Weinstein exactness of nearby Lagrangians and the Lagrangian C^0 flux conjecture

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Abstract

This paper is concerned with the following problem: given a Lagrangian L and a Hamiltonian diffeomorphism φ such that $\varphi(L)$ is in a small neighbourhood U of L, does there exist another Hamiltonian isotopy from L to $\varphi(L)$ supported in U? On one side, we construct an irrational counterexample in any symplectic manifold of dimension at least six. On the other side, we answer an *a priori* weaker form of the question by the positive in many cases when L satisfies some rationality condition. The techniques that we develop also have applications for the Lagrangian counterpart of the C^0 flux conjecture. In turn, these results have many applications, in particular, to understand C^0-rigidity phenomena of Hamiltonian diffeomorphisms and the space of Lagrangians with a given rationality constant.

1 Introduction

Note: For the sake of conciseness, we will refer throughout the paper to a ***closed connected Lagrangian submanifold of a connected symplectic manifold without boundary*** as a "Lagrangian in a symplectic manifold."

This paper aims to study the local topological properties of natural sets of Lagrangians, most notably the Hamiltonian and symplectic orbits of a given Lagrangian L, respectively

$$\mathcal{L}\mathrm{Ham}(L) := \mathrm{Ham}(M) \cdot L = \{\varphi(L) \,|\, \varphi \in \mathrm{Ham}(M)\}\,,$$
$$\mathcal{L}\mathrm{Symp}_0(L) := \mathrm{Symp}_0(M) \cdot L = \{\psi(L) \,|\, \psi \in \mathrm{Symp}_0(M)\}\,.$$

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To do this, we first fix a metric g on the underlying manifold L. Recall that the Weinstein neighbourhood theorem ensures that there exist $r > 0$ and a symplectomorphism $\Psi : D_r^*L \to \mathcal{W}_r(L)$ from the codisk bundle of L of radius r to a neighbourhood $\mathcal{W}_r(L)$ of L in M which maps the 0-section to L.

Therefore, understanding $\mathcal{L}\mathrm{Ham}(L)$ ***locally*** is intimately related to the nearby Lagrangian conjecture (or NLC for short), which completely characterizes Lagrangians which are in the Hamiltonian orbit of the 0-section in T^*L. Indeed, it states that those are precisely the exact Lagrangians. It is known to hold for S^1, S^2 [Hin04], $\mathbb{RP}^2$ [HPW16, Ada22], and $\mathbb{T}^2$ [RGI16]. Without restriction on the diffeomorphism type, the most advanced result in the direction of the NLC states that the natural projection $\pi : T^*L \to L$ induces a (simple) homotopy equivalence between any exact closed Lagrangian and the 0-section [AK18]. This latter result will play a crucial role in our study of the local structure of $\mathcal{L}\mathrm{Ham}(L)$

Inspired by this conjecture, we propose that if $L' \in \mathcal{L}\mathrm{Ham}(L)$ is close to L, then there is an accordingly small Hamiltonian isotopy from L to L'. More precisely, we make the following conjecture.

Conjecture A (Strong conjecture) *Let L be a Lagrangian in a symplectic manifold M. There exists a neighbourhood U of L with the following property. If L' is Hamiltonian isotopic to L in M and $L' \subseteq U$, then there exists a Hamiltonian isotopy $\{\varphi_t\}_{t\in[0,1]}$ supported in U such that $\varphi_1(L) = L'$.*

That this conjecture holds would imply that the Hamiltonian orbit $\mathcal{L}\mathrm{Ham}(L)$ of a Lagrangian L is locally path connected via Hamiltonian isotopies. Although this statement is new in the Lagrangian context, there are some results towards its Hamiltonian counterpart. More precisely, the group $\mathrm{Ham}_c(M)$ of compactly supported Hamiltonian diffeomorphisms of a symplectic manifold M is locally path connected in the C^0 topology (via Hamiltonian isotopies) if M is a closed surface or the open ball B^{2n}. The former case follows from Fathi's work on homeomorphisms preserving a volume form [Fat80] and the folkloric fact that a path of such homeomorphisms on a closed surface can be C^0 approximated by a path of symplectomorphisms (see [Oh06] for a proof). The latter case was proved by Seyfaddini [Sey13].

Note that the local path connectedness of $\mathrm{Ham}(M)$ implies Conjecture A for graphs of symplectomorphisms of M. However, even if Conjecture A holds for all graphs in $M \times M$, it does not imply local path connectedness of $\mathrm{Ham}(M)$, since the Hamiltonian isotopy given by the conjecture is not necessarily through graphs.

In this work, we prove the existence of neighbourhoods of ***local exactness*** for several classes of Lagrangians, by which we mean a Weinstein neighbourhood $\mathcal{W}(L)$ of a Lagrangian L so that any Lagrangian Hamiltonian isotopic to L included in this neighbourhood is exact in $\mathcal{W}(L)$. When the NLC is known to hold for L, we can then deduce that L satisfies the strong conjecture above.

This indicates that it is reasonable to believe that the strong conjecture is extremely hard to prove in general. However, the following weaker form of the conjecture holds and might be more easily provable in full generality.

Conjecture B (Weak conjecture) *Let L be a displaceable Lagrangian in a symplectic manifold M. There exists a neighbourhood U of L with the following property. If L' is Hamiltonian isotopic to L in M and $L' \subseteq U$, then $L \cap L' \neq \emptyset$.*

The Lagrangians which admit a neighbourhood of ***local exactness*** as described above obviously satisfy the weak conjecture. In fact, such Lagrangians must intersect in at least $\sum_{i=0}^{n} \beta_i(L) \geq 2$ points, where $\beta_i(L)$ is the i-th Betti number of L.

Actually, our methods allow us to prove that a large class of Lagrangians satisfy a slightly strengthened version of the weak conjecture, namely that if L' is the image of L under any ***symplectomorphism*** and $L' \subseteq U$, then $L \cap L' \neq \emptyset$. Note that we do not conjecture that this always holds (since it obviously does not). In what follows, we will refer to satisfying the weak conjecture (respectively its symplectic version) as ***having a neighbourhood of Hamiltonian nondisplacement*** (respectively of ***symplectic nondisplacement***).

1.1 Main results

Our first result provides a counterexample to Conjectures A and B (and to Conjecture C below, as well as its C^1 variant). This shows that there is no hope to prove the above conjectures in full generality.

Theorem 1 *In any symplectic manifold of dimension $2n \geq 6$, there exists a Lagrangian torus whose Hamiltonian orbit*

(i) *is not closed in Hausdorff topology inside the set of Lagrangian tori,*

(ii) *admits arbitrarily Hausdorff-close disjoint elements.*

Both claims actually hold for any reasonable notion of C^1 topology — see Section 2 below.

The existence of such tori follows directly from the characterization of product tori in the Hamiltonian orbit of a given product Lagrangian torus in $\mathbb{C}^n$ by Chekanov [Che96] and in large enough balls by Chekanov and Schlenk [CS16]. We give the details in Section 2 below.

We make the crucial observation that these Lagrangian tori are not ***rational***, and we turn to specific families of Lagrangians which do satisfy either one or both of the conjectures above.

1.1.1 Existence of nondisplacement neighbourhoods

We start with a simple exercise, which inspires our approach to the above conjectures.

Proposition 2 *Conjecture A holds for any closed embedded curve L in an orientable surface M.*

Its proof, detailed in Appendix A, can be roughly summarized as follows.

Sketch of proof. Fix a Weinstein neighbourhood $\Psi : D_r^*S^1 \to \mathcal{W}_r(L)$ of L, and let $L' = \varphi(L) \subseteq \mathcal{W}_r(L)$ for some symplectomorphism φ of M. Define

$$\tau := \inf\{\omega(u) > 0 \mid u \in H_2(M, L; \mathbb{Z})\} \in [0, +\infty],$$

where we set $\tau = +\infty$ if $\omega(H_2(M, L; \mathbb{Z})) = 0$.

If $\tau > 0$, the proof then follows in two steps.

(1) Show that, if $r > 0$ is such that $\mathcal{W}_r(L)$ has area smaller than τ, then L' must be weakly exact in $\mathcal{W}_r(L)$.

(2) Show that if L' is weakly exact in $\mathcal{W}_r(L)$, then it must in fact be exact.

Since the nearby Lagrangian conjecture is known to hold for the circle, this yields Conjecture A when $\tau > 0$.

If $\tau = 0$, simple geometric considerations due to the low dimension of the situation allow us to easily adapt the proof above. □

Inspired by this simple case, we develop a criterion which should ensure the existence of neighbourhoods of nondisplacement for Lagrangians in arbitrary dimensions. Namely, we define a notion of ***homological rationality***, or H-rationality for short.

Definition *Let L be a Lagrangian of a symplectic manifold (M, ω). We say that L is* H**-rational** *in (M, ω) if $\omega\big(H_2(M, L; \mathbb{Z})\big) = \tau\mathbb{Z}$ for some $\tau \geq 0$. We then call τ the* H**-rationality constant** *of L. When $\tau = 0$, we say that L is* H**-exact** *in (M, ω).*

Note that H-rationality (respectively H-exactness) is the homological version of the usual notion of rationality (respectively weak exactness). In fact, as we shall see in Section 3.2 below, those conditions are equivalent in many important cases, e.g. when $\pi_1(M) = 0$.

We denote by $\mathcal{L}(\tau)$ the space of all H-rational Lagrangian submanifolds of M which have H-rationality constant $\tau \geq 0$, and by $\mathcal{L}(L, \tau)$ its subspace formed by those Lagrangians which have the same diffeomorphism type as L.

We now adapt the two steps of the 1-dimensional case to higher dimensions. First, we prove the existence of ***neighbourhoods of homological exactness*** for several classes of Lagrangians — this is Theorem 3 below. Second, we show that such neighbourhoods lead to ***Weinstein neighbourhoods of exactness*** for Hamiltonian isotopic Lagrangians. We also get such neighbourhoods for Lagrangians with a given H-rationality constant under an extra homological condition — this is the content of Theorem 4.

Theorem 3 *Suppose that L is a Lagrangian submanifold of (M, ω) which satisfies one of the following:*

(*a*) *the underlying manifold has $H_1(L; \mathbb{R}) = \mathbb{R}$ and admits a Lagrangian embedding in a Liouville domain W with $SH(W) = 0$,*

(*b*) *L is a Klein bottle.*

For every $\tau \geq 0$, there exists a Weinstein neighbourhood $\mathcal{W}(L)$ of L such that all $L' \in \mathcal{L}(L, \tau)$ included in $\mathcal{W}(L)$ is H-exact in $\mathcal{W}(L)$.

Theorem 4 *Let L be a H-rational Lagrangian submanifold of (M,ω) and let $L' \in \mathcal{L}\mathrm{Ham}(L)$ be a Lagrangian included in a Weinstein neighbourhood $\mathcal{W}_r(L)$ of size $r > 0$ such that L' is H-exact in $\mathcal{W}_r(L)$. Then, for a maybe smaller r, L' is exact in $\mathcal{W}_r(L)$.*

Moreover, if the inclusion of L into M induces the 0-map $H_1(L;\mathbb{R}) \to H_1(M;\mathbb{R})$, then the same result holds with $\mathcal{L}\mathrm{Ham}(L)$ replaced by $\mathcal{L}(\tau)$, where τ is the H-rationality constant of L.

Remarks 1. *In many instances, the constraints on L and L' can actually be weakened. We refer the interested reader to Section 3.2 for specifics.*

Corollary 5 *A H-rational Lagrangian L satisfying the assumptions of Theorem 3 admits a Hamiltonian non-displacement neighbourhood. If furthermore $H_1(L;\mathbb{R}) \to H_1(M;\mathbb{R})$ is zero, then it also admits a neighbourhood of symplectic non-displacement.*

In other words, such a Lagrangian L satisfies Conjecture B (or its slightly strengthened symplectic version). Furthermore, if the Nearby Lagrangian Conjecture holds in T^*L, we also get Conjecture A.

Remarks 2. *Theorem 3 above also holds for n-dimensional Lagrangian tori. This can be proven along the same lines as for case (a) by bounding our variant of the Cieliebak–Mohnke capacities by the McDuff–Siegel higher capacities. However, the latter being much harder to handle, the proof becomes much more involved and will appear in further work.*

Interestingly enough:

1. *In dimension 2, we can prove that*
   - (a) *A* **displaceable** *rational Lagrangian 2-torus in a simply-connected Darboux chart admits a neighbourhood of symplectic nondisplacement, i.e. satisfies the symplectic version of Conjecture B.*
   - (b) *A* **nondisplaceable** *Lagrangian 2-torus L whose fundamental class $[L]$ does not vanish in $H_2(M;\mathbb{Z})$, satisfies Conjecture A.*

   Because the proofs of these results do not fit in the general framework developed here, we prove them in Appendix B, where we also state them precisely.

2. *In their work in progress on the C^0 flux conjecture for Hamiltonian diffeomorphisms, Atallah and Shelukhin [AS24a] get a similar result of local exactness for graphs in $M \times M$ of C^0-small Hamiltonian diffeomorphisms for M closed. In this case, they do not require that M be rational, contrary to our setup.*

3. *As we were finalizing this paper, Atallah and Shelukhin also informed us [AS24b] that they had results similar to Corollary 5. In particular, they established a version of Theorem 4 whose proof is strikingly similar to ours. They also proved a version of Theorem 3 for rational Lagrangians L of the form $\mathbb{T}^n \times Q$, where Q is simply connected, under the assumption that $\pi_1(L) \to \pi_1(M)$ is zero. It appears that combining their methods and ours leads to the following fact: if L and L' satisfy the conclusion of Theorem 3, then so does $L \times L'$. We want to investigate this exciting direction together.*

1.1.2 Lagrangian flux conjectures

We now move on to another conjecture about Lagrangian submanifolds.

Conjecture C (Lagrangian C^0 flux conjecture) *Let L be a Lagrangian in a symplectic manifold M. Its Hamiltonian orbit $\mathcal{L}\mathrm{Ham}(L)$ is Hausdorff-closed in the space $\mathcal{L}\mathrm{Lag}(L)$ of all Lagrangians which are Lagrangian isotopic to L.*

As far as the authors know, this version of the conjecture has not been studied previously — we will talk about its C^1 cousin, which has been studied, below. The name that we give it here is in analogy to the famous C^0 flux conjecture for Hamiltonian diffeomorphisms, which states that the group $\mathrm{Ham}(M)$ of Hamiltonian diffeomorphisms of a closed manifold M is C^0 closed in the identity component $\mathrm{Symp}_0(M)$ of the group of symplectomorphisms of M. This conjecture is only known to hold in some fairly specific case [LMP98, Buh15]. This is in stark contrast with its C^1 cousin, which is known to hold in full generality [Ono06].

We note that, similarly to Conjecture A above, the Lagrangian C^0 flux conjecture does not imply the one for Hamiltonian diffeomorphisms. Indeed, suppose that $\{\varphi_i\} \subseteq \mathrm{Ham}(M)$ C^0-converges to $\psi \in \mathrm{Symp}_0(M)$. Then, all that the Lagrangian flux conjecture ensures is that there is some Hamiltonian diffeomorphism Φ of $M \times M$ such that $\mathrm{graph}(\psi) = \Phi(\Delta)$, where $\Delta \subseteq M \times M$ is the diagonal. However, we cannot be sure that Φ can be chosen of the form $\mathbb{1} \times \varphi$ for some $\varphi \in \mathrm{Ham}(M)$, which is what the flux conjecture for Hamiltonian diffeomorphisms would require.

To study this conjecture, we can use the techniques developed to prove Theorems 3 and 4. In fact, they imply the following continuity result.

Theorem 6 *Let $\{L_i\}$ be a sequence of H-rational Lagrangians of a tame symplectic manifold M such that*

(*i*) *$\{L_i\}$ Hausdorff-converges to a n-dimensional smooth submanifold L;*

(*ii*) *$\inf \tau_i > 0$, where τ_i denotes the H-rationality constant of L_i.*

Then, L is itself Lagrangian.

