RANDOM COMPRESSION OF AN INTEGER PARTITION

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Abstract. The objective of this note is to explain how to compute the asymptotics of a random integer partition obtained by compression of a large integer partition. The procedure of random compression is related to the operation of restriction of an irreducible representation of $\mathfrak{S}(N)$ to a smaller symmetric group, and the method of observables yields the limit shape of the random compressed partition.

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1. The model of random compression

In this first section, we present the combinatorial model of random compression of an integer partition, and we detail the link with the representation theory of the symmetric groups.

1.1. **Integer partitions and standard tableaux.**

Given a positive integer N, recall that an *integer partition* of size N is a sequence $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell)$ of positive integers which is non-increasing and such that $|\lambda| = \sum_{i=1}^{\ell} \lambda_i = N$. We shall denote $\ell = \ell(\lambda)$ the length of a partition, which is its number of non-zero parts. A partition is usually represented by its *Young diagram*, which is the array of boxes with λ_1 boxes on the first row, λ_2 boxes on the second row, *etc.*; see Figure 1 for an example. We shall denote $\mathfrak{Y}(N)$ the set of integer partitions with size N, and

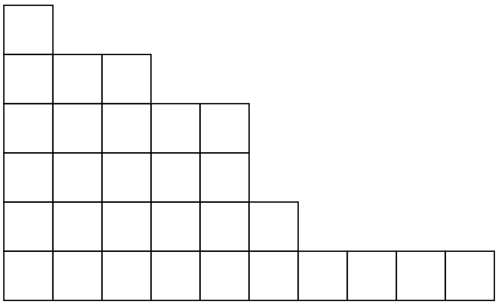

Figure 1. The Young diagram of the integer partition $\lambda = (10, 6, 5, 5, 3, 1)$ with size $|\lambda| = 30$.

$\mathfrak{Y} = \bigsqcup_{N \in \mathbb{N}} \mathfrak{Y}(N)$. A *standard tableau* with shape $\lambda \in \mathfrak{Y}(N)$ is a numbering of the cells of its Young diagram by positive integers in $[\![1, N]\!]$, in such a way that the rows and columns of the tableau are strictly increasing; see for instance Figure 2. The set of standard tableaux with shape

11									
10	17	24							
6	12	18	25	30					
4	9	15	23	28					
3	5	8	16	20	27				
1	2	7	13	14	19	21	22	26	29

Figure 2. A standard Young tableau with shape $(10, 6, 5, 5, 3, 1)$.

λ will be denoted $\mathrm{ST}(\lambda)$, and there is a combinatorial formula for the number of Young tableaux with shape λ:

$$\mathrm{card}\,\mathrm{ST}(\lambda) = \frac{n!}{\prod_{\square \in \lambda} h(\square)},$$

where the product in the denominator runs over the cells of the Young diagram of λ, and $h(\square)$ is the *hook-length* of the cell $\square$, which is the number of cells of the hook based at $\square$ and connecting it horizontally and vertically to the top-right border of the Young diagram; see Figure 3. For a proof of the hook-length formula, see for instance [GNW79]. Notice on the other hand that $\mathrm{card}\,\mathrm{ST}(\lambda)$ is the number of sequences $(\emptyset \nearrow \lambda^{(1)} \nearrow \lambda^{(2)} \nearrow \cdots \nearrow \lambda^{(N)} = \lambda)$, where for each i the notation $\lambda^{(i)} \nearrow \lambda^{(i+1)}$ means that the integer partition $\lambda^{(i+1)}$ with size $i+1$ is obtained from $\lambda^{(i)}$ by adding one cell at the edge of this partition. Indeed, such a sequence clearly determines uniquely a standard tableau with shape λ.

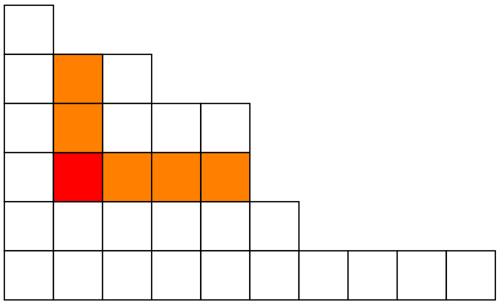

FIGURE 3. Hook-length of a cell in a Young diagram: here, $h(\square) = 1 + 2 + 3 = 6$.

1.2. **Random compression and Poisson random compression.** Given a standard tableau T with size N and $k \leq N$, we denote $T^{(k)}$ the subtableau of T which consists in the cells labeled by the integers in $[\![1, k]\!]$, and $\lambda^{(k)}$ the shape of this integer partition, which has size k. For instance, if T is the standard tableau from Figure 2, then $\lambda^{(13)} = (4, 3, 2, 2, 1, 1)$.

Definition 1.1 (Random compression). *Fix an integer partition $\lambda \in \mathfrak{Y}(N)$, and a real parameter $t \in (0, 1)$. The random compression with parameter t of λ is the random integer partition $\mathrm{RC}_t(\lambda)$ with size $k = \lfloor tN \rfloor$ which is obtained by choosing a random standard tableau T uniformly in $\mathrm{ST}(\lambda)$, and by looking at the shape $\lambda^{(k)}$ of the subtableau $T^{(k)}$.*

In the next paragraph, we shall give a representation-theoretic interpretation of this procedure. For a reason which will be given in Subsection 3.3, it is also interesting to introduce a version of the random compression where the size k is itself random.

Definition 1.2 (Poisson random compression). *In the same setting as before, the Poisson random compression with parameter t of λ is the random integer partition $\mathrm{PRC}_t(\lambda)$ obtained by the following procedure:*

- *One chooses as before T randomly and uniformly in $\mathrm{ST}(\lambda)$.*
- *One picks at random N independent uniformly distributed points $x_1, \ldots, x_N$ in $(0, 1)$, and one denotes k the number of x_i's which are smaller than t. Obviously, k follows a binomial distribution $\mathcal{B}(N, t)$.*
- *Finally, one takes as before $\mathrm{PRC}_t(\lambda) = \lambda^{(k)} = \mathrm{shape}(T^{(k)})$.*

It is expected that the Poisson version of the random compression yields determinantal point processes; see Theorem 3.6 for a result in this direction.

1.3. **Link with the representation theory of symmetric groups.** Denote $\mathfrak{S}(N)$ the symmetric group of order N, which has cardinality $N!$. The isomorphism classes of the irreducible (linear, complex) representations of $\mathfrak{S}(N)$ are in bijection with the integer partitions of size N, and they can be labeled in such a way that $\dim V^\lambda = \mathrm{card}\,\mathrm{ST}(\lambda)$; see [Mél17, Chapters 2 and 3] for a detailed account of this representation theory and of the combinatorics of tableaux that is related to it. In particular, one has the identity

$$N! = \sum_{\lambda \in \mathfrak{Y}(N)} (\mathrm{card}\,\mathrm{ST}(\lambda))^2,$$

which comes from the isomorphism of algebras

$$\mathbb{C}\mathfrak{S}(N) \simeq_{\mathrm{iso}} \bigoplus_{\lambda \in \mathfrak{Y}(N)} \mathrm{End}(V^\lambda),$$

itself a standard fact from the representation theory of finite groups (see [Mél17, Theorem 1.14]). In the sequel, we shall simply write $\dim\lambda$ for the dimension of V^λ; the uniform measure on $\mathrm{ST}(\lambda)$ gives a weight $\frac{1}{\dim\lambda}$ to each standard tableau with shape λ, and $\dim\lambda$ is given by the hook-length formula. The identity $\dim\lambda = \mathrm{card}\,\mathrm{ST}(\lambda)$ can be made much more precise: there exists a linear basis $(e_T)_{T\in\mathrm{ST}(\lambda)}$ of the irreducible representation V^λ of $\mathfrak{S}(N)$ such that, for any $k \leq N$ and any standard tableau T, the subspace $\mathbb{C}\mathfrak{S}(k)(e_T) \subset V^\lambda$ is an irreducible representation of $\mathfrak{S}(k)$ with type $\lambda^{(k)} = \mathrm{shape}(T^{(k)})$. This is the so-called *Gelfand–Tsetlin* basis of V^λ, see [Mél17, Chapter 8]. A linear basis of $\mathbb{C}\mathfrak{S}(k)(e_T)$ is given by:

$$\mathbb{C}\mathfrak{S}(k)(e_T) = \bigoplus_{\substack{T' \text{ with the same entries} \\ k+1,\ldots,N \text{ as the tableau } T}} \mathbb{C}e'_T.$$

This leads to the following:

Proposition 1.3 (Distribution of a random compression)**.** *Let λ be an integer partition with size N, $t \in (0,1)$ be a real parameter. With $k = \lfloor tN \rfloor$, we expand V^λ as a direct sum of irreducible $\mathbb{C}\mathfrak{S}(k)$-modules:*

$$V^\lambda = \bigoplus_{\substack{\mu\subset\lambda \\ \mu\in\mathfrak{Y}(k)}} m_\mu^\lambda\, V^\mu,$$

where the m_μ^λ are integer multiplicities. The notation $\mu \subset \lambda$ means that the Young diagram of μ is included in the Young diagram of λ.

