1. Cartier's theorem

The goal is to prove Cartier's theorem :

Theorem 1.0.1. *Let k be a field of characteristic 0. Let $G \to \mathrm{Spec}\ k$ be a group scheme which is locally of finite type over $\mathrm{Spec}\ k$. Then G is smooth.*

We first see that the assumption on the characteristic of k is necessary.

Exercise 1.0.2. Let p be a prime number. Prove that the group functor μ_p over $\mathrm{Spec}\ \mathbb{Z}$ which maps a scheme T to the group $\mu_p(T) = \{f \in \mathrm{H}^0(T, \mathscr{O}_T),\ f^p = 1\}$ is representable by a group scheme μ_p. Prove that $\mu_p|_{\mathrm{Spec}\ \mathbb{F}_p}$ is not smooth.

We let $e : \mathrm{Spec}\ k \to G$ be the unit section. Let $\omega_G = e^\star \Omega^1_{G/k}$. This is a finite dimensional k-vector space.

Exercise 1.0.3. By considering the map $G \times G \to G \times G$, $(x, y) \mapsto (xy^{-1}, y)$, prove that $\Omega^1_{G/k} = \omega_G \otimes_k \mathscr{O}_G$.

We will now prove the following proposition:

Proposition 1.0.4. *Let k be a algebraically closed field of characteristic 0. Let A be a k-algebra of finite type. Let $\mathfrak{m}$ be a maximal ideal of A. Assume that $\Omega^1_{A/k} \otimes_A A_\mathfrak{m}$ is a free $A_\mathfrak{m}$-module. Then A is smooth at $\mathfrak{m}$.*

When we say that A is smooth at $\mathfrak{m}$, we mean that there is $f \in A$, $f(\mathfrak{m}) \neq 0$ such that $A[1/f]$ is smooth.

Exercise 1.0.5. Prove that the proposition implies Cartier's theorem.

Show that the assumption on the characteristic is necessary in proposition 1.0.4 :

Exercise 1.0.6. Take $k = \mathbb{F}_p(t)$ and $A = \mathbb{F}_p(t)[X]/X^p - t$. Prove that $\Omega^1_{A/k}$ is a free A-module but that A is not smooth.

Exercise 1.0.7. Prove that $\Omega^1_{A/k} \otimes_A A/\mathfrak{m} = \mathfrak{m}/\mathfrak{m}^2$.

Let $x_1, \cdots, x_n \in \mathfrak{m}$, inducing a basis of $\mathfrak{m}/\mathfrak{m}^2$.

Exercise 1.0.8. Prove that there exists derivation $D_1, \cdots, D_n$ of $A_\mathfrak{m}$ with the property that $D_i(x_j) = 0 \mod \mathfrak{m}$ if $i \neq j$ and $D_i(x_i) = 1 \mod \mathfrak{m}$

We let $\hat{A}_\mathfrak{m} = \lim_n A/\mathfrak{m}^n$ be the completion of A at $\mathfrak{m}$.

Exercise 1.0.9. Prove that $D_i(\mathfrak{m}^k) \subseteq \mathfrak{m}^{k-1}$. Deduce that D_i extends by continuity to a derivation of $\hat{A}_\mathfrak{m}$.

Exercise 1.0.10. Prove that the map $\alpha : k[[X_1, \cdots, X_n]] \to \hat{A}_\mathfrak{m}$, sending X_i to x_i is surjective.

We define a map $\beta : \hat{A}_\mathfrak{m} \to k[[X_1, \cdots, X_n]]$, sending $a \in \hat{A}_\mathfrak{m}$ to

$$\sum_{(k_1,\cdots,k_n)\in\mathbb{Z}^n_{\geq 0}} \frac{1}{k_1! \cdots k_n!} D_1^{k_1} D_2^{k_2} \cdots D_n^{k_n}(a) X_1^{k_1} X_2^{k_2} \cdots X_n^{k_n}.$$

Exercise 1.0.11. Prove that β is an algebra homomorphism. Show that $\beta \circ \alpha$ is an automorphism. Deduce that α is an isomorphism.

