{\mathrm{mot}}^{\star}\mathrm{TC}(X) \longrightarrow \mathrm{Fil}_{\mathrm{mot}}^{\star}\mathrm{TC}(\pi_0(X))\Big)$$

is $\mathbb{N}$-indexed. To prove this, we use Proposition 4.30. For every prime number p, the filtration $\mathrm{Fil}_{\mathrm{BMS}}^{\star}\mathrm{TC}(X;\mathbb{Z}_p)$ is $\mathbb{N}$-indexed on polynomial $\mathbb{Z}$-algebras by the previous paragraph, and hence on general qcqs derived schemes by Zariski descent and Theorem 2.25 (2). In particular, the filtration

$$\mathrm{cofib}\Big(\prod_{p\in\mathbb{P}}\mathrm{Fil}_{\mathrm{BMS}}^{\star}\mathrm{TC}(X;\mathbb{Z}_p) \longrightarrow \prod_{p\in\mathbb{P}}\mathrm{Fil}_{\mathrm{BMS}}^{\star}\mathrm{TC}(\pi_0(X);\mathbb{Z}_p)\Big)$$

is $\mathbb{N}$-indexed. The filtration

$$\mathrm{cofib}\Big(\mathrm{Fil}_{\mathrm{HKR}}^{\star}\mathrm{HC}^-(X_{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{Fil}_{\mathrm{HKR}}^{\star}\mathrm{HC}^-(\pi_0(X)_{\mathbb{Q}}/\mathbb{Q})\Big)$$

is $\mathbb{N}$-indexed by [EM23, Theorem 4.39]. Similarly, using Theorem 4.41 and Proposition 4.44 as in the proof of Corollary 4.46, the natural map

$$\mathrm{Fil}_{\mathrm{HKR}}^{\star}\big(\prod_{p\in\mathbb{P}}{}'\mathrm{HH}(X;\mathbb{Q}_p)\big)^{t\mathrm{S}^1} \longrightarrow \mathrm{Fil}_{\mathrm{HKR}}^{\star}\big(\prod_{p\in\mathbb{P}}{}'\mathrm{HH}(\pi_0(X);\mathbb{Q}_p)\big)^{t\mathrm{S}^1}$$

is an equivalence of filtered spectra. In particular, the filtration

$$\mathrm{cofib}\Big(\mathrm{Fil}_{\mathrm{HKR}}^{\star}\big(\prod_{p\in\mathbb{P}}{}'\mathrm{HH}(X;\mathbb{Q}_p)\big)^{h\mathrm{S}^1} \longrightarrow \mathrm{Fil}_{\mathrm{HKR}}^{\star}\big(\prod_{p\in\mathbb{P}}{}'\mathrm{HH}(\pi_0(X);\mathbb{Q}_p)\big)^{h\mathrm{S}^1}\Big)$$

is naturally identified with the filtration

$$\mathrm{cofib}\Big(\mathrm{Fil}_{\mathrm{HKR}}^{\star}\big(\prod_{p\in\mathbb{P}}{}'\mathrm{HH}(X;\mathbb{Q}_p)\big)_{h\mathrm{S}^1} \longrightarrow \mathrm{Fil}_{\mathrm{HKR}}^{\star}\big(\prod_{p\in\mathbb{P}}{}'\mathrm{HH}(\pi_0(X);\mathbb{Q}_p)\big)_{h\mathrm{S}^1}\Big)[1],$$

which is $\mathbb{N}$-indexed by Remark 4.20. $\qquad\square$

Corollary 4.48. *Let X be a qcqs scheme. Then for every integer $i < 0$, the motivic complex $\mathbb{Z}(i)^{\mathrm{mot}}(X) \in \mathcal{D}(X)$ is zero.*

Proof. This is a direct consequence of Proposition 4.47. $\qquad\square$

Proposition 4.49. *Let $d \geq 0$ be an integer, and X be a qcqs scheme of valuative dimension at most d. Then for every integer $i \in \mathbb{Z}$, the fibre of the natural map of spectra*

$$\mathrm{Fil}^i_{\mathrm{mot}}\mathrm{TC}(X) \longrightarrow \mathrm{Fil}^i_{\mathrm{mot}}L_{\mathrm{cdh}}\mathrm{TC}(X)$$

is in cohomological degrees at most $-i + d + 2$. In particular, the filtration formed by these spectra for all integers $i \in \mathbb{Z}$, and the rationalisation of this filtration, are complete.

Proof. The last statement is a consequence of the first, as the connectivity bound for a given filtration induces the same connectivity bound for its rationalisation. For the connectivity bound, we use Proposition 4.30 and Corollary 4.31 to compare the spectra $\mathrm{Fil}^i_{\mathrm{mot}}\mathrm{TC}(X)$ and $\mathrm{Fil}^i_{\mathrm{mot}}L_{\mathrm{cdh}}\mathrm{TC}(X)$.

The presheaf $\prod_{p \in \mathbb{P}} \mathrm{Fil}^i_{\mathrm{BMS}}\mathrm{TC}(X; \mathbb{Z}_p)$ takes values in cohomological degrees at most $-i+1$ on affine schemes (Lemma 2.21). In particular, the fibre of the natural map of spectra

$$\prod_{p \in \mathbb{P}} \mathrm{Fil}^i_{\mathrm{BMS}}\mathrm{TC}(X; \mathbb{Z}_p) \longrightarrow \big(L_{\mathrm{cdh}} \prod_{p \in \mathbb{P}} \mathrm{Fil}^i_{\mathrm{BMS}}\mathrm{TC}(-; \mathbb{Z}_p)\big)(X)$$

is in cohomological degrees at most $-i + d + 2$, as each term is in cohomological degrees at most $-i + d + 1$ ([CM21, Theorem 3.12] and [EHIK21, Theorem 2.4.15]).

By Proposition 4.42, the fibre of the natural map of spectra

$$\mathrm{Fil}^i_{\mathrm{HKR}}\mathrm{HC}^-(X_{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \big(L_{\mathrm{cdh}}\mathrm{Fil}^i_{\mathrm{HKR}}\mathrm{HC}^-(-_{\mathbb{Q}}/\mathbb{Q})\big)(X)$$

is naturally identified with the fibre of the natural map of spectra

$$\mathrm{Fil}^{i-1}_{\mathrm{HKR}}\mathrm{HC}(X_{\mathbb{Q}}/\mathbb{Q})[1] \longrightarrow \big(L_{\mathrm{cdh}}\mathrm{Fil}^{i-1}_{\mathrm{HKR}}\mathrm{HC}(-_{\mathbb{Q}}/\mathbb{Q})[1]\big)(X).$$

The presheaf $\mathrm{Fil}^{i-1}_{\mathrm{HKR}}\mathrm{HC}(-_{\mathbb{Q}}/\mathbb{Q})$ takes values in cohomological degrees at most $-i-1$ (*e.g.*, by Lemma 2.9 and the description of the graded pieces Remark 2.7), so the previous fibre is in cohomological degrees at most $-i + d - 1$ ([CM21, Theorem 3.12] and [EHIK21, Theorem 2.4.15]).

Similarly, by Corollary 4.46, the fibre of the natural map of spectra

$$\mathrm{Fil}^i_{\mathrm{HKR}}\big(\prod_{p \in \mathbb{P}}{}'\mathrm{HH}(X; \mathbb{Q}_p)\big)^{h\mathrm{S}^1} \longrightarrow \big(L_{\mathrm{cdh}}\mathrm{Fil}^i_{\mathrm{HKR}}\big(\prod_{p \in \mathbb{P}}{}'\mathrm{HH}(-; \mathbb{Q}_p)\big)^{h\mathrm{S}^1}\big)(X)$$

is naturally identified with the fibre of the natural map of spectra

$$\mathrm{Fil}^{i-1}_{\mathrm{HKR}}\big(\prod_{p \in \mathbb{P}}{}'\mathrm{HH}(X; \mathbb{Q}_p)\big)_{h\mathrm{S}^1}[1] \longrightarrow \big(L_{\mathrm{cdh}}\mathrm{Fil}^{i-1}_{\mathrm{HKR}}\big(\prod_{p \in \mathbb{P}}{}'\mathrm{HH}(-; \mathbb{Q}_p)\big)_{h\mathrm{S}^1}[1]\big)(X)$$

The presheaf $\mathrm{Fil}^{i-1}_{\mathrm{HKR}}\big(\prod'_{p \in \mathbb{P}}\mathrm{HH}(-; \mathbb{Q}_p)\big)_{h\mathrm{S}^1}$ takes values in cohomological degrees at most $-i - 1$ (Remark 4.20), so the previous fibre is in cohomological degrees at most $-i + d - 1$ ([CM21, Theorem 3.12] and [EHIK21, Theorem 2.4.15]).

The previous three connectivity results imply the desired result. □

Proposition 4.50 (Completeness of the motivic filtration). *Let $d \geq 0$ be an integer, and X be a qcqs scheme of valuative dimension at most d. Then for every integer $i \in \mathbb{Z}$, the spectrum $\mathrm{Fil}^i_{\mathrm{mot}}\mathrm{K}(X)$ is in cohomological degrees at most $-i + d + 2$. In particular, the motivic filtration $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X)$, and its rationalisation $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X; \mathbb{Q})$, are complete.*

Proof. As in the proof of Proposition 4.49, the last statement is a consequence of the first. By Definition 3.17, there is a natural fibre sequence of spectra

$$\mathrm{fib}\big(\mathrm{Fil}^i_{\mathrm{mot}}\mathrm{TC}(X) \longrightarrow \mathrm{Fil}^i_{\mathrm{mot}}L_{\mathrm{cdh}}\mathrm{TC}(X)\big) \longrightarrow \mathrm{Fil}^i_{\mathrm{mot}}\mathrm{K}(X) \longrightarrow \mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{KH}(X).$$

The left term of this fibre sequence is in cohomological degrees at most $-i + d + 2$ by Proposition 4.49, and the right term is in cohomological degrees at most $-i + d$ by [BEM], hence the desired result. □

Remark 4.51 (Non-noetherian Weibel vanishing). Let X be a qcqs scheme of valuative dimension at most d. The same argument as in Proposition 4.50 implies that the negative K-groups $\mathrm{K}_{-n}(X)$ vanish for integers $n > d + 2$. This is a weak form of Weibel vanishing in the non-noetherian case.

Remark 4.52 (Motivic Weibel vanishing). Let X be a qcqs scheme of dimension at most d. Proposition 4.50 implies that for every integer $i \in \mathbb{Z}$, the motivic complex $\mathbb{Z}(i)^{\mathrm{mot}}(X) \in \mathcal{D}(\mathbb{Z})$ is in degrees at most $i + d + 2$. When X is noetherian (in which case the valuative and Krull dimensions coincide), we will prove that it is even in degrees at most $i + d$ (Theorem 9.27).

Remark 4.53. A map of finitary presheaves of filtered spectra on qcqs schemes, which are filtration-complete on finite-dimensional noetherian schemes, is an equivalence if and only if it is an equivalence on graded pieces. In light of Propositions 4.59 and 4.50, we will formulate most of our results at the level of motivic cohomology, although they can often be promoted to results on the associated filtered spectra.

Corollary 4.54 (Completeness of $\mathrm{Fil}^{\star}_{\mathrm{mot}} L_{\mathrm{cdh}} \mathrm{TC}$). *Let X be a qcqs scheme of finite valuative dimension. Then the filtrations $\mathrm{Fil}^{\star}_{\mathrm{mot}} L_{\mathrm{cdh}} \mathrm{TC}(X)$ and $\mathrm{Fil}^{\star}_{\mathrm{mot}} L_{\mathrm{cdh}} \mathrm{TC}(X; \mathbb{Q})$ are complete.*

Proof. This is a consequence of Propositions 2.35 and 4.49. $\qquad\square$

Remark 4.55. The filtrations $\mathrm{Fil}^{\star}_{\mathrm{mot}} \mathrm{TC}(X)$ and $\mathrm{Fil}^{\star}_{\mathrm{mot}} L_{\mathrm{cdh}} \mathrm{TC}(X)$ do not satisfy separately a connectivity bound similar to that of Proposition 4.49.

Corollary 4.56 (Atiyah–Hirzebruch spectral sequence). *Let X be a qcqs derived scheme. The motivic filtration $\mathrm{Fil}^{\star}_{\mathrm{mot}} \mathrm{K}(X)$ on non-connective algebraic K-theory $\mathrm{K}(X)$ induces a natural Atiyah–Hirzebruch spectral sequence*

$$E_2^{i,j} = \mathrm{H}^{i-j}_{\mathrm{mot}}(X, \mathbb{Z}(-j)) \Longrightarrow \mathrm{K}_{-i-j}(X).$$

If X is a qcqs classical scheme of finite valuative dimension, then this Atiyah–Hirzebruch spectral sequence is convergent.

Proof. The first statement is a consequence of the fact the motivic filtration $\mathrm{Fil}^{\star}_{\mathrm{mot}} \mathrm{K}(X)$ is $\mathbb{N}$-indexed (Proposition 4.47). The second statement is a consequence of the connectivity bound for this motivic filtration (Proposition 4.50). $\qquad\square$

The main consequence of Propositions 4.47 and 4.50 that we will use is the following result.

Corollary 4.57. *Let X be a qcqs derived scheme. Then for every integer $i \geq 0$, there exists a natural equivalence of spectra*

$$\mathrm{K}(X; \mathbb{Q}) \simeq \big(\bigoplus_{0 \leq j < i} \mathbb{Q}(i)^{\mathrm{mot}}(X)[2i] \big) \oplus \mathrm{Fil}^i_{\mathrm{mot}} \mathrm{K}(X; \mathbb{Q}).$$

Proof. The motivic filtration $\mathrm{Fil}^{\star}_{\mathrm{mot}} \mathrm{K}(X; \mathbb{Q})$ is $\mathbb{N}$-indexed by Proposition 4.47. The result on qcqs classical schemes is then a consequence of Lemma 4.12 and Remark 4.13, where the necessary hypotheses are satisfied by Proposition 4.50 and Corollary 4.11.

Assume now that X is a general qcqs derived scheme. Again by Lemma 4.12, where the necessary hypotheses are satisfied for the filtration $\mathrm{Fil}^{\star}_{\mathrm{mot}} \mathrm{TC}(-)$ by Proposition 2.35 and Corollary 4.8, there is a natural equivalence of spectra

$$\mathrm{Fil}^0_{\mathrm{mot}} \mathrm{TC}(X; \mathbb{Q}) \simeq \big(\bigoplus_{0 \leq j < i} \mathbb{Q}(j)^{\mathrm{TC}}(X)[2j] \big) \oplus \mathrm{Fil}^i_{\mathrm{mot}} \mathrm{TC}(X; \mathbb{Q}).$$

The result is then a consequence of Definition 3.18 and the previous case, where the equivalences

$$\mathrm{Fil}^0_{\mathrm{mot}} \mathrm{K}(\pi_0(X); \mathbb{Q}) \simeq \bigoplus_{0 \leq j < i} \mathbb{Q}(j)^{\mathrm{mot}}(\pi_0(X))[2j] \oplus \mathrm{Fil}^i_{\mathrm{mot}} \mathrm{K}(\pi_0(X); \mathbb{Q})$$

and

$$\mathrm{Fil}^0_{\mathrm{mot}} \mathrm{TC}(\pi_0(X); \mathbb{Q}) \simeq \big(\bigoplus_{0 \leq j < i} \mathbb{Q}(j)^{\mathrm{TC}}(\pi_0(X))[2j] \big) \oplus \mathrm{Fil}^i_{\mathrm{mot}} \mathrm{TC}(X; \mathbb{Q})$$

50

are compatible, by construction, with the natural map

$$\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(\pi_0(X);\mathbb{Q}) \longrightarrow \mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(\pi_0(X);\mathbb{Q})$$

of Definition 3.17. $\qquad\square$

Lemma 4.58. *Let τ be the Zariski, Nisnevich, or cdh topology, and $(F_i)_{i\in I}$ be a direct system of presheaves. Then the natural map of presheaves*

$$\varinjlim_i L_\tau\, F_i \longrightarrow L_\tau \varinjlim_i F_i$$

is an equivalence. In particular, the sheafification functor L_τ sends finitary presheaves to finitary τ sheaves.

Proof. As a left adjoint, the sheafification functor L_τ, from presheaves to τ sheaves, commutes with all colimits. Being a sheaf for the topology τ is detected using only finite limits, so the inclusion functor from τ sheaves to presheaves commutes with filtered colimits. Composing the previous two functors then implies that the functor L_τ, from presheaves to presheaves, commutes with filtered colimits.

To prove the second statement, let F be a finitary presheaf, and $(R_i)_{i\in I}$ be a direct system of commutative rings. The fact that the natural map

$$\varinjlim_i L_\tau\, F(R_i) \longrightarrow L_\tau F(\varinjlim_i R_i)$$

is an equivalence is a consequence of the finitariness of F, and of the first statement applied to the direct system of presheaves $\big(F(-_{R_i})\big)_{i\in I}$. $\qquad\square$

Proposition 4.59 (The motivic filtration is finitary). *Let $i\in\mathbb{Z}$ be an integer. Then the presheaf*

$$\mathrm{Fil}^i_{\mathrm{mot}}\mathrm{K}(-):\mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{Sp}$$

is a finitary Nisnevich sheaf, i.e., it satisfies descent for the Nisnevich topology and commutes with filtered colimits of rings.

Proof. It suffices to prove the result modulo p for every prime number p, and rationally. Algebraic K-theory is a finitary Nisnevich sheaf on qcqs derived schemes ([CMNN20, Proposition A.15]). The presheaf $\mathrm{Fil}^i_{\mathrm{mot}}\mathrm{K}(-;\mathbb{Q})$ is a natural direct summand of rationalised algebraic K-theory (Corollary 4.57), so it also is a finitary Nisnevich sheaf. To prove the result modulo a prime number p, note that the presheaf $\mathrm{Fil}^i_{\mathrm{cdh}}\mathrm{KH}(-)$ is a finitary cdh sheaf ([BEM]), and in particular a finitary Nisnevich sheaf. By Theorem 2.25, the presheaf $\mathrm{Fil}^i_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{F}_p)$ is a finitary Nisnevich sheaf. By Lemma 4.58, this implies that the presheaf $L_{\mathrm{cdh}}\mathrm{Fil}^i_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{F}_p)$ is a finitary Nisnevich sheaf, and the result modulo p is then a consequence of Proposition 3.24. $\qquad\square$

Corollary 4.60. *Let $i\in\mathbb{Z}$ be an integer. Then the presheaf*

$$\mathbb{Z}(i)^{\mathrm{mot}}(-):\mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathcal{D}(\mathbb{Z})$$

is a finitary Nisnevich sheaf.

Proof. This is a direct consequence of Proposition 4.59. $\qquad\square$

Proof of Theorem 4.1. The motivic filtration $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(-)$ is finitary on qcqs schemes (Proposition 4.59), and, for every noetherian scheme X of finite dimension d and every integer $i\in\mathbb{Z}$, the spectrum $\mathrm{Fil}^i_{\mathrm{mot}}\mathrm{K}(X;\mathbb{Q})$ is in cohomological degrees at most $-i+d+2$ (Proposition 4.50). Proposition 4.15 then implies that there exists a natural multiplicative equivalence of filtered spectra

$$\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X;\mathbb{Q}) \simeq \bigoplus_{j\geq\star}\mathbb{Q}(j)^{\mathrm{mot}}(X)[2j].$$

The same argument as in Corollary 4.57 then implies the result for general qcqs derived schemes. $\qquad\square$

Corollary 4.61. *Let X be a qcqs derived scheme. Then there exists a natural equivalence of spectra*

$$\mathrm{K}(X;\mathbb{Q}) \simeq \bigoplus_{i \geq 0} \mathbb{Q}(i)^{\mathrm{mot}}(X)[2i].$$

Proof. The motivic filtration $\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{K}(X;\mathbb{Q})$ is $\mathbb{N}$-indexed by Proposition 4.47. The result is then a consequence of the rational splitting Theorem 4.1. $\square$

4.6 Rational structure of motivic cohomology

In this subsection, we finish the proof of Theorem 4.2. We first use an argument of Weibel to prove the following result at the level of K-theory. We then use the rational splitting Corollary 4.57 to prove a filtered version of this result, which reduces the proof of Theorem 4.2 to the case of characteristic zero.

Lemma 4.62. *Let X be a qcqs scheme. Then the natural commutative diagram*

$$
\begin{array}{ccc}
\mathrm{K}(X;\mathbb{Q}) & \longrightarrow & \mathrm{K}(X_{\mathbb{Q}};\mathbb{Q}) \\
\downarrow & & \downarrow \\
\mathrm{KH}(X;\mathbb{Q}) & \longrightarrow & \mathrm{KH}(X_{\mathbb{Q}};\mathbb{Q})
\end{array}
$$

is a cartesian square of spectra.

Proof. By Zariski descent, it suffices to prove the result for affine schemes $X = \mathrm{Spec}(R)$. For every integer $n \in \mathbb{Z}$, let

$$\mathrm{NK}_n(R) := \mathrm{coker}\big(\mathrm{K}_n(R) \longrightarrow \mathrm{K}_n(R[T])\big)$$

be the nth NK-group of R. By [Wei81, Corollary 6.4] (see also [TT90, Exercise 9.12], where some unnecessary hypotheses in Weibel's result are removed), there is a natural isomorphism of abelian groups

$$\mathrm{NK}_n(R) \otimes_{\mathbb{Z}} \mathbb{Q} \overset{\cong}{\longrightarrow} \mathrm{NK}_n(R \otimes_{\mathbb{Z}} \mathbb{Q})$$

for every integer $n \in \mathbb{Z}$ and every commutative ring R. For every commutative ring R, the homotopy groups of the fibre $\mathrm{K}^{\mathrm{W}}(R)$ of the map of spectra $\mathrm{K}(R) \to \mathrm{KH}(R)$ have a natural exhaustive complete filtration with graded pieces given by the iterated NK-groups of R. In particular, for every integer $n \in \mathbb{Z}$ and every commutative ring R, the natural map

$$\mathrm{K}^{\mathrm{W}}(R;\mathbb{Q}) \longrightarrow \mathrm{K}^{\mathrm{W}}(R \otimes_{\mathbb{Z}} \mathbb{Q};\mathbb{Q})$$

is an equivalence of spectra, which implies the desired result. $\square$

Corollary 4.63. *Let X be a qcqs scheme. Then the natural commutative diagram*

$$
\begin{array}{ccc}
\mathrm{K}(X;\mathbb{Q}) & \longrightarrow & \mathrm{HC}^{-}(X_{\mathbb{Q}}/\mathbb{Q}) \\
\downarrow & & \downarrow \\
\mathrm{KH}(X;\mathbb{Q}) & \longrightarrow & L_{\mathrm{cdh}}\mathrm{HC}^{-}(-_{\mathbb{Q}}/\mathbb{Q})(X)
\end{array}
$$

is a cartesian square of spectra.

Proof. The result for qcqs $\mathbb{Q}$-schemes X is due to Cortiñas–Haesemeyer–Schlichting–Weibel [CHSW08, CHW08] (see also Theorem 1.1). The general result is then a consequence of Lemma 4.62. $\square$

We record for completeness the following result, which is well-known in characteristic zero ([CHSW08, CHW08]).

Corollary 4.64. *Let X be a qcqs scheme. Then there is a natural fibre sequence of spectra*

$$\mathrm{K}(X;\mathbb{Q}) \longrightarrow \mathrm{KH}(X;\mathbb{Q}) \longrightarrow \mathrm{cofib}\big(\mathrm{HC}(X_{\mathbb{Q}}/\mathbb{Q}) \longrightarrow L_{\mathrm{cdh}}\mathrm{HC}(-_{\mathbb{Q}}/\mathbb{Q})(X)\big)[1].$$

Proof. By Theorem 1.1 and Lemma 4.62, the natural commutative diagram

$$
\begin{array}{ccc}
\mathrm{K}(X;\mathbb{Q}) & \longrightarrow & \mathrm{HC}^-(X_{\mathbb{Q}}/\mathbb{Q}) \\
\downarrow & & \downarrow \\
\mathrm{KH}(X;\mathbb{Q}) & \longrightarrow & L_{\mathrm{cdh}}\mathrm{HC}^-(-_{\mathbb{Q}}/\mathbb{Q})(X)
\end{array}
$$

is a cartesian square of spectra. By construction, there is moreover a natural commutative diagram of spectra

$$
\begin{array}{ccccc}
\mathrm{HC}^-(X_{\mathbb{Q}}/\mathbb{Q}) & \longrightarrow & \mathrm{HP}(X_{\mathbb{Q}}/\mathbb{Q}) & \longrightarrow & \mathrm{HC}(X_{\mathbb{Q}}/\mathbb{Q})[2] \\
\downarrow & & \downarrow & & \downarrow \\
L_{\mathrm{cdh}}\mathrm{HC}^-(-_{\mathbb{Q}}/\mathbb{Q})(X) & \longrightarrow & L_{\mathrm{cdh}}\mathrm{HP}(-_{\mathbb{Q}}/\mathbb{Q})(X) & \longrightarrow & L_{\mathrm{cdh}}\mathrm{HC}(-_{\mathbb{Q}}/\mathbb{Q})(X)[2],
\end{array}
$$

where the horizontal lines are fibre sequences. The middle vertical line of this diagram is an equivalence ([CHSW08, Corollary 3.13], see also [LT19, Corollary A.6]), so the cofibre of the left vertical map is naturally identified with the spectrum

$$
\mathrm{cofib}\big(\mathrm{HC}(X_{\mathbb{Q}}/\mathbb{Q}) \longrightarrow L_{\mathrm{cdh}}\mathrm{HC}(-_{\mathbb{Q}}/\mathbb{Q})(X)\big)[1].
$$

$\square$

The following result is a filtered refinement of Lemma 4.62.

Corollary 4.65. *Let X be a qcqs scheme. Then the natural commutative diagram*

$$
\begin{array}{ccc}
\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X;\mathbb{Q}) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X_{\mathbb{Q}};\mathbb{Q}) \\
\downarrow & & \downarrow \\
\mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{KH}(X;\mathbb{Q}) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{KH}(X_{\mathbb{Q}};\mathbb{Q})
\end{array}
$$

is a cartesian square of filtered spectra.

Proof. By Corollary 4.57, for every integer $i \geq 0$, the spectrum $\mathrm{Fil}^i_{\mathrm{mot}}\mathrm{K}(X;\mathbb{Q})$ is naturally a direct summand of the spectrum $\mathrm{K}(X;\mathbb{Q})$. The same applies to the other three filtrations of the cartesian square, by noting that the motivic filtration $\mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{KH}(-)$ also is $\mathbb{N}$-indexed ([BEM]). The compatibility between the several filtrations is automatic from the construction of the splittings. So the result is a consequence of Lemma 4.62. $\square$

The following result is a filtered refinement of Corollary 4.63.

Theorem 4.66. *Let X be a qcqs scheme. Then the natural commutative diagram*

$$
\begin{array}{ccc}
\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X;\mathbb{Q}) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X_{\mathbb{Q}}/\mathbb{Q}) \\
\downarrow & & \downarrow \\
\mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{KH}(X;\mathbb{Q}) & \longrightarrow & L_{\mathrm{cdh}}\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(-_{\mathbb{Q}}/\mathbb{Q})(X)
\end{array}
$$

is a cartesian square of filtered spectra.

Proof. This is a consequence of Corollary 4.65, where the filtration $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(-)$ of qcqs $\mathbb{Q}$-schemes is naturally identified with the filtration $\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(-/\mathbb{Q})$ (Remark 2.31). $\square$

The following result is the rational part of Theorem B.

Corollary 4.67. *Let X be a qcqs scheme. Then for every integer $i \in \mathbb{Z}$, the natural commutative diagram*

$$\begin{array}{ccc}
\mathbb{Q}(i)^{\mathrm{mot}}(X) & \longrightarrow & R\Gamma_{\mathrm{Zar}}\big(X, \widehat{\mathbb{L}\Omega}^{\geq i}_{-\mathbb{Q}/\mathbb{Q}}\big) \\
\downarrow & & \downarrow \\
\mathbb{Q}(i)^{\mathrm{cdh}}(X) & \longrightarrow & R\Gamma_{\mathrm{cdh}}\big(X, \widehat{\mathbb{L}\Omega}^{\geq i}_{-\mathbb{Q}/\mathbb{Q}}\big)
\end{array}$$

is a cartesian square in the derived category $\mathcal{D}(\mathbb{Q})$.

Proof. This is a direct consequence of Theorem 4.66. $\qquad\square$

Proof of Theorem 4.2. By Theorem 4.66, the natural commutative diagram

$$\begin{array}{ccc}
\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{K}(X;\mathbb{Q}) & \longrightarrow & \mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}^{-}(X_{\mathbb{Q}}/\mathbb{Q}) \\
\downarrow & & \downarrow \\
\mathrm{Fil}^{\star}_{\mathrm{cdh}}\mathrm{KH}(X;\mathbb{Q}) & \longrightarrow & L_{\mathrm{cdh}}\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}^{-}(-_{\mathbb{Q}}/\mathbb{Q})(X)
\end{array}$$

is a cartesian square of filtered spectra. By Definition 2.6 and Remark 2.10, there is a natural commutative diagram of filtered spectra

$$\begin{array}{ccccc}
\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}^{-}(X_{\mathbb{Q}}/\mathbb{Q}) & \longrightarrow & \mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HP}(X_{\mathbb{Q}}/\mathbb{Q}) & \longrightarrow & \mathrm{Fil}^{\star-1}_{\mathrm{HKR}}\mathrm{HC}(X_{\mathbb{Q}}/\mathbb{Q})[2] \\
\downarrow & & \downarrow & & \downarrow \\
L_{\mathrm{cdh}}\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}^{-}(-_{\mathbb{Q}}/\mathbb{Q})(X) & \rightarrow & L_{\mathrm{cdh}}\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HP}(-_{\mathbb{Q}}/\mathbb{Q})(X) & \rightarrow & L_{\mathrm{cdh}}\mathrm{Fil}^{\star-1}_{\mathrm{HKR}}\mathrm{HC}(-_{\mathbb{Q}}/\mathbb{Q})(X)[2]
\end{array}$$

where the horizontal lines are fibre sequences. The middle vertical map of this diagram is an equivalence (Proposition 4.42), so the cofibre of the left vertical map is naturally identified with the filtered spectrum

$$\mathrm{cofib}\big(\mathrm{Fil}^{\star-1}_{\mathrm{HKR}}\mathrm{HC}(X_{\mathbb{Q}}/\mathbb{Q}) \longrightarrow L_{\mathrm{cdh}}\mathrm{Fil}^{\star-1}_{\mathrm{HKR}}\mathrm{HC}(-_{\mathbb{Q}}/\mathbb{Q})(X)\big)[1].$$

$\qquad\square$

The following result was proved by Elmanto–Morrow [EM23] for qcqs schemes X over $\mathbb{Q}$.

Corollary 4.68 (Rational motivic cohomology)**.** *Let X be qcqs scheme. Then for every integer $i \in \mathbb{Z}$, there is a natural fibre sequence*

$$\mathbb{Q}(i)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Q}(i)^{\mathrm{cdh}}(X) \longrightarrow \mathrm{cofib}\Big(R\Gamma_{\mathrm{Zar}}\big(X, \mathbb{L}\Omega^{<i}_{-\mathbb{Q}/\mathbb{Q}}\big) \longrightarrow R\Gamma_{\mathrm{cdh}}\big(X, \Omega^{<i}_{-\mathbb{Q}/\mathbb{Q}}\big)\Big)[-1]$$

in the derived category $\mathcal{D}(\mathbb{Q})$.

Proof. For every valuation ring extension V of $\mathbb{Q}$, the natural map

$$\mathbb{L}\Omega^{<i}_{V/\mathbb{Q}} \longrightarrow \Omega^{<i}_{V/\mathbb{Q}}$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Q})$ by results of Gabber–Ramero ([GR03, Theorem 6.5.8 (ii) and Corollary 6.5.21]). The presheaves $R\Gamma_{\mathrm{cdh}}(-, \mathbb{L}\Omega^{<i}_{-\mathbb{Q}/\mathbb{Q}})$ and $R\Gamma_{\mathrm{cdh}}(-, \Omega^{<i}_{-\mathbb{Q}/\mathbb{Q}})$ are finitary cdh sheaves on qcqs schemes, so the natural map

$$R\Gamma_{\mathrm{cdh}}\big(-, \mathbb{L}\Omega^{<i}_{-\mathbb{Q}/\mathbb{Q}}\big) \longrightarrow R\Gamma_{\mathrm{cdh}}\big(-, \Omega^{<i}_{-\mathbb{Q}/\mathbb{Q}}\big)$$

is an equivalence ([EHIK21, Corollary 2.4.19]). The result then follows from Theorem 4.2. $\qquad\square$

Example 4.69 (Weight zero motivic cohomology)**.** For every qcqs scheme X, the natural map

$$\mathbb{Z}(0)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Z}(0)^{\mathrm{cdh}}(X) \simeq R\Gamma_{\mathrm{cdh}}(X, \mathbb{Z}),$$

where the last idenfication is [BEM], is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$. Indeed, it suffices to prove the result rationally, and modulo p for every prime number p. The result rationally is a consequence of Corollary 4.68. For every prime number p, the presheaf $\mathbb{F}_p(0)^{\mathrm{BMS}}$ is naturally identified with the presheaf $R\Gamma_{\text{ét}}(-, \mathbb{F}_p)$ ([BMS19, Proposition 7.16]) which is a cdh sheaf on qcqs schemes ([BM21, Theorem 5.4]), so the result modulo p is a consequence of Corollary 3.25.

4.7 A global Beilinson fibre square

In this subsection we prove Theorem 4.72, which is a rewriting of the Beilinson fibre square of Antieau–Mathew–Morrow–Nikolaus [AMMN22, Theorem 6.17] in terms of the rigid-analytic derived de Rham cohomology of Section 4.2. Note that our statements are formulated in the generality of derived schemes, and that the functor $-_{\mathbb{F}_p}$ is then the derived base change from $\mathbb{Z}$ to $\mathbb{F}_p$. The results in [AMMN22] are stated in the generality of p-torsionfree commutative rings, on which derived and classical reduction modulo p coincide.

Construction 4.70 (The map χ). Let $i \in \mathbb{Z}$ be an integer. Following [AMMN22], we construct for every qcqs derived scheme X a natural map

$$\chi : \Big(\prod_{p \in \mathbb{P}} \mathbb{Z}_p(i)^{\mathrm{BMS}}(X_{\mathbb{F}_p}) \Big)_{\mathbb{Q}} \longrightarrow \Big(\prod_{p \in \mathbb{P}} R\Gamma_{\mathrm{Zar}}\big(X, (\mathbb{L}\Omega_{-/\mathbb{Z}})_p^{\wedge}\big) \Big)_{\mathbb{Q}}$$

in the derived category $\mathcal{D}(\mathbb{Q})$.

(1) ($p \leq i+1$) Let p be a prime number. By [AMMN22, Theorem 6.17], there exists an integer $N \geq 0$ depending only on i and a natural map

$$\chi : \mathbb{Z}_p(i)^{\mathrm{BMS}}(R/p) \longrightarrow \frac{1}{p^N}\big(\mathbb{L}\Omega_{R/\mathbb{Z}}\big)_p^{\wedge}$$

on p-torsionfree p-quasisyntomic rings R, and in particular on polynomial $\mathbb{Z}$-algebras R. The functors $\mathbb{Z}_p(i)^{\mathrm{BMS}}(-/p)$ and $\frac{1}{p^N}(\mathbb{L}\Omega_{-/\mathbb{Z}})_p^{\wedge}$, as functors from animated commutative rings to p-complete objects in the derived category $\mathcal{D}(\mathbb{Z})$, are left Kan extended from polynomial $\mathbb{Z}$-algebras (Corollary 2.26 (2) and by construction, respectively). Left Kan extending the previous map then induces a natural map

$$\chi : \mathbb{Z}_p(i)^{\mathrm{BMS}}(R/p) \longrightarrow \frac{1}{p^N}\big(\mathbb{L}\Omega_{R/\mathbb{Z}}\big)_p^{\wedge}$$

on animated commutative rings R, where the reduction modulo p is the derived one. Inverting p and Zariski sheafifying induces a natural map

$$\chi : \mathbb{Q}_p(i)^{\mathrm{BMS}}(X_{\mathbb{F}_p}) \longrightarrow R\Gamma_{\mathrm{Zar}}\big(X, (\mathbb{L}\Omega_{-/\mathbb{Z}})_p^{\wedge}[\tfrac{1}{p}]\big)$$

in the derived category $\mathcal{D}(\mathbb{Q})$, on general qcqs derived schemes X.

