GEOMETRIC REPRESENTATION THEORY

1. Introduction

In this course we will use techniques from algebraic geometry in order to understand representations of algebraic groups.

Let us consider in this introduction the group $\mathrm{SL}_2 = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix}, ad - bc = 1\}$. We let B be the upper triangular Borel. This group naturally acts on the right on the projective line $\mathbb{P}^1$ by the formula

$$[X, Y].\begin{pmatrix} a & b \\ c & d \end{pmatrix} = [aX + cY, bX + dY].$$

In fact, $\mathbb{P}^1 = B\backslash \mathrm{SL}_2$. The Picard group of $\mathbb{P}^1$ is $\mathbb{Z}$ and for every $n \in \mathbb{Z}$, we can construct a line bundle $\mathscr{O}_{\mathbb{P}^1}(n)$. For example, using the Proj construction, we have that $\mathbb{P}^1 = \mathrm{Proj}\ k[X, Y]$ and $\mathscr{O}_{\mathbb{P}^1}(n)$ is associated to the graded module $\oplus_{k\geq 0} M_k$ where $M_k = k[X, Y]_{n+k}$ is the set of homogeneous polynomials of degree $n+k$. The group SL_2 also acts on these modules and therefore acts equivariantly on the sheaf. When $n \geq 0$, $\mathrm{H}^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(n)) = \oplus_{p+q=n} kX^pY^q$. This is the irreducible $n + 1$-dimensional representation of SL_2. In fact, we have :

Theorem 1.0.1. *For any $n \geq -1$, $\mathrm{H}^1(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(n)) = 0$. For any $n \leq -1$, and $\mathrm{H}^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(n)) = 0$. Moreover, we have a (non-canonical) isomorphism of SL_2-representations : $\mathrm{H}^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(n)) = \mathrm{H}^1(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(2-n))$.*

There is a connection between the sheaves $\mathscr{O}_{\mathbb{P}^1}(n)$, $\mathscr{O}_{\mathbb{P}^1}(2-n)$ and the representation $\mathrm{Sym}^n k^2$. As we have seen, we can obtain the representation by taking cohomology. In fact it is also possible to recover the sheaves from the representation by applying a certain localization functor.

Theorem 1.0.2. *There are two sheaves of "twisted" differential operators $\mathcal{D}_{-n}$ and $\mathcal{D}_{n-2}$ on $\mathbb{P}^1$ (locally, they look like the Weyl algebra $k[X, \partial_X]$ but the gluing data is non trivial) such that*

$$\mathcal{D}_{-n} \otimes_{U(\mathfrak{sl}_2)} \mathrm{Sym}^n k^2 = \mathscr{O}_{\mathbb{P}^1}(n)$$

and

$$\mathcal{D}_{n-2} \otimes^L_{U(\mathfrak{sl}_2)} \mathrm{Sym}^n k^2 = \mathscr{O}_{\mathbb{P}^1}(2-n)[1]$$

We next want to illustrate that having a sheaf on a space, rather than just a representation allows for many interesting constructions. We consider the stratification by B-orbits where $w_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$:

$$\mathbb{P}^1 = B\backslash Bw_0B \coprod B\backslash B = \mathbb{A}^1 \coprod \{\infty\}.$$

Let $\pi : \mathrm{SL}_2 \to \mathbb{P}^1$ be the projection. Here $\mathbb{A}^1 = \pi(w_0U)$ (where U is the unipotent radical in B), and $\{\infty\} = \pi(1)$. In terms of coordinates, $\mathbb{A}^1 = \mathrm{Spec}\ k[Y/X]$, and $\infty = [0, 1]$. We have an exact triangle :

$$(i_\infty)_\star i^!_\infty \mathscr{O}_{\mathbb{P}^1}(n) \to \mathscr{O}_{\mathbb{P}^1}(n) \to j_\star j^\star \mathscr{O}_{\mathbb{P}^1}(n) \stackrel{+1}{\to}$$

for $i_\infty : \{\infty\} \to \mathbb{P}^1 \leftarrow \mathbb{A}^1 : j$. We deduce that the following complex computes the cohomology:

$$\mathcal{C}ous(n) : 0 \to \mathrm{H}^0(\mathbb{A}^1, \mathscr{O}_{\mathbb{P}^1}(n)) \to \mathrm{H}^1_\infty(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(n)) \to 0$$

If $n \geq 0$, we have a short exact sequence $0 \to \mathrm{Sym}^n k^2 \to \mathrm{H}^0(\mathbb{A}^1, \mathscr{O}_{\mathbb{P}^1}(n)) \to \mathrm{H}^1_\infty(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(n)) \to 0$. If $n \leq -2$, we have a short exact sequence $0 \to \mathrm{H}^0(\mathbb{A}^1, \mathscr{O}_{\mathbb{P}^1}(n)) \to \mathrm{H}^1_\infty(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(n)) \to \mathrm{Sym}^n k^2 \to 0$.

We claim that there is an action of $\mathfrak{sl}_2$ on $\mathcal{C}ous(n)$ and the two above exact sequence are the "famous" dual BGG and BGG resolution of $\mathrm{Sym}^n k^2$.

Let us in fact compute everything. We have an isomorphism $\mathrm{H}^0(\mathbb{A}^1, \mathscr{O}_{\mathbb{P}^1}(n)) = X^n k[Y/X]$. One easily computes the action of T. For $t = \mathrm{diag}(t, t^{-1}) \in T$, we have $t.X^n(X/Y)^s = t^{n-2s}X^n(X/Y)^s$. Therefore the weights of T on $\mathrm{H}^0(\mathbb{A}^1, \mathscr{O}_{\mathbb{P}^1}(n))$ are $n, n-2, n-4, \cdots$.

We can also compute $\mathrm{H}^1_\infty(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(n)))$. Let $\bar{U}$ be the opposite unipotent radical. We find that $\bar{U}$ maps isomorphically via π to a neighborhood $(\mathbb{A}^1)'$ of $\{\infty\} \in \mathbb{P}^1$. We have a short exact sequence:

$$0 \to \mathrm{H}^0((\mathbb{A}^1)', \mathscr{O}_{\mathbb{P}^1}(n)) \to \mathrm{H}^0((\mathbb{A}^1)' \setminus \{\infty\}, \mathscr{O}_{\mathbb{P}^1}(n)) \to \mathrm{H}^1_\infty(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(n))) \to 0$$

Moreover, $\mathrm{H}^0((\mathbb{A}^1)', \mathscr{O}_{\mathbb{P}^1}(n)) = Y^n k[X/Y]$ and $\mathrm{H}^0((\mathbb{A}^1)' \setminus \{\infty\}, \mathscr{O}_{\mathbb{P}^1}(n)) = Y^n k[X/Y, Y/X]$ so that $\mathrm{H}^1_\infty(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(n)) = Y^n k[X/Y, Y/X]/k[X/Y]$. The weights of T are $-n-2, -n-4, \cdots$.

We deduce that $\mathcal{C}ous(n)$ is given by the following complex:

$$0 \to X^n k[Y/X] \to Y^n k[X/Y, Y/X]/k[X/Y] \to 0$$

Let us finally examine all the actions we have on this complex. We have an action of B, where $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.Y/X = Y/X + t$.

There is no action of $\bar{U}$ since $\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}.Y/X = \frac{Y/X}{1+tY/X}$. We can however differentiate this action to get an action of $\mathfrak{u}$ where $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = -(Y/X)^2\partial_{Y/X}$.

The goal of this course will be to generalize these constructions from SL_2 to an arbitrary reductive group G and prove versions of theorems 1.0.1 and 1.0.2 in this setting.

2. Recollections on schemes

2.1. Affine Schemes. Let A be a commutative ring. We define $\mathrm{Spec}\ A = \{\text{prime ideals of } A\}$. We equip $\mathrm{Spec}\ A$ with the Zariski topology. A basis of open are the $\{D(f)\}_{f \in A}$ where $D(f) = \mathrm{Spec}\ A[1/f] \hookrightarrow \mathrm{Spec}\ A$.

We construct a sheaf of rings $\mathscr{O}_{\mathrm{Spec}\ A}$ on the topological space $\mathrm{Spec}\ A$ by putting $\mathscr{O}_{\mathrm{Spec}\ A}(D[f]) = A[1/f]$. That this defines a sheaf follows from the following proposition.

Proposition 2.1.1. *Let $f_1, \cdots, f_n \in A$ be such that $(f_1, \cdots, f_n) = A$. Then the following sequence is exact :*

$$0 \to A \to \prod_i A[1/f_i] \to \prod_{i,j} A[1/f_i f_j]$$

where the first map is the diagonal map $a \mapsto (a)_i$ and the second map if $(f_i) \mapsto (f_{i,j})$ where $f_{i,j} = f_i - f_j$.

The pair $(\mathrm{Spec}\ A, \mathscr{O}_{\mathrm{Spec}\ A})$ is an affine scheme. Any ring morphism $f : A \to B$ induces a map of topological spaces $f : \mathrm{Spec}\ B \to \mathrm{Spec}\ A$ and a map of sheaves $\mathscr{O}_{\mathrm{Spec}\ A} \to f_\star \mathscr{O}_{\mathrm{Spec}\ B}$.

2.2. Schemes.

Definition 2.2.1. A locally ringed space $(X, \mathscr{O}_X)$ is a pair consisting of a topological space X and a sheaf of rings $\mathscr{O}_X$ over X with the property that for all $x \in X$, the stalk $\mathscr{O}_{X,x}$ is a local ring. A map $f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ of locally ringed spaces is a map $f : X \to Y$ of topological spaces together with a map of sheaves of rings :

$$f^\star \mathscr{O}_Y \to \mathscr{O}_X$$

such that for all $x \in X$, the map $\mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$ is a local ring map.

Definition 2.2.2. A scheme is a locally ringed space $(X, \mathscr{O}_X)$ which is locally isomorphic to an affine scheme.

Schemes are therefore a full subcategory of the category of locally ringed spaces. Inside the category of schemes, we have the full subcategory of affine schemes.

Proposition 2.2.3. *The category of affine schemes is equivalent to the opposite category of rings via the quasi-inverse functors* $(X, \mathscr{O}_X) \to \mathrm{H}^0(X, \mathscr{O}_X)$ *and* $A \to (\mathrm{Spec}\ A, \mathscr{O}_{\mathrm{Spec}\ A})$*, which are respectively left and right adjoints of the other.*

Remark 2.2.4. *This proposition explains why we insist on working with locally ringed spaces and not just ringed spaces. Let* k *be a field and let* $\mathrm{Spec}\ k[[T]]$ *be the affine scheme. This has a special point* s *and a generic point* η*. Consider the map* $\mathrm{Spec}\ k((T)) \to \mathrm{Spec}\ k[[T]]$ *obtained by sending* $(0) = \mathrm{Spec}\ k((T))$ *to* s*. This induces a map of ringed spaces, but not of locally ringed spaces. The point is that the map* $k[[T]] = \mathscr{O}_{\mathrm{Spec}\ k[[T]],s} \to k((T)) = \mathscr{O}_{\mathrm{Spec}\ k((T)),0}$ *is not a local map. The good map* $\mathrm{Spec}\ k((T)) \to \mathrm{Spec}\ k[[T]]$ *is the one induced by applying* Spec *to the map* $k[[T]] \to k((T))$ *and it sends* (0) *to* η*.*

One often fixes a base scheme S and consider the category of S-schemes Sch/S. This is the category whose objects are given by a scheme X together with a "structural" morphism $X \to S$. Maps $X \to Y$ between two objects of Sch/S is a map of schemes which respects the structural morphisms.

