5. Plancherel measures and representations of finite groups

We are looking for a simple proof of the LSKV law of large numbers.

$\lambda_n \sim Pl_n$ Plancherel measure on $\mathbb{Y}(n)$.

ω_{λ_n} = continuous function associated to the Young diagram λ_n

$\omega_n(s) = \dfrac{\omega(\sqrt{n}\,s)}{\sqrt{n}}$; $\displaystyle\int \frac{\omega_n(s) - |s|}{2}\, ds = 1$.

$\omega_n \xrightarrow[\mathbb{P},\ \|.\|_\infty]{} \Omega$; $\Omega(s) = \begin{cases} \frac{2}{\pi}\left(s \arcsin \frac{s}{2} + \sqrt{4-s^2}\right) & \text{if } |s| \leq 2. \\ |s| & \text{if } |s| > 2. \end{cases}$

<u>idea</u>: consider functionals $F(\omega)$ in an algebra of observables $\mathcal{O}$ such that $\omega_n \to \Omega \iff \forall F \in \mathcal{O},\ F(\omega_n) \to F(\Omega)$ and use a method of moments on the random variables $F(\omega_n)$.

1. Geometric observables of partitions.

$$\underset{k \geq 2}{\tilde{p}_k(\lambda)} = k(k-1) \int_{\mathbb{R}} \sigma_\lambda(s)\, s^{k-2}\, ds \qquad \sigma_\lambda(s) = \frac{\omega_\lambda(s) - |s|}{2}.$$

$$= -k \int \sigma_\lambda'(s)\, s^{k-1}\, ds = \int \sigma_\lambda''(s)\, s^k\, ds.$$

<u>Lemma</u>: suppose that $(\omega_n)_{n \in \mathbb{N}}$ is a sequence of continuous Young diagrams with:

functions $\mathbb{R} \to \mathbb{R}_+$, 1-Lipschitz, $= |s|$ for $|s|$ large enough

1. $\exists M$: $\forall |s| \geq M$, $\forall n \in \mathbb{N}$, $\omega_n(s) = |s|$

2. $\tilde{p}_k(\omega_n) \underset{n \to +\infty}{\longrightarrow} \tilde{p}_k(\omega) \quad \forall k \geq 2$.

Then, $\omega_n \underset{\|\cdot\|_\infty}{\longrightarrow} \omega$.

<u>Proof</u>: The set of continuous Young diagrams with support $\subset [-M, M]$ is compact (Arzelà-Ascoli)

If $\bar{\omega}$ is a limit of a convergent subsequence of $(\omega_n)_{n \in \mathbb{N}}$, then:

$\forall k \geq 2$, $\tilde{p}_k(\bar{\omega}) = \tilde{p}_k(\omega)$.

$\underset{\text{Stone Weierstrass}}{\Longrightarrow} \quad \forall f \in C([-M, M])$, $\int f(s) (\bar{\omega} - \omega)(s)\, ds = 0$

$\Longrightarrow \int (\bar{\omega} - \omega)^2 (s)\, ds = 0$, so $\bar{\omega} = \omega$.

The unicity of accumulation point implies that $\omega_n \underset{\|\cdot\|_\infty}{\longrightarrow} \omega$. $\square$.

<u>Remarks</u>: ① With very high probability, if $\lambda_n \sim Pl_n$, then ω_n is supported by $[-2e, 2e]$.

Indeed:

$$\mathbb{P}[\underbrace{\ell_n}_{=(by\,\omega)\ \lambda_{n,1}} \geq K] \leq \frac{1}{n!}\binom{n}{K}^2 (n-K)! = \frac{n!}{K!^2\, (n-K)!} \leq \frac{n^K}{K!^2}$$

and if $K = c\sqrt{n}$, then the RHS is equivalent to

$$\frac{n^{c\sqrt{n}}\, e^{2c\sqrt{n}}}{(c\sqrt{n})^{2c\sqrt{n}}\,(2\pi c\sqrt{n})} = 2\pi c\sqrt{n} \left(\frac{e}{c}\right)^{2c\sqrt{n}} \ll 1 \text{ for } c > e, \text{ say } c = 2e$$

② $$\tilde{p}_k(\Omega) = k(k-1) \int_{-2}^{2} s^{k-2}\, \frac{\Omega(s) - |s|}{2}\, ds$$

$$\underset{(IPP)}{=} -k \int_{-2}^{2} s^{k-1} \left(\frac{2}{\pi} \arcsin \frac{s}{2} - \operatorname{sgn}(s) \right) ds$$

