3. Schur measures

Previously, we considered random infinite tableaux $\emptyset \nearrow \lambda_1 \nearrow \lambda_2 \nearrow \cdots$ with the property of central measures:

- conditionally to $\lambda_n \nearrow \lambda_{n+1} \nearrow \cdots$, the law of $T = \emptyset \nearrow \lambda_1 \nearrow \cdots \nearrow \lambda_n$ is uniform on $ST(\lambda_n)$.

$\iff$ the measures Π_n of the random partitions λ_n satisfy

$$\frac{\Pi_n(\lambda)}{|ST(\lambda)|} = \sum_{\Lambda : \lambda \nearrow \Lambda} \frac{\Pi_{n+1}(\Lambda)}{|ST(\Lambda)|}.$$

We claim that the extremal central measures are related to Schur functions.

1. Further properties of Schur functions.

① Pieri rule: $\quad p_1 \cdot s_\lambda = \sum_{\Lambda : \lambda \nearrow \Lambda} s_\Lambda .$

proof: With a sufficient number N of variables: $(N \geq |\lambda| + 1)$.

$$\begin{aligned}
&p_1(x_1, \ldots, x_N)\, a_{\lambda+\rho_N}(x_1, \ldots, x_N) \\
&= \sum_{j=1}^{N} \sum_{\sigma \in \mathfrak{S}(N)} x_j\, \varepsilon(\sigma) \prod_{i=1}^{N} x^{\lambda_i + N - \sigma(i)} \\
&= \sum_{j=1}^{N} \underbrace{a_{(\lambda+e_j)+\rho_N}(x_1, \ldots, x_N)}_{0 \text{ if adding a box on the } j\text{-th row of } \lambda \text{ does not produce an integer partition.}} \\
&= \sum_{\Lambda : \lambda \nearrow \Lambda} a_{\Lambda+\rho_N}(x_1, \ldots, x_N) \qquad \square
\end{aligned}$$

Therefore, if $\Psi : \mathrm{Sym} \to \mathbb{R}$ is a morphism of algebras with

1. $\Psi(p_1) = 1$
2. $\Psi(s_\lambda) \geq 0 \quad \forall \lambda \in \mathbb{Y}$,

then $\Pi_n(\lambda) = |ST(\lambda)|\, \Psi(s_\lambda)$ is a central measure.

② Hall scalar product on Sym.

The Hall scalar product is the unique $\langle\cdot|\cdot\rangle$ on Sym which makes $(s_\lambda)_{\lambda\in\mathbb{Y}}$ an orthonormal basis.

<u>lemma</u>: Consider two graded basis $(u_\lambda)_{\lambda\in\mathbb{Y}}$ and $(v_\lambda)_{\lambda\in\mathbb{Y}}$ of Sym.
We have $\langle u_\lambda | v_\rho\rangle = \delta_{\lambda\rho}$ iff $\displaystyle\sum_{\lambda\in\mathbb{Y}} u_\lambda(X)\, v_\lambda(Y) = \prod_{i,j} \frac{1}{1-x_iy_j}$.

Indeed, the formal power series rewrites in terms of Schur functions:

$$u_\lambda(X) = \sum_{|\mu|=|\lambda|} U_{\lambda\mu}\, s_\mu(X) \quad ; \qquad v_\rho(Y) = \sum_{|\rho|=|\eta|} V_{\rho\eta}\, s_\eta(Y)$$

and $\langle u_\lambda | v_\rho\rangle$

$$= \sum_\mu U_{\lambda\mu} V_{\rho\mu} = (U^tV)_{\lambda\rho}.$$

$$\text{LHS} = \sum_{\lambda,\mu,\eta} U_{\lambda\mu} V_{\lambda\eta}\, s_\mu(X)\, s_\eta(Y)$$

$$= \sum_{\mu,\eta} (U^tV)_{\mu\eta}\, s_\mu(X)\, s_\eta(Y)$$

$$= \prod \frac{1}{1-x_iy_j} \quad \text{iff} \quad (U^tV)_{\mu\eta} = \mathbb{1}_{(\mu\eta)}$$

$\iff U = V^{-1}$, is an orthogonal matrix.