Moreover, if L_i is H-exact in a Weinstein neighbourhood $\mathcal{W}(L)$ for i large, then $\tau_i \equiv \tau$ for larger i, and τ is the H-rationality constant of L. This is in particular the case if the L_i's respect the hypotheses of Theorem 3.

By ***tame***, we mean that M admits an almost structure J making $g_J := \omega(\cdot, J\cdot)$ into a complete Riemannian metric whose sectional curvature is bounded and whose injectivity radius is bounded away from zero.

The first part of the theorem is a fairly direct application of Laudenbach and Sikorav's result on the displaceability of non-Lagrangian submanifolds [LS94] — we mostly write it here for the reader's convenience. Furthermore, the second part of the theorem is very reminiscent of Theorem 1 of [MO21] — the proof is in fact very inspired by what appears in that paper. The strength of our result is that it applies to sequences $\{L_i = \varphi_i(L)\}$ where the sequence of Hamiltonian diffeomorphisms $\{\varphi_i\}$ need not C^0-converge. See Section 5.2 for more details.

Before moving on to corollaries of this result, note that, in the formulation above, one could also ask for C^0-closure of $\mathrm{Ham}(M)$ in larger groups than $\mathrm{Symp}_0(M)$, most notably in $\mathrm{Symp}(M)$ or $\mathrm{Diff}(M)$. Here, $\mathrm{Symp}(M)$ denotes the group of symplectomorphisms of M and $\mathrm{Diff}(M)$, the group of its diffeomorphisms. For $\mathrm{Symp}(M)$, this is harder to prove, since it is not known if $\mathrm{Symp}_0(M)$ is C^0-closed in $\mathrm{Symp}(M)$. However, the celebrated result from Gromov [Gro85] and Eliashberg [Eli87] shows that $\mathrm{Symp}(M)$ is C^0-closed in $\mathrm{Diff}(M)$. Therefore, the closure in $\mathrm{Symp}(M)$ is equivalent to that in $\mathrm{Diff}(M)$.

Following this logic, we can replace $\mathcal{L}\mathrm{Lag}(L)$ in the above with larger spaces. Most notably, we will also be interested in the spaces $\mathsf{SMan}(L)$, of all submanifolds of M with the same diffeomorphism type as L, and SMan_n, of all n-dimensional submanifolds of M^{2n}. By Theorem 6, closure of $\mathcal{L}\mathrm{Ham}(L)$ in the two latter spaces is equivalent to closure in the subspace formed by Lagrangian submanifolds.

To address these many spaces, we will make use of the following weaker form of Theorem 4.

Proposition 7 *Let L be a H-rational Lagrangian submanifold of M with H-rationality constant τ. There is some $r_0 > 0$ with the following property. Assume that $L' \in \mathcal{L}(\tau)$ is a Lagrangian included in a Weinstein neighbourhood $\mathcal{W}_r(L)$ of size $r \in (0, r_0]$ such that L' is H-exact in $\mathcal{W}_r(L)$. Then, there is a symplectic isotopy $\{\psi_t\}_{t\in[0,1]}$ of M such that $\psi_1(L')$ is exact in $\mathcal{W}_r(L)$. Furthermore, the size of the isotopy is controlled by r.*

The last sentence corresponds in actuality to a precise estimate on the flux of the Lagrangian isotopy $\{\psi_t(L')\}$, but we do not want to get into all the details here. We refer the interested reader to Section 5.3.

Combining Theorem 3, Theorem 4, and Proposition 7, we thus get the following — again, the precise proof is in Section 5.3.

Corollary 8 *Let L be a H-rational Lagrangian in a tame symplectic manifold M. Suppose that*

(i) *L satisfies the hypotheses of Theorem 3 or $H_1(L;\mathbb{R}) = 0$*

(ii) *and the nearby Lagrangian conjecture holds in T^*L.*

Then, $\mathcal{L}\mathrm{Ham}(L)$ and $\mathcal{L}\mathrm{Symp}_0(L)$ are Hausdorff-closed in $\mathsf{SMan}(L)$.

Remarks 3. *Note that the tameness condition on M can be dropped if one is only interested in closedness in the subspace $\mathcal{L}(L)$ of $\mathsf{SMan}(L)$ consisting of the Lagrangians. Indeed, tameness is only used to ensure that the limit of Lagrangians is still a Lagrangian.*

Likewise, one can, in some contexts, upgrade from $\mathsf{SMan}(L)$ to SMan_n. For example, if $n = 2$, this is the case. Indeed, any H-exact Lagrangian in the cotangent bundle of a surface has the same diffeomorphism type as that surface (see Lemma 24 below). This is a nontrivial update: Polterovich [Pol93] constructed Lagrangian tori in the cotangent bundle of any flat manifold; these tori can be made to be arbitrarily close to the zero-section. We discuss these examples in more details at the very end of Section 3.2.

We will explore in Section 1.2 below examples where these conditions are all satisfied.

The Lagrangian C^1 flux conjecture A natural variant of Conjecture C is obtained by replacing closedness in the Hausdorff metric with closedness in the C^1 topology. We call this the Lagrangian C^1 flux conjecture.

By C^1 topology, we mean the one constructed as follows. Fix a Riemannian metric g on M. We say that a closed connected half-dimensional submanifold N' is ε-C^1-close to another one N if N' is in a tubular neighbourhood of N and there is a normal vector field ν along N such that $\|\nu\| < \varepsilon$ and $\exp(\nu(N)) = N'$. We then set

$$B(N, \varepsilon) := \left\{ N' \,\middle|\, N' \text{ is } \varepsilon\text{-}C^1\text{-close to } N \text{ \& vice-versa} \right\}.$$

The C^1 topology on SMan_n is then the topology generated by the $B(N, \varepsilon)$'s. One can easily check that this is independent of the choice of Riemannian metric.

With our methods, we get the following.

Corollary 9 *Let L be a H-rational Lagrangian in a tame symplectic manifold M. Then, $\mathcal{L}\mathrm{Ham}(L)$ and $\mathcal{L}\mathrm{Symp}_0(L)$ are C^1-closed in SMan_n.*

The reason that we don't need hypothesis (i) of Corollary 8 is because Lagrangians with are C^1-close of L are graphs of 1-forms in $\mathcal{W}(L)$, and graphs are necessarily H-exact. Likewise, hypothesis (ii) is not needed since exact graphs are Hamiltonian isotopic to the zero-section in $\mathcal{W}(L)$. Note that C^1-close Lagrangians are necessarily diffeomorphic so that closure in SMan_n is the same as closure in $\mathsf{SMan}(L)$.

The Lagrangian C^1 flux conjecture has been studied previously by Ono [Ono08] and Solomon [Sol13] in the case when M is closed or a cotangent bundle. They proved that it holds when L has Maslov class zero and is unobstructed in the sense of [FOOO09] and when the so-called Lagrangian flux group of L is discrete, respectively. When L is H-rational, the Lagrangian flux group is automatically discrete. Therefore, our improvement with regards to Solomon's result is that we allow M to be open — otherwise, we only have proved a subcase. As for Ono's, our condition is somewhat orthogonal to his: he needs no bad disks, but we ask for a lot of them.

1.2 Examples

We give a few examples where Conjecture B follows from the results above. Note that, as long as $\omega(H_2(M;\mathbb{Z}))$, an arbitrarily C^1-small perturbation of a Lagrangian will be H-rational. Therefore, we do not care if the precise examples below are H-rational or not.

We start by giving a few examples of Lagrangians satisfying Case *(a)* of Theorem 3 and natural situations when they are displaceable.

First of all, note that for any Liouville domain V, $W = V \times \mathbb{D}$ is such that $SH(W) = 0$, since that domain is displaceable in its completion. In particular, Case *(a)* covers the case $L = Q \times S^1$ with $H_1(Q;\mathbb{R}) = 0$. Note that these are displaceable since $\mathbb{D}$ is displaceable in $\mathbb{C}$.

If $TQ \otimes \mathbb{C}$ is additionally assumed to be trivial, then L embeds as a Lagrangian in $\mathbb{C}^n$ by the Gromov–Lees h-principle [Gro70, Lee76] and a result of Audin, Lalonde, and Polterovich [ALP94]. In particular, $L = S^{n-1} \times S^1$,

$n \geq 3$ is an example of such a manifold. In particular, we have examples of L satisfying Case *(a)* in any symplectic manifold of dimension at least 6. In fact, any symplectic manifold containing a displaceable Lagrangian S^n, e.g. the full flag manifold of $\mathbb{C}^3$ [Pab15], will contain such Lagrangians, because T^*S^n has Lagrangians $S^{n-1} \times S^1$ arbitrarily close to the zero-section.

In another direction, Ekholm, Eliashberg, Murphy, and Smith [EEMS13] showed that, given any 3-manifold Q, $L = Q\#(S^1 \times S^2)$ embeds as a Lagrangian in $\mathbb{C}^3$. But, by the van Kampen theorem, $\pi_1(L) = \pi_1(Q) * \pi_1(S^1 \times S^2)$, so that $H_1(L;\mathbb{R}) = H_1(Q;\mathbb{R}) \oplus \mathbb{R}$. Therefore, Case *(i)* covers $L = Q\#(S^1 \times S^2)$ with $H_1(Q;\mathbb{R}) = 0$, e.g. Q can be a (connected sum of) lens spaces.

To resume the above discussion, we have the following.

Corollary 10 *Conjecture B holds for displaceable H-rational Lagrangians of the form* $Q \times S^1$ *or* $Q\#(S^1 \times S^2)$ *with* $H_1(Q;\mathbb{R}) = 0$ *and, in the latter case,* $\dim Q = 3$.

There are of course examples which do not fit within this pattern. For example, it is well known that the Lagrangian Grassmannian Λ_n admits a Lagrangian embedding in $\mathrm{Sym}(\mathbb{C}^n) = \mathbb{C}^{n(n+1)/2}$ (see, for example, [ALP94]). This is an example of Case *(a)*, since $\pi_1(\Lambda_n) = \mathbb{Z}$.

When it comes to Case *(b)* of Theorem 3, there is one main example: the Lagrangian Klein bottle in $S^2 \times \mathbb{C}$. It is obtained from the usual Lagrangian Klein bottle in $S^2 \times S^2$ (see, for example, [Eva22]) by removing a point on the second copy of S^2 and identifying $S^2 \setminus \{pt\}$ with $\mathbb{D} \subseteq \mathbb{C}$. Again, it is displaceable, because $\mathbb{D}$ is. In fact, the Klein bottle can even be made to be monotone. To resume, we have the following.

Corollary 11 *Conjecture B holds for displaceable H-rational Lagrangian Klein bottles. There exist such Lagrangians — and a monotone one — in* $S^2 \times \mathbb{C}$.

To conclude with Conjecture B, we go back to Remark 2: using different methods, we can prove that the conjecture holds for rational 2-tori in simply connected Darboux charts. Using Theorem C of [RGI16], we, in particular, get the following result.

Corollary 12 *Conjecture B holds for displaceable rational* 2*-tori in* $\mathbb{C}^2$, $S^2 \times S^2$, $\mathbb{C}P^2$, *and blow-ups of* $\mathbb{C}P^2$.

We conclude with one additional case when we can establish Conjecture A: when L is a 2-sphere or a projective plane. Indeed, any other such Lagrangian in $\mathcal{W}(L)$ is then automatically exact in that neighbourhood, so there is no need for Theorems 3 or 4. Since the NLC is known to hold in T^*S^2 [Hin04] and $T^*\mathbb{R}P^2$ [HPW16], we thus directly get the following.

Corollary 13 *Conjectures A and C hold for Lagrangian* 2*-spheres or projective planes.*

1.3 Applications

We end this introduction with several applications of our results. These are divided in four parts: additional rigidity results on Lagrangians with regards to Hamiltonian diffeomorphisms, further study on the local topology of $\mathcal{L}\mathrm{Ham}(L)$, new results on the space of (H-)rational Lagrangians with a fixed rationality constant, and some computations of numerical invariants. The next to last part has further implications when it comes to the space of all Lagrangians of a given symplectic manifold.

This last part of the introduction is intended to be almost completely self-contained, using the results above as black boxes (except for a couple of references to further results when finer technical variants are needed.)

C^0 rigidity of Hamiltonian diffeomorphisms There is a natural variant of Conjecture B where we ask not that L' be close to L, but rather that the Hamiltonian diffeomorphism sending L to L' be small. More precisely, we can make the following conjecture.

Conjecture D *Let L be a displaceable Lagrangian in a symplectic manifold M. There exists $\delta > 0$ with the following property. If φ is a Hamiltonian diffeomorphism of M and $d_{C^0}(\mathbb{1}, \varphi) < \delta$, then $L \cap \varphi(L) \neq \varnothing$.*

In other words, any Hamiltonian diffeomorphism displacing L is uniformly C^0-bounded away from 0.

The existence of such a bound is not at all trivial: if L is a displaceable n-dimensional submanifold which is not Lagrangian, then it can be displaced by an arbitrarily C^0-small Hamiltonian diffeomorphism [LS94]. Moreover, this does not follow from the fact that Lagrangians have positive displacement energy, since there are Hamiltonian diffeomorphisms which are arbitrarily C^0-small, but arbitrarily Hofer-large.

However, this is not expected to be the case for the spectral metric, that is, C^0-small Hamiltonian diffeomorphisms should also have small spectral norm. More precisely, Conjecture D follows from the fact that Lagrangians have positive spectral displacement energy [AAC23] in the cases where it is known that the spectral metric is C^0-continuous, i.e. when M is $\mathbb{C}^n$ [Vit92], a closed surface [Sey13], closed and symplectically aspherical [BHS21], $\mathbb{C}P^n$ [She22], or closed and negative monotone [Kaw22].

In the context of this paper, Conjecture D is implied by Conjecture A or by the Hamiltonian version of Conjecture B above when it holds. However, it turns out to be much easier to prove than either one of these conjectures. More precisely, we have the following lemma.

Lemma 14 *For every Lagrangian L, there exists $\delta > 0$ with the following property. Suppose that $\psi : M \to M$ is a map such that $d_{C^0}(\mathbb{1}, \psi) < \delta$ and $\psi(L)$ is Lagrangian. Then, $\psi(L)$ is H-exact in some $\mathcal{W}(L)$.*

Proof. Take a Riemannian metric g on M which corresponds to a Sasaki metric on T^*L on a Weinstein neighbourhood $\mathcal{W}(L)$. With such a metric, the geodesics starting at L and going to $L' = \psi(L)$ stay in $\mathcal{W}(L)$ (see Lemma A.4 of [Cha24] for example). Therefore, if we assume that δ is smaller than the injectivity radius $r_{\mathrm{inj}}(TM|_L)$ of the Riemannian exponential of g restricted to $TM|_L$, we get for every $x \in L$ a unique geodesic $\gamma_x : [0,1] \to M$ such that $\gamma_x(0) = x$, $\gamma_x(1) = \psi(x)$, and $\gamma_x([0,1]) \subseteq \mathcal{W}(L)$. Moreover, γ_x smoothly depends on x. Therefore, $(x,t) \mapsto \gamma_x(t)$ defines a smooth homotopy in $\mathcal{W}(L)$ from the inclusion $\iota : L \hookrightarrow \mathcal{W}(L)$ to $\varphi\iota$. Since ι is a homotopy equivalence, then so must be $\varphi\iota$. In particular, $H_2(\mathcal{W}(L), L') = 0$, and L' is H-exact. □

Then, it suffices to use Theorem 4 to get the following.

Corollary 15 *Conjecture D holds for H-rational Lagrangians.*

Likewise, we get a rigidity result for sequences of Hamiltonian or symplectic diffeomorphisms from Theorem 6 and Corollary 8.