(2) ($p \geq i+2$) For prime numbers p such that $p \geq i+2$, there actually exists a natural map

$$\chi : \mathbb{Z}_p(i)^{\mathrm{BMS}}(R/p) \longrightarrow \big(\mathbb{L}\Omega_{R/\mathbb{Z}}\big)_p^{\wedge}$$

on p-torsionfree p-quasisyntomic rings R ([AMMN22, Theorem 6.17]), and in particular on polynomial $\mathbb{Z}$-algebras R. Left Kan extending this map again induces a natural map

$$\chi : \mathbb{Z}_p(i)^{\mathrm{BMS}}(R/p) \longrightarrow \big(\mathbb{L}\Omega_{R/\mathbb{Z}}\big)_p^{\wedge}$$

on animated commutative rings R. Taking the product over all primes $p \geq i+2$, and then rationalisation and Zariski sheafification, induces a natural map

$$\chi : \Big(\prod_{p \in \mathbb{P}_{\geq i+2}} \mathbb{Z}_p(i)^{\mathrm{BMS}}(X_{\mathbb{F}_p}) \Big)_{\mathbb{Q}} \longrightarrow \Big(\prod_{p \in \mathbb{P}_{\geq i+2}} R\Gamma_{\mathrm{Zar}}\big(X, (\mathbb{L}\Omega_{-/\mathbb{Z}})_p^{\wedge}\big) \Big)_{\mathbb{Q}}$$

in the derived category $\mathcal{D}(\mathbb{Q})$, on general qcqs derived schemes X.

(3) (general construction) For every qcqs derived scheme X, define the desired natural map χ as the product of the map χ of (2) with the finite product over prime numbers $p \leq i+1$ of the map χ of (1).

Theorem 4.71 (Beilinson fibre square, after [AMMN22])**.** *Let X be a qcqs derived scheme. Then for every integer $i \in \mathbb{Z}$, the natural diagram*

$$
\begin{array}{ccc}
\left(\prod_{p \in \mathbb{P}} \mathbb{Z}_p(i)^{\mathrm{BMS}}(X) \right)_{\mathbb{Q}} & \longrightarrow & \left(\prod_{p \in \mathbb{P}} R\Gamma_{\mathrm{Zar}}\big(X, (\mathbb{L}\Omega^{\geq i}_{-/\mathbb{Z}})^{\wedge}_p\big) \right)_{\mathbb{Q}} \\
\downarrow & & \downarrow \\
\left(\prod_{p \in \mathbb{P}} \mathbb{Z}_p(i)^{\mathrm{BMS}}(X_{\mathbb{F}_p}) \right)_{\mathbb{Q}} & \xrightarrow{\ \chi\ } & \left(\prod_{p \in \mathbb{P}} R\Gamma_{\mathrm{Zar}}\big(X, (\mathbb{L}\Omega_{-/\mathbb{Z}})^{\wedge}_p\big) \right)_{\mathbb{Q}}
\end{array}
$$

in the derived category $\mathcal{D}(\mathbb{Q})$ is commutative, with total cofibre naturally identified with the complex

$$
\left(\prod_{p \in \mathbb{P}} R\Gamma_{\mathrm{Zar}}\big(X, \mathbb{L}\Omega^{<i}_{-_{\mathbb{F}_p}/\mathbb{Z}}\big) \right)_{\mathbb{Q}} \in \mathcal{D}(\mathbb{Q}).
$$

Proof. This is a consequence of Construction 4.70 and [AMMN22, Theorem 6.17]. $\qquad\square$

In the following result, the map $\widehat{\chi}$ is defined as the composite of the map χ of Construction 4.70 with the natural maps

$$
\big(\prod_{p \in \mathbb{P}} R\Gamma_{\mathrm{Zar}}\big(X, (\mathbb{L}\Omega_{-/\mathbb{Z}})^{\wedge}_p\big) \big)_{\mathbb{Q}} \to \big(\prod_{p \in \mathbb{P}} R\Gamma_{\mathrm{Zar}}\big(X, (\widehat{\mathbb{L}\Omega}_{-/\mathbb{Z}})^{\wedge}_p\big) \big)_{\mathbb{Q}} \to R\Gamma_{\mathrm{Zar}}\big(X, {\prod_{p \in \mathbb{P}}}' \widehat{\underline{\mathbb{L}\Omega}}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p} \big)
$$

induced by Hodge-completion and Remark 4.27.

Theorem 4.72. *Let X be a qcqs derived scheme. Then for every integer $i \in \mathbb{Z}$, there exists a natural commutative square*

$$
\begin{array}{ccc}
\mathbb{Q}(i)^{\mathrm{TC}}(X) & \longrightarrow & R\Gamma_{\mathrm{Zar}}\big(X, \widehat{\mathbb{L}\Omega}^{\geq i}_{-_{\mathbb{Q}}/\mathbb{Q}}\big) \\
\downarrow & & \downarrow \\
\big(\prod_{p \in \mathbb{P}} \mathbb{Z}_p(i)^{\mathrm{BMS}}(X_{\mathbb{F}_p}) \big)_{\mathbb{Q}} & \xrightarrow{\ \widehat{\chi}\ } & R\Gamma_{\mathrm{Zar}}\big(X, {\prod_{p \in \mathbb{P}}}' \widehat{\underline{\mathbb{L}\Omega}}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p} \big)
\end{array}
$$

in the derived category $\mathcal{D}(\mathbb{Q})$, whose total cofibre is naturally identified with the complex

$$
\left(\prod_{p \in \mathbb{P}} R\Gamma_{\mathrm{Zar}}\big(X, \mathbb{L}\Omega^{<i}_{-_{\mathbb{F}_p}/\mathbb{Z}}\big) \right)_{\mathbb{Q}} \in \mathcal{D}(\mathbb{Q}).
$$

Proof. By construction, there exists a natural commutative diagram

$$
\begin{array}{ccccc}
\mathbb{Q}(i)^{\mathrm{TC}}(X) & \longrightarrow & R\Gamma_{\mathrm{Zar}}\big(X, (\widehat{\mathbb{L}\Omega}^{\geq i}_{-/\mathbb{Z}})_{\mathbb{Q}}\big) & \longrightarrow & R\Gamma_{\mathrm{Zar}}\big(X, \widehat{\mathbb{L}\Omega}^{\geq i}_{-_{\mathbb{Q}}/\mathbb{Q}}\big) \\
\downarrow & & \downarrow & & \downarrow \\
\big(\prod_{p \in \mathbb{P}} \mathbb{Z}_p(i)^{\mathrm{BMS}}(X) \big)_{\mathbb{Q}} & \longrightarrow & \big(\prod_{p \in \mathbb{P}} R\Gamma_{\mathrm{Zar}}\big(X, (\widehat{\mathbb{L}\Omega}^{\geq i}_{-/\mathbb{Z}})^{\wedge}_p\big) \big)_{\mathbb{Q}} & \longrightarrow & R\Gamma_{\mathrm{Zar}}\big(X, {\prod_{p \in \mathbb{P}}}' \widehat{\underline{\mathbb{L}\Omega}}^{\geq i}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p} \big) \\
\downarrow & & \downarrow & & \downarrow \\
\big(\prod_{p \in \mathbb{P}} \mathbb{Z}_p(i)^{\mathrm{BMS}}(X_{\mathbb{F}_p}) \big)_{\mathbb{Q}} & \longrightarrow & \big(\prod_{p \in \mathbb{P}} R\Gamma_{\mathrm{Zar}}\big(X, (\widehat{\mathbb{L}\Omega}_{-/\mathbb{Z}})^{\wedge}_p\big) \big)_{\mathbb{Q}} & \longrightarrow & R\Gamma_{\mathrm{Zar}}\big(X, {\prod_{p \in \mathbb{P}}}' \widehat{\underline{\mathbb{L}\Omega}}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p} \big)
\end{array}
$$

in the derived category $\mathcal{D}(\mathbb{Q})$, where all the inner squares are cartesian expect the left bottom one, and where the commutativity for the left bottom square is part of Theorem 4.71. The desired total cofibre is then naturally identified with the total cofibre of the left bottom square, and the result is then a consequence of Theorem 4.71. $\qquad\square$

Theorem 4.72 means in particular that the complex $\mathbb{Q}(i)^{\mathrm{TC}}(X)$ can be expressed purely in terms of characteristic zero, characteristic p, and rigid-analytic information.

Corollary 4.73. *Let p be a prime number, and X be a qcqs derived $\mathbb{Z}_{(p)}$-scheme. Then for every integer $i \in \mathbb{Z}$, there exists a natural cartesian square*

$$
\begin{array}{ccc}
\mathbb{Q}(i)^{\mathrm{TC}}(X) & \longrightarrow & R\Gamma_{\mathrm{Zar}}\big(X, \widehat{\mathbb{L}\Omega}^{\geq i}_{-_{\mathbb{Q}}/\mathbb{Q}}\big) \\
\downarrow & & \downarrow \\
\mathbb{Q}_p(i)^{\mathrm{BMS}}(X_{\mathbb{F}_p}) & \xrightarrow{\widehat{\chi}} & R\Gamma_{\mathrm{Zar}}\big(X, \widehat{\underline{\mathbb{L}\Omega}}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p}\big)
\end{array}
$$

in the derived category $\mathcal{D}(\mathbb{Q})$.

Proof. The base change $X_{\mathbb{F}_\ell}$ is zero for every prime number ℓ different from p. The complex $\mathbb{L}\Omega^{\leq i}_{X_{\mathbb{F}_p}/\mathbb{Z}}$ is $\mathbb{F}_p$-linear, so its rationalisation vanishes. The result is then a consequence of Theorem 4.72 and Remark 4.23. $\qquad\square$

The following result is an analogue of Theorem 4.72 at the level of filtered objects.

Theorem 4.74. *Let X be qcqs derived scheme. Then there exists a natural commutative square of filtered spectra*

$$
\begin{array}{ccc}
\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(X;\mathbb{Q}) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X_{\mathbb{Q}}/\mathbb{Q}) \\
\downarrow & & \downarrow \\
\Big(\prod_{p\in\mathbb{P}}\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(X_{\mathbb{F}_p})\Big)_{\mathbb{Q}} & \xrightarrow{\widehat{\chi}} & \mathrm{Fil}^\star_{\mathrm{HKR}}\Big(\prod'_{p\in\mathbb{P}}\mathrm{HH}(X;\mathbb{Q}_p)\Big)^{t\mathrm{S}^1}
\end{array},
$$

whose total cofibre is naturally identified with the filtered spectrum $\big(\prod_{p\in\mathbb{P}}\mathrm{Fil}^{\star-1}_{\mathrm{HKR}}\mathrm{HC}(X_{\mathbb{F}_p})\big)_{\mathbb{Q}}[2]$.

Proof. The construction of the map χ in the proof of [AMMN22, Theorem 6.17] adapts readily to define a map at the filtered level instead of graded pieces. The proof is then the same as in Construction 4.70 and Theorem 4.72. $\qquad\square$

Question 4.75. Given the results of the previous sections, and in particular Corollary 4.46 and Theorem 4.74, it is a natural question –to which we do not know the answer– to ask whether the presheaf

$$
\Big(\prod_{p\in\mathbb{P}}\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-_{\mathbb{F}_p})\Big)_{\mathbb{Q}} : \mathrm{Sch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}
$$

is a cdh sheaf, where $-_{\mathbb{F}_p}$ again means derived base change from $\mathbb{Z}$ to $\mathbb{F}_p$.

5 p-ADIC STRUCTURE OF MOTIVIC COHOMOLOGY

In this section, we give a description of motivic cohomology with finite coefficients in terms of Bhatt–Lurie's syntomic cohomology (Theorem 5.10).

5.1 Comparison to étale cohomology

In this subsection, we construct a natural comparison map, called the Beilinson–Lichtenbaum comparison map, from the motivic cohomology of a scheme to the étale cohomology of its generic fibre (Definition 5.3). We then use this comparison map to establish a complete description of ℓ-adic motivic cohomology in terms of étale cohomology (Theorem 5.5).

We use the following important result of Deligne to construct the Beilinson–Lichtenbaum comparison map.

Theorem 5.1 ([BM21]). *Let p be a prime number, and $k \geq 1$ be an integer. Then for every integer $i \geq 0$, the presheaf*

$$R\Gamma_{\text{ét}}(-[\tfrac{1}{p}], \mu_{p^k}^{\otimes i}) : \text{Sch}^{\text{qcqs,op}} \longrightarrow \mathcal{D}(\mathbb{Z}/p^k)$$

is a cdh sheaf.

Proof. By [BM21, Theorem 5.4], the presheaf $R\Gamma_{\text{ét}}(-[\tfrac{1}{p}], \mu_{p^k}^{\otimes i})$ is an arc sheaf on qcqs schemes, and the arc topology is finer than the cdh topology.[7] $\qquad\square$

Definition 5.2 (Cdh-local Beilinson–Lichtenbaum comparison map). Let p be a prime number, and $k \geq 1$ be an integer. For any integer $i \geq 0$, the *cdh-local Beilinson–Lichtenbaum comparison map* is the map

$$\mathbb{Z}/p^k(i)^{\text{cdh}}(-) \longrightarrow R\Gamma_{\text{ét}}(-[\tfrac{1}{p}], \mu_{p^k}^{\otimes i})$$

of functors from (the opposite category of) qcqs derived schemes to the derived category $\mathcal{D}(\mathbb{Z}/p^k)$ defined as the composite

$$\mathbb{Z}/p^k(i)^{\text{cdh}}(-) \longrightarrow \mathbb{Z}/p^k(i)^{\text{cdh}}(-[\tfrac{1}{p}]) \simeq \left(L_{\text{cdh}}\tau^{\leq i}R\Gamma_{\text{ét}}(-, \mu_{p^k}^{\otimes i})\right)(-[\tfrac{1}{p}]) \longrightarrow R\Gamma_{\text{ét}}(-[\tfrac{1}{p}], \mu_{p^k}^{\otimes i})$$

where the first map is induced by base change from $\mathbb{Z}$ to $\mathbb{Z}[\tfrac{1}{p}]$, the equivalence is [BEM], and the last map is induced by the natural transformation $\tau^{\leq i} \to \text{id}$ and Theorem 5.1.

Definition 5.3 (Beilinson–Lichtenbaum comparison map). Let p be a prime number, and $k \geq 1$ be an integer. For any integer $i \geq 0$, the *Beilinson–Lichtenbaum comparison map* (or *motivic-étale comparison map*) is the map

$$\mathbb{Z}/p^k(i)^{\text{mot}}(-) \longrightarrow R\Gamma_{\text{ét}}(-[\tfrac{1}{p}], \mu_{p^k}^{\otimes i})$$

of functors from (the opposite category of) qcqs schemes to the category $\mathcal{D}(\mathbb{Z}/p^k)$ defined as the composite

$$\mathbb{Z}/p^k(i)^{\text{mot}}(-) \longrightarrow \mathbb{Z}/p^k(i)^{\text{cdh}}(-) \longrightarrow R\Gamma_{\text{ét}}(-[\tfrac{1}{p}], \mu_{p^k}^{\otimes i})$$

where the first map is cdh sheafification and the second map is the cdh-local Beilinson–Lichtenbaum comparison map of Definition 5.2.

Remark 5.4. Let p be a prime number, $k \geq 1$ be an integer, R be a commutative ring, and R_p^h be the p-henselisation of R. Then for every integer $i \geq 0$, the natural diagram

$$
\begin{array}{ccccc}
\mathbb{Z}/p^k(i)^{\text{mot}}(\text{Spec}(R)) & \longrightarrow & \mathbb{Z}/p^k(i)^{\text{cdh}}(\text{Spec}(R)) & \longrightarrow & R\Gamma_{\text{ét}}\left(\text{Spec}(R[\tfrac{1}{p}]), \mu_{p^k}^{\otimes i}\right) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{Z}/p^k(i)^{\text{mot}}(\text{Spec}(R_p^h)) & \longrightarrow & \mathbb{Z}/p^k(i)^{\text{cdh}}(\text{Spec}(R_p^h)) & \longrightarrow & R\Gamma_{\text{ét}}\left(\text{Spec}(R_p^h[\tfrac{1}{p}]), \mu_{p^k}^{\otimes i}\right) \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle\text{id}} \\
\mathbb{Z}/p^k(i)^{\text{BMS}}(\text{Spec}(R_p^h)) & \longrightarrow & \left(L_{\text{cdh}}\mathbb{Z}/p^k(i)^{\text{BMS}}\right)(\text{Spec}(R_p^h)) & \longrightarrow & R\Gamma_{\text{ét}}\left(\text{Spec}(R_p^h[\tfrac{1}{p}]), \mu_{p^k}^{\otimes i}\right),
\end{array}
$$

---

[7] More precisely, the arc topology is finer than the v topology, which is finer than the h topology, which is finer than the cdh topology (see [BM21, EHIK21]).

is a commutative diagram in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$, where the top horizontal right map and the middle horizontal right map are given by Definition 5.2, and the bottom horizontal right map is induced by [BL22, Theorem 8.3.1]. This statement is a consequence of the naturality of the constructions, except for the commutativity of the bottom right square, which is proved in [BEM].

Theorem 5.5 (ℓ-adic motivic cohomology). *Let p be a prime number, X be a qcqs scheme over $\mathbb{Z}[\frac{1}{p}]$, and $k \geq 1$ be an integer. Then for every integer $i \geq 0$, the Beilinson–Lichtenbaum comparison for classical motivic cohomology induces a natural equivalence*

$$\mathbb{Z}/p^k(i)^{\mathrm{mot}}(X) \simeq \left(L_{\mathrm{cdh}} \tau^{\leq i} R\Gamma_{\text{ét}}(-, \mu_{p^k}^{\otimes i})\right)(X)$$

in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$.

Proof. The syntomic complex $\mathbb{Z}/p^k(i)^{\mathrm{BMS}}$ and its cdh sheafification $L_{\mathrm{cdh}} \mathbb{Z}/p^k(i)^{\mathrm{BMS}}$ vanish on qcqs derived $\mathbb{Z}[\frac{1}{p}]$-schemes. In particular, the natural map

$$\mathbb{Z}/p^k(i)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{cdh}}(X)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$ (Proposition 3.24). The Beilinson–Lichtenbaum comparison for classical motivic cohomology induces a natural equivalence

$$\mathbb{Z}/p^k(i)^{\mathrm{cdh}}(X) \xleftarrow{\sim} \left(L_{\mathrm{cdh}} \tau^{\leq i} R\Gamma_{\text{ét}}(-, \mu_{p^k}^{\otimes i})\right)(X)$$

in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$ ([BEM]). The desired equivalence is the composite of the previous two equivalences. $\square$

Corollary 5.6. *Let p be a prime number, X be a qcqs $\mathbb{Z}[\frac{1}{p}]$-scheme, and $k \geq 1$ be an integer. Then for every integer $i \geq 0$, there is a natural equivalence*

$$\tau^{\leq i} \mathbb{Z}/p^k(i)^{\mathrm{mot}}(X) \simeq \tau^{\leq i} R\Gamma_{\text{ét}}(X, \mu_{p^k}^{\otimes i})$$

in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$.

Proof. The natural map

$$\tau^{\leq i}\left(L_{\mathrm{cdh}} \tau^{\leq i} R\Gamma_{\text{ét}}(-, \mu_{p^k}^{\otimes i})\right)(X) \longrightarrow \tau^{\leq i}\left(L_{\mathrm{cdh}} R\Gamma_{\text{ét}}(-, \mu_{p^k}^{\otimes i})\right)(X)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$. The result is then a consequence of Theorems 5.1 and 5.5. $\square$

5.2 Comparison to syntomic cohomology

In this subsection, we study motivic cohomology with finite coefficients. Our main result is a computation of p-adic motivic cohomology in terms of syntomic cohomology (Theorem 5.10).

Notation 5.7 (Syntomic cohomology of derived scheme, after Bhatt–Lurie [BL22]). Let X be a qcqs derived scheme, p be a prime number, and $i \in \mathbb{Z}$ be an integer. We denote by

$$\mathbb{Z}_p(i)^{\mathrm{syn}}(X) \in \mathcal{D}(\mathbb{Z}_p)$$

the *syntomic cohomology* of X, as defined in [BL22, Section 8.4]. For every integer $k \geq 1$, we also denote by $\mathbb{Z}/p^k(i)^{\mathrm{syn}}(X)$ the derived reduction modulo p^k of the previous complex. In particular, the presheaf $\mathbb{Z}/p^k(i)^{\mathrm{syn}}$ is a Zariski sheaf, whose restriction to animated commutative rings is left Kan extended from smooth $\mathbb{Z}$-algebras, and such that on classical commutative rings R, there is, by definition, a natural cartesian square

$$\begin{array}{ccc}
\mathbb{Z}/p^k(i)^{\mathrm{syn}}(\mathrm{Spec}(R)) & \longrightarrow & R\Gamma_{\text{ét}}(\mathrm{Spec}(R[\frac{1}{p}]), \mu_{p^k}^{\otimes i}) \\
\downarrow & & \downarrow \\
\mathbb{Z}/p^k(i)^{\mathrm{BMS}}(\mathrm{Spec}(R_p^h)) & \longrightarrow & R\Gamma_{\text{ét}}(\mathrm{Spec}(R_p^h[\frac{1}{p}]), \mu_{p^k}^{\otimes i})
\end{array}$$

in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$, where R_p^h is the p-henselisation of the commutative ring R, and the bottom map is the map of [BL22, Theorem 8.3.1].

Construction 5.8 (Motivic-syntomic comparison map)**.** Let p be a prime number, and $k \geq 1$ be an integer. For any integer $i \geq 0$, the *motivic-syntomic comparison map* is the map

$$\mathbb{Z}/p^k(i)^{\mathrm{mot}}(-) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{syn}}(-)$$

of functors from (the opposite category of) qcqs schemes to the derived category $\mathcal{D}(\mathbb{Z}/p^k)$ defined as the Zariski sheafification of the map on commutative rings R induced by the natural commutative diagram

$$\begin{array}{ccc} \mathbb{Z}/p^k(i)^{\mathrm{mot}}(\mathrm{Spec}(R)) & \longrightarrow & R\Gamma_{\text{ét}}\big(\mathrm{Spec}(R[\tfrac{1}{p}]), \mu_{p^k}^{\otimes i}\big) \\ \downarrow & & \downarrow \\ \mathbb{Z}/p^k(i)^{\mathrm{BMS}}(\mathrm{Spec}(R_p^h)) & \longrightarrow & R\Gamma_{\text{ét}}\big(\mathrm{Spec}(R_p^h[\tfrac{1}{p}]), \mu_{p^k}^{\otimes i}\big) \end{array}$$

of Remark 5.4.

The following cartesian square, where the bottom horizontal map was described independently in [BEM], can be seen as an alternative definition of p-adic motivic cohomology of qcqs schemes (see Corollary 3.25).

Proposition 5.9. *Let X be a qcqs scheme, p be a prime number, and $k \geq 1$ be an integer. Then for every integer $i \geq 0$, the commutative diagram*

$$\begin{array}{ccc} \mathbb{Z}/p^k(i)^{\mathrm{mot}}(X) & \longrightarrow & \mathbb{Z}/p^k(i)^{\mathrm{syn}}(X) \\ \downarrow & & \downarrow \\ \mathbb{Z}/p^k(i)^{\mathrm{cdh}}(X) & \longrightarrow & \big(L_{\mathrm{cdh}}\,\mathbb{Z}/p^k(i)^{\mathrm{syn}}\big)(X) \end{array}$$

where the top horizontal map is the motivic-syntomic comparison map of Construction 5.8 and the vertical maps are cdh sheafification, is a cartesian square in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$.

Proof. By [BL22, Remark 8.4.4], there is a natural fibre sequence

$$R\Gamma_{\text{ét}}(X, j_!\mu_{p^k}^{\otimes i}) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{syn}}(X) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{BMS}}(X)$$

in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$. The first term of this fibre sequence satisfies arc descent by [BM21, Theorem 5.4], hence in particular cdh descent. The result is then a consequence of Corollary 3.25. $\quad\square$

The following result is a mixed characteristic generalisation of Elmanto–Morrow's fundamental fibre sequence for motivic cohomology of characteristic p schemes ([EM23, Corollary 4.32]).

Theorem 5.10 (*p-adic motivic cohomology*)**.** *Let X be a qcqs scheme, p be a prime number, and $k \geq 1$ be an integer. Then for every integer $i \geq 0$, there is a natural fibre sequence*

$$\mathbb{Z}/p^k(i)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{syn}}(X) \longrightarrow \big(L_{\mathrm{cdh}}\tau^{>i}\,\mathbb{Z}/p^k(i)^{\mathrm{syn}}\big)(X)$$

in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$. In particular, the fibre of the motivic-syntomic comparison map is in degrees at least $i+2$.

Proof. By [BEM], there is a natural equivalence

$$\mathbb{Z}/p^k(i)^{\mathrm{cdh}}(X) \simeq \big(L_{\mathrm{cdh}}\tau^{\leq i}\,\mathbb{Z}/p^k(i)^{\mathrm{syn}}\big)(X)$$

in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$. The result is then a consequence of Proposition 5.9. $\quad\square$

Corollary 5.11. *Let X be a qcqs scheme, p be a prime number, and $k \geq 1$ be an integer. Then for every integer $i \geq 0$, the motivic-syntomic comparison map induces a natural equivalence*

$$\tau^{\leq i} \mathbb{Z}/p^k(i)^{\mathrm{mot}}(X) \xrightarrow{\sim} \tau^{\leq i} \mathbb{Z}/p^k(i)^{\mathrm{syn}}(X)$$

in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$.

Proof. This is a consequence of Theorem 5.10. $\qquad\square$

Algebraic K-theory being $\mathbb{A}^1$-invariant on regular schemes, one could expect the classical $\mathbb{A}^1$-invariant motivic cohomology to be a good theory for general regular schemes, not only schemes that are smooth over a Dedekind domain. However, most of the results on classical motivic cohomology in mixed characteristic are proved only in the smooth case, as consequences of the Gersten conjecture proved by Geisser [Gei04]. This is the case for the Beilinson–Lichtenbaum conjecture, which compares motivic cohomology with finite coefficients to étale cohomology. Combined with Theorem D (6), the following result extends the analogous result for classical motivic cohomology to the regular case.

Corollary 5.12 (Beilinson–Lichtenbaum conjecture for F-smooth schemes)**.** *Let p be a prime number, X be a p-torsionfree F-smooth scheme (e.g., a regular scheme flat over $\mathbb{Z}$), and $k \geq 1$ be an integer. Then for every integer $i \geq 0$, the fibre of the Beilinson–Lichtenbaum comparison map*

$$\mathbb{Z}/p^k(i)^{\mathrm{mot}}(X) \longrightarrow R\Gamma_{\mathrm{\acute{e}t}}\left(X[\tfrac{1}{p}], \mu_{p^k}^{\otimes i}\right)$$

is in degrees at least $i + 1$.

Proof. By [BM23, Theorem 1.8], the fibre of the natural map

$$\mathbb{Z}/p^k(i)^{\mathrm{syn}}(X) \longrightarrow R\Gamma_{\mathrm{\acute{e}t}}\left(X[\tfrac{1}{p}], \mu_{p^k}^{\otimes i}\right)$$

is in degrees at least $i + 1$. The result is then a consequence of Theorem 5.10. $\qquad\square$

Corollary 5.13. *Let p be a prime number, X be a p-torsionfree F-smooth scheme, and $k \geq 1$ be an integer. Then the fibre of the natural map*

$$\mathbb{Z}/p^k(i)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{mot}}\left(X[\tfrac{1}{p}]\right)$$

is in degrees at least $i + 1$.

Proof. By construction, the Beilinson–Lichtenbaum comparison map

$$\mathbb{Z}/p^k(i)^{\mathrm{mot}}(X) \longrightarrow R\Gamma_{\mathrm{\acute{e}t}}\left(X[\tfrac{1}{p}], \mu_{p^k}^{\otimes i}\right)$$

naturally factors as the composite

$$\mathbb{Z}/p^k(i)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{mot}}\left(X[\tfrac{1}{p}]\right) \longrightarrow R\Gamma_{\mathrm{\acute{e}t}}\left(X[\tfrac{1}{p}], \mu_{p^k}^{\otimes i}\right).$$

The fibre of the second map is in degrees at least $i + 1$ by Corollary 5.6, and the fibre of the composite is in degrees at least $i + 1$ by Corollary 5.12, so the fibre of the first map is in degrees at least $i + 1$. $\quad\square$

6   COMPARISON TO CLASSICAL MOTIVIC COHOMOLOGY

For smooth schemes over a field, Elmanto–Morrow proved that the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$ coincide with the classical motivic complexes $\mathbb{Z}(i)^{\mathrm{cla}}$ ([EM23, Corollary 6.4]). Their proof uses Gersten injectivity and the projective bundle formula to reduce the statement to the case of fields, and relies on Gabber's presentation lemma ([CTHK97, Theorem 3.1.1]), which is unknown in mixed characteristic. In this section, we prove partial results comparing the complexes $\mathbb{Z}(i)^{\mathrm{mot}}$ and $\mathbb{Z}(i)^{\mathrm{cla}}$ in mixed characteristic.

6.1   Comparison to classical motivic cohomology in low degrees

In this subsection, we prove that the classical-motivic comparison map $\mathbb{Z}(i)^{\mathrm{cla}} \to \mathbb{Z}(i)^{\mathrm{mot}}$ (Definition 3.22) is an equivalence with rational or ℓ-adic coefficients, and integrally in degrees at most $i+1$.

Proposition 6.1. *Let p be a prime number, B be a Dedekind domain such that every characteristic p residue field of B is perfect, and X be a smooth scheme over B. Then for any integers $i \geq 0$ and $k \geq 1$, the fibre of the classical-motivic comparison map*

$$\mathbb{Z}/p^k(i)^{\mathrm{cla}}(X) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{mot}}(X)$$

is in degrees at least $i+2$ in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$. If p is moreover invertible in X, then this fibre vanishes, i.e., the previous classical-motivic comparison map is an equivalence.

Proof. If p is invertible in the scheme X, then the composite map

$$\mathbb{Z}/p^k(i)^{\mathrm{cla}}(X) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{cdh}}(X)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$ by [BEM]. The right map is also an equivalence by Remark 3.26, so the left map is an equivalence. In general, consider the natural commutative diagram

$$
\begin{array}{ccc}
\mathbb{Z}/p^k(i)^{\mathrm{cla}}(X) & \longrightarrow & \left(L_{\mathrm{Nis}}\tau^{\leq i}\,\mathbb{Z}/p^k(i)^{\mathrm{syn}}\right)(X) \\
\downarrow & & \downarrow \\
\mathbb{Z}/p^k(i)^{\mathrm{mot}}(X) & \longrightarrow & \mathbb{Z}/p^k(i)^{\mathrm{syn}}(X)
\end{array}
$$

in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$. The top horizontal map is an equivalence by [Gei04, Theorems 1.2 (2) and 1.3] and [BM23, Theorem 5.8]. The fibre of the right vertical map is naturally identified with $\left(L_{\mathrm{Nis}}\tau^{>i}\,\mathbb{Z}/p^k(i)^{\mathrm{syn}}\right)(X)[-1]$, and is thus in degrees at least $i+2$. The fibre of the bottom horizontal map is in degrees at least $i+2$ by Theorem 5.10. So the fibre of the left vertical map is in degrees at least $i+2$. $\qquad\square$

Proposition 6.2. *Let B be a mixed characteristic Dedekind domain, and X be a smooth scheme over B. Then for every integer $i \geq 0$, the classical-motivic comparison map*

$$\mathbb{Q}(i)^{\mathrm{cla}}(X) \longrightarrow \mathbb{Q}(i)^{\mathrm{mot}}(X)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Q})$.

Proof. This is a consequence of the rational splitting of algebraic K-theory induced by Adams operations. More precisely, we use the splitting induced by Lemma 4.12 for the filtrations $\mathrm{Fil}_{\mathrm{cla}}^{\star}\mathrm{K}(-;\mathbb{Q})$ (which is $\mathbb{N}$-indexed by construction) and $\mathrm{Fil}_{\mathrm{mot}}^{\star}\mathrm{K}(-;\mathbb{Q})$ (which is $\mathbb{N}$-indexed by Proposition 4.47). These decompositions are compatible with the classical-motivic comparison map because of the compatibility between the associated Adams operations (Section 4.1). $\qquad\square$

Theorem 6.3 (Comparison to classical motivic cohomology)**.** *Let B be a mixed characteristic Dedekind domain such that every residue field of B is perfect, and X be a smooth scheme over B. Then for every integer $i \geq 0$, the fibre of the classical-motivic comparison map*

$$\mathbb{Z}(i)^{\mathrm{cla}}(X) \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(X)$$

is in degrees at least $i+3$.

Proof. Let $F(X) \in \mathcal{D}(\mathbb{Z})$ be the fibre of the classical-motivic comparison map

$$\mathbb{Z}(i)^{\mathrm{cla}}(X) \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(X).$$

We want to prove that for every integer $k \leq i + 2$, the abelian group $\mathrm{H}^k(F(X))$ is zero. By Proposition 6.2, the abelian group $\mathrm{H}^k(F(X))$ is torsion for every integer $k \in \mathbb{Z}$. For every prime number p and every integer $k \in \mathbb{Z}$, there is a natural short exact sequence of abelian groups

$$0 \longrightarrow \mathrm{H}^k(F(X))/p \longrightarrow \mathrm{H}^k(F(X)/p) \longrightarrow \mathrm{H}^{k+1}(F(X))[p] \longrightarrow 0.$$

By Proposition 6.1, for every prime number p, the abelian group $\mathrm{H}^k(F(X)/p)$ is zero if $k \leq i+1$, hence the abelian group $\mathrm{H}^k(F(X))[p]$ is zero for every integer $k \leq i + 2$. This implies the desired result. $\quad\square$

Corollary 6.4. *Let B be a mixed characteristic Dedekind domain such that every residue field of B is perfect. Then for every integer $i \geq 0$, the classical-motivic comparison map induces an equivalence of $\mathcal{D}(\mathbb{Z})$-valued presheaves*

$$\mathbb{Z}(i)^{\mathrm{cla}}(-) \longrightarrow \left(L_{\mathrm{Zar}} \tau^{\leq i} \, \mathbb{Z}(i)^{\mathrm{mot}} \right)(-)$$

on essentially smooth B-schemes.

Proof. The classical motivic complex $\mathbb{Z}(i)^{\mathrm{cla}}$ is a Zariski sheaf which is locally supported in degrees at most i ([Gei04, Corollary 4.4]), so the result is a consequence of Theorem 6.3. $\quad\square$

6.2 Comparison to classical motivic cohomology in low dimensions

In this subsection, we study in more detail the defect for the classical-motivic comparison map to be an equivalence (Theorem 6.9), and prove that this defect vanishes on smooth schemes of dimension less than or equal to one over a mixed characteristic Dedekind domain (Corollary 6.10).

Lemma 6.5. *Let B be a commutative ring, π be an element of B, $\mathcal{C}$ be a presentable ∞-category, and $F : \mathrm{Alg}_B \to \mathcal{C}$ be a finitary and rigid functor. If the functor F is zero on $B[\frac{1}{\pi}]$-algebas, then for every qcqs B-scheme X, the natural map*

$$\left(L_{\mathrm{cdh}} F \right)(X) \longrightarrow \left(L_{\mathrm{cdh}} F \right)(X_{B/\pi})$$

is an equivalence in the ∞-category $\mathcal{C}$.