<u>example 1</u>: $$\sum_\lambda h_\lambda(X)\, m_\lambda(Y) = \sum_I h_I(X)\, Y^I = \prod_j \left(\sum_k h_k(X)(y_j)^k\right) = \prod_{i,j} \frac{1}{1-x_iy_j}.$$

<u>example 2</u>: set $z_\lambda = \prod_{s\geq 1} s^{m_s(\lambda)}\, m_s(\lambda)!$

Then,
$$\sum_\lambda \frac{p_\lambda(X)p_\lambda(Y)}{z_\lambda} = \sum_\lambda \frac{p_\lambda(XY)}{z_\lambda} = \exp\left(\sum_{k=1}^{\infty} \frac{p_k}{k}(XY)\right)$$
$$= \exp\left(\sum_{i,j} \log\left(\frac{1}{1-x_iy_j}\right)\right) = \prod_{i,j} \frac{1}{1-x_iy_j}.$$

③ expansion of Schur functions on semistandard tableaux:

Denote x^T the product of variables x_i, i appearing in a cell of a semistandard tableau T.

Then: $$s_\lambda(x_1, x_2, \dots, x_N) = \sum_{T \in SST(\lambda, N)} x^T.$$

proof: the Hall scalar product allows one to define skew Schur functions $s_{\lambda\backslash\mu}$ with $|\lambda| \geq |\mu|$:

$$\forall e,\ \langle s_{\lambda\backslash\mu} | s_e \rangle = \langle s_\lambda | s_\mu s_e \rangle;\quad \deg s_{\lambda\backslash\mu} = |\lambda| - |\mu| \text{ if } s_{\lambda\backslash\mu} \neq 0.$$

We have a generalisation of the Jacobi-Trudy formula:

$$s_{\lambda\backslash\mu} = \det(h_{\lambda_i - i - \mu_j + j})$$

(this implies that if $\mu \not\subseteq \lambda$, then $s_{\lambda\backslash\mu} = 0$).

We have $$\sum_\lambda s_{\lambda\backslash\mu}(X) s_\lambda(Y) = \sum_{\lambda, e} \langle s_{\lambda\backslash\mu} | s_e \rangle s_e(X) s_\lambda(Y)$$
$$= \sum_{\lambda, e} \langle s_\lambda | s_\mu s_e \rangle s_e(X) s_\lambda(Y)$$
$$= \sum_e s_e(X) s_\mu(Y) s_e(Y) = C(X,Y) s_\mu(Y).$$
$$= \sum_\upsilon h_\upsilon(X) m_\upsilon(Y) s_\mu(Y)$$

$$\Rightarrow \sum_\lambda s_{\lambda\backslash\mu}(X) a_{\lambda+e_N}(y_1, \dots, y_N)$$
$$= \sum_I h_I(X) Y^I a_{\mu+e_N}(y_1 \cdots y_N)$$

$$\Rightarrow s_{\lambda\backslash\mu}(X) = [y^{\lambda+e_N}]\Big(\sum_{\substack{I \\ \sigma \in S(N)}} \varepsilon(\sigma)\, y^{I + \sigma(\mu + e_N)} h_I(X)\Big)$$
$$= \det(h_{(\lambda_i - i) - (\mu_j - j)}).$$
$$= \text{(with a bit more work)} \det(e_{(\lambda_i' - i) - (\mu_j' - j)}).$$

$H(t)E(-t) = 1 \rightsquigarrow$

In particular, if $X = \{x_1\}$, then $s_{\lambda\backslash\mu}(x_1) = \mathbb{1}_{\lambda\backslash\mu \text{ is an horizontal strip}}\, (x_1)^{|\lambda|-|\mu|}$

This implies the expansion, because $s_\lambda(X+Y) = \sum_\mu s_{\lambda\backslash\mu}(X)\, s_\mu(Y)$.