Corollary 16 *Let $\{\psi_i\}$ be a sequence of symplectomorphisms with (weak) C^0 limit $\psi \in C^0(M, M)$, and let $L \in \mathcal{L}(\tau)$. If $\psi(L)$ is a smooth n-submanifold, then $\psi(L) \in \mathcal{L}(\tau)$.*

If, furthermore, the NLC holds on T^*L and

(a) if $\{\psi_i\} \subseteq \mathrm{Ham}(M)$, then $\psi(L) \in \mathcal{L}\mathrm{Ham}(L)$;

(b) if $\{\psi_i\} \subseteq \mathrm{Symp}_0(M)$, then $\psi(L) \in \mathcal{L}\mathrm{Symp}_0(L)$.

Note that a similar result about the continuity of the area spectrum under C^0 limits was shown by Membrez and Opshtein [MO21].

Local contractibility of $\mathcal{L}\mathrm{Ham}(L)$ Even though we need the NLC in all cases where we can prove Conjecture A, its implication that $\mathcal{L}\mathrm{Ham}(L)$ is locally path connected turns out to be easier to prove. More precisely, we get the following.

Corollary 17 *Suppose that L is H-rational and respects the hypotheses of Theorem 3. Then $\mathcal{L}\mathrm{Ham}(L)$ is locally contractible in the Hausdorff metric.*

Note that this is not quite the type of results we mentioned earlier in the introduction. Indeed, we do not claim that the Hausdorff-continuous path from L' to L in a small neighbourhood of L is generated by an actual Hamiltonian isotopy, simply that it stays at all time in $\mathcal{L}\mathrm{Ham}(L)$.

Proof. Note that it suffices to prove this statement at L. Fix a Weinstein neighbourhood $\Psi : D_r^*L \to \mathcal{W}(L)$ as given by the conclusions of Theorems 3 and 4. Then, every $L' \in \mathcal{L}\mathrm{Ham}(L)$ which is in $\mathcal{W}(L)$ is exact in that neighbourhood. We can thus take

$$(L', t) \mapsto \Psi(t\Psi^{-1}(L'))$$

to be the contraction. Indeed, exactness in $\mathcal{W}(L)$ ensures that this is a Hamiltonian isotopy for all $t > 0$. But exactness also implies that the projection $\Psi^{-1}(L') \to T^*L \to L$ is a homotopy equivalence [AK18]. In particular, that projection must be surjective, otherwise $H_n(L') \to H_n(L) \neq 0$ would be zero. Therefore, L' being close to L implies that L is close to L'. This mean that the Hausdorff limit of $\Psi(t\Psi^{-1}(L'))$ as $t \to 0$ is precisely L, i.e. the above contraction is indeed continuous. □

Spaces of Lagrangians with fixed H-rationality constant We now turn our attention to the space $\mathcal{L}(L,\tau)$ of all Lagrangians of M with the diffeomorphism type of L and H-rationality constant τ.

From Theorems 3 and 4, we get the following.

Corollary 18 *Let L be a H-rational Lagrangian in a tame symplectic manifold, and denote by τ its H-rationality constant. Then, $\mathcal{L}\mathrm{Symp}_0(L)$ is open in $\mathcal{L}(L,\tau)$ in the C^1 topology. If moreover L respects the hypotheses of Theorem 3 or $H_1(L;\mathbb{R}) = 0$ and the NLC holds on T^*L, then the same holds in the Hausdorff topology.*

Proof. Note that it suffices to prove that there is an open neighbourhood of L in $\mathcal{L}(L,\tau)$ which is fully in $\mathcal{L}\mathrm{Symp}_0(L)$. Let thus $\Psi : D_r^*L \to \mathcal{W}_r(L)$ be the Weinstein neighbourhood given by Proposition 7. Then, every graph in $\mathcal{W}(L)$ must be, up to a global symplectic isotopy, exact. Since exact graphs are Hamiltonian isotopic to the zero-section, such a graph must thus be in $\mathcal{L}\mathrm{Symp}_0(L)$. This proves the C^1 case.

For the Hausdorff case, suppose that r is also small enough so that Theorem 3 and Proposition 7 hold in $\mathcal{W}_r(L)$. Then, any $L' \in \mathcal{L}(L,\tau)$ such that $L' \subseteq \mathcal{W}(L)$ must be, up to some global symplectic isotopy, exact in $\mathcal{W}(L)$ — we still denote by L' its image under the isotopy. As in the proof of Corollary 17, we note that the path $t \mapsto \Psi(t\Psi^{-1}(L'))$, $t \in [0,1]$, is continuous in the Hausdorff metric. Furthermore, it is a Hamiltonian isotopy for all $t > 0$. In particular, L must be in the Hausdorff closure of $\mathcal{L}\mathrm{Ham}(L') \subseteq \mathcal{L}\mathrm{Symp}_0(L')$. But $\mathcal{L}\mathrm{Symp}_0(L')$ is Hausdorff closed by Corollary 8 and the hypotheses on L. Therefore, $L' \in \mathcal{L}\mathrm{Symp}_0(L)$. □

Putting this result with the Lagrangian flux conjecture, we get the following result.

Corollary 19 *Let L and τ be as above. The (path) connected components of $\mathcal{L}(L,\tau)$ in the C^1 topology are precisely the orbits of $\mathrm{Symp}_0(M)$. In particular, the quotient $\mathcal{L}(L,\tau)/\mathrm{Symp}_0(M)$ is discrete in the induced topology. If moreover L respects the hypotheses of Theorem 3 or $H_1(L;\mathbb{R}) = 0$ and the NLC holds on T^*L, then the same holds in the Hausdorff metric.*

For example, this means that a ρ-monotone Clifford torus can never be reached from a Chekanov torus (or any monotone special torus) by a C^1-continuous path in $\mathcal{L}(\mathbb{T}^2, 2\rho)$. Contrast this with the fact that all these tori are Lagrangian isotopic [RGI16].

Proof. Combining Corollaries 9 and 18, we get that for all $L \in \mathcal{L}(L,\tau)$, the orbit $\mathcal{L}\mathrm{Symp}_0(L)$ is both closed and open in $\mathcal{L}(L,\tau)$ with the C^1 topology. Therefore, $\mathcal{L}\mathrm{Symp}_0(L)$ must be a union of connected components of $L \in \mathcal{L}(L,\tau)$ by point-set topology. Since $\mathcal{L}\mathrm{Symp}_0(L)$ is obviously path connected, it must be both a connected component and a path connected component of $L \in \mathcal{L}(L,\tau)$. The proof in the Hausdorff setting is completely analogous. □

Note that, when $H_1(L;\mathbb{R}) \to H_1(M;\mathbb{R})$ is zero, the role of $\mathcal{L}\mathrm{Symp}_0(L)$ in the above proof can be replaced by $\mathcal{L}\mathrm{Ham}(L)$. In particular, both $\mathcal{L}\mathrm{Symp}_0(L)$ and $\mathcal{L}\mathrm{Ham}(L)$ are the connected component of $\mathcal{L}(\tau)$ containing L in the C^1 topology, so they must be equal. This can be seen as a generalization that $\mathrm{Symp}_0(M) = \mathrm{Ham}(M)$ for closed manifolds with $H_1(M;\mathbb{R}) = 0$.

Corollary 20 *Let L be H-rational and such that $H_1(L;\mathbb{R}) \to H_1(M;\mathbb{R})$ is zero. There is a symplectic isotopy $\{\psi_t\}$ of M such that $\psi_1(L) = L'$ if and only if there is a Hamiltonian isotopy $\{\varphi_t\}$ such that $\varphi_1(L) = L'$. In other words, $\mathcal{L}\mathrm{Symp}_0(L) = \mathcal{L}\mathrm{Ham}(L)$.*

We also have the following.

Corollary 21 *The space $\cup_{\tau\geq 0}\mathcal{L}(L,\tau)/\mathrm{Symp}_0(M)$ is Hausdorff in the topology induced by the C^1 topology. In particular, the quotient $\mathcal{L}\mathrm{Lag}(L)/\mathrm{Symp}_0(M)$ can only be non-Hausdorff at orbits corresponding to H-irrational Lagrangians. The same holds for $\mathrm{Symp}_0(M)$ replaced by $\mathrm{Ham}(M)$.*

The part on $\mathrm{Symp}_0(M)$ follows directly from Corollary 19. The part with $\mathrm{Ham}(M)$ is a finer result that also makes use of the local description of L in $\mathcal{L}(L,\tau)$ given by Corollary 43 below.

It has been proven by Ono [Ono08] and Solomon [Sol13] that the quotient $\mathcal{L}\mathrm{Lag}(L)/\mathrm{Ham}(M)$ is Hausdorff in the C^1 topology in different settings. Most notably, they both ask that $H_1(L;\mathbb{R}) \to H_1(M;\mathbb{R})$ be injective, which makes L automatically H-exact. Corollary 21 shows the difficulty of relaxing the condition that $H_1(L;\mathbb{R}) \to H_1(M;\mathbb{R})$ be injective: H-irrational Lagrangians can create non-Hausdorff points in the quotient. In fact, Theorem 1 shows that in dimension $2n \geq 6$, this always happens. That this is a problem was already mentioned by Ono in his work on the subject.

Quantitative symplectic topology When Theorem 3 holds, it allows for a new measurement associated with a Lagrangian embedding $Q \hookrightarrow M$ with image L and a Riemannian metric g on Q:

$$c^e_{(M,L)}(Q,g) := \sup\left\{r \geq 0 \,\middle|\, \text{all } L' \in \mathcal{L}\mathrm{Ham}(L) \text{ in } \mathcal{W}^g_r(L) \text{ are exact}\right\}.$$

By writing $\mathcal{W}^g_r(L)$, we want to underline that it is the image of a Weinstein neighbourhood $\Psi : D^*_rQ \to \mathcal{W}_r(L)$, where the radius r of the codisk bundle is computed using g. We write $c^e_{(M,L)}(Q,g) = 0$ if L has no neighbourhood of local exactness, e.g. for the example given by Theorem 1.

Note that $c^e_{(M,L)}(Q,g)$ is invariant under symplectomorphisms, so it is truly a symplectic quantity. Furthermore, $c^e_{(M,L)}(Q,g)$ is bounded from above by the size of the largest Weinstein neighbourhood of L in M, i.e. by the relative capacity

$$c^{\mathcal{W}}_{(M,L)}(Q,g) := \sup\left\{r > 0 \,\middle|\, L \text{ admits a neighbourhood } \mathcal{W}^g_r(L)\right\}.$$

This can in turn be bounded in terms of Poisson bracket invariants of L in M [MO21].

Going through the proof of Proposition 2 (see Appendix A for details) gives the following quantitative counterpart.

Corollary 22 *Let L be a closed curve in a surface M. If L bounds an embedded disk, let A be the smallest area of such a disk. If there are no such disks, we set $A = +\infty$. We* *have that*

$$c^e_{(M,L)}(S^1, g_0) = \min\left\{\frac{A}{2}, c^{\mathcal{W}}_{(M,L)}(S^1, g_0)\right\},$$

where g_0 is the flat metric so that S^1 has length 1*.*

Note that $\frac{A}{2}$ is precisely half the radius of the largest Weinstein neighbourhood of the circle $T(A)$ enclosing area A in $\mathbb{C}$, i.e. $c^e_{(\mathbb{C},S^1(A))}(S^1, g_0) = \frac{1}{2}c^{\mathcal{W}}_{(\mathbb{C},S^1(A))}(S^1, g_0)$.

In general, however, it is hard to get an estimate on $c^e_{(M,L)}(Q, g)$, as it is hard to get one on the neighbourhood for which Theorem 3 holds. One exception to this is when $Q = K$ is the Klein bottle: in this case, the theorem holds on every Weinstein neighbourhood (see Theorem 39 below). Therefore, the bound comes only from the proof of Theorem 4 — more precisely, from Lemma 40 and Proposition 41. In particular, we have the following bound.

Corollary 23 *Let L be a H-rational Lagrangian Klein bottle with H-rationality constant τ. We have that*

$$c^e_{(M,L)}(K, g) \geq \min\left\{\frac{\tau}{\ell^{\min}_g(\beta)}, c^{\mathcal{W}}_{(M,L)}(K, g)\right\},$$

where $\ell^{\min}_g(\beta)$ denotes the minimal length in g of a curve representing the generator β of the free factor of $H_1(K;\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_2$.

Remarks 4*. There are of course many variations of $c^e_{(M,L)}(Q, g)$ that one could take. For example, one could be interested in $c^A_{(M,L)}(Q, g)$ or $c^B_{(M,L)}(Q, g)$, the largest neighbourhood on which Conjecture A or Conjecture B, respectively, holds. However, if one believes in the NLC, then we should always have $c^A = c^e$. Moreover, we have not found an example where $c^B \neq c^{\mathcal{W}}$. Therefore, c^e seems to be the more fruitful version of the relative capacity.*

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2 An example to counter them all (but one)

We now explain the construction of the Lagrangian tori from Theorem 1. These tori support the fact that we need in general to require some type of ***rationality*** condition on our Lagrangians for Conjectures A, B or C to hold.

We start with the case when $M = \mathbb{C}^3$. Consider the product torus $L = T(1, 2, 1+\alpha) := T(1) \times T(2) \times T(1+\alpha)$, where $\alpha > 0$ is an irrational number and $T(A) \subseteq \mathbb{C}$ denotes the round circle enclosing area $A > 0$. By work of Chekanov [Che96], another product torus $T(a, b+a, c+a)$ with $a, b, c > 0$ is Hamiltonian isotopic to L in $\mathbb{C}^3$ if and only if $a = 1$ and $\mathrm{span}_{\mathbb{Z}}\{b, c\} = \mathrm{span}_{\mathbb{Z}}\{1, \alpha\} =: G$.

Since α is irrational, Dirichlet's approximation theorem ensures that there exist relatively prime integers p and q such that $|p + q\alpha| < \varepsilon$ for any $\varepsilon > 0$. This shows that $\mathrm{span}_{\mathbb{Z}}\{1, \alpha\}$ is dense in $\mathbb{R}$. We now want to show that we may take b and c arbitrarily small and still have them generate G. Fix $\varepsilon > 0$, and take $b = p + q\alpha$ as above.

Note that $c = r + s\alpha$ will be such that $\mathrm{span}_{\mathbb{Z}}\{b, c\} = G$ if and only if the matrix

$$\begin{pmatrix} p & r \\ q & s \end{pmatrix}$$

has determinant ± 1. Pick r_0 and s_0 such that the corresponding matrix has determinant ± 1 — it exists since p and q are relatively prime. Without loss of generality, we may suppose that $p + q\alpha$ and $r_0 + s_0\alpha$ are positive. Note that if r and s are of the form $r = r_0 - ip$ and $s = s_0 - iq$ for some $i \in \mathbb{Z}$, then the corresponding matrix also has determinant ± 1. But we have that

$$|r + s\alpha| = (r_0 + s_0\alpha) - i(p + q\alpha) \qquad \forall i \leq \frac{r_0 + s_0\alpha}{p + q\alpha}.$$

Therefore, if we take $i = \lfloor \frac{r_0 + s_0\alpha}{p + q\alpha} \rfloor$, then we have that

$$|c| \leq (p + q\alpha) < \varepsilon,$$

which proves that b and c may be taken arbitrarily small and still generate G.

This means that we can take $T(1, 1+b, 1+c)$ which are all in the Hamiltonian orbit of L but are arbitrarily C^1-close to the monotone torus $T(1, 1, 1)$. Therefore, without the H-rational hypothesis on L, not even the Lagrangian C^1 flux conjecture is true in $\mathbb{C}^3$.

Note that a similar argument as above actually implies that the set of $b, c > 0$ such that $T(1, 1+b, 1+c)$ is Hamiltonian isotopic to L is dense in $\mathbb{R}^2_{>0}$. This means that any neighbourhood U of such a torus $T(1, 1+b, 1+c)$ contains infinitely many $T(1, 1+b', 1+c')$ in the same Hamiltonian orbit. But $T(1, 1+b, 1+c) \cap T(1, 1+b', 1+c') = \emptyset$ if $b \neq b'$ or $c \neq c'$. Therefore, without the H-rational hypothesis on L, not even Conjecture B is true in $\mathbb{C}^3$.