Proof. Covers in a site are stable under base change, and the cdh sheafification of a finitary presheaf is finitary, so the presheaf

$$(L_{\mathrm{cdh}} F)(-_{B/\pi})$$

is a finitary cdh sheaf on qcqs B-schemes. It then suffices to prove that for every henselian valuation ring V which is a B-algebra, the natural map

$$F(V) \longrightarrow \left(L_{\mathrm{cdh}} F \right)(V/\pi)$$

is an equivalence in the ∞-category $\mathcal{C}$. The presheaf $L_{\mathrm{cdh}} F$ is a finitary cdh sheaf, so it is invariant under nilpotent extensions. In particular, the natural map

$$\left(L_{\mathrm{cdh}} F \right)(V/\pi) \longrightarrow \left(L_{\mathrm{cdh}} F \right)(V/\sqrt{(\pi)})$$

is an equivalence in the ∞-category $\mathcal{C}$. The quotient of a henselian valuation ring by one of its prime ideals is a henselian valuation ring, so the target of the previous map is naturally identified with the object $F(V/\sqrt{(\pi)}) \in \mathcal{C}$. We finally prove that the natural map

$$F(V) \longrightarrow F(V/\sqrt{(\pi)})$$

is an equivalence in the ∞-category $\mathcal{C}$. If π is invertible in the henselian valuation ring V, then both terms are zero by hypothesis on the functor F. And if π is not invertible in V, then in particular the henselian local ring V is π-henselian, and the result is a consequence of rigidity for the functor F. $\quad\square$

Corollary 6.6. *Let p be a prime number, B be a discrete valuation ring of mixed characteristic $(0, p)$, π be a uniformizer of B, and X be a qcqs B-scheme. Then for any integers $i \geq 0$ and $k \geq 1$, the natural map*

$$\left(L_{\mathrm{cdh}}\tau^{>i}\,\mathbb{Z}\,/p^k(i)^{\mathrm{BMS}}\right)(X) \longrightarrow \left(L_{\mathrm{cdh}}\tau^{>i}\,\mathbb{Z}\,/p^k(i)^{\mathrm{BMS}}\right)(X_{B/\pi})$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z}\,/p^k)$.

Proof. The functor

$$\tau^{>i}\,\mathbb{Z}\,/p^k(i)^{\mathrm{BMS}} : \mathrm{Alg}_B \longrightarrow \mathcal{D}(\mathbb{Z}\,/p^k)$$

is finitary (Theorem 2.25 (2)) and rigid (Theorem 2.27). The functor $\mathbb{Z}\,/p^k(i)^{\mathrm{BMS}}$ is moreover zero on $\mathbb{Z}[\frac{1}{p}]$-algebras (Remark 2.16), so the result is a consequence of Lemma 6.5. $\qquad\square$

Remark 6.7. One can prove similarly that Corollary 6.6 holds for B a general valuation ring of mixed characteristic $(0, p)$, where the base change to the characteristic p field B/π is replaced by the base change to the characteristic p valuation ring $B/\sqrt{(p)}$.

Proposition 6.8. *Let p be a prime number, B be a discrete valuation ring of mixed characteristic $(0, p)$, and R be a henselian local ind-smooth B-algebra of residue characteristic p. Then for any integers $i \geq 0$ and $k \geq 1$, the natural map*

$$\tau^{>i}\,\mathbb{Z}\,/p^k(i)^{\mathrm{BMS}}(R) \longrightarrow \left(L_{\mathrm{cdh}}\tau^{>i}\,\mathbb{Z}\,/p^k(i)^{\mathrm{BMS}}\right)(R)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z}\,/p^k)$.

Proof. Let π be a uniformizer of the discrete valuation ring B, and consider the commutative diagram

$$
\begin{array}{ccc}
\tau^{>i}\,\mathbb{Z}\,/p^k(i)^{\mathrm{BMS}}(R) & \longrightarrow & \left(L_{\mathrm{cdh}}\tau^{>i}\,\mathbb{Z}\,/p^k(i)^{\mathrm{BMS}}\right)(R) \\
\downarrow & & \downarrow \\
\tau^{>i}\,\mathbb{Z}\,/p^k(i)^{\mathrm{BMS}}(R/\pi) & \longrightarrow & \left(L_{\mathrm{cdh}}\tau^{>i}\,\mathbb{Z}\,/p^k(i)^{\mathrm{BMS}}\right)(R/\pi)
\end{array}
$$

in the derived category $\mathcal{D}(\mathbb{Z}\,/p^k)$. The commutative ring R is p-henselian, hence π-henselian, so the left vertical map is an equivalence by Theorem 2.27. The right vertical map is an equivalence by Corollary 6.6. We prove now that the bottom horizontal map is an equivalence. The commutative ring R/π is a henselian local ind-smooth algebra over the field B/π, so the classical-motivic comparison map

$$\mathbb{Z}\,/p^k(i)^{\mathrm{cla}}(R/\pi) \longrightarrow \mathbb{Z}\,/p^k(i)^{\mathrm{mot}}(R/\pi)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z}\,/p^k)$ ([EM23, Corollary 6.4]). The complex $\mathbb{Z}\,/p^k(i)^{\mathrm{cla}}(R) \in \mathcal{D}(\mathbb{Z}\,/p^k)$ is moreover in degrees at most i since the commutative ring R is henselian local, so this is equivalent to the fact that the natural map

$$\tau^{>i}\,\mathbb{Z}\,/p^k(i)^{\mathrm{BMS}}(R/\pi) \longrightarrow \left(L_{\mathrm{cdh}}\tau^{>i}\,\mathbb{Z}\,/p^k(i)^{\mathrm{BMS}}\right)(R/\pi)$$

is an equivalence in the derived $\mathcal{D}(\mathbb{Z}\,/p^k)$ (Theorem 5.10). $\qquad\square$

Theorem 6.9. *Let p be a prime number, B be a mixed characteristic Dedekind domain such that every characteristic p residue field of B is perfect, and R be a henselian local ind-smooth B-algebra of residue characteristic p. Then for any integers $i \geq 0$ and $k \geq 1$, the fibre of the classical-motivic comparison map*

$$\mathbb{Z}\,/p^k(i)^{\mathrm{cla}}(R) \longrightarrow \mathbb{Z}\,/p^k(i)^{\mathrm{mot}}(R)$$

is in degrees at least $i + 2$, given by the complex

$$\left(L_{\mathrm{cdh}}\tau^{\leq i}R\Gamma_{\mathrm{\acute{e}t}}(-, j_!\mu_{p^k}^{\otimes i})\right)(R)[-1] \in \mathcal{D}(\mathbb{Z}\,/p^k).$$

Proof. The fact that the fibre of the classical-motivic comparison map is in degrees at least $i + 2$ is a special case of Proposition 6.1. The henselian local ring R is local for the Nisnevich topology, so the proof of Proposition 6.1 implies that the complex $\mathbb{Z}/p^k(i)^{\mathrm{cla}}(R) \in \mathcal{D}(\mathbb{Z}/p^k)$ is moreover in degrees at most i, and this fibre is naturally identified with the complex $\left(\tau^{>i}\mathbb{Z}/p^k(i)^{\mathrm{mot}}(R)\right)[-1] \in \mathcal{D}(\mathbb{Z}/p^k)$. By Theorem 5.10, this complex is in turn naturally identified with the complex

$$\mathrm{fib}\left(\tau^{>i}\mathbb{Z}/p^k(i)^{\mathrm{syn}}(R) \longrightarrow \left(L_{\mathrm{cdh}}\tau^{>i}\mathbb{Z}/p^k(i)^{\mathrm{syn}}(R)\right)[-1] \in \mathcal{D}(\mathbb{Z}/p^k).$$

Consider the commutative diagram

$$
\begin{array}{ccc}
\tau^{>i}\mathbb{Z}/p^k(i)^{\mathrm{syn}}(R) & \longrightarrow & \left(L_{\mathrm{cdh}}\tau^{>i}\mathbb{Z}/p^k(i)^{\mathrm{syn}}\right)(R) \\
\downarrow & & \downarrow \\
\tau^{>i}\mathbb{Z}/p^k(i)^{\mathrm{BMS}}(R) & \longrightarrow & \left(L_{\mathrm{cdh}}\tau^{>i}\mathbb{Z}/p^k(i)^{\mathrm{BMS}}\right)(R)
\end{array}
$$

in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$. The commutative ring R is p-henselian, so the left vertical map is an equivalence (Notation 5.7). The commutative ring R is local, so there exists a prime ideal $\mathfrak{p}$ of the Dedekind domain B such that $p \in \mathfrak{p}$ and R is essentially smooth over the localisation $B_{\mathfrak{p}}$. The local ring $B_{\mathfrak{p}}$ is a discrete valuation ring of mixed characteristic $(0, p)$, so the bottom horizontal map is an equivalence (Proposition 6.8). The fibre of the top horizontal map is then naturally identified with the (-1)-cohomological shift of the fibre of the right vertical map. By Lemma 8.8 (applied for $j = i$ and after cdh sheafification), this fibre is naturally identified with the complex

$$\left(L_{\mathrm{cdh}}\tau^{>i}R\Gamma_{\mathrm{\acute{e}t}}(-, j_!\mu_{p^k}^{\otimes i})\right)(R) \in \mathcal{D}(\mathbb{Z}/p^k).$$

The functor $R\Gamma_{\mathrm{\acute{e}t}}(-, j_!\mu_{p^k}^{\otimes i})$ is rigid ([Gab94], see also [BM21, Corollary 1.18 (1)]) and satisfies cdh descent ([BM21, Theorem 5.4]), so the object

$$\left(L_{\mathrm{cdh}}R\Gamma_{\mathrm{\acute{e}t}}(-, j_!\mu_{p^k}^{\otimes i})\right)(R)$$

is zero in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$ for the p-henselian commutative ring R. In particular, there is a natural equivalence

$$\left(L_{\mathrm{cdh}}\tau^{>i}R\Gamma_{\mathrm{\acute{e}t}}(-, j_!\mu_{p^k}^{\otimes i})\right)(R) \simeq \left(L_{\mathrm{cdh}}\tau^{\leq i}R\Gamma_{\mathrm{\acute{e}t}}(-, j_!\mu_{p^k}^{\otimes i})\right)(R)[1]$$

in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$, which implies the desired equivalence. □

Corollary 6.10. *Let B be a mixed characteristic Dedekind domain such that every residue field of B is perfect, and X be a smooth B-scheme of dimension less than or equal to one over B. Then for every integer $i \geq 0$, the classical-motivic comparison map*

$$\mathbb{Z}(i)^{\mathrm{cla}}(X) \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(X)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$.

Proof. The presheaves $\mathbb{Z}(i)^{\mathrm{cla}}$ and $\mathbb{Z}(i)^{\mathrm{mot}}$ are finitary Nisnevich sheaves, hence it suffices to prove the result on the henselisation of every local ring of the scheme X. Let R be such a henselian local ring, which is then a henselian local ind-smooth B-algebra of Krull dimension at most three. The result rationally is a special case of Proposition 6.2, so it suffices to prove the result modulo p, for every prime number p. Let p be a prime number. If p is invertible in the local ring R, this is Proposition 6.1. Assume now that p is not invertible in the henselian local ring R, and in particular that the ring R is p-henselian. By Theorem 6.9, the fibre of the classical-motivic comparison map

$$\mathbb{F}_p(i)^{\mathrm{cla}}(R) \longrightarrow \mathbb{F}_p(i)^{\mathrm{mot}}(R)$$

is in degrees at least $i + 2$, given by the complex

$$\left(L_{\mathrm{cdh}}\tau^{\leq i}R\Gamma_{\mathrm{\acute{e}t}}(-, j_!\mu_p^{\otimes i})\right)(R)[-1] \in \mathcal{D}(\mathbb{F}_p).$$

The commutative ring R is a noetherian ring of Krull dimension at most two, hence of valuative dimension at most two. So this complex is also in degrees at most $i + 3$ ([EHIK21, Theorem 2.4.15]), and it is thus only in degrees $i + 2$ and $i + 3$. The complexes $\mathbb{F}_p(i)^{\mathrm{cla}}(R)$ and $\mathbb{F}_p(i)^{\mathrm{mot}}(R)$ (for $i \geq 0$) are the graded pieces of the $\mathbb{N}$-indexed complete filtrations $\mathrm{Fil}^{\star}_{\mathrm{cla}}\mathrm{K}(R; \mathbb{F}_p)$ and $\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{K}(X; \mathbb{F}_p)$ on the algebraic K-theory of R. In particular, there is natural spectral sequence

$$E_2^{i,j} = \mathrm{H}^{i-j}\big(\big(L_{\mathrm{cdh}}\tau^{\leq -j}R\Gamma_{\mathrm{\acute{e}t}}(-, j_!\mu_p^{\otimes(-j)})\big)(R)\big) \implies 0.$$

The previous cohomological bound implies that all the differentials in this spectral sequence are zero, so this spectral sequence degenerates. This implies the desired equivalence. $\square$

7 COMPARISON TO MILNOR K-THEORY AND LISSE MOTIVIC COHOMOLOGY

In this section, we study the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$ on local rings. We prove that these are left Kan extended, in degrees at most i, from local essentially smooth $\mathbb{Z}$-algebras (Theorem 7.11). Via the comparison to classical motivic cohomology in the smooth case (Theorem 6.3), this means that the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$ are controlled, up to degree i, by classical motivic cohomology (Corollary 7.12). We use the latter result to construct a comparison map from the ith (improved) Milnor K-group of a general local ring A to the motivic cohomology group $\mathrm{H}^i_{\mathrm{mot}}(A, \mathbb{Z}(i))$, and prove that this map is an isomorphism with finite coefficients (Theorem 7.21).

7.1 Comparison to lisse motivic cohomology

In this subsection, we prove a comparison between motivic cohomology and lisse motivic cohomology on general local rings (Corollary 7.12), which generalises to mixed characteristic the analogous comparison result of Elmanto–Morrow over a field ([EM23, Theorem 7.7]). To do so, we use the following comparison map.

Definition 7.1 (Lisse-motivic comparison map)**.** For every integer $i \in \mathbb{Z}$, the *lisse-motivic comparison map* is the map

$$\mathbb{Z}(i)^{\mathrm{lisse}}(-) \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(-)$$

of functors from animated commutative rings to the derived category $\mathcal{D}(\mathbb{Z})$ defined as the composite

$$\left(L_{\mathrm{AniRings}/\mathrm{Sm}_{\mathbb{Z}}} \mathbb{Z}(i)^{\mathrm{cla}}\right)(-) \longrightarrow \left(L_{\mathrm{AniRings}/\mathrm{Sm}_{\mathbb{Z}}} \mathbb{Z}(i)^{\mathrm{mot}}\right)(-) \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(-),$$

where the first map is the map induced by Definition 3.22 and the second map is the canonical map.

Lemma 7.2. *For every integer $i \geq 0$, the functor*

$$\tau^{\leq 2i} \mathbb{Q}(i)^{\mathrm{mot}}(-) : \mathrm{AniRings} \longrightarrow \mathcal{D}(\mathbb{Q})$$

is left Kan extended from smooth $\mathbb{Z}$-algebras.

Proof. By [EHK⁺20, Example 1.0.6], connective algebraic K-theory

$$\tau^{\leq 0} \mathrm{K}(-; \mathbb{Q}) : \mathrm{AniRings} \longrightarrow \mathcal{D}(\mathbb{Q})$$

is left Kan extended from smooth $\mathbb{Z}$-algebras. By Corollary 4.57, this implies that the functor

$$\bigoplus_{i \geq 0} \tau^{\leq 0}\left(\mathbb{Q}(i)^{\mathrm{mot}}(-)[2i]\right) : \mathrm{AniRings} \longrightarrow \mathcal{D}(\mathbb{Q})$$

is left Kan extended from smooth $\mathbb{Z}$-algebras, which is equivalent to the desired result. $\qquad\square$

Corollary 7.3. *Let R be an animated commutative ring. Then for every integer $i \geq 0$, the lisse-motivic comparison map induces a natural equivalence*

$$\mathbb{Q}(i)^{\mathrm{lisse}}(R) \overset{\sim}{\longrightarrow} \tau^{\leq 2i} \mathbb{Q}(i)^{\mathrm{mot}}(R)$$

in the derived category $\mathcal{D}(\mathbb{Q})$.

Proof. If R is a smooth $\mathbb{Z}$-algebra, the result is a consequence of Proposition 6.2 and the fact that, by construction, the classical motivic complex $\mathbb{Z}(i)^{\mathrm{cla}}(R) \in \mathcal{D}(\mathbb{Z})$ is in degrees at most $2i$. In general, this is then a consequence of Lemma 7.2. $\qquad\square$

Proposition 7.4. *Let R be a local ring. Then for every integer $i \geq 0$, the lisse-motivic comparison map induces a natural equivalence*

$$\mathbb{Q}(i)^{\mathrm{lisse}}(R) \longrightarrow \tau^{\leq i} \mathbb{Q}(i)^{\mathrm{mot}}(R)$$

in the derived category $\mathcal{D}(\mathbb{Q})$. Moreover, the motivic cohomology group $\mathrm{H}^j_{\mathrm{mot}}(R, \mathbb{Q}(i))$ is zero for $i < j \leq 2i$.

Proof. The classical motivic complex $\mathbb{Z}(i)^{\mathrm{cla}}(-)$ is Zariski-locally in degrees at most i ([Gei04, Corollary 4.4]). By taking left Kan extension, this implies that the lisse motivic complex $\mathbb{Z}(i)^{\mathrm{lisse}}(-)$ is also Zariski-locally in degrees at most i. In particular, the lisse motivic complex $\mathbb{Q}(i)^{\mathrm{lisse}}(R)$ is in degrees at most i. The result is then a consequence of Corollary 7.3. $\qquad\square$

Remark 7.5. By Drinfeld's theorem ([Dri06, Theorem 3.7]), the K-group $\mathrm{K}_{-1}(R)$ vanishes for every henselian local ring R. By Corollary 4.57, this implies that for every integer $i \geq 0$, the motivic cohomology group $\mathrm{H}_{\mathrm{mot}}^{2i+1}(R, \mathbb{Q}(i))$ is zero, *i.e.*, that the motivic cohomology group $\mathrm{H}_{\mathrm{mot}}^{2i+1}(R, \mathbb{Z}(i))$ is torsion.

Corollary 7.6. *For every integer $i \geq 0$, the functor $\tau^{\leq i}\, \mathbb{Q}(i)^{\mathrm{mot}}$, from local rings to the derived category $\mathcal{D}(\mathbb{Q})$, is left Kan extended from local essentially smooth $\mathbb{Z}$-algebras.*

Proof. This is a consequence of Lemma 7.2 and Proposition 7.4. $\qquad\square$

Proposition 7.7. *Let p be a prime number, and k be an integer. Then for every integer $i \geq 0$, the functor*

$$\tau^{\leq i}\, \mathbb{Z}\,/p^k(i)^{\mathrm{mot}}(-) : \mathrm{Rings} \longrightarrow \mathcal{D}(\mathbb{Z}\,/p^k)$$

is left Kan extended from smooth $\mathbb{Z}$-algebras.

Proof. By Theorem 5.10, this is equivalent to the fact that the functor $\tau^{\leq i}\, \mathbb{Z}\,/p^k(i)^{\mathrm{syn}}(-)$ on commutative rings is left Kan extended from smooth $\mathbb{Z}$-algebras. The functor $\mathbb{Z}\,/p^k(i)^{\mathrm{syn}}(-)$ is left Kan extended from smooth $\mathbb{Z}$-algebras (Notation 5.7), so this is equivalent to the fact that the functor $\tau^{>i}\, \mathbb{Z}\,/p^k(i)^{\mathrm{syn}}(-)$ on commutative rings is left Kan extended from smooth $\mathbb{Z}$-algebras. By [EM23, Lemma 7.6], it then suffices to prove that the functor $\tau^{>i}\, \mathbb{Z}\,/p^k(i)^{\mathrm{syn}}(-)$ is rigid. To prove this, consider the fibre sequence of $\mathcal{D}(\mathbb{Z}\,/p^k)$-valued functors

$$R\Gamma_{\mathrm{\acute{e}t}}(-, j_!\mu_{p^k}^{\otimes i}) \longrightarrow \mathbb{Z}\,/p^k(i)^{\mathrm{syn}}(-) \longrightarrow \mathbb{Z}\,/p^k(i)^{\mathrm{BMS}}(-)$$

on commutative rings ([BL22, Remark 8.4.4]). By rigidity for étale cohomology ([Gab94], see also [BM21, Corollary 1.18 (1)]), the first term of this fibre sequence is rigid. The desired result is then a consequence of Theorem 2.27. $\qquad\square$

Corollary 7.8. *Let p be a prime number, and $k \geq 1$ be an integer. Then for every integer $i \geq 0$, the functor $\tau^{\leq i}\, \mathbb{Z}\,/p^k(i)^{\mathrm{mot}}$, from local rings to the derived category $\mathcal{D}(\mathbb{Z}\,/p^k)$, is left Kan extended from local essentially smooth $\mathbb{Z}$-algebras.*

Proof. This is a consequence of Proposition 7.7. $\qquad\square$

Lemma 7.9. *Let R be a local ring, p be a prime number, and $k \geq 1$ be an integer. Then for every integer $i \geq 0$, the natural map of abelian groups*

$$\mathrm{H}_{\mathrm{mot}}^i(R, \mathbb{Z}(i)) \longrightarrow \mathrm{H}_{\mathrm{mot}}^i(R, \mathbb{Z}\,/p^k(i))$$

is surjective.

Proof. Let $P \to R$ be a henselian surjection, where P is a local ind-smooth $\mathbb{Z}$-algebra. By Corollary 7.8, the functor $\tau^{\leq i}\, \mathbb{Z}\,/p^k(i)^{\mathrm{mot}}$ is left Kan extended on local rings from local essentially smooth $\mathbb{Z}$-algebras, so the natural map of abelian groups

$$\mathrm{H}_{\mathrm{mot}}^i(P, \mathbb{Z}\,/p^k(i)) \longrightarrow \mathrm{H}_{\mathrm{mot}}^i(R, \mathbb{Z}\,/p^k(i))$$

is surjective. That is, the right vertical map in the commutative diagram of abelian groups

$$\begin{array}{ccc} \mathrm{H}_{\mathrm{mot}}^i(P, \mathbb{Z}(i)) & \longrightarrow & \mathrm{H}_{\mathrm{mot}}^i(P, \mathbb{Z}\,/p^k(i)) \\ \downarrow & & \downarrow \\ \mathrm{H}_{\mathrm{mot}}^i(R, \mathbb{Z}(i)) & \longrightarrow & \mathrm{H}_{\mathrm{mot}}^i(R, \mathbb{Z}\,/p^k(i)) \end{array}$$

is surjective. To prove that the bottom horizontal map is surjective, it thus suffices to prove that the top vertical map is surjective. The local ring P is a filtered colimit of local essentially smooth $\mathbb{Z}$-algebras, so it suffices to prove that this top vertical map is surjective for local essentially smooth $\mathbb{Z}$-algebras. To prove this, it suffices to prove that the motivic complex $\mathbb{Z}(i)^{\mathrm{mot}}(-)$ is zero in degree $i+1$ on local essentially smooth $\mathbb{Z}$-algebras. The classical motivic complex $\mathbb{Z}(i)^{\mathrm{cla}}(-)$ is Zariski-locally in degrees at most i ([Gei04, Corollary 4.4]), so this is a consequence of Theorem 6.3. □

Corollary 7.10. *Let R be a local ring. Then for every integer $i \geq 1$, the motivic cohomology group* $\mathrm{H}^{i+1}_{\mathrm{mot}}(R, \mathbb{Z}(i))$ *is zero. If the local ring R is moreover henselian, then the motivic cohomology group* $\mathrm{H}^1_{\mathrm{mot}}(R, \mathbb{Z}(0))$ *is zero.*

Proof. By Lemma 7.9 and the short exact sequence of abelian groups

$$0 \longrightarrow \mathrm{H}^i_{\mathrm{mot}}(R, \mathbb{Z}(i))/p \longrightarrow \mathrm{H}^i_{\mathrm{mot}}(R, \mathbb{F}_p(i)) \longrightarrow \mathrm{H}^{i+1}_{\mathrm{mot}}(R, \mathbb{Z}(i))[p] \longrightarrow 0$$

for every prime number p and every integer $i \geq 0$, the abelian group $\mathrm{H}^{i+1}_{\mathrm{mot}}(R, \mathbb{Z}(i))$ is torsionfree. By Proposition 7.4 if $i \geq 1$, and by Remark 7.5 if $i = 0$ and R is henselian, it is also torsion, so it is zero. □

Theorem 7.11. *For every integer $i \geq 0$, the functor $\tau^{\leq i} \mathbb{Z}(i)^{\mathrm{mot}}$, from local rings to the derived category $\mathcal{D}(\mathbb{Z})$, is left Kan extended from local essentially smooth $\mathbb{Z}$-algebras.*

Proof. It suffices to prove the result rationally, and modulo p for every prime number p. The result rationally is Corollary 7.6. Let p be a prime number. For every local ring R, the natural map of abelian groups

$$\mathrm{H}^i_{\mathrm{mot}}(R, \mathbb{Z}(i)) \longrightarrow \mathrm{H}^i_{\mathrm{mot}}(R, \mathbb{F}_p(i))$$

is surjective by Lemma 7.9, so the natural map

$$\left(\tau^{\leq i} \mathbb{Z}(i)^{\mathrm{mot}}(R) \right)/p \longrightarrow \tau^{\leq i} \mathbb{F}_p(i)^{\mathrm{mot}}(R)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{F}_p)$. The result modulo p is then Corollary 7.8. □

Note that the proof of Theorem 7.11 is similar to the proof of Elmanto–Morrow in equicharacteristic. The following consequence, however, uses the comparison to classical motivic cohomology Theorem 6.3. The proof of the latter, in any characteristic, is somehow simpler than the proof of Elmanto–Morrow's (stronger) comparison result to classical motivic cohomology: in particular, it does not use a presentation lemma, or the projective bundle formula. The proof of Corollary 7.12 then provides an alternative argument to the proof of [EM23, Theorem 7.7].

Corollary 7.12 (Comparison to lisse motivic cohomology). *Let R be a local ring. Then for every integer $i \geq 0$, the lisse-motivic comparison map induces a natural equivalence*

$$\mathbb{Z}(i)^{\mathrm{lisse}}(R) \xrightarrow{\sim} \tau^{\leq i} \mathbb{Z}(i)^{\mathrm{mot}}(R)$$

in the derived category $\mathcal{D}(\mathbb{Z})$.

Proof. The classical motivic complex $\mathbb{Z}(i)^{\mathrm{cla}}(-)$ is Zariski-locally in degrees at most i ([Gei04, Corollary 4.4]). The result is then a consequence of Theorems 6.3 and 7.11. □

In the rest of this section, we restrict our attention to henselian local rings, in order to describe the motivic cohomology group $\mathrm{H}^2_{\mathrm{mot}}(-, \mathbb{Z}(1))$.

Lemma 7.13. *Let R be a henselian local ring, and p be a prime number. Then for any integers $i \geq 0$ and $k \geq 1$, the motivic cohomology group $\mathrm{H}^{i+1}_{\mathrm{mot}}(R, \mathbb{Z}/p^k(i))$ is zero.*

Proof. By Theorem 5.10, the motivic cohomology group $\mathrm{H}^{i+1}_{\mathrm{mot}}(R, \mathbb{Z}/p^k(i))$ is naturally identified with the kernel of the natural map of abelian groups

$$\mathrm{H}^{i+1}(\mathbb{Z}/p^k(i)^{\mathrm{syn}}(R)) \longrightarrow \mathrm{H}^{i+1}((L_{\mathrm{cdh}}\tau^{>i}\mathbb{Z}/p^k(i)^{\mathrm{syn}})(R))$$

for every commutative ring R. If R is henselian local, let $\mathfrak{m}$ be its maximal ideal, and consider the natural commutative diagram

$$
\begin{array}{ccc}
\mathrm{H}^{i+1}(\mathbb{Z}/p^k(i)^{\mathrm{syn}}(R)) & \longrightarrow & \mathrm{H}^{i+1}((L_{\mathrm{cdh}}\tau^{>i}\mathbb{Z}/p^k(i)^{\mathrm{syn}})(R)) \\
\downarrow & & \downarrow \\
\mathrm{H}^{i+1}(\mathbb{Z}/p^k(i)^{\mathrm{syn}}(R/\mathfrak{m})) & \longrightarrow & \mathrm{H}^{i+1}((L_{\mathrm{cdh}}\tau^{>i}\mathbb{Z}/p^k(i)^{\mathrm{syn}})(R/\mathfrak{m}))
\end{array}
$$

of abelian groups. The functor $\tau^{>i}\mathbb{Z}/p^k(i)^{\mathrm{syn}}$ is rigid (proof of Proposition 7.7), so the left vertical map is an isomorphism. The field $R/\mathfrak{m}$ is a local ring for the cdh topology, so the bottom horizontal map is an isomorphism. In particular, the top horizontal map is injective. $\qquad\square$

Proposition 7.14. *Let R be a henselian local ring. Then for every integer $i \geq 1$, the motivic cohomology group $\mathrm{H}^{i+2}_{\mathrm{mot}}(R, \mathbb{Z}(i))$ is zero.*

Proof. By Lemma 7.13 and the short exact sequence of abelian groups

$$0 \longrightarrow \mathrm{H}^{i+1}_{\mathrm{mot}}(R, \mathbb{Z}(i))/p \longrightarrow \mathrm{H}^{i+1}_{\mathrm{mot}}(R, \mathbb{F}_p(i)) \longrightarrow \mathrm{H}^{i+2}_{\mathrm{mot}}(R, \mathbb{Z}(i))[p] \longrightarrow 0$$

for every prime number p, the abelian group $\mathrm{H}^{i+2}_{\mathrm{mot}}(R, \mathbb{Z}(i))$ is torsionfree. By Proposition 7.4 if $i \geq 2$, and by Remark 7.5 if $i = 1$, it is also torsion, so it is zero. $\qquad\square$

The following example is a consequence of Example 3.9, Corollary 7.12, and Proposition 7.14.

Example 7.15. For every qcqs scheme X, the natural map

$$R\Gamma_{\mathrm{Nis}}(X, \mathbb{G}_m)[-1] \longrightarrow \mathbb{Z}(1)^{\mathrm{mot}}(X),$$

defined as the Nisnevich sheafification of the lisse-motivic comparison map (see also Definition 8.1), is an isomorphism in degrees at most three. That is, the motivic complex $\mathbb{Z}(1)^{\mathrm{mot}}(X)$ vanishes in degrees at most zero, and there are natural isomorphisms of abelian groups

$$\mathrm{H}^1_{\mathrm{mot}}(X, \mathbb{Z}(1)) \cong \mathcal{O}(X)^\times, \quad \mathrm{H}^2_{\mathrm{mot}}(X, \mathbb{Z}(1)) \cong \mathrm{Pic}(X), \quad \mathrm{H}^3_{\mathrm{mot}}(X, \mathbb{Z}(1)) \cong \mathrm{H}^2_{\mathrm{Nis}}(X, \mathbb{G}_m).$$

7.2 Comparison to Milnor K-theory

In this subsection, we construct, for every integer $i \geq 1$, a symbol map

$$\mathrm{K}^{\mathrm{M}}_i(A) \longrightarrow \mathrm{H}^i_{\mathrm{mot}}(A, \mathbb{Z}(i))$$

for local rings A (Definition 7.18), through which we compare the Milnor K-groups to motivic cohomology (Theorem 7.21). Note that the arguments in Lemmas 7.17 and 7.19 are very similar to that of [EM23, Section 7], except for the Gersten injectivity for classical motivic cohomology, which is unknown integrally in mixed characteristic.

For every commutative ring A, the lisse-motivic comparison map (Definition 7.1 and Example 3.9) induces on H^1 a natural isomorphism of abelian groups

$$A^\times \xrightarrow{\cong} \mathrm{H}^1_{\mathrm{mot}}(A, \mathbb{Z}(1)).$$

By multiplicativity of the motivic complexes, this induces, for every integer $i \geq 0$, a symbol map of abelian groups

$$(A^\times)^{\otimes i} \longrightarrow \mathrm{H}^i_{\mathrm{mot}}(A, \mathbb{Z}(i)).$$

Lemma 7.16. *For every local essentially smooth $\mathbb{Z}$-algebra A, there is a natural isomorphism*

$$\mathrm{H}^2_{\mathrm{mot}}(A, \mathbb{Z}(2)) \cong \mathrm{K}_2(A)$$

of abelian groups.

Proof. By Theorem 6.3, the classical-motivic comparison map

$$\mathrm{H}^2_{\mathrm{cla}}(A, \mathbb{Z}(2)) \longrightarrow \mathrm{H}^2_{\mathrm{mot}}(A, \mathbb{Z}(2))$$

is an isomorphism of abelian groups. The result is then a consequence of the Atiyah–Hirzebruch spectral sequence for classical motivic cohomology (Remark 3.3), where we use that the classical motivic complex $\mathbb{Z}(1)^{\mathrm{cla}}(A) \in \mathcal{D}(\mathbb{Z})$ is concentrated in degree one (Example 3.5 and [Gei04, Corollary 4.4]). $\square$

Lemma 7.17. *Let A be a local ring. Then for every integer $i \geq 0$, the natural map of abelian groups*

$$(A^{\times})^{\otimes i} \longrightarrow \mathrm{H}^i_{\mathrm{mot}}(A, \mathbb{Z}(i))$$

induced by the lisse-motivic comparison map factors through the Milnor K-group $\mathrm{K}^{\mathrm{M}}_i(A)$.

Proof. By definition of the Milnor K-groups, it suffices to prove that the symbol map respects the Steinberg relations. Let $a \in A$ be an element such that a and $1 - a$ are units in A. By multiplicativity of the motivic complexes, it suffices to consider the case $i = 2$ and to prove that $a \otimes (1 - a)$ is sent to zero via the symbol map. Let $\mathbb{Z}[t] \to A$ be the ring homomorphism sending t to a, and let $\mathfrak{p} \subset \mathbb{Z}[t]$ be the prime ideal defined as the inverse image of the maximal ideal of A via this ring homomorphism. By naturality of the symbol map, the diagram of abelian groups

$$
\begin{array}{ccc}
(\mathbb{Z}[t]_{\mathfrak{p}})^{\times} \otimes_{\mathbb{Z}} (\mathbb{Z}[t]_{\mathfrak{p}})^{\times} & \longrightarrow & \mathrm{H}^2_{\mathrm{mot}}(\mathbb{Z}[t]_{\mathfrak{p}}, \mathbb{Z}(2)) \\
\downarrow & & \downarrow \\
A^{\times} \otimes_{\mathbb{Z}} A^{\times} & \longrightarrow & \mathrm{H}^2_{\mathrm{mot}}(A, \mathbb{Z}(2))
\end{array}
$$

is commutative. It then suffices to prove that the top horizontal arrow of this diagram sends $t \otimes (1-t)$ to zero. The local ring $\mathbb{Z}[t]_{\mathfrak{p}}$ is essentially smooth over $\mathbb{Z}$, so the right vertical map of the commutative diagram of abelian groups

$$
\begin{array}{ccc}
(\mathbb{Z}[t]_{\mathfrak{p}})^{\times} \otimes_{\mathbb{Z}} (\mathbb{Z}[t]_{\mathfrak{p}})^{\times} & \longrightarrow & \mathrm{H}^2_{\mathrm{mot}}(\mathbb{Z}[t]_{\mathfrak{p}}, \mathbb{Z}(2)) \\
\downarrow & & \downarrow \\
(\mathrm{Frac}(\mathbb{Z}[t]_{\mathfrak{p}}))^{\times} \otimes_{\mathbb{Z}} (\mathrm{Frac}(\mathbb{Z}[t]_{\mathfrak{p}}))^{\times} & \longrightarrow & \mathrm{H}^2_{\mathrm{mot}}(\mathrm{Frac}(\mathbb{Z}[t]_{\mathfrak{p}}), \mathbb{Z}(2))
\end{array}
$$

is injective. Indeed, by Lemma 7.16 this is equivalent to the fact that the natural map

$$\mathrm{K}_2(A) \longrightarrow \mathrm{K}_2(\mathrm{Frac}(A))$$

is injective, and this is the Gersten injectivity for K_2 ([GL87, Corollary 6] and [DS75, Theorem 2.2]). It then suffices to prove that the bottom horizontal map of this diagram sends $t \otimes (1 - t)$ to zero. This is a consequence of the fact that the symbol map to classical motivic cohomology respects the Steinberg relation for fields ([NS89], see also [Tot92]). $\square$

Definition 7.18 (Symbol map). Let A be a local ring. For every integer $i \geq 0$, the *symbol map*

$$\mathrm{K}^{\mathrm{M}}_i(A) \longrightarrow \mathrm{H}^i_{\mathrm{mot}}(A, \mathbb{Z}(i))$$

is the natural map of abelian groups of Lemma 7.17.