Remark 2.2.5. $Sch = Sch/\mathbb{Z}$.

One is often led to impose finiteness conditions. Here is a brutal list of the most common finiteness conditions:

Finiteness conditions on a scheme :

(1) A scheme is quasi-compact if its underlying topological space is quasi compact.
(2) Quasi separated if the intersection of two quasi-compact subsets is quasi-compact.
(3) Locally noetherian : each point as an open affine neighborhood $\mathrm{Spec}\ R$ with R noetherian.
(4) Noetherian : quasi compact and locally noetherian.

Fineteness conditions on a morphism $f : X \to S$.

(1) quasi-compact : for any quasi compact open $U \hookrightarrow S$, $f^{-1}(U)$ is quasi-compact.
(2) quasi-separated : the diagonal $X \to X \times_S X$ is quasi-compact.
(3) separated : the diagonal is a closed immersion.
(4) locally of finite type : for every point $x \in X$ there are open affine $x \in \mathrm{Spec}\ R \hookrightarrow X$ and $\mathrm{Spec}\ A \hookrightarrow S$ with $f(\mathrm{Spec}\ R) \subseteq \mathrm{Spec}\ A$ and R is a finite type A-algebra.
(5) locally of finite presentation : same as before with R a finite presentation A-algebra.
(6) finite type : locally of finite type + quasi-compact.
(7) finite presentation : locally of finite presentation + quasi-compact + quasi-separated.

2.3. **Sheaves.** In the case of a ring A, we have the abelian category $\mathrm{Mod}(A)$ of A-modules and its full subcategory $Mod_f(A)$ of finite type A-modules. The category $Mod_f(A)$ is abelian if A is Noetherian. To $M \in \mathrm{Mod}(A)$, we can associate a sheaf of $\mathscr{O}_{\mathrm{Spec}\ A}$-modules over $\mathrm{Spec}\ A$, denoted by $\widetilde{M}$ and defined by the rule that $\widetilde{M}(D(f)) = M \otimes_A A[1/f]$. That this defines a sheaf follows from:

Proposition 2.3.1. *Let* $f_1, \cdots, f_n \in A$ *be such that* $(f_1, \cdots, f_n) = A$*. Then the following sequence is exact :*

$$0 \to M \to \prod_i M[1/f_i] \to \prod_{i,j} M[1/f_i f_j]$$

where the first map is the diagonal map $m \mapsto (m)_i$ *and the second map if* $(m_i) \mapsto (m_{i,j})$ *where* $m_{i,j} = m_i - m_j$*.*

Definition 2.3.2. Let X be a scheme and let $\mathscr{F}$ be a sheaf of $\mathscr{O}_X$-modules. The sheaf $\mathscr{F}$ is quasi-coherent if there is a covering $X = \cup \text{Spec } A_i$ and A_i-modules M_i such that $\mathscr{F}|_{\text{Spec } A_i} = \widetilde{M_i}$. The sheaf is called coherent if theres is a covering as before such that the modules M_i are finite A_i-modules.

We denote by $QCoh(X)$ the category of quasi-coherent sheaves on a scheme X and $Coh(X)$ the category of coherent sheaves on X. This category $QCoh(X)$ is abelian. The category $Coh(X)$ is also abelian if X is locally Noetherian.

Remark 2.3.3. *One finds in the literature several definitions of coherent sheaves on general schemes, which all agree in the locally Noetherian case. We have chosen the simplest one.*

Proposition 2.3.4. *Let* Spec A *be an affine scheme. The category* $QCoh(\text{Spec } A)$ *is equivalent to the category* $Mod(A)$*, and the category* $Coh(\text{Spec } A)$ *is equivalent to the category* $Mod_f(A)$ *of finite* A*-modules via the quasi-inverse functors :* $\mathscr{F} \to \mathrm{H}^0(\text{Spec } A, \mathscr{F})$ *and* $M \to \widetilde{M}$.

2.4. **Functor of points.** To any scheme X we attach a functor of points :

$$\begin{array}{rcl} X(-) : Sch^{opp} & \to & SETS \\ T & \mapsto & X(T) \end{array}$$

Lemma 2.4.1 (Yoneda)**.** *The functor* $Sch \to Func(Sch^{opp}, SETS)$ *is fully faithful.*

Definition 2.4.2. A functor $F : Sch^{opp} \to SETS$ is representable if it is in the essential image of the Yoneda functor.

2.5. **Fibre products.** [Reference, [Har77], II, thm. 3.3] Let X, Y, S be schemes and $f : X \to S$, $g : Y \to S$ be maps. Then there is a scheme $X \times_S Y$ called the fibre product of X and Y over S. It fits in a commutative diagram :

$$\begin{array}{ccc} X \times_S Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & S \end{array}$$

and satisfies the following universal property:

$$\mathrm{Hom}(-, X \times_S Y) = \mathrm{Hom}(-, X) \times_{\mathrm{Hom}(-,S)} \mathrm{Hom}(-, Y).$$

In the affine case $X = \text{Spec } A, Y = \text{Spec } B, S = \text{Spec } R$ then $X \times_S Y = \text{Spec } (A \otimes_R B)$ which is in particular affine. The general case is obtained by gluing.

2.6. **Sites.**

Definition 2.6.1. A site is a category $\mathcal{C}$ and a collection $Cov(\mathcal{C})$ of families of morphisms with fixed target (called coverings) satisfying the following axioms :

(1) An isomorphism $\phi : V \to U$ is a covering,
(2) If $\{\phi_i : U_i \to U\}_I$ is a covering, and $\{\phi_{i,j} : U_{i,j} \to U_i\}_j$ is a covering then $\{\phi_i \circ \phi_{i,j} : U_{i,j} \to U\}_{i,j}$ is a covering.
(3) If $\{Ui \to U\}_{i \in I}$ is a covering and $V \to U$ is a morphism in $\mathcal{C}$, then $\forall i$ the fiber product $U_i \times_U V$ exists in $\mathcal{C}$, and $\{U_i \times_U V \to V\}_{i \in I}$ is a covering.

Definition 2.6.2. A presheaf F on a site $\mathcal{C}$ is a functor $\mathcal{C}^{op} \to SET$. A presheaf F is a sheaf if for any covering $\{\phi_i : U_i \to U\}_{i \in I}$, the diagram:

$$F(U) \to \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_U U_j)$$

is exact. If the morphism $F(U) \to \prod_i F(U_i)$ is simply injective, the presheaf is said to be separated. A morphism of presheaves is simply a natural transformation of functors. Define $\mathrm{Sh}(\mathcal{C})$ to be the full subcategory of $Func(\mathcal{C}^{op}, SET)$ whose objects are sheaves.

Before giving an example of site in the theory of schemes we mention a few examples:

Example 2.6.3.
1. Let X be a topological space. Let $Op(X)$ be the category of open subsets of X, ordered by inclusion. Coverings are jointly surjective maps. A sheaf on $Op(X)$ is a sheaf in the usual sense, *ie* a topological sheaf.
2. Let $SETS$ be the category of sets. We turn it into a site by declaring that the coverings are the jointly surjective maps.
3. Let Top be the category of topological spaces. Coverings are open coverings.
4. Let $CompTop$ be the category of compact Hausdorff topological spaces. Coverings are finite collections of maps, jointly surjective. A sheaf on $CompTop$ for this topology is called a "condensed set".

2.7. **The** $fppf$ **topology.** Recall that and R-module M is flat if the functor on $Mod(R) : M \otimes_R -$ is exact.

Definition 2.7.1. A morphism $f : X \to S$ is flat if for all $x \in X$, $\mathscr{O}_{X,x}$ is flat over $\mathscr{O}_{S,f(x)}$.

Proposition 2.7.2. *A morphism of affine schemes $X = \mathrm{Spec}\ A \to S = \mathrm{Spec}\, R$ is flat if and only if A is R-flat.*

Proof. If $R \to A$ is flat then for all $x \in \mathrm{Spec}\ A$ mapping to $y \in \mathrm{Spec}\ R$ and any R_y-module M we have that $A_x \otimes_{R_y} M = A_x \otimes_A A \otimes_R M$. Thus $A_x \otimes_{R_y} -$ is exact. Conversely, assume that A_x is R_y-flat for all x. Let $0 \to I \to R$ be an inclusion. Let $0 \to K \to I \otimes_R A \to A$. We see that for all $x \in \mathrm{Spec}\ A$, $K_x = 0$ thus $K = 0$. □

Definition 2.7.3. A family of morphisms $\{\phi_i : U_i \to X\}_{i \in I}$ is an $fppf$ covering if each ϕ_i is flat and locally of finite presentation and $X = \cup_i \phi_i(U_i)$.

Proposition 2.7.4. *Sch_{fppf} is a site.*

Proof. This follows from the fact that a composition of flat morphisms is flat and that the base change of a flat morphism is flat. □

Theorem 2.7.5. *Let X be a scheme. The functor of points $X(-)$ is an fppf sheaf.*

2.8. **Differentials and smoothness.**

2.8.1. *The module of differentials.* Let R be a ring and let A be an R-algebra. For any A-module M an R-derivation from A to M is an R-linear map $D : A \to M$ such that $D(ab) = aD(b) + bD(a)$ for all $(a, b) \in A^2$. There is a universal A-module Ω^1_A/R equipped with a derivation $d : A \to \Omega^1_{A/R}$ for which $\mathrm{Der}_R(A, M) = \mathrm{Hom}_A(\Omega^1_{A/R}, M)$ for any A-module M. There is a construction by generators an relations

$$\Omega^1_{A/R} = \oplus_{a \in A} A da / \langle d(ra) = r da\ \forall (r, a) \in R \times A,\ d(ab) = adb + bda,\ \forall (a, b) \in A \times A \rangle.$$

Here is a second construction. We can also consider the exact sequence $0 \to I \to A \otimes_R A \to A \to 0$ and we let $\Omega^1_{A/R} = I/I^2$, and let $d : A \to I/I^2$ be $d(f) = 1 \otimes f - f \otimes 1$. To see that $d : A \to I/I^2$ is universal, let M be an A-module and let $D : A \to M$ be a derivation. Consider $1 \otimes D : A \otimes_R A \to M$ be the linearization. One checks that $1 \otimes D(I^2) = 0$ and we can consider the A-linear map $1 \otimes D : I/I^2 \to M$. We recover D as the composition $A \to I/I^2 \to M$.

