# 6. The Frobenius-Schur isomorphism

We defined previously the spectral measure of a representation $V = \bigoplus_{\lambda \in \hat{G}} m_\lambda V^\lambda$ of a finite group $G$.

$$\mathbb{P}_V[\lambda] = \frac{m_\lambda \dim V^\lambda}{\dim V}.$$

<u>observation</u>: under $\mathbb{P}_V$, there are a lot of random variables with known expectation.

<u>Definition</u>: The character of $V$ is $ch^V(g) = \mathrm{tr}\, \rho(g)$.
The normalised character is $\chi^V(g) = \frac{ch^V(g)}{ch^V(e_G)} = \frac{1}{\dim V} ch^V(g)$.

The irreducible characters of $G$ $ch^\lambda$ form an orthonormal basis of $Z(\mathbb{C}G)$.
Under the spectral measure $\mathbb{P}_V$,

$$\mathbb{E}_V[\chi^\lambda(g)] = \sum \frac{m_\lambda ch^\lambda(g)}{\dim V} = \frac{ch^V(g)}{\dim V} = \chi^V(g).$$

<u>idea</u>: use these relations to have observables of random partitions $\lambda \sim \mathbb{P}_n$ with known expectations / moments.

<u>prerequisite</u>: have a good understanding of the bijection $\widehat{\mathfrak{S}(n)} \longleftrightarrow \mathcal{Y}(n)$ and of the irreducible characters of $\mathfrak{S}(n)$.

## 1. The Grothendieck ring of representations

- Given a finite group $G$, its Grothendieck group of representations is:

$$R(G) = \bigoplus_{\lambda \in \hat{G}} \mathbb{R} V^\lambda$$

$\rightsquigarrow$ formal linear combinations of irreducible representations, including the true representations $V = \sum_{\lambda \in \hat{G}} m_\lambda V^\lambda$.

We endow $R(G)$ with the scalar product $\langle V^\lambda | V^\mu \rangle = \mathbb{1}_{(\lambda = \mu)}$.

There is an isometry

$$\mathbb{C} \otimes_{\mathbb{R}} R(G) \to Z(\mathbb{C}G)$$
$$V^\lambda \mapsto ch^\lambda$$

If all the (irreducible) characters of $G$ take real values, then this isometry comes from the isometry
$$\begin{aligned} R(G) &\longrightarrow Z(\mathbb{R}G) \\ V^\lambda &\longmapsto ch^\lambda. \end{aligned}$$

<u>Lemma</u>: This works for $G = \mathfrak{S}(n)$.

<u>Proof</u>: $\forall g, \exists \underbrace{n}_{\text{order of } g} \geq 1 \;|\; g^n = 1 \implies \rho^\lambda(g)$ is a unitary matrix whose eigenvalues are $n$-th roots of unity.

So, $ch^\lambda(g)$ is a sum of $n$-th roots of unity. It is an algebraic integer in $\mathcal{O}_k$, $k = \mathbb{Q}[e^{\frac{2i\pi}{n}}]$.

$Gal(k, \mathbb{Q}) = (\mathbb{Z}/n\mathbb{Z})^*$ : if $m \wedge n = 1$, $m$ acts on $k$ by sending $\zeta$ to $\zeta^m$.

However, $g$ and $g^m$ are conjugated in $\mathfrak{S}(n)$, so $ch^\lambda(g) = ch^\lambda(g^m)$
$$= m \cdot ch^\lambda(g).$$

Therefore, $ch^\lambda(g)$ is an algebraic integer in $k^{Gal(k,\mathbb{Q})} = \mathbb{Q}$, hence an integer $\square$.