Following [Ker10], for A a local ring and $i \geq 0$ an integer, we denote by $\widehat{\mathrm{K}}^{\mathrm{M}}_i(A)$ the ith *improved Milnor K-group* of A.

Lemma 7.19. *Let A be a local ring. Then for every integer $i \geq 0$, the symbol map*

$$\mathrm{K}_i^{\mathrm{M}}(A) \longrightarrow \mathrm{H}_{\mathrm{mot}}^i(A, \mathbb{Z}(i))$$

factors through the improved Milnor K-group $\widehat{\mathrm{K}}_i^{\mathrm{M}}(A)$.

Proof. Let $M_i \geq 1$ be the integer defined in [Ker10]. If the residue field of the local ring A has at least M_i elements, then the natural map

$$\mathrm{K}_i^{\mathrm{M}}(A) \longrightarrow \widehat{\mathrm{K}}_i^{\mathrm{M}}(A)$$

is an isomorphism of abelian groups ([Ker10, Proposition 10 (5)]). Assume now that the residue field of the local ring A has less than M_i elements. We want to prove that the symbol map

$$\mathrm{K}_i^{\mathrm{M}}(A) \longrightarrow \mathrm{H}_{\mathrm{mot}}^i(A, \mathbb{Z}(i))$$

factors through the surjective map $\mathrm{K}_i^{\mathrm{M}}(A) \to \widehat{\mathrm{K}}_i^{\mathrm{M}}(A)$, *i.e.*, that every element of the abelian group $\ker(\mathrm{K}_i^{\mathrm{M}}(A) \to \widehat{\mathrm{K}}_i^{\mathrm{M}}(A))$ is sent to zero by the previous symbol map. Let $\mathfrak{m}$ be the maximal ideal of the local ring A, and p be its residue characteristic. The residue field $A/\mathfrak{m}$ of the local ring A is isomorphic to a finite extension $\mathbb{F}_q$ of $\mathbb{F}_p$. Let $\ell \geq 1$ be an integer which is coprime to the degree of this extension, and such that $p^\ell \geq M_i$. As a tensor product of finite field extensions of coprime degree, the commutative ring $\mathbb{F}_q \otimes_{\mathbb{F}_p} \mathbb{F}_{p^\ell}$ is a field. Let V be the finite étale extension of $\mathbb{Z}_{(p)}$ corresponding to the field extension $\mathbb{F}_{p^\ell}$ of $\mathbb{F}_p$. The commutative ring $A' := A \otimes_{\mathbb{Z}_{(p)}} V$ is finite over the local ring A, and the quotient $A'/\mathfrak{m}A'$ is a field, so the commutative ring A' is a local A-algebra, whose residue field has at least M_i elements.

Let $P_\bullet \to A$ be a simplicial resolution of the local ring A where each term P_m is a local ind-smooth $\mathbb{Z}$-algebra, and each face map $P_{m+1} \to P_m$ is a henselian surjection. By Theorem 7.11, there is then a natural equivalence

$$\operatorname*{colim}_m \tau^{\leq i} \mathbb{Z}(i)^{\mathrm{mot}}(P_m) \xrightarrow{\sim} \tau^{\leq i} \mathbb{Z}(i)^{\mathrm{mot}}(A)$$

in the derived category $\mathcal{D}(\mathbb{Z})$. In particular, this equivalence induces a natural isomorphism

$$\operatorname{coeq}\bigl(\mathrm{H}_{\mathrm{mot}}^i(P_1, \mathbb{Z}(i)) \rightrightarrows \mathrm{H}_{\mathrm{mot}}^i(P_0, \mathbb{Z}(i))\bigr) \xrightarrow{\cong} \mathrm{H}_{\mathrm{mot}}^i(A, \mathbb{Z}(i))$$

of abelian groups, where the motivic cohomology groups in the left term are naturally identified with classical motivic cohomology groups by Theorem 6.3. Similarly, $P_\bullet \otimes_{\mathbb{Z}_{(p)}} V \to A'$ is a simplicial resolution of the local ring A' where each term $P_m \otimes_{\mathbb{Z}_{(p)}} V$ is an ind-smooth V-algebra, and each face map $P_{m+1} \otimes_{\mathbb{Z}_{(p)}} V \to P_m \otimes_{\mathbb{Z}_{(p)}} V$ is a henselian surjection, so there is a natural isomorphism

$$\operatorname{coeq}\bigl(\mathrm{H}_{\mathrm{mot}}^i(P_1 \otimes_{\mathbb{Z}_{(p)}} V, \mathbb{Z}(i)) \rightrightarrows \mathrm{H}_{\mathrm{mot}}^i(P_0 \otimes_{\mathbb{Z}_{(p)}} V, \mathbb{Z}(i))\bigr) \xrightarrow{\cong} \mathrm{H}_{\mathrm{mot}}^i(A', \mathbb{Z}(i))$$

of abelian groups, where the motivic cohomology groups of the left term are naturally identified with classical motivic cohomology groups. Classical motivic cohomology of smooth schemes over a mixed characteristic Dedekind domain admits functorial transfer maps along finite étale morphisms, so the previous two isomorphisms induce a transfer map

$$N_\ell : \mathrm{H}_{\mathrm{mot}}^i(A', \mathbb{Z}(i)) \longrightarrow \mathrm{H}_{\mathrm{mot}}^i(A, \mathbb{Z}(i))$$

such that pre-composition with the natural map $\mathrm{H}_{\mathrm{mot}}^i(A, \mathbb{Z}(i)) \to \mathrm{H}_{\mathrm{mot}}^i(A', \mathbb{Z}(i))$ is multiplication by ℓ. In particular, the kernel of the natural map $\mathrm{H}_{\mathrm{mot}}^i(A, \mathbb{Z}(i)) \to \mathrm{H}_{\mathrm{mot}}^i(A', \mathbb{Z}(i))$ is ℓ-torsion.

Consider the commutative diagram

$$
\begin{array}{ccc}
\mathrm{K}_i^{\mathrm{M}}(A) & \longrightarrow & \mathrm{K}_i^{\mathrm{M}}(A') \\
\downarrow & & \downarrow \\
\mathrm{H}_{\mathrm{mot}}^i(A, \mathbb{Z}(i)) & \longrightarrow & \mathrm{H}_{\mathrm{mot}}^i(A', \mathbb{Z}(i))
\end{array}
$$

of abelian groups, and let x be an element of the abelian group $\ker(\mathrm{K}_i^{\mathrm{M}}(A) \to \widehat{\mathrm{K}}_i^{\mathrm{M}}(A))$. The residue field of the local ring A' has at least M_i elements, so the natural map $\mathrm{K}_i^{\mathrm{M}}(A') \to \widehat{\mathrm{K}}_i^{\mathrm{M}}(A')$ is an isomorphism ([Ker10, Proposition 10 (5)]), and x is sent to zero by the top horizontal map. In particular, the image of x by the left vertical map is in the kernel of the bottom horizontal map, and is thus ℓ-torsion by the previous paragraph. Let $\ell' \geq 1$ be an integer which is coprime to ℓ and to the degree of $\mathbb{F}_q$ over $\mathbb{F}_p$, and such that $p^{\ell'} \geq M_i$. The previous argument for this integer ℓ' implies that the image of x by the left vertical map is also ℓ'-torsion, hence it is zero. $\qquad\square$

Conjecture 7.20. Let A be a local ring. Then for every integer $i \geq 0$, the natural map

$$\widehat{\mathrm{K}}_i^{\mathrm{M}}(A) \longrightarrow \mathrm{H}_{\mathrm{mot}}^i(A, \mathbb{Z}(i))$$

induced by Lemma 7.19 is an isomorphism of abelian groups.

The previous conjecture was proved by Elmanto–Morrow for equicharacteristic local rings ([EM23, Theorem 7.12]). Their proof uses as an input the analogous result in the smooth case for classical motivic cohomology, which is unknown in mixed characteristic (see Remark 7.22).

Theorem 7.21 (Singular Nesterenko–Suslin isomorphism with finite coefficients). *Let A be a henselian local ring. Then for any integers $i \geq 0$ and $n \geq 1$, the natural map*

$$\widehat{\mathrm{K}}_i^{\mathrm{M}}(A)/n \longrightarrow \mathrm{H}_{\mathrm{mot}}^i(A, \mathbb{Z}(i))/n$$

is an isomorphism of abelian groups.

Proof. If the local ring A contains a field, then the natural map

$$\widehat{\mathrm{K}}_i^{\mathrm{M}}(A) \longrightarrow \mathrm{H}_{\mathrm{mot}}^i(A, \mathbb{Z}(i))$$

is an isomorphism of abelian groups ([EM23, Theorem 7.12]). Otherwise, A is a henselian local ring of mixed characteristic $(0, p)$ for some prime number p. In particular, the local ring A is p-henselian. If p does not divide the integer n, then consider the commutative diagram

$$
\begin{array}{ccc}
\widehat{\mathrm{K}}_i^{\mathrm{M}}(A)/n & \longrightarrow & \mathrm{H}_{\mathrm{mot}}^i(A, \mathbb{Z}(i))/n \\
\downarrow & & \downarrow \\
\widehat{\mathrm{K}}_i^{\mathrm{M}}(A/p)/n & \longrightarrow & \mathrm{H}_{\mathrm{mot}}^i(A/p, \mathbb{Z}(i))/n
\end{array}
$$

of abelian groups. The left vertical map is an isomorphism by [Ker10, Proposition 10 (7)]. By Lemma 7.9 and Corollary 5.6, the right vertical map is naturally identified with the natural map of abelian groups

$$\mathrm{H}_{\mathrm{\acute{e}t}}^i(A, \mu_n^{\otimes i}) \longrightarrow \mathrm{H}_{\mathrm{\acute{e}t}}^i(A/p, \mu_n^{\otimes i}),$$

which is an isomorphism by rigidity of étale cohomology ([Gab94], see also [BM21, Corollary 1.18 (1)]). The local ring A/p is an $\mathbb{F}_p$-algebra, so the bottom horizontal map is an isomorphism ([EM23, Theorem 7.12]), and the result is true in this case. It then suffices to prove that for every integer $k \geq 1$, the natural map

$$\widehat{\mathrm{K}}_i^{\mathrm{M}}(A)/p^k \longrightarrow \mathrm{H}_{\mathrm{mot}}^i(A, \mathbb{Z}(i))/p^k$$

is an isomorphism of abelian groups. By Lemma 7.9 and Corollary 5.11, the natural map

$$\mathrm{H}_{\mathrm{mot}}^i(A, \mathbb{Z}(i))/p^k \longrightarrow \mathrm{H}^i(\mathbb{Z}/p^k(i)^{\mathrm{BMS}}(A))$$

is an isomorphism of abelian groups. By [LM23, Theorem 3.1], the composite map

$$\widehat{\mathrm{K}}_i^{\mathrm{M}}(A)/p^k \longrightarrow \mathrm{H}_{\mathrm{mot}}^i(A, \mathbb{Z}(i))/p^k \longrightarrow \mathrm{H}^i(\mathbb{Z}/p^k(i)^{\mathrm{BMS}}(A))$$

is an isomorphism of abelian groups, hence the desired result. $\qquad\square$

Remark 7.22. If Conjecture 7.20 is true for local ind-smooth $\mathbb{Z}$-algebras, then the left Kan extension properties [LM23, Proposition 1.17] and Theorem 7.11 imply that Conjecture 7.20 is true for all local rings. See the proof of [EM23, Theorem 7.12] for more details.

Remark 7.23. Let A be a local essentially smooth $\mathbb{Z}$-algebra, $i \geq 0$ be an integer, and consider the commutative diagram

$$
\begin{array}{ccc}
\widehat{\mathrm{K}}_i^{\mathrm{M}}(A) & \longrightarrow & \mathrm{H}_{\mathrm{mot}}^i(A, \mathbb{Z}(i)) \\
\downarrow & & \downarrow \\
\widehat{\mathrm{K}}_i^{\mathrm{M}}(\mathrm{Frac}(A)) & \longrightarrow & \mathrm{H}_{\mathrm{mot}}^i(\mathrm{Frac}(A), \mathbb{Z}(i))
\end{array}
$$

of abelian groups. The bottom horizontal map is an isomorphism by the Nesterenko–Suslin isomorphism for fields ([NS89], see also [EM23, Theorem 7.12]). The left vertical being injective then implies that the top horizontal map is injective. That is, the Gersten injectivity conjecture for the improved Milnor K-groups would imply the injectivity part of Conjecture 7.20. Knowing the Gersten injectivity conjecture for the motivic cohomology group $\mathrm{H}_{\mathrm{mot}}^i(-, \mathbb{Z}(i))$ would imply that these two facts are equivalent. See [Lüd22] for related results on the Gersten conjecture for improved Milnor K-groups.

8 THE PROJECTIVE BUNDLE FORMULA

In this section, we prove that the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$ satisfy the projective bundle formula (Theorem 8.16) and regular blowup excision (Theorem 8.15). This implies in particular that the presheaves $\mathbb{Z}(i)^{\mathrm{mot}}$ fit within the recent theory of non-$\mathbb{A}^1$-invariant motives of Annala–Iwasa [AI23] and Annala–Hoyois–Iwasa [AHI23, AHI24].

8.1 First Chern classes

In this subsection, we construct the motivic first Chern class (Definition 8.1) in order to formulate the projective bundle formula (Theorem 8.16).

Definition 8.1 (Motivic first Chern class)**.** Let X be a qcqs derived scheme. The *motivic first Chern class* is the natural map
$$c_1^{\mathrm{mot}} : R\Gamma_{\mathrm{Nis}}(X, \mathbb{G}_m)[-1] \longrightarrow \mathbb{Z}(1)^{\mathrm{mot}}(X),$$
in the derived category $\mathcal{D}(\mathbb{Z})$, defined as the Nisnevich sheafification of the natural map of presheaves
$$\left(\tau^{\leq 1} R\Gamma_{\mathrm{Zar}}(-, \mathbb{G}_m)\right)[-1] \longrightarrow \mathbb{Z}(1)^{\mathrm{mot}}(-)$$
induced by Definition 7.1 and Example 3.9. We also denote by
$$c_1^{\mathrm{mot}} : \mathrm{Pic}(X) \longrightarrow \mathrm{H}_{\mathrm{mot}}^2(X, \mathbb{Z}(1))$$
the map induced on H^2.

Remark 8.2. The motivic first Chern class of Definition 8.1 is uniquely determined by its naturality, and the fact that it is given by the map of Definition 3.22 on smooth $\mathbb{Z}$-schemes.

For every qcqs scheme X, the line bundle $\mathcal{O}(1) \in \mathrm{Pic}(\mathbb{P}_X^1)$, via the multiplicative structure of the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$, induces, for every integer $i \in \mathbb{Z}$, a natural map
$$c_1^{\mathrm{mot}}(\mathcal{O}(1)) : \mathbb{Z}(i-1)^{\mathrm{mot}}(\mathbb{P}_X^1)[-2] \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(\mathbb{P}_X^1)$$
in the derived category $\mathcal{D}(\mathbb{Z})$. If $\pi : \mathbb{P}_X^1 \to X$ is the canonical projection map, this in turn induces a natural map
$$\pi^* \oplus c_1^{\mathrm{mot}}(\mathcal{O}(1))\pi^* : \mathbb{Z}(i)^{\mathrm{mot}}(X) \oplus \mathbb{Z}(i-1)^{\mathrm{mot}}(X)[-2] \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(\mathbb{P}_X^1) \tag{2}$$
in the derived category $\mathcal{D}(\mathbb{Z})$. The aim of the following section is to prove that this map is an equivalence (Theorem 8.7). To prove such an equivalence, we will need compatibilities with other first Chern classes.

Construction 8.3 ($\mathbb{P}^1$-bundle formula for additive invariants)**.** Following [EM23, Section 5.1], every additive invariant E of $\mathbb{Z}$-linear ∞-categories in the sense of [HSS17, Definition 5.11] admits a natural first Chern class, inducing a natural map of spectra
$$\pi^* \oplus (1 - c_1)(\mathcal{O}(-1))\pi^* : E(X) \oplus E(X) \longrightarrow E(\mathbb{P}_X^1).$$
For every additive invariant of $\mathbb{Z}$-linear ∞-categories, this map is an equivalence ([EM23, Lemma 5.6]).

Remark 8.4 (Compatibility with filtrations)**.** If the additive invariant E of $\mathbb{Z}$-linear ∞-categories, seen as a presheaf of spectra on qcqs schemes, admits a multiplicative filtered refinement $\mathrm{Fil}^\star E$ which is a multiplicative filtered module over the lisse motivic filtration $\mathrm{Fil}_{\mathrm{lisse}}^\star \mathrm{K}^{\mathrm{conn}}$ (Definition 3.7),[8] then this map has a natural filtered refinement
$$\pi^* \oplus (1 - c_1)(\mathcal{O}(-1))\pi^* : \mathrm{Fil}^\star E(X) \oplus \mathrm{Fil}^{\star-1} E(X) \longrightarrow \mathrm{Fil}^\star E(\mathbb{P}_X^1)$$
by [EM23, Construction 5.11 and Lemma 5.12]. If E is algebraic K-theory, equipped with the motivic filtration $\mathrm{Fil}_{\mathrm{mot}}^\star \mathrm{K}$ (Definition 3.19), the argument of [EM23, Lemma 5.12] and Remark 8.2 imply that this map recovers, up to a shift, the map (2) on graded pieces.

---

[8]The lisse motivic filtration $\mathrm{Fil}_{\mathrm{lisse}}^\star \mathrm{K}^{\mathrm{conn}}$ is usually defined only on affine schemes. Here the argument works if $\mathrm{Fil}^\star E$ is a Zariski sheaf of filtered spectra on qcqs schemes, and if its restriction to affine schemes is a multiplicative filtered module over the lisse motivic filtration $\mathrm{Fil}_{\mathrm{lisse}}^\star \mathrm{K}^{\mathrm{conn}}$.

Example 8.5 (Compatibility with cdh-local motivic cohomology). The multiplicative cdh-local filtration $\mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{KH}$ (Definition 3.10) is naturally a module over the multiplicative filtration $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}$ (*e.g.*, because cdh sheafification preserves multiplicative structures), so the first Chern class for the cdh-local motivic complexes of [BEM] is compatible with the motivic first Chern class of Definition 8.1.

Example 8.6 (Compatibility with syntomic cohomology). Let X be a qcqs scheme, and p be a prime number. By [BEM], the syntomic first Chern class of [BL22, Section 7] is compatible with the motivic first Chern class of Definition 8.1 via the motivic-syntomic comparison map (Construction 5.8). Note here that the motivic-syntomic comparison map can be seen as the map induced on graded pieces from a multiplicative map of filtered spectra

$$\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X;\mathbb{Z}_p) \longrightarrow \mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}^{\mathrm{Sel}}(X;\mathbb{Z}_p),$$

where the target is the filtration on p-completed Selmer K-theory ([BL22, Remark 8.4.3]). These motivic and syntomic first Chern classes then coincide with the first Chern classes coming from the additive invariants $\mathrm{K}(-;\mathbb{Z}_p)$ and $\mathrm{K}^{\mathrm{Sel}}(-;\mathbb{Z}_p)$ (Remark 8.4).

8.2 $\mathbb{P}^1$-bundle formula

In this subsection, we prove that the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$ satisfy the $\mathbb{P}^1$-bundle formula on qcqs schemes (Theorem 8.7). Note that the $\mathbb{P}^1$-bundle formula is unknown for the cdh-local motivic complexes $\mathbb{Z}(i)^{\mathrm{cdh}}$ on general qcqs schemes.[9] The cartesian square of Remark 3.20 thus cannot be used directly to prove the $\mathbb{P}^1$-bundle formula for the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$, as was done by Elmanto–Morrow in equicharacteristic ([EM23, Section 5]). Instead, we use in a crucial way our main result on p-adic motivic cohomology (Theorem 5.10), and a degeneration argument using Selmer K-theory.

Theorem 8.7 ($\mathbb{P}^1$-bundle formula). *Let X be a qcqs scheme, and $\pi : \mathbb{P}^1_X \to X$ be the canonical projection map. Then for every integer $i \in \mathbb{Z}$, the natural map*

$$\pi^* \oplus c_1^{\mathrm{mot}}(\mathcal{O}(1))\pi^* : \mathbb{Z}(i)^{\mathrm{mot}}(X) \oplus \mathbb{Z}(i-1)^{\mathrm{mot}}(X)[-2] \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(\mathbb{P}^1_X)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$.

Using Theorem 5.10, the proof of Theorem 8.7 will reduce to the proof of a similar equivalence for the cdh sheaves $\big(L_{\mathrm{cdh}}\tau^{>i}\,\mathbb{F}_p(i)^{\mathrm{syn}}\big)(-)$ (Proposition 8.14). Most of this section is devoted to the study of these cdh sheaves.

Lemma 8.8. *Let p be a prime number. Then for any integers $i, j \geq 0$ and $k \geq 1$, the natural sequence*

$$L_{\mathrm{Nis}}\tau^{>j}R\Gamma_{\mathrm{\acute{e}t}}(-, j_!\mu_{p^k}^{\otimes i}) \longrightarrow \big(L_{\mathrm{Nis}}\tau^{>j}\,\mathbb{Z}/p^k(i)^{\mathrm{syn}}\big)(-) \longrightarrow \big(L_{\mathrm{Nis}}\tau^{>j}\,\mathbb{Z}/p^k(i)^{\mathrm{BMS}}\big)(-)$$

is a fibre sequence of $\mathcal{D}(\mathbb{Z}/p^k)$-valued presheaves on qcqs schemes.

Proof. The three presheaves are finitary Nisnevich sheaves, so it suffices to prove the result on henselian local rings ([CM21, Corollary 3.27 and Example 4.31]). Let A be a henselian local ring. By [BL22, Remark 8.4.4], there is a natural fibre sequence

$$R\Gamma_{\mathrm{\acute{e}t}}(A, j_!\mu_{p^k}^{\otimes i}) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{syn}}(A) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{BMS}}(A)$$

in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$, so it suffices to prove that the natural map

$$\mathbb{Z}/p^k(i)^{\mathrm{syn}}(A) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{BMS}}(A)$$

is surjective in degree j. If p is invertible in the henselian valuation ring A, the target of this map is zero. If p is not invertible in A, then the valuation ring A is p-henselian, and this map is an equivalence (Notation 5.7). □

---

[9]More precisely, Bachmann–Elmanto–Morrow proved in [BEM] that if a qcqs scheme X satisfies the condition $\mathrm{Val}(X)$ (Section 10.2), then the cdh-local motivic complexes $\mathbb{Z}(i)^{\mathrm{cdh}}$ satisfy the $\mathbb{P}^1$-bundle formula at X. In particular, the $\mathbb{P}^1$-bundle formula is known for these complexes over a field, and over a mixed characteristic perfectoid valuation ring by the results of [Bou23], but not on general qcqs schemes.

Lemma 8.9. *Let p be a prime number, and V be a rank one henselian valuation ring of mixed characteristic $(0, p)$. Then for any integers $i \geq 0$ and $k \geq 1$, the complex*

$$\left(L_{\mathrm{cdh}} \tau^{>i} R\Gamma_{\text{ét}}(-, j_! \mu_{p^k}^{\otimes i}) \right)(\mathbb{P}_V^1) \in \mathcal{D}(\mathbb{Z}/p^k)$$

is concentrated in degree $i + 1$.[10]

Proof. The presheaf $L_{\mathrm{cdh}} \tau^{>i} R\Gamma_{\text{ét}}(-, j_! \mu_{p^k}^{\otimes i})$ is the cdh sheafification of a presheaf taking values in degrees at least $i + 1$, so it takes values in degrees at least $i + 1$. To prove that the complex

$$\left(L_{\mathrm{cdh}} \tau^{>i} R\Gamma_{\text{ét}}(-, j_! \mu_{p^k}^{\otimes i}) \right)(\mathbb{P}_V^1) \in \mathcal{D}(\mathbb{Z}/p^k)$$

is in degrees at most $i + 1$, consider the fibre sequence

$$\left(L_{\mathrm{cdh}} \tau^{\leq i} R\Gamma_{\text{ét}}(-, j_! \mu_{p^k}^{\otimes i}) \right)(\mathbb{P}_V^1) \longrightarrow R\Gamma_{\text{ét}}(\mathbb{P}_V^1, j_! \mu_{p^k}^{\otimes i}) \longrightarrow \left(L_{\mathrm{cdh}} \tau^{>i} R\Gamma_{\text{ét}}(-, j_! \mu_{p^k}^{\otimes i}) \right)(\mathbb{P}_V^1)$$

in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$, which is a consequence of arc descent for the presheaf $R\Gamma_{\text{ét}}(-, j_! \mu_{p^k}^{\otimes i})$ ([BM21, Theorem 1.8]). The scheme $\mathbb{P}_V^1$ has valuative dimension two, so the complex

$$\left(L_{\mathrm{cdh}} \tau^{\leq i} R\Gamma_{\text{ét}}(-, j_! \mu_{p^k}^{\otimes i}) \right)(\mathbb{P}_V^1) \in \mathcal{D}(\mathbb{Z}/p^k)$$

is in degrees at most $i + 2$ ([EHIK21, Theorem 2.4.15]). By the $\mathbb{P}^1$-bundle formula for the presheaves $R\Gamma_{\text{ét}}(-, j_! \mu_{p^k}^{\otimes i})$ ([BL22, proof of Theorem 9.1.1]), there is a natural equivalence

$$R\Gamma_{\text{ét}}(V, j_! \mu_{p^k}^{\otimes i}) \oplus R\Gamma_{\text{ét}}(V, j_! \mu_{p^k}^{\otimes(i-1)})[-2] \longrightarrow R\Gamma_{\text{ét}}(\mathbb{P}_V^1, j_! \mu_{p^k}^{\otimes i})$$

in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$. The functors $R\Gamma_{\text{ét}}(-, j_! \mu_{p^k}^{\otimes i})$ and $R\Gamma_{\text{ét}}(-, j_! \mu_{p^k}^{\otimes(i-1)})$ are moreover rigid ([Gab94], see also [BM21, Corollary 1.18 (1)]) and the valuation ring V is p-henselian, so the complex $R\Gamma_{\text{ét}}(\mathbb{P}_V^1, j_! \mu_{p^k}^{\otimes i}) \in \mathcal{D}(\mathbb{Z}/p^k)$ is zero. This implies that the complex $\left(L_{\mathrm{cdh}} \tau^{>i} R\Gamma_{\text{ét}}(-, j_! \mu_{p^k}^{\otimes i}) \right)(\mathbb{P}_V^1)$ is naturally identified with the complex

$$\left(L_{\mathrm{cdh}} \tau^{\leq i} R\Gamma_{\text{ét}}(-, j_! \mu_{p^k}^{\otimes i}) \right)(\mathbb{P}_V^1)[1] \in \mathcal{D}(\mathbb{Z}/p^k),$$

and is thus in degrees at most $i + 1$. $\qquad\square$

Following [EHIK21], we say that a $\mathcal{D}(\mathbb{Z})$-valued presheaf on qcqs schemes satisfies *henselian v-excision* if for every henselian valuation ring V and every prime ideal $\mathfrak{p}$ of V, this presheaf sends the bicartesian square of commutative rings

$$\begin{array}{ccc} V & \longrightarrow & V_{\mathfrak{p}} \\ \downarrow & & \downarrow \\ V/\mathfrak{p} & \longrightarrow & V_{\mathfrak{p}}/\mathfrak{p}V_{\mathfrak{p}} \end{array}$$

to a cartesian square. Note that in the previous bicartesian square, all the commutative rings are henselian valuation rings by [EHIK21, Lemma 3.3.5]. The following lemma explains how to use henselian v-excision to prove that a map of cdh sheaves is an equivalence.

Lemma 8.10. *Let S be a qcqs scheme of finite valuative dimension, $\mathcal{C}$ be a ∞-category which is compactly generated by cotruncated objects, and $F, G : \mathrm{Sch}_S^{\mathrm{qcqs,op}} \to \mathcal{C}$ be finitary cdh sheaves satisfying henselian v-excision. Then a map of presheaves $F \to G$ is an equivalence of presheaves if and only if the map $F(V) \to G(V)$ is an equivalence in $\mathcal{C}$ for every henselian valuation ring V of rank at most one with a map $\mathrm{Spec}(V) \to S$.*

---

[10] We will prove, at the end of Proposition 8.12, that this complex is actually zero.

Proof. By [EHIK21, Proposition 3.1.8 (2)], a map $F \to G$ is an equivalence of presheaves if and only if it is an equivalence on henselian valuation rings over S. The presheaves F and G being finitary, this is equivalent to the fact that it is an equivalence on henselian valuation rings of finite rank over S. By induction, and using henselian v-excision, this is in turn equivalent to the fact that it is an equivalence on henselian valuation rings of rank at most one over S. □

Lemma 8.11. *Let p be a prime number. Then for any integers $i \geq 0$ and $k \geq 1$, the $\mathcal{D}(\mathbb{Z}/p^k)$-valued presheaves $\left(L_{\mathrm{cdh}}\tau^{>i}\,\mathbb{Z}/p^k(i)^{\mathrm{syn}}\right)(-)$ and $\left(L_{\mathrm{cdh}}\tau^{>i}\,\mathbb{Z}/p^k(i)^{\mathrm{syn}}\right)(\mathbb{P}^1_-)$ are finitary cdh sheaves on qcqs schemes, and satisfy henselian v-excision.*

Proof. The presheaf $\mathbb{Z}/p^k(i)^{\mathrm{syn}}$ is finitary, and the cdh sheafification of a finitary presheaf is a finitary cdh sheaf (Lemma 4.58), so the presheaf $\left(L_{\mathrm{cdh}}\tau^{>i}\,\mathbb{Z}/p^k(i)^{\mathrm{syn}}\right)(-)$ is a finitary cdh sheaf. Covers in a site are stable under base change, so the presheaf

$$\left(L_{\mathrm{cdh}}\tau^{>i}\,\mathbb{Z}/p^k(i)^{\mathrm{syn}}\right)(\mathbb{P}^1_-)$$

is also a finitary cdh sheaf. Henselian valuation rings are local rings for the cdh topology, and the presheaves $\tau^{>i}R\Gamma_{\mathrm{\acute{e}t}}(-, j_!\mu_{p^k}^{\otimes i})$ and $\tau^{>i}\,\mathbb{Z}/p^k(i)^{\mathrm{BMS}}$ are rigid ([Gab94] and Theorem 2.27), so the presheaves

$$\left(L_{\mathrm{cdh}}\tau^{>i}R\Gamma_{\mathrm{\acute{e}t}}(-, j_!\mu_{p^k}^{\otimes i})\right)(-) \quad \text{and} \quad \left(L_{\mathrm{cdh}}\tau^{>i}\,\mathbb{Z}/p^k(i)^{\mathrm{BMS}}\right)(-)$$

satisfy henselian v-excision. By Lemma 8.8 (applied for $j = i$ and after cdh sheafification), the presheaf $\left(L_{\mathrm{cdh}}\tau^{>i}\,\mathbb{Z}/p^k(i)^{\mathrm{syn}}\right)(-)$ then satisfies henselian v-excision. Finally, the presheaf

$$\left(L_{\mathrm{cdh}}\tau^{>i}\,\mathbb{Z}/p^k(i)^{\mathrm{syn}}\right)(\mathbb{P}^1_-)$$

satisfies henselian v-excision, as a consequence of [EHIK21, Lemma 3.3.7], and henselian v-excision for the presheaf $\left(L_{\mathrm{cdh}}\tau^{>i}\,\mathbb{Z}/p^k(i)^{\mathrm{syn}}\right)(-)$. □

For every qcqs scheme X, the compatibilities between the motivic and syntomic first Chern classes of Section 8.1 imply that the natural diagram

$$\begin{array}{ccc} \mathbb{Z}/p^k(i)^{\mathrm{mot}}(X) \oplus \mathbb{Z}/p^k(i-1)^{\mathrm{mot}}(X)[-2] & \xrightarrow{\pi^* \oplus c_1^{\mathrm{mot}}(\mathcal{O}(1))\pi^*} & \mathbb{Z}/p^k(i)^{\mathrm{mot}}(\mathbb{P}^1_X) \\ \downarrow & & \downarrow \\ \mathbb{Z}/p^k(i)^{\mathrm{syn}}(X) \oplus \mathbb{Z}/p^k(i-1)^{\mathrm{syn}}(X)[-2] & \xrightarrow{\pi^* \oplus c_1^{\mathrm{syn}}(\mathcal{O}(1))\pi^*} & \mathbb{Z}/p^k(i)^{\mathrm{syn}}(\mathbb{P}^1_X) \end{array}$$

is commutative. We define the natural map

$$\left(L_{\mathrm{cdh}}\tau^{>i}\,\mathbb{Z}/p^k(i)^{\mathrm{syn}}\right)(X) \oplus \left(L_{\mathrm{cdh}}\tau^{>i-1}\,\mathbb{Z}/p^k(i-1)^{\mathrm{syn}}\right)(X)[-2] \longrightarrow \left(L_{\mathrm{cdh}}\tau^{>i}\,\mathbb{Z}/p^k(i)^{\mathrm{syn}}\right)(\mathbb{P}^1_X)$$

in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$ as the map induced, via Theorem 5.10, by taking cofibres along the vertical maps of this commutative diagram.

Proposition 8.12. *Let p be a prime number, and V be a rank one henselian valuation ring of mixed characteristic $(0, p)$. Then for any integers $i \geq 0$ and $k \geq 1$, the natural map*

$$\tau^{>i}\,\mathbb{Z}/p^k(i)^{\mathrm{syn}}(V) \oplus \left(\tau^{>i-1}\,\mathbb{Z}/p^k(i-1)^{\mathrm{syn}}(V)\right)[-2] \longrightarrow \left(L_{\mathrm{cdh}}\tau^{>i}\,\mathbb{Z}/p^k(i)^{\mathrm{syn}}\right)(\mathbb{P}^1_V)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$.

Proof. The valuation ring V is p-henselian, so the natural maps

$$\tau^{>i}\,\mathbb{Z}/p^k(i)^{\mathrm{syn}}(V) \longrightarrow \tau^{>i}\,\mathbb{Z}/p^k(i)^{\mathrm{BMS}}(V)$$

and

$$\left(\tau^{>i-1}\,\mathbb{Z}/p^k(i-1)^{\mathrm{syn}}(V)\right)[-2] \longrightarrow \left(\tau^{>i-1}\,\mathbb{Z}/p^k(i)^{\mathrm{BMS}}(V)\right)[-2]$$

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are equivalences in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$. We first prove that the induced map

$$\tau^{>i} \mathbb{Z}/p^k(i)^{\mathrm{BMS}}(V) \oplus \big(\tau^{>i-1} \mathbb{Z}/p^k(i-1)^{\mathrm{BMS}}(V)\big)[-2] \longrightarrow \big(L_{\mathrm{cdh}}\tau^{>i} \mathbb{Z}/p^k(i)^{\mathrm{BMS}}\big)(\mathbb{P}^1_V)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$. Let κ be the residue field of V. By the rigidity property Theorem 2.27, the natural maps

$$\tau^{>i} \mathbb{Z}/p^k(i)^{\mathrm{BMS}}(V) \longrightarrow \tau^{>i} \mathbb{Z}/p^k(i)^{\mathrm{BMS}}(\kappa)$$

and

$$\big(\tau^{>i-1} \mathbb{Z}/p^k(i-1)^{\mathrm{BMS}}(V)\big)[-2] \longrightarrow \big(\tau^{>i-1} \mathbb{Z}/p^k(i-1)^{\mathrm{BMS}}(\kappa)\big)[-2]$$

are equivalences in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$. By Corollary 6.6, the natural map

$$\big(L_{\mathrm{cdh}}\tau^{>i} \mathbb{Z}/p^k(i)^{\mathrm{BMS}}\big)(\mathbb{P}^1_V) \longrightarrow \big(L_{\mathrm{cdh}}\tau^{>i} \mathbb{Z}/p^k(i)^{\mathrm{BMS}}\big)(\mathbb{P}^1_{V/p})$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$. The presheaf $(L_{\mathrm{cdh}}\tau^{>i} \mathbb{Z}/p^k(i)^{\mathrm{BMS}})(\mathbb{P}^1_-)$ is moreover a finitary cdh sheaf, so it is invariant under nilpotent extensions. In particular, the natural map

$$\big(L_{\mathrm{cdh}}\tau^{>i} \mathbb{Z}/p^k(i)^{\mathrm{BMS}}\big)(\mathbb{P}^1_{V/p}) \longrightarrow \big(L_{\mathrm{cdh}}\tau^{>i} \mathbb{Z}/p^k(i)^{\mathrm{BMS}}\big)(\mathbb{P}^1_\kappa)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$. It then suffices to prove that the natural map

$$\tau^{>i} \mathbb{Z}/p^k(i)^{\mathrm{BMS}}(\kappa) \oplus \big(\tau^{>i-1} \mathbb{Z}/p^k(i-1)^{\mathrm{BMS}}(\kappa)\big)[-2] \longrightarrow \big(L_{\mathrm{cdh}}\tau^{>i} \mathbb{Z}/p^k(i)^{\mathrm{BMS}}\big)(\mathbb{P}^1_\kappa)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$, and this is a consequence of the $\mathbb{P}^1$-bundle formula on characteristic p fields for the presheaves $\mathbb{Z}/p^k(i)^{\mathrm{cdh}}$ ([BEM]) and $L_{\mathrm{cdh}} \mathbb{Z}/p^k(i)^{\mathrm{BMS}}$ ([EM23, Lemma 5.17]).