2.8.2. *Two exact sequences.*

Lemma 2.8.1. *If $A \to B$ is a map of R-algebras, we have an exact sequence :*

$$\Omega^1_{A/R} \otimes_A B \to \Omega^1_{B/R} \to \Omega^1_{B/A} \to 0$$

Proof. It suffices to check that for any B-module M, the sequence :

$$0 \to \mathrm{Der}_A(B, M) \to \mathrm{Der}_R(B, M) \to \mathrm{Der}_R(A, M)$$

is exact. □

Lemma 2.8.2. *If $A \stackrel{\alpha}{\to} B$ is a surjective map with kernel I, we have :*

$$I/I^2 \stackrel{d}{\to} \Omega^1_{A/R} \otimes B \to \Omega^1_{B/R} \to 0$$

If $A \to B$ has a splitting $B \to A$ as algebras, then

$$0 \to I/I^2 \stackrel{d}{\to} \Omega^1_{A/R} \otimes B \to \Omega^1_{B/R} \to 0.$$

Proof. It suffices to check that for any B-module M, the sequence :

$$0 \to \mathrm{Der}_R(B, M) \to \mathrm{Der}_R(A, M) \to \mathrm{Hom}_A(I/I^2, M)$$

is exact. In case we have a splitting, we check that the map is onto. Indeed, we have $A/I^2 = B \oplus I/I^2$. Given $D \in \mathrm{Hom}_A(I/I^2, M)$, we can extend it to a derivation on $B \oplus I/I^2$ by $D(b+i) = D(i)$. □

Example 2.8.3. We have that $\Omega^1_{R[T_1,\cdots,T_n]/R} = \oplus_{i=1}^n R[T_1, \cdots, T_n]dT_i$. Indeed, one checks that the map $\oplus_{i=1}^n R[T_1, \cdots, T_n]dT_i \to \Omega^1_{R[T_1,\cdots,T_n]/R}$ is surjective using the presentation. We have the derivation $\partial_{T_i} : R[T_1, \cdots, T_n] \to R[T_1, \cdots, T_n]$ and they give linear maps : $\partial_{T_i} : \Omega^1_{R[T_1,\cdots,T_n]/R} \to R[T_1, \cdots, T_n]$ with the property that $\partial_{T_i}(dT_j) = \delta_{i,j}$. We deduce that $\{dT_1, \cdots, dT_n\}$ are indeed a basis of the differentials.

Example 2.8.4. Let $A = R[T_1, \cdots, T_n]/(P_1, \cdots, P_r)$. Then $\Omega^1_{A/R} = \oplus_{i=1}^n AdT_i/(dP_1, \cdots, dP_r)$.

2.8.3. *The naive cotangent complex.* Let B be an R-algebra of finite presentation. This means that we have an exact sequence $0 \to I \to A \stackrel{\alpha}{\to} B \to 0$ where A is a polynomial algebra over R and I is a finitely generated ideal. To any such presentation, we can associate the complex : $C(\alpha) : I/I^2 \stackrel{d}{\to} \Omega^1_{A/R} \otimes B$.

Lemma 2.8.5. *For any two presentations α, α', the complexes $C(\alpha)$ and $C(\alpha')$ are homotopic.*

Proof. We first prove that if we have a map of presentations :

$$\begin{array}{ccccccc} 0 & \longrightarrow & I & \longrightarrow & A & \stackrel{\alpha}{\longrightarrow} & B \\ \downarrow & & \downarrow & & \downarrow \lambda & & \downarrow Id \\ 0 & \longrightarrow & I' & \longrightarrow & A' & \stackrel{\alpha'}{\longrightarrow} & B \end{array}$$

we get a map $\lambda : C(\alpha) \to C(\alpha')$.

Second we show that if λ and λ' are two maps of presentation, λ and λ' are homotopic from $C(\alpha)$ to $C(\alpha')$. The homotopy is provided by the map $\lambda - \lambda' : A \to I'/(I')^2$ which is a derivation.

Third, we show that given any two presentations, there is a map between them. It follows that we have maps $C(\alpha) \to C(\alpha')$ and $C(\alpha') \to C(\alpha)$ and both compositions are homotopic to the identity. □

Definition 2.8.6. A ring morphism $R \to B$ is smooth if it is of finite presentation and for any presentation α, the complex $C(\alpha) : I/I^2 \xrightarrow{d} \Omega^1_{A/R} \otimes B$ is injective with projective cokernel. A ring morphism $R \to B$ is étale if it is smooth and the Naive cotangent complex is quasi-isomorphic to 0.

Proposition 2.8.7. *(1) Let $R \to B$ and $B \to B'$ be smooth (resp. étale) morphisms. Then $R \to B'$ is smooth (resp. étale).*
(2) Let $R \to B$ and $R \to B'$ be smooth (resp. étale) morphisms. Then $R \to B \otimes_R B'$ is smooth (resp. étale).

Proof. Take a presentation $\alpha : R[T_1, \cdots T_n] \to B$ with kernel I, and a presentation $\beta : R[T_1, \cdots, T_n, X_1, \cdots, X_r] \to B'$ with kernel J inducing a presentation $\gamma : B[X_1, \cdots, X_r] \to B'$ with kernel K.

We get a commutative diagram :

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \oplus_i B' dT_i & \longrightarrow & \oplus_i B' dT_i \oplus \oplus_j B' dX_j & \longrightarrow & \oplus_j B' dX_j & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & I/I^2 \otimes_B B' & \longrightarrow & J/J^2 & \longrightarrow & K/K^2 & \longrightarrow & 0
\end{array}$$

From which we deduce that the middle map is injective with projective cokernel. The second point is left to the reader. □

2.8.4. *Standard smooth morphisms.*

Definition 2.8.8. An R algebra A is called standard smooth if it has a presentation $R[T_1, \cdots, T_n]/(f_1, \cdots, f_c)$ where the Jacobian matrix $(\partial_{T_i} f_j)_{1 \leq i,j \leq c}$ is invertible in A.

Lemma 2.8.9. *A standard smooth R-algebra A is smooth.*

Proof. Indeed, we observe that $df_1, \cdots, df_c, dT_{c+1}, \cdots, dT_n$ is a basis of $\Omega^1_{R[T_1, \cdots, T_n]} \otimes_R A$. □

Lemma 2.8.10. *Let A be a standard smooth algebra. There is an étale map $R[X_1, \cdots, X_t] \to A$.*

Proof. We consider a standard presentation : $R[T_1, \cdots, T_n]/(f_1, \cdots, f_c)$. We just take $X_1 = T_{c+1}, \cdots, X_t = T_n$. □

Lemma 2.8.11. *A smooth R-algebra A admits a Zariski cover $\operatorname{Spec} A = \cup_i \operatorname{Spec} A[1/f_i]$ where $A[1/f_i]$ is a standard smooth R-algebra.*

Proof. Let A be a smooth R-algebra. We take a presentation $0 \to I \to R[T_1, \cdots, T_n] \to A \to 0$. For any $f \in A$, with lift $\widetilde{f}$, $0 \to (I, T_{n+1}\widetilde{f} - 1) \to R[T_1, \cdots, T_n, T_{n+1}] \to A[1/f] \to 0$ is a presentation. We also observe that $(I, T_{n+1}\widetilde{f} - 1)/(I, T_{n+1}\widetilde{f} - 1) = I/I^2 \otimes_A A[1/f] \oplus A[1/f](f dT_{n+1} - df T_{n+1})$. Let $x \in \operatorname{Spec} \neg\dagger A$. Since I/I^2 is projective, there exists $f \in A$ such that $I/I^2 \otimes_A A[1/f]$ is free and $f(x) \neq 0$. We can replace A by $A[1/f]$ and assume I/I^2 is free. It has a basis $(f_1, \cdots, f_c)$. Pick a lift $(h_1, \cdots, h_c)$ in I. By Nakayama, $h \in I$ such that $(1 + h)(h_1, \cdots, h_c) \subseteq I$. We see that $0 \to (h_1, \cdots, h_c, T_{n+1}(1 + h) - 1) \to R[T_1, \cdots, T_{n+1}] \to A \to 0$ is a presentation. Thus we can assume that I is generated by $(f_1, \cdots, f_c)$ which map to a basis of I/I^2. We see that one of the minors of size c of $(\partial_{T_i} f_j)_{1 \leq j \leq c, 1 \leq i \leq n}$ is non-zero. Making one more localization, and possibly reordering, we can assume that $(\partial_{T_i} f_j)_{1 \leq i,j \leq c}$ is invertible in A. □

2.9. **Smoothness and flatness.**

Proposition 2.9.1. *A smooth morphism $R \to B$ is flat.*

Proof. See [Sta13] TAG 00TA. Note that syntomic morphisms are flat by definition. □

Proposition 2.9.2. *(1) Let R be a field. A morphism $R \to B$ is étale if and ony if B is a product of finitely many finite separable field extensions of R.*
(2) Let R be a ring. A morphism $R \to B$ is étale if and only if it is of finite presentation, flat, and for all prime ideal $\mathfrak{p}$ in R, $k(\mathfrak{p}) \to B \otimes_R k(\mathfrak{p})$ is étale.

Proof. First, assume that R is a field and $B = R[x]/P(x)$ with $(P(x), P'(x)) = 1$. Then $R \to B$ is étale (the naive cotangent complex is given by $B \overset{P'(x)}{\to} B$). In the other direction, we may assume that R is algebraically closed. Then one needs to see that if $R \to B$ is étale, then B is finite over R and reduced. See [Sta13] TAG 00U3. For the second point, see [Sta13] TAG 00U6. ☐

2.9.1. *Smooth morphism.* If $X \to S$ is a map of schemes, we let $\Omega^1_{X/S}$ be the quasi-coherent sheaf over X of relative differentials. If X/S is locally of finite type, this sheaf is coherent. One possible definition is to consider the locally closed immersion $\Delta : X \to X \times_S X$, factor it as the composite of a closed immersion, with ideal $\mathscr{I}$ and open immersion $X \hookrightarrow W \hookrightarrow X \times_S X$ and to let $\Omega^1_{X/S} = \Delta^\star \mathscr{I}/\mathscr{I}^2$. We can also check that for $R \to A$ and $f \in A$, $\Omega^1_{A/R} \otimes_A A_f = \Omega^1_{A_f/R}$, so that the construction of $\Omega^1_{A/R}$ is compatible with Zariski localization.

Definition 2.9.3. A morphism $f : X \to S$ is smooth at $x \in X$ is x has an affine neighboorhood $\operatorname{Spec} B$ over an open $\operatorname{Spec}\ R$ of S containing $f(x)$ and $R \to B$ is a smooth map of rings.

Definition 2.9.4. A morphism is smooth if it is smooth at all points.

The rank of $\Omega^1_{X/S}$ is called the relative dimension of f.

Definition 2.9.5. A morphism is étale if it is smooth of relative dimension zero.

Proposition 2.9.6. *A morphism $f : X \to S$ is étale if*
(1) it is locally of finite presentation,
(2) it is flat,
(3) for all $s \in S$, the fiber X_s is a disjoint union of spectra of finite separable extension of $k(s)$.

2.10. **The Tangent sheaf and the Zariski tangent space.** Assume that $X \to S$ is locally of finite type.

Definition 2.10.1. The tangent sheaf is $T_{X/S} = \underline{Hom}(\Omega^1_{X/S}, \mathscr{O}_{X/S})$.

Proposition 2.10.2. *Assume that $S = \operatorname{Spec}\ R$. Then $\mathrm{H}^0(X, T_{X/S})$ identifies with the group of automorphisms of $X \times_{\operatorname{Spec}\ R} \operatorname{Spec}\ R[\varepsilon]$ which induce the identity on X.*

Proof. Let ϕ be such automorphism. Since $X \times_{\operatorname{Spec}\ R} \operatorname{Spec}\ R[\varepsilon]$ and X have the same open subsets, ϕ will preserve any affine cover. We can assume that X is affine, say $X = \operatorname{Spec}\ A$. We therefore have $\phi : A[\varepsilon] \to A[\varepsilon]$. We have $\phi(a) = a + \varepsilon D(a)$. We check that D is in $\operatorname{Der}_R(A, A) = \operatorname{Hom}_A(\Omega^1_{A/R}, A)$. ☐

Assume that $S = \operatorname{Spec} R$. Let $X \to S$ be a scheme and let $x : S \to X$ be an S-point. Let I_x be the ideal sheaf of the immersion. We let $V(I_x^2)$ be the first neighborhood of x. We remark that I_x/I_x^2 is supported on S and corresponds to an R-module still denoted by I_x/I_x^2. It follows that $V(I_x^2) = \operatorname{Spec}(R \oplus I_x/I_x^2)$.