(1) The multiplicity m_μ^λ is the number of skew standard tableaux of shape $\lambda \setminus \mu$, $\lambda \setminus \mu$ being the skew partition with size $N-k$ whose Young diagram consists in the cells of the diagram λ that do not belong to the diagram of μ.

(2) The spectral measure

$$\mathbb{P}[\mu] = \frac{m_\mu^\lambda \dim\mu}{\dim\lambda}$$

is the distribution of the random compression $\mathrm{RC}_t(\lambda)$.

Proof. We have just explained that any skew standard tableau of size $N-k$ and with outer shape λ corresponds to an irreducible $\mathbb{C}\mathfrak{S}(k)$-submodule of V^λ. Those that give a component with type μ have inner shape μ, whence the first item of proposition. Now, the spectral measure of the restriction from $\mathfrak{S}(n)$ to $\mathfrak{S}(k)$ corresponds to the following procedure: pick at random an element $T \in \mathrm{ST}(\lambda)$, and consider the isomorphism class of $\mathbb{C}\mathfrak{S}(k)(e_T)$. By the previous discussion, this isomorphism class is $\mathrm{shape}(T^{(k)})$, so by definition we obtain the random variable $\mathrm{RC}_t(\lambda)$. □

The previous proposition is a particular case of a general construction: given a representation V of $\mathfrak{S}(N)$, its decomposition in irreducible components $V = \bigoplus_{\lambda\in\mathfrak{Y}(N)} m_\lambda V^\lambda$ yields the *spectral measure*

$$\mathbb{P}_V[\lambda] = \frac{m_\lambda \dim\lambda}{\dim V},$$

which is an interesting way for choosing at random an integer partition $\lambda \in \mathfrak{Y}(N)$. The random compression is the case where $V = \mathrm{Res}^{\mathfrak{S}(N)}_{\mathfrak{S}(\lfloor tN\rfloor)}(V^\lambda)$. Let us look at some important other examples, in relation to the procedure of random compression.

Example 1.4 (Plancherel measure). The *Plancherel measures* PL_N are the spectral measures of the regular representations $\mathbb{C}\mathfrak{S}(N)$ of the symmetric groups. They are given by

$$\mathrm{PL}_N[\lambda] = \frac{(\dim\lambda)^2}{N!}$$

for $\lambda \in \mathfrak{Y}(N)$. They have a property of stability with respect to random compression: for any N and t,

$$(\mathrm{RC}_t)_* \,\mathrm{PL}_N = \mathrm{PL}_{\lfloor tN \rfloor}.$$

Indeed, given $k \leq N$, $\mu \in \mathfrak{Y}(k)$ and $\lambda \in \mathfrak{Y}(N)$, by Frobenius' reciprocity, the multiplicity m_μ^λ of V^μ in $\mathrm{Res}_{\mathfrak{S}(k)}^{\mathfrak{S}(N)}(V^\lambda)$ is equal to the multiplicity of V^λ in $\mathrm{Ind}_{\mathfrak{S}(k)}^{\mathfrak{S}(N)}(V^\mu)$. Therefore,

$$\sum_{\substack{\mu \subset \lambda \\ \lambda \in \mathfrak{Y}(N)}} m_\mu^\lambda \dim \lambda = \dim \left(\mathrm{Ind}_{\mathfrak{S}(k)}^{\mathfrak{S}(N)}(V^\mu)\right) = \frac{N!}{k!} \dim \mu.$$

As a consequence, with $k = \lfloor tN \rfloor$, we obtain

$$((\mathrm{RC}_t)_* \mathrm{PL}_N)\,[\mu] = \sum_{\substack{\mu \subset \lambda \\ \lambda \in \mathfrak{Y}(N)}} \frac{m_\mu^\lambda \dim \mu}{\dim \lambda} \frac{(\dim \lambda)^2}{N!} = \frac{(\dim \mu)^2}{k!} = \mathrm{PL}_k[\mu].$$

With an algebraic point of view, the stability property is related to the fact that $\mathrm{Res}_{\mathfrak{S}(k)}^{\mathfrak{S}(N)}(\mathbb{C}\mathfrak{S}(N))$ is a multiple of the representation $\mathbb{C}\mathfrak{S}(k)$:

$$\mathrm{Res}_{\mathfrak{S}(k)}^{\mathfrak{S}(N)}(\mathbb{C}\mathfrak{S}(N)) \simeq_{\mathrm{iso}} \frac{N!}{k!}\, \mathbb{C}\mathfrak{S}(k).$$

Example 1.5 (Schur–Weyl measure). The *Schur–Weyl measures* $\mathrm{SW}_{M,N}$ are the spectral measures of the permutation representation of $\mathfrak{S}(N)$ on the space of tensors $(\mathbb{C}^M)^{\otimes N}$. The Schur–Weyl duality (see [Mél17, Section 2.5]) ensures that the commutant of the action of $\mathfrak{S}(N)$ is the algebra spanned by the diagonal action of the general linear group $\mathrm{GL}(M,\mathbb{C})$, and yields the decomposition in $\mathrm{GL}(M,\mathbb{C}) \times \mathfrak{S}(N)$-bimodules

$$(\mathbb{C}^M)^{\otimes N} = \bigoplus_{\substack{\lambda \in \mathfrak{Y}(N) \\ \ell(\lambda) \leq N}} U^\lambda \otimes_{\mathbb{C}} V^\lambda,$$

where U^λ is the irreducible representation of $\mathrm{GL}(M,\mathbb{C})$ with highest weight λ. Again, the Schur–Weyl measures have a property of stability with respect to random compression:

$$(\mathrm{RC}_t)_* \,\mathrm{SW}_{M,N} = \mathrm{SW}_{M,\lfloor tN \rfloor}.$$

Indeed, we clearly have the algebraic identity

$$\mathrm{Res}_{\mathfrak{S}(k)}^{\mathfrak{S}(N)}\left((\mathbb{C}^M)^{\otimes N}\right) \simeq_{\mathrm{iso}} M^{N-k}\,(\mathbb{C}^M)^{\otimes k}.$$

2. Observables of Young diagrams

Our objective is to understand the behavior of a random compressed partition $\mathrm{RC}_t(\lambda)$ when $\lambda = \lambda_N$ has size N and grows in such a way that the rows and columns of λ are *balanced*: there are $O(\sqrt{N})$ non-empty rows and $O(\sqrt{N})$ non-empty columns in λ_N. In Subsection 2.1, we explain the adequate point of view in order to deal with such partitions, and we introduce the notion of the transition measure μ_λ of an integer partition λ. Then, in Subsection 2.2, we introduce the free cumulants of a Young diagram, which play an essential role in the discussion of the asymptotics. We shall see in Section 3 that the random compression $\mathrm{RC}_t(\lambda)$ of a large balanced integer partition has with high probability its transition measure close to the t-free compression of the transition measure of λ.

2.1. **Young curves and transition measures.** Given an integer partition λ with size N, the *Russian convention* for drawing it consists in taking the Young diagram of λ and rotating it by 45 degrees, and then look at it as a part of the xy-plane, the origin corresponding to the first cell of the Young diagram, and the cells being drawn as squares with area 2. One also adds the half-lines $y = \pm x$, and one looks at the upper boundary of this figure (in red on Figure 4). This curve $\omega_\lambda : \mathbb{R} \to \mathbb{R}_+$ is affine by parts, with slopes ± 1, and we have $\omega_\lambda(s) = |s|$ for s large enough.