We prove now that the natural map

$$\tau^{>i} \mathbb{Z}/p^k(i)^{\mathrm{syn}}(V) \oplus \big(\tau^{>i-1} \mathbb{Z}/p^k(i-1)^{\mathrm{syn}}(V)\big)[-2] \longrightarrow \big(L_{\mathrm{cdh}}\tau^{>i} \mathbb{Z}/p^k(i)^{\mathrm{syn}}\big)(\mathbb{P}^1_V)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$. By Lemma 8.8 (applied for $j = i$ and after cdh sheafification), we just proved that the cofibre of this map is naturally identified with the complex

$$\big(L_{\mathrm{cdh}}\tau^{>i} R\Gamma_{\mathrm{\acute{e}t}}(-, j_!\mu_{p^k}^{\otimes i})\big)(\mathbb{P}^1_V) \in \mathcal{D}(\mathbb{Z}/p^k).$$

By Example 8.6, these complexes, indexed by integers $i \geq 0$, form the graded pieces of the filtered spectrum defined as the cofibre of the natural map of filtered spectra

$$\mathrm{Fil}^\star_{\mathrm{mot}} \mathrm{K}(X; \mathbb{Z}/p^k) \longrightarrow \mathrm{Fil}^\star_{\mathrm{mot}} \mathrm{K}^{\mathrm{Sel}}(X; \mathbb{Z}/p^k).$$

The cofibre of the natural map $\mathrm{K}(-; \mathbb{Z}/p^k) \to \mathrm{K}^{\mathrm{Sel}}(-; \mathbb{Z}/p^k)$, as a cofibre of two additive invariants of $\mathbb{Z}$-linear ∞-categories, is an additive invariant of $\mathbb{Z}$-linear ∞-categories and, as such, satisfies the $\mathbb{P}^1$-bundle formula ([EM23, Lemma 5.6]). This filtration then induces a spectral sequence

$$E_2^{i,j} = \mathrm{H}^{i-j}\big(\big(L_{\mathrm{cdh}}\tau^{>-j} R\Gamma_{\mathrm{\acute{e}t}}(-, j_!\mu_{p^k}^{\otimes(-j)})\big)(\mathbb{P}^1_V)\big) \implies 0.$$

By Lemma 8.9, for every integer $i \geq 0$, the complex $\big(L_{\mathrm{cdh}}\tau^{>i} R\Gamma_{\mathrm{\acute{e}t}}(-, j_!\mu_{p^k}^{\otimes i})\big)(\mathbb{P}^1_V) \in \mathcal{D}(\mathbb{Z}/p^k)$ is concentrated in degree $i+1$, so this spectral sequence degenerates. This implies the desired equivalence. $\square$

Remark 8.13. The proof of Proposition 8.12 uses a reduction to the case of fields of characteristic p, where the result is a consequence of the $\mathbb{P}^1$-bundle formula for the presheaves $\mathbb{Z}/p^k(i)^{\mathrm{cdh}}$ ([BEM]) and $L_{\mathrm{cdh}} \mathbb{Z}/p^k(i)^{\mathrm{BMS}}$ ([EM23, Lemma 5.17]). It is however possible to bypass these two results and prove directly the $\mathbb{P}^1$-bundle formula on fields of characteristic p for the presheaves $L_{\mathrm{cdh}}\tau^{>i} \mathbb{Z}/p^k(i)^{\mathrm{BMS}}$, by imitating the degeneration argument of [EM23, Lemma 5.17].

Proposition 8.14. *Let X be a qcqs scheme, and p be a prime number. Then for any integers $i \geq 0$ and $k \geq 1$, the natural map*

$$\left(L_{\mathrm{cdh}}\tau^{>i}\,\mathbb{Z}/p^k(i)^{\mathrm{syn}}\right)(X) \oplus \left(L_{\mathrm{cdh}}\tau^{>i-1}\,\mathbb{Z}/p^k(i-1)^{\mathrm{syn}}\right)(X)[-2] \longrightarrow \left(L_{\mathrm{cdh}}\tau^{>i}\,\mathbb{Z}/p^k(i)^{\mathrm{syn}}\right)(\mathbb{P}^1_X)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$.

Proof. The presheaves $L_{\mathrm{cdh}}\tau^{>i}\,\mathbb{Z}/p^k(i)^{\mathrm{syn}}$ and $\left(L_{\mathrm{cdh}}\tau^{>i}\,\mathbb{Z}/p^k(i)^{\mathrm{syn}}\right)(\mathbb{P}^1_-)$ are finitary cdh sheaves on qcqs schemes, and satisfy henselian v-excision (Lemma 8.11). It then suffices to prove the desired equivalence for henselian valuation rings of rank at most one (Lemma 8.10). Let V be a henselian valuation ring of rank at most one.

If p is invertible in the valuation ring V, then this is equivalent to proving that the natural map

$$\tau^{>i}R\Gamma_{\mathrm{\acute{e}t}}(V, \mu_{p^k}^{\otimes i}) \oplus \left(\tau^{>i-1}R\Gamma_{\mathrm{\acute{e}t}}(V, \mu_{p^k}^{\otimes(i-1)})\right)[-2] \longrightarrow \left(L_{\mathrm{cdh}}\tau^{>i}R\Gamma_{\mathrm{\acute{e}t}}(-, \mu_{p^k}^{\otimes i})\right)(\mathbb{P}^1_V)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$. For every integer $i \geq 0$, there is a fibre sequence of $\mathcal{D}(\mathbb{Z}/p^k)$-valued presheaves

$$\mathbb{Z}/p^k(i)^{\mathrm{cdh}}(-) \longrightarrow R\Gamma_{\mathrm{\acute{e}t}}(-, \mu_{p^k}^{\otimes i}) \longrightarrow L_{\mathrm{cdh}}\tau^{>i}R\Gamma_{\mathrm{\acute{e}t}}(-, \mu_{p^k}^{\otimes i})$$

on qcqs $\mathbb{Z}[\frac{1}{p}]$-schemes ([BEM]). The desired equivalence is then a consequence of the $\mathbb{P}^1$-bundle formula on qcqs $\mathbb{Z}[\frac{1}{p}]$-schemes for the presheaf $\mathbb{Z}/p^k(i)^{\mathrm{cdh}}$ ([BEM]) and the presheaf $R\Gamma_{\mathrm{\acute{e}t}}(-, \mu_{p^k}^{\otimes i})$ ([BL22, proof of Theorem 9.1.1]).

If p is zero in the valuation ring V, then this is a consequence of the $\mathbb{P}^1$-bundle formula on qcqs $\mathbb{F}_p$-schemes for the presheaves $\mathbb{Z}/p^k(i)^{\mathrm{cdh}}$ ([BEM]) and $L_{\mathrm{cdh}}\,\mathbb{Z}/p^k(i)^{\mathrm{BMS}}$ ([EM23, Theorem 5.14]).

If p is neither invertible nor zero in the valuation ring V, then V is a rank one henselian valuation ring of mixed characteristic $(0, p)$, and the result is Proposition 8.12. $\qquad\square$

Proof of Theorem 8.7. It suffices to prove the result rationally, and modulo p for every prime number p. Rationally, the Atiyah–Hirzebruch spectral sequence degenerates (Theorem 4.1), so the result is a consequence of the $\mathbb{P}^1$-bundle formula for algebraic K-theory (Section 8.1). Let p be a prime number. By Theorem 5.10, for every integer $i \in \mathbb{Z}$, there is a fibre sequence of $\mathcal{D}(\mathbb{F}_p)$-valued presheaves on qcqs schemes

$$\mathbb{F}_p(i)^{\mathrm{mot}}(-) \longrightarrow \mathbb{F}_p(i)^{\mathrm{syn}}(-) \longrightarrow \left(L_{\mathrm{cdh}}\tau^{>i}\,\mathbb{F}_p(i)^{\mathrm{syn}}\right)(-).$$

By [BL22, Theorem 9.1.1], the natural map

$$\pi^* \oplus c_1^{\mathrm{syn}}(\mathcal{O}(1))\pi^* : \mathbb{F}_p(i)^{\mathrm{syn}}(X) \oplus \mathbb{F}_p(i-1)^{\mathrm{syn}}(X)[-2] \longrightarrow \mathbb{F}_p(i)^{\mathrm{syn}}(\mathbb{P}^1_X)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{F}_p)$. The result modulo p is then a consequence of Proposition 8.14. $\qquad\square$

8.3 Regular blowup and projective bundle formulae

In this subsection, we prove the regular blowup formula for the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$ (Theorem 8.15). By an argument of Annala–Iwasa, this and the $\mathbb{P}^1$-bundle formula imply the general projective bundle formula for the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$ (Theorem 8.16).

Theorem 8.15 (Regular blowup formula). *Let $Y \to Z$ be a regular closed immersion of qcqs schemes.[11] Then for every integer $i \geq 0$, the commutative diagram*

$$
\begin{array}{ccc}
\mathbb{Z}(i)^{\mathrm{mot}}(Z) & \longrightarrow & \mathbb{Z}(i)^{\mathrm{mot}}(Y) \\
\downarrow & & \downarrow \\
\mathbb{Z}(i)^{\mathrm{mot}}(\mathrm{Bl}_Y(Z)) & \longrightarrow & \mathbb{Z}(i)^{\mathrm{mot}}(\mathrm{Bl}_Y(Z) \times_Z Y)
\end{array}
$$

is a cartesian square in the derived category $\mathcal{D}(\mathbb{Z})$.

---

[11] A morphism $Y \to Z$ is a regular closed immersion if it is a closed immersion, and if Z admits an affine open cover such that Y is defined by a regular sequence on each of the corresponding affine schemes.

Proof. It suffices to prove the result rationally, and modulo p for every prime number p. By definition, a cdh sheaf sends an abstract blowup square to a cartesian square, and in particular satisfies the regular blowup formula. By Corollary 4.68, the regular blowup formula for the presheaf $\mathbb{Q}(i)^{\mathrm{mot}}$ is then equivalent to the regular blowup formula for the presheaf $R\Gamma_{\mathrm{Zar}}(-, \mathbb{L}\Omega^{\leq i}_{-_{\mathbb{Q}}/\mathbb{Q}})$. And the regular blowup formula for the presheaf $R\Gamma_{\mathrm{Zar}}(-, \mathbb{L}\Omega^{\leq i}_{-_{\mathbb{Q}}/\mathbb{Q}})$ is a consequence of the fact that for every integer $j \geq 0$, the presheaf $R\Gamma_{\mathrm{Zar}}(-, \mathbb{L}^{j}_{-_{/\mathbb{Z}}} \otimes_{\mathbb{Z}} \mathbb{Q})$ satisfies the regular blowup formula ([BL22, Lemma 9.4.3]).

Let p be a prime number. Similarly, Corollary 3.25 implies that the regular blowup formula for the presheaf $\mathbb{F}_p(i)^{\mathrm{mot}}$ is equivalent to the regular blowup formula for the presheaf $\mathbb{F}_p(i)^{\mathrm{BMS}}$. By [AMMN22, Corollary 5.31], there exists an integer $m \geq 0$ and an equivalence of presheaves

$$\mathbb{F}_p(i)^{\mathrm{BMS}}(-) \xrightarrow{\sim} \mathrm{fib}\Big(\mathrm{can} - \phi_i : (\mathcal{N}^{\geq i}\mathbb{\Delta}_{-}\{i\}/\mathcal{N}^{\geq i+m}\mathbb{\Delta}_{-}\{i\})/p \longrightarrow (\mathbb{\Delta}_{-}\{i\}/\mathcal{N}^{\geq i+m}\mathbb{\Delta}_{-}\{i\})/p\Big).$$

In particular, it suffices to prove that for every integer $j \geq 0$, the presheaf $\mathcal{N}^{j}\mathbb{\Delta}_{-}/p$ satisfies the regular blowup formula. By [BL22, Remark 5.5.8 and Example 4.7.8], there is a fibre sequence of presheaves

$$\mathcal{N}^{j}\mathbb{\Delta}_{-}\{i\}/p \longrightarrow \mathrm{Fil}^{\mathrm{conj}}_{j}\overline{\mathbb{\Delta}}_{-/\mathbb{Z}_p[\![\widetilde{p}]\!]}/p \xrightarrow{\Theta+j} \mathrm{Fil}^{\mathrm{conj}}_{j-1}\overline{\mathbb{\Delta}}_{-/\mathbb{Z}_p[\![\widetilde{p}]\!]}/p.$$

The presheaves $\mathrm{Fil}^{\mathrm{conj}}_{j}\overline{\mathbb{\Delta}}_{-/\mathbb{Z}_p[\![\widetilde{p}]\!]}/p$ and $\mathrm{Fil}^{\mathrm{conj}}_{j-1}\overline{\mathbb{\Delta}}_{-/\mathbb{Z}_p[\![\widetilde{p}]\!]}/p$ have finite filtrations with graded pieces given by modulo p powers of the cotangent complex, and the result is then a consequence of the regular blowup formula for powers of the cotangent complex ([BL22, Lemma 9.4.3]). $\square$

Theorem 8.16 (Projective bundle formula). *Let X be a qcqs scheme, $r \geq 1$ be an integer, $\mathcal{E}$ be a vector bundle of rank $r + 1$ on X, and $\pi : \mathbb{P}_X(\mathcal{E}) \to X$ be the projectivisation of $\mathcal{E}$. Then for every integer $i \in \mathbb{Z}$, the natural map*

$$\sum_{j=0}^{r} c_1^{\mathrm{mot}}(\mathcal{O}(1))^j \pi^* : \bigoplus_{j=0}^{r} \mathbb{Z}(i-j)^{\mathrm{mot}}(X)[-2j] \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(\mathbb{P}_X(\mathcal{E}))$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$.

Proof. By Zariski descent, it suffices to consider the case where the vector bundle $\mathcal{E}$ is given by $\mathbb{A}_X^{r+1}$, *i.e.*, to prove that the natural map

$$\sum_{j=0}^{r} c_1^{\mathrm{mot}}(\mathcal{O}(1))^j \pi^* : \bigoplus_{j=0}^{r} \mathbb{Z}(i-j)^{\mathrm{mot}}(X)[-2j] \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(\mathbb{P}_X^r)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$. The presheaves $\mathbb{Z}(i)^{\mathrm{mot}}$ satisfy the $\mathbb{P}^1$-bundle formula (Theorem 8.7). Moreover, for every qcqs scheme X and every integer $m \geq 0$, they send the blowup square

$$\begin{array}{ccc} \mathbb{P}_X^m & \longrightarrow & \mathrm{Bl}_X(\mathbb{A}_X^{m+1}) \\ \downarrow & & \downarrow \\ X & \xrightarrow{\ 0\ } & \mathbb{A}_X^{m+1} \end{array}$$

to a cartesian square in the derived category $\mathcal{D}(\mathbb{Z})$ (Theorem 8.15, in the special case where the regular closed immersion $Y \to Z$ is the zero section $X \to \mathbb{A}_X^{m+1}$). By the argument of [AI23, Lemma 3.3.5], these two properties imply, by induction, the desired projective bundle formula. $\square$

In the following result, denote by $\mathbb{Z}(i)_X^{\mathrm{mot}} : \mathrm{Sm}_X^{\mathrm{op}} \to \mathcal{D}(\mathbb{Z})$ the Zariski sheaves on smooth schemes over X induced by restriction of the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$.

Corollary 8.17 (Motivic cohomology is represented in motivic spectra). *For every qcqs scheme X, the motivic complexes $\{\mathbb{Z}(i)_X^{\mathrm{mot}}\}_{i \in \mathbb{Z}}$ are represented by a $\mathbb{P}^1$-motivic spectrum in the sense of [AI23].*

Proof. By definition of $\mathbb{P}^1$-motivic spectra, this is a consequence of elementary blowup excision (which is a special case of Theorem 8.15) and the $\mathbb{P}^1$-bundle formula (Theorem 8.7). $\square$

9 MOTIVIC WEIBEL VANISHING AND PRO CDH DESCENT

In this section, we study the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$ on noetherian schemes. We prove a general vanishing result which refines Weibel's vanishing conjecture on negative K-groups (Theorem 9.27), and prove that they coincide with Kelly–Saito's pro cdh motivic complexes $\mathbb{Z}(i)^{\mathrm{procdh}}$ (Theorem 9.32). The key input for both these results is the fact that the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$ satisfy pro cdh excision (Theorem 9.23), *i.e.*, that they send abstract blowup squares to pro cartesian squares.

Notation 9.1 (Abstract blowup square)**.** An *abstract blowup square* (of noetherian schemes) is a cartesian square

$$
\begin{array}{ccc}
Y' & \longrightarrow & X' \\
\downarrow & & \downarrow \\
Y & \longrightarrow & X
\end{array}
\tag{3}
$$

of qcqs schemes (resp. of noetherian schemes) such that $X' \to X$ is proper and finitely presented, $Y \to X$ is a finitely presented closed immersion, and the induced map $X' \setminus Y' \to X \setminus Y$ is an isomorphism. In this context, we also denote, for every integer $r \geq 0$, by rY (resp. rY') the $r-1^{\mathrm{st}}$ infinitesimal thickening of Y inside X (resp. of Y' inside X').

We use Kelly–Saito's recent definition in [KS24] of the pro cdh topology to encode the fact that a Nisnevich sheaf (*e.g.*, the motivic complex $\mathbb{Z}(i)^{\mathrm{mot}}$) satisfies pro cdh excision. Kelly–Saito proved in particular that if S has finite valuative dimension and noetherian topological space, then the pro cdh topos of S is hypercomplete and has enough points. For our purposes, the following definition will be used only for noetherian schemes S.

Definition 9.2 (Pro cdh descent, after [KS24])**.** Let S be a qcqs scheme. A *pro cdh sheaf* on finitely presented S-schemes is a presheaf

$$
F : \mathrm{Sch}_S^{\mathrm{fp,op}} \longrightarrow \mathcal{D}(\mathbb{Z})
$$

satisfying Nisnevich descent, and such that for every abstract blowup square of finitely presented S-schemes (3), the natural commutative diagram

$$
\begin{array}{ccc}
F(X) & \longrightarrow & F(X') \\
\downarrow & & \downarrow \\
\{F(rY)\}_r & \longrightarrow & \{F(rY')\}_r
\end{array}
$$

is a weakly cartesian square of pro objects in the derived category $\mathcal{D}(\mathbb{Z})$.[12]

9.1 Pro cdh descent for the cotangent complex

In this subsection, we review the pro cdh descent for powers of the cotangent complex on noetherian schemes (Proposition 9.9). On finite-dimensional noetherian schemes, this is [Mor16, Theorem 2.10]. On general noetherian schemes, the proof follows the sketch presented in [EM23, proof of Lemma 8.5]. In particular, the arguments are exactly as in [Mor16], except for the following generalisation of Grothendieck's formal functions theorem ([Gro61, Corollary 4.1.7]), where the finite dimensionality hypothesis is removed. We give some details for the sake of completeness.

For every commutative ring A, recall that a pro A-module $\{M_r\}_r$ is zero if for every index r, there exists an index $r' \geq r$ such that the map $M_{r'} \to M_r$ is the zero map. Similarly, a map $\{M_r\}_r \to \{N_r\}_r$ of pro A-modules is an isomorphism if its kernel and cokernel are zero pro A-modules. We say that a pro object $\{C_r\}_r$ in the derived category $\mathcal{D}(A)$ is weakly zero if all its cohomology groups are zero pro

---

[12]By this, we mean that all the cohomology groups of the total fibre of this commutative square are zero as pro abelian groups. All the presheaves F that we will consider (most importantly, the presheaves $\mathbb{Z}(i)^{\mathrm{mot}}$) are bounded above on noetherian schemes, by a constant depending only on the dimension of their input (for the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$, this is Proposition 4.50); this definition of weakly cartesian square will then be equivalent to being weakly cartesian in the stable ∞-category of pro objects in the derived category $\mathcal{D}(\mathbb{Z})$, in the sense of [LT19, Definition 2.27].

A-modules. Note that all the pro complexes that we will consider are uniformly bounded above, so this definition is equivalent to being weakly zero in the stable ∞-category of pro objects in the derived category $\mathcal{D}(A)$ ([LT19, Definition 2.27]). Similarly, we say that a map $\{C_r\}_r \to \{C'_r\}_r$ of pro objects in the derived category $\mathcal{D}(A)$ is a weak equivalence if its fibre is weakly zero as a pro object in the derived category $\mathcal{D}(A)$.

Lemma 9.3 (Formal functions theorem, after Lurie [Lur19]). *Let A be a noetherian commutative ring, I be an ideal of A, X be a proper scheme over $\mathrm{Spec}(A)$, and $X_I^\wedge$ be the formal completion of X along the vanishing locus of I. Then for every coherent sheaf $\mathcal{F}$ over X, the natural map*

$$R\Gamma_{\mathrm{Zar}}(X, \mathcal{F}) \longrightarrow R\Gamma_{\mathrm{Zar}}(X_I^\wedge, \mathcal{F}_I^\wedge)$$

where $\mathcal{F}_I^\wedge$ is the pullback of $\mathcal{F}$ along the natural map $X_I^\wedge \to X$, exhibits the target as the I-adic completion[13] of the source in the derived category $\mathcal{D}(A)$. More precisely, the natural map

$$\{R\Gamma_{\mathrm{Zar}}(X, \mathcal{F})/I^r\}_r \longrightarrow \{R\Gamma_{\mathrm{Zar}}(X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/I^r), \mathcal{F} \otimes_{\mathcal{O}_X}^{\mathbb{L}} \mathcal{O}_X/I^r\mathcal{O}_X)\}_r$$

is a weak equivalence of pro objects in the derived category $\mathcal{D}(A)$.

Proof. The first statement is a special case of [Lur19, Lemma 8.5.1.1]. The second statement, although *a priori* stronger, follows by an examination of the previous proof (and in particular, the proof of [Lur19, Lemma 8.1.2.3]). $\square$

Lemma 9.4. *Let A be a noetherian commutative ring, and X be a proper scheme over $\mathrm{Spec}(A)$. Then for any integers $j \geq 0$ and $n \in \mathbb{Z}$, the A-module $\mathrm{H}_{\mathrm{Zar}}^n(X, \mathbb{L}_{-/A}^j)$ is finitely generated.*

Proof. The scheme X is of finite type over $\mathrm{Spec}(A)$, so the $\mathcal{O}_X$-module $\mathcal{H}_{\mathrm{Zar}}^n(-, \mathbb{L}_{-/A}^j)$ is coherent. Because X is proper over $\mathrm{Spec}(A)$, its cohomology groups are thus finitely generated A-modules. $\square$

Corollary 9.5. *Let A be a noetherian commutative ring, I be an ideal of A, and X be a noetherian scheme which is proper over $\mathrm{Spec}(A)$. Then for any integers $j \geq 0$ and $n \in \mathbb{Z}$, the natural map*

$$\{\mathrm{H}_{\mathrm{Zar}}^n(X, \mathbb{L}_{-/A}^j)/I^r\}_r \longrightarrow \{\mathrm{H}_{\mathrm{Zar}}^n(X, \mathbb{L}_{-/A}^j \otimes_{\mathcal{O}_X}^{\mathbb{L}} \mathcal{O}_X/I^r\mathcal{O}_X)\}_r$$

is an isomorphism of pro A-modules.

Proof. By Lemma 9.4 and its proof, all the terms in the hypercohomology spectral sequence

$$E_2^{p,q} = \mathrm{H}_{\mathrm{Zar}}^p(X, \mathcal{H}_{\mathrm{Zar}}^q(-, \mathbb{L}_{-/R}^j)) \Longrightarrow \mathrm{H}_{\mathrm{Zar}}^{p+q}(X, \mathbb{L}_{-/A}^j)$$

are finitely generated A-modules. The functor $\{- \otimes_A A/I^r\}_r$ is exact on the category of finitely generated A-modules ([Mor16, Theorem 1.1 (ii)]), so it induces a spectral sequence of pro A-modules

$$E_2^{p,q} = \{\mathrm{H}_{\mathrm{Zar}}^p(X, \mathcal{H}_{\mathrm{Zar}}^q(-, \mathbb{L}_{-/A}^j))/I^r\}_r \Longrightarrow \{\mathrm{H}_{\mathrm{Zar}}^{p+q}(X, \mathbb{L}_{-/A}^j)/I^r\}_r.$$

It then suffices to prove that the natural map of pro A-modules

$$\{\mathrm{H}_{\mathrm{Zar}}^p(X, \mathcal{H}_{\mathrm{Zar}}^q(-, \mathbb{L}_{-/A}^j))/I^r\}_r \longrightarrow \{\mathrm{H}_{\mathrm{Zar}}^p(X, \mathcal{H}_{\mathrm{Zar}}^q(-, \mathbb{L}_{-/A}^j \otimes_{\mathcal{O}_X}^{\mathbb{L}} \mathcal{O}_X/I^r\mathcal{O}_X))\}_r$$

is an isomorphism for all integers $p, q \geq 0$. The natural map

$$\{\mathrm{H}_{\mathrm{Zar}}^p(X, \mathcal{H}_{\mathrm{Zar}}^q(-, \mathbb{L}_{-/A}^j))/I^r\}_r \longrightarrow \{\mathrm{H}_{\mathrm{Zar}}^p(X, \mathcal{H}_{\mathrm{Zar}}^q(-, \mathbb{L}_{-/A}^j) \otimes_A^{\mathbb{L}} A/I^r)\}_r$$

is an isomorphism by Lemma 9.3 applied to the coherent sheaf $\mathcal{H}_{\mathrm{Zar}}^q(-, \mathbb{L}_{-/A})$ on X, and the natural map

$$\{\mathrm{H}_{\mathrm{Zar}}^p(X, \mathcal{H}_{\mathrm{Zar}}^q(-, \mathbb{L}_{-/A}^j) \otimes_A^{\mathbb{L}} A/I^r)\}_r \longrightarrow \{\mathrm{H}_{\mathrm{Zar}}^p(X, \mathcal{H}_{\mathrm{Zar}}^q(-, \mathbb{L}_{-/A}^j \otimes_{\mathcal{O}_X}^{\mathbb{L}} \mathcal{O}_X/I^r\mathcal{O}_X))\}_r$$

is an isomorphism by [Mor16, Lemma 2.3]. $\square$

---

[13]The cohomology groups of these coherent sheaves are finitely generated A-modules (because X is proper over $\mathrm{Spec}(A)$), so the derived I-adic completion and the classical I-adic completion coincide in this context.

Lemma 9.6. *Let A be a noetherian commutative ring, I be an ideal of A, and X be a proper scheme over $\mathrm{Spec}(A)$ such that the induced map $X \setminus V(I\mathcal{O}_X) \to \mathrm{Spec}(A) \setminus V(I)$ is an isomorphism.*

(1) For any integers $j \geq 0$ and $n \in \mathbb{Z}$, the natural map

$$\{\mathrm{H}^n_{\mathrm{Zar}}(X, \mathbb{L}^j_{-/A} \otimes^{\mathbb{L}}_{\mathcal{O}_X} I^r \mathcal{O}_X)\}_r \longrightarrow \{I^r \mathrm{H}^n_{\mathrm{Zar}}(X, \mathbb{L}^j_{-/A})\}_r$$

is an isomorphism of pro A-modules.

(2) For any integers $j \geq 0$ and $n \in \mathbb{Z}$ such that $(j, n) \neq (0, 0)$, the A-module $\mathrm{H}^n_{\mathrm{Zar}}(X, \mathbb{L}^j_{-/A})$ is killed by a power of I; in particular, the pro A-module $\{I^r \mathrm{H}^n_{\mathrm{Zar}}(X, \mathbb{L}^j_{-/A})\}_r$ is zero.

Proof. (1) The short exact sequence $0 \to \{I^r \mathcal{O}_X\}_r \to \mathcal{O}_X \to \{\mathcal{O}_X / I^r \mathcal{O}_X\}_r \to 0$ of pro $\mathcal{O}_X$-modules induces a long exact sequence

$$\cdots \to \mathrm{H}^n_{\mathrm{Zar}}(X, \mathbb{L}^j_{-/A} \otimes^{\mathbb{L}}_{\mathcal{O}_X} I^r \mathcal{O}_X)\}_r \to \mathrm{H}^n_{\mathrm{Zar}}(X, \mathbb{L}^j_{-/A}) \to \{\mathrm{H}^n_{\mathrm{Zar}}(X, \mathbb{L}^j_{-/A} \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{O}_X / I^r \mathcal{O}_X)\}_r \to \cdots$$

of pro A-modules. By Corollary 9.5, the boundary maps of this long exact sequence vanish, hence the natural map

$$\{\mathrm{H}^n_{\mathrm{Zar}}(X, \mathbb{L}^j_{-/A} \otimes^{\mathbb{L}}_{\mathcal{O}_X} I^r \mathcal{O}_X)\}_r \longrightarrow \{I^r \mathrm{H}^n_{\mathrm{Zar}}(X, \mathbb{L}^j_{-/A})\}_r$$

is an isomorphism of pro A-modules.
(2) By Lemma 9.4, the A-module $\mathrm{H}^n_{\mathrm{Zar}}(X, \mathbb{L}^j_{-/A})$ is finitely generated. Because the map

$$X \setminus V(I\mathcal{O}_X) \to \mathrm{Spec}(A) \setminus V(I)$$

is an isomorphism, this A-module is moreover supported on $V(I)$ if $(j, n) \neq (0, 0)$. If $(j, n) \neq (0, 0)$, this implies that the A-module $\mathrm{H}^n_{\mathrm{Zar}}(X, \mathbb{L}^j_{-/A})$ is killed by a power of I. $\qquad\square$

Corollary 9.7. *Let A be a noetherian commutative ring, I be an ideal of A, and X be a proper scheme over $\mathrm{Spec}(A)$ such that the induced map $X \setminus V(I\mathcal{O}_X) \to \mathrm{Spec}(A) \setminus V(I)$ is an isomorphism. Then for every integer $j \geq 0$, the natural map*

$$\{\mathbb{L}^j_{A/\mathbb{Z}} \otimes^{\mathbb{L}}_A I^r\}_r \longrightarrow \{R\Gamma_{\mathrm{Zar}}(X, \mathbb{L}^j_{-/\mathbb{Z}} \otimes^{\mathbb{L}}_{\mathcal{O}_X} I^r \mathcal{O}_X)\}_r$$

is a weak equivalence of pro objects in the derived category $\mathcal{D}(A)$.

Proof. By Lemma 9.6, and for any integers $n, a, b \in \mathbb{Z}$, the pro A-module

$$\{\mathrm{H}^n(\mathbb{L}^j_{A/\mathbb{Z}} \otimes^{\mathbb{L}}_A \mathrm{H}^a_{\mathrm{Zar}}(X, \mathbb{L}^b_{-/A} \otimes^{\mathbb{L}}_{\mathcal{O}_X} I^r \mathcal{O}_X))\}_r$$

is zero, except if $(a, b) = (0, 0)$. By transitivity for the powers of the cotangent complex (see the proof of [Mor16, Lemma 2.8 (ii)] for more details), this implies that the natural map

$$\{\mathrm{H}^n(\mathbb{L}^j_{A/\mathbb{Z}} \otimes^{\mathbb{L}}_A \mathrm{H}^0_{\mathrm{Zar}}(X, I^r \mathcal{O}_X))\}_r \longrightarrow \{\mathrm{H}^n_{\mathrm{Zar}}(X, \mathbb{L}^j_{-/\mathbb{Z}} \otimes^{\mathbb{L}}_{\mathcal{O}_X} I^r \mathcal{O}_X)\}_r$$

is an isomorphism of pro A-modules. Let B be the A-algebra $\mathrm{H}^0_{\mathrm{Zar}}(X, \mathcal{O}_X)$. Applying Lemma 9.6 (1) for $j = n = 0$, it then suffices to prove that the natural map $\{I^r\}_r \to \{I^r B\}_r$ is an isomorphism of pro A-modules. The A-algebra B is finite and isomorphic to A away from the vanishing locus of I, so the kernel and cokernel of the structure map $A \to B$ are killed by a power of I. The result is then a formal consequence of [Mor16, Theorem 1.1 (ii)]. $\qquad\square$

Lemma 9.8. *Let $Y \to X$ be a closed immersion of noetherian schemes, and $\mathcal{I}$ be the associated ideal sheaf on X. Then for every integer $j \geq 0$, the natural map*

$$\{R\Gamma_{\mathrm{Zar}}(X, \mathbb{L}^j_{-/\mathbb{Z}} \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{O}_X / \mathcal{I}^r)\}_r \longrightarrow \{R\Gamma_{\mathrm{Zar}}(rY, \mathbb{L}^j_{-/\mathbb{Z}})\}_r$$

is a weak equivalence of pro objects in the derived category $\mathcal{D}(A)$.

Proof. The scheme X is noetherian, hence quasi-separated, so we may assume by induction that X is affine. In this case, the result is [Mor18a, Corollary 4.5 (ii)]. □

Proposition 9.9. *Let $j \geq 0$ be an integer. Then for every abstract blowup square of noetherian schemes (3), the natural commutative diagram*

$$
\begin{array}{ccc}
R\Gamma_{\mathrm{Zar}}(X, \mathbb{L}^j_{-/\mathbb{Z}}) & \longrightarrow & R\Gamma_{\mathrm{Zar}}(X', \mathbb{L}^j_{-/\mathbb{Z}}) \\
\downarrow & & \downarrow \\
\{R\Gamma_{\mathrm{Zar}}(rY, \mathbb{L}^j_{-/\mathbb{Z}})\}_r & \longrightarrow & \{R\Gamma_{\mathrm{Zar}}(rY', \mathbb{L}^j_{-/\mathbb{Z}})\}_r
\end{array}
$$

is a weakly cartesian square of pro objects in the derived category $\mathcal{D}(\mathbb{Z})$. In particular, the presheaf $R\Gamma_{\mathrm{Zar}}(-, \mathbb{L}^j_{-/\mathbb{Z}})$ is a pro cdh sheaf on noetherian schemes.