Proposition 2.10.3. *Consider the map $r : X(R[\varepsilon]) \to X(R)$ induced by the map $R[\varepsilon] \to R$, $a + \varepsilon b \mapsto a$. Then $r^{-1}(x) = \operatorname{Hom}_R(I_x/I_x^2, R)$.*

Proof. An element of $r^{-1}(x)$ corresponds to a morphism $R \oplus I_x/I_x^2 \to R[\varepsilon]$. ☐

We call $\operatorname{Hom}(I_x/I_x^2, R)$ the Zariski tangent space at x. We can spell out the connection with $T_{X/S}$.

Lemma 2.10.4. *There is a canonical map $x^\star T_{X/S} \to \mathrm{Hom}_R(I_x/I_x^2, R)$. If X/S is smooth, this map is an isomorphism.*

Proof. Let us assume $X = \mathrm{Spec}\, A$ is affine. We have $x^\star T_{X/S} = \mathrm{Hom}_A(\Omega^1_{A/R}, A) \otimes_A R \to \mathrm{Hom}_A(\Omega^1_{A/R}, R) = \mathrm{Hom}_R(R \otimes_A \Omega^1_{A/R}, R) = \mathrm{Hom}_R(I_x/I_x^2, R)$. In the smooth case, the first map is an isomorphism. □

2.11. **Differential operators.** Let $X \to S$ be a map of schemes. Let $\mathcal{P}_{X/S} = \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_S} \mathscr{O}_X$ and $\mathcal{P}^n_{X/S} = \mathcal{P}_{X/S}/I^{n+1}$ where I is the kernel of the map $\mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_S} \mathscr{O}_X \to \mathscr{O}_X$. This quasi-coherent sheaf has two structures of $\mathscr{O}_X$-modules given by left and right multiplication.

Lemma 2.11.1. *The map $X \times_S X \times_S X \to X \times_S X$, $(x, y, z) \mapsto (x, y)$ induces a map $\delta : \mathcal{P}^{n+m}_{X/S} \to \mathcal{P}^n_{X/S} \otimes_{\mathscr{O}_X} \mathcal{P}^m_{X/S}$, given by $a \otimes b \mapsto a \otimes 1 \otimes 1 \otimes b$.*

Proof. We have a map $\delta : \mathcal{P}_{X/S} \to \mathcal{P}_{X/S} \otimes_{\mathscr{O}_X} \mathcal{P}_{X/S}$, given by $a \otimes b \mapsto a \otimes 1 \otimes 1 \otimes b$. This map sends the ideal I to $I \otimes \mathcal{P}_{X/S} + \mathcal{P}_{X/S} \otimes I$. It sends I^{m+n+1} to $\sum_{a+b=m+n+1} (I \otimes \mathcal{P}_{X/S})^a (I \otimes \mathcal{P}_{X/S})^b = I^a \otimes I^b$. We see that $I^a \otimes I^b \subseteq I^{n+1} \otimes \mathcal{P}_{X/S} + \mathcal{P}_{X/S} \otimes I^{m+1}$. □

Remark 2.11.2. *Geometrically, the lemma says that if (x, y) are closed to order m and (y, z) are closed to order n, the (x, z) are closed to order $m + n$.*

We want to understand the structure of $\mathcal{P}^m_{X/S}$.

Definition 2.11.3.
1. Let A be a ring. Let $f_1, \cdots, f_c$ be elements of A. We say that the elements $f_1, \cdots, f_c$ define a regular sequence if f_i is not a zero divisor in $A/(f_1, \cdots, f_{i-1})$ for all $0 \le i \le c$.
2. Let X be a scheme. An ideal $I \subseteq \mathscr{O}_X$ is called regular, if for any $x \in X$, there is an open affine U and elements $f_1, \cdots, f_c$ which generate $I(U)$ and form a regular sequence in $\mathscr{O}_X(U)$.
3. An immersion of schemes $Z \to X$ is called regular if there exists an open $U \subseteq X$ such that Z can be defined by a regular ideal I in U.

Lemma 2.11.4 ([Sta13], Tag 00LN)**.** *Let A be a ring and $f_1, \cdots, f_c$ be a regular sequence defining an ideal I. The map $A/I[X_1, \cdots, X_c] \to \oplus_{n \ge 0} I^n/I^{n+1}$, sending $\prod X_i^{e_i}$ to $\prod f_i^{e_i} \mod I^{n+1}$ is an isomorphism.*

Proposition 2.11.5. *Assume that X is smooth and that X admits an étale map $X \to \mathbb{A}^n_S$. Let x_i be the coordinates on $\mathbb{A}^n_S$ and $\xi_i = 1 \otimes x_i - x_i \otimes 1 \in \mathcal{P}_{X/S}$. Then $\mathcal{P}^m_{X/S}$ is the free $\mathscr{O}_X$-module with basis the $\prod_i \xi_i^{\alpha_i}$ with $\sum \alpha_i \le m$.*

Proof.
1. The map $X \to X \times_S X$ is a regular immersion (see [Sta13], Tag 067U). This implies that $\oplus_k I^k/I^{k+1} = \mathrm{Sym}(I/I^2)$.
2. The elements $\xi_i \in I$ map to a basis of I/I^2 by our assumption.
3. We prove by induction on m that the map $\oplus_\alpha \mathscr{O}_X \xi^\alpha \to \mathcal{P}^m_{X/S}$ is an isomorphism.

□

Let $\mathcal{E}$ and $\mathcal{F}$ be quasi-coherent sheaves over X. Let $D : \mathcal{E} \to \mathcal{F}$ be an $f^{-1}(\mathscr{O}_S)$-linear operator. One can linearize it by considering $1 \otimes D : \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_S} \mathcal{E} \to \mathcal{F}$. Alternatively, we see that $\mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_S} \mathcal{E} = \mathcal{P}_{X/S} \otimes_{\mathscr{O}_X} \mathcal{E}$.

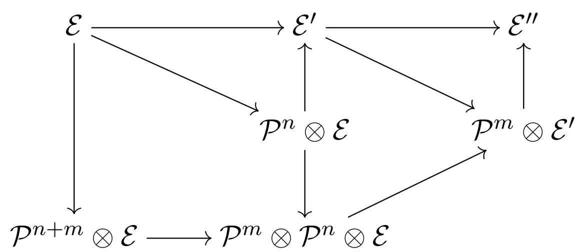
Definition 2.11.6. We say that D is a differential operator of order $\le n$ is we have a factorization:

$$\begin{array}{ccc} \mathcal{E} & \longrightarrow & \mathcal{F} \\ \downarrow & \nearrow & \\ \mathcal{P}^n_{X/S} \otimes \mathcal{E} & & \end{array}$$

Remark 2.11.7. *D is of order 0 if and only if it is a linear map.*

Lemma 2.11.8. *Let $D : \mathcal{E} \to \mathcal{E}'$ and $D' : \mathcal{E}' \to \mathcal{E}''$ be differential operators of order $\leq n$ and $\leq m$. Then $D' \circ D$ is of order $\leq m + n$.*

Proof. We have maps $\delta : \mathcal{P}_{X/S} \to \mathcal{P}_{X/S} \otimes_{\mathscr{O}_X} \mathcal{P}_{X/S}$, given by $a \otimes b \mapsto a \otimes 1 \otimes b$. It induces maps $\mathcal{P}^{n+m}_{X/S} \to \mathcal{P}^m_{X/S} \otimes_{\mathscr{O}_X} \mathcal{P}^m_{X/S}$. We now consider the diagram :



The top maps are $e \mapsto D(e) \mapsto D'D(e)$. The map $\mathcal{P}^n \otimes \mathcal{E} \to \mathcal{E}$ is $a \otimes b \otimes e \mapsto aD(be)$. The map $\mathcal{P}^n \otimes \mathcal{E} \to \mathcal{P}^m \otimes \mathcal{P}^n \otimes \mathcal{E}$ is $a \otimes b \otimes e \mapsto 1 \otimes a \otimes b \otimes e$. The map $\mathcal{P}^{n+m} \otimes \mathcal{E} \to \mathcal{P}^m \otimes \mathcal{P}^n \otimes \mathcal{E}$ is $a' \otimes b \otimes e \mapsto a' \otimes 1 \otimes b \otimes e$. The map $\mathcal{P}^m \otimes \mathcal{P}^n \otimes \mathcal{E} \to \mathcal{P}^m \otimes \mathcal{E}'$ is $a \otimes b \otimes c \otimes e \mapsto a \otimes b \otimes D(ce)$. ☐

Definition 2.11.9. We let $\mathcal{D}_{X/S}$ be the ring of differential operators on $\mathscr{O}_X$.

This is a graded ring with $(\mathcal{D}_{X/S})_n = \mathrm{Hom}_{\mathscr{O}_X}(\mathcal{P}^n_{X/S}, \mathscr{O}_X)$ are the differential operators of order $\leq n$. This is a subsheaf of $\mathrm{End}_{f^{-1}(\mathscr{O}_S)}(\mathscr{O}_X)$.

Lemma 2.11.10. *(1) $(\mathcal{D}_{X/S})_0 = \mathscr{O}_X$*
(2) $(\mathcal{D}_{X/S})_1 = \mathscr{O}_X \oplus T_X$.

Proof. The first point is clear. For the second point, we observe that $\mathcal{P}^1_{X/S} = \Omega^1_{X/S} \oplus \mathscr{O}_X$. ☐

Remark 2.11.11. *We see that a section D of $\mathrm{End}_{f^{-1}(\mathscr{O}_S)}(\mathscr{O}_X)$ is a differential operator of order m if and only if $D((f_1 \otimes 1 - 1 \otimes f_1)....(f_m \otimes 1 - 1 \otimes f_m)) = 0$ for any local sections $(f_i)_{1 \leq i \leq m}$ in $\mathscr{O}_X$. Observe that $D((f_1 \otimes 1 - 1 \otimes f_1)....(f_m \otimes 1 - 1 \otimes f_m)) = [D, f_1](f_2 \otimes 1 - 1 \otimes f_2)....(f_m \otimes 1 - 1 \otimes f_m))$. We deduce that D is of order $\leq m$ if and only if $[D, f]$ is of order $\leq m - 1$ for all $f \in \mathscr{O}_X$.*

Lemma 2.11.12. *If D is of order $\leq n$ and D' is of order $\leq m$, then $[D, D']$ is of order $\leq m+n-1$.*

Proof. We prove the lemma by induction on the order of D' and D, using the above remark. We have $[[D, D'], f] = [[D, f], D'] + [D, [D', f]]$. ☐

We define $\mathrm{gr}(\mathcal{D}_{X/S}) = \oplus (\mathcal{D}_{X/S})_n / (\mathcal{D}_{X/S})_{n-1}$.