If k is odd, then $\tilde{p}_k(\Omega) = 0$. Otherwise,

$$\tilde{p}_{2k}(\Omega) = 2k \int_0^2 s^{2k-1} \left(1 - \frac{2}{\pi} \arcsin \frac{s}{2}\right) ds$$

$$= 2^{2k+1} k \int_0^{\pi/2} (\sin \theta)^{2k-1} \cos\theta \left(1 - \frac{2\theta}{\pi}\right) d\theta$$

$$= \frac{2^{2k+1}}{\pi} \int_0^{\pi/2} (\sin\theta)^{2k} \, d\theta = \binom{2k}{k} \quad \text{Wallis' integral.}$$

$\Rightarrow$ to prove LSKV, it suffices to show that $\tilde{p}_k(\omega_n) \xrightarrow[\mathbb{P}]{} 1_{k \text{ even}} \binom{k}{k/2}$.

problem: set $\mathcal{O} = \mathbb{R}[\tilde{p}_2, \tilde{p}_3, \dots]$. If $F \in \mathcal{O}$, how to compute $\mathbb{E}[F(\omega_n)]$?

expectation under the Plancherel measure $\mathbb{P}_n$.

2. Representations of finite groups.

The Plancherel measures and more generally the central measures are spectral measures of representations or traces on the symmetric groups $\mathfrak{S}(n)$.

Definitions: Let G be a finite group.

A representation of G is a pair (V, ρ) where V is a (finite-dimensional) complex vector space and ρ is a morphism of groups $\rho : G \to GL(V)$.

It allows G to act linearly on V: $\quad g \cdot v = \rho(g)(v)$.

examples:
1. trivial representation: $V = \mathbb{C}$; $\rho(g) = \mathrm{id}_{\mathbb{C}} \ \forall g$.
2. permutation representation of $\mathfrak{S}(n)$: $V = \mathbb{C}^n$, $\sigma \cdot (x_1, \dots, x_n) = (x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(n)})$
3. subrepresentation: $W \subset \mathbb{C}^n$, $W = \{x \in \mathbb{C}^n \mid x_1 + x_2 + \dots + x_n = 0\}$.
4. regular representation of G finite group.

$$\mathbb{C}G = \left\{ \text{formal linear combinations } \sum_{g \in G} c_g \, g \right\}$$

$h \cdot \sum c_g g = \sum c_g \cdot (hg)$.

⑤ direct sum of representations: if (V_1, ρ_1) and (V_2, ρ_2) are two representations, then $(V_1 \oplus V_2, \rho_1 \oplus \rho_2)$ is a representation:

$$g \cdot (v_1 + v_2) = g \cdot_1 v_1 + g \cdot_2 v_2.$$

<u>objective:</u> describe all the representations of a finite group G.

<u>Definition</u> A morphism of representations between (V_1, ρ_1) and (V_2, ρ_2) is a linear map $\phi : V_1 \to V_2$ such that

$$\phi(g \cdot_1 v) = g \cdot_2 \phi(v). \iff \text{commutativity of the diagrams}$$

$$\begin{tikzcd} V_1 \arrow[r, "\rho_1(g)"] \arrow[d, "\phi"'] & V_1 \arrow[d, "\phi"] \\ V_2 \arrow[r, "\rho_2(g)"'] & V_2 \end{tikzcd}$$

An isomorphism of representations is a morphism of representations which is also a linear bijection (then, the inverse is also a morphism).

<u>Lemma</u>: The image and the kernel of a morphism of representations are subrepresentations

<u>Def</u> A representation (V, ρ) of G is said irreducible if its only subrepresentations are $\{0\}$ and V, and if $\dim V \geq 1$.

<u>Lemma</u> If W is a subrepresentation of a representation V with $\{0\} \subsetneq W \subsetneq V$, then there exists a complement subspace Z which is a subrepresentation:

$V = W \oplus Z$ direct sum of representations.

<u>Proof:</u> by taking means over G, we can construct a Hermitian scalar product $\langle \cdot | \cdot \rangle$ such that G acts by isometries.

Then, the orthogonal complement of W is a subrepresentation Z. □

As a consequence, we can split any representation V as a direct sum of irreducible representations (Maschke theorem).

$$V = \bigoplus_{i=1}^{r} m_i V_i, \quad \text{the } V_i\text{'s non isomorphic irreducible representations of } G.$$

unicity of the decomposition?

Lemma (Schur). Consider the spaces of morphisms $\hom_G(V,W)$.

① if V and W are irreducibles, then $\dim_{\mathbb{C}} \hom_G(V,W) = \mathbb{1}_{V \underset{iso}{\simeq} W}$.