<u>Definition</u>: The Grothendieck ring of representations of the symmetric groups is
$$R = \bigoplus_{n \in \mathbb{N}} R(\mathfrak{S}(n)).$$

structure of graded ring? we need to define a product
$$R(\mathfrak{S}(n)) \otimes R(\mathfrak{S}(m)) \longrightarrow R(\mathfrak{S}(n+m)).$$

- induction and restriction of representations.

given $H \subset G$, there is an obvious restriction map $Res^G_H : R(G) \to R(H)$
$$(V, \rho) \mapsto (V, \rho_{|H}).$$

The converse construction is a bit more complicated. It relies on the identification
$$\{\text{representations of a finite group } G\} \iff \{\text{modules over the algebra } \mathbb{C}G\}.$$

Indeed, given $(V, \rho)$, $\mathbb{C}G$ acts on $V$ by: $\left(\sum c_g g\right) \cdot v = \sum c_g \rho(g)(v)$.
If $H \subset G$, then $\mathbb{C}H$ is a subalgebra of $\mathbb{C}G$, which allows one to use tensor products to extend the base ring:

$$\mathrm{Ind}_H^G(V) = \mathbb{C}G \underset{\mathbb{C}H}{\otimes} V.$$

Notice that $\mathrm{Res}_H^G$ does not change the dimension, but $\dim \mathrm{Ind}_H^G(V) = \dfrac{|G|}{|H|} \dim V$.

Formula for induced characters:
fix a family of representatives $g_1 H, \dots, g_n H$ of $G/H$.
An orthonormal basis $e_1, \dots, e_d$ of $V$.

A linear basis of $\mathrm{Ind}_H^G(V)$ consists in the $g_i \otimes e_j$. We choose a scalar product which makes it orthonormal.

$$ch^{\mathrm{Ind}_H^G(V)}(g) = \sum_{i,j} \langle \underbrace{g g_i}_{= g_k h} \otimes e_j \mid g_i \otimes e_j \rangle$$

$= g_k h$ for a unique $k$, $h \in H$; the scalar product $= 0$ if $k \neq i$.

$$= \sum_{\substack{i : g_i^{-1} g g_i \in H \\ j}} \langle h e_j \mid e_j \rangle = \sum_{i : g_i^{-1} g g_i \in H} ch^V(g_i^{-1} g g_i).$$

For $V^\lambda \in \widehat{\mathfrak{S}(n)}$ and $V^\mu \in \widehat{\mathfrak{S}(m)}$, we define

$$V^\lambda \times V^\mu = \mathrm{Ind}_{\mathfrak{S}(n) \times \mathfrak{S}(m)}^{\mathfrak{S}(n+m)} (V^\lambda \otimes V^\mu) \in R(\mathfrak{S}(n+m)).$$

↓ representation of the product group

So, $R(\mathfrak{S})$ is a graded algebra with a scalar product ...

## 2. The Frobenius isomorphism.

Consider the characteristic map:

$$\Psi : Z(R\mathfrak{S}(n)) \longrightarrow \mathrm{Sym}.$$
$$C_\mu \longmapsto \frac{p_\mu}{z_\mu}.$$

It extends to a linear map $R(\mathfrak{S}) \longrightarrow \mathrm{Sym}$.

<u>Theorem</u>: This is an isometric isomorphism of (graded) algebras.

<u>Proof</u>: isometry: The $C_\mu$'s and the $p_\mu$'s are orthogonal bases

$$\|C_\mu\|^2 = \frac{1}{n!} \operatorname{card} C_\mu = \frac{1}{z_\mu}.$$

$$\left\|\frac{p_\mu}{z_\mu}\right\|^2 = \frac{\|p_\mu\|^2}{{z_\mu}^2} = \frac{z_\mu}{{z_\mu}^2} = \frac{1}{z_\mu}.$$

morphism of algebras: if $\lambda \in \widehat{\mathfrak{S}}(m)$ and $\rho \in \widehat{\mathfrak{S}}(n)$, then

$$ch^\lambda \times ch^\rho = \sum_{\sigma \in \mathfrak{S}(m+n)} \sum_{\substack{\tau_i^{-1}\sigma\tau_i \in \mathfrak{S}(m)\times\mathfrak{S}(n) \\ = (\sigma_1, \sigma_2)}} ch^\lambda(\sigma_1)\, ch^\rho(\sigma_2) \cdot \sigma$$