We now explain how to generalize the result to any manifold of dimension $2n \geq 6$. First note that by taking a product with $T(1)^{n-3}$, we get a counterexample to our conjectures in $\mathbb{C}^n$ whenever $n \geq 3$. Furthermore, by Theorem 1.1(ii) of [CS16], the Hamiltonian isotopy from $T(1, \ldots, 1, 1+b, 1+c)$ to $T(1, \ldots, 1, 1+b', 1+c')$ can be taken to be fully supported in the ball $B^{2n}(A)$ of capacity $A = n+1+\max\{b+c, b'+c'\}$, i.e. of radius $\sqrt{\frac{A}{\pi}}$. In particular, for b, b', c, and c' small enough, it can be supported in the ball of capacity $n+2$. Therefore, we get a counterexample in $M = B^{2n}(n+2)$ But then, by simply rescaling the ball, we get a counterexample in the ball $B^{2n}(A)$ for any $A > 0$. By the Darboux theorem, any symplectic manifold M^{2n} admits a symplectic embedding of the ball $B^{2n}(A)$ for A small enough, which gives the counterexample for every M with $\dim M \geq 6$.

Remarks 5. *Interestingly enough, the above counterexample does not work in dimension* 4. *Indeed, Chekanov's classification of product tori implies that every product torus* L *in* $\mathbb{C}^2$ *has a* C^1 *neighbourhood* U *such that* $\mathcal{L}\mathrm{Ham}(L) \cap U = \{L\}$. *In particular, the* C^1 *version of Conjecture A holds for* L*, and if its Hamiltonian orbit is not closed, then the limit cannot be a product or a Chekanov torus. By Theorem* 1.3 *of* [CS16], *the same holds for product tori in small enough Darboux balls in subtame symplectically aspherical symplectic* 4*-manifolds.*

However, we can use Theorem 1.5 *of* [CS16] *to construct — in a similar fashion as above — a counterexample to Conjecture B in any* (*spherically*) *irrational symplectic* 4*-fold.*

3 Relations between homological rationality and exactness and their standard counterparts

In this section, we discuss relations between standard rationality/exactness and H-rationality/H-exactness. In Section 3.1, we prove the general following fact: for a closed Lagrangian of the cotangent bundle, being H-exact is equivalent to being isotopic to an exact Lagrangian. In Section 3.2, we first explain some specific situations in which H-rationality reduces to rationality. We then however give an example which illustrates why we generally work with the H-rationality condition rather than the standard rationality one.

3.1 The central lemma

The following lemma will prove to be quite useful in order to prove the main results of this work.

Lemma 24 *Let* L *be a closed Lagrangian in* T^*Q*, the following are equivalent*:

(*i*) *L is isotopic to an exact Lagrangian via Lagrangian submanifolds*;

(*ii*) *L is symplectically isotopic to an exact Lagrangian*;

(*iii*) *L is H-exact*;

(*iv*) *the composition* $L \to T^*Q \to Q$ *is a homotopy equivalence.*

Proof. Let $i^* : H^1(T^*Q;\mathbb{R}) \to H^1(L;\mathbb{R})$ be induced by the inclusion $i : L \to T^*Q$. H-exactness of L ensures that the form $i^*\lambda$ is closed so that it defines a class $[i^*\lambda] \in H^1(L;\mathbb{R})$. Because the canonical projection $\pi : T^*Q \to Q$ is a homotopy equivalence, there exists a closed 1-form $\sigma \in \Omega^1(Q)$ such that $[i^*\lambda] = [i^*(\pi^*\sigma)]$.

Now, σ induces a fibrewise symplectomorphism ψ_σ of T^*Q which satisfies $[\psi_\sigma^*(i^*\lambda)] = 0$ so that ψ_σ maps L to an exact Lagrangian. This shows that *(iii)* yields *(ii)*, which obviously yields *(i)*.

Note also that *(i)* implies that the inclusion $L \hookrightarrow T^*Q$ is homotopic to the inclusion of an exact Lagrangian. But, when L is exact, the composition $L \to T^*Q \to Q$ is a (simple) homotopy equivalence [AK18], i.e. *(iv)* holds.

Finally, if *(iv)* holds, then $H_2(T^*Q, L) = 0$, and we have *(iii)*. □

3.2 From rationality to H-rationality

Obviously, H-rationality implies usual rationality, i.e. $\omega(H_2(M, L))$ being discrete implies that $\omega(\pi_2(M, L))$ also is. Furthermore, in many cases, these conditions are equivalent. This is the case, for example, when $\pi_1(M) = 0$. Indeed, in this case, the relative Hurewicz morphism $\pi_2(M, L) \to H_2(M, L; \mathbb{Z})$ can be shown to be surjective. Expanding on this idea, we get the following.

Lemma 25 *Suppose that* $[\pi_1(M), \pi_1(M)]$ *is finite. Then, we have that*

$$N\omega(H_2(M, L; \mathbb{Z})) \subseteq \omega(\pi_2(M, L)) + \omega(H_2(M; \mathbb{Z})),$$

where N *is the order of* $[\pi_1(M), \pi_1(M)]$*. In particular, if* $\pi_1(M)$ *is abelian, then we have equality.*

Proof. We consider the following commutative diagram.

$$\begin{array}{ccccccc}
\pi_2(M) & \xrightarrow{j} & \pi_2(M, L) & \xrightarrow{\partial} & \pi_1(L) & \xrightarrow{i} & \pi_1(M) \\
\downarrow{\scriptstyle h_2} & & \downarrow{\scriptstyle h_2''} & & \downarrow{\scriptstyle h_1'} & & \downarrow{\scriptstyle h_1} \\
H_2(M) & \xrightarrow{j} & H_2(M, L) & \xrightarrow{\partial} & H_1(L) & \xrightarrow{i} & H_1(M)
\end{array}$$

Here, the rows are the long exact sequences of the pair (M, L) in homotopy and homology with integer coefficients, respectively, and the columns are the various Hurewicz morphisms; it commutes by naturality of the Hurewicz map. We make the abuse of notation of using the same symbols for morphisms in homotopy and homology, since it will be clear from the context which one we are using when.

The proof follows from a straightforward diagram chasing argument, but we still give the details. Let $A \in H_2(M, L)$. Since h_1' is surjective — the Hurewicz morphism in first degree is simply the abelianization morphism — there is some $a \in \pi_1(L)$ such that $\partial(A) = h_1'(a)$. But note that

$$h_1 i(a) = i h_1'(a) = i\partial(A) = 0$$

by exactness at $H_1(L)$. Therefore, $i(a) \in \operatorname{Ker} h_1 = [\pi_1(M), \pi_1(M)]$. By hypothesis, the order N of $[\pi_1(M), \pi_1(M)]$ is finite, so that $i(Na) = 0$. Therefore, there is some $u \in \pi_2(M, L)$ such that $\partial(u) = Na$. But note that

$$\partial(NA - h_2''(u)) = h_1'(Na) - h_1'(\partial u) = 0.$$

By exactness at $H_2(M, L)$, there is thus some $B \in H_2(M)$ such that $NA = j(B) + h_2''(u)$. To conclude, we only note that $\omega(j(B)) = \omega(B)$ and $\omega(h_2''(u)) = \omega(u)$. □

From the above lemma, we directly get the following result.

Corollary 26 *If $[\pi_1(M), \pi_1(M)]$ is finite and $\omega(H_2(M;\mathbb{Z})) \subseteq \omega(\pi_2(M))$, then every rational Lagrangian with rationality constant τ is H-rational with rationality constant $\frac{k}{N}\tau$ for some $k \in \mathbb{N}$. Moreover, if $\pi_1(M)$ is abelian, we have $k = N = 1$. In particular, in all those cases, every weakly exact Lagrangian is H-exact.*

Note that this corollary recovers the statement at the start of the subsection that H-rationality and rationality are the same when $\pi_1(M) = 0$. However, what is perhaps most interesting is the case $M = D_r^*L$. In that case, the condition on $\omega(H_2(M;\mathbb{Z}))$ is automatically satisfied, since ω is exact, and the condition on the commutator subgroup of $\pi_1(M)$ becomes that $[\pi_1(L), \pi_1(L)]$ be finite. We thus get a new version of Theorem 4.

Corollary 27 *Let L be a rational Lagrangian submanifold of (M, ω) such that $[\pi_1(L), \pi_1(L)]$ is finite, and let $L' \in \mathcal{L}\mathrm{Ham}(L)$ be a Lagrangian included in a Weinstein neighbourhood $\mathcal{W}_r(L)$ of size $r > 0$ such that L' is weakly exact in $\mathcal{W}_r(L)$. Then, for a maybe smaller r, L' is exact in $\mathcal{W}_r(L)$.*

Since H-exactness implies weak exactness, Theorem 3 also gives a neighbourhood of weak exactness. However, one could also work directly with rationality (see Remark 6 below). Furthermore, there is also a version of Theorem 6 in terms of usual rationality. Therefore, we also have a rational version of the weak Lagrangian C^0 flux conjecture.

Corollary 28 *Let L be a rational Lagrangian in a symplectic manifold M such that $[\pi_1(L), \pi_1(L)]$ is finite. Suppose that*

(*i*) *L satisfies the hypotheses of Theorem 3 or $H_1(L;\mathbb{R}) = 0$,*

(*ii*) *and the nearby Lagrangian conjecture holds in T^*L.*

Then, $\mathcal{L}\mathrm{Ham}(L)$ and $\mathcal{L}\mathrm{Symp}_0(L)$ is Hausdorff-closed in $\mathsf{SMan}(L)$*.*

We end this subsection with an example which showcases the need for $[\pi_1(L), \pi_1(L)]$ to be finite. This also exemplifies why we are working with H-rational Lagrangians and not just rational Lagrangians.

Example. In [Pol93], Polterovich constructs for any vector $v \in \mathbb{R}^n$ and any flat manifold Q a Lagrangian torus L_v in T^*Q. This torus has the property that

(*i*) *for a contractible open $U \subseteq Q$, $L_v \cap T^*Q|_U = U \times \{v\} \subseteq U \times \mathbb{R}^n$;*

(*ii*) *the map $L_v \to Q$ given by restriction of $\pi : T^*Q \to Q$ is a covering.*

We concentrate our efforts on the simplest case: $n = 2$ and $Q = K$ is the Klein bottle. In that case, $L_v \to K$ is the 2:1 cover.

First note that L_v is weakly exact in T^*K. To see this, denote by $p : \mathbb{T}^2 \to K$ the 2:1 cover and take $\widetilde{p} : T^*\mathbb{T}^2 \to T^*K$ to be its lift using the flat metrics on $\mathbb{T}^2$ and K. Point (i) gives that $\widetilde{p}^{-1}(L_v) = \mathbb{T}^2 \times \{v\} \subseteq T^*\mathbb{T}^2 = \mathbb{T}^2 \times \mathbb{R}^2$. But any disk u with boundary along L_v admits a lift $\widetilde{u}$ in $T^*\mathbb{T}^2$ with boundary along $\mathbb{T}^2 \times \{v\}$. Since $\mathbb{T}^2 \times \{v\} \hookrightarrow T^*\mathbb{T}^2$ is a homotopy equivalence, $\pi_2(T^*\mathbb{T}^2, \mathbb{T}^2 \times \{v\}) = 0$, and the lift $\widetilde{u}$ is contractible. But then, so must be u, and we have that $\pi_2(T^*K, L_v) = 0$.

On the other hand, L_v is **not** *H-exact. Indeed, let $\gamma : S^1 \to K$ be a loop admitting a lift to L_v, that is, $[\gamma] \in p_*(\pi_1(\mathbb{T}^2))$. Since $L_v \to K$ is a 2:1 cover, there are two lifts $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ of γ. Furthermore, each lift $\widetilde{\gamma}_i$ defines a cylinder C_i in T^*K by taking*

$C_i(s,t) = t\widetilde{\gamma}_i(s)$, $(s,t) \in S^1 \times [0,1]$. Note that $\partial C_i = \widetilde{\gamma}_i \sqcup -\gamma$, where the minus sign denotes the reversal of orientation. Therefore, $C := C_1 \cup_\gamma -C_2$ is a cylinder in T^*K with boundary along L_v. Furthermore, it has area

$$\omega_0(C) = \lambda_0(\widetilde{\gamma}_1) - \lambda_0(-\widetilde{\gamma}_2) = 2\int_{S^1} \langle v, \dot{\gamma}(s)\rangle ds,$$

where $\langle\cdot,\cdot\rangle$ denotes the Euclidean scalar product. In particular, if we take γ to be a simple loop corresponding to a straight line in the fundamental domain of $\mathbb{R}^2$ defining $K = \mathbb{R}^2/\pi_1(K)$ and v to be positively proportional to $\dot{\gamma}$, then $\omega_0(C) = 2|v| > 0$. Therefore, such an L_v is indeed not H-exact.

Finally, note that, as $v \longrightarrow 0$, $L_v \longrightarrow K$ in the Hausdorff metric. Therefore, however small we take a neighbourhood of the zero-section of T^*K, there is a weakly exact Lagrangian in that neighbourhood which is not exact. Therefore, unlike in Theorem 4, there is not an equivalent to Corollary 27 with $\mathcal{L}\mathrm{Ham}(L)$ replaced by the space of all τ-rational Lagrangians if $[\pi_1(L), \pi_1(L)]$ is not finite. In particular, most applications in the introduction do not have equivalents in spaces of τ-rational Lagrangians. That being said, it is entirely possible that Corollary 27 holds in $\mathcal{L}\mathrm{Ham}(L)$ without the hypothesis on $[\pi_1(L), \pi_1(L)]$ — it is however beyond the scope of the techniques presented in this paper. In other words, with the present techniques, it is unclear whether Conjecture B is true for rational Lagrangians in general, even if we restrict to those diffeomorphism types covered by Theorem 3.

4 Proof of Theorem 3

We now turn our attention to Theorem 3. To obtain a proof, we introduce some capacities inspired by work of Cieliebak and Mohnke [CM18] (Section 4.1 below), and we explain how their finiteness implies the theorem (Section 4.2). We conclude by an explicit proof of Theorem 3 in the case of Klein bottles which gives a better estimate in that case (Section 4.3). Note that the methods developed here will also be central to the proof of Theorem 6.

4.1 Some capacities *à la* Cieliebak–Mohnke

In [CM18], Cieliebak and Mohnke introduce — and compute in some cases — a capacity which measures, in a given domain, the largest possible area of a minimal disk with boundary along a Lagrangian torus. We start by introducing a small tweak in their definition, which will turn out to be quite useful in our setting.

Let Q be a closed connected n-manifold. For any $2n$-dimensional symplectic manifold (X, ω), we define two classes of Lagrangians:

$$\mathscr{L}_Q(X) := \{L = \mathrm{Im}(f : Q \hookrightarrow X) \mid f^*\omega = 0,\ \omega(H_2(X, L; \mathbb{Z})) \neq 0\}$$
$$\mathscr{L}_Q^0(X) := \{L = f(Q) \in \mathscr{L}_Q(X) \mid H_1(f) \otimes \mathbb{R} = 0\},$$

where $H_1(f) \otimes \mathbb{R}$ is the map induced by f on first homology with real coefficients. In turn, this defines two capacities:

$$c_Q(X) := \sup\{A^H_{\min}(L,X) \mid L \in \mathscr{L}_Q(X)\} \in [0,+\infty] \quad \text{and}$$
$$c^0_Q(X) := \sup\{A^H_{\min}(L,X) \mid L \in \mathscr{L}^0_Q(X)\} \in [0,+\infty],$$

where

$$A^H_{\min}(L,X) := \inf\{\omega(u) \mid u \in H_2(X,L;\mathbb{Z}), \omega(u) > 0\}.$$

We take the convention that $c_Q(X) = 0$ (respectively $c^0_Q(X) = 0$) if $\mathscr{L}_Q(X) = \varnothing$ (respectively $\mathscr{L}^0_Q(X) = \varnothing$). Obviously, we have that $c^0_Q \leq c_Q$. Finally, we set

$$c_{\text{all}}(X) := \sup c_Q(X) \quad \text{and}$$
$$c^0_{\text{all}}(X) := \sup c^0_Q(X),$$

where the supremum runs over all closed connected n-dimensional manifolds.