② $\hom_G(\bullet, W)$ is additive

③ $\hom_G(\mathbb{C}G, \bullet)$ is the functor $\mathrm{Repr}_G \to \text{Vector Spaces}_{\mathbb{C}}$
$$(V,\rho) \mapsto V.$$

Proof: ① Let $\phi : V \to W$ be a morphism between two representations.
If ϕ is not an isomorphism, then $\ker\phi \neq \{0\} \Rightarrow \ker\phi = V$
or $\operatorname{im}\phi \subsetneq W \Rightarrow \operatorname{im}\phi = \{0\}$

Both cases imply that $\phi = 0$.
So, if V and W are not isomorphic, then $\hom_G(V,W) = \{0\}$.
If $V \underset{iso}{=} W$, then we can regard ϕ as a complex endomorphism of V.
Let λ be an eigenvalue of ϕ. $\phi - \lambda\, \mathrm{id}_V$ is a non-injective morphism,
so $\phi - \lambda\,\mathrm{id}_V = 0 \iff \phi = \lambda\,\mathrm{id}_V$.
So, if $V \underset{iso}{=} W$, then $\hom_G(V,W) = \mathbb{C}\,\mathrm{id}_V$.

② obvious.

③. We consider the map
$$\Psi : \hom_G(\mathbb{C}G, V) \longrightarrow V$$
$$\phi \longmapsto \phi(e_G)$$
It is injective: if $\Psi(\phi) = 0$, then $\forall \sum c_g g$,
$$\phi(\sum c_g g) = \phi(\sum c_g\, g.e_G) = \sum c_g\, g. \underbrace{\phi(e_G)}_{=0} = 0.$$
so $\phi = 0$.
It is surjective: if $v \in V$, then v is obtained by the morphism of representations
$\phi : \sum c_g g \longmapsto \sum c_g\, g.v$.

(we leave the functorial aspects as an exercise). $\square$

Corollary Set $\hat{G} = \{$ isomorphism classes of irreducible representations of $G\}$.
$$\lambda = (V^\lambda, \rho^\lambda)$$

① Any representation of G splits uniquely as a direct sum

$$V = \bigoplus_{\lambda \in \widehat{G}} m_\lambda V^\lambda .$$

② $\widehat{G}$ is finite, and we have the decomposition $\mathbb{C}G = \bigoplus_{\lambda \in \widehat{G}} (\dim \lambda) V^\lambda$.

③ In particular, $|G| = \sum_{\lambda \in \widehat{G}} (\dim \lambda)^2$.

Proof: ① $m_\lambda = \dim \hom_G(V, V^\lambda)$ by ① and ② of the Schur lemma; this implies the unicity.

② and ③ if $\mathbb{C}G = \sum_{\lambda \in \widehat{G}} d_\lambda V^\lambda$, then $d_\lambda = \dim \hom_G(\mathbb{C}G, V^\lambda)$
$= \dim V^\lambda$. $\square$.

Definition: the spectral measure of a representation V of G is the probability measure on $\widehat{G}$ given by:

$$\mathbb{P}_V[\lambda] = \frac{m_\lambda \dim \lambda}{\dim V} \quad \text{if } V = \bigoplus_{\lambda \in \widehat{G}} m_\lambda V^\lambda .$$

The Plancherel measure of a group G is the spectral measure of the regular representation $\mathbb{C}G$.

$$\mathrm{Pl}_G(\lambda) = \frac{(\dim \lambda)^2}{|G|} .$$

Fact: there exists a bijection $\widehat{\mathfrak{S}(n)} \to \mathcal{Y}(n)$ which makes correspond the Plancherel measure of $\mathfrak{S}(n)$ with the Plancherel measure $\mathrm{Pl}_n(\lambda) = \frac{|\mathrm{ST}(\lambda)|^2}{n!}$.

3. The Fourier isomorphism.

We endow $\mathbb{C}G$ with a structure of algebra:

$$\left(\sum c_g \cdot g\right) * \left(\sum d_h \cdot h\right) = \sum_{g,h} c_g d_h \cdot (gh) .$$

$\mathbb{C}G$ is a complex algebra with dimension $|G|$.
For each $\lambda \in \widehat{G}$, there is a morphism of algebras

$$\rho^\lambda : \mathbb{C}G \to \mathrm{End}(V^\lambda)$$
$$\sum c_g g \mapsto \sum c_g \rho^\lambda(g) .$$

Definition: The Fourier transform of G is the map

$$\mathbb{C}G \longrightarrow \bigoplus_{\lambda \in \hat{G}} \mathrm{End}(V^\lambda).$$

$$x \longmapsto \sum_{\lambda \in \hat{G}} (\rho^\lambda(x) = \hat{x}(\lambda))$$

We endow $\mathbb{C}G$ with the scalar product $\langle \sum c_g \cdot g \mid \sum d_g \cdot g \rangle = \dfrac{\sum \overline{c_g}\, d_g}{|G|}$.