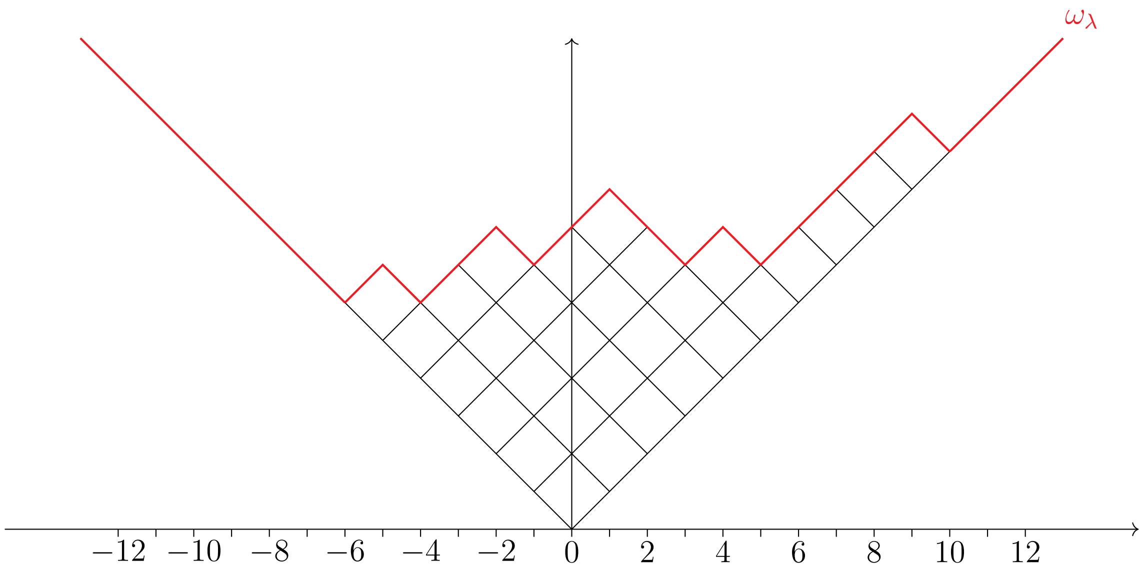

FIGURE 4. The Young curve ω_λ associated to integer partition $\lambda = (10, 6, 5, 5, 3, 1)$.

We call ω_λ the *Young curve* of the partition λ. It is also convenient to consider $\sigma_\lambda(s) = \frac{\omega_\lambda(s)-|s|}{2}$, which is a compactly supported positive continuous function. We have

$$\int_{\mathbb{R}} \sigma_\lambda(s)\, ds = |\lambda|.$$

The curves ω_λ and σ_λ are particularly useful in order to deal with growing sequences of Young diagrams and limit shapes thereof. The functions ω_λ belong to the space $\mathscr{Y}$ of continuous functions $\omega : \mathbb{R}_+ \to \mathbb{R}$ which are equal to $|s|$ for $|s|$ large enough, and which are Lipschitz with constant 1. For $u > 0$, this space is stable by a renormalisation

$$\omega(\cdot) \mapsto \frac{\omega(\sqrt{u}\,\cdot)}{\sqrt{u}},$$

which multiplies the areas $\int_{\mathbb{R}} \frac{\omega(s)-|s|}{2}\, ds$ by a factor $\frac{1}{u}$. On the other hand, if $\omega \in \mathscr{Y}$, then one can define its *transition measure* as follows. We first introduce the generating function of ω:

$$G_\omega(z) = \frac{1}{z} \exp\left(-\int_{\mathbb{R}} \frac{\sigma'(s)}{z-s}\, ds\right),$$

where as before $\sigma(s) = \frac{\omega(s)-|s|}{2}$. The function G_ω is well defined and holomorphic on the upper half-plane $\mathbb{C}_+$, and it has the following properties:

(1) For any $z \in \mathbb{C}_+$, $G_\omega(z) \in \mathbb{C}_-$ (negative imaginary part).

(2) We have $\lim_{y\to\infty} \mathrm{i}y\, G_\lambda(\mathrm{i}y) = 1$.

General results from the theory of complex functions ensure then that G_ω is the Cauchy transform of a probability measure: there exists a unique probability measure μ_ω on $\mathbb{R}$ such that

$$G_\omega(z) = \int_{\mathbb{R}} \frac{\mu_\omega(ds)}{z-s}.$$

Conversely, any compactly supported probability measure μ_ω corresponds in this way to a Young curve in $\mathscr{Y}$, and the convergence of all the moments of a sequence of compactly supported probability measures $\mu_N \to_{N\to\infty} \mu$ implies the uniform convergence $\omega_N \to_{N\to\infty} \omega$ over $\mathbb{R}$ of the corresponding Young curves; see [Mél17, Section 7.4]. In the sequel, if $\lambda \in \mathfrak{Y}(N)$, we denote $\mu_\lambda = \mu_{\omega_\lambda}$. In this particular case, the transition measure is discrete and supported by integers. More precisely, consider the spectral measure of $\mathrm{Ind}_{\mathfrak{S}(N)}^{\mathfrak{S}(N+1)}(V^\lambda)$. The Pieri rules for induction of irreducible representations of symmetric groups yield:

$$\mathrm{Ind}_{\mathfrak{S}(N)}^{\mathfrak{S}(N+1)}(V^\lambda) = \bigoplus_{\lambda \nearrow \Lambda} V^\Lambda,$$

where the sum runs over integer partitions which are obtained from λ by adding exactly one cell to the Young diagram. Therefore,

$$\mathbb{P}_{\mathrm{Ind}_{\mathfrak{S}(N)}^{\mathfrak{S}(N+1)}(V^\lambda)}[\Lambda] = \frac{\dim \Lambda}{(N+1)\,\dim \lambda}.$$

Let us associate to each possible integer partition Λ the abscissa x_Λ of the corner at which the cell $\Lambda \setminus \lambda$ is added (by drawing the Young diagram with the Russian convention). Then, one can show that

$$\mu_\lambda = \sum_{\lambda \nearrow \Lambda} \frac{\dim \Lambda}{(N+1)\,\dim \lambda}\,\delta_{x_\Lambda}.$$

This is the origin of the terminology of transition measure: μ_λ drives the random process of adding randomly cells to a Young diagram λ.

Example 2.1 (Large random partitions under the Plancherel measure). The *Logan–Shepp–Kerov–Vershik curve* [LS77; KV77] is the Young curve $\Omega \in \mathscr{Y}$ defined by:

$$\Omega(s) = \begin{cases} \frac{2}{\pi}\,\left(s\,\arcsin\left(\frac{s}{2}\right) + \sqrt{4-s^2}\right) & \text{if } |s| < 2, \\ |s| & \text{if } |s| \geq 2. \end{cases}$$

It is related to the Plancherel measures of the symmetric groups by the following law of large numbers: if λ_N is a random partition chosen under the Plancherel measure PL_N, and if

$$\omega_N(s) = \frac{\omega_{\lambda_N}(\sqrt{N}s)}{\sqrt{N}},$$

then ω_N converges in probability towards Ω as N goes to infinity; see Figure 5 for an illustration of this result.

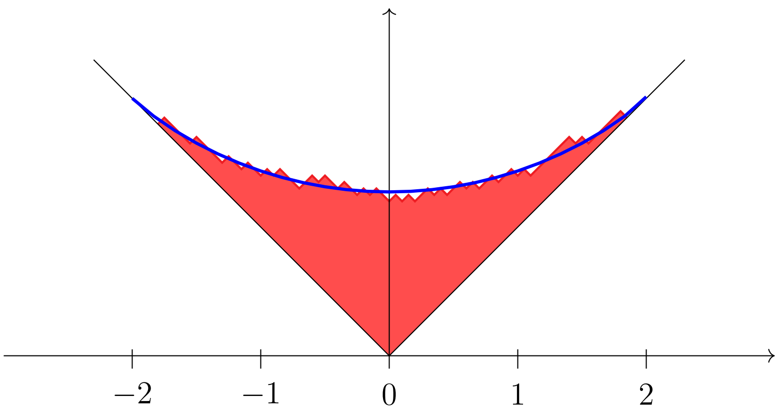

FIGURE 5. A random integer partition of size $N = 400$ under the Plancherel measure; the limit shape (in blue) is the Logan–Shepp–Kerov–Vershik curve.