$\{\tau_i\}_{i\in I}$ representatives of the quotient

By decomposition over the basis of irreducible characters, we get similarly for $\mu \in \mathfrak{Y}(m)$ and $\nu \in \mathfrak{Y}(n)$:

$$C_\mu \times C_\nu = \sum_{\sigma \in \mathfrak{S}(m+n)} \sum_{\tau_i^{-1}\sigma\tau_i = (\sigma_1, \sigma_2)} 1_{\sigma_1 \in C_\mu} 1_{\sigma_2 \in C_\nu} \cdot \sigma$$

remark: any element that appears has cycle type $\mu \sqcup \nu$.

if $\sigma \in C_{\mu\sqcup\nu}$, then $\#\{i : \tau_i^{-1}\sigma\tau_i \in \mathfrak{S}(m)\times\mathfrak{S}(n)\}$

$$= \frac{1}{m!\,n!} \#\{g \in \mathfrak{S}(m+n) \mid g^{-1}\sigma g \in \mathfrak{S}(m)\times\mathfrak{S}(n)\}$$

$$= \underbrace{\frac{\operatorname{card} C_\mu}{m!}}_{\frac{1}{z_\mu}} \times \underbrace{\frac{\operatorname{card} C_\nu}{n!}}_{\frac{1}{z_\nu}} \times \underbrace{\operatorname{card} \text{Centralizer } \sigma}_{z_{\mu\sqcup\nu}}$$

$$C_\mu \times C_\nu = \frac{z_{\mu\sqcup\nu}}{z_\mu z_\nu} C_{\mu\sqcup\nu}.$$ $\Psi$ is a morphism of graded algebras. $\square$.

## 3. Frobenius formula for the characters of the symmetric group.

**Theorem:** There is a way to label the irreducible representations of $\mathfrak{S}(n)$ by integer partitions with size $n$ such that $\Psi(V^\lambda) = s_\lambda$. Moreover, $\dim V^\lambda = |ST(\lambda)|$.

**proof:** First, notice that $\Psi(V_{trivial}) = \sum_{\mu \vdash n} \Psi(C_\mu) = \sum_{\mu \vdash n} \frac{p_\mu}{z_\mu} = h_n = s_{(n)}$.

by the Jacobi-Trudy formula,

$$\Psi^{-1}(s_\lambda) = \Psi^{-1}(\det(h_{\lambda_i - i + j})) = \sum_{\sigma \in \mathfrak{S}(\ell)} \varepsilon(\sigma) \prod_{i:1 \to \ell}^{R(\lambda)} V_{trivial}(\mathfrak{S}_{\lambda_i - i + \sigma(i)}).$$

is a sum with coefficients in $\mathbb{Z}$ of representations of $\mathfrak{S}(n)$.

However, $\|s_\lambda\|^2 = 1$

$\|V_{irreducible}\|^2 = 1$ $\Longrightarrow$ $\Psi^{-1}(s_\lambda) = \pm_\lambda V^\lambda$ for some indexing.

Finally, $(p_1)^n = \sum_{\lambda \in \mathfrak{Y}(n)} |ST(\lambda)| \, s_\lambda$ and $\mathbb{C}\mathfrak{S}(n) = \mathrm{Ind}_1^{\mathfrak{S}(n)}(1) = \sum_{\lambda \in \widehat{\mathfrak{S}}(n)} (\dim V^\lambda) \, V^\lambda$,

so $\Psi^{-1}(s_\lambda) = + V^\lambda$ and $\dim V^\lambda = |ST(\lambda)|$. $\square$.

**Application:** $ch^\lambda(\sigma_\mu) = \frac{\langle ch^\lambda \mid C_\mu \rangle \, n!}{\mathrm{card}\, C_\mu} = \langle ch^\lambda \mid z_\mu C_\mu \rangle$

$= \langle s_\lambda \mid p_\mu \rangle$.

So $p_\mu = \sum_{|\lambda| = |\mu|} ch^\lambda(\mu) \, s_\lambda$ ; $s_\lambda = \sum_{|\mu| = |\lambda|} ch^\lambda(\mu) \frac{p_\mu}{z_\mu}$.

$\leadsto$ going back to the original problem.