Proof. The scheme X is noetherian, hence quasi-separated, so we may assume by induction that X is affine, given by the spectrum of a noetherian commutative ring A. Let I be the ideal of A defining the closed subscheme Y of $\mathrm{Spec}(A)$. By Lemma 9.8, the desired statement is equivalent to the fact that the commutative diagram

$$
\begin{array}{ccc}
\mathbb{L}^j_{A/\mathbb{Z}} & \longrightarrow & R\Gamma_{\mathrm{Zar}}(X', \mathbb{L}^j_{-/\mathbb{Z}}) \\
\downarrow & & \downarrow \\
\{\mathbb{L}^j_{A/\mathbb{Z}} \otimes^{\mathbb{L}}_A A/I^r\}_r & \longrightarrow & \{R\Gamma_{\mathrm{Zar}}(X', \mathbb{L}^j_{-/\mathbb{Z}} \otimes^{\mathbb{L}}_{\mathcal{O}_{X'}} \mathcal{O}_{X'}/I^r\mathcal{O}_{X'})\}_r
\end{array}
$$

*is a weakly cartesian square of pro objects in the derived category $\mathcal{D}(\mathbb{Z})$. Taking fibres along the vertical maps, this is exactly Corollary 9.7. □

We now use Proposition 9.9 to prove pro cdh descent for variants of the cotangent complex (Corollary 9.12). In the following two lemmas, we consider inverse systems of objects in the derived category $\mathcal{D}(\mathbb{Z})$. We say that an inverse system $(C_r)_r$ in the derived category $\mathcal{D}(\mathbb{Z})$ is *essentially zero* if for every index r and every integer $n \in \mathbb{Z}$, there exists an index $r' \geq r$ such that the map $\mathrm{H}^n(C_{r'}) \to \mathrm{H}^n(C_r)$ is the zero map. In particular, an inverse system $(C_r)_r$ in the derived category $\mathcal{D}(\mathbb{Z})$ is essentially zero if and only if the associated pro object $\{C_r\}_r$ in the derived category $\mathcal{D}(\mathbb{Z})$ is weakly zero.

Lemma 9.10. *Let $(C_r)_r$ be an inverse system in the derived category $\mathcal{D}(\mathbb{Z})$. If $(C_r)_r$ is essentially zero, then $\left(\prod_{p \in \mathbb{P}} C_r/p\right)_r$ is essentially zero.*

Proof. Assume that the inverse system $(C_r)_r$ is essentially zero. Let r_0 be an index of this inverse system, $n \in \mathbb{Z}$ be an integer, and p be a prime number. We will use repeatedly that for every index r, there is a natural short exact sequence

$$
0 \longrightarrow \mathrm{H}^n(C_r)/p \longrightarrow \mathrm{H}^n(C_r/p) \longrightarrow \mathrm{H}^{n+1}(C_r)[p] \longrightarrow 0
$$

of abelian groups. Let $r_1 \geq r_0$ be an index such that the map $\mathrm{H}^n(C_{r_1}) \to \mathrm{H}^n(C_{r_0})$ is the zero map. Then for every index $r \geq r_1$, the map $\mathrm{H}^n(C_r)/p \to \mathrm{H}^n(C_{r_0})/p$ is the zero map, and the map $\mathrm{H}^n(C_r/p) \to \mathrm{H}^n(C_{r_0}/p)$ thus factors through the map $\mathrm{H}^n(C_r/p) \to \mathrm{H}^{n+1}(C_r)[p]$. Let $r_2 \geq r_1$ be an index such that the map $\mathrm{H}^{n+1}(C_{r_2}) \to \mathrm{H}^{n+1}(C_{r_1})$ is the zero map. Then the map

$$
\mathrm{H}^{n+1}(C_{r_2})[p] \to \mathrm{H}^{n+1}(C_{r_1})[p]
$$

is the zero map. By construction, the map

$$
\mathrm{H}^n(C_{r_2}/p) \longrightarrow \mathrm{H}^n(C_{r_0}/p)
$$

factors as

$$
\mathrm{H}^n(C_{r_2}/p) \longrightarrow \mathrm{H}^{n+1}(C_{r_2})[p] \xrightarrow{\ 0\ } \mathrm{H}^{n+1}(C_{r_1})[p] \longrightarrow \mathrm{H}^n(C_{r_0}/p),
$$

and is thus also the zero map. The index r_2 does not depend on the prime number p, so the map

$$\prod_{p \in \mathbb{P}} \mathrm{H}^n(C_{r_2}/p) \longrightarrow \prod_{p \in \mathbb{P}} \mathrm{H}^n(C_{r_0}/p)$$

is the zero map, and the inverse system $\left(\prod_{p \in \mathbb{P}} C_r/p \right)_r$ is essentially zero. $\qquad \square$

Lemma 9.11. *Let $(C_r)_r$ be an inverse system in the derived category $\mathcal{D}(\mathbb{Z})$. If $(C_r)_r$ is essentially zero, then $\left(\prod_{p \in \mathbb{P}} (C_r)_p^{\wedge} \right)_r$ is essentially zero.*

Proof. Assume that the inverse system $(C_r)_r$ is essentially zero. Let r_0 be an index of this inverse system, $n \in \mathbb{Z}$ be an integer, and p be a prime number. We will use repeatedly that for every index $r \geq 0$, there is a short exact sequence

$$0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}_p}(\mathbb{Q}_p / \mathbb{Z}_p, \mathrm{H}^n(C_r)) \longrightarrow \mathrm{H}^n((C_r)_p^{\wedge}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p / \mathbb{Z}_p, \mathrm{H}^{n+1}(C_r)) \longrightarrow 0$$

of abelian groups. Let $r_1 \geq r_0$ be an index such that the map $\mathrm{H}^n(C_{r_1}) \to \mathrm{H}^n(C_{r_0})$ is the zero map. Then for every index $r \geq r_1$, the map $\mathrm{Ext}^1_{\mathbb{Z}_p}(\mathbb{Q}_p / \mathbb{Z}_p, \mathrm{H}^n(C_r)) \to \mathrm{Ext}^1_{\mathbb{Z}_p}(\mathbb{Q}_p / \mathbb{Z}_p, \mathrm{H}^n(C_{r_0}))$ is the zero map, and the map $\mathrm{H}^n((C_r)_p^{\wedge}) \to \mathrm{H}^n((C_{r_0})_p^{\wedge})$ thus factors through the map

$$\mathrm{H}^n((C_r)_p^{\wedge}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p / \mathbb{Z}_p, \mathrm{H}^{n+1}(C_r)).$$

Let $r_2 \geq r_1$ be an index such that the map $\mathrm{H}^{n+1}(C_{r_2}) \to \mathrm{H}^{n+1}(C_{r_1})$ is the zero map. Then the map $\mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p / \mathbb{Z}_p, \mathrm{H}^{n+1}(C_{r_2})) \to \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p / \mathbb{Z}_p, \mathrm{H}^{n+1}(C_{r_1}))$ is the zero map. By construction, the map

$$\mathrm{H}^n((C_{r_2})_p^{\wedge}) \longrightarrow \mathrm{H}^n((C_{r_0})_p^{\wedge})$$

factors as

$$\mathrm{H}^n((C_{r_2})_p^{\wedge}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p / \mathbb{Z}_p, \mathrm{H}^{n+1}(C_{r_2})) \overset{0}{\longrightarrow} \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p / \mathbb{Z}_p, \mathrm{H}^{n+1}(C_{r_1})) \longrightarrow \mathrm{H}^n((C_{r_0})_p^{\wedge}),$$

and is thus the zero map. The index r_2 does not depend on the prime number p, so the map

$$\prod_{p \in \mathbb{P}} \mathrm{H}^n((C_{r_2})_p^{\wedge}) \longrightarrow \prod_{p \in \mathbb{P}} \mathrm{H}^n((C_{r_0})_p^{\wedge})$$

is the zero map, and the inverse system $\left(\prod_{p \in \mathbb{P}} (C_r)_p^{\wedge} \right)_r$ is essentially zero. $\qquad \square$

Corollary 9.12. *Let $j \geq 0$ be an integer, and let F be one of the presheaves*

$$R\Gamma_{\mathrm{Zar}}\left(-, \mathbb{L}^j_{-/\mathbb{Z}}\right), \;\; R\Gamma_{\mathrm{Zar}}\left(-, \prod_{p \in \mathbb{P}} \mathbb{L}^j_{-_{\mathbb{F}_p}/\mathbb{F}_p}\right), \;\; R\Gamma_{\mathrm{Zar}}\left(-, \prod_{p \in \mathbb{P}} (\mathbb{L}^j_{-/\mathbb{Z}})_p^{\wedge}\right), \;\; \text{and } R\Gamma_{\mathrm{Zar}}\left(-, \mathbb{L}^j_{-_{\mathbb{Q}}/\mathbb{Q}}\right),$$

where $-_{\mathbb{F}_p}$ is the derived base change from $\mathbb{Z}$ to $\mathbb{F}_p$. Then the presheaf F is a pro cdh sheaf on noetherian schemes.

Proof. The presheaf F is a Nisnevich sheaf, so the result is equivalent to proving that F sends an abstract blowup square of noetherian schemes to a weakly cartesian square of pro objects in the derived category $\mathcal{D}(\mathbb{Z})$. For $R\Gamma_{\mathrm{Zar}}\left(-, \mathbb{L}^j_{-/\mathbb{Z}}\right)$, this is Proposition 9.9. For $R\Gamma_{\mathrm{Zar}}\left(-, \prod_{p \in \mathbb{P}} \mathbb{L}^j_{-_{\mathbb{F}_p}/\mathbb{F}_p}\right)$, this is a formal consequence of Proposition 9.9 and Lemma 9.10. For $R\Gamma_{\mathrm{Zar}}\left(-, \prod_{p \in \mathbb{P}} (\mathbb{L}^j_{-/\mathbb{Z}})_p^{\wedge}\right)$, this is similarly a formal consequence of Proposition 9.9 and Lemma 9.11. And for $R\Gamma_{\mathrm{Zar}}\left(-, \mathbb{L}^j_{-_{\mathbb{Q}}/\mathbb{Q}}\right)$, this is a consequence of Proposition 9.9 and the fact that the rationalisation of a zero pro system of abelian groups is zero. $\qquad \square$

9.2 Pro cdh descent for motivic cohomology

In this subsection, we prove pro cdh descent for the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$ (Theorem 9.23). We use Corollary 4.32 to decompose the proof into several steps, which ultimately all rely on Corollary 9.12. We start with the following rational results.

Proposition 9.13 ([EM23]). *For every integer $i \geq 0$, the presheaf $R\Gamma_{\mathrm{Zar}}\big(-, \widehat{\mathbb{L}\Omega}^{\geq i}_{-_{\mathbb{Q}}/\mathbb{Q}}\big)$ is a pro cdh sheaf on noetherian schemes.*

Proof. This is a part of [EM23, proof of Theorem 8.2]. More precisely, one uses Proposition 4.42 to reduce the proof to a finite number of powers of the cotangent complex relative to $\mathbb{Q}$, where this is Corollary 9.12. $\qquad\square$

The following result is a rigid-analytic variant of Proposition 9.13.

Proposition 9.14. *For every integer $i \geq 0$, the presheaf $R\Gamma_{\mathrm{Zar}}\big(-, \prod'_{p\in\mathbb{P}} \widehat{\mathbb{L}\Omega}^{\geq i}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p}\big)$ is a pro cdh sheaf on noetherian schemes.*

Proof. By Remark 4.28, there is a fibre sequence of presheaves

$$R\Gamma_{\mathrm{Zar}}\Big(-, \prod_{p\in\mathbb{P}}{}' \widehat{\mathbb{L}\Omega}^{\geq i}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p}\Big) \longrightarrow R\Gamma_{\mathrm{Zar}}\Big(-, \prod_{p\in\mathbb{P}}{}' \widehat{\mathbb{L}\Omega}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p}\Big) \longrightarrow R\Gamma_{\mathrm{Zar}}\Big(-, \big(\prod_{p\in\mathbb{P}} \big(\mathbb{L}\Omega^{<i}_{-/\mathbb{Z}}\big)^{\wedge}_p\big)_{\mathbb{Q}}\Big)$$

on qcqs derived schemes, and in particular on noetherian schemes. By Corollary 4.43, the presheaf $R\Gamma_{\mathrm{Zar}}\big(-, \prod'_{p\in\mathbb{P}} \widehat{\mathbb{L}\Omega}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p}\big)$ is a cdh sheaf on noetherian schemes, so it is a pro cdh sheaf on noetherian schemes. The presheaf $R\Gamma_{\mathrm{Zar}}\big(-, \big(\prod_{p\in\mathbb{P}} \big(\mathbb{L}\Omega^{<i}_{-/\mathbb{Z}}\big)^{\wedge}_p\big)_{\mathbb{Q}}\big)$ has a finite filtration with graded pieces given by the presheaves $R\Gamma_{\mathrm{Zar}}\big(-, \big(\prod_{p\in\mathbb{P}} \big(\mathbb{L}^j_{-/\mathbb{Z}}\big)^{\wedge}_p\big)_{\mathbb{Q}}\big)$ $(0 \leq j < i)$. These presheaves are pro cdh sheaves on noetherian schemes by Corollary 9.12, so the presheaf $R\Gamma_{\mathrm{Zar}}\big(-, \big(\prod_{p\in\mathbb{P}} \big(\mathbb{L}\Omega^{<i}_{-/\mathbb{Z}}\big)^{\wedge}_p\big)_{\mathbb{Q}}\big)$ is a pro cdh sheaf on noetherian schemes. This implies that the presheaf $R\Gamma_{\mathrm{Zar}}\big(-, \prod'_{p\in\mathbb{P}} \widehat{\mathbb{L}\Omega}^{\geq i}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p}\big)$ is a pro cdh sheaf on noetherian schemes. $\qquad\square$

Proposition 9.15. *For every integer $i \geq 0$, the presheaf $\mathbb{Q}(i)^{\mathrm{mot}}$ is a pro cdh sheaf on noetherian schemes.*

Proof. By Corollary 4.68, there is a fibre sequence of presheaves

$$\mathbb{Q}(i)^{\mathrm{mot}}(-) \longrightarrow \mathbb{Q}(i)^{\mathrm{cdh}}(-) \longrightarrow \mathrm{cofib}\Big(R\Gamma_{\mathrm{Zar}}\big(-, \mathbb{L}\Omega^{<i}_{-_{\mathbb{Q}}/\mathbb{Q}}\big) \longrightarrow R\Gamma_{\mathrm{cdh}}\big(-, \Omega^{<i}_{-_{\mathbb{Q}}/\mathbb{Q}}\big)\Big)[-1]$$

on qcqs derived schemes, and in particular on noetherian schemes. Cdh sheaves are in particular pro cdh sheaves, so it suffices to prove that the presheaf $R\Gamma_{\mathrm{Zar}}\big(X, \mathbb{L}\Omega^{<i}_{-_{\mathbb{Q}}/\mathbb{Q}}\big)$ is a pro cdh sheaf on noetherian schemes. This presheaf has a finite filtration with graded pieces given by the presheaves $R\Gamma_{\mathrm{Zar}}\big(-, \mathbb{L}^j_{-_{\mathbb{Q}}/\mathbb{Q}}\big)$ $(0 \leq j < i)$, so the result is a consequence of Corollary 9.12.

Alternatively, one can prove this result by using Corollary 4.57 and pro cdh descent for algebraic K-theory ([KST18, Theorem A]). $\qquad\square$

Corollary 9.16. *For every integer $i \geq 0$, the presheaf $\mathbb{Q}(i)^{\mathrm{TC}}$ is a pro cdh sheaf on noetherian schemes.*

Proof. By Remark 3.20, the presheaf $\mathbb{Q}(i)^{\mathrm{TC}}$ is a pro cdh sheaf on noetherian schemes if and only if the presheaf $\mathbb{Q}(i)^{\mathrm{mot}}$ is a pro cdh sheaf on noetherian schemes. The result is then a consequence of Proposition 9.15. $\qquad\square$

By Remark 3.20, the presheaf $\mathbb{Q}(i)^{\mathrm{TC}}$ is a pro cdh sheaf on noetherian schemes if and only if the presheaf $\mathbb{Q}(i)^{\mathrm{mot}}$ is a pro cdh sheaf on noetherian schemes. One can then prove Proposition 9.15 alternatively by using Corollary 4.57 and pro cdh descent for algebraic K-theory ([KST18, Theorem A]).

We now turn our attention to Bhatt–Morrow–Scholze's syntomic complexes $\mathbb{Z}_p(i)^{\mathrm{BMS}}$.

Corollary 9.17. *For every integer $i \geq 0$, the presheaf $\left(\prod_{p \in \mathbb{P}} \mathbb{Z}_p(i)^{\mathrm{BMS}}\right)_{\mathbb{Q}}$ is a pro cdh sheaf on noetherian schemes.*

Proof. Rationalising the cartesian square of Corollary 4.32 yields a cartesian square of presheaves

$$
\begin{array}{ccc}
\mathbb{Q}(i)^{\mathrm{TC}}(-) & \longrightarrow & R\Gamma_{\mathrm{Zar}}\left(-, \widehat{\mathbb{L}\Omega}^{\geq i}_{-\mathbb{Q}/\mathbb{Q}}\right) \\
\downarrow & & \downarrow \\
\left(\prod_{p \in \mathbb{P}} \mathbb{Z}_p(i)^{\mathrm{BMS}}(-)\right)_{\mathbb{Q}} & \longrightarrow & R\Gamma_{\mathrm{Zar}}\left(-, \prod'_{p \in \mathbb{P}} \widehat{\mathbb{L}\Omega}^{\geq i}_{-\mathbb{Q}_p/\mathbb{Q}_p}\right)
\end{array}
$$

on qcqs derived schemes, and in particular on noetherian schemes. The other three presheaves of this cartesian square being pro cdh sheaves on noetherian schemes (Propositions 9.13, 9.14, and 9.15), the bottom left presheaf is also a pro cdh sheaf on noetherian schemes. $\qquad\square$

Lemma 9.18. *Let p be a prime number. Then for every integer $i \geq 0$, the presheaf $\mathbb{F}_p(i)^{\mathrm{BMS}}$ is a pro cdh sheaf on noetherian schemes.*

Proof. By [AMMN22, Corollary 5.31], there exists an integer $m \geq 0$ and an equivalence of presheaves[14]

$$
\mathbb{F}_p(i)^{\mathrm{BMS}}(-) \xrightarrow{\sim} \mathrm{fib}\left(\mathrm{can} - \phi_i : (\mathcal{N}^{\geq i}\mathbb{\Delta}_-\{i\}/\mathcal{N}^{\geq i+m}\mathbb{\Delta}_-\{i\})/p \longrightarrow (\mathbb{\Delta}_-\{i\}/\mathcal{N}^{\geq i+m}\mathbb{\Delta}_-\{i\})/p\right).
$$

In particular, it suffices to prove that for every integer $j \geq 0$, the presheaf $\mathcal{N}^j\mathbb{\Delta}_-/p$ is a pro cdh sheaf on noetherian schemes. By [BL22, Remark 5.5.8 and Example 4.7.8], there is a fibre sequence of presheaves

$$
\mathcal{N}^j\mathbb{\Delta}_-\{i\}/p \longrightarrow \mathrm{Fil}^{\mathrm{conj}}_j \overline{\mathbb{\Delta}}_{-/\mathbb{Z}_p[\![\widetilde{p}]\!]}/p \xrightarrow{\Theta+j} \mathrm{Fil}^{\mathrm{conj}}_{j-1} \overline{\mathbb{\Delta}}_{-/\mathbb{Z}_p[\![\widetilde{p}]\!]}/p.
$$

The presheaves $\mathrm{Fil}^{\mathrm{conj}}_j \overline{\mathbb{\Delta}}_{-/\mathbb{Z}_p[\![\widetilde{p}]\!]}/p$ and $\mathrm{Fil}^{\mathrm{conj}}_{j-1} \overline{\mathbb{\Delta}}_{-/\mathbb{Z}_p[\![\widetilde{p}]\!]}/p$ have finite filtrations with graded pieces given by modulo p powers of the cotangent complex. The result is then a consequence of Corollary 9.12. $\quad\square$

Lemma 9.19. *Let A be an abelian group of the form $A = \prod_{p \in \mathbb{P}} A_p$, where A_p is a derived p-complete abelian group. If A is torsion, then A is bounded torsion (i.e., there exists an integer $N \geq 1$ such that A is N-torsion).*

Proof. Assume that the abelian group A is torsion. Then for every prime number p, the abelian group A_p is torsion and derived p-complete, hence it is bounded p-power torsion by [Bha19, Theorem 1.1]. Let S be the set of prime numbers p such that A_p is not the zero group. Then there exists an inclusion of abelian groups $\prod_{p \in S} \mathbb{F}_p \subseteq A$, and, if S is infinite, then $\prod_{p \in S} \mathbb{F}_p$ is not torsion. So S is finite, and, as a finite product of bounded torsion abelian groups, the abelian group A is bounded torsion. $\quad\square$

Proposition 9.20. *For every integer $i \geq 0$, the presheaf $\prod_{p \in \mathbb{P}} \mathbb{Z}_p(i)^{\mathrm{BMS}}$ is a pro cdh sheaf on noetherian schemes.*

Proof. Fix an abstract blowup square of noetherian schemes (3). Let $\{C_r\}_r$ be the pro object in the derived category $\mathcal{D}(\mathbb{Z})$ defined as the total fibre of the commutative square obtained by applying the presheaf $\prod_{p \in \mathbb{P}} \mathbb{Z}_p(i)^{\mathrm{BMS}}$ to this abstract blowup square. We want to prove that $\{C_r\}_r$ is weakly zero. By Corollary 9.17, its rationalisation $\{C_r \otimes_{\mathbb{Z}} \mathbb{Q}\}_r$ is weakly zero.

Let $r_0 \geq 0$ and $n \in \mathbb{Z}$ be integers. Let $r_1 \geq r_0$ be an integer such that the map

$$
\mathrm{H}^n(C_{r_1}) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \mathrm{H}^n(C_{r_0}) \otimes_{\mathbb{Z}} \mathbb{Q}
$$

---

[14]Prismatic cohomology was first defined on p-complete p-quasisyntomic rings ([BMS19, BS22]), and then generalised to arbitrary animated commutative rings by taking the left Kan extension from polynomial $\mathbb{Z}$-algebras, and imposing that it depends only on the derived p-completion of its input ([AMMN22, BL22]). On noetherian rings R, the derived and classical p-completions agree, so the prismatic cohomology of R is naturally identified with the prismatic cohomology of the classical p-completion of R.

is the zero map. We now construct an integer $r_2 \geq r_1$ such that the map

$$\mathrm{H}^n(C_{r_2}) \longrightarrow \mathrm{H}^n(C_{r_0})$$

is the zero map. By Lemma 9.18, and for every prime number p, the pro abelian group $\{\mathrm{H}^n(C_r/p)\}_r$ is zero, which implies that the pro abelian group $\{\mathrm{H}^n(C_r)/p\}_r$ is zero. By induction, this implies that for every integer $N \geq 1$, the pro abelian group $\{\mathrm{H}^n(C_r)/N\}_r$ is zero. By construction, the cohomology groups $\mathrm{H}^n(C_r)$ ($r \geq 0$) are naturally products, indexed by prime numbers p, of derived p-complete abelian groups. The kernel and cokernel of a map of derived p-complete abelian groups are derived p-complete abelian groups. So the image A_{r_0} of the map $\mathrm{H}^n(C_{r_1}) \to \mathrm{H}^n(C_{r_0})$ is a product, indexed by prime numbers p, of derived p-complete abelian groups. This abelian group A_{r_0} is also torsion by definition of the integer r_1, so Lemma 9.19 implies that there exists an integer $N \geq 1$ such that A_{r_0} is N-torsion. Let $r_2 \geq r_1$ be an integer such that the map $\mathrm{H}^n(C_{r_2})/N \to \mathrm{H}^n(C_{r_1})/N$ is the zero map. Then the map

$$\mathrm{H}^n(C_{r_2}) \longrightarrow \mathrm{H}^n(C_{r_0})$$

factors as

$$\mathrm{H}^n(C_{r_2}) \longrightarrow \mathrm{H}^n(C_{r_2})/N \xrightarrow{0} \mathrm{H}^n(C_{r_1})/N \longrightarrow A_{r_0} \subseteq \mathrm{H}^n(C_{r_0}),$$

and is thus the zero map, which concludes the proof. $\square$

Corollary 9.21. *Let p be a prime number. Then for every integer $i \geq 0$, the presheaf $\mathbb{Z}_p(i)^{\mathrm{BMS}}$ is a pro cdh sheaf on noetherian schemes.*

Proof. The presheaf $\mathbb{Z}_p(i)^{\mathrm{BMS}}$ is a direct summand of the presheaf $\prod_{\ell \in \mathbb{P}} \mathbb{Z}_\ell(i)^{\mathrm{BMS}}$, so the result is a consequence of Proposition 9.20. $\square$

Proposition 9.22. *For every integer $i \geq 0$, the presheaf $\mathbb{Z}(i)^{\mathrm{TC}}$ is a pro cdh sheaf on noetherian schemes.*

Proof. By Corollary 4.32, there is a cartesian square of presheaves

$$\begin{array}{ccc} \mathbb{Z}(i)^{\mathrm{TC}} & \longrightarrow & R\Gamma_{\mathrm{Zar}}\big(-, \widehat{\mathbb{L}\Omega}^{\geq i}_{-\mathbb{Q}/\mathbb{Q}}\big) \\ \downarrow & & \downarrow \\ \prod_{p \in \mathbb{P}} \mathbb{Z}_p(i)^{\mathrm{BMS}} & \longrightarrow & R\Gamma_{\mathrm{Zar}}(-, \prod'_{p \in \mathbb{P}} \widehat{\mathbb{L}\Omega}^{\geq i}_{-\mathbb{Q}_p/\mathbb{Q}_p}) \end{array}$$

on qcqs derived schemes, and in particular on noetherian schemes. The presheaves

$$R\Gamma_{\mathrm{Zar}}\big(-, \widehat{\mathbb{L}\Omega}^{\geq i}_{-\mathbb{Q}/\mathbb{Q}}\big), \ R\Gamma_{\mathrm{Zar}}\big(-, \prod_{p \in \mathbb{P}}{}' \widehat{\mathbb{L}\Omega}^{\geq i}_{-\mathbb{Q}_p/\mathbb{Q}_p}\big), \text{ and } \prod_{p \in \mathbb{P}} \mathbb{Z}_p(i)^{\mathrm{BMS}}$$

are pro cdh sheaves on noetherian schemes by Propositions 9.13, 9.14, and 9.20 respectively. So the presheaf $\mathbb{Z}(i)^{\mathrm{TC}}$ is a pro cdh sheaf on noetherian schemes. $\square$

The following result was proved on noetherian schemes over a field by Elmanto–Morrow [EM23].

Theorem 9.23 (Pro cdh descent)**.** *For every integer $i \geq 0$, the motivic complex $\mathbb{Z}(i)^{\mathrm{mot}}$ is a pro cdh sheaf on noetherian schemes.*

Proof. By Remark 3.20, there is a cartesian square of presheaves

$$\begin{array}{ccc} \mathbb{Z}(i)^{\mathrm{mot}} & \longrightarrow & \mathbb{Z}(i)^{\mathrm{TC}} \\ \downarrow & & \downarrow \\ \mathbb{Z}(i)^{\mathrm{cdh}} & \longrightarrow & L_{\mathrm{cdh}}\, \mathbb{Z}(i)^{\mathrm{TC}} \end{array}$$

on qcqs schemes, and in particular on noetherian schemes. The presheaf $\mathbb{Z}(i)^{\mathrm{TC}}$ is a pro cdh sheaf on noetherian schemes by Proposition 9.22. The presheaves $\mathbb{Z}(i)^{\mathrm{cdh}}$ and $L_{\mathrm{cdh}}\, \mathbb{Z}(i)^{\mathrm{TC}}$ are cdh sheaves on noetherian schemes by construction, hence pro cdh sheaves on noetherian schemes. So the presheaf $\mathbb{Z}(i)^{\mathrm{mot}}$ is a pro cdh sheaf. $\square$

Remark 9.24 (Pro cdh descent for algebraic K-theory). The arguments to prove Theorem 9.23 can be adapted to give a new proof of the pro cdh descent for algebraic K-theory of Kerz–Strunk–Tamme [KST18]. More precisely, by Theorem 1.1, pro cdh descent for algebraic K-theory is equivalent to pro cdh descent for TC. By Corollary 4.64, the result rationally reduces to the pro cdh descent for HC, which is proved by Morrow ([Mor16, Theorem 0.2]). The result mod p is similar to that of Lemma 9.18, where the Nygaard filtration and the relative prismatic cohomology are replaced by the Tate filtration and by relative THH; the pro cdh descent for relative THH then reduces to the pro cdh descent for powers of the cotangent complex by [AMMN22, Section 5.2]. Following Section 4.2, there is a natural cartesian square

$$
\begin{array}{ccc}
\mathrm{TC}(-) & \longrightarrow & \mathrm{HC}^{-}(-_{\mathbb{Q}}/\mathbb{Q}) \\
\downarrow & & \downarrow \\
\prod_{p\in\mathbb{P}}\mathrm{TC}(-;\mathbb{Z}_p) & \longrightarrow & \left(\prod_{p\in\mathbb{P}}'\mathrm{HH}(-;\mathbb{Q}_p)\right)^{h\mathrm{S}^1}.
\end{array}
$$

Using the cdh descent for the presheaves $\mathrm{HP}(-_{\mathbb{Q}}/\mathbb{Q})$ ([LT19]) and $\left(\prod_{p\in\mathbb{P}}'\mathrm{HH}(-;\mathbb{Q}_p)\right)^{t\mathrm{S}^1}$ (Corollary 4.43), the pro cdh descent of the two right terms reduces to the pro cdh descent for HC. The integral statement is then similarly a consequence of Lemma 9.19.

9.3 Motivic Weibel vanishing

In this subsection, we prove Theorem 9.27, which is a motivic refinement of Weibel's vanishing conjecture on negative K-groups ([KST18, Theorem B (i)]).

Lemma 9.25. *Let V be a henselian valuation ring. Then for every integer $i \geq 0$, the motivic complex $\mathbb{Z}(i)^{\mathrm{mot}}(V) \in \mathcal{D}(\mathbb{Z})$ is in degrees at most i.*

Proof. Henselian valuation rings are local rings for the cdh topology, so the natural maps

$$\mathbb{Z}(i)^{\mathrm{mot}}(V) \longrightarrow \mathbb{Z}(i)^{\mathrm{cdh}}(V) \longleftarrow \mathbb{Z}(i)^{\mathrm{lisse}}(V)$$

are equivalences in the derived category $\mathcal{D}(\mathbb{Z})$ (Remark 3.23 and Definition 3.10). $\qquad\square$

Lemma 9.26. *Let A be a local ring, and I be a nil ideal of A. Then for every integer $i \geq 0$, the fibre of the natural map*

$$\mathbb{Z}(i)^{\mathrm{mot}}(A) \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(A/I)$$

is in degrees at most i.

Proof. We first prove the result rationally, and modulo p for every prime number p. Any finitary cdh sheaf is invariant under nil extensions. By Corollary 4.68, the result after rationalisation is thus equivalent to the fact that the fibre of the natural map

$$\mathbb{L}\Omega^{<i}_{(A_{\mathbb{Q}})/\mathbb{Q}}[-1] \longrightarrow \mathbb{L}\Omega^{<i}_{((A/I)_{\mathbb{Q}})/\mathbb{Q}}[-1]$$

is in degrees at most i. Both terms of this map are in degrees at most i. In degree i, this map is given by the natural map

$$\Omega^{i-1}_{(A_{\mathbb{Q}})/\mathbb{Q}} \longrightarrow \Omega^{i-1}_{((A/I)_{\mathbb{Q}})/\mathbb{Q}},$$

which is surjective as the $\mathbb{Q}$-algebra $(A/I)_{\mathbb{Q}}$ is a quotient of the $\mathbb{Q}$-algebra $A_{\mathbb{Q}}$. Let p be a prime number. By Corollary 3.25, the result modulo p is equivalent to the fact that the fibre of the natural map

$$\mathbb{F}_p(i)^{\mathrm{BMS}}(A) \longrightarrow \mathbb{F}_p(i)^{\mathrm{BMS}}(A/I)$$

is in degrees at most i. The pair (A, I) is henselian, so this is a consequence of Theorem 2.27.

By the previous rational statement, the fibre $F \in \mathcal{D}(\mathbb{Z})$ of the natural map

$$\mathbb{Z}(i)^{\mathrm{mot}}(A) \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(A/I)$$

has torsion cohomology groups in degrees at least $i+1$. By the short exact sequence of abelian groups

$$0 \longrightarrow \mathrm{H}^j(F)/p \longrightarrow \mathrm{H}^j(F/p) \longrightarrow \mathrm{H}^{j+1}(F)[p] \longrightarrow 0$$

for every prime number p and every integer $j \geq i+1$, the previous torsion statement implies that these cohomology groups are also torsionfree, hence zero, in degrees at least $i+2$. It then remains to prove that the abelian group $\mathrm{H}^{i+1}(F)$ is zero. By Corollary 7.10 and its proof, the abelian group $\mathrm{H}_{\mathrm{mot}}^{i+1}(A, \mathbb{Z}(i))$ is torsionfree, so it suffices to prove that the natural map of abelian groups $\mathrm{H}_{\mathrm{mot}}^i(A, \mathbb{Z}(i)) \to \mathrm{H}_{\mathrm{mot}}^i(A/I, \mathbb{Z}(i))$ is surjective. Let P be a local ind-smooth $\mathbb{Z}$-algebra with a surjective map $P \to A$. By Theorem 7.11 (see also Lemma 7.19 for a related argument), and because $P \to A/I$ is also a surjection from a local ind-smooth $\mathbb{Z}$-algebra, the composite map of abelian groups

$$\mathrm{H}_{\mathrm{mot}}^i(P, \mathbb{Z}(i)) \longrightarrow \mathrm{H}_{\mathrm{mot}}^i(A, \mathbb{Z}(i)) \longrightarrow \mathrm{H}_{\mathrm{mot}}^i(A/I, \mathbb{Z}(i))$$

is surjective, so the right map is surjective, as desired. □

Theorem 9.27 (Motivic Weibel vanishing). *Let $d \geq 0$ be an integer, and X be a noetherian scheme of dimension at most d. Then for every integer $i \geq 0$, the motivic complex $\mathbb{Z}(i)^{\mathrm{mot}}(X) \in \mathcal{D}(\mathbb{Z})$ is in degrees at most $i+d$.*

Proof. The presheaf $\mathbb{Z}(i)^{\mathrm{mot}} : \mathrm{Sch}^{\mathrm{qcqs,op}} \to \mathcal{D}(\mathbb{Z})$ satisfies the following properties:

(1) it is finitary (Corollary 4.60);

(2) it satisfies pro cdh descent on noetherian schemes (Theorem 9.23);

(3) for every henselian valuation ring V, the complex $\mathbb{Z}(i)^{\mathrm{mot}}(V)$ is in degrees at most i (Lemma 9.25);

(4) for every noetherian local ring A and every nilpotent ideal I of A, the fibre of the natural map $\mathbb{Z}(i)^{\mathrm{mot}}(A) \to \mathbb{Z}(i)^{\mathrm{mot}}(A/I)$ is in degrees at most i (Lemma 9.26).

By [BEM], the presheaf $\mathbb{Z}(i)^{\mathrm{cdh}} : \mathrm{Sch}^{\mathrm{qcqs,op}} \to \mathcal{D}(\mathbb{Z})$ is a finitary cdh sheaf which is in degrees at most i on henselian valuation rings, hence it also satisfies the previous properties.

By [EM23, Proposition 8.10] applied to the presheaf $\mathrm{fib}\big(\mathbb{Z}(i)^{\mathrm{mot}} \to \mathbb{Z}(i)^{\mathrm{cdh}}\big)[i]$, this implies that for every noetherian scheme X of dimension at most d, the complex

$$\mathrm{fib}\big(\mathbb{Z}(i)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Z}(i)^{\mathrm{cdh}}(X)\big)$$

is in degrees at most $i+d$. The complex $\mathbb{Z}(i)^{\mathrm{cdh}}(X)$ is also in degrees at most $i+d$ ([BEM]), so the complex $\mathbb{Z}(i)^{\mathrm{mot}}(X)$ is in degrees at most $i+d$. □

Remark 9.28 (Relation to Weibel's K-theoretic vanishing conjecture). Let X be a noetherian scheme of dimension at most d. Theorem 9.27 states that the Atiyah–Hirzebruch spectral sequence

$$E_2^{i,j} = \mathrm{H}_{\mathrm{mot}}^{i-j}(X, \mathbb{Z}(-j)) \Longrightarrow \mathrm{K}_{-i-j}(X)$$

is supported in the left half plane $x \leq d$: see the following representation of the E_2 page, where $\mathrm{H}^j(i)$ denotes the motivic cohomology group $\mathrm{H}_{\mathrm{mot}}^j(X, \mathbb{Z}(i))$.