The LSKV curve is deeply connected to the Wigner semicircle law $\mu_{\mathrm{semicircle}}(ds) = 1_{|s|<2}\,\frac{\sqrt{4-s^2}}{2\pi}\,ds$, which is its transition measure:

$$G_\Omega(z) = \frac{2}{z+\sqrt{z^2-4}} = \int_{\mathbb{R}} \frac{\mu_{\mathrm{semicircle}}(ds)}{z-s}.$$

2.2. **Free cumulants of a Young diagram.** Given a compactly supported probability measure μ on $\mathbb{R}$, its Cauchy transform $G_\mu(z)$ is equivalent to $\frac{1}{z}$ for $z \in \mathbb{C}_+$ and $|z| \to +\infty$, and it can be expanded in a series of powers of $\frac{1}{z}$:

$$G_\mu(z) = \sum_{k=0}^{\infty} \frac{M_k(\mu)}{z^{k+1}},$$

where $M_k(\mu) = \int_{\mathbb{R}} s^k\,\mu(ds)$ is the k-th moment of μ. The map $z \mapsto G_\mu(z)$ maps bijectively a neighborhood of ∞ in the complex plane to a neighborhood of 0, and it can be locally inverted by a map

$$K_\mu(w) = \frac{1}{w} + \sum_{k=1}^{\infty} R_k(\mu)\,w^{k-1}$$

so that $K_\mu(G_\mu(z)) = z$ for $|z|$ large enough. The coefficients $(R_k(\mu))_{k\geq 1}$ are called the *free cumulants* of the measure μ. For the link between these quantities and free probability theory, see for instance [NS06] and [Mél17, Section 9.1]. The free cumulants are related to the moments $(M_k(\mu))_{k\geq 1}$ by the combinatorics of *non-crossing partitions*. Consider the set $\mathfrak{N}(k)$ of non-crossing partitions of size k: they are the set partitions π of $[\![1,k]\!]$ such that one cannot find two distinct parts π_i and π_j and elements $a,c \in \pi_i$ and $b,d \in \pi_j$ with $a<b<c<d$. If $\pi = \pi_1 \sqcup \pi_2 \sqcup \cdots \sqcup \pi_\ell$, we denote $M_\pi(\mu) = \prod_{i=1}^{\ell} M_{|\pi_i|}(\mu)$ and $R_\pi(\mu) = \prod_{i=1}^{\ell} R_{|\pi_i|}(\mu)$. Then, for any $k \geq 1$,

$$M_k(\mu) = \sum_{\pi \in \mathfrak{N}(k)} R_\pi(\mu),$$

and this relation can be inverted by using the Möbius function of the lattice of non-crossing partitions.

The free cumulants of a partition λ are defined by using the transition measure μ_λ:

$$R_k(\lambda) = R_k(\mu_\lambda).$$

We can of course extend the definition to Young curves $\omega \in \mathscr{Y}$. The free cumulants are extremely powerful observables in order to prove that a family of random partitions $(\lambda_N)_{N\in\mathbb{N}}$ chosen under a family of spectral measures of representations $(\mathbb{P}_N)_{N\in\mathbb{N}}$ admits a limit shape after rescaling the Young curves by a factor $\sqrt{N}$. This technique is due to Biane [Bia98; Bia01], who proved a deep connection between the analytic properties of the Young curve ω_λ (*via* its transition measure and its free cumulants), and the algebraic properties of the irreducible representation V^λ of $\mathfrak{S}(N)$. On the analytic side, note that if sc_u is the scaling of Young curves which multiplies the areas by a factor $\frac{1}{u}$, then

$$G_{\mathrm{sc}_u(\omega)}(z) = \frac{1}{z}\,\exp\left(-\int_{\mathbb{R}} \frac{(\mathrm{sc}_u(\sigma))'(s)}{z-s}\,ds\right) = \frac{1}{z}\,\exp\left(-\int_{\mathbb{R}} \frac{\sigma'(s)}{\sqrt{u}z-s}\,ds\right) = \sqrt{u}\,G_\omega(\sqrt{u}z),$$

so $\mu_{\mathrm{sc}_u(\omega)}(B) = \mu_\omega(\sqrt{u}\,B)$ for any Borel subset $B \subset \mathbb{R}$. As a consequence, for any $k \geq 1$,

$$M_k(\mu_{\mathrm{sc}_u(\omega)}) = u^{-\frac{k}{2}}\,M_k(\mu_\omega);$$
$$R_k(\mu_{\mathrm{sc}_u(\omega)}) = u^{-\frac{k}{2}}\,R_k(\mu_\omega).$$

Therefore, if one has a sequence $(\lambda_N)_{N\in\mathbb{N}}$ of random integer partitions such that

$$\forall k \geq 1,\quad N^{-\frac{k}{2}}\,R_k(\lambda_N) \to_{\mathbb{P}} R_k(\omega)$$

for some Young curve ω, then the scaled Young diagrams $\mathrm{sc}_{\sqrt{N}}(\omega_{\lambda_N})$ converge in probability towards the limit shape ω. Now, the free cumulants of large integer partitions are asymptotically equivalent to the *renormalised character values*, as a consequence of the following important result (see [Bia98, Theorem 1.3], [IO02, Section 10] and [Mél17, Theorem 9.20]):

Theorem 2.2 (Biane, Ivanov–Olshanski). *For $\lambda \in \mathfrak{Y}(N)$ and $k \geq 1$, we define*

$$\varSigma_k(\lambda) = \begin{cases} N^{\downarrow k}\, \frac{\mathrm{tr}\, \phi^\lambda(c_k)}{\dim \lambda} & \text{if } N \geq k, \\ 0 & \text{if } N < k. \end{cases},$$

where $N^{\downarrow k} = N(N-1)\cdots(N-k+1)$, c_k is a k-cycle and ϕ^λ is the defining morphism of the irreducible representation V^λ of $\mathfrak{S}(N)$. This renormalised character value can be expanded as a polynomial in the free cumulants of λ. Moreover, if we define a gradation $\deg R_k = k$, then

$$\varSigma_k = R_{k+1} + \text{polynomial in } R_2, R_3, \ldots, R_k \text{ of degree smaller than } k.$$

As a consequence, the criterion of convergence after scaling for a sequence $(\lambda_N)_{N\in\mathbb{N}}$ of random integer partitions becomes:

$$\forall k \geq 1,\;\; N^{-\frac{k+1}{2}}\, \varSigma_k(\lambda_N) \to_{\mathbb{P}} R_{k+1}(\omega).$$

A method of moments can be used in order to prove this convergence in probability of the scaled character values. Actually, in many cases, one can even prove Gaussian fluctuations for the scaled character values; see in particular [IO02; Śni06a; Śni06b]. Notice that when λ_N is chosen according to a spectral measure $\mathbb{P}_{V_N}$, the expectation of $\varSigma_k(\lambda_N)$ is given by:

$$\mathbb{E}_{V_N}[\varSigma_k] = \frac{N^{\downarrow k}}{\dim V_N} \sum_{\lambda \in \mathfrak{Y}(N)} m_\lambda^{\lambda_N}\, \mathrm{tr}\, \phi^\lambda(c_k) = N^{\downarrow k} \frac{\mathrm{tr}\, \phi^{V_N}(c_k)}{\dim V_N}.$$

Thus, its computation is immediate if one knows the characters of the representations V_N. For the computation of the higher moments (in particular the second one), one can use a similar technique with observables in the algebra spanned by the functions $\varSigma_k$; see [Mél17, Chapter 7] for a detailed presentation of this algebra, whose construction is due to Kerov and Olshanski [KO94].

Example 2.3 (Logan–Shepp–Kerov–Vershik law of large numbers). With this technology, the proof of the law of large numbers for Plancherel measures is almost immediate. Indeed, if $\lambda_N \sim \mathrm{PL}_N$, then $\mathbb{E}[\varSigma_k(\lambda_N)] = N\, 1_{k=1}$, so we have the convergence of the expectations

$$N^{-\frac{k+1}{2}}\, \mathbb{E}[\varSigma_k(\lambda_N)] \to_{N\to\infty} \begin{cases} 1 & \text{if } k = 1, \\ 0 & \text{if } k \geq 2. \end{cases}$$

By looking at the second moments, one can prove that the convergence of the expectations is also a convergence in probability. Therefore, the scaled random diagrams $\mathrm{sc}_{\sqrt{N}}(\lambda_N)$ converge in probability towards the unique Young curve Ω whose free cumulants are:

$$R_k(\Omega) = \begin{cases} 1 & \text{if } k = 2, \\ 0 & \text{otherwise.} \end{cases}$$

This is the LSKV curve, since $G_\Omega(z) = \frac{2}{z+\sqrt{z^2-4}}$ and $K_\Omega(w) = \frac{1}{w} + w$.