Corollary 2.11.13. *We have that $\mathrm{gr}(\mathcal{D}_{X/S})$ is a commutative algebra.*

We now put ourselves in the setting of proposition 2.11.5. We assume that X is smooth, and that X admits an étale map $X \to \mathbb{A}^n_S$. Let x_i be the coordinates on $\mathbb{A}^n_S$ and $\xi_i = 1 \otimes x_i - x_i \otimes 1 \in \mathcal{P}_{X/S}$. In the above situation, for any $q = (q_1, \cdots, q_n)$ we let $\xi^q = \prod_i \xi_i^{q_i}$. If $\sum q_i \leq m$, let D_q be the differential operator of order $\leq m$ which satisfies $D_q(\xi^q) = 1$ and $D_q(\xi^{q'}) = 0$ for $q' \neq q$. The D_q form a basis of $\mathcal{D}_{X/S}$.

Remark 2.11.14. *In particular, for $m = 1$, we see that D_q corresponding to $q = (0, \cdots, 0, 1, 0, \cdots, 0)$ with 1 in i-th position is ∂_{x_i}. Indeed, $\partial_{x_i}(1 \otimes x_i - x_i \otimes 1) = \partial_{x_i}(x_i) - x_i \partial_{x_i}(1) = 1$.*

We now need to understand the composition.

Lemma 2.11.15. *We have $D_q \circ D_{q'} = \frac{(q+q')!}{q!q'!} D_{q+q'}$.*

Proof. We have $\delta(\xi_i) = x_i \otimes 1 \otimes 1 - 1 \otimes 1 \otimes x_i = \xi_i \otimes 1 + 1 \otimes \xi_i$. We deduce that $\delta(\xi^p) = \sum_{i+j=p} \frac{q!}{i!(p-i)!}(\xi^{p-i} \otimes 1)(1 \otimes \xi^i)$. Applying $Id \otimes D_{q'}$ first, we find 0, unless $i = q'$. Then apply D_q, we see that we must have $p - q' = q$ otherwise we get 0. □

Proposition 2.11.16. *Assume that X is smooth over S and that S is a $\mathbb{Q}$-scheme. Then $\mathcal{D}_{X/S}$ is the ring generated by $\mathscr{O}_X$ and $T_{X/S}$, subject to the following relations :*

(1) $f_1.f_2 = f_1 f_2$, $f_i \in \mathscr{O}_X$,
(2) $f.D = fD$, $f \in \mathscr{O}_X$, $D \in T_X$,
(3) $D_1.D_2 - D_2.D_1 = [D_1, D_2]$,
(4) $f.D - D.f = D(f)$.

In particular, $\mathrm{gr}(\mathcal{D}_{X/S}) = \mathrm{Sym}(T_X)$.

Proof. The above description shows that $\mathcal{D}_{X/S} = \oplus_{(i_1,\cdots,i_n)} \mathscr{O}_X \prod_{l=1}^n \partial_{x_l}^{i_l}$ and

$$(\mathcal{D}_{X/S})_k = \oplus_{(i_1,\cdots,i_n),\sum i_l \leq k} \mathscr{O}_X \prod_{l=1}^n \partial_{x_l}^{i_l}.$$

This implies that the map $T_{X/S} \to \mathrm{gr}((\mathcal{D}_{X/S}))$ induces an isomorphism $\mathrm{gr}((\mathcal{D}_{X/S})) = \mathrm{Sym}(T_X)$. Let $\mathcal{A}$ be the algebra generated by $\mathscr{O}_X$ and $T_{X/S}$ as above. We have a map $\mathcal{A} \to \mathcal{D}_{X/S}$. We can turn $\mathcal{A}$ into a graded algebra by declaring that elements of $T_{X/S}$ have degree ≤ 1 and elements of $\mathscr{O}_X$ have degree ≤ 0. Now we have $\mathrm{Sym}(T_X) \to \mathrm{gr}(\mathcal{A}) \to \mathrm{gr}((\mathcal{D}_{X/S}))$ and the composite is an isomorphism, while the first map is onto. We deduce that $\mathrm{gr}(\mathcal{A}) \to \mathrm{gr}((\mathcal{D}_{X/S}))$ is an isomorphism, hence that $\mathcal{A} \to \mathcal{D}_{X/S}$ is an isomorphism. □

We also need to understand the (left) stalks of $\mathcal{D}_{X/S}$.

Lemma 2.11.17. *Assume that $S = \mathrm{Spec}\ k$ where k is a field. Let $x : S \to X$ be a section. We have a map : $x^\star \mathcal{D}_{X/S} \to \mathrm{colim}_n \mathrm{Hom}_k(\mathscr{O}_{X,x}/\mathfrak{m}_x^n, k)$. This map is an isomorphism if $X \to S$ is smooth.*

Proof. We claim that $x^\star \mathcal{P}^m_{X/S} = \mathscr{O}_{X,x}/\mathfrak{m}_x^{m+1}$. We reduce to the affine case, $X = \mathrm{Spec}\ A$. We have $R \otimes_A (A \otimes_R A) = A$. The ideal $R \otimes_A I$ maps to the maximal ideal $\mathfrak{m}_x$. Now we have $x^\star \mathcal{D}_{X/S} = \mathrm{colim}_n \mathrm{Hom}(\mathcal{P}^m_{X/S}, \mathscr{O}_X) \otimes_{\mathscr{O}_X} k \to \mathrm{colim}_n \mathrm{Hom}_k(\mathscr{O}_{X,x}/\mathfrak{m}_x^n, k)$ where the map is an isomorphism in the smooth case. □

Corollary 2.11.18. *Assume that k is of characteristic 0 and X/S is smooth. We have that $x^\star \mathcal{D}_{X/S} = \oplus_{(i_1,\cdots,i_n)} k \prod_{l=1}^n \partial_{x_l}^{i_l}$ where ∂_{x_l} are a basis of the Tangent space at x. The isomorphism takes $\prod_l \partial_{x_l}^{i_l}$ to the map $f \mapsto (\prod_l \partial_{x_l}^{i_l} f)(x)$ where $f \in \mathscr{O}_{X,x}$.*

2.12. **Dimension.** Let X be a scheme. A closed subset Z is called irreducible if it is non-empty and whenever $Z = Z_1 \cup Z_2$ where Z_1 and Z_2 are closed, then $Z_1 = Z$ or $Z_2 = Z$. Equivalently, this means that any non-empty open subset of Z is dense.

Lemma 2.12.1. *Let U be an open affine subset of X. We have a bijection between irreducible closed subsets Z of X such that $Z \cap U \neq \emptyset$ and irreducible closed subsets Z of U. The maps are given by $Z \mapsto Z \cap U$ and $Z \mapsto \overline{Z}$.*

Proof. Let Z be closed in U. We claim that $\overline{Z} \cap U = Z$. Indeed, let $x \in U \setminus Z$. Then we can find a function f in the ideal defining U such that $x \in D(f)$. Thus, $D(f)$ is open in X. And $\overline{Z} \subseteq D(f)^c$, which proves that $x \notin \overline{Z} \cap U$. If Z is irreducible, then we claim that $\overline{Z}$ is irreducible. Otherwise, $\overline{Z} = Z_1 \cup Z_2$. Then $\overline{Z} \cap U = Z = Z_1 \cap U \cup Z_2 \cap U$. So, we can assume that $Z = Z_1 \cap U$ and $\overline{Z} = Z_1$. Let Z be a irreducible subset of X be such that $Z \cap U \neq \emptyset$. We claim that $Z = \overline{Z \cap U}$. Indeed, $Z = \overline{Z \cap U} \cup U^c \cap Z$ and Z is irreducible. We also deduce that $Z \cap U$ is irreducible. Indeed, if $Z_1 \cup Z_2 = Z \cap U$, then $\overline{Z_1} \cup \overline{Z_2} = Z$ so that $\overline{Z}_1 \subseteq \overline{Z}_2$ (or conversely) and therefore $Z_1 \subseteq Z_2$.
□

Lemma 2.12.2. *Any irreducible closed subset of X has a unique generic point.*

Proof. Let Z be an irreducible closed subset. Take U affine open such that $Z \cap U \neq \emptyset$. Let ξ be the generic point of $Z \cap U$. Then the closure of ξ in X is Z.Let ξ and ξ' be two points in X with the same closure. Let U be an affine open containing ξ. If $\xi' \notin U$, we get that $\xi' \in X \setminus U$, a contradiction. So $\xi, \xi' \in U$. But then $\xi = \xi'$ correspond to the same prime ideal. □

We say that X is noetherian if every open subset is quasi-compact.

Lemma 2.12.3. *If X is noetherian, then any closed subset is a finite union of irreducible closed subsets.*

Proof. Let Z be a smallest closed subset which is not a finite union of irreducible closed subset (exists by noetherian assumption). Then Z is not irreducible, hence $Z = Z_1 \cup Z_2$ where Z_i are strictly included in Z. Then Z_i are finite union of irreducible components. Thus Z is a finite union of irreducible closed subsets. □

We let $\dim(X)$ be the maximal length of chain of irreducible closed subsets $Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n$ (this chain has length n) in X. When A is a ring we let $\dim(A) = \dim(\mathrm{Spec}\ A)$.

Lemma 2.12.4. *Let X be a scheme. Then* $\dim(X) = \sup_{x \in X} \dim(\mathscr{O}_{X,x})$.

Proof. Let $\xi_0 \to \xi_1 \to \xi_2 \cdots \to \xi_n$ be a chain of specializations in $\mathscr{O}_{X,x}$. They define a chain of prime ideals in any affine open containing x and thus a chain of irreducible closed subsets in X. Therefore $\dim(\mathscr{O}_{X,x}) \leq \dim(X)$. Conversely let $\xi_1 \to \xi_2 \cdots \to \xi_n$ be a chain of specializations in X. These define a chain of specializations in $\mathscr{O}_{X,x}$ with $x = \xi_n$. □

We let $\mathrm{codim}_X(x) = \dim(\mathscr{O}_{X,x})$.

We now assume that $(A, \mathfrak{m})$ be a noetherian local ring. A system of parameters of A is a sequence of elements $(a_1, \cdots, a_n)$ such that $\sqrt{(a_1, \cdots, a_n)} = \mathfrak{m}$.

Theorem 2.12.5 ([Sta13], Tag 00KQ)**.** *The minimal number of elements defining a system of parameters is the dimension of A.*

Corollary 2.12.6. *Let A be a noetherian scheme, then* $\dim A[X] = \dim A + 1$.

Proof. Let $\mathfrak{p}_1 \subseteq \cdots \subseteq \mathfrak{p}_n$ be a sequence of prime ideals in A. Then $\mathfrak{p}_1 \subseteq \cdots \subseteq \mathfrak{p}_n \subseteq (\mathfrak{p}_n, X)$ is a sequence of lenght $n+1$. Thus, $\dim A[X] \geq \dim A + 1$. Conversely, let $x \in \mathrm{Spec}\ A[X]$ mapping to y. Let $(f_1, \cdots, f_n)$ be a sequence of parameters of y. Observe that $A[X]_x/\mathfrak{m}_y$ is a localization of $k(y)[X]$ at a prime ideal. Thus, it is either a field, if which case $(f_1, \cdots, f_n)$ is a sequence of parameters of x as well, or x defines a closed point corresponding to some irreducible polynomial $P(X) \in k(y)[X]$. In that case, $(f_1, \cdots, f_n, \widetilde{P})$ is a sequence of parameters at x. □

Corollary 2.12.7. *Let k be a field, then* $\dim\ k[X_1, \cdots, X_n] = n$. *More precisely, for any closed point x,* $\dim\ k[X_1, \cdots, X_n]_x = n$.