Remarks 6. *The main differences between our definition and Cieliebak–Mohnke's are that we work with homology instead of homotopy, we allow any Q and not only tori, and we only look at Lagrangians which do bound some homology class with nonvanishing area. The latter is central to our argument, as we will mainly be interested in the case $X = D^*Q$, but such a manifold obviously admits an exact Lagrangian Q. Therefore, without this restriction, $c_Q(D^*Q)$ would be infinite for trivial reasons, which runs counter to our purpose here.*

However, we could develop an entirely analogous theory using homotopy. With it, we would get a version of Theorem 3 for rational Lagrangians, i.e. a neighbourhood of weak exactness. However, this is in general not enough to get a neighbourhood of exactness as in Theorem 4 — see Section 3.2 for a discussion as to when that is the case.

The following properties follow directly from the definition of the capacities.

Lemma 29 *Let c denote c_Q, c^0_Q, c_{all}, or c^0_{all}. We have the two following properties.*

(*i*) *For all $\alpha \neq 0$, we have that $c(X, \alpha\omega) = |\alpha| c(X,\omega)$.*

(*ii*) *If there is a 0-codimensional symplectic embedding $\iota : X \hookrightarrow X'$ such that $H_2(X', \iota(X); \mathbb{R}) = 0$, then $c(X) \leq c(X')$.*

The problem with the monotonicity property (ii) when $H_2(X', \iota(X); \mathbb{R}) \neq 0$ is that there could then be homology classes in X' with smaller area than those in X — thus inverting the expected direction of the inequality. However, the capacity c^0_Q partially goes around that issue.

Lemma 30 *If there exists a 0-codimensional symplectic embedding $\iota : X \hookrightarrow X'$ and X' is exact, then $c^0_Q(X) \leq Bc^0_Q(X')$, where $B \geq 1$ only depends on the torsion part of $H_1(X;\mathbb{Z})$.*

Proof. Let λ' be a primitive of the symplectic form of ω' on X'. Then, $\lambda = \iota^*\lambda'$ is a primitive of ω on X. Fix $L = f(Q) \in \mathscr{L}^0_Q(X)$. Since $H_1(f) \otimes \mathbb{R} = 0$, we must have that $f_*(H_1(Q;\mathbb{Z}))$ is a torsion subgroup of $H_1(X;\mathbb{Z})$. Take B to be the order of the torsion of $H_1(X;\mathbb{Z})$ if it is nonzero, i.e. if

$$H_1(X;\mathbb{Z}) = \mathbb{Z}^b \oplus \mathbb{Z}_{p_1^{k_1}} \oplus \cdots \oplus \mathbb{Z}_{p_\ell^{k_\ell}},$$

then $B = p_1^{k_1} \dots p_\ell^{k_\ell}$. If $H_1(X;\mathbb{Z})$ has no torsion, then we simply set $B = 1$. We thus have $B \cdot f_*(H_1(Q;\mathbb{Z})) = 0$. By the homology long exact sequence of the pair (X, L), this is equivalent to saying that $\partial H_2(X, L;\mathbb{Z}) \supseteq B \cdot H_1(L;\mathbb{Z})$. Therefore, we have that

$$\begin{aligned}
A^H_{\min}(L, X) &= \inf_{\substack{u \in H_2(X,L;\mathbb{Z}) \\ \omega(u)>0}} \omega(u) \\
&= \inf_{\substack{a \in \partial H_2(X,L;\mathbb{Z}) \\ \lambda(a)>0}} \lambda(a) \\
&\le \inf_{\substack{a \in B\cdot H_1(L;\mathbb{Z}) \\ \lambda(a)>0}} \lambda(a) \\
&= B \cdot \inf_{\substack{a \in H_1(L;\mathbb{Z}) \\ \lambda(a)>0}} \lambda(a) \\
&= B \cdot \inf_{\substack{a' \in H_1(\iota(L);\mathbb{Z}) \\ \lambda'(a')>0}} \lambda'(a') \\
&\le B \cdot \inf_{\substack{a' \in \partial H_2(X',\iota(L);\mathbb{Z}) \\ \lambda'(a')>0}} \lambda'(a') \\
&= B \cdot A^H_{\min}(\iota(L), X').
\end{aligned}$$

Since $\iota(\mathscr{L}^0_Q(X)) \subseteq \mathscr{L}^0_Q(X')$, this gives the desired inequality. □

We note that $c_Q(X)$ and $c^0_Q(X)$ are equal whenever $H_1(X;\mathbb{R}) = 0$. However, there is another important case where they also coincide.

Lemma 31 *If* $\dim H_1(Q;\mathbb{R}) = 1$ *and X is exact, we have that* $c_Q(X) = c^0_Q(X)$.

Proof. We fix Q and X as above. We can assume that there is a Lagrangian embedding $f : Q \hookrightarrow X$; otherwise both capacities are equal to 0. Since $\dim H_1(Q;\mathbb{R}) = 1$, $H_1(f) \otimes \mathbb{R}$ is either 0 or injective. Suppose that it is injective. By the long exact sequence in homology, we then get that the boundary map $\partial : H_2(X, L;\mathbb{R}) \to H_1(L;\mathbb{R})$ is zero, where $L = f(Q)$. Since $\omega(H_2(X, L;\mathbb{R})) = \lambda(\partial(H_2(M, L;\mathbb{R})))$ whenever $\omega = d\lambda$, we then conclude that L is H-exact. In particular, $L \notin \mathscr{L}_Q(X)$. Therefore, we have that $\mathscr{L}_Q(X) = \mathscr{L}^0_Q(X)$, which implies the result. □

We end this short list of properties of our capacities by proving that they behave relatively well under products.

Lemma 32 *Suppose that Q' admits a H-exact Lagrangian embedding in X'. Then,* $c_Q(X) \le c_{Q\times Q'}(X \times X')$. *In particular,* $c_{\mathrm{all}}(X) \le c_{\mathrm{all}}(X \times X')$ *as soon as X' admits a H-exact Lagrangian.*

If Q' admits any Lagrangian embedding in an exact X' and $H_1(Q';\mathbb{R}) = 0$, then we have that $c^0_Q(X) \leq c^0_{Q\times Q'}(X \times X')$. In particular, $c^0_{\text{all}}(X) \leq c^0_{\text{all}}(X \times X')$ as soon as X' admits a Lagrangian with vanishing first Betti number.

Proof. Let L be the image of a Lagrangian embedding of Q in X, and let L' be the image of a H-exact Lagrangian embedding in X'. Note that we can suppose that L bounds some homology class, otherwise the inequality is trivial. Let thus $v : (\Sigma, \partial\Sigma) \to (X \times X', L \times L')$ for some compact surface Σ with boundary. Projecting on each component gives maps $u : (\Sigma, \partial\Sigma) \to (X, L)$ and $u' : (\Sigma, \partial\Sigma) \to (X', L')$. Furthermore, if ω is the symplectic form of X and ω' of X', we then have that

$$(\omega \oplus \omega')(v) = \omega(u) + \omega(u') = \omega(u),$$

since L' is H-exact. Taking infima over all v, we thus get

$$c_{Q\times Q'}(X \times X') \geq A^H_{\min}(L \times L', X \times X') = \inf_{\substack{u=pr_1\circ v\\ \omega(u)>0}} \omega(u) \geq A^H_{\min}(L, X).$$

We then get the inequality by taking the supremum over all possible L's.

The case $H_1(Q';\mathbb{R}) = 0$ is proven in much the same way. Indeed, exactness of X' along with $H_1(Q';\mathbb{R}) = 0$ ensures that we also have $(\omega \oplus \omega')(v) = \omega(u)$. Furthermore, the vanishing of the first Betti number ensures that $H_1(Q \times Q';\mathbb{R}) \to H_1(X \times X';\mathbb{R})$ vanishes if and only if $H_1(Q;\mathbb{R}) \to H_1(X;\mathbb{R})$ does. □

4.2 Finiteness of the capacities

Having enunciated the main properties of our capacities, we now explain how one can get Theorem 3 from their finiteness. To do this, we first formulate a meta result.

Proposition 33 *Let L be a Lagrangian in a symplectic manifold M. Suppose that $c_Q(D^*_R L)$ is finite for some $R > 0$. For every $\tau \geq 0$, there exists a Weinstein neighbourhood $\mathcal{W}(L)$ of L in M, such that all $L' \in \mathcal{L}(Q, \tau)$ included in $\mathcal{W}(L)$ is H-exact in $\mathcal{W}(L)$.*

If $c_{\text{all}}(D^*_R L)$ is finite for some $R > 0$, then the same holds for all $L' \in \mathcal{L}(\tau)$.

We recall that $\mathcal{L}(\tau)$ denotes the space of τ-H-rational Lagrangians of M; $\mathcal{L}(Q, \tau)$ is the subspace of those Lagrangians which have the diffeomorphism type of Q.

Proof. We only prove the case $c_Q(D^*_R L) < \infty$, as the case $c_{\text{all}}(D^*_R L) < \infty$ is completely analogous. To do so, we make the following two observations.

(1) If $c_Q(D^*_R L) < \infty$, then $\lim_{r\to 0} c_Q(D^*_r L) = 0$.

(2) If $L' \in \mathcal{L}(Q, \tau)$ and $L' \subseteq \mathcal{W}_r(L)$, then $c_Q(D^*_r L) \geq \tau$ whenever we have that $\omega(H_2(\mathcal{W}_r(L), L')) \neq 0$.

Obviously, the theorem follows directly from these two observations.

The first observation follows directly from Property (i) of Lemma 29. Indeed, we have that

$$\lim_{r\to 0} c_Q(D^*_r L) = \lim_{r\to 0} \frac{r}{R} c_Q(D^*_R L) = 0.$$

Here, we have made use of the fact that $(D^*_r L, \omega_0)$ is symplectomorphic to $(D^*_{r/a} L, a\omega_0)$ via the map $(q,p) \mapsto (q, ap)$. Note that Property (ii) of Lemma 29 implies that our capacity is invariant under symplectomorphisms.

For the second observation, take $L' \in \mathcal{L}(Q,\tau)$ such that $L' \subseteq \mathcal{W}_r(L)$, and suppose there is some class $u \in H_2(\mathcal{W}_r(L), L')$ such that $\omega(u) \neq 0$. Without loss of generality, we may suppose that $\omega(u) > 0$. By definition of H-rationality, there is some $k \in \mathbb{Z}$ such that $\omega(A) = k\tau$. Since $\omega(u) > 0$, $k \geq 1$. Therefore, $\omega(u) \geq \tau$. Taking the infimum over all possible u's, we get

$$\tau \leq A^H_{\min}(L', \mathcal{W}_r(L)) \leq c_Q(\mathcal{W}_r(L)) = c_Q(D^*_r L).$$

Again, we have made use of the fact that c_Q is invariant under symplectomorphisms. □

Therefore, proving Theorem 3 reduces to proving finiteness of some capacity in cotangent bundles. In general, this turns out to be nontrivial, since even $c_{\mathbb{T}^n}(X)$ — the best-behaved version of our capacities — is only well understood when X is a convex or concave toric domain, which is far from the case we need. We will explore this further down, but we already note some interesting cases where finiteness is achievable.

Proposition 34 *If $c_{Q\times Q'}(D^*_R(Q\times Q')) < \infty$, then we have that $c_Q(D^*_R Q) < \infty$ and $c_{Q'}(D^*_R Q') < \infty$.*

Proof. It follows from Lemma 32 that

$$c_Q(D^*_R Q) \leq c_{Q\times Q'}(D^*_R Q \times D^*_R Q').$$

But $D^*_R Q \times D^*_R Q'$ embeds symplectically in $D^*_{2R}(Q\times Q')$ and that embedding is a homotopy equivalence. The proposition then follows directly from Property (ii) of Lemma 29, since finiteness for some $R > 0$ implies finiteness for every $R > 0$ by Property (i) of that lemma. □

Note that if L is a displaceable Lagrangian in a tame symplectic manifold, $A^H_{\min}(L)$ is a lower bound for its displacement energy — this follows from Chekanov's famous estimate [Che98]. In particular, $c_{\text{all}}(B^{2n})$ is bounded by the displacement energy of B^{2n}, and thus it is finite. Zhou [Zho20] proved a broad generalization of this result using a truncated version of Viterbo's transfer map.

Theorem 35 ([Zho20]) *Let X be a Liouville domain with $SH(X) = 0$. We have that $c_{\text{all}}(X) < \infty$.*

Note that $SH(D^*_R L) \neq 0$ because of Viterbo's isomorphism [Vit99]. Therefore, Zhou's theorem never directly implies Theorem 3. However, in some cases, we still manage to compare $c_Q(D^*L)$ to $c_{\text{all}}(X)$ as we shall see below.

Remarks 7. Zhou actually works with homotopy — not homology like us — and allows for the possibility of weakly exact Lagrangians. He also allows some nonexact Liouville domains, but it will not be needed here. Therefore, his result is actually more general than what is cited here.

Case $\dim H_1(Q;\mathbb{R}) = 1$**.** We now turn our attention to the capacity c^0_Q. Recall that from Lemma 31, $c^0_Q(X) = c_Q(X)$ whenever $\dim H_1(Q;\mathbb{R}) = 1$ and X is exact. However, studying directly c^0_Q allows us to show the following general result.

Theorem 36 *Let L be a Lagrangian submanifold of M. Suppose that, as an abstract manifold, L admits a Lagrangian embedding in a Liouville domain W with $SH(W) = 0$. For every $\tau \geq 0$, there exists a Weinstein neighbourhood $\mathcal{W}(L)$ of L in M, such that if $L' \in \mathcal{L}(\tau)$ is included in $\mathcal{W}(L)$, then the map*

$$H_1(L';\mathbb{R}) \xrightarrow{\pi_*} H_1(L;\mathbb{R})$$

induced by the projection $\pi : L' \to L$ is nonzero.

In turn, this follows from a variant of Proposition 33 and a proof of the finiteness of $c^0_{\mathrm{all}}(D^*_R L)$ for L as in the theorem. More precisely, we need the following two results.

Proposition 37 *Let L be a Lagrangian in a symplectic manifold M. Suppose that $c^0_Q(D^*_R L)$ is finite for some $R > 0$. For every $\tau \geq 0$, there exists a Weinstein neighbourhood $\mathcal{W}(L)$ of L in M, such that all $L' \in \mathcal{L}(Q,\tau)$ included in $\mathcal{W}(L)$ has nontrivial morphism $\pi_* : H_1(L';\mathbb{R}) \to H_1(L;\mathbb{R})$ if $H_1(L;\mathbb{R}) \neq 0$.*

If $c^0_{\mathrm{all}}(D^*_R L)$ is finite for some $R > 0$, then the same holds for all $L' \in \mathcal{L}(\tau)$.

Proof. The proof is essentially that of Proposition 33, except that Observation (2) is replaced with the following.

(2′) If $L' \in \mathcal{L}(Q,\tau)$, $L' \subseteq \mathcal{W}_r(L)$, and $H_1(L';\mathbb{R}) \to H_1(\mathcal{W}(L);\mathbb{R})$ is zero, then $c^0_Q(D^*_r L) \geq \tau$ whenever we have that $\omega(H_2(\mathcal{W}_r(L), L')) \neq 0$.

Therefore, for r small enough, all $L' \in \mathcal{L}(Q,\tau)$ such that $L' \subseteq \mathcal{W}_r(L)$ must be either H-exact or induce a nonzero map $H_1(L';\mathbb{R}) \to H_1(\mathcal{W}_r(L);\mathbb{R})$.