3. Asymptotics of the random compressed partitions

In the sequel, we fix a sequence $(\lambda_N)_{N\in\mathbb{N}}$ of integer partitions which can be random and which have roughly size N (for instance, we can assume that $\frac{|\lambda_N|}{N}$ converges in probability to 1). We say that $(\lambda_N)_{N\in\mathbb{N}}$ has *limit shape* $\omega \in \mathscr{Y}$ if we have the convergence in probability

$$\mathrm{sc}_{\sqrt{N}}(\omega_{\lambda_N}) \to_{\mathbb{P}} \omega,$$

the topology on $\mathscr{Y}$ being as before the topology of uniform convergence. By the previous discussion, this is equivalent to the convergence in probability of the scaled random free cumulants, or of the scaled character values. The limit shape satisfies $R_2(\omega) = \int_{\mathbb{R}} \frac{\omega(s)-|s|}{2}\, ds = 1$. We fix such a limit shape, and we are interested in the behavior of $\mathrm{RC}_t(\lambda_N)$, and in the existence of a limit shape ω_t for it.

3.1. Free compression of a probability measure and the law of large numbers.

Given a compactly supported probability measure μ on $\mathbb{R}$, its R-transform is the function $R_\mu(w) = K_\mu(w) - \frac{1}{w} = \sum_{k=1}^{\infty} R_k(\mu)\, w^{k-1}$; its coefficients are the free cumulants of μ. The *free compression* of μ with parameter $t \in (0,1)$ is the unique (compactly supported) probability measure $\pi_t(\mu)$ such that:

$$R_{\pi_t(\mu)}(w) = R_\mu(tw).$$

Equivalently, $R_k(\pi_t(\mu)) = t^{k-1}\, R_k(\mu)$ for any $k \geq 1$. The t-free compression of a Young curve ω is the Young curve $\pi_t(\omega)$ such that $\mu_{\pi_t(\omega)} = \pi_t(\mu_\omega)$. This modifies the area $R_2(\cdot)$ by a factor t, so if ω is normalised to have area 1, then $\pi_t(\omega)$ has area t.

Theorem 3.1 (Biane). *Let $(\lambda_N)_{N\in\mathbb{N}}$ be a sequence of random partitions which has a limit shape $\omega \in \mathscr{Y}$. For any $t \in (0,1)$, $(\mathrm{RC}_t(\lambda_N))_{N\in\mathbb{N}}$ has limit shape $\pi_t(\omega)$:*

$$\mathrm{sc}_{\sqrt{N}}(\omega_{\mathrm{RC}_t(\lambda_N)}) \to_{\mathbb{P}} \pi_t(\omega).$$

The same result holds for the Poisson random compression:

$$\mathrm{sc}_{\sqrt{N}}(\omega_{\mathrm{PRC}_t(\lambda_N)}) \to_{\mathbb{P}} \pi_t(\omega).$$

The proof of the theorem is almost trivial from the previous discussion if instead of the convergence in probability $N^{-\frac{k}{2}}\, R_k(\lambda_N) \to R_k(\omega)$, we assume a stronger convergence in joint moments (this hypothesis is natural in the setting of random partitions stemming from spectral measures of representations). Indeed, let us extend the definition of the renormalised character values by allowing products of disjoint cycles: for $\lambda \in \mathfrak{Y}(N)$ and $\rho \in \mathfrak{Y}(k)$, we set

$$\varSigma_\rho(\lambda) = N^{\downarrow k}\, \frac{\mathrm{tr}\, \phi^\lambda(c_\rho)}{\dim \lambda},$$

where c_ρ is a permutation with cycle type ρ, and the falling factorial vanishes if $N < k$. Given a parameter t such that $\lfloor tN \rfloor \geq k$, we can assume that $c_\rho \in \mathfrak{S}(\lfloor tN \rfloor)$, and then,

$$\begin{aligned}
\mathbb{E}\,[\varSigma_\rho(\mathrm{RC}_t(\lambda_N))] &= \sum_{\substack{\lambda\in\mathfrak{Y}(N)\\ \nu\in\mathfrak{Y}(\lfloor tN\rfloor)}} \mathbb{P}[\lambda_N = \lambda,\ \mathrm{RC}_t(\lambda) = \nu]\, (\lfloor tN \rfloor)^{\downarrow k}\, \frac{\mathrm{tr}\, \phi^\nu(c_\rho)}{\dim \nu} \\
&= (\lfloor tN \rfloor)^{\downarrow k} \sum_{\lambda\in\mathfrak{Y}(N)} \mathbb{P}[\lambda_N = \lambda]\, \frac{\mathrm{tr}\, \phi^\lambda(c_\rho)}{\dim \lambda} \\
&= t^k\, \left(1 + O(N^{-1})\right)\, \mathbb{E}[\varSigma_\rho(\lambda_N)].
\end{aligned}$$

Here, we have assumed that λ_N is exactly of size N, but this is not really important. Now, $\varSigma_\rho = \prod_{i=1}^{\ell(\rho)} R_{\rho_i+1} +$ remainder, where the remainder is a polynomial of degree smaller than $\rho + \ell(\rho) - 1$

in the free cumulants. As a consequence, for any family $(k_1, \ldots, k_\ell)$ of positive integers, we have

$$\mathbb{E}\left[\prod_{i=1}^{\ell} R_{k_i}(\mathrm{RC}_t(\lambda_N))\right] = \mathbb{E}\left[\prod_{i=1}^{\ell} t^{k_i-1}\, R_{k_i}(\lambda_N)\right] \left(1 + O(N^{-\frac{1}{2}})\right).$$

By assumption, the right-hand side is asymptotic as N goes to infinity to $N^{\sum_{i=1}^{\ell} \frac{k_i}{2}} \prod_{i=1}^{\ell} R_{k_i}(\pi_t(\omega))$; this ends the proof.

Example 3.2 (Large random partitions under Schur–Weyl measures). For $c \in \mathbb{R}_+^*$, let us introduce the *Marcenko–Pastur distribution*

$$\mu_{\mathrm{MP},c}(ds) = \begin{cases} \frac{\sqrt{4-(s-c)^2}}{2\pi(1+sc)}\, 1_{s\in[c-2,c+2]}\, ds & \text{if } c \in [0,1], \\ \frac{\sqrt{4-(s-c)^2}}{2\pi(1+sc)}\, 1_{s\in[c-2,c+2]}\, ds + \left(1 - \frac{1}{c^2}\right) \delta_{-\frac{1}{c}}(ds) & \text{if } c > 1. \end{cases}$$

Notice that if $c = 0$, then one recovers the Wigner semicircle distribution. We refer to [Mél17, Figure 13.4] for a representation of the densities of the Marcenko–Pastur distributions. The Cauchy transforms of these probability measures are:

$$G_{\mu_{\mathrm{MP},c}}(z) = \frac{2}{z + c + \sqrt{(z-c)^2 - 4}},$$

and the inverses of these functions are

$$K_{\mu_{\mathrm{MP},c}}(w) = \frac{1}{w} + \frac{w}{1 - cw} = \frac{1}{w} + \sum_{k=2}^{\infty} c^{k-2}\, w^{k-1}.$$

Therefore, $R_k(\mu_{\mathrm{MP},c}) = c^{k-2}\, 1_{k\geq 2}$, and the Marcenko–Pastur distributions behave well with respect to free compression:

$$\frac{R_k(\pi_t(\mu_{\mathrm{MP},c}))}{t^{\frac{k}{2}}} = t^{\frac{k}{2}-1}\, c^{k-2}\, 1_{k\geq 2} = \left(\sqrt{t}\, c\right)^{k-2} 1_{k\geq 2} = R_k(\mu_{\mathrm{MP},\sqrt{t}\,c});$$
$$(\pi_t(\mu_{\mathrm{MP},c}))\left(\sqrt{t}\, B\right) = \mu_{\mathrm{MP},\sqrt{t}\,c}(B).$$

for any Borel subset $B \subset \mathbb{R}$. Consider now a random integer partition λ_N under the Schur–Weyl measure $\mathrm{SW}_{\lfloor c^{-1}\sqrt{N}\rfloor,N}$. For any integer partition ρ, one computes

$$\mathbb{E}[\Sigma_\rho(\lambda_N)] = N^{\downarrow|\rho|} \left(\lfloor c^{-1}\sqrt{N}\rfloor\right)^{\ell(\rho)-|\rho|} \simeq c^{|\rho|-\ell(\rho)}\, N^{\frac{|\rho|+\ell(\rho)}{2}},$$

so $R_k(\mathrm{sc}_{\sqrt{N}}(\omega_{\lambda_N})) \to_{\mathbb{P}} c^{k-2}$ for any k. Hence, λ_N has a limit shape Ω_c, which is the Young curve with transition measure $\mu_{\mathrm{MP},c}$. An explicit formula for Ω_c is given in [Bia01], see also [Mél17, Figure 13.3]. In this framework, the stability of the Marcenko–Pastur distributions with respect to free compression is the asymptotic counterpart of the stability of the Schur–Weyl measures with respect to random compression. More generally, given any sequence of random integer partitions such that

$$\mathrm{sc}_{\sqrt{N}}(\omega_{\lambda_N}) \to \Omega_c,$$

we have

$$\mathrm{sc}_{\sqrt{tN}}(\omega_{\mathrm{RC}_t(\lambda_N)}) \to \Omega_{\sqrt{t}c}$$

for any parameter $t \in (0,1)$.