<u>proof</u>: $s_\lambda(X+Y) = [s_\lambda(Z)] \sum_\lambda s_\lambda(X+Y) s_\lambda(Z)$

$$= [s_\lambda(Z)]\left(C(X+Y, Z)\right)$$
$$= [s_\lambda(Z)]\left(C(X,Z)\, C(Y,Z)\right)$$
$$= [s_\lambda(Z)]\left(\sum_{\nu,\mu} s_\nu(X)\, s_\mu(Y)\, s_\nu(Z) s_\mu(Z)\right).$$
$$= \sum_\mu s_{\lambda\backslash\mu}(X)\, s_\mu(Y). \qquad \square.$$

2. The Thoma simplex indices extremal central measures.

$\Omega = \{(\alpha, \beta)$ pair of non increasing non negative sequences with $\sum_{i=1}^\infty \alpha_i + \beta_i = 1 - \gamma \leq 1\}$

We set $p_1(\alpha,\beta) = 1$

$p_{k\geq 2}(\alpha,\beta) = \sum_{i=1}^\infty (\alpha_i)^k + (-1)^{k-1}(\beta_i)$.

This defines a morphism of algebras $\mathrm{Sym} = \mathbb{R}[p_1, p_2, \dots] \to \mathbb{R}$ with $\Psi_{\alpha,\beta}(p_1) = 1$.

<u>Theorem</u>: $\forall (\alpha,\beta) \in \Omega$, $\forall \lambda \in \mathbb{Y}$, $\Psi_{\alpha,\beta}(s_\lambda) = s_\lambda(\alpha,\beta) \geq 0$.

At this point, it is very hard to give a complete proof of this fact; some arguments rely on the connection between Schur functions and the représentations of $\mathfrak{S}(n)$.

<u>Sketch of proof</u>. 1) the Schur positivity is kept by addition of morphisms on the power sums $p_{k\geq 1}$.

This is related to the fact that a product of two Schur functions is a <u>non-negative</u> linear combination of Schur functions.

2) if $X = \alpha = (\alpha_1, \alpha_2, \dots)$ is a summable sequence with positive coefficients, then $p_{k\geq 1}(\alpha) = \sum_{i=1}^{\infty} (\alpha_i)^k$ is Schur positive, because we then have $s_\lambda(\alpha) = \sum_{T \in SST(\lambda)} \alpha^T \geq 0$.

3) if $\left| \begin{array}{l} X = \beta \text{ is a summable sequence with positive coefficients} \\ p_{k\geq 1}(\overline{\beta}) = (-1)^{k-1} \sum_{i=1}^{\infty} (\beta_i)^k \end{array} \right.$

then $s_\lambda(\overline{\beta}) \geq 0 \ \forall \lambda \in \mathbb{Y}$.

Indeed, by stability by sum, it suffices to treat the case of $\beta = (\beta_1) = (\beta_1, 0, \dots)$.

$$
\begin{aligned}
s_\lambda(\overline{\beta}) &= [s_\lambda(Z)]\Big(\sum_\lambda s_\lambda(\overline{\beta}) s_\lambda(Z)\Big) \\
&= [s_\lambda(Z)]\Big(\exp\Big(\sum_{k=1}^{\infty} (-1)^{k-1} \beta^k \frac{p_k(Z)}{k}\Big)\Big) \\
&= [s_\lambda(Z)]\Big(\prod_{i=1}^{\infty} (1 + \beta z_i)\Big) \\
&= [s_\lambda(Z)]\Big(\sum_{k=0}^{\infty} \beta^k e_k(Z)\Big)
\end{aligned}
$$

However: $h_k(Z) = s_{(k)}(Z)$ (clear by Jacobi Trudy)

$e_k(Z) = s_{(1^k)}(Z)$

$$\frac{(\det(z_i^{k-j}))_{1\leq i,j\leq k}}{\det(z_i^{k-1-j})_{1\leq i,j\leq k}} = z_1 z_2 \cdots z_k \text{ if } Z = (z_1 \dots z_k) = e_k(Z)$$

So $s_\lambda(\overline{\beta}) = \mathbb{1}_{(\lambda = 1^k)} \beta^k \geq 0$.