By the central lemma 24, H-exactness in $\mathcal{W}_r(L)$ implies that the projection $\pi : L' \to L$ is a homotopy equivalence. Therefore, it also implies that the map $H_1(L';\mathbb{R}) \to H_1(\mathcal{W}_r(L);\mathbb{R})$ is nonzero whenever $H_1(\mathcal{W}_r(L);\mathbb{R}) \neq 0$. Since the projection $\mathcal{W}_r(L) \to L$ is a homotopy equivalence, this implies the result. □

Lemma 38 *Let L be a manifold which admits a Lagrangian embedding in W with $SH(W) = 0$. Then,*

$$c^0_{\mathrm{all}}(D^*_R L) < \infty$$

for some $R > 0$.

Proof. By the Weinstein neighbourhood theorem, there is some $R > 0$ such that $D^*_R L$ embeds symplectically in W. Therefore, we have that

$$c^0_{\mathrm{all}}(D^*_R L) \leq B c^0_{\mathrm{all}}(W) < \infty$$

where $B = B(L)$ is the constant of Lemma 30. The latter finiteness is that of Theorem 35 above. □

We now get pretty directly proofs of Theorems 36 and 3, Case *(a)*.

Proof of Theorem 36. Combining Proposition 37 and Lemma 38 gives the existence of a neighbourhood $\mathcal{W}(L)$ such that whenever $L' \in \mathcal{L}(\tau)$ is in $\mathcal{W}(L)$, then $H_1(L';\mathbb{R}) \to H_1(L;\mathbb{R})$ is nonzero if $H_1(L;\mathbb{R}) \neq 0$. But by the Viterbo transfer morphism, $H_1(L;\mathbb{R}) \neq 0$ since it embeds as a Lagrangian in a Liouville domain with $SH(W) = 0$ (see [Rit13]). ◻

Proof of Theorem 3, Case (a). Whenever $\dim H_1(Q;\mathbb{R}) = 1$ and L admits a Lagrangian embedding in W with $SH(W) = 0$, then

$$c_Q(D_R^*L) = c_Q^0(D_R^*L) \leq c_{\mathrm{all}}^0(D_R^*L) < \infty,$$

where the first equality is Lemma 31 and the last inequality is Lemma 38. Case *(a)* of Theorem 3 then follows from Proposition 33. ◻

Remarks 8. *In view of Lemma 32, it actually suffices to prove finiteness of c^0 for $Q \times Q'$ admitting an embedding in W with $SH(W) = 0$, where $H_1(Q';\mathbb{R}) = 0$. However, we cannot find an example of a Q with $H_1(Q;\mathbb{R}) = \mathbb{R}$ such that $Q \times Q'$ admits such an embedding but not Q by itself.*

4.3 Lagrangian Klein bottles in cotangent bundles

We now focus our efforts on proving Case (c) of Theorem 3, i.e. the case of the Klein bottle K. The proof is fairly different from previous cases, as it relies on the classification of Lagrangian Klein bottles in T^*K, which turns out to be doable by a direct computation. It does however rely on the deep fact that there is no Lagrangian Klein bottle in $\mathbb{C}^2$ [She09].

Theorem 39 *Every Lagrangian Klein bottle in T^*K is H-exact. In other words, $c_K(T^*K) = 0$.*

Proof. Let L be a Lagrangian Klein bottle in T^*K. We equip K and the 2-torus $\mathbb{T}^2$ with the flat metric, so that the covering $p : \mathbb{T}^2 \to K$ is a local isometry. By rescaling if necessary, we can suppose that $L \subseteq D_r^*K$ for r arbitrarily small. In particular, we may choose r small enough so that there exists a Weinstein neighbourhood $\Psi : D_r^*\mathbb{T}^2 \to \mathbb{C}^2$ of the standard Clifford torus $S^1 \times S^1$.

Using the flat metric on $\mathbb{T}^2$ and K, the 2:1 covering $p : \mathbb{T}^2 \to K$ lifts to another 2:1 covering $\widetilde{p} : T^*\mathbb{T}^2 \to T^*K$ which is also a local isometry and symplectomorphism. Therefore, $\widetilde{L} := \widetilde{p}^{-1}(L)$ must be a (possibly disconnected) Lagrangian submanifold of $D_r^*\mathbb{T}^2$. Since $\widetilde{p}|_{\widetilde{L}}$ is also a 2:1 covering, $\widetilde{L}$ must either be two disconnected copies of a Klein bottle or a 2-torus. However, if the former was the case, then each connected component of $\Psi(\widetilde{L})$ would be a Lagrangian Klein bottle in $\mathbb{C}^2$, which does not exist [She09]. Therefore, $\widetilde{L}$ must be a 2-torus. In other words, the composition

$$\mathbb{T}^2 \xrightarrow{2:1} L \xhookrightarrow{i} T^*K$$

admits a lift to $T^*\mathbb{T}^2$, but the composition

$$K \xrightarrow{\sim} L \overset{i}{\hookrightarrow} T^*K$$

does not.

We now interpret these statements in algebraic terms. To do so, we first look at the fundamental groups $\pi_1(T^*K) = \langle a, b | ab = b^{-1}a \rangle$ and $\pi_1(L) = \langle a', b' | a'b' = (b')^{-1}a' \rangle$. With these presentations, the subgroups associated to the coverings $T^*\mathbb{T}^2 \to T^*K$ and $\mathbb{T}^2 \to L$ are those generated by $\{a^2, b\}$ and $\{(a')^2, b'\}$, respectively. Denote $i_*(a') = a^k b^\ell$ and $i_*(b') = a^m b^n$. Here, we have made use of the presentation above to conclude that any element of $\pi_1(T^*K)$ can be written in that way. Given the lifting criterion for coverings, the fact that the composition $\mathbb{T}^2 \to L \to T^*K$ admits a lift is equivalent to m being even. Indeed, we have that

$$i_* \left((a')^2\right) = (i_*(a'))^2 = a^{2k} b^{(1+(-1)^k)\ell},$$

so that this element always admits a lift to $T^*\mathbb{T}^2$. In turn, this forces k to be odd, since the composition $K \to L \to T^*K$ does not admit a lift. In particular, k is nonzero. But a generates the free factor and b the torsion factor of $H_1(T^*K; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_2$ under the Hurewicz morphism (and analogously for a' and b' in $H_1(L; \mathbb{Z})$). Therefore, i induces a monomorphism $i_* : H_1(L; \mathbb{Z})^{\text{free}} \to H_1(T^*K; \mathbb{Z})^{\text{free}}$ between the free part of the homologies. But then $i_* : H_1(L; \mathbb{R}) \to H_1(T^*K; \mathbb{R})$ is also injective. By the long exact sequence in homology, this implies that the boundary map $\partial : H_2(T^*K, L; \mathbb{R}) \to H_1(L; \mathbb{R})$ is zero. Since $\omega_0(H_2(T^*K, L)) = \lambda_0(\partial(H_2(T^*K, L)))$, L must be H-exact. □

5 Theorem 4 and the C^0 Lagrangian flux conjecture

In this section, we first prove Theorem 4 (Section 5.1). We then give a short proof of Theorem 6 (Section 5.2), which follows almost directly from the proof of Theorem 4. Finally, we prove a refined version of Proposition 7 and use it to properly show Corollary 8 (Section 5.3).

5.1 Proof of Theorem 4

We consider a H-rational Lagrangian submanifold L of (M, ω) of rationality constant $\tau \geq 0$. We fix a Riemannian metric g on L and a Weinstein neighbourhood $\mathcal{W}_r(L)$ in M of size $r > 0$. Let $L' \in \mathcal{L}(\tau)$ be a Lagrangian entirely contained and H-exact in $\mathcal{W}_r(L)$.

We want to prove that there exists $r' > 0$, such that L' is exact in $\mathcal{W}_{r'}(L)$ whenever one of the following conditions hold:

(a) $L' \in \mathcal{L}\mathrm{Ham}(L)$, or

(b) the map $H_1(i) \otimes \mathbb{R}$ induced by the inclusion $i : L' \hookrightarrow M$ vanishes.

To do so, we first claim that under any of these assumptions, the rationality constant of L' in $\mathcal{W}_r(L)$, seen as a subset of T^*L, is a fraction of that of L' in M.

Lemma 40 *Let L and L' be as above, and denote by $\Psi : D_r^*L \to \mathcal{W}_r(L)$ a Weinstein neighbourhood of L. There exists an integer $k = k(M, L)$ such that $\lambda_0(H_1(\Psi^{-1}(L'))) \subseteq \frac{\tau}{k}\mathbb{Z}$.*

This lemma, whose proof we postpone to § 5.1.2 below, directly shows that, when $\tau = 0$, H-exactness yields exactness.

When $\tau > 0$, we conclude by using the following additional estimate.

Proposition 41 *Let $L \hookrightarrow (D^*_r L, d\lambda_0)$ be a Lagrangian embedding whose image L' is H-exact. We have that*

$$\forall \beta' \in H_1(L'), \qquad |\lambda_0(\beta')| \leq r\ell_g^{\min}(\pi_*\beta')$$

where $\ell_g^{\min}(\beta)$ denotes the length of the shortest geodesic loop for g in L representing the class β.

Indeed, we choose a basis $\{\beta'_1, \ldots \beta'_m\}$ of $H_1(L')$ and we fix $r' < \frac{\tau}{k\ell}$ where

$$\ell = \max\{\ell_g^{\min}(\pi_*\beta'_i) \,|\, 1 \leq i \leq m\}\,.$$

The proposition above gives that, for all i, $|\lambda_0(\beta'_i)| \leq r'\ell < \frac{\tau}{k}$. Because of Lemma 40, we then get that λ_0 vanishes on $H_1(L')$, which proves the exactness of L'.

It only remains to prove the lemma and proposition above to conclude the proof of Theorem 4.

5.1.1 Proof of Proposition 41

We start with the proposition. First, let us remark that when $L = \mathbb{T}^n$, the estimate follows directly from Eliashberg's result on the shape of subsets of $T^*\mathbb{T}^n$ [Eli91]. With the additional hypothesis that L is also contained in a Weinstein neighbourhood of L', this is a result of Membrez and Opshtein [MO21]. However, as they themselves point out, there should be a proof of this result without their additional constraint using the theory of graph selectors — they even sketch out a proof, which we mostly follow here.

Proof of Proposition 41. In Theorem 6.1 of [PPS03], Paternain, Polterovich, and Siburg show that, for every Lagrangian submanifold $L' \subseteq T^*L$ Lagrangian isotopic to the zero-section and every fiberwise-convex neighbourhood W of L', there is a closed 1-form σ of L such that $\mathrm{graph}(\sigma) \subseteq W$ and $[\sigma] = [\lambda_0|_{L'}]$. However, inspecting the proof of that statement, we see that all that is truly required is the existence of a symplectic isotopy preserving fibres sending L' to an exact Lagrangian submanifold admitting a graph selector — we refer to that paper for the definition of a graph selector. On the one hand, we have shown in Lemma 24 that H-exact Lagrangians in T^*L indeed have associated symplectic isotopies preserving fibres which send them to exact ones. On the other hand, it is now known that every exact Lagrangian submanifold of T^*L admits a graph selector. This was proven using Floer theory by Amorim, Oh, and Dos Santos [AOS18] and using microlocal sheaves by Guillermou [Gui23]. Therefore, the result applies as is in our case.

But it follows from this that

$$\left\{[\iota^*\lambda_0] \,\middle|\, \iota : L \hookrightarrow D^*_r L \text{ is } H\text{-exact}\right\} = \left\{[\sigma] \in H^1(L;\mathbb{R}) \,\middle|\, |\sigma| < r\right\}.$$

In particular, for every H-exact Lagrangian embedding $\iota : L \hookrightarrow D^*_r L$ and every loop $\gamma : S^1 \to L$, we have that

$$|\lambda_0(\iota \circ \gamma)| < r\ell_g(\gamma),$$

where ℓ_g denotes the length in the metric g. By taking the infimum over all loops representing a class $\beta = \pi_* \beta'$, we get the desired inequality. □

5.1.2 Proof of Lemma 40

Recall that L is a H-rational Lagrangian with rationality constant $\tau \geq 0$, that $\Psi : D^*_r L \to \mathcal{W}_r(L)$ is a Weinstein neighbourhood of L in M of size $r > 0$, and that $L' \in \mathcal{L}(\tau)$ is a Lagrangian entirely contained and H-exact in $\mathcal{W}_r(L)$. The lemma states that, under one of the following conditions,

(a) $L' \in \mathcal{L}\mathrm{Ham}(L)$, or

(b) the map $H_1(i) \otimes \mathbb{R}$ induced by the inclusion $i : L' \hookrightarrow M$ vanishes

there exists an integer $k = k(M, L)$ such that $\lambda_0(H_1(\Psi^{-1}(L'))) \subseteq \frac{\tau}{k}\mathbb{Z}$.

For convenience, we denote by $\overline{X}$ the object in T^*L corresponding to X via Ψ^{-1}, e.g. $\Psi^{-1}(L) = \overline{L}$.

Proof of Lemma 40. Fix a representative $\overline{\beta} : S^1 \to \overline{L}$ of a class in $H_1(\overline{L})$. Since $\overline{L'}$ is H-exact in $D^*_r L$, the projection $\overline{L'} \hookrightarrow T^*L \to \overline{L}$ is a homotopy equivalence by Lemma 24. Therefore, there exist a loop $\overline{\beta'}$ in $\overline{L'}$ and a cylinder $\overline{C}$ in $D^*_r L$ such that $\pi_*(\overline{\beta'}) = \overline{\beta}$ and $\partial\overline{C} = \overline{\beta'} \sqcup (-\overline{\beta})$. By Stokes Theorem and exactness of the 0-section $\overline{L}$ in T^*L, we thus have that

$$\omega(C) = d\lambda_0(\overline{C}) = \lambda_0(\overline{\beta'}) - \lambda_0(\overline{\beta}) = \lambda_0(\overline{\beta'})\,.$$

In case (a), take a Hamiltonian isotopy $\{\varphi_t\}_{t\in[0,1]}$ starting at identity and such that $\varphi_1(L) = L'$. Then $C'(s,t) := \varphi_t^{-1}(\beta'(s))$ defines a cylinder in M and $C'' := C \cup_{\beta'} C'$ represents a class in $H_2(M, L)$. In particular, $\omega(C) + \omega(C') = \omega(C'') \in \tau\mathbb{Z}$. But note that, since $\{\varphi_t^{-1}\}$ is Hamiltonian,

$$\omega(C') = \mathrm{Flux}(\{\varphi_t^{-1}\})(\beta') = 0\,.$$

Therefore, $\omega(C) = \lambda_0(\overline{\beta'}) \in \tau\mathbb{Z}$, and we can take $k = 1$.

In case (b), note that $H_1(L;\mathbb{R}) \to H_1(M;\mathbb{R})$ being zero is equivalent to $H_1(L) \to H_1(M)$ being finite, since $H_1(\,\cdot\,;\mathbb{R}) = H_1(\,\cdot\,) \otimes \mathbb{R}$. By the long exact sequence of the pair (M, L), this is in turn equivalent to $H_2(M, L) \to H_1(L)$ having finite cokernel, whose size we denote by k. Then, $k\beta$ bounds some $u \in H_2(M, L)$, and we have that

$$k\lambda_0(\overline{\beta'}) = k\omega(C) = \omega(u\#kC) - \omega(u) \in \tau\mathbb{Z},$$

because $u\#kC \in H_2(M, L')$, and L and L' belong to $\mathcal{L}(\tau)$. Therefore, $\lambda_0|_{L'}$ must take values in $\frac{\tau}{k}\mathbb{Z}$. □

5.2 Proof of Theorem 6

We now turn our attention to Theorem 6 on limits of H-rational Lagrangians. As we shall see, the theorem follows pretty directly from the techniques that we developed to prove Theorem 3 and 4.

Proof of Theorem 6. We start with the first part of the statement: if L_i converges to L with L smooth and n-dimensional and $L_i \in \mathcal{L}(\tau_i)$ with $\inf \tau_i > 0$, then L is Lagrangian. This follows pretty directly from Laudenbach and Sikorav's result on displacement of non-Lagrangians [LS94].