2.12.1. *Dimensions of k-schemes.* Let $A = k[X_1, \cdots, X_n]/(P_1, \cdots, P_r)$. Let x be a closed point of $\mathrm{Spec}\ A$.

Lemma 2.12.8. *We have* $\dim A_x \geq n - r$.

Proof. If $f_1, \cdots, f_k$ is a system of parameters of x in $\mathrm{Spec}\ A$, then $(\widetilde{f_1}, \cdots, \widetilde{f_k}, P_1, \cdots, P_r)$ is a system of parameters of x in $\mathrm{Spec}\ k[X_1, \cdots, X_n]$. Thus $k + r \geq n$. □

Theorem 2.12.9 ([Sta13], Tag 02JN)**.** *If $P_1, \cdots, P_r$ are a regular sequence in $k[X_1, \cdots, X_n]_x$, then* $\dim(k[X_1, \cdots, X_n]/(P_1, \cdots, P_r))_x = n - r$.

Theorem 2.12.10. *Assume that $X \to \mathrm{Spec}\ k$ is a smooth scheme, then* $\dim(X) = \dim(\mathscr{O}_{X,x})$ *for any closed point x is the rank of $\Omega^1_{X/k}$.*

Proof. Locally, X is standard smooth, of the form $\mathrm{Spec}\ k[X_1, \cdots, X_n]/(P_1, \cdots, P_r)$. Then $P_1, \cdots, P_r$ is a regular sequence by [Sta13] Tag 067U. □

2.12.2. *Constructibility theorem.* Let X be a qcqs scheme. A subset of X is called constructible if it is a finite union of sets of the form $U \cap Z$ where U is a quasi-compact open and Z is a closed subset with quasi-compact complement.

Theorem 2.12.11. *Let $f : X \to Y$ be a morphisms where f is locally of finite presentation, and X, Y are quasi-compact, quasi-separated. Then the image of any constructible set is constructible.*

Corollary 2.12.12. *Assume that X is noetherian. The image $f(X)$ contains an open subset of its closure.*

Proof. This means that $f(X) = \cup_{i\in I} U_i \cap Z_i$ where I is finite, U_i and Z_i are quasi-compact opens and complement of quasi-compact opens respectively. We can also suppose that Z_i is irreducible and U_i dense in Z_i. We claim that $f(X)$ contains an open subset of its closure. Let I' be a minimal set such that $\cup_{i\in I'} Z_i = \cup_{i\in I} Z_i$. Let $U'_i = U_i \setminus \{\cup_{j\neq i \in I'} Z_j\}$. Then U'_i is still dense in Z_i and open in $\cup_i Z_i$. Thus, $\cup_{i\in I'} U'_i = V$ is dense open in the closure of $f(X)$ □

2.12.3. *Generic flatness theorem.* Let us first recall Noether's normalization lemma.

Lemma 2.12.13 ([Sta13], Tag 07NA). *Let $R \to R'$ be an injective map of algebras with R a domain. There exists $f \in R \neq 0$ and an integer d such that we have a factorization $R_f \to R_f[T_1, \cdots, T_d] \hookrightarrow R'_f$ with R'_f finite over $R_f[T_1, \cdots, T_d]$.*

Theorem 2.12.14. *Let S be a noetherian scheme. Let $X \to S$ be a morphism of finite type, with S reduced. There is an open dense $U \subseteq S$ such that $X|_U \to U$ is flat.*

This is a consequence of the following.

Proposition 2.12.15. *Let A be a noetherian integral ring, let B be an A-algebra of finite type. Let M be a finite B-module. There exists $f \in A \setminus \{0\}$ such that M_f is free over A_f.*

Proof. Let $K = \mathrm{Frac}(A)$. The proof is by induction on the dimension of the support of $M \otimes K$. We also note that if we have an exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ and the lemma holds for M_1 and M_3 it holds for M_2. Suppose that $M \otimes K$ is zero. Let $m_1, \cdots, m_n$ be generators of M as a B-module. There exists $f \in A \neq 0$ such that $fm_1 = \cdots = fm_n = 0$. Therefore $M \otimes_A A_f = 0$. In general, we recall that M is a successive extension of modules of the form $B/\mathfrak{p}$ where $\mathfrak{p}$ is a prime ideal. We reduce to the case that $M = B$ is a domain. By Noether normalization, there exists $f \in A$ and $b_1, \cdots, b_n \in B$ such that $A_f \to A_f[b_1, \cdots, b_n] \to B_f$ where $A_f[b_1, \cdots, b_n]$ is a polynomial algebra and $A_f[b_1, \cdots, b_n] \to B$ is finite. We let r be the generic rank of M over $A_f[b_1, \cdots, b_n]$. We have a map $0 \to A_f[b_1, \cdots, b_n]^r \to M \to T \to 0$ and the dimension of the support of $T \otimes K$ is less than n. □

Lemma 2.12.16. *Let A be a Noetherian ring. Let M be a finite type A-module. Then M is a finite successive extension of modules of the shape $A/\mathfrak{p}$ where $\mathfrak{p}$ is a prime ideal.*

Proof. First, we see that M is a finite successive extension of modules of the shape A/I by induction on the number of generators of M. So we reduce to $M = A/I$. Now we consider an ideal I which is maximal among ideals with the property that A/I is not a successive extension of modules of the shape $A/\mathfrak{p}$. By contradiction, assume $I \neq A$. Clearly, I is not prime, so there is a, b with $ab \in I$ but $a, b \notin I$. Replace A/I by A. We have $0 \to A/ann(a) \to A \to A/a \to 0$. But both $A/ann(a)$ and A/a have the property. So does A. □

2.12.4. *Generically smooth.*

Proposition 2.12.17. *Let $X \to \mathrm{Spec}\ k$ be a finite type morphism. Assume that X is geometrically reduced. Then X is generically smooth.*

Proof. We can suppose $X = \text{Spec } A$ with A a domain. By Noether normalization, we have $k \to k[x_1, \cdots, x_n] \to A$ where $k[x_1, \cdots, x_n] \to A$ is injective and finite. Passing to the generic point, we have $A \otimes_{k[x_1,\cdots,x_n]} k(x_1, \cdots, x_n)$ is an étale $k(x_1, \cdots, x_n)$-algebra. Indeed, $A \otimes_{k[x_1,\cdots,x_n]} k(x_1, \cdots, x_n)$ is reduced, so it must be a product of finite field extensions of $k(x_1, \cdots, x_n)$. Being geometricall reduced implies these are étale extension. We deduce that there is a map $k[x_1, \cdots, x_n, y_1, \cdots, y_r]/[P_1, \cdots, P_r] \to A$ which induces an isomorphism over $k(x_1, \cdots, x_n)$. This implies that there is $f \in k[x_1, \cdots, x_n]$ such that $k[x_1, \cdots, x_n, y_1, \cdots, y_r]/[P_1, \cdots, P_r][1/f] \simeq A[1/f]$. We look at the determinant d of the Jacobian $(\partial_{Y_j} P_i)_{1 \leq i,j \leq r}$. We have $d \in k[x_1, \cdots, x_n, y_1, \cdots, y_r]/[P_1, \cdots, P_r]$ and $V(d)$ is nowhere dense (indeed, d is invertible over $k(x_1, \cdots, x_n)$). We deduce that

$$k[x_1, \cdots, x_n][1/f][y_1, \cdots, y_r, y_{r+1}]/(P_1, \cdots, P_r, y_{r+1}d - 1) = A[1/fd]$$

is standard étale over $k[x_1, \cdots, x_n][1/f]$. □

3. Group schemes

3.1. **Group Schemes.** We now work in Sch/S. A group scheme $G \to S$ is a scheme equipped with the following additional structure : $m : G \times_S G \to G$, $e : S \to G$, $\iota : G \to G$, satisfying associativity, neutral element and inverse axioms. Alternatively, a group scheme is a group functor $Fun(\text{Sch}^{op}, Gr)$ that is representable by a scheme G. A morphism of Group schemes is a morphism of schemes, compatible with the group structure. When G is affine we can completely describe this extra structure in ring theoretic terms.

Definition 3.1.1. Let R be a base ring. An Hopf algebra A over R is a commutative ring equipped with a comultiplication $m^\star : A \to A \otimes_R A$, counit $e^\star : A \to R$ and coinverse $\iota^\star : A \to A$ and satisfy the following axioms :

(1) Co-associative

$$\begin{array}{ccc} A & \xrightarrow{m^\star} & A \otimes A \\ \downarrow{\scriptstyle m^\star} & & \downarrow{\scriptstyle 1 \otimes m^\star} \\ A \otimes A & \xrightarrow{m^\star \otimes 1} & A \otimes A \otimes A \end{array}$$

(2) Inverse $A \stackrel{m^\star}{\to} A \otimes A \stackrel{Id \otimes \iota^\star}{\to} A \otimes A \to A$ and $A \stackrel{m^\star}{\to} A \otimes A \stackrel{\iota^\star \otimes Id}{\to} A \otimes A \to A$ are the identity map.

(3) Neutral element $A \stackrel{m^\star}{\to} A \otimes A \stackrel{Id \otimes e^\star}{\to} A$ and $A \stackrel{m^\star}{\to} A \otimes A \stackrel{e^\star \otimes Id}{\to} A$ are the identity map.

Proposition 3.1.2. *Hopf algebras over R are anti-equivalent to afffine group schemes over* Spec R

Example 3.1.3.
(1) We have $\mathbb{G}_a = \text{Spec } R[X]$, with $m^\star(X) = 1 \otimes X + X \otimes 1$.
(2) We have $\mathbb{G}_a = \text{Spec } R[X, X^{-1}]$, with $m^\star(X) = X \otimes X$.
(3) We have $\text{GL}_n = \text{Spec } R[X_{i,j}, 1 \leq i, j \leq n][1/\det]$ with $m^\star(X_{i,j}) = \sum_l X_{i,l} \otimes X_{l,j}$.

3.2. **Action.** Let G be a group scheme and let X be a scheme. A left action of G on X is a morphism : $G \times X \to X$ such that the action is associative and the unit acts trivially. Equivalently, for any S-scheme T, we have an action $G(T) \times X(T) \to X(T)$, functorially in T.

3.3. **Representations.** Let us assume $S = \text{Spec } R$. Let M be an R-module. We can associate to M the functor $\underline{M}$ on R-algebras such that $\underline{M}(A) = A \otimes_R M$. A representation of G on M is a map of functors $G \times \underline{M} \to \underline{M}$ such that for all $A \in Alg/R$, $G(A) \times \underline{M}(A) \to \underline{M}(A)$ defines an A-linear action of $G(A)$ on $\underline{M}(A)$. Let us denote by $GL(\underline{M})$ the group functor which sends an R-algebra B to the group $GL(M \otimes_R B)$ of B-linear automorphisms of $M \otimes_R B$. A representation on M is therefore a group functor map $G \to GL(\underline{M})$. If $M = A^n$ is a finite free module, a representation on M is the same as a group scheme homomorphism $G \to \text{GL}_n$. We let $Mod_G(R)$ be the category of representations of G on R-modules. Assume that $G = \text{Spec } A$ is affine, with A an Hopf algebra.