4) $(\alpha, \beta) \in \Omega$ corresponds to the specialisation $\alpha + \overline{\beta} + \gamma E$ with $p_1(E) = 1$, $p_{k\geq 2}(E) = 0$.

Schur positive: $s_\lambda(E) = \dfrac{|ST(\lambda)|}{n!}$ by using the Pieri rule. $\square$

<u>Kerov-Vershik ring theorem</u>: Let $\varphi : Sym \to \mathbb{R}$ be a linear function such that

- $\varphi(p_1 s_\lambda) = \sum_{\Lambda : \lambda \nearrow \Lambda} s_\Lambda$ Pieri rule
- $\varphi(s_\lambda) \geq 0 \quad \forall \lambda$.
- $\varphi(p_1) = 1$.

We associate to φ the central measure $\pi_n(\lambda) = |ST(\lambda)|\, \varphi(s_\lambda)$.
Then, $(\pi_n)_{n \in \mathbb{N}}$ is an extremal central measure iff φ is a morphism of algebras:
$$\varphi(fg) = \varphi(f)\varphi(g) \quad \forall f, g \in Sym.$$

<u>proof</u>: $\Rightarrow$ Suppose that φ corresponds to an extremal central measure. We can view φ as a linear map on the quotient ring $A = Sym/(p_1 - 1)$, positive on the cone K spanned by the projections of the Schur functions.

Let $a, b \in A$. We want to prove that $\varphi(ab) = \varphi(a)\varphi(b)$. The ring A is spanned by K, so we can assume that $a, b \in K$. Also, since
$$(p_1)^n = \sum_{\lambda \in \mathbb{Y}(n)} |ST(\lambda)|\, s_\lambda$$
$$1_A = \sum_{\lambda \in \mathbb{Y}(n)} |ST(\lambda)|\, \overline{s_\lambda},$$
every element of K is smaller than a multiple of 1_A.

So we can assume that $1_A - b \in K$. Let us then write:
$$\varphi(a) = \varphi(ab) + \varphi(a(1_A - b)).$$

1st case: $\varphi(b) \neq 0$, $\varphi(1_A - b) \neq 0$. Then,
$$\varphi(a) = \varphi_b(a) + \varphi_{1-b}(a) \quad \text{with} \quad \varphi_c(a) = \frac{\varphi(ac)}{\varphi(c)}.$$
two maps on A corresponding to central measures.

By extremality, $\varphi = \varphi_b$, so $\varphi(ab) = \varphi(a)\varphi(b)$.

2nd case: $\varphi(b) = 0$ or $\varphi(1_A - b) = 0$. Let us treat the first case.
We then have $a \leq \lambda 1_A$, so $ab \leq \lambda b$ and $\varphi(ab) \leq \lambda \varphi(b) = 0$.

$\Leftarrow$ Suppose that Ψ is multiplicative. We decompose Ψ as an integral over the set of extremal points: Choquet theorem

$$\Psi = \int_{\mathrm{Extr}} \phi \, \mathbb{P}[d\phi].$$ (This decomposition is not unique in general!)

For $\lambda \in \mathbb{Y}$, set $X_\lambda = \phi(s_\lambda)$ under $\mathbb{P}$. We have $\mathbb{E}[X_\lambda] = \Psi(s_\lambda)$.

We also have:

$$\mathbb{E}[(X_\lambda)^2] = \mathbb{E}[\phi(s_\lambda)\phi(s_\lambda)]$$

$$\underset{\substack{\text{(extremal points yield} \\ \text{multiplicative functions)}}}{=} \mathbb{E}[\phi(s_\lambda^2)] = \Psi(s_\lambda^2) = \Psi(s_\lambda)^2 = \mathbb{E}[X_\lambda]^2.$$

So, $\mathrm{var}\, X_\lambda = 0$, and X_λ is constant.