3.2. **Random point process associated to a random partition.** The result of the previous paragraph has a *global* nature, since it concerns limit shapes of random integer partitions. Indeed, when looking at the scaled shape of a partition λ_N of size N, we forget everything that might happen in a region of the boundary ω_{λ_N} of size $o(\sqrt{N})$. This makes one wonder whether it is also possible to obtain some *local* information on the random partitions $\mathrm{RC}_t(\lambda_N)$, either in the bulk of the limit shape or at the edge. By local we mean in a region of size $o(\sqrt{N})$, or even $O(1)$. The right way to think about this is to introduce the *descent coordinates* of a partition, and to study the corresponding random point processes. Set $\mathbb{Z}' = \mathbb{Z} + \frac{1}{2}$, and for $\lambda \in \mathfrak{Y}$, let us define

$$M_\lambda = \left\{\lambda_i - i + \frac{1}{2}\right\},$$

which is an infinite configuration of points in $\mathbb{Z}'$ with the property that

$$\mathrm{card}(\mathbb{Z}'_- \setminus M_\lambda) = \mathrm{card}(\mathbb{Z}'_+ \cap M_\lambda) < +\infty.$$

The configuration M_λ is called the set of descent coordinates of the partition λ, because it is obtained by looking at the Young diagram of λ drawn with the Russian convention, and by projecting on the x-axis the middles of the decreasing segments of the boundary ω_λ; see Figure 6 for an example.

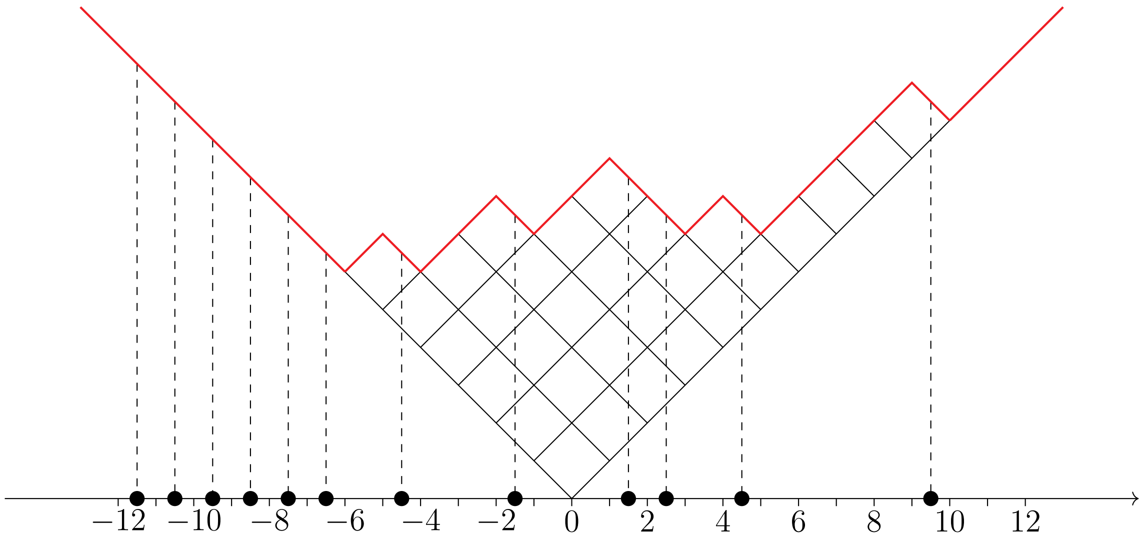

FIGURE 6. The configuration $M_\lambda \subset \mathbb{Z}'$ associated to the integer partition $\lambda = (10, 6, 5, 5, 3, 1)$.

If λ is random, then we can introduce the correlation functions

$$\rho(X) = \mathbb{P}[X \subset M_\lambda]$$

for X (finite) subset of $\mathbb{Z}'$. In certain situations, these correlation functions happen to be given by determinants: there exists a (Hermitian) kernel $K : (\mathbb{Z}')^2 \to \mathbb{C}$ such that

$$\rho(X) = \det(K(x, y))_{x,y \in X}$$

for any X finite subset of $\mathbb{Z}'$. We then say that M_λ is a *determinantal point process* with kernel K; in the sequel, we identify the configuration M_λ with the discrete measure $\sum_{x \in M_\lambda} \delta_x$. In the setting of random partitions, the following result ensures that a large class of models of random partitions yield determinantal point processes; see [Oko01; Oko02].

Theorem 3.3 (Okounkov). *Let X and Y be two specialisations of the algebra of symmetric functions* Sym *which are non-negative on the basis of Schur functions. The Schur measure on $\mathfrak{Y}$ with parameters X and Y is given by*

$$\mathbb{P}_{X,Y}[\lambda] = \exp\left(-\sum_{k=1}^{\infty} \frac{p_k(X)\, p_k(Y)}{k}\right) s_\lambda(X)\, s_\lambda(Y).$$

If λ is chosen randomly according to $\mathbb{P}_{X,Y}$, then M_λ is determinantal, and its kernel is given by the following generating series:

$$\mathscr{K}_{X,Y}(z,w) = \sum_{x,y \in \mathbb{Z}'} K(x,y)\, z^x\, w^{-y} = \frac{\sqrt{zw}}{z-w}\, \frac{J_{X,Y}(z)}{J_{X,Y}(w)},$$

where

$$J_{X,Y}(z) = \exp\left(\sum_{k=1}^{\infty} \frac{p_k(X)}{k}\, z^k - \sum_{k=1}^{\infty} \frac{p_k(Y)}{k}\, z^{-k}\right).$$

The formula for $\mathscr{K}_{X,Y}(z,w)$ provides a representation of the kernel $K(x,y)$ as a double contour integral:

$$K(x,y) = \frac{1}{(2\mathrm{i}\pi)^2} \oint\!\!\oint_{|z|>|w|} \frac{1}{z-w}\, \frac{J_{X,Y}(z)}{J_{X,Y}(w)}\, \frac{dz\, dw}{\sqrt{zw}},$$

and the asymptotics of the kernel in the case where $x = x_N$, $y = y_N$, $X = X_N$ and $Y = Y_N$ can usually be obtained by saddle point analysis.

Example 3.4 (Local asymptotics of the Plancherel measures). Consider the Poissonised Plancherel measure $\mathrm{PL}_{\{\theta\}} = \sum_{N=0}^{\infty} \frac{\mathrm{e}^{-\theta}\,\theta^N}{N!}\, \mathrm{PL}_N$, where θ is a positive real number. The exponential specialisation E of the algebra Sym is defined by

$$p_1(E) = 1 \qquad ; \qquad p_{k\geq 2}(E) = 0.$$

The Frobenius–Schur change of basis formula between Schur functions and power sums shows then that

$$s_\lambda(E) = \frac{\dim \lambda}{|\lambda|!}$$

for any integer partition λ. It follows that $\mathrm{PL}_{\{\theta\}}$ can be identified as the Schur measure with parameters $X = Y = \sqrt{\theta}\, E$. In this case,

$$J_{X,Y}(z) = \exp\left(\sqrt{\theta}\,(z - z^{-1})\right) = \sum_{n\in\mathbb{Z}} J_n(2\sqrt{\theta})\, z^n$$

is the generating series of the Bessel functions $J_n(2\sqrt{\theta})$, $n \in \mathbb{Z}$. We therefore have:

$$K_{\{\theta\}}(x,y) = \frac{1}{(2\mathrm{i}\pi)^2} \oint\!\!\oint_{|z|>|w|} \frac{1}{z-w}\, \exp\left(\sqrt{\theta}(z - z^{-1} - w + w^{-1})\right)\, z^{-x-\frac{1}{2}}\, w^{y-\frac{1}{2}}\, dz\, dw.$$