Definition 3.3.1. A co-module is an R-module M equipped with an A-linear map $\Delta : M \to M \otimes A$ which satisfies the axioms :

(1)

$$\begin{array}{ccc} M & \xrightarrow{\Delta} & M \otimes A \\ \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \Delta \otimes Id} \\ M \otimes A & \xrightarrow{Id \otimes m^\star} & M \otimes A \otimes A \end{array}$$

(2) $M \to M \otimes A \stackrel{Id \otimes e^\star}{\to} M$ is the identity map.

Proposition 3.3.2. *The category $Mod_G(R)$ is equivalent to the category of co-modules.*

Proof. Given a co-module $\Delta : M \to M \otimes A$, and a B and A-algebra, we produce a map $G(B) \to \mathrm{GL}(M \otimes_A B)$ as follows. Let $g \in G(B)$, corresponding to $g : A \to B$. We have an A-linear map $M \to M \otimes_R A \stackrel{Id \otimes g}{\to} M \otimes_R B$ which extends to a B-linear map $\Theta_g : M \otimes_A B \to M \otimes_A B$. The associativity axiom implies that $\Theta_g \circ \Theta_h = \Theta_{gh}$. The following is commutative :

$$\begin{array}{ccccc} M & \xrightarrow{\Delta} & M \otimes A & \xrightarrow{Id \otimes m^\star} & M \otimes A \otimes A \\ \downarrow & & \downarrow{\scriptstyle Id \otimes g} & & \downarrow{\scriptstyle Id \otimes (g \otimes h)} \\ M \otimes_R B & \xrightarrow{\Theta_g} & M \otimes_R B & \xrightarrow{\Theta_h} & M \otimes_R B \end{array}$$

In particular, $\Theta_{g^{-1}} \circ \Theta_g = \Theta_e = \mathrm{Id}$ so Θ_g is an automorphism. Thus, the co-module gives a group action on $\underline{M}$. Conversely, assume we have a group action. Let $un \in G(A)$ be the universal element (corresponding to the identity morphism $A \to A$. Then we set $\Delta : M \to M \otimes_R A$ be the action of the universal element. This is the co-module structure. □

Example 3.3.3.
(1) If $G = \mathrm{Spec}\ A$ is affine, we can consider the regular representation : we take $M = A$ itself. We claim that the map $G \to GL(\underline{A})$ is injective (as a map of group functors).
(2) The category $Mod_{\mathbb{G}_m}(R)$ is equivalent to the category of $\mathbb{Z}$-graded modules.
(3) The category $Mod_{\mathbb{G}_a}(R)$ is more complicated. When R is a $\mathbb{Q}$-algebra, a representation of $\mathbb{G}_a$ on a module M is equivalent to the data of an endomorphism E of M which is locally nilpotent.

3.4. The Lie algebra of a group scheme.

Let $G \to S$ be a group scheme. For any R-algebra B, we let

$$1 \to \mathrm{Lie}(G)(B) \to G(B[\varepsilon]) \to G(B) \to 1$$

We see that this defines a group functor $\mathrm{Lie}(G)(-)$ on R algebras. Let us put $\mathrm{Lie}(G) := \mathrm{Lie}(G)(R)$. Let us also put $\omega_G = e^\star \Omega^1_{G/S}$. We get a functor $\mathrm{Gr}/S \to \mathrm{Mod}_R$, $G \mapsto \mathrm{Lie}(G) := \mathrm{Lie}(G)(R)$.

Remark 3.4.1. *This definition of Lie algebra applies more generally to any group functor (not necessarily representable).*

Theorem 3.4.2. *We have an isomorphism of groups* $\mathrm{Lie}(G)(B) = \mathrm{Hom}_R(\omega_G, B)$.

Proof. By proposition 2.10.3, $\mathrm{Lie}(G)(B) = \mathrm{Hom}_R(\omega_G, B)$. The RHS carries a natural group law (call it $\star$). The multiplication $m : G \times G$ on the group induces a map $\mathrm{Lie}(G)(B) \oplus \mathrm{Lie}(G)(B) \to \mathrm{Lie}(G)(B)$ which gives a second group law $\circ$ compatible with $\star$. To show that these two group law agree, we use the lemma below. □

Lemma 3.4.3. *Let X be a set. We assume that X has two group structures, $\star$ and $\circ$ and that $(a \star b) \circ (a' \star b') = (a \circ a') \star (b \circ b')$. Then $\star = \circ$ are commutative group laws.*

Proof. We first check that the units $1_\star$ and $1_\circ$ agree :

$$\begin{aligned} 1_\star &= (1_\circ \circ 1_\star) \star (1_\star \circ 1_\circ) \\ &= (1_\circ \star 1_\star) \circ (1_\circ \star 1_\circ) \\ &= 1_\circ \end{aligned}$$

We deduce that:

$$\begin{aligned} a \star b &= (a \circ 1) \star (1 \circ b) \\ &= a \circ b \end{aligned}$$

Finally, we have:

$$\begin{aligned} a \star b &= (1 \circ a) \star (b \circ 1) \\ &= b \circ a \end{aligned}$$

□

If $x \in \mathrm{Lie}(G)(B)$, we let $e^{\varepsilon x}$ be its image in $G(B[\varepsilon])$.

Remark 3.4.4. *When ω_G is finite projective, then $\mathrm{Lie}(G)(B) = \mathrm{Lie}(G) \otimes_R B$. This is true if R is a field, and G is of finite type. We often simply restrict to this case.*

Example 3.4.5. (1) If we take M an R module, then $\mathrm{Lie}(\underline{M}) = M$.
(2) If M is an R-module, then $\mathrm{End}(M) = \mathrm{Lie}(\mathrm{GL}(\underline{M}))$ via the map sending N to $Id + \varepsilon N$. Remark that $Id + \varepsilon N$ has inverse $Id - \varepsilon N$.

3.5. **Lie bracket.** Consider a linear representation ρ of a group G on a module M. This induces a group morphism $d\rho : \mathrm{Lie}(G) \to \mathrm{End}(M)$, with the property that $\rho(e^{\varepsilon x}) = 1 + \varepsilon \mathrm{d}\rho(x)$.

We now assume that $\mathrm{Lie}(G)(B) = \mathrm{Lie}(G) \otimes_R B$ as in remark 3.4.4. We have a linear adjoint representation of G on $\mathrm{Lie}(G)$ denoted by $Ad : G \to \mathrm{GL}(\underline{\mathrm{Lie}(G)})$. Indeed we look at the exact sequence :

$$1 \to \mathrm{Lie}(G)(B) \to G(B[\varepsilon]) \to G(B) \to 1$$

and the group $G(B)$ acts by conjugation on $\mathrm{Lie}(G)(B)$. We justify that the elements of $G(B)$ act B-linearly. For $g \in G(B)$, we get an map $g : G_B \to G_B$, $h \mapsto ghg^{-1}$. By functoriality, this induces a B-linear map on the tangent space (which is the map we are considering). By derivation, we get $ad : \mathrm{Lie}(G) \to \mathrm{End}(\mathrm{Lie}(G))$. We define a Lie bracket by $ad(x)(y) = [x, y]$.

Consider the ring $R[\varepsilon, \varepsilon'] = R[X, X']/(X^2, (X')^2)$. It contains the subrings $R[\varepsilon]$, $R[\varepsilon']$ and $R[\varepsilon\varepsilon']$. We also have an exact sequence $0 \to \varepsilon' R[\varepsilon] \to R[\varepsilon', \varepsilon] \to R[\varepsilon] \to 0$.

Lemma 3.5.1. *Let $x, y \in \mathrm{Lie}(G)$. We have*

$$e^{\varepsilon x} e^{\varepsilon' y} e^{-\varepsilon x} e^{-\varepsilon' y} = e^{\varepsilon\varepsilon'[x,y]}.$$

Proof. Consider the long exact sequence :

$1 \to \mathrm{Lie}(G)(R[\varepsilon']) \overset{e^{\varepsilon'(-)}}{\to} G(R[\varepsilon, \varepsilon']) \to G(R[\varepsilon]) \to 1$, so that $e^{\varepsilon x} \in G(R[\varepsilon])$. We have

$$\begin{aligned} e^{\varepsilon x} e^{\varepsilon' y} e^{-\varepsilon x} &= Ad(e^{\varepsilon x})(e^{\varepsilon' y}) \\ &= (Id + \varepsilon ad(x))(e^{\varepsilon' y}) \\ &= e^{\varepsilon' y + \varepsilon\varepsilon'[x,y]} \end{aligned}$$

□

Corollary 3.5.2. *Let M be a finite projective R-module. The Lie bracket on $\mathrm{End}(M)$ is given by $[x, y] = xy - yx$.*

Proposition 3.5.3. *For any representation ρ of G on M, the map $d\rho : Lie(G) \to \mathrm{End}(M)$ is compatible with the Lie bracket.*

Proof. We work in $R[\varepsilon, \varepsilon']$. We have $e^{\varepsilon x}e^{\varepsilon' y}e^{-\varepsilon x}e^{-\varepsilon' y} = e^{\varepsilon\varepsilon'[x,y]}$. Applying ρ we get $Id+\varepsilon\varepsilon' ad([x,y]) = (Id+\varepsilon ad(x))(Id+\varepsilon' ad(y))(Id-\varepsilon ad(x))(Id-\varepsilon' ad(y))$. □

3.6. **Lie algebra and derivations.** If X is a scheme, we can define a group functor $\mathrm{Aut}(X)$ by $\mathrm{Aut}(X)(B) = \mathrm{Aut}(X \times \mathrm{Spec}\ B/\mathrm{Spec}\ B)$. We can in particular consider $\mathrm{Lie}(\mathrm{Aut})(X)$.

Proposition 3.6.1. $\mathrm{Lie}(\mathrm{Aut})(X) = \mathrm{Der}(X)$.

Proof. See proposition 2.10.2. □

Let G be a group scheme acting on the right on X. Then we get a map $G \to \mathrm{Aut}(X)$ and a corresponding map $\mathrm{Lie}(G) \to \mathrm{Der}(X)$. We can make this more explicit. Let $x \in \mathrm{Lie}(G)$. Then we have a map $e^{\varepsilon x} : X \times_R R[\varepsilon] \to X \times_R R[\varepsilon]$.

If f is a local function on X, and $m \in X$, then $f(me^{\varepsilon x}) = f(m) + \varepsilon D_x(f)(m)$.

Proposition 3.6.2. *The map* $\mathrm{Lie}(G) \to \mathrm{Der}(X)$ *is compatible with Lie bracket.*

Proof. We compute :

$$\begin{aligned}
f(me^{-\varepsilon' y}) &= f(m) - \varepsilon' D_y(f)(m) \\
f(me^{-\varepsilon x}e^{-\varepsilon' y}) &= f(m) - \varepsilon' D_y(f)(m) - \varepsilon D_x f(m) + \varepsilon\varepsilon' D_x D_y(f)(m) \\
f(me^{\varepsilon' y}e^{-\varepsilon x}e^{-\varepsilon' y}) &= f(m) - \varepsilon' D_y(f)(m) - \varepsilon D_x f(m) + \varepsilon\varepsilon' D_x D_y(f)(m) + \varepsilon' D_y f(m) - \varepsilon'\varepsilon D_y D_x f(m) \\
&= f(m) - \varepsilon D_x f(m) + \varepsilon\varepsilon' D_x D_y(f)(m) - \varepsilon'\varepsilon D_y D_x f(m) \\
f(me^{\varepsilon x}e^{\varepsilon' y}e^{-\varepsilon x}e^{-\varepsilon' y}) &= f(m) - \varepsilon D_x f(m) + \varepsilon\varepsilon' D_x D_y(f)(m) - \varepsilon'\varepsilon D_y D_x f(m) + \varepsilon D_x f(m) \\
&= f(m) + \varepsilon\varepsilon'[D_x, D_y]f(m)
\end{aligned}$$

□

We can consider the action of G on itself by right translation, $\star_r$ and by left translation $\star_l$.