$\cdots$	0	0	0	0	$\cdots$	0	0	0	0
$\cdots$	0	0	$\mathrm{H}^0(0)$	$\mathrm{H}^1(0)$	$\cdots$	$\mathrm{H}^{d-2}(0)$	$\mathrm{H}^{d-1}(0)$	$\mathrm{H}^d(0)$	0
$\cdots$	0	$\mathrm{H}^0(1)$	$\mathrm{H}^1(1)$	$\mathrm{H}^2(1)$	$\cdots$	$\mathrm{H}^{d-1}(1)$	$\mathrm{H}^d(1)$	$\mathrm{H}^{d+1}(1)$	0
$\cdots$	$\mathrm{H}^0(2)$	$\mathrm{H}^1(2)$	$\mathrm{H}^2(2)$	$\mathrm{H}^3(2)$	$\cdots$	$\mathrm{H}^d(2)$	$\mathrm{H}^{d+1}(2)$	$\mathrm{H}^{d+2}(2)$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$

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In particular, the negative K-groups $\mathrm{K}_{-i-j}(X)$ vanish for $-i-j < -d$ (this is Weibel's vanishing conjecture on algebraic K-theory), and there is a natural edge map isomorphism

$$\mathrm{K}_{-d}(X) \cong \mathrm{H}^d_{\mathrm{mot}}(X, \mathbb{Z}(0))$$

of abelian groups. Using the description of weight zero motivic cohomology (Example 4.69), the latter result recovers the known description of $\mathrm{K}_{-d}(X)$ ([KST18, Corollary D]). Note that Theorem 9.27 is however not a new proof of these results of Kerz–Strunk–Tamme, as our Atiyah–Hirzebruch spectral sequence relating motivic cohomology and algebraic K-theory relies on Theorem 1.1, which itself relies on the results in [KST18].

Remark 9.29. Let X be a noetherian scheme of dimension at most d. Then for every integer $i \geq 0$, the proof of Theorem 9.27 also implies that the natural map

$$\mathbb{Z}(i)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Z}(i)^{\mathrm{cdh}}(X)$$

is surjective on H^{i+d}. For $i = 0$, this map is even an isomorphism on H^d (actually on all cohomology groups, by Example 4.69), thus recovering Weibel's conjecture that the map $\mathrm{K}_{-d}(X) \to \mathrm{KH}_{-d}(X)$ is an isomorphism [Wei80, KST18].

The following result is a description of the group K_{-d+1}, similar to the description of the group K_{-d} predicted by Weibel (Remark 9.28).

Corollary 9.30. *Let $d \geq 0$ be an integer, and X be a noetherian scheme of dimension at most d. Then there is a natural exact sequence*

$$\mathrm{H}^{d-2}_{\mathrm{cdh}}(X, \mathbb{Z}) \xrightarrow{\delta} \mathrm{H}^{d+1}_{\mathrm{mot}}(X, \mathbb{Z}(1)) \longrightarrow \mathrm{K}_{-d+1}(X) \longrightarrow \mathrm{H}^{d-1}_{\mathrm{cdh}}(X, \mathbb{Z}) \longrightarrow 0$$

of abelian groups, where δ is the differential map coming from the E_2-page of the Atiyah–Hirzebruch spectral sequence (Corollary 4.56). Moreover, for every integer $m \geq 2$, if m is invertible in X, then the image of the map $(m-1)\delta$ is m-power torsion.

Proof. The motivic complex $\mathbb{Z}(i)^{\mathrm{mot}}(X)$ is zero for $i < 0$ (Corollary 4.48), and is naturally identified with the complex $R\Gamma_{\mathrm{cdh}}(X, \mathbb{Z})$ for $i = 0$ (Example 4.69). The first statement is then a consequence of the Atiyah–Hirzebruch spectral sequence (Corollary 4.56) and of motivic Weibel's vanishing (Theorem 9.27). The second statement is a consequence of the compatibility of the map δ with the Adams operation ψ^m (Construction 4.10). More precisely, Corollary 4.11 implies that the induced map

$$\delta : \mathrm{H}^{d-2}_{\mathrm{cdh}}(X, \mathbb{Z})[\tfrac{1}{m}] \longrightarrow \mathrm{H}^{d+1}_{\mathrm{mot}}(X, \mathbb{Z}(1))[\tfrac{1}{m}]$$

satisfies the equation $\delta = m\delta$, *i.e.*, $(m-1)\delta = 0$, which implies the desired result. $\square$

9.4 Comparison to pro cdh motivic cohomology

In this subsection, we compare the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$ to Kelly–Saito's pro cdh motivic complexes $\mathbb{Z}(i)^{\mathrm{procdh}}$ (Theorem 9.32). In equicharacteristic, this is [KS24, Corollary 1.11] (see also [EM23, Theorem 1.15]). Our proof is structurally the same, although finitariness and pro cdh descent in mixed characteristic rely on the main results of Section 4, and our proof of the comparison to lisse motivic cohomology is different in mixed characteristic (see comment before Corollary 7.12).

Lemma 9.31. *Let R be a nil extension of a henselian valuation ring, i.e., a commutative ring R whose quotient R/I by its ideal of nilpotent elements I is a henselian valuation ring. Then for every integer $i \geq 0$, the lisse-motivic comparison map*

$$\mathbb{Z}(i)^{\mathrm{lisse}}(R) \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(R)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$.

Proof. By Corollary 7.12, it suffices to prove that the complex $\mathbb{Z}(i)^{\mathrm{mot}}(R) \in \mathcal{D}(\mathbb{Z})$ is in degrees at most i. Let I be the ideal of nilpotent elements of the commutative ring R. Using the natural fibre sequence

$$\mathrm{fib}\big(\mathbb{Z}(i)^{\mathrm{mot}}(R) \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(R/I)\big) \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(R) \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(R/I)$$

in the derived category $\mathcal{D}(\mathbb{Z})$, the result is then a consequence of Lemmas 9.25 and 9.26. $\qquad\square$

Theorem 9.32 (Comparison to pro cdh motivic cohomology)**.** *Let X be a noetherian scheme. Then for every integer $i \geq 0$, the lisse-motivic comparison map induces a natural equivalence*

$$\mathbb{Z}(i)^{\mathrm{procdh}}(X) \xrightarrow{\sim} \mathbb{Z}(i)^{\mathrm{mot}}(X)$$

in the derived category $\mathcal{D}(\mathbb{Z})$.

Proof. The presheaf $\mathbb{Z}(i)^{\mathrm{mot}} : \mathrm{Sch}^{\mathrm{qcqs,op}} \to \mathcal{D}(\mathbb{Z})$ satisfies the following properties:

(1) it is finitary (Corollary 4.60);

(2) it satisfies pro cdh descent on noetherian schemes (Theorem 9.23);

(3) for every pro cdh local ring R, the lisse-motivic comparison map

$$\mathbb{Z}(i)^{\mathrm{lisse}}(R) \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(R)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$ (Lemma 9.31 and [KS24, Proposition 1.7]).

By [KS24, Theorem 9.7],[15] this implies that for every noetherian scheme X, the lisse-motivic comparison map induces a natural equivalence

$$\mathbb{Z}(i)^{\mathrm{procdh}}(X) \xrightarrow{\sim} \mathbb{Z}(i)^{\mathrm{mot}}(X)$$

in the derived category $\mathcal{D}(\mathbb{Z})$. $\qquad\square$

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[15]This theorem is stated for schemes over a field, but the proof works over any noetherian commutative ring.

10 $\mathbb{A}^1$-INVARIANT MOTIVIC COHOMOLOGY-INVARIANT MOTIVIC COHOMOLOGY

The theory of classical motivic cohomology of smooth schemes over a mixed characteristic Dedekind domain [Blo86, Lev01, Gei04], as a theory of $\mathbb{A}^1$-invariant motivic cohomology, admits a natural generalisation to general qcqs schemes. More precisely, Spitzweck constructed in [Spi18], for every qcqs scheme X, an $\mathbb{A}^1$-motivic spectrum $H\mathbb{Z}^{\mathrm{Spi}} \in \mathrm{SH}(\mathbb{Z})$, which represents Bloch's cycle complexes on smooth $\mathbb{Z}$-schemes, and whose pullback to $\mathrm{SH}(B)$, for B a field or a mixed characteristic Dedekind domain, still represents Bloch's cycle complexes on smooth B-schemes. Bachmann then proved in [Bac22] that Spitzweck's construction coincides with the zeroth slice of the homotopy K-theory motivic spectrum $\mathrm{KGL} \in \mathrm{SH}(\mathbb{Z})$. Finally, Bachmann–Elmanto–Morrow recently proved in [BEM] that the slice filtration is compatible with arbitrary pullbacks, thus defining a well-behaved $\mathbb{A}^1$-motivic spectrum $H\mathbb{Z}_X \in \mathrm{SH}(X)$ for arbitrary qcqs schemes X. The associated $\mathbb{A}^1$-invariant motivic complexes

$$\mathbb{Z}(i)^{\mathbb{A}^1}(X) \in \mathcal{D}(\mathbb{Z})$$

are related to the homotopy K-theory $\mathrm{KH}(X)$ by an $\mathbb{A}^1$-invariant Atiyah–Hirzebruch spectral sequence. For our purposes, we will only use that there is a natural map

$$\mathbb{Z}(i)^{\mathrm{cdh}}(X) \longrightarrow \mathbb{Z}(i)^{\mathbb{A}^1}(X)$$

which exhibits the target as the $\mathbb{A}^1$-localisation of the source, and which is an equivalence if the qcqs scheme X satisfies the condition $\mathrm{Val}(X)$ (see Definition 10.12 and Theorem 10.15).

10.1 Comparison to $\mathbb{A}^1$-invariant motivic cohomology

In this subsection, we prove that the $\mathbb{A}^1$-localisation of the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$ recover Bachmann–Elmanto–Morrow's $\mathbb{A}^1$-invariant motivic complexes $\mathbb{Z}(i)^{\mathbb{A}^1}$ (Theorem 10.5). This is a motivic refinement of [Elm21, Theorem 1.0.1], and implies that the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$ recover the classical motivic complexes $\mathbb{Z}(i)^{\mathrm{cla}}$ on smooth schemes over a mixed characteristic Dedekind domain after $\mathbb{A}^1$-localisation (Corollary 10.9).

Lemma 10.1. *Let $\mathcal{C}$ be a presentable stable ∞-category, and $F : \mathrm{Sch}^{\mathrm{qcqs,op}} \to \mathcal{C}$ be a presheaf. Then the natural map*

$$L_{\mathbb{A}^1} L_{\mathrm{cdh}} F \longrightarrow L_{\mathbb{A}^1} L_{\mathrm{cdh}} L_{\mathbb{A}^1} F$$

induced by $\mathbb{A}^1$-localisation is an equivalence of $\mathcal{C}$-valued presheaves.

Proof. A filtered colimit of cdh sheaves is a cdh sheaf (Lemma 4.58). In particular, the $\mathbb{A}^1$-localisation of a cdh sheaf is a cdh sheaf, so the natural composites

$$L_{\mathbb{A}^1} L_{\mathrm{cdh}} F \longrightarrow L_{\mathbb{A}^1} L_{\mathrm{cdh}} L_{\mathbb{A}^1} F \longrightarrow L_{\mathbb{A}^1} L_{\mathrm{cdh}} L_{\mathbb{A}^1} L_{\mathrm{cdh}} F$$

and

$$L_{\mathbb{A}^1} L_{\mathrm{cdh}} L_{\mathbb{A}^1} F \longrightarrow L_{\mathbb{A}^1} L_{\mathrm{cdh}} L_{\mathbb{A}^1} L_{\mathrm{cdh}} F \longrightarrow L_{\mathbb{A}^1} L_{\mathrm{cdh}} L_{\mathbb{A}^1} L_{\mathrm{cdh}} L_{\mathbb{A}^1} F$$

are equivalences in the ∞-category $\mathcal{C}$. This implies the desired result. $\square$

Lemma 10.2. *For every integer $i \geq 0$, the $\mathbb{A}^1$-localisation of the presheaf*

$$R\Gamma_{\mathrm{Zar}}\big(-, \mathbb{L}\Omega^{<i}_{-_{\mathbb{Q}}/\mathbb{Q}}\big) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathcal{D}(\mathbb{Q})$$

is zero.

Proof. By Zariski descent, it suffices to prove the result on animated commutative rings. The functor $\mathbb{L}\Omega^{<i}_{-_{\mathbb{Q}}/\mathbb{Q}}$, from animated commutative rings to the derived category $\mathcal{D}(\mathbb{Q})$, is left Kan extended from polynomial $\mathbb{Z}$-algebras. Equivalently, it commutes with sifted colimits. This property being preserved by $\mathbb{A}^1$-localisation, the functor $L_{\mathbb{A}^1} \mathbb{L}\Omega^{<i}_{-_{\mathbb{Q}}/\mathbb{Q}}$ is also left Kan extended from polynomial $\mathbb{Z}$-algebras. As it is also constant on polynomial $\mathbb{Z}$-algebras, and zero on the zero ring, it is the zero functor. $\square$

Corollary 10.3. *Let X be a qcqs derived scheme. Then for every integer $i \geq 0$, the natural map*

$$\left(L_{\mathbb{A}^1}\widehat{\mathbb{L}\Omega}^{\geq i}_{-\mathbb{Q}/\mathbb{Q}}\right)(X) \longrightarrow \left(L_{\mathbb{A}^1}\widehat{\mathbb{L}\Omega}_{-\mathbb{Q}/\mathbb{Q}}\right)(X)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Q})$.

Proof. There is a natural fibre sequence

$$\left(L_{\mathbb{A}^1}\widehat{\mathbb{L}\Omega}^{\geq i}_{-\mathbb{Q}/\mathbb{Q}}\right)(X) \longrightarrow \left(L_{\mathbb{A}^1}\widehat{\mathbb{L}\Omega}_{-\mathbb{Q}/\mathbb{Q}}\right)(X) \longrightarrow \left(L_{\mathbb{A}^1}\mathbb{L}\Omega^{<i}_{-\mathbb{Q}/\mathbb{Q}}\right)$$

in the derived category $\mathcal{D}(\mathbb{Q})$. The result is then a consequence of Lemma 10.2. $\square$

Lemma 10.4. *Let p be a prime number. Then for every integer $i \geq 0$, the $\mathbb{A}^1$-localisation of the presheaf*

$$\mathbb{F}_p(i)^{\mathrm{BMS}}(-) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathcal{D}(\mathbb{F}_p)$$

is zero.

Proof. The presheaf $\mathbb{F}_p(i)^{\mathrm{BMS}}$ is a Zariski sheaf, and its restriction to animated commutative rings is left Kan extended from polynomial $\mathbb{Z}$-algebras (Corollary 2.26). The result then follows by the same argument as in Lemma 10.2. $\square$

Theorem 10.5. *Let X be a qcqs scheme. Then for every integer $i \geq 0$, the natural map*

$$\left(L_{\mathbb{A}^1}\,\mathbb{Z}(i)^{\mathrm{mot}}\right)(X) \longrightarrow \left(L_{\mathbb{A}^1}\,\mathbb{Z}(i)^{\mathrm{cdh}}\right)(X) \simeq \mathbb{Z}(i)^{\mathbb{A}^1}(X)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$.

Proof. It suffices to prove the result rationally, and modulo p for every prime number p. By Lemma 10.2, the object

$$\left(L_{\mathbb{A}^1}R\Gamma_{\mathrm{Zar}}\left(X,\mathbb{L}\Omega^{<i}_{-\mathbb{Q}/\mathbb{Q}}\right)\right)(X)$$

is zero in the derived category $\mathcal{D}(\mathbb{Q})$. Lemma 10.1 then implies that the object

$$\left(L_{\mathbb{A}^1}R\Gamma_{\mathrm{cdh}}\left(-,\mathbb{L}\Omega^{<i}_{-\mathbb{Q}/\mathbb{Q}}\right)\right)(X)$$

is zero in the derived category $\mathcal{D}(\mathbb{Q})$. In particular, the natural map

$$\left(L_{\mathbb{A}^1}R\Gamma_{\mathrm{Zar}}\left(-,\mathbb{L}\Omega^{<i}_{-\mathbb{Q}/\mathbb{Q}}\right)\right)(X) \longrightarrow \left(L_{\mathbb{A}^1}R\Gamma_{\mathrm{cdh}}\left(-,\mathbb{L}\Omega^{<i}_{-\mathbb{Q}/\mathbb{Q}}\right)\right)(X)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Q})$, which implies the desired result rationally by Corollary 4.68. Similarly, for every prime number p, the syntomic complex $\mathbb{F}_p(i)^{\mathrm{BMS}}$ vanishes after $\mathbb{A}^1$-localisation by Lemma 10.4, which implies the desired result modulo p by Lemma 10.1 and Corollary 3.25. $\square$

Remark 10.6. Let X be a qcqs derived scheme. One can prove, using similar arguments and Corollary 4.32, that there is a natural fibre sequence

$$\left(L_{\mathbb{A}^1}\,\mathbb{Z}(i)^{\mathrm{TC}}\right)(X) \longrightarrow R\Gamma_{\mathrm{Zar}}\left(X,\widehat{\mathbb{L}\Omega}_{-\mathbb{Q}/\mathbb{Q}}\right) \longrightarrow R\Gamma_{\mathrm{Zar}}\left(X,\prod_{p\in\mathbb{P}}{}'\widehat{\mathbb{L}\Omega}_{-\mathbb{Q}_p/\mathbb{Q}_p}\right)$$

in the derived category $\mathcal{D}(\mathbb{Z})$. Note that this implies Theorem 10.5 on qcqs classical schemes, via the cdh descent results [EM23, Lemma 4.5] and Corollary 4.45.

Theorem 10.7. *Let X be a smooth scheme over a mixed characteristic Dedekind domain. Then the natural map*

$$\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(X) \longrightarrow \left(L_{\mathbb{A}^1}\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}\right)(X)$$

is an equivalence of filtered spectra.

Proof. By Proposition 6.2 and its proof, this natural map is an equivalence rationally, so it suffices to prove that it is an equivalence modulo every prime number p. Let p be a prime number. By [BEM], the natural map

$$\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(X) \longrightarrow \big(L_{\mathbb{A}^1}\mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{KH}\big)(X)$$

is an equivalence of filtered spectra, so it suffices to prove that the natural map

$$\big(L_{\mathbb{A}^1}\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}\big)(X)/p \longrightarrow \big(L_{\mathbb{A}^1}\mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{KH}\big)(X)/p$$

is an equivalence of filtered spectra. By Proposition 3.24, this is equivalent to the fact that the natural map

$$\big(L_{\mathbb{A}^1}\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{F}_p)\big)(X) \longrightarrow \big(L_{\mathrm{A}^1}L_{\mathrm{cdh}}\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{F}_p)\big)(X)$$

is an equivalence of filtered spectra. The filtered spectrum $\big(L_{\mathbb{A}^1}\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{F}_p)\big)(X)$ is complete by the connectivity bound of Lemma 2.21, and its graded pieces are zero by Lemma 10.4, so it is zero. By Lemma 10.1, the target $\big(L_{\mathrm{A}^1}L_{\mathrm{cdh}}\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{F}_p)\big)(X)$ of the previous map is then also zero, and this map is in particular an equivalence. $\qquad\square$

Remark 10.8. We expect that the $\mathbb{A}^1$-localisation in Theorem 10.7 is not necessary, *i.e.*, that the natural map

$$\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X) \longrightarrow \big(L_{\mathbb{A}^1}\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}\big)(X)$$

is an equivalence of filtered spectra for smooth schemes X over a torsionfree Dedekind domain. By Theorem 10.5, this is equivalent to the fact that the composite

$$\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X) \xrightarrow{L_{\mathrm{cdh}}} \mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{KH}(X) \xrightarrow{L_{\mathbb{A}^1}} \big(L_{\mathbb{A}^1}\mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{KH}\big)(X)$$

is an equivalence of filtered spectra, where the second map is expected to be an equivalence for every qcqs scheme X ([BEM]).

Corollary 10.9. *Let X be a smooth scheme over a mixed characteristic Dedekind domain. Then for every integer $i \geq 0$, there is a natural equivalence*

$$z^i(X,\bullet)[-2i] \simeq \big(L_{\mathbb{A}^1}\mathbb{Z}(i)^{\mathrm{mot}}\big)(X)$$

in the derived category $\mathcal{D}(\mathbb{Z})$.

Proof. This is a consequence of Theorem 10.7. $\qquad\square$

10.2 F-smoothness of valuation rings

In this subsection, we formulate the key hypothesis used in the work of Bachmann–Elmanto–Morrow [BEM] on $\mathbb{A}^1$-invariant motivic cohomology (Definition 10.12). This hypothesis relies on the following conjecture, implicit in the work of Bhatt–Mathew [BM23] on F-smoothness.

Conjecture 10.10 (Bhatt–Mathew). Every valuation ring is F-smooth.

The following theorem summarizes the known cases of Conjecture 10.10. Note that the notion of F-smoothness implicitly depends on a fixed prime number p, and that a commutative ring with bounded p-power torsion is F-smooth if and only if its p-completion is F-smooth.

Theorem 10.11. *Let V be a valuation ring, and p be a prime number.*

(1) *(Bhatt–Mathew [BM23]) If V is a discrete valuation ring, then V is F-smooth.*

(2) *(Gabber–Ramero [GR03], Gabber [KST21], Kelly–Morrow [KM21]) If V is a valuation ring extension of $\mathbb{F}_p$, then V is F-smooth.*

(3) *(Bouis [Bou23]) If V is a valuation ring extension of a p-torsionfree perfectoid valuation ring of mixed characteristic $(0,p)$, then V is F-smooth.*

Definition 10.12. (1) Let p be a prime number (on which the notion of F-smoothness implicitly depends). A qcqs scheme X *satisfies the condition* $\mathrm{Val}(X, p)$ if every valuation ring V with a map $\mathrm{Spec}(V) \to X$ is F-smooth.

(2) A qcqs scheme X *satisfies the condition* $\mathrm{Val}(X)$ if it satisfies the condition $\mathrm{Val}(X, p)$ for every prime number p.

Examples 10.13. Let p be a prime number.

(1) Every qcqs $\mathbb{Z}[\frac{1}{p}]$-scheme satisfies the condition $\mathrm{Val}(X, p)$ because every valuation ring in which p is invertible is vacuously F-smooth. Consequently, every qcqs $\mathbb{Q}$-scheme satisfies the condition $\mathrm{Val}(X)$.

(2) Every qcqs $\mathbb{F}_p$-scheme satisfies the condition $\mathrm{Val}(X)$ (Theorem 10.11 (2)). More generally, for every integer $N \geq 1$, every qcqs $\mathbb{Z}/N$-scheme satisfies the condition $\mathrm{Val}(X)$.

(3) Every qcqs V-scheme, where V is a commutative ring with bounded p-power torsion and whose p-completion is a perfectoid valuation ring of mixed characteristic $(0, p)$, satisfies the condition $\mathrm{Val}(X)$ (Theorem 10.11 (3) for the condition $\mathrm{Val}(X, p)$, and (1) for the condition $\mathrm{Val}(X, \ell)$ at all other primes ℓ).

Remark 10.14. The results in [BEM] that are conditional on some hypothesis on valuation rings actually assume that valuation rings satisfy a suitable syntomic-étale comparison theorem. Such a condition is weaker than that of Definition 10.12, as this syntomic-étale comparison is a consequence of F-smoothness ([BM23, Theorem 1.8]). Note that all known cases of this syntomic-étale comparison for mixed characteristic valuation rings are proved as consequences of F-smoothness.

10.3 Motivic regularity

By Quillen's fundamental theorem of algebraic K-theory, the algebraic K-theory of a regular noetherian ring is $\mathbb{A}^1$-invariant. Vorst conjectured a partial converse of this result, *i.e.*, that an essentially finite type algebra over a field whose K-groups are $\mathbb{A}^1$-invariant is regular ([Vor79]). The conjecture was proved in characteristic zero by Cortiñas–Haesemeyer–Weibel ([CHW08]). Kerz–Strunk–Tamme then proved a variant of the conjecture in positive characteristic ([KST21, Theorem A]), and asked if Vorst's conjecture holds for general excellent noetherian rings ([KST21, Question D]).

In this subsection, we study the extent to which these questions have a natural analogue in motivic cohomology. More precisely, given a qcqs scheme X and an integer $i \geq 0$, and in light of Theorem 10.5, we will be interested in when the natural map

$$\mathbb{Z}(i)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Z}(i)^{\mathbb{A}^1}(X)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$. We will use repeatedly the following result of [BEM].

Theorem 10.15 ([BEM]). *Let $i \geq 0$ be an integer.*

(1) For every qcqs scheme X, the natural map

$$\mathbb{Q}(i)^{\mathrm{cdh}}(X) \longrightarrow \mathbb{Q}(i)^{\mathbb{A}^1}(X)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Q})$.

(2) For every prime number p and every qcqs $\mathbb{Z}[\frac{1}{p}]$-scheme X, the natural map

$$\mathbb{F}_p(i)^{\mathrm{cdh}}(X) \longrightarrow \mathbb{F}_p(i)^{\mathbb{A}^1}(X)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{F}_p)$.

(3) *For every qcqs scheme X satisfying the condition* $\mathrm{Val}(X)$ *(Definition 10.12), the natural map*

$$\mathbb{Z}(i)^{\mathrm{cdh}}(X) \longrightarrow \mathbb{Z}(i)^{\mathbb{A}^1}(X)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$.

Theorem 10.16 (Rational motivic regularity). *Let X be a qcqs scheme. Then for every integer $i \geq 0$, the following are equivalent:*

(1) *the natural map* $\mathbb{Q}(i)^{\mathrm{mot}}(X) \to \mathbb{Q}(i)^{\mathbb{A}^1}(X)$ *is an equivalence in the derived category* $\mathcal{D}(\mathbb{Q})$;

(2) *the natural map* $\mathbb{Q}(i)^{\mathrm{mot}}(X_{\mathbb{Q}}) \to \mathbb{Q}(i)^{\mathbb{A}^1}(X_{\mathbb{Q}})$ *is an equivalence in the derived category* $\mathcal{D}(\mathbb{Q})$.

Moreover, (1) *or* (2) *for all integers $i \geq 0$ is equivalent to each of the following statements:*

(3) *for every integer $j \geq 0$, the natural map* $R\Gamma_{\mathrm{Zar}}(X, \mathbb{L}^j_{-_{\mathbb{Q}}/\mathbb{Q}}) \to R\Gamma_{\mathrm{cdh}}(X, \Omega^j_{-_{\mathbb{Q}}/\mathbb{Q}})$ *is an equivalence in the derived category* $\mathcal{D}(\mathbb{Q})$;

(4) *the natural map* $\mathrm{K}(X; \mathbb{Q}) \to \mathrm{KH}(X; \mathbb{Q})$ *is an equivalence of spectra;*

(5) *the natural map* $\mathrm{HC}(X_{\mathbb{Q}}/\mathbb{Q}) \to L_{\mathrm{cdh}}\mathrm{HC}(-_{\mathbb{Q}}/\mathbb{Q})(X)$ *is an equivalence of spectra.*

Proof. (4) and (5) are equivalent by Corollary 4.68. Theorem 10.15 (1) and the Adams decompositions Corollary 4.57 and

$$\mathrm{KH}(X; \mathbb{Q}) \simeq \bigoplus_{i \geq 0} \mathbb{Q}(i)^{\mathrm{cdh}}(X)$$

then imply that (1) for all integers $i \geq 0$ and (4) are equivalent. By Corollary 4.68, (1) is equivalent to the fact that the natural map

$$R\Gamma_{\mathrm{Zar}}\big(X, \mathbb{L}\Omega^{<i}_{-_{\mathbb{Q}}/\mathbb{Q}}\big) \longrightarrow R\Gamma_{\mathrm{cdh}}\big(X, \Omega^{<i}_{-_{\mathbb{Q}}/\mathbb{Q}}\big)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Q})$, which is in turn equivalent to (2), and implies that (1) for all integers $i \geq 0$ is equivalent to (3). $\square$

Corollary 10.17 (Motivic regularity in characteristic zero). *Let X be a qcqs $\mathbb{Q}$-scheme. Then the following are equivalent:*

(1) *for every integer $i \geq 0$, the natural map* $\mathbb{Z}(i)^{\mathrm{mot}}(X) \to \mathbb{Z}(i)^{\mathbb{A}^1}(X)$ *is an equivalence in the derived category* $\mathcal{D}(\mathbb{Z})$;

(2) *for every integer $j \geq 0$, the natural map* $R\Gamma_{\mathrm{Zar}}(X, \mathbb{L}^j_{-/\mathbb{Q}}) \to R\Gamma_{\mathrm{cdh}}(X, \Omega^j_{-/\mathbb{Q}})$ *is an equivalence in the derived category* $\mathcal{D}(\mathbb{Q})$;

(3) *the natural map* $\mathrm{K}(X) \to \mathrm{KH}(X)$ *is an equivalence of spectra;*

(4) *the natural map* $\mathrm{HC}(X/\mathbb{Q}) \to L_{\mathrm{cdh}}\mathrm{HC}(-/\mathbb{Q})(X)$ *is an equivalence of spectra.*

Proof. For every prime number p, the natural map

$$\mathrm{K}(X; \mathbb{F}_p) \longrightarrow \mathrm{KH}(X; \mathbb{F}_p)$$

is an equivalence of spectra on qcqs $\mathbb{Q}$-schemes ([Wei89, Proposition 1.6]). Similarly, for any prime number p and integer $i \geq 0$, the natural map

$$\mathbb{F}_p(i)^{\mathrm{mot}}(X) \longrightarrow \mathbb{F}_p(i)^{\mathrm{cdh}}(X)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{F}_p)$ (Remark 3.26). In particular, the natural map

$$\mathrm{K}(X) \longrightarrow \mathrm{KH}(X) \quad (\text{resp. } \mathbb{Z}(i)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Z}(i)^{\mathrm{cdh}}(X))$$

is an equivalence of spectra (resp. in the derived category $\mathcal{D}(\mathbb{Z})$) if and only if the natural map

$$\mathrm{K}(X; \mathbb{Q}) \longrightarrow \mathrm{KH}(X; \mathbb{Q}) \quad (\text{resp. } \mathbb{Q}(i)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Q}(i)^{\mathrm{cdh}}(X))$$

is an equivalence of spectra (resp. in the derived category $\mathcal{D}(\mathbb{Q})$). The result then follows from Theorems 10.15 (1) and 10.16. $\square$

Proposition 10.18 (ℓ-adic motivic regularity)**.** *Let ℓ be a prime number, and X be a qcqs $\mathbb{Z}[\frac{1}{\ell}]$-scheme. Then for any integers $i \geq 0$ and $k \geq 1$, the natural map*

$$\mathbb{Z}/\ell^k(i)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Z}/\ell^k(i)^{\mathbb{A}^1}(X)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z}/\ell^k)$.

Proof. This is a consequence of Remark 3.26 and Theorem 10.15 (2). $\qquad\square$

Recall the following known result at the level of K-theory in positive characteristic.

Corollary 10.19. *Let $N \geq 1$ be an integer, and X be a qcqs $\mathbb{Z}/N$-scheme. Then the natural map $\mathrm{K}(X) \to \mathrm{KH}(X)$ is an equivalence of spectra if and only if for every prime number p dividing N, the natural map $\mathrm{TC}(X; \mathbb{F}_p) \to L_{\mathrm{cdh}} \mathrm{TC}(-; \mathbb{F}_p)(X)$ is an equivalence of spectra.*

Proof. For every prime number p, the functor $\mathrm{TC}(-; \mathbb{F}_p)$ is zero on qcqs $\mathbb{Z}[\frac{1}{p}]$-schemes. The natural map $\mathrm{K}(X) \to \mathrm{KH}(X)$ is an equivalence if and only if it is an equivalence rationally, and modulo p for every prime number p. The result is then a consequence of Theorems 1.1 and 10.16 (4)-(5). $\qquad\square$

The following result is a motivic analogue of Corollary 10.19.

Corollary 10.20 (Motivic regularity in positive characteristic)**.** *Let $N \geq 1$ and $i \geq 0$ be integers, and X be a qcqs $\mathbb{Z}/N$-scheme. Then the natural map $\mathbb{Z}(i)^{\mathrm{mot}}(X) \to \mathbb{Z}(i)^{\mathbb{A}^1}(X)$ is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$ if and only if for every prime number p dividing N, the natural map $\mathbb{F}_p(i)^{\mathrm{BMS}}(X) \to (L_{\mathrm{cdh}} \mathbb{F}_p(i)^{\mathrm{BMS}})(X)$ is an equivalence in the derived category $\mathcal{D}(\mathbb{F}_p)$.*

Proof. By Theorem 10.15 (3) and Example 10.13 (2), the natural map

$$\mathbb{Z}(i)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Z}(i)^{\mathbb{A}^1}(X)$$

is an equivalence if and only if the natural map $\mathbb{Z}(i)^{\mathrm{mot}}(X) \to \mathbb{Z}(i)^{\mathrm{cdh}}(X)$ is an equivalence. The natural map $\mathbb{Z}(i)^{\mathrm{mot}}(X) \to \mathbb{Z}(i)^{\mathrm{cdh}}(X)$ is an equivalence if and only if it is an equivalence rationally, and modulo p for every prime number p. For every prime number p, the functor $\mathbb{F}_p(i)^{\mathrm{BMS}}$ is zero on qcqs $\mathbb{Z}[\frac{1}{p}]$-schemes. The result is then a consequence of Corollary 3.25 and Theorem 10.16 (1)-(2). $\quad\square$

The following result is [EM23, Theorem 6.1 (2)], where we use Theorem 10.15 and Example 10.13 (2) to identify cdh-local and $\mathbb{A}^1$-invariant motivic cohomologies in characteristic p.

Theorem 10.21 ([EM23])**.** *Let p be a prime number, and X be an ind-regular $\mathbb{F}_p$-scheme. Then for every integer $i \geq 0$, the natural map*

$$\mathbb{Z}(i)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Z}(i)^{\mathbb{A}^1}(X)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$.

Corollary 10.22. *Let p be a prime number, V be a characteristic p perfect valuation ring, and X be an ind-smooth scheme over V. Then for every integer $i \geq 0$, the natural map*

$$\mathbb{Z}(i)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Z}(i)^{\mathbb{A}^1}(X)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$.

Proof. By [AD21, Proposition 4.1.1], the perfect valuation ring V is the filtered colimit of its smooth $\mathbb{F}_p$-subalgebras. So the scheme X is in particular ind-smooth over $\mathbb{F}_p$, and the result is a consequence of Theorem 10.21. $\qquad\square$

We now study motivic regularity in mixed characteristic.

Proposition 10.23 (Motivic regularity of valuation rings)**.** *Let V be a henselian valuation ring. If V satisfies the condition $\mathrm{Val}(V)$ (e.g., if V is an extension of a perfectoid valuation ring), then for every integer $i \geq 0$, the natural map*

$$\mathbb{Z}(i)^{\mathrm{mot}}(V) \longrightarrow \mathbb{Z}(i)^{\mathbb{A}^1}(V)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$.

Proof. If the ring V satisfies the condition $\mathrm{Val}(V)$, the natural map

$$\mathbb{Z}(i)^{\mathrm{mot}}(V) \longrightarrow \mathbb{Z}(i)^{\mathbb{A}^1}(V)$$

is naturally identified with the natural map

$$\mathbb{Z}(i)^{\mathrm{mot}}(V) \longrightarrow \mathbb{Z}(i)^{\mathrm{cdh}}(V)$$

in the derived category $\mathcal{D}(\mathbb{Z})$ (Theorem 10.15 (3)). This latter map is an equivalence by Remark 3.23, and because henselian valuation rings are local for the cdh topology. $\qquad\square$

Proposition 10.24 ([BEM])**.** *Let p be a prime number, S be a qcqs scheme of finite valuative dimension and satisfying the condition $\mathrm{Val}(S, p)$, and X be a qcqs S-scheme. Then for any integers $i \geq 0$ and $k \geq 1$, the fibre of the cdh-local Beilinson–Lichtenbaum comparison map (Definition 5.2)*

$$\mathbb{Z}/p^k(i)^{\mathrm{cdh}}(X) \longrightarrow R\Gamma_{\mathrm{ét}}(X[\tfrac{1}{p}], \mu_{p^k}^{\otimes i})$$

in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$ is in degrees at least i.