We define the algebraic observables of partitions as follows:

$$\Sigma_\mu(\lambda) = \begin{cases} n^{\downarrow k} \, \chi^\lambda(\sigma_\mu) & \text{if } n = |\lambda| \geq |\mu| = k, \\ 0 & \text{otherwise.} \end{cases}$$

We shall see next time the following:

Theorem: 1) The $\Sigma_\mu$'s form an algebra of functions on $\mathbb{Y}$.

2) Actually, this algebra $\mathcal{P}$ is also $\mathcal{P} = \mathbb{C}[\tilde{p}_2, \tilde{p}_3, \dots \tilde{p}_k, \dots]$

3) $\mathcal{P}$ is graded by $\deg \Sigma_\mu = |\mu|$

and also by $\mathrm{wt}\, \Sigma_\mu = |\mu| + \ell(\mu)$.

4) We have $\mathrm{wt}(\tilde{p}_k) = k$.

5) We have

$$\tilde{p}_k = \sum_{\mu : |\mu| + \ell(\mu) = k} \frac{k^{\downarrow \ell(\mu)}}{\prod_{i \geq 1} (m_i(\mu))!} \Sigma_\mu + \text{terms with weight} \leq k.$$

preliminary: formulas for $\dim \lambda$ and $\Sigma_k(\lambda)$.

$$\dim \lambda = \langle s_\lambda | p_1^n \rangle = [a_{\lambda+e}] \left( p_1(x_1, \dots, x_n)^n \, a_e(x_1 \dots x_n) \right)$$

$e = (n-1, n-2, \dots, 0)$

$$= [x^{\lambda+e}] \left( \left(\sum_{i=1}^n x_i\right)^n \sum_{\sigma \in S(n)} \varepsilon(\sigma) x_1^{n-\sigma(1)} \cdots x_n^{n-\sigma(n)} \right)$$

$$= [x^{\lambda+e}] \sum_{\substack{\sigma \in S(n) \\ k_1 + \dots + k_n = n}} \frac{n! \, \varepsilon(\sigma)}{k_1! \dots k_n!} x_1^{n+k_1-\sigma(1)} \cdots x_n^{n+k_n-\sigma(n)}$$

$\mu = \lambda + e$

$k_i = \mu_i - n + \sigma(i)$

$$= \sum_{\sigma \in S(n)} \varepsilon(\sigma) \, n! \prod_{i=1}^n \frac{1}{\mu_i - n + \sigma(i)!}$$

$$= \frac{n!}{\prod_{i=1}^n \mu_i!} \det \left( \mu_i^{\downarrow n-j} \right)$$

$$= \frac{n!}{\prod_{i=1}^n \mu_i!} \det \left( \mu_i^{n-j} \right) = \frac{n!}{\prod_{i=1}^n \mu_i!} \Delta(\mu_1, \dots, \mu_n).$$

Similarly:

$$\mathrm{ch}^\lambda(c_k) = \langle s_\lambda \mid p_1^{n-k} p_k \rangle$$

$$= [x^{\lambda+e}]\left(\left(\sum_{i=1}^n x_i\right)^{n-k} \sum_{i=1}^n x_i^k \sum_{\sigma \in S(n)} \varepsilon(\sigma)\, x_1^{n-\sigma(1)} \cdots x_n^{n-\sigma(n)}\right)$$

$$= \sum_{i=1}^n [x^{\mu-ke_i}] \sum_{\substack{\sigma \\ k_1+\cdots+k_n=n-k}} \varepsilon(\sigma) \frac{n-k!}{k_1!\cdots k_n!}\, x_1^{n+k_1-\sigma(1)} \cdots x_n^{n+k_n-\sigma(n)}$$

$$n^{\downarrow k}\, \mathrm{ch}^\lambda(c_k) = \sum_{i=1}^n \frac{n!}{\mu-ke_i!}\, \Delta(\mu-ke_i).$$

$$\Sigma_k(\lambda) = \sum_{i=1}^n \mu_i^{\downarrow k} \prod_{j\neq i} \frac{\mu_i-\mu_j-k}{\mu_i-\mu_j}.$$