This expression can be used in order to recover the Logan–Shepp–Kerov–Vershik law of large numbers, and to determine the local asymptotics of the random partitions under the Poissonised Plancherel measures. First, as θ goes to infinity, we have the following limiting result when

$$\frac{x_\theta}{\sqrt{\theta}} \to s_0 \in (-2,2) \qquad ; \qquad \frac{y_\theta}{\sqrt{\theta}} \to s_0 \in (-2,2) \qquad ; \qquad x_\theta - y_\theta = x - y \in \mathbb{Z}.$$

The kernel $K_{\{\theta\}}(x_\theta, y_\theta)$ converges then towards the discrete sine kernel:

$$K_{\{\theta\}}(x_\theta, y_\theta) \to_{\theta\to\infty} K^{\mathrm{dsine},\phi_0}(x,y) = \frac{\sin \phi_0(x-y)}{\pi(x-y)},$$

where $\phi_0 = \arccos(\frac{s_0}{2})$; see [BOO00, Theorem 3] and [Oko02, Section 3]. Therefore, the discrete determinantal point process

$$M_{\lambda,\{\theta\},s_0}(B \in \mathbb{Z}') = M_{\lambda,\mathrm{PL}_{\{\theta\}}}(\lfloor s_0\sqrt{\theta}\rfloor + B)$$

converges towards a translation-invariant determinantal point process on $\mathbb{Z}'$. An analogue result of convergence of determinantal point processes can be established at the edge of the region $(-2\sqrt{\theta}, 2\sqrt{\theta})$, after an adequate renormalisation: one obtains the Airy determinantal point process, see [BOO00, Theorem 4]. On the other hand, the local convergence of the point process of the descent coordinates allows one to recover the Logan–Shepp–Kerov–Vershik law of large numbers. Indeed, by taking $x_\theta = y_\theta$, one sees that the density of descending segments around $s_0\sqrt{\theta}$ tends to $\frac{\phi_0}{\pi} = \frac{1}{\pi}\arccos\left(\frac{s_0}{2}\right)$. Therefore, the limiting derivative of $\mathrm{sc}_{\sqrt{\theta}}(\omega_\lambda)$ at $s = s_0 \in (-2,2)$ is

$$1 - \frac{2}{\pi}\arccos\left(\frac{s}{2}\right) = \frac{2}{\pi}\arcsin\left(\frac{s}{2}\right).$$

This is precisely the derivative at s of the LSKV curve Ω.

Example 3.5 (Local asymptotics of the Schur–Weyl measures). Consider similarly the Poissonised Schur–Weyl measure

$$\mathrm{SW}_{c,\{\theta\}} = \sum_{N=0}^{\infty} \frac{\mathrm{e}^{-\theta}\,\theta^N}{N!}\,\mathrm{SW}_{\lfloor c^{-1}\sqrt{\theta}\rfloor,N},$$

where $c \in \mathbb{R}_+^*$ is a fixed parameter. In the following, we denote $\lfloor c^{-1}\sqrt{\theta}\rfloor = (c_\theta)^{-1}\sqrt{\theta}$; as θ goes to infinity, c_θ converges to c. For $\lambda \in \mathfrak{Y}(N)$, we have

$$\frac{\mathrm{SW}_{(c_\theta)^{-1}\sqrt{\theta},N}[\lambda]}{N!} = \frac{\dim\lambda}{N!}\,\frac{s_\lambda(1^{(c_\theta)^{-1}\sqrt{\theta}})}{((c_\theta)^{-1}\sqrt{\theta})^N} = \frac{s_\lambda(E)\,s_\lambda(1^{(c_\theta)^{-1}\sqrt{\theta}})}{((c_\theta)^{-1}\sqrt{\theta})^N},$$

where 1^M denotes the alphabet $(1_1, 1_2, \ldots, 1_M, 0, 0, \ldots)$. Therefore,

$$\mathrm{SW}_{c,\{\theta\}}[\lambda] = \mathrm{e}^{-\theta}\,s_\lambda(\sqrt{\theta}\,E)\,s_\lambda((c_\theta)^{(c_\theta)^{-1}\sqrt{\theta}})$$

is the Schur measure with parameters $X = \sqrt{\theta}\,E$ and $Y = (c_\theta)^{(c_\theta)^{-1}\sqrt{\theta}}$. The associated function $J_{X,Y}$ is given by $J_{X,Y}(z) = \exp(\sqrt{\theta}(z + (c_\theta)^{-1}\log(1 - c_\theta z^{-1})))$. Note that we recover the Plancherel case by making c_θ go to 0. As before, we obtain a double contour integral:

$$K_{c,\{\theta\}}(x,y)$$
$$= \frac{1}{(2\mathrm{i}\pi)^2}\oint\!\!\oint_{|z|>|w|>c_\theta} \exp\left(\sqrt{\theta}\left(z + \frac{1}{c_\theta}\log\left(1 - \frac{c_\theta}{z}\right) - w - \frac{1}{c_\theta}\log\left(1 - \frac{c_\theta}{w}\right)\right)\right)\frac{z^{-x-\frac{1}{2}}\,w^{y-\frac{1}{2}}}{z-w}\,dz\,dw.$$

In the sequel, we drop the index θ from c_θ in order to simplify the notations. Consider parameters $x_\theta, y_\theta \in \mathbb{Z}'$ such that

$$\frac{x_\theta}{\sqrt{\theta}} \to s_0 \in (c-2, c+2) \quad ; \quad \frac{y_\theta}{\sqrt{\theta}} \to s_0 \in (c-2, c+2) \quad ; \quad x_\theta - y_\theta = x - y \in \mathbb{Z}.$$

We have

$$K_{c,\{\theta\}}(x_\theta, y_\theta) = \frac{1}{(2\mathrm{i}\pi)^2}\oint\!\!\oint \frac{1}{(z-w)\sqrt{zw}}\exp\left(\sqrt{\theta}\,F\left(z, \frac{x_\theta}{\sqrt{\theta}}\right) - \sqrt{\theta}\,F\left(w, \frac{y_\theta}{\sqrt{\theta}}\right)\right)dz\,dw,$$

where $F(z,t) = z + c^{-1}\log(1 - cz^{-1}) - t\,\log z$. The two critical poins of $F(\cdot, s_0)$ are

$$z = c + \mathrm{e}^{\pm\mathrm{i}\phi_0}, \quad \text{with } \phi_0 = \arccos\left(\frac{t_0 - c}{2}\right).$$

By deforming the paths of integration exactly as in [Oko02, Section 3.2] and picking up the residues, one obtains:

$$\lim_{\theta\to\infty} K_{c,\{\theta\}}(x_\theta, y_\theta) = \frac{1}{2\mathrm{i}\pi}\oint_{\gamma: c+\mathrm{e}^{-\mathrm{i}\phi_0} \to c+\mathrm{e}^{\mathrm{i}\phi_0}} \frac{1}{z^{x-y+1}}\,dz,$$

where the path of integration γ if the arc of circle with center c, radius 1 and connecting the two critical points. We can deform this path and take instead the arc of circle with center 0 and radius $1 + cs_0$; the integral is then easy to compute, hence,

$$K_{\{\theta\}}(x_\theta, y_\theta) \to_{\theta\to\infty} K^{\mathrm{dsine},\psi_0}(x,y)$$

where $\psi_0 = \arccos(\frac{c+s_0}{2\sqrt{1+cs_0}})$. By the same arguments as for the Plancherel measure, we see that the limiting derivative of $\mathrm{sc}_{\sqrt{\theta}}(\omega_\lambda)$ at $s \in (c-2, c+2)$ is

$$\frac{2}{\pi} \arcsin\left(\frac{c+s}{2\sqrt{1+cs}}\right),$$

which is the derivative of Ω_c; hence, one recovers the law of large numbers (at least in the interval $(c-2, c+2)$).

3.3. **Random point process associated to a random tableau.** If λ is an integer partition and $T \in \mathrm{ST}(\lambda)$, then by looking at the associated sequence of integer partitions ($\emptyset = \lambda^{(0)} \nearrow \lambda^{(1)} \nearrow \cdots \nearrow \lambda^{(N)} = \lambda$), one obtains a family of non-intersecting paths which connects the "empty" configuration $M_\emptyset = \mathbb{Z}'_-$ to the configuration of descents M_λ. Each insertion of a cell corresponds to a move of one of the paths to the right; see Figure 7.