Indeed, suppose L is not Lagrangian. Then, $L \times S^1 \subseteq M \times T^*S^1$ is also not Lagrangian and its normal bundle admits a nowhere vanishing section. Therefore, it follows from [LS94] that, for every $\varepsilon > 0$, there is a Hamiltonian diffeomorphism φ of $M \times T^*S^1$ such that $\varphi(L \times S^1) \cap L \times S^1 = \emptyset$ and with Hofer norm $\|\varphi\|_H < \varepsilon$. But then, there is a neighbourhood U of $L \times S^1$ such that $\varphi(U) \cap U = \emptyset$. In particular, for i large enough, $\varphi(L_i \times S^1) \cap (L_i \times S^1) = \emptyset$. Therefore, if $e(L_i \times S^1)$ is the displacement energy of $L_i \times S^1$, we have that

$$\varepsilon \geq \limsup e(L_i \times S^1) \geq \limsup \tau_i \geq \inf \tau_i > 0,$$

where the second inequality follows from Chekanov's estimate on displacement energy [Che98]. We get a contradiction by taking the limit $\varepsilon \to 0$.

The second part — that is, for when we know that the L_i's are H-exact in $\mathcal{W}(L)$ for i large — follows from the proof of Theorem 4. Indeed, the proof of Lemma 40 gives that $\lambda_0|_{L_i}$ takes values in $\tau_i\mathbb{Z}$ on the image of the boundary morphism $\partial_i : H_2(M, L_i) \to H_1(L_i)$. Here, we identify L_i with its preimage in T^*L under a Weinstein neighbourhood of L. Just like in Theorem 4, we can thus use Proposition 41 to conclude that $\lambda_0(\partial_i(H_2(M, L_i))) = 0$ if $L_i \subseteq \mathcal{W}_r(L)$ for r small enough, i.e. for i large enough. Note that r may be taken independently of i since $\inf \tau_i > 0$. But then, this means that, for all $A \in H_2(M, L_i)$,

$$\omega(A) = \omega(A\#C) - \omega(C) = \omega(A\#C) - \lambda_0(\partial C) = \omega(A\#C) \in \omega(H_2(M, L)),$$

where C is the usual (union of) cylinder in T^*L from ∂A to $\pi(\partial A)$ and $\pi : T^*L \to L$ the canonical projection. Therefore, we have that $\omega(H_2(M, L_i)) \subseteq \omega(H_2(M, L))$. But H-exactness implies that $L_i \to T^*L \to L$ is a homotopy equivalence by Lemma 24. Therefore, for every $A \in H_2(M, L)$, ∂A admits a lift $a \in H_1(L_i)$. We can then again create a cylinder C from ∂A to a and run the above argument to get $\omega(H_2(M, L)) \subseteq \omega(H_2(M, L_i))$ for i large. Therefore, we have that

$$\omega(H_2(M, L)) = \omega(H_2(M, L_i)) = \tau_i\mathbb{Z}$$

for large i. This is only possible if τ_i is independent of i for i large. □

5.3 Proof of Proposition 7

We now turn to the proof of Proposition 7, i.e. the partial result one gets instead of Theorem 4 when one does not know that $H_1(L; \mathbb{R}) \to H_1(M; \mathbb{R})$ is zero. In fact, we prove the following stronger statement.

Proposition 42 *Let L be a H-rational Lagrangian submanifold of M with H-rationality constant τ. There is some $r_0 > 0$ and some $C > 0$ with the following property. Assume that $L' \in \mathcal{L}(\tau)$ is a Lagrangian included in a Weinstein neighbourhood $\mathcal{W}_r(L)$ of size $r \in (0, r_0]$ such that L' is H-exact in $\mathcal{W}_r(L)$. Then, there is a symplectic isotopy $\{\psi_t\}_{t\in[0,1]}$ of M with $|\mathrm{Flux}(\{\psi_t(L')\})| \le Cr$ such that $\psi_1(L')$ is exact in $\mathcal{W}_r(L)$.*

By $\mathrm{Flux}(\{L_t\}) \in H^1(L;\mathbb{R})$, we mean the Lagrangian flux of the Lagrangian isotopy $\{L_t\}$; it is defined as follow. Take $F : L \times [0,1] \to M$ such that $F(L,t) = L_t$. Then, $F^*\omega = \alpha_t \wedge dt$ for some time-dependent 1-form α_t on L, and we set $\mathrm{Flux}(\{L_t\})(\gamma) := \int_0^1 \alpha_t(\gamma)dt$ for any loop $\gamma : S^1 \to L$. This is precisely the area swept by γ through the isotopy — in particular, it is independent of the parametrization F of $\{L_t\}$.

Proof. Denote by V the image of the boundary map $H_2(M, L';\mathbb{R}) \to H_1(L';\mathbb{R})$. Pick a complement W of V in $H_1(L';\mathbb{R})$, and take loops $\{\gamma_1, \dots, \gamma_k\}$ which induce a basis of W. Similarly to Section 5.2 above, the proof of Theorem 4 still implies that $\lambda_0|_{L'}(V) = 0$ for r small enough. Therefore, we can take r_0 to ensure this is true for all $r \le r_0$.

We divide our isotopy in two parts. First, we consider the Lagrangian isotopy $F : t \mapsto [(\alpha-1)t+1]\cdot L'$ induced by the multiplication along the fibers of T^*L, where $\alpha \in [0,1]$. A direct computation gives that $F^*\omega = (\alpha-1)\lambda_0|_{L'} \wedge dt$, so that the flux associated to the isotopy is $(\alpha-1)[\lambda_0|_{L'}]$. Note that, by the above paragraph, this cohomology class is in the annihilator V^0 of V, which we can identify with the dual W^* of W in $H^1(L';\mathbb{R}) = \mathrm{Hom}(H_1(L';\mathbb{R}),\mathbb{R})$.

Second, take a closed 1-form σ on L such that $\sigma(V) = 0$ and $\sigma(\pi\circ\gamma_i) = \lambda_0(\gamma_i)$ for all i. It exists, since the projection $L' \to L$ is a homotopy equivalence by Lemma 24. Consider the symplectic isotopy $\{\psi'_t\}$ of T^*L generated by X such that $\iota_X\omega_0 = -\pi^*\sigma$, where $\pi : T^*L \to L$ is the canonical projection. It is easy to check that

(i) $\psi'_1(L')$ is exact in T^*L,

(ii) if $L' \subseteq D^*_rL$, then $\psi'_t(L') \subseteq D^*_{r+|\sigma|}L$ for all $t \in [0,1]$,

(iii) $\mathrm{Flux}(\{\psi'_t(L')\}) = (\iota')^*\mathrm{Flux}(\{\psi'_t\}) = -(\iota')^*\pi^*[\sigma] = -[\lambda_0|_{L'}]$.

We have made here the slight abuse of notation of identifying L' with its preimage in T^*L via the Weinstein neighbourhood. Again, *(iii)* implies that the flux of the isotopy is in W^*.

The Lagrangian isotopy $\{L'_t\}$ from L' to an exact Lagrangian L'' that we are interested in is the (smoothing of the) concatenation of Lagrangian isotopies as above. More precisely, start with $L' \subseteq D^*_rL$ and σ as above. Then, the first half of the isotopy is given by the scaling from L' to $\alpha L'$ for $\alpha = \frac{r}{r+|\sigma|}$. Note that then, $\alpha\sigma$ is a closed 1-form on L having the same properties as above for the Lagrangian $\alpha L'$. We thus get from it a symplectic isotopy $\{\psi'_t\}$ with properties *(i)*–*(iii)* for $\alpha L'$. In particular, $\psi'_t(\alpha L') \subseteq D^*_{\alpha r+|\alpha\sigma|}L = D^*_rL$ and $\mathrm{Flux}(\{\psi'_t(L')\}) = -\alpha[\lambda_0|_{L'}]$. Therefore,

$$\mathrm{Flux}(\{L'_t\}) = (\alpha-1)[\lambda_0|_{L'}] - \alpha[\lambda_0|_{L'}] = -[\lambda_0|_{L'}] \in W^*,$$

where we have made use of the additivity of the flux under concatenation. Furthermore, Proposition 41 then implies that $|\mathrm{Flux}(\{L'_t\})| \leq r \max_i \ell_g^{\min}(\gamma_i)$, and it suffices to take $C := \max_i \ell_g^{\min}(\gamma_i)$.

We now show how $\{L'_t\}$ comes from a symplectic isotopy of M — this is essentially Lemma 6.6 of [Sol13]. Note that in the splitting $H^1(L';\mathbb{R}) = V^* \oplus W^*$, W^* corresponds to the image of the restriction homomorphism $\Psi^* : H^1(M;\mathbb{R}) \to H^1(\mathcal{W}_r(L);\mathbb{R})$ under the restriction isomorphism $H^1(\mathcal{W}_r(L);\mathbb{R}) \to H^1(L';\mathbb{R})$. Here, we make use of the fact that L' is isotopic to an exact Lagrangian of T^*L, so that the inclusion $L' \to \mathcal{W}(L)$ induces an isomorphism on cohomology. In particular, since $[\lambda_0|_{L'}]$ belongs to W^*, there is a closed 1-form θ' of M such that $\theta'|_{L'} = \lambda_0|_{L'} + dF$ for some function $F : L' \to \mathbb{R}$. We then pick an extension $F' : M \to \mathbb{R}$ of F and set $\theta := \theta' - dF'$. Taking $\{\psi_t\}$ generated by θ gives the desired symplectic isotopy in M. □

Corollary 43 *By taking r_0 smaller if necessary, we have the following. If we have that* $\mathrm{Flux}(\{\psi_t(L')\}) \neq 0$, *then L' and $\psi_1(L')$ are in different Hamiltonian isotopy class in M.*

Moreover, if the NLC holds on T^*L, then $L', L'' \in \mathcal{L}(\tau)$ with $L', L'' \subseteq \mathcal{W}_r(L)$, $r \leq r_0$, are Hamiltonian isotopic in M if and only if their associated isotopy to an exact Lagrangian has the same flux.

Proof. Suppose that there is a Hamiltonian isotopy $\{\varphi_t\}$ of M sending L' to $\psi_1(L')$. Then, the concatenation $\{L''_t\}$ of $\{\psi_t(L')\}$ and $\{\varphi_t^{-1}(\psi_1(L'))\}$ is a loop, so that $\mathrm{Flux}(\{L''_t\}) \in H^1(L';\tau\mathbb{Z})$. Indeed, for every loop γ of L', $\mathrm{Flux}(\{L''_t\})(\gamma) \in \tau\mathbb{Z}$, since it is the area of a cylinder with boundary in L'. If we take $r_0 < \frac{\tau}{C}$, then this is only possible if $\mathrm{Flux}(\{L''_t\}) = 0$. Since the flux of a Hamiltonian isotopy is zero, this implies the first result.

If the NLC holds on T^*L, we get an extension $\{\psi_t\}_{t\in[0,2]}$ of $\{\psi_t\}_{t\in[0,1]}$ to a symplectic isotopy with $\psi_2(L') = L$ and same flux. Let $\{\psi'_t\}_{t\in[0,2]}$ be the corresponding isotopy for L''. If L' and L'' are Hamiltonian isotopic, we can construct a loop similarly to above using that Hamiltonian isotopy, $\{\psi_t\}$ and $\{\psi'_t\}$. We then again get that the flux of this loop is zero, so that $\mathrm{Flux}(\{\psi_t(L')\}) = \mathrm{Flux}(\{\psi'_t(L'')\})$. If the fluxes are the same, then extension and concatenation as above give a symplectic isotopy in T^*L from L' to L'' with zero flux. By Proposition 2.3 of [Ono08] or Lemma 6.7 of [Sol13], that isotopy must be Hamiltonian. □

We now give a proper proof of the Lagrangian C^0 flux conjecture, i.e. Corollary 8.

Proof. The closedness of $\mathcal{L}\mathrm{Symp}_0(L)$ follows directly from Proposition 42 together with Theorems 3 and 6. For the closedness of $\mathcal{L}\mathrm{Ham}(L)$, take a sequence $\{L_i\}$ in that space with limit $L_0 \in \mathsf{SMan}(L)$. By Theorem 6, L_0 is a H-rational Lagrangian with same rationality constant as the L_i's — the L_i's respect the hypotheses of Theorem 3, so that they are H-exact in $\mathcal{W}(L_0)$ for i large. Since all the L_i's are Hamiltonian isotopic to each other, their associated symplectic isotopy from Proposition 42 must all have the same flux by Corollary 43. But by that proposition, that flux must tend to 0 as $L_i \to L_0$. Therefore, for i large, there is a symplectic isotopy in T^*L_0 sending L_i to L_0 with zero flux; again, we suppose that the NLC holds here. By Proposition 2.3 of [Ono08] or Lemma 6.7 of [Sol13], that isotopy must be Hamiltonian, and we have closure. □

Remarks 9. *If NLC holds for T^*L, Corollary 43 actually allows us to identify a Hausdorff neighbourhood of L in $\mathcal{L}(\tau)$ with a neighbourhood of $(L,0)$ in $\mathcal{L}\mathrm{Ham}(L) \times W^*$, where we recall that W is a complement of the image of the boundary map $H_2(M,L;\mathbb{R}) \to H_1(L;\mathbb{R})$. We do not know how much this extends to a global homeomorphism.*

A Closed embedded loops satisfy Conjecture A

Let L be a closed embedded loop in a symplectic surface and define

$$\tau := \inf\{\omega(u) > 0 \mid u \in H_2(M,L;\mathbb{Z})\} \in [0,+\infty],$$

where we set $\tau = +\infty$ if $\omega(H_2(M,L;\mathbb{Z})) = 0$.

Remarks 10. *Note that L is H-rational but not H-exact if and only if $\tau \in (0,+\infty)$ and that, in this case, τ as defined above coincides with its H-rationality constant. However, L is H-exact when $\tau = +\infty$, and non-H-rational when $\tau = 0$. In particular, $\tau \in (0,+\infty]$ ensures that $\omega(H_2(M,L;\mathbb{Z}))$ is discrete, and that it is generated by τ whenever τ is finite.*

We fix a metric on L and a Weinstein neighbourhood $\Psi : D_r^*S^1 \to \mathcal{W}_r(L)$ of L. Let $L' = \varphi(L) \subseteq \mathcal{W}_r(L)$ for some Hamiltonian diffeomorphism φ of M.

We start with the case $\tau \in (0,+\infty]$. In this case, we prove that L' is necessarily exact in $\mathcal{W}_r(L)$ for a small enough $r > 0$ thanks to the two steps below. The conclusion then follows from the nearby Lagrangian conjecture.

Step 1:** H**-exactness. Fix $r > 0$ such that $\mathrm{Area}(\mathcal{W}_r(L)) < \tau$, and suppose that L' bounds a surface u whose image is contained in $\mathcal{W}_r(L)$. Since $\mathcal{W}_r(L)$ is a cylinder, u must be a disk. Without loss of generality, the boundary of u is mapped with degree 1 to L'. Since $\mathcal{W}_r(L)$ is exact, we necessarily have that $\omega(u) = |\lambda_0(L')|$, which is precisely the area of the contractible region bounded by L' in $\mathcal{W}_r(L)$. Hence, $\omega(u) \leq \mathrm{Area}(\mathcal{W}_r(L)) < \tau$. Because φ preserves the area, we get that $\omega(u) = 0$ by definition of τ. Hence, L' is H-exact in $\mathcal{W}_r(L)$.

Step 2: exactness. Now, L' being H-exact in $\mathcal{W}_r(L)$ ensures that the projection $L' \to \mathcal{W}_r(L) \to L$ is a homotopy equivalence. Therefore, there is a cylinder C in $\mathcal{W}_r(L)$ such that $\partial C = L \sqcup L'$ and $|\omega(C)| < \mathrm{Area}(\mathcal{W}_r(L))$. But if $\{\varphi_t\}$ is a Hamiltonian isotopy sending L' to L, $C' = \cup_t \varphi_t(L')$ defines a cylinder with boundary $\partial C' = L' \sqcup L$. On the one hand, we have that $\omega(C \cup_{L'} C') \in \tau\mathbb{Z}$, where we make the abuse of notation that $+\infty\mathbb{Z} = 0$. But on the other hand, $\omega(C') = \mathrm{Flux}(\varphi_t)([L']) = 0$, since $\{\varphi_t\}$ is Hamiltonian, and $\omega(C) = \lambda_0(L') - \lambda_0(L) = \lambda_0(L')$. Therefore, we have that $|\lambda_0(L')| < \tau$ and $\lambda_0(L') \in \tau\mathbb{Z}$, i.e. L' is exact in $\mathcal{W}_r(L)$.