Proof. The presheaves $\mathbb{Z}/p^k(i)^{\mathrm{cdh}}(-)$ and $R\Gamma_{\mathrm{ét}}(-[\tfrac{1}{p}], \mu_{p^k}^{\otimes i})$ are finitary cdh sheaves on qcqs S-schemes ([BEM] and Theorem 5.1), so it suffices to prove the result on henselian valuation rings V with a map $\mathrm{Spec}(V) \to S$ ([EHIK21, Corollary 2.4.19]). If the henselian valuation ring V is p-torsionfree, the condition $\mathrm{Val}(S, p)$ and Corollary 5.12 imply that the fibre of the Beilinson–Lichtenbaum comparison map

$$\mathbb{Z}/p^k(i)^{\mathrm{mot}}(V) \longrightarrow R\Gamma_{\mathrm{ét}}(V[\tfrac{1}{p}], \mu_{p^k}^{\otimes i})$$

is in degrees at least $i+1$. If the henselian valuation ring V is not p-torsionfree, then it is an $\mathbb{F}_p$-algebra, and it is F-smooth (Theorem 10.11 (2)). In particular, there is a natural equivalence

$$\mathbb{Z}/p^k(i)^{\mathrm{BMS}}(V) \xrightarrow{\ \sim\ } R\Gamma_{\mathrm{ét}}(V, W_k\Omega_{\log}^i)[-i]$$

in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$ ([LM23, Proposition 5.1 (ii)]), and Theorem 5.10 implies that the fibre of the Beilinson–Lichtenbaum comparison map is in degrees at least i. $\qquad\square$

Theorem 10.25. *Let p be a prime number, V be a valuation ring whose p-completion is perfectoid, and X be a p-torsionfree F-smooth scheme over $\mathrm{Spec}(V)$. Then for any integers $i \geq 0$ and $k \geq 1$, the fibre of the natural map*

$$\mathbb{Z}/p^k(i)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Z}/p^k(i)^{\mathbb{A}^1}(X)$$

in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$ is in degrees at least $i+1$.

Proof. By Theorem 10.15 (3) and Example 10.13 (3), this is equivalent to the fact that the fibre of the left vertical map in the natural commutative diagram

$$
\begin{array}{ccc}
\mathbb{Z}/p^k(i)^{\mathrm{mot}}(X) & \longrightarrow & R\Gamma_{\mathrm{ét}}(X[\tfrac{1}{p}], \mu_{p^k}^{\otimes i}) \\
\downarrow & & \downarrow \\
\mathbb{Z}/p^k(i)^{\mathrm{cdh}}(X) & \longrightarrow & \left(L_{\mathrm{cdh}} R\Gamma_{\mathrm{ét}}(-[\tfrac{1}{p}], \mu_{p^k}^{\otimes i})\right)(X)
\end{array}
$$

is in degrees at least $i+1$, *i.e.*, that it is an isomorphism on cohomology groups in degrees less than or equal to $i-1$, and injective in degree i. The right vertical map is an equivalence (Theorem 5.1). The

fibre of the top horizontal map is in degrees at least $i + 1$ (Corollary 5.12). The fibre of the bottom horizontal map is in degrees at least i (Proposition 10.24 and Theorem 10.11 (3)). This implies the desired result, by the following elementary argument. On cohomology groups in degrees at most $i - 2$, all but the left vertical map are isomorphisms, so the left vertical map is an isomorphism. In degree $i - 1$, the top horizontal and right vertical maps are isomorphisms, so the left vertical map is injective; the bottom horizontal map is moreover injective, so the left vertical map is an isomorphism. In degree i, the top horizontal map is injective and the right vertical map is an isomorphism, so the left vertical map is injective. $\qquad\square$

Corollary 10.26. *Let V be a valuation ring whose p-completion is perfectoid for every prime number p,*[16] *and X be an ind-smooth V-scheme. Then for every integer $i \geq 0$, the fibre of the natural map*

$$\mathbb{Z}(i)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Z}(i)^{\mathbb{A}^1}(X)$$

in the derived category $\mathcal{D}(\mathbb{Z})$ is in degrees at least $i + 2$.

Proof. By [AMM22, Corollary 2.3 and Proposition 2.4.2], the natural map

$$\mathrm{K}(X) \longrightarrow \mathrm{KH}(X)$$

is an equivalence of spectra for every ind-smooth scheme over a general valuation ring V. By Theorem 10.1, this implies that the natural map

$$\mathbb{Q}(i)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Q}(i)^{\mathbb{A}^1}(X)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Q})$. That is, the cohomology groups of the fibre of the natural map

$$\mathbb{Z}(i)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Z}(i)^{\mathbb{A}^1}(X)$$

are torsion. In degrees less than or equal to $i + 2$, and if the p-completion of the valuation ring V is perfectoid for every prime number p, then these cohomology groups are also p-torsionfree for every prime number p by Theorem 10.25, and are thus zero. $\qquad\square$

---

[16]Note that this condition is vacuously true for every prime number p different from the residue characteristic of the valuation ring V.

11   EXAMPLES

In this subsection, we revisit certain known results on algebraic K-theory in terms of the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$.

11.1   Perfect and semiperfect rings

Let p be a prime number. It was proved by Kratzer [Kra80, Corollary 5.5] that for every perfect $\mathbb{F}_p$-algebra R and every integer $n \geq 1$, the K-group $\mathrm{K}_n(R)$ is uniquely p-divisible (see also [AMM22] for a mixed characteristic generalisation). It was also proved by Kelly–Morrow that for every $\mathbb{F}_p$-algebra R with perfection R_{perf}, the natural map $\mathrm{K}(R) \to \mathrm{K}(R_{\mathrm{perf}})$ is an equivalence after inverting p ([KM21, Lemma 4.1], see also [EK20, Example 2.1.11] and [Cou23, Theorem 3.1.2 and Proposition 3.3.1] for different proofs). The following result is a motivic refinement of these two facts.

Theorem 11.1 (Motivic cohomology of perfect $\mathbb{F}_p$-schemes, after [EM23]). *Let X be a qcqs $\mathbb{F}_p$-scheme.*

(1) For every integer $i \geq 0$, the natural map

$$\mathbb{Z}(i)^{\mathrm{mot}}(X)[\tfrac{1}{p}] \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(X_{\mathrm{perf}})[\tfrac{1}{p}]$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z}[\tfrac{1}{p}])$.

(2) For every integer $i \geq 1$, the natural map

$$\mathbb{Z}(i)^{\mathrm{mot}}(X_{\mathrm{perf}}) \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(X_{\mathrm{perf}})[\tfrac{1}{p}]$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$.

Proof. By [EM23, Theorem 4.24 (5)],[17] for every integer $i \geq 0$, the natural map

$$\phi_X^* : \mathbb{Z}(i)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(X)$$

induced by the absolute Frobenius $\phi_X : X \to X$ of a qcqs $\mathbb{F}_p$-scheme X is multiplication by p^i. In particular, this natural map is an equivalence after inverting p, and (1) is a consequence of this and the fact that the presheaf $\mathbb{Z}(i)^{\mathrm{mot}}$ is finitary ([EM23, Theorem 4.24 (4)]). Similarly, the same result applied to the perfect $\mathbb{F}_p$-scheme X_{perf} implies that multiplication by p^i on the complex $\mathbb{Z}(i)^{\mathrm{mot}}(X_{\mathrm{perf}}) \in \mathcal{D}(\mathbb{Z})$ is an equivalence. If $i \geq 1$, this is equivalent to the fact that the natural map

$$\mathbb{Z}(i)^{\mathrm{mot}}(X_{\mathrm{perf}}) \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(X_{\mathrm{perf}})[\tfrac{1}{p}]$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$. $\qquad\square$

Remark 11.2 (Negative K-groups of perfect $\mathbb{F}_p$-algebras). It is possible to construct examples of perfect $\mathbb{F}_p$-algebras whose negative K-groups are not p-divisible ([Cou23, Section 3.3]). Theorem 11.1 (2) states that the only non-p-divisible information in the negative K-groups of a perfect $\mathbb{F}_p$-algebra R actually come from weight zero motivic cohomology, *i.e.*, from the complex $R\Gamma_{\mathrm{cdh}}(R, \mathbb{Z})$ (Example 4.69).

Recall that a $\mathbb{F}_p$-algebra is *semiperfect* if its Frobenius is surjective.

Corollary 11.3 (Motivic cohomology of semiperfect $\mathbb{F}_p$-algebras). *Let S be a semiperfect $\mathbb{F}_p$-algebra. Then for every integer $i \geq 1$, the natural commutative diagram*

$$
\begin{array}{ccc}
\mathbb{Z}(i)^{\mathrm{mot}}(S) & \longrightarrow & \mathbb{Z}_p(i)^{\mathrm{syn}}(S) \\
\downarrow & & \downarrow \\
\mathbb{Z}(i)^{\mathrm{mot}}(S_{\mathrm{perf}}) & \longrightarrow & \mathbb{Z}_p(i)^{\mathrm{syn}}(S_{\mathrm{perf}})
\end{array}
$$

is a cartesian square in the derived category $\mathcal{D}(\mathbb{Z})$.

---

[17]This result is proved as a consequence of the same result in classical motivic cohomology [GL00] and in syntomic cohomology [AMMN22], and ultimately goes back to the fact that the Frobenius acts by multiplication by p^i on the logarithmic de Rham–Witt sheaf $W\Omega_{\mathrm{log}}^i$.

Proof. It suffices to prove the result modulo p, and after inverting p. After inverting p, the vertical maps become equivalences by Theorem 11.1 (1) (and the same argument for syntomic cohomology). We prove now the result modulo p. By Theorem 11.1 (2) (and the same argument for syntomic cohomology), the bottom terms of the commutative diagram are zero modulo p, so it suffices to prove that the natural map

$$\mathbb{F}_p(i)^{\mathrm{mot}}(S) \longrightarrow \mathbb{F}_p(i)^{\mathrm{syn}}(S)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{F}_p)$. By [EM23, Corollary 4.32] (see also Theorem 5.10 for a mixed characteristic generalisation), this is equivalent to the fact that

$$R\Gamma_{\mathrm{cdh}}(S, \widetilde{\nu}(i))[-i-1] \simeq 0$$

in the derived category $\mathcal{D}(\mathbb{F}_p)$. By definition, the Frobenius map $\phi_S : S \to S$ is surjective, and has nilpotent kernel. The presheaf $R\Gamma_{\mathrm{cdh}}(-, \widetilde{\nu}(i))[-i-1]$ is a finitary cdh sheaf, so the natural map

$$R\Gamma_{\mathrm{cdh}}(S, \widetilde{\nu}(i))[-i-1] \longrightarrow R\Gamma_{\mathrm{cdh}}(S_{\mathrm{perf}}, \widetilde{\nu}(i))[-i-1]$$

is then an equivalence in the derived category $\mathcal{D}(\mathbb{F}_p)$. The target of this map is zero by Theorem 11.1 (2) (where we use that $i \geq 1$, and the same argument for syntomic cohomology), and applying [EM23, Corollary 4.32] to the perfect $\mathbb{F}_p$-algebra S_{perf}. □

11.2 Finite chain rings

Finite chain rings are commutative rings $\mathcal{O}_K/\pi^n$, where $\mathcal{O}_K$ is a mixed characteristic discrete valuation ring with finite residue field, π is a uniformizer of $\mathcal{O}_K$, and $n \geq 1$ is an integer. Examples of finite chain rings thus include finite fields, rings of the form $\mathbb{Z}/p^n$, and truncated polynomials over a finite field.

Lemma 11.4. *Let $\mathcal{O}_K$ be a discrete valuation ring of mixed characteristic $(0, p)$ and with finite residue field $\mathbb{F}_q$, π be a uniformizer of $\mathcal{O}_K$, and $n \geq 1$ be an integer. Then for every integer $i \geq 0$, there is a natural equivalence*

$$\mathbb{Z}(i)^{\mathrm{mot}}(\mathcal{O}_K/\pi^n) \simeq \begin{cases} \mathbb{Z}[0] & \text{if } i = 0 \\ \mathbb{Z}_p(i)^{\mathrm{BMS}}(\mathcal{O}_K/\pi^n) \oplus \mathbb{Z}(i)^{\mathrm{mot}}(\mathbb{F}_q)[\tfrac{1}{p}] & \text{if } i \geq 1 \end{cases}$$

in the derived category $\mathcal{D}(\mathbb{Z})$.

Proof. The result for $i = 0$ follows from the equivalences

$$\mathbb{Z}(0)^{\mathrm{mot}}(\mathcal{O}_K/\pi^n) \simeq R\Gamma_{\mathrm{cdh}}(\mathcal{O}_K/\pi^n, \mathbb{Z}) \simeq R\Gamma_{\mathrm{cdh}}(\mathbb{F}_q, \mathbb{Z}) \simeq \mathbb{Z}[0]$$

in the derived category $\mathcal{D}(\mathbb{Z})$, the first equivalence being Example 4.69, the second equivalence being nilpotent invariance of cdh sheaves, and the last equivalence being a consequence of the fact that fields are local for the cdh topology.

For every integer $i \geq 0$, the commutative diagram

$$\begin{array}{ccc}
\mathbb{Z}(i)^{\mathrm{mot}}(\mathcal{O}_K/\pi^n) & \longrightarrow & \mathbb{Z}_p(i)^{\mathrm{BMS}}(\mathcal{O}_K/\pi^n) \\
\downarrow & & \downarrow \\
\mathbb{Z}(i)^{\mathrm{mot}}(\mathbb{F}_q) & \longrightarrow & \mathbb{Z}_p(i)^{\mathrm{BMS}}(\mathbb{F}_q)
\end{array}$$

is a cartesian square in the derived category $\mathcal{D}(\mathbb{Z})$ (Theorem 3.28). If $i \geq 1$, the bottom right term vanishes (use for instance the description of Bhatt–Morrow–Scholze's syntomic cohomology in characteristic p in terms of logarithmic de Rham–Witt forms), and there is a natural equivalence

$$\mathbb{Z}(i)^{\mathrm{mot}}(\mathbb{F}_q) \xrightarrow{\sim} \mathbb{Z}(i)^{\mathrm{mot}}(\mathbb{F}_q)[\tfrac{1}{p}]$$

in the derived category $\mathcal{D}(\mathbb{Z})$ (by a classical result in motivic cohomology, see also Theorem 11.1 (2) for a more general statement), hence the desired result. □

Proposition 11.5. *Let $\mathcal{O}_K$ be a mixed characteristic discrete valuation ring with finite residue field, π be a uniformizer of $\mathcal{O}_K$, and $n \geq 1$ be an integer. Then for every integer $m \in \mathbb{Z}$, there is a natural isomorphism*

$$
K_m(\mathcal{O}_K/\pi^n) \cong \left\{ \begin{array}{ll} \mathbb{Z} & \text{if } m = 0 \\ \mathrm{H}^1_{\mathrm{mot}}(\mathcal{O}_K/\pi^n, \mathbb{Z}(i)) & \text{if } m = 2i-1,\ i \geq 1 \\ \mathrm{H}^2_{\mathrm{mot}}(\mathcal{O}_K/\pi^n, \mathbb{Z}(i)) & \text{if } m = 2i-2,\ i \geq 2 \\ 0 & \text{if } m < 0 \end{array} \right.
$$

of abelian groups.

Proof. Let p be the residue characteristic of the discrete valuation ring $\mathcal{O}_K$. The result with p-adic coefficients is [AKN24, Corollary 2.16]. The result with $\mathbb{Z}[\frac{1}{p}]$-coefficients reduces to the case $n = 1$, where the result follows from the description of the (classical) motivic cohomology of finite fields. The integral result is then a consequence of Lemma 11.4. $\qquad\square$

Theorem 11.6 (Motivic cohomology of finite chain rings, after [AKN24]). *Let $\mathcal{O}_K$ be a discrete valuation ring of mixed characteristic $(0, p)$ and with finite residue field $\mathbb{F}_q$, π be a uniformizer of $\mathcal{O}_K$, and $n \geq 1$ be an integer. Then for every integer $i \geq 4p^n$,[18] the motivic complex*

$$
\mathbb{Z}(i)^{\mathrm{mot}}(\mathcal{O}_K/\pi^n) \in \mathcal{D}(\mathbb{Z})
$$

is concentrated in degree one, where it is given by a group of order $(q^i - 1)q^{i(n-1)}$.

Proof. This is a consequence of Lemma 11.4, the classical computation of the motivic cohomology of $\mathbb{F}_q$, and [AKN24, Theorem 1.4 and Proposition 1.5]. $\qquad\square$

Remark 11.7 (Nilpotence of v_1). Antieau–Krause–Nikolaus also determine the nilpotence degree of the element v_1 in the mod p syntomic cohomology of $\mathbb{Z}/p^n$ ([AKN24, Theorem 1.8]). This is a refinement of the key result in the study of $K(1)$-local K-theory of Bhatt–Clausen–Mathew [BCM20]. Note that this result on the nilpotence degree of v_1 can be reformulated, via Lemma 11.4, as a statement on the mod p motivic cohomology of $\mathbb{Z}/p^n$.

11.3 Valuation rings

Recall that a valuation ring is an integral domain V such that for any elements f and g in V, either $f \in gV$ or $g \in fV$. In recent years, valuation rings have been used as a way to bypass resolution of singularities, in order to adapt arguments from characteristic zero to more general contexts [KST21, KM21, Bou23, BEM]. In this subsection, we describe the motivic cohomology of valuation rings (Theorems 11.8 and 11.13). We start with the following result, stating that the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$, on henselian valuation rings, have a description purely in terms of algebraic cycles. See [EM23, Section 9] for related results over a field.

Theorem 11.8. *Let V be a henselian valuation ring. Then for every integer $i \geq 0$, the motivic complex $\mathbb{Z}(i)^{\mathrm{mot}}(V) \in \mathcal{D}(\mathbb{Z})$ is in degrees at most i, and the lisse-motivic comparison map (Definition 7.1)*

$$
\mathbb{Z}(i)^{\mathrm{lisse}}(V) \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(V)
$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$.

Proof. The second statement already appears in the proof of Lemma 9.25. As in Lemma 9.25 or Corollary 7.12, the first statement is then a consequence of [Gei04, Corollary 4.4]. $\qquad\square$

Example 11.9. Let V be a henselian valuation ring. By Example 4.69, there is a natural equivalence

$$
\mathbb{Z}(0)^{\mathrm{mot}}(V) \simeq \mathbb{Z}[0]
$$

---

[18]Note that this is not an optimal lower bound on the integer i. See [AKN24, Theorem 1.4] for a more precise result, in terms of the ramification index of $\mathcal{O}_K$.

in the derived category $\mathcal{D}(\mathbb{Z})$. Similarly, Theorem 11.8, Example 3.9, and the fact that the Picard group of a local ring is zero, imply that the motivic complex $\mathbb{Z}(1)^{\mathrm{mot}}(V) \in \mathcal{D}(\mathbb{Z})$ is concentrated in degree one, where it is given by

$$\mathrm{H}^1_{\mathrm{mot}}(V, \mathbb{Z}(1)) \cong V^\times.$$

We now apply the results of the previous sections to give an alternative description of the motivic cohomology of valuation rings with finite coefficients. The following proposition will be used to reformulate the results of [Bou23] on syntomic cohomology in terms of motivic cohomology.

Proposition 11.10. *Let p be a prime number, and V be a henselian valuation ring. Then for any integers $i \geq 0$ and $k \geq 1$, there is a natural equivalence*

$$\mathbb{Z}/p^k(i)^{\mathrm{mot}}(V) \xrightarrow{\sim} \tau^{\leq i} \mathbb{Z}/p^k(i)^{\mathrm{syn}}(V)$$

in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$.

Proof. Henselian valuation rings are local rings for the cdh topology, so this is a consequence of Theorem 5.10. □

The following result is an analogue for valuation rings of Geisser–Levine's description of motivic cohomology of smooth $\mathbb{F}_p$-algebras [GL00]. It can be deduced from the results of Kelly–Morrow [KM21] and Elmanto–Morrow [EM23].

Theorem 11.11. *Let p be a prime number, and V be a henselian valuation ring of characteristic p. Then for any integers $i \geq 0$ and $k \geq 1$, there is a natural equivalence*

$$\mathbb{Z}/p^k(i)^{\mathrm{mot}}(V) \xrightarrow{\sim} W_k\Omega^i_{V,\log}[-i]$$

in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$.

Proof. Valuation rings of characteristic p are Cartier smooth over $\mathbb{F}_p$ by results of Gabber–Ramero and Gabber ([Bou23, Theorem 3.4]), so this is a consequence of [LM23, Proposition 5.1 (ii)] and Proposition 11.10. □

We then prove a mixed characteristic version of Theorem 11.11, starting with the following ℓ-adic general result.

Proposition 11.12. *Let p be a prime number, and V be a henselian valuation ring such that p is invertible in V. Then for any integers $i \geq 0$ and $k \geq 1$, the Beilinson–Lichtenbaum comparison map (Definition 5.3) naturally factors through an equivalence*

$$\mathbb{Z}/p^k(i)^{\mathrm{mot}}(V) \xrightarrow{\sim} \tau^{\leq i} R\Gamma_{\mathrm{\acute{e}t}}(\mathrm{Spec}(V), \mu_{p^k}^{\otimes i})$$

in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$.

Proof. By Proposition 11.10, the motivic complex $\mathbb{Z}/p^k(i)^{\mathrm{mot}}(V) \in \mathcal{D}(\mathbb{Z}/p^k)$ is in degrees at most i, so the result is a consequence of Corollary 5.6. □

The following result generalises Proposition 11.12 when p is not necessarily invertible in the valuation ring V, at least over a perfectoid base.

Theorem 11.13 (Motivic cohomology of valuation rings with finite coefficients)**.** *Let p be a prime number, V_0 be a p-torsionfree valuation ring whose p-completion is a perfectoid ring, and V be a henselian valuation ring extension of V_0. Then for any integers $i \geq 0$ and $k \geq 1$, the Beilinson–Lichtenbaum comparison map (Definition 5.3) induces a natural map*

$$\mathbb{Z}/p^k(i)^{\mathrm{mot}}(V) \longrightarrow \tau^{\leq i} R\Gamma_{\mathrm{\acute{e}t}}(\mathrm{Spec}(V[\tfrac{1}{p}]), \mu_{p^k}^{\otimes i})$$

in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$, which is an isomorphism in degrees less than or equal to $i-1$. On H^i, this map is injective, with image generated by symbols, via the symbol map

$$(V^\times)^{\otimes i} \to \mathrm{H}^i_{\mathrm{\acute{e}t}}(\mathrm{Spec}(V[\tfrac{1}{p}]), \mu_{p^k}^{\otimes i}).$$

Proof. The fact that the Beilinson–Lichtenbaum comparison map factors through the complex

$$\tau^{\leq i} R\Gamma_{\text{ét}}(\operatorname{Spec}(V[\tfrac{1}{p}]), \mu_{p^k}^{\otimes i}) \in \mathcal{D}(\mathbb{Z}/p^k)$$

is a consequence of Proposition 11.10. The isomorphism in degrees less than or equal to $i-1$ and the injectivity in degree i of this map are then a consequence of [Bou23, Theorems 3.1 and 4.12]. The last statement is a consequence of the isomorphism

$$\widehat{\mathrm{K}}_i^{\mathrm{M}}(V)/p^k \xrightarrow{\cong} \mathrm{H}_{\mathrm{mot}}^i(V, \mathbb{Z}/p^k(i))$$

of abelian groups (Theorem 7.21 and Corollary 7.10). $\qquad\square$

Remark 11.14. The generation by symbols appearing in Theorem 11.13 was also studied in the context of syntomic cohomology of general p-torsionfree F-smooth schemes by Bhatt–Mathew [BM23]. Note that all valuation rings are conjecturally F-smooth (see Conjecture 10.10), and that the proof of Theorem 11.13 adapts more generally to any henselian F-smooth valuation ring.

11.4 $\mathbb{C}^\star$-algebras

By Gelfand representation theorem, the commutative $\mathbb{C}^\star$-algebras are exactly the algebras of continuous complex-valued functions $\mathscr{C}(X;\mathbb{C})$ on a compact Hausdorff space X. An important theorem of Cortiñas–Thom states that commutative $\mathbb{C}^\star$-algebras are K-regular ([CT12, Theorem 1.5]). This result was further generalised recently by Aoki to all smooth algebras over commutative $\mathbb{C}^\star$-algebras, and over a general local field ([Aok24, Theorem 8.7]). The following result is a motivic analogue of the latter result.

Theorem 11.15 ($\mathbb{C}^\star$-algebras are motivically regular, after [CT12, Aok24]). *Let X be a compact Hausdorff space, F be a characteristic zero local field, and A be a smooth $\mathscr{C}(X;F)$-algebra. Then for any integers $i \geq 0$ and $n \geq 0$, the natural map*

$$\mathbb{Z}(i)^{\mathrm{mot}}(A) \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(A[T_1, \ldots, T_n])$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$.

Proof. By [Aok24, Theorem 8.7 (2)], the natural map

$$\mathrm{K}(A[T_1, \ldots, T_n]) \longrightarrow \mathrm{KH}(A[T_1, \ldots, T_n])$$

is an equivalence of spectra for every integer $n \geq 0$. By Corollary 10.17, this implies that the vertical maps in the commutative diagram

$$\begin{array}{ccc}
\mathbb{Z}(i)^{\mathrm{mot}}(A) & \longrightarrow & \mathbb{Z}(i)^{\mathrm{mot}}(A[T_1, \ldots, T_n]) \\
\downarrow & & \downarrow \\
\mathbb{Z}(i)^{\mathbb{A}^1}(A) & \longrightarrow & \mathbb{Z}(i)^{\mathbb{A}^1}(A[T_1, \ldots, T_n])
\end{array}$$

are equivalences in the derived category $\mathcal{D}(\mathbb{Z})$. The bottom horizontal map is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$ by definition of the presheaf $\mathbb{Z}(i)^{\mathbb{A}^1}$ ([BEM], see also Section 10). So the top horizontal map is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$. $\qquad\square$

11.5 Truncated polynomials

In this subsection, we study the motivic cohomology of truncated polynomials, *i.e.*, the motivic cohomology of commutative rings of the form $R[x]/(x^e)$. Given a $\mathcal{D}(\mathbb{Z})$-valued functor $F(-)$, a commutative ring R, and an integer $e \geq 1$, we use the notation

$$F(R[x]/(x^e), (x)) := \mathrm{fib}(F(R[x]/(x^e)) \longrightarrow F(R)),$$

where the map is induced by the canonical projection $R[x]/(x^e) \to R$.

The relative K-theory $\mathrm{K}(k[x]/(x^e), (x))$ of truncated polynomials over a perfect field k of positive characteristic was computed by Hesselholt–Madsen [HM97b, HM97a], using topological restriction homology. Their calculation was reproved by Speirs [Spe20] using Nikolaus–Scholze's approach to topological cyclic homology [NS18], and by Mathew [Mat22] and Sulyma [Sul23] using Bhatt–Morrow–Scholze's filtration on topological cyclic homology [BMS19]. This last approach was then extended to mixed characteristic by Riggenbach [Rig22]. More precisely, Riggenbach used computations in prismatic cohomology to extend the previous result to a computation of the p-adic relative K-theory $\mathrm{K}(R[x]/(x^e), (x); \mathbb{Z}_p)$ of perfectoid rings R, and also reproved the p-adic part of the known description of $\mathrm{K}(\mathbb{Z}[x]/(x^e), (x))$, originally due to Angeltveit–Gerhardt–Hesselholt [AGH09].

This recent progress would seem to indicate that K-theory calculations using equivariant stable homotopy may be pushed further by using cohomological techniques. Note however that the calculations in [Mat22, Sul23, Rig22] are purely p-adic ones, as they rely on (instances of) prismatic cohomology. In fact, all of the previous integral calculations in mixed characteristic (*i.e.*, for R the ring of integers of a number field) rely on a rational result of Soulé [Sou81] and Staffeldt [Sta85], who compute the ranks of the associated relative K-groups using equivariant homotopy theory. In this subsection, we revisit and extend this rational computation, and discuss some natural motivic refinements of the previous results.

All of the above calculations use trace methods, via the Dundas–Goodwillie–McCarthy theorem. We first state the corresponding results at the level of cohomology theories.

Lemma 11.16. *Let R be a commutative ring, and $e \geq 1$ be an integer. Then for every integer $i \geq 0$, the natural map*

$$\mathbb{Z}(i)^{\mathrm{mot}}(R[x]/(x^e), (x)) \longrightarrow \mathbb{Z}(i)^{\mathrm{TC}}(R[x]/(x^e), (x))$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$.

Proof. This is a direct consequence of Remark 3.20, and the fact that cdh sheaves are invariant under nilpotent extensions. $\square$

Corollary 11.17. *Let R be a commutative ring, $e \geq 1$ be an integer, and p be a prime number. Then for every integer $i \geq 0$, the natural map*

$$\mathbb{Z}_p(i)^{\mathrm{mot}}(R[x]/(x^e), (x)) \longrightarrow \mathbb{Z}_p(i)^{\mathrm{BMS}}(R[x]/(x^e), (x))$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z}_p)$.

Proof. This is a consequence of Lemma 11.16. $\square$

Corollary 11.18. *Let R be a commutative ring, and $e \geq 1$ be an integer. Then for every integer $i \geq 0$, there is a natural equivalence*

$$\mathbb{Q}(i)^{\mathrm{mot}}(R[x]/(x^e), (x)) \simeq \mathbb{L}\Omega^{<i}_{(R[x]/(x^e), (x))_{\mathbb{Q}}/\mathbb{Q}}[-1]$$

in the derived category $\mathcal{D}(\mathbb{Q})$.

Proof. This is a consequence of Lemma 11.16 and cdh descent for the presheaf $\widehat{\mathbb{L}\Omega}_{-/\mathbb{Q}}$ on commutative $\mathbb{Q}$-algebras ([EM23, Lemma 4.5]). $\square$

Lemma 11.19. *For every commutative ring R and integer $e \geq 1$, the object*

$$\mathbb{Z}(0)^{\mathrm{mot}}(R[x]/(x^e), (x))$$

is zero in the derived category $\mathcal{D}(\mathbb{Z})$.

Proof. This is a consequence of the fact that the motivic complex $\mathbb{Z}(0)^{\mathrm{mot}}$ is a cdh sheaf (Example 4.69). $\square$

Lemma 11.20. *For any integers $e \geq 1$ and $i \geq 0$, the complex*

$$\mathbb{L}\Omega^{\leq i}_{(\mathbb{Q}[x]/(x^e),(x))/\mathbb{Q}} \in \mathcal{D}(\mathbb{Q})$$

is concentrated in degree zero, given by a $\mathbb{Q}$-vector space of dimension $e - 1$.

Proof. This follows from a standard argument using the natural grading of the $\mathbb{Q}$-algebra $\mathbb{Q}[x]/(x^e)$ and the $\mathbb{Q}$-linear derivation $d : \mathbb{Q}[x]/(x^e) \to \mathbb{Q}[x]/(x^e)$ given by $d(x^j) = jx^j$; see for instance the proof of [Sta85, Proposition 5]. $\square$

Theorem 11.21. *Let R be a commutative ring such that the cotangent complex $\mathbb{L}_{(R \otimes_{\mathbb{Z}} \mathbb{Q})/\mathbb{Q}}$ vanishes (e.g., if $R \otimes_{\mathbb{Z}} \mathbb{Q}$ is ind-étale over $\mathbb{Q}$),[19] and $e \geq 1$ be an integer. Then for every integer $i \geq 1$, there is a natural equivalence*

$$\mathbb{Q}(i)^{\mathrm{mot}}(R[x]/(x^e),(x)) \simeq (R \otimes_{\mathbb{Z}} \mathbb{Q})^{e-1}[-1]$$

in the derived category $\mathcal{D}(\mathbb{Q})$.

Proof. By Corollary 11.18, there is a natural equivalence

$$\mathbb{Q}(i)^{\mathrm{mot}}(R[x]/(x^e),(x)) \simeq \mathbb{L}\Omega^{<i}_{(R[x]/(x^e),(x))_{\mathbb{Q}}/\mathbb{Q}}[-1]$$

in the derived category $\mathcal{D}(\mathbb{Q})$. By the Künneth formula for derived de Rham cohomology, and because all the positive powers of the cotangent complex $\mathbb{L}_{(R \otimes_{\mathbb{Z}} \mathbb{Q})/\mathbb{Q}}$ vanish, there is a natural equivalence

$$\mathbb{L}\Omega^{<i}_{(R[x]/(x^e),(x))_{\mathbb{Q}}/\mathbb{Q}} \simeq \mathbb{L}\Omega^{<i}_{(\mathbb{Q}[x]/(x^e),(x))/\mathbb{Q}} \otimes_{\mathbb{Q}} R$$

in the derived category $\mathcal{D}(\mathbb{Q})$. The result is then a consequence of Lemma 11.20. $\square$

When R is the ring of integers of a number field, the following result is due to Soulé [Sou81] when $e = 2$, and to Staffeldt [Sta85] for $e \geq 2$ a general integer. Their proof uses rational homotopy theory, and ultimately reduces to a computation in cyclic homology.

Corollary 11.22. *Let R be a commutative ring such that the cotangent complex $\mathbb{L}_{(R \otimes_{\mathbb{Z}} \mathbb{Q})/\mathbb{Q}}$ vanishes, and $e \geq 1$ be an integer. Then for every integer $n \in \mathbb{Z}$, there is a natural isomorphism*

$$\mathrm{K}_n(R[x]/(x^e),(x);\mathbb{Q}) \cong \left\{ \begin{array}{ll} (R \otimes_{\mathbb{Z}} \mathbb{Q})^{e-1} & \text{if } n \text{ is odd and } n \geq 1 \\ 0 & \text{otherwise} \end{array} \right.$$

of abelian groups.

Proof. This is a consequence of Theorem 11.21 and Corollary 4.57. $\square$

Remark 11.23. Let K be a number field, $\mathcal{O}_K$ be its ring of integers, and $e \geq 1$ be an integer. The orders in the torsion part of the relative K-theory $\mathrm{K}(\mathcal{O}_K[x]/(x^e),(x))$ were completely determined in [Rig22, Remark 1.8]. It would be interesting to use this result and Theorem 11.21 to obtain an integral description of the relative motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}(\mathcal{O}_K[x]/(x^e),(x))$ for all $i \geq 0$. This would in particular reprove and generalise the result for $K = \mathbb{Q}$ of Angeltveit–Gerhardt–Hesselholt [AGH09].

We also deduce from the work of Riggenbach the following motivic interpretation of the analogous result in K-theory ([Rig22, Theorem 1.1]).

Theorem 11.24 (Truncated polynomials over perfectoids, after [Rig22])**.** *Let R be a perfectoid ring, and $e \geq 1$ be an integer. Then for every integer $i \geq 1$, there is a natural equivalence*

$$\mathbb{Z}_p(i)^{\mathrm{mot}}(R[x]/(x^e),(x)) \simeq \mathbb{W}_{ei}(R)/V_e\mathbb{W}_i(R)[-1]$$

in the derived category $\mathcal{D}(\mathbb{Z}_p)$, where $\mathbb{W}(R)$ denotes the big Witt vectors of R, and V the associated Verschiebung operator.

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[19]See [MM22] for more on this condition.

Proof. This is a consequence of [Rig22, proof of Corollary 6.5] and Corollary 11.17. □

Remark 11.25 (Cuspidal curves)**.** The algebraic K-theory of cuspidal curves (*i.e.*, curves that are defined by an equation of the form $y^a - x^b$, for $a, b \geq 2$ coprime integers) was completely determined over a perfect $\mathbb{F}_p$-algebra by Hesselholt–Nikolaus [HN20], using Nikolaus–Scholze's approach [NS18] to topological cyclic homology. This result was then generalised to mixed characteristic perfectoid rings by Riggenbach [Rig23], ultimately relying on computations in relative topological Hochschild homology. It would seem that the associated Atiyah–Hirzebruch spectral sequence should degenerate in this context, thus providing a similar computation of the motivic cohomology of cuspidal curves. An interesting question would be whether these results can be reproved, or even extended to more general base rings, using techniques from prismatic cohomology and derived de Rham cohomology.

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