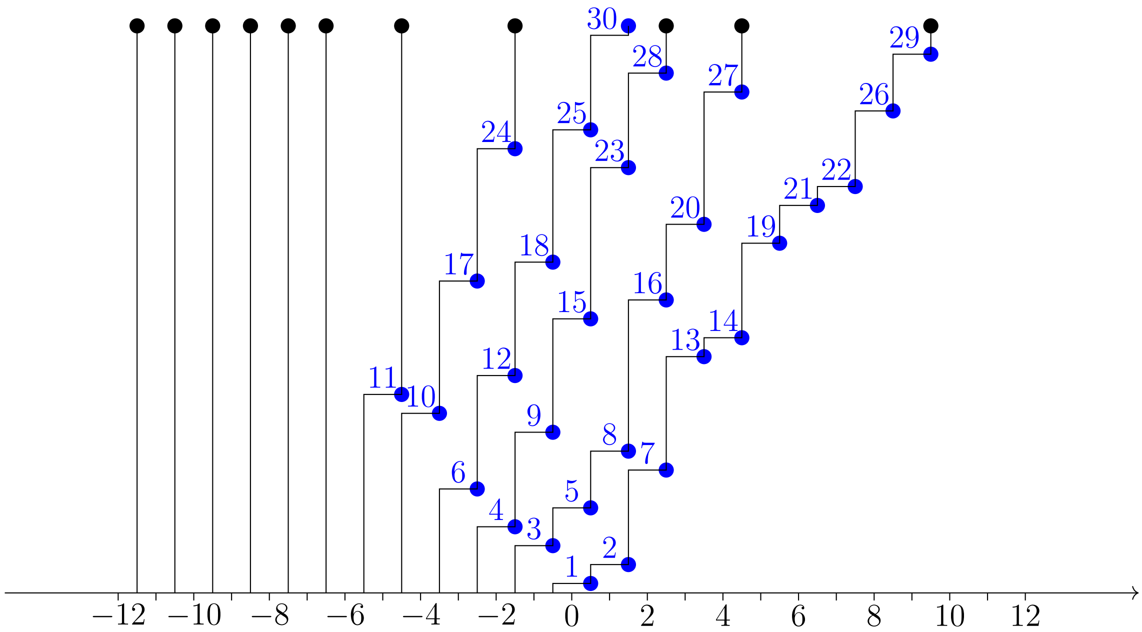

FIGURE 7. The set of non-intersecting paths associated to the standard tableau from Figure 2; it is encoded by the configuration of points M_T (in blue).

We encode this set of intersecting paths by the set $M_T \subset (\mathbb{Z}' \times \mathbb{Z}_+)$ of the coordinates of the right moves; the first coordinates is the location of the right move, whereas the second coordinate is its time. For a tableau of size N, we obtain a finite configuration of N points. We can extend the definition to increasing sequences of partitions

$$\emptyset = \lambda^0 \nearrow \lambda^{t_1} \nearrow \lambda^{t_2} \nearrow \cdots \nearrow \lambda^{t_N} = \lambda,$$

where the times $t_1 < t_2 < \cdots < t_N$ are in $\mathbb{R}_+$; the configuration M_T is then a finite subset of $\mathbb{Z}' \times \mathbb{R}_+$. In this setting, if $t \in \mathbb{R}_+$, we denote $\lambda_t = \lambda^{t_i}$ and $T_t = (\lambda^0 \nearrow \cdots \nearrow \lambda^{t_i})$, where $t_i \leq t < t_{i+1}$. The *Poisson standard tableau* with shape λ is obtained by picking N independent points in $[0, 1]$, by reordering them so as to obtain the sequence of times $t_1 < \cdots < t_N$, and by choosing the random tableau T uniformly in $\mathrm{ST}(\lambda)$. We then denote $T = \mathrm{PST}(\lambda)$. By definition,

$$\mathrm{shape}((\mathrm{PST}(\lambda))_t) = \mathrm{PRC}_t(\lambda)$$

for any parameter $t \in (0, 1)$. The following result ensures that if T is a Poisson standard tableau, then M_T is determinantal on $\mathbb{Z}' \times (0, 1)$ (see [GR17, Theorem 1.5]):

Theorem 3.6 (Gorin–Rahman). *Fix $\lambda \in \mathfrak{Y}$, and denote M_T the point process on $\mathbb{Z}' \times [0,1]$ associated to $T = \mathrm{PST}(\lambda)$. It is determinantal with kernel*

$$K_\lambda((x,t_1),(y,t_2)) = 1_{x>y,\, t_1<t_2}\, \frac{(t_1-t_2)^{x-y-1}}{(x-y-1)!}$$
$$+ \frac{1}{(2\mathrm{i}\pi)^2} \oint\!\!\oint \frac{H_\lambda(y+z)}{H_\lambda(x-1-w)}\, \frac{\left(\frac{\Gamma(y+z+\frac{1}{2})}{\Gamma(z+1)}\right)}{\left(\frac{\Gamma(x-w-\frac{1}{2})}{\Gamma(-w)}\right)}\, \frac{(1-t_2)^z\,(1-t_1)^w}{z+w+y-x+1}\, dz\, dw,$$

where H_λ is the Frobenius generating series of the diagram λ [Mél17, p. 343], defined by

$$H_\lambda(u) = \prod_{i=1}^{\infty} \frac{u+i-\frac{1}{2}}{u-\lambda_i+i-\frac{1}{2}} = \frac{\prod_{t\in M_\emptyset} u-t}{\prod_{t \in M_\lambda} u-t},$$

and the double contour integral runs over two paths γ_z and γ_w which are drawn in Figure 8. Here, the reference measure is the tensor product of the counting measure on $\mathbb{Z}'$ by the Lebesgue measure on $[0,1]$.

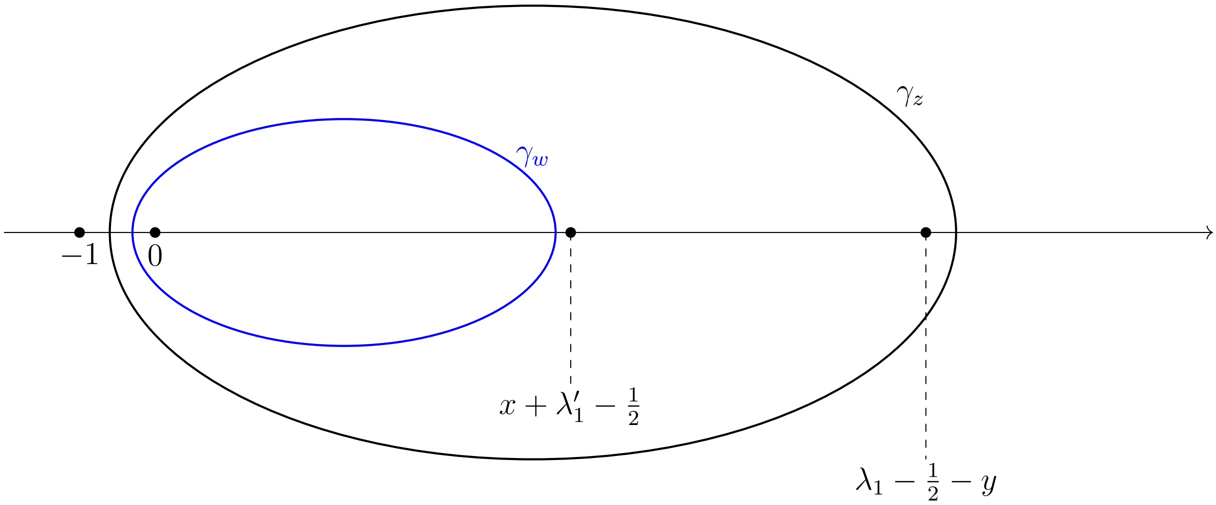

FIGURE 8. The contours of integration for the kernel K_λ (they enclose only the integers in $[0, x+\lambda_1'-\frac{3}{2}]$ and in $[0, \lambda_1-\frac{1}{2}-y]$, and they do not cross).

Remark 3.7. The generating function H_λ is related to the generating function G_λ by:

$$G_\lambda(z) = \frac{1}{z}\, \frac{H_\lambda(z-\frac{1}{2})}{H_\lambda(z+\frac{1}{2})}.$$

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