- each space $\mathrm{End}(V^\lambda)$ with the scalar product

$$\langle U \mid V \rangle = \frac{d_\lambda}{|G|^2} \operatorname{tr}(U^* V)$$

Theorem The Fourier transform is an isomorphism of algebras and isometry.

Proof We already know that it is a morphism of algebras, so it suffices to prove the isometry property.

Let us compute $\langle \rho^\lambda_{ij} \mid \rho^\mu_{kl} \rangle$, $i,j \leq \dim \lambda$, $k,l \leq \dim \mu$

$\hookrightarrow$ matrices in orthonormal bases of V^λ and V^μ.

$$= \frac{1}{|G|} \sum_{g \in G} \overline{\rho^\lambda_{ij}(g)}\, \rho^\mu_{kl}(g) = \frac{1}{|G|} \sum_{g \in G} \overline{\langle e_i^\lambda \mid g \cdot e_j^\lambda \rangle} \langle e_k^\mu \mid g \cdot e_l^\mu \rangle.$$

$$= \langle e_k^\mu \mid \underbrace{\frac{1}{|G|} \sum_{g \in G} g \cdot e_l^\mu \rangle \langle g \cdot e_j^\lambda}_{T:\, V^\lambda \to V^\mu.} \mid e_i^\lambda \rangle$$

We remark that T is a morphism of representations:

$$T(h \cdot v) = \frac{1}{|G|} \sum_{g \in G} |g \cdot e_l^\mu \rangle \langle g \cdot e_j^\lambda \mid h \cdot v \rangle$$

$$\langle g \cdot e_j^\lambda \mid h \cdot v \rangle = \langle h^{-1} g \cdot e_j^\lambda \mid v \rangle$$

$$= h \cdot \frac{1}{|G|} \sum_{g \in G} |h^{-1} g \cdot e_l^\mu \rangle \langle h^{-1} g \cdot e_j^\lambda \mid v \rangle = h \cdot T(v).$$

So, $T = 0$ if $\lambda \neq \mu$. If $\lambda = \mu$, then $T = c \cdot \mathrm{id}_{V^\lambda}$ and $\mathrm{tr}\, T = \dim \lambda \cdot c$.

However, $\mathrm{tr}\, T = \sum_i \langle e_i^\lambda | T | e_i^\lambda \rangle = \frac{1}{|G|} \sum_{\substack{g \in G \\ i \leq \dim \lambda}} \langle e_i^\lambda | g \cdot e_\ell^\lambda \rangle \langle g \cdot e_j^\lambda | e_i^\lambda \rangle$

$$= \frac{1}{|G|} \sum_{g \in G} \langle g \cdot e_j^\lambda | g \cdot e_\ell^\lambda \rangle$$

$$= \mathbb{1}_{j=\ell}.$$

$$\Rightarrow T = \frac{1}{\dim \lambda} \mathbb{1}_{j=\ell} \, \mathrm{id}_{V^\lambda}$$

and $\langle \rho_{ij}^\lambda | \rho_{k\ell}^\mu \rangle = \dfrac{\mathbb{1}_{\lambda=\mu,\, i=k,\, j=\ell}}{\dim \lambda}$.

Therefore, (ρ_{ij}^λ) is an orthogonal basis of $\mathbb{C}G$. Moreover,

$$\widehat{\rho_{ij}^\lambda}(\mu) = \Big(\sum_{g \in G} \rho_{ij}^\lambda(g)\, \rho_{k\ell}^\mu(g) \Big)_{k,\ell}$$

$$= \frac{|G|}{\dim \lambda} \mathbb{1}_{\lambda^* = \mu} \, E_{ij}^\mu,$$ which has the right norm. $\square$

<u>Corollary</u>: $Z(\mathbb{C}G) \simeq \bigoplus_{\lambda \in \widehat{G}} Z(\mathrm{End}(V_\lambda)) = \bigoplus_{\lambda \in \widehat{G}} \mathbb{C}\, \mathrm{id}_{V^\lambda}$

which implies that $|\widehat{G}|$ = number of conjugacy classes of G.

When $G = \mathfrak{S}(n)$, the conjugacy classes are labelled by integer partitions, so there is a bijection:

$$\widehat{\mathfrak{S}(n)} \Longleftrightarrow \mathcal{Y}(n).$$