1. The Bruhat stratification for GL_3

We let G be the group GL_3 over a field k. We let $\mathbb{P}^2$ be the projective space of dimension 2 over Spec k. Over $\mathbb{P}^2$ we have an invertible sheaf $\mathscr{O}_{\mathbb{P}^2}(1)$ and it comes with a surjective map $\mathscr{O}^3_{\mathbb{P}^2} \to \mathscr{O}_{\mathbb{P}^2}(1)$.

If R is a k algebra, morphisms $f : \mathrm{Spec}\ R \to \mathbb{P}^2$ correspond bijectively to isomorphism classes of rank 1 projective R-module L, together with a surjective map $\Psi : R^3 \to L$. Via this correspondence, $L = \mathrm{H}^0(\mathrm{Spec}\ R, f^\star \mathscr{O}_{\mathbb{P}^2}(1))$ and $R^3 = \mathrm{H}^0(\mathrm{Spec}\ R, f^\star \mathscr{O}^3_{\mathbb{P}^2})$.

We also fix an isomorphism between R^3 and $\mathrm{Hom}(R^3, R)$ (by using the dual basis of the canonical basis).

Exercise 1.0.1. Using duality, show that the data of a rank 1 projective R-module L, together with a surjective map $R^3 \to L$ is equivalent to the data of a rank 2 projective R-module V, together with a surjective map $R^3 \to V$.

We deduce that $\mathbb{P}^2(\mathrm{Spec}\ R)$ is also the set of isomorphism classes of a rank 1 projective R-module M together with an injective map $\psi : M \to R^3$ such that R^3/M is projective of rank 2.

The group G acts on $\mathbb{P}^2$ via the rule : $(g \in \mathrm{G}(R), (M \hookrightarrow R^3)) \mapsto (g^{-1}(M) \hookrightarrow R^3)$. We also let $y_{can} : k \to k^3$ be the k-point given by the inclusion $x \mapsto (x, 0, 0)$.

Exercise 1.0.2. Describe the stablizor P of y_{can} inside G.

Exercise 1.0.3. Prove that the map $G \to \mathbb{P}^2$, $g \mapsto y_{can}.g$ is an epimorphism of fppf sheaves (in fact it is even an epimorphism for the Zariski topology).

Exercise 1.0.4. Deduce that $P\backslash G \to \mathbb{P}^2$ is an isomorphism (as fppf sheaves).

Consider the subfunctor X of $\mathbb{P}^2$ given by the condition that $M \subseteq R^2 \hookrightarrow R^3$ where $R^2 \hookrightarrow R^3$ is the embedding described by $(r_1, r_2) \mapsto (r_1, r_2, 0)$.

Exercise 1.0.5. Prove that X is a closed subscheme of $\mathbb{P}^2$ isomorphic to $\mathbb{P}^1$.

We let B be the upper triangular Borel.

Exercise 1.0.6. Prove that $\{y_{can}\}$, $X \setminus \{y_{can}\}$ and $\mathbb{P}^2 \setminus X$ are the orbits for the action of B on $\mathbb{P}^2$.

Exercise 1.0.7. Prove that $X \setminus \{y_{can}\} \simeq \mathbb{A}^1$ and $\mathbb{P}^2 \setminus X \simeq \mathbb{A}^2$ and describe the closure relations

Let R be a k-algebra. A full flag in R^3 consists of a chain of submodules $0 \subseteq L_1 \subseteq L_2 \subseteq R^3$, where $\psi_1 : L_1 \to R^3$ is an injection with projective cokernel of rank 2 and $\psi_2 : L_2 \to R^3$ is an injection with projective cokernel of rank 1.

We now consider the functor FL on k-algebras, which associates to an algebra R, isomorphism classes of full flags on R^3. The group G acts on FL via $(g \in G(R), 0 \subseteq L_1 \subseteq L_2 \subseteq R^3) \mapsto (0 \subseteq g^{-1}(L_1) \subseteq g^{-1}(L_2) \subseteq R^3)$.

Exercise 1.0.8. Prove that FL identifies naturally with a closed subscheme of $\mathbb{P}^2 \times \mathbb{P}^2$.

We let x_{can} be the k-point of FL corresponding to $ke_1 \subseteq ke_1 \oplus ke_2$ inside k^3, where e_1, e_2, e_3 is the canonical basis.

Exercise 1.0.9. Prove that the map $G \to FL$, $x_{can} \mapsto x_{can}g$ is an epimorphisms of sheaves.

Exercise 1.0.10. Deduce that $B\backslash G \to FL$ is an isomorphism.

For any permutation $\sigma \in \mathcal{S}_3$, we consider the flag $x_{can}\sigma$, given by $ke_{\sigma^{-1}(1)} \subseteq ke_{\sigma^{-1}(1)} \oplus ke_{\sigma^{-1}(2)}$. We let X_σ be the B-orbit of $x_{can}.\sigma$.

We consider the subfunctor Z of FL given by the condition that $L_1 = Re_1$.

Exercise 1.0.11. Prove that $Z \simeq \mathbb{P}^1$ and that $X_{(2,3)} \to Z$ is dense.

We consider the subfunctor Z' of FL given by the condition that $L_2 = Re_1 \oplus Re_2$.

Exercise 1.0.12. Prove that $Z' \simeq \mathbb{P}^1$ and that $X_{(1,2)} \to Z'$ is dense.

We consider the subfunctor Z'' of FL given by the condition that $e_1 \in L_2$.

Exercise 1.0.13. Prove that we have a map $Z'' \to \mathbb{P}^1$, given by $(L_1, L_2) \mapsto (L_2/Re_1 \hookrightarrow Re_2 \oplus Re_3)$. Prove that the fibers of $Z'' \to \mathbb{P}^1$ are also isomorphic to $\mathbb{P}^1$.

Exercise 1.0.14. Prove that we have a map $X_{(1,2,3)} \to Z''$ withe dense image. Prove that $X_{(1,2)} \to Z''$.

We consider the subfunctor Z''' of FL given by the condition that $L_1 \subseteq Re_1 \oplus Re_2$.

Exercise 1.0.15. Prove that we have a map $Z''' \to \mathbb{P}^1$, given by $(L_1, L_2) \mapsto (L_1 \hookrightarrow Re_1 \oplus Re_2$. Prove that the fibers of $Z'' \to \mathbb{P}^1$ are also isomorphic to $\mathbb{P}^1$.

Exercise 1.0.16. Prove that we have a map $X_{(1,3,2)} \to Z"'$ withe dense image. Prove that $X_{(2,3)} \to Z'''$.

Exercise 1.0.17. Can you draw a diagram with vertices the $\{X_\sigma\}_{\sigma \in \mathcal{S}_3}$ and an edge from $X_{\sigma'}$ to X_σ if X_σ lies in the closure of $X_{\sigma'}$?