This shows that $\hat{A}_\mathfrak{m}$ is formally smooth. We indicate one way to deduce smoothness from there, using the concept of regular ring.

Definition 1.0.12. We say that a local noetherian algebra $(A, \mathfrak{m})$ is regular if $\dim(A) = \dim_{k(\mathfrak{m})}(\mathfrak{m}/\mathfrak{m}^2)$.

We admit that a regular ring is a domain. We will prove the following proposition.

Proposition 1.0.13. *Assume that k is algebraically closed. Let A be a k-algebra of finite type. Let $\mathfrak{m}$ be a maximal ideal. Then A is smooth at $\mathfrak{m}$ if and only if $A_{\mathfrak{m}}$ is regular.*

Exercise 1.0.14. Prove the direct implication.

We now prove the reverse implication.

Exercise 1.0.15. Prove that there exists a presentation $k[x_1, \cdots, x_r]/I \to A$ where $\mathfrak{m}' = (x_i, 1 \leq i \leq r)$ maps to $\mathfrak{m}$.

Exercise 1.0.16. Prove that we can find elements $f_1, \cdots, f_l \in I \cap \mathfrak{m}'$ with $f_1, \cdots f_l$ map to a basis of the kernel of $\mathfrak{m}'/(\mathfrak{m}')^2 \to \mathfrak{m}/\mathfrak{m}^2$.

Exercise 1.0.17. Prove by induction on s that $k[x_1, \cdots, x_r]/(f_1, \cdots, f_s)$ is regular at $\mathfrak{m}'/(f_1, \cdots, f_s)$ of dimension $r - s$.

Exercise 1.0.18. Prove that the map $k[x_1, \cdots, x_r]/(f_1, \cdots, f_l) \to A$ induces an isomorphism after localizing at $\mathfrak{m}'/(f_1, \cdots, f_l)$ and $\mathfrak{m}$ (compare dimensions). Deduce that there exists $h \in k[x_1, \cdots, x_r]$ with $h(\mathfrak{m}') \neq 0$ such that $k[x_1, \cdots, x_r]/(f_1, \cdots, f_l)[1/h] \to A[1/h]$ is an isomorphism.

Exercise 1.0.19. Prove that $k[x_1, \cdots, x_r]/(f_1, \cdots, f_l)$ is smooth at $\mathfrak{m}'/(f_1, \cdots, f_l)$ and conclude the proof of proposition 1.0.13.

We will now prove proposition 1.0.4.

Exercise 1.0.20. Conclude that $\hat{A}_{\mathfrak{m}}$ is regular and show that $A_{\mathfrak{m}}$ is regular.

2. Representations of $\mathbb{G}_m$

Let R be a ring. Let $\mathbb{G}_m$ be the multiplicative group, with underlying ring of functions $R[T, T^{-1}]$. The Hopf algebra structure is given by $m^\star(T) = T \otimes T$. We want to prove the following theorem.

Theorem 2.0.1. *The category of representations of $\mathbb{G}_m$ on R-modules is equivalent to the category of $\mathbb{Z}$-graded R-modules.*

A $\mathbb{Z}$-graded R-module M is simply an R-module equipped with a decomposition $M = \oplus_{n \in \mathbb{Z}} M_n$ where M_n is an R-module. Given $m \in M$, we let $m = \sum_n m_n$ where $m_n \in M_n$.

Conversely, a representation of $\mathbb{G}_m$ is the data of an $R[T, T^{-1}]$-comodule.

Exercise 2.0.2. Let $M = \oplus_{n \in \mathbb{Z}} M_n$ be a graded R-module. Define a map $\Delta : M \to M \otimes_R R[T, T^{-1}]$, by $m \mapsto \sum_{n \in \mathbb{Z}} m_n T^n$. Prove that M equipped with Δ is a co-module.

Exercise 2.0.3. Let M be a $R[T, T^{-1}]$-comodule. For any $m \in M$, we have $\Delta(m) = \sum m_n T^n$. Prove that the map $p_n : M \to M$ defined by $m \mapsto m_n$ is a projector $p_n^2 = p_n$ and that $p_n p_m = 0$ if $n \neq m$. Conclude.