For the non-H-rational case $\tau = 0$, note that M is necessarily closed, since $H_2(M;\mathbb{Z}) = 0$ for open M, and thus $\tau > 0$. Likewise, we can suppose that L separates M, since we have $\tau = +\infty$ otherwise. Then, the two regions A and B of $M \setminus L$ generates $H_2(M,L;\mathbb{Z})$. Therefore, the result follows as above by taking $\mathrm{Area}(\mathcal{W}_r(L)) < \min\{\omega(A), \omega(B)\}$.

B The case of Lagrangian 2-tori

B.1 Displaceable Lagrangian 2-tori satisfy Conjecture B

Theorem 44 *Let L be a displaceable rational Lagrangian torus in a 4-dimensional symplectic manifold (M, ω) without boundary. Suppose that L is included in a simply connected Darboux chart U.*

If $L' \in \mathcal{L}\mathrm{Symp}(L)$ is contained in a small enough Weinstein neighbourhood of L, then $L \cap L' \neq \emptyset$. Moreover, if L and L' intersect transversely, then $\#(L \cap L') \geq 4$.

The structure of the proof may be divided into four steps.

(1) We first construct an exact symplectomorphism Ψ from an open V of $\mathbb{C}^2$ to a neighborhood W of L in U sending the standard product torus $S^1(r_1) \times S^1(r_2)$ to L for some $r_1, r_2 > 0$.

(2) Using the hypotheses on U, we show that whenever L' is Hamiltonian isotopic to L and contained in W, we may take $r_1 = r_2 = r$ and the symplectic action class $[\lambda_0]$ of $\Psi^{-1}(L')$ then takes values $\pi r^2 \mathbb{Z}$.

(3) We use a result of Dimitroglou Rizell [Riz21] to conclude that $\Psi^{-1}(L)$ and $\Psi^{-1}(L')$ are Hamiltonian isotopic with an isotopy supported in an appropriate Euclidean ball.

(4) Finally, we make use of the fact that a large monotone product torus is not displaceable in the Euclidean ball to conclude that $\Psi^{-1}(L')$ must intersect $\Psi^{-1}(L) = S^1(r) \times S^1(r)$.

We now begin with the proof of Theorem 44. The first step consists of proving the following lemma.

Lemma 45 *Let L be a Lagrangian torus of a 4-manifold M without boundary, and let U be an open neighbourhood of L such that $\omega|_U = d\lambda$. Take a basis $\{b_1, b_2\}$ of $\pi_1(L) = \mathbb{Z}^2$ and $r_1, r_2 > 0$ such that $\lambda(b_i) = \pi r_i^2$, and consider the product torus*

$$S^1(r_1) \times S^1(r_2) := \left\{ (z_1, z_2) \in \mathbb{C}^2 \,\middle|\, |z_i| = r_i \right\}.$$

There exist open neighborhoods V of $S^1(r_1) \times S^1(r_2)$ in $\mathbb{C}^2$ and W of L in U, a symplectomorphism $\Psi : V \longrightarrow W$ sending $S^1(r_1) \times S^1(r_2)$ to L, and a function $F : V \longrightarrow \mathbb{R}$ such that

$$\Psi^* \lambda = \lambda_0 + dF,$$

where λ_0 is the standard Liouville form of $\mathbb{C}^2$.

Proof. Take Weinstein neighborhoods $\Psi_0 : D^*_{\rho_0}\mathbb{T}^2 \longrightarrow \mathbb{C}^2$ and $\Psi_1 : D^*_{\rho_1}\mathbb{T}^2 \longrightarrow U$ of $S^1(r_1) \times S^1(r_2)$ and L, respectively. We then get a symplectomorphism $\Psi' := \Psi_1 \circ \Psi_0^{-1} : V' \longrightarrow W'$ from some open neighborhood V' of $S^1(r_1) \times S^1(r_2)$ to some open neighborhood W' of L. Recall that every diffeomorphism of $\mathbb{T}^2$ lifts to a symplectomorphism of $T^*\mathbb{T}^2$. Therefore, by precomposing Ψ_1 by such a lift if necessary, we may suppose that Ψ' sends $S^1(r_1) \times \{1\}$ to b_1 and $\{1\} \times S^1(r_2)$ to b_2. It then follows directly that $(\Psi')^*[\lambda] = [\lambda_0]$ in $H^1(\mathbb{T}^2; \mathbb{R})$.

Therefore, there is a function $f : S^1(r_1) \times S^1(r_2) \to \mathbb{R}$ which satisfies $(\Psi')^*\lambda|_{S^1(r_1)\times S^1(r_2)} = \lambda_0|_{S^1(r_1)\times S^1(r_2)} + df$. We now wish to extend f to a function $\tilde{F} : \mathbb{C}^2 \to \mathbb{R}$ so that this equality stands on $TU|_{S^1(r_1)\times S^1(r_2)}$. To do so, take an orthonormal frame (X_1, X_2) of $T(S^1(r_1) \times S^1(r_2))$ which exists because the tangent bundle of the torus is trivial. Then, $(Y_1 = J_0X_1, Y_2 = J_0X_2)$ is an orthonormal frame of the normal bundle of $S^1(r_1) \times S^1(r_2)$ in $\mathbb{C}^2$. We take on a small enough tubular neighborhood of $S^1(r_1) \times S^1(r_2)$

$$\tilde{F}\left(x + \sum_{i=1}^{2} y_i Y_i\right) := f(x) + \sum_{i=1}^{2} y_i \left((\Psi')^*\lambda - \lambda_0\right)(Y_i(x))$$

and extend $\tilde{F}$ to $\mathbb{C}^2$. In is then easy to see that $\tilde{F}$ extends f and has the right differential along $S^1(r_1) \times S^1(r_2)$.

We then conclude using Moser's trick. More precisely, take $\{\varphi_t\}$ to be the flow of the vector field X on V' defined via $\iota_X\omega_0 = \lambda_0 - (\Psi')^*\lambda + d\tilde{F}$. Note that $X = 0$ along $S^1(r_1) \times S^1(r_2)$ by construction, so that φ_t is the identity on that torus for all t and is well defined up to time $t = 1$ on some neighborhood V of it. By making V smaller if necessary, we may suppose that $\varphi_t(V) \subseteq V'$ for all $t \in [0, 1]$. If we set $\alpha_t := t(\Psi')^*\lambda + (1-t)\lambda_0$, we get that

$$\frac{d}{dt}\varphi_t^*\alpha_t = \varphi_t^* d\iota_X\alpha_t + d\tilde{F}.$$

Therefore, integrating from 0 to 1, we get

$$\varphi_1^*(\Psi')^*\lambda - \lambda_0 = d\left(\int_0^1 \varphi_t^*(\iota_X\alpha_t)dt + \tilde{F}\right),$$

i.e. the proposition holds for $\Psi := \Psi' \circ \varphi_1$ and $F := \int_0^1 \varphi_t^*(\iota_X\alpha_t)dt + \tilde{F}$. □

We start with the second step. We begin by showing that if L is rational and $\varphi : U \to M$ is as in Theorem 44, then there exists a basis $\{b_1, b_2\}$ of $\pi(L)$ such that $\lambda(b_1) = \lambda(b_2) > 0$, where $\lambda := \varphi_*\lambda_0$. To do so, first consider the homotopy long exact sequence of the pair $(\varphi(U), L)$:

$$\cdots \quad \pi_2(\varphi(U)) \longrightarrow \pi_2(\varphi(U), L) \xrightarrow{\partial} \pi_1(L) \xrightarrow{\iota} \pi_1(\varphi(U)) \longrightarrow \cdots \tag{1}$$

Since U is simply connected and φ is an embedding, $\pi_1(\varphi(U)) = 0$. Therefore, the boundary operator $\pi_2(\varphi(U), L) \to \pi_1(L)$ is surjective. Since $\omega(u) = \lambda(\partial u)$ for all disks u in $\varphi(U)$ and L is rational, this implies that $\lambda(\pi_1(L)) \subseteq \tau\mathbb{Z}$, where τ is the rationality constant of L. Because every subgroup of a cyclic group is itself cyclic, either L is λ-exact or there exists a positive integer n such that $\lambda(\pi_1(L)) = n\tau\mathbb{Z}$. However, L cannot be exact, otherwise $\varphi^{-1}(L)$ would be an exact Lagrangian torus in $\mathbb{C}^2$, which does not exist [Gro85].

Now fix an identification $\pi_1(L) = \mathbb{Z}^2$, and take $b_1 = (b_{11}, b_{12})$ to be any element such that $\lambda(b_1) = n\tau$. Note that $gcd(b_{11}, b_{12}) = 1$. Otherwise, there would be some integer $k \geq 2$ such that $\frac{1}{k}b_1 \in \mathbb{Z}^2$ and $0 < \lambda(\frac{1}{k}b_1) < n\tau$, which is of course not possible. Therefore, there exists integers m_1 and m_2 such that

$m_1 b_{11} + m_2 b_{12} = 1$. In particular, $\{b_1, (m_1, m_2)\}$ is a basis of $\mathbb{Z}^2$. Let m be such that $\lambda(m_1, m_2) = mn\tau$, and take $b_2 := (m_1, m_2) + (1 - m)b_1$.

We now fix $r > 0$ such that $n\tau = \pi r^2$ and take Ψ given by the basis $\{b_1, b_2\}$ through Lemma 45. We also suppose that the Lagrangian submanifold L' is in W and fix a symplectomorphism ψ of M sending L to L'.

It is now time to show that $[\lambda_0] \in H^1(\Psi^{-1}(L'); \mathbb{R})$ takes value in the cyclic group $\pi r^2\mathbb{Z}$. To see this, note that if b_1, b_2, u_1, and u_2 are as above, then we have that

$$\lambda(\psi_* b_i) = \omega(\psi_* u_i) = \omega(u_i) = \lambda(b_i) = \pi r^2.$$

Here, we have made use of the fact that ψ preserves ω and that ∂ and ψ_* commute, so that $\partial\psi_* u_i = \psi_* b_i$. However, $\psi_* : \pi_1(L) \to \pi_1(L')$ is an isomorphism. Therefore, $\{\psi_* b_1, \psi_* b_2\}$ is a basis of $\pi_1(L')$. In particular, it follows that $[\lambda|_{L'}]$ takes values $\pi r^2\mathbb{Z}$. However, $[\lambda|_{L'}]$ and $[\lambda_0|_{\Psi^{-1}(L')}]$ take the same values by Lemma 45. This completes the second step.

For the third step, we first recall the precise theorem of Dimitroglou Rizell we will need — we only rephrase it for a Euclidean ball of arbitrary radius.

Theorem (Theorem 1.1(1) of [Riz21]) *Let $L' \subseteq B^4(R)$ be a Lagrangian torus inside the open Euclidean ball of radius R whose symplectic action class takes the values $\pi r^2\mathbb{Z}$ on $H^1(L)$, where $R \le \sqrt{3}r$. There exists a Hamiltonian isotopy inside the ball which takes L to the standard monotone product torus $S^1(r) \times S^1(r)$ if and only if it is disjoint from the interior of some symplectic embedding of the closed 4-ball $D^4(\sqrt{\frac{2}{3}}R)$ in $B^4(R)$.*

But note that if we pick $R \in (\sqrt{2}r, \sqrt{3}r)$, then $\Psi^{-1}(L) = S^1(r) \times S^1(r)$ is disjoint from the closed Euclidean ball $D^4(\sqrt{\frac{2}{3}}R)$, because $|z_1|^2 + |z_2|^2 = 2r^2 > \frac{2}{3}R^2$ for all $(z_1, z_2) \in S^1(r) \times S^1(r)$. Furthermore, by making the open W of Lemma 45 — and thus also the open V — smaller if necessary, we can assume that $\Psi^{-1}(L')$ is also disjoint from that closed ball. Therefore, by Rizell's theorem, $\Psi^{-1}(L')$ is Hamiltonian isotopic to $S^1(r) \times S^1(r)$ in $B^4(R)$.

To conclude the proof, just note that $S^1(r) \times S^1(r)$ is not displaceable in $B^4(R)$ if $r \ge \frac{R}{\sqrt{3}}$. This follows from the result of Biran–Entov–Polterovich [BEP04] that $S^1(\frac{1}{\sqrt{3}}) \times S^1(\frac{1}{\sqrt{3}})$ is not displaceable in $B^4(1)$ — and thus *a fortiori* not displaceable in $B^4(s)$ for $s \le 1$ — by rescaling.

Likewise, the estimate on the number of intersection points between L and L' follows from the fact that $\Psi^{-1}(L')$ is Hamiltonian isotopic to $S^1(r) \times S^1(r)$ in $B^4(R)$ and from the computation of the Floer cohomology of $S^1 \times S^1$ in $\mathbb{C}P^2$ by Cho [Cho04].

Remarks 11. *We only have made use of the fact that $U' = \varphi(U)$ is symplectomorphic to an open of $\mathbb{C}^2$ to make sure that L is not exact in U'. In fact, a bit more work allows us to conclude that Theorem 44 holds for any rational Lagrangian tori L in a U' such that*

(*i*) $\omega|_{U'} = d\lambda$;

(*ii*) $L \subseteq U'$ *is not λ-exact*;

(iii) $\pi_1(U')$ *is finite.*

Likewise, if M is itself exact and ψ is Hamiltonian, then the second step follows quite directly. Since it is the only place where we make use of the fact that L is in a nice Darboux chart, it follows that the theorem also holds in this case.

In particular, Conjecture B holds for displaceable rational Lagrangian tori for slightly more general U or in exact symplectic 4-manifolds.

B.2 Non-displaceable Lagrangian 2-tori satisfy Conjecture A

Proposition 46 *Let L be a Lagrangian 2-torus with $[L] \neq 0 \in H_2(M;\mathbb{Z})$. Suppose furthermore that*

(*a*) *either L is nondisplaceable;*

(*b*) *L is H-rational.*

Then, Conjecture A holds for L.

Remarks 12. *It has been proven by Albers [Alb05, Alb10] for $\mathbb{K} = \mathbb{Z}_2$ and by Entov and Polterovich [EP09] for $\mathbb{K} = \mathbb{C}$ that $[L] = 0 \in H_2(M;\mathbb{K})$ when L is monotone and displaceable. Therefore, in a lot of examples — probably all — nondisplaceability follows from $[L] \neq 0$.*

Proof. The result follows Dimitroglou Rizell's version of the nearby Lagrangian conjecture [Riz19]. Indeed, if $L' = \varphi(L) \subseteq \mathcal{W}_r(L)$ for some Hamiltonian diffeomorphism, we then have that $[L'] = \varphi_*[L] \neq 0$. Therefore, that L' must represent a nonzero homology class in D_r^*L. From the above-mentioned version of the nearby Lagrangian conjecture, there is thus a Hamiltonian isotopy supported in D_r^*L from L' to the graph of a 1-form σ.

To conclude in Case *(a)*, note that we could suppose that σ had no zeroes if it were not exact. But then, we would have displaced L from itself, which would be in contradiction with the nondisplaceability hypothesis. Therefore, σ must be exact, and we have a Hamiltonian isotopy supported in $\mathcal{W}(L)$ from L' to L.

In Case *(b)*, note that being isotopic to the graph of a 1-form ensures that L' is H-exact in $\mathcal{W}(L)$. Therefore, it suffices to take r small enough so that Theorem 4 applies. Then, one concludes using the NLC [RGI16, Riz19] ◻

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