We have a map $G^{op} \to \mathrm{Aut}(G)$, $g \mapsto g \star_r (-)$.

Lemma 3.6.3. *The map* $G^{op} \to \mathrm{Aut}(G)$ *induces an isomorphism of* G^{op} *on the subspace of* $Aut^{\star_l G}(G)$*, of automorphisms which commute with left translation.*

Proof. We define a map $Aut^{\star_l G}(G) \to G$, by $\phi \mapsto \phi(e)$. Observe that $\phi(g) = g\phi(e)$ and $\phi' \circ \phi \mapsto \phi(e)\phi'(e)$. □

Corollary 3.6.4. *The map* $\mathrm{Lie}(G) \to \mathrm{Der}(G)$ *identifies* $\mathrm{Lie}(G)$ *with the space of left invariant derivations.*

Assume G is $\mathrm{Spec}\ A$. Then we can make explicit what are the left invariant derivation. Let $D : A \to A$ be a derivation. For any element $g \in G(T)$. We have a map $g : A \otimes T \to A \otimes T$ given by left translation. Then we ask that $g^{-1} \circ D \otimes 1 \circ g = D \otimes 1$.

3.7. **Lie algebras : general.** A Lie algebra $\mathfrak{g}$ over a ring R is an R-module $\mathfrak{g}$ endowed with a braket $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. Such that :

(1) $[,]$ is bilinear over R,
(2) $[X, X] = 0$ for all $X \in \mathfrak{g}$.
(3) (Jacobi identity) For all $X, Y, Z \in \mathfrak{g}$, $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]]$.

Example 3.7.1. (1) If M is an R-module, then $\mathrm{End}_R(M)$, endowed with the braket $[X, Y] = XY - YX$ is a Lie algebra.
(2) If A is an R-algebra, then $\mathrm{Der}_R(A)$ is a sub-Lie algebra of $\mathrm{End}_R(A)$.
(3) If X is a $\mathrm{Spec}\ R$-scheme, then $Der(X) = \mathrm{H}^0(X, T_{X/S})$ is a Lie algebra.

Corollary 3.7.2. *Let G be a group scheme. Then* $\mathrm{Lie}(G)$ *with its bracket* $[,]$ *is a Lie algebra.*

Proof. Indeed, $\mathrm{Lie}(G)$ is a sub-Lie algebra of $Der(G)$. □

3.8. **Affine algebraic groups over a field.** We fix a field k. All schemes are over Spec k.

Definition 3.8.1. An algebraic group is a group scheme G which is of finite type over Spec k. An affine algebraic group G is an affine group scheme over Spec k which is of finite type.

Concretely, this means that $G = \text{Spec } A$ where A is a k-algebra of finite type and an Hopf algebra.

3.8.1. *Smoothness.* The following result is known as Cartier's theorem.

Theorem 3.8.2. *Assume that k is of characteristic 0. Let $G \to \text{Spec } k$ be a group scheme, locally of finite type. Then G is smooth over* Spec k.

3.8.2. *Subgroups.*

Lemma 3.8.3. *Let $U \subseteq G$ be a dense open subscheme. Then $U \times U$ maps surjectively onto G.*

Proof. We can suppose $k = \bar{k}$. It suffices to see that $U.U(\bar{k}) \to G$ which is open in G, contains all k-points. Let $g \in G(k)$. Then $U \cap gU^{-1}$ is again dense open (indeed, U and gU^{-1} contain all generic points). Thus, there are points $v, w \in U(\bar{k})$ such that $v = gw^{-1}$. □

Let G be an algebraic group and let $i : H \hookrightarrow G$ be an algebraic subgroup.

Lemma 3.8.4. *The image of H is a closed subset of G.*

Proof. We claim that the image $i(H)$ of H in G is a closed subspace. In order to prove this, we can assume that $k = \bar{k}$. The image $i(H)$ in G is constructible. This implies that $i(H)$ contains a subset V which is dense and open in $\overline{i(H)}$. Then $H(\bar{k}).i^{-1}(V) = H$. We deduce that $i(H)$ is open in $\overline{i(H)}$. Note that $\overline{i(H)}$ is a closed subgroup of G. The above lemma shows that $i(H) = \overline{i(H)}$. □

We assume that the map $H \to G$ is a monomorphism of sheaves.

Lemma 3.8.5. *The group H is a closed subgroup of G.*

Proof. We first claim that $i : H \to G$ is injective and induces isomorphisms on residue fields. Consider a point $g \in G$ and look at the fiber $H_g \to g$. Then H_g is a subfunctor of g. In particular $H_g \times_g H_g \to H_g$ is an isomorphism. This means that H_g has a unique point. So $H_g = \text{Spec } A$ for some artinian alegbra and $A \otimes_{k(g)} A = A$ which implies that A has dimension 1 (as a $k(g)$-vector space). So $A = k(g)$.

We next claim that there exists a dense open V of G such that $i^{-1}(V) \to V$ is a closed immersion. Let h be a generic point of H mapping to $h \in G$. Consider the map $\mathscr{O}_{G,h} \mapsto \mathscr{O}_{H,h}$. Since $\mathscr{O}_{H,h}$ is Artinian, and they have the same residue field, we deduce that $\mathscr{O}_{G,h} \to \mathscr{O}_{H,h}$ is surjective. Let Spec A be an irreducible open subset of H mapping to Spec B open in G. We have a map $B \to B/I \hookrightarrow A$. Moreover B/I and A have the same generic point ξ and $(B/I)_\xi = A_\xi$. Let $x_1, \cdots, x_n$ be generators of A as a B-algebra. There exists $f \in B$ with $f(\xi) \neq 0$ such that $fx_i \in B$. We deduce that $B[1/f]/I \to A[1/f]/I$ is an isomorphism. By translation, this finally implies that H is a closed subgroup of G. □

3.8.3. *Existence of representations.* Let G be an affine algebraic group and H a subgroup of G (necessarily closed in G and affine).

Lemma 3.8.6. *There exists a finite dimensional representation $\rho : G \to GL(V)$ with the property that H is the stabilizer of a line L.*

Proof. We let A be the algebra of G and I the ideal of H. We consider the representation of G on A. We first claim that H is exactly the stabilizor of I. If $h \in H(R)$, and $f \in I \otimes R$, we have $h.f = f(-h)$. It is clear that if f vanishes on H, then so does $h.f$. Conversely, if $g \in G(R)$ is such that $g.f = f(-g) \in I \otimes R$ for all f, then $f(g) = 0$ for all f. Thus $g \in H(R)$.

We now let $W \subseteq A$ be finite dimesional k-vector space, with the property that V generates A as an algebra and $W \cap I$ generates I as an ideal.

We claim that there exists $W \subseteq X$ such that X is a finite dimensional representation of G. Let (a_i) be a k-basis of A. We have $\Delta(x) = \sum_i x_i \otimes a_i$ and $\sum_i \Delta(x_i) \otimes a_i = \sum_i x_i \otimes \Delta(a_i) = \sum_{i,j} x_i \otimes b_{ij} \otimes a_j$. We deduce that $\Delta(x_i) = \sum x_j \otimes b_{ji}$. Thus, we let X be the space generated by x_i's. Let n be the dimension of $X \cap I$. Then H is stabilizer of $X \cap I$. We finally consider $\Lambda^n X$ and $\Lambda^n X \cap I$.

□

Lemma 3.8.7. *If H is normal, there exists a finite dimensional representation $\rho : G \to GL(V)$ with the property that H is the kernel.*

Proof. Let us take the representation given by the last lemma. Let us consider $\bigoplus_{g \in G(\bar{k})} g.L$. Picking representatives, we can write this space $\oplus_{g_i} g_i L$. This is a representation of G. Moreover, H preserves each of these lines. We can replace V by $\oplus_{g_i} g_i L$. We now consider the composition $Ad \circ \rho : G \to GL(\mathrm{End}(V))$. Clearly, H is in the kernel. Any element in the kernel of Ad is a scalar in $GL(V)$. A scalar in $GL(V)$ preserves L and therefore the kernel of $Ad \circ \rho$ consists of elements of G which stabilize L. This is inside H. □

3.9. **Quotient.** Our goal is to prove the following theorems.

Theorem 3.9.1. *Let G be an algebraic group acting on a scheme of finite type X. For any $x \in X(k)$, the orbit map $G \to X$, $g \mapsto gx$ factors through an immersion $G/H \to X$ where H is the stabilizor of x. Moreover, if G is smooth, the orbit is smooth.*

We only give the proof in characteristic 0. Therefore we know that G is smooth (hence reduced).

Proof. We let H be the pre-image of x under the orbit map. This is obviously a closed subgroup H of G. The map $orb : G \to X$ has constructible image. Thus there exists $V \subseteq \overline{orb(G)}$ dense open such that $V \subseteq orb(G)$. We deduce that $orb^{-1}(V) \subseteq G$ is open. Assuming $k = \bar{k}$ we deduce that $G(k) orb^{-1}(V) = G$ so that $orb(G)$ is open in its closure. We now equip $\overline{orb(G)}$ with the reduced scheme structure and this induces a scheme structure on $orb(G)$. Since G is smooth, the map $G \to X$ factors through a map $G \to \overline{orb(G)} \to X$. By generic flatness, there is a dense open W of $orb(G)$ such that $orb^{-1}(W) \to W$ is flat. Using group translation, we deduce that $G \to orb(G)$ is flat. This is thus an fppf cover. Moreover, $G \times_{orb(G)} G = G \times H$. Thus, $orb(G) = G/H$. Note that $orb(G)$ is geometrically reduded, hence generically smooth. By homogenity it is smooth. □

Theorem 3.9.2. *Let G be an affine algebraic group. Let H be a closed subgroup. The fppf quotient G/H is representable.*

Proof. We pick a representation $\rho : G \to GL(V)$ with V finite dimensional k-vector space. We let n be the dimension of V and we let $\mathrm{Gr}(n, 1)$ be the Grassmanian of lines in V. This is the functor which sends a k-scheme T to isomorphism classes of exact sequence $0 \to \mathcal{L} \to \mathscr{O}_T \otimes_k V \to \mathscr{G} \to 0$ where $\mathcal{L}$ is an invertible sheaf and $\mathscr{G}$ is locally free of rank $n-1$. The group $GL(V)$ acts on $\mathrm{Gr}(n, 1)$ and so does G. The line L in V defines a k-point. And G/H is represented by the orbit of L in $\mathrm{Gr}(n, 1)$. □

Theorem 3.9.3. *Let G be an affine algebraic group. Let H be a normal subgroup, then G/H is an affine subgroup.*

Proof. We consider a representation $\rho : G \to GL(V)$ with kernel H. The image of ρ is closed, so that $G/H \to \rho(G)$ identifies with a closed subgroup of $GL(V)$. Hence it is affine. □

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