$\Longrightarrow$ $\mathbb{P}$ is concentrated on one point, and Ψ is extremal $\square$.

missing: the interpretation of the parameters α and β w.r.t. the sequence of random partitions $(\lambda_n)_{n \in \mathbb{N}}$.

3. Schur measures and their determinantal expression.

The Cauchy formula leads one to investigate a new family of measures on partitions.

Suppose given X, Y morphisms of algebras $\mathrm{Sym} \to \mathbb{R}$ with $s_\lambda(X) \geq 0$, $s_\lambda(Y) \geq 0$ $\forall \lambda \in \mathbb{Y}$.

If the series $\sum_{k=1}^{\infty} \frac{p_k(X)\, p_k(Y)}{k}$ is convergent, then

$$\sum_{\lambda \in \mathbb{Y}} s_\lambda(X)\, s_\lambda(Y) = \exp\left(\sum_{k=1}^{\infty} \frac{p_k(X)\, p_k(Y)}{k}\right) = C(X, Y).$$

<u>Definition</u>: The Schur measure with parameters $X, Y \in \mathbb{R}_+\Omega$ is the

probability measure on $\mathbb{Y} = \bigsqcup_{n \in \mathbb{N}} \mathbb{Y}(n)$ given by

$$\mathbb{S}_{X,Y}[\lambda] = \frac{s_\lambda(X)\, s_\lambda(Y)}{C(X,Y)}.$$

<u>example</u>: Fix $\theta > 0$, and consider the central measure associated to a parameter $\omega = (\alpha, \beta) \in \Omega$.

$$\pi_n(\lambda) = |ST(\lambda)|\, s_\lambda(\alpha, \beta).$$

Let $\mathbb{P}_{\theta,\omega}$ be the Poissonised version of the measures π_n:

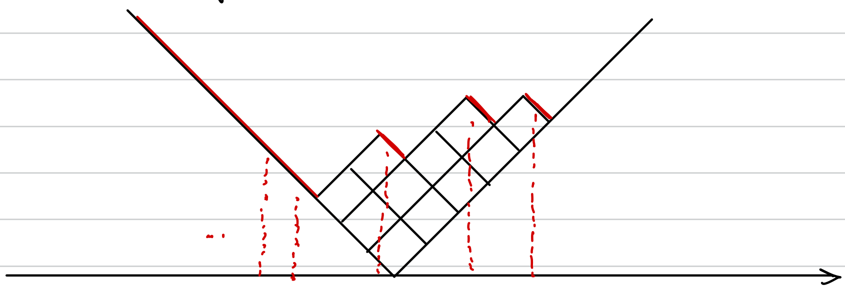
$$\mathbb{P}_{\theta,\omega}[\lambda] = \mathrm{Poisson}(\theta)[n]\, \pi_n[\lambda]$$
$$= e^{-\theta} \frac{\theta^n}{n!} |ST(\lambda)|\, s_\lambda(\omega).$$

This is the Schur measure associated to the parameters $X = \theta E$ and $Y = \omega$.

<u>example</u>: In particular, the Poissonised Plancherel measure

$$PPl_\theta[\lambda] = e^{-\theta} \frac{\theta^n}{n!} \frac{|ST(\lambda)|^2}{n!} \text{ is } \mathbb{S}_{\theta E, E} = \mathbb{S}_{\sqrt{\theta} E, \sqrt{\theta} E}.$$

We associate to a partition $\lambda \in \mathbb{Y}$ its descent set $D(\lambda) = \left(\lambda_i - i + \frac{1}{2}\right)_{i \geq 1}$



This is a subset of $\mathbb{Z}' = \mathbb{Z} + \frac{1}{2}$ with the property

$$|\mathbb{Z}'_+ \cap D(\lambda)| = |\mathbb{Z}'_- \setminus \mathbb{Z}' \cap D(\lambda)| < +\infty.$$

<u>Theorem</u> (Okounkov, 2000). Consider a random partition $\lambda \sim \mathbb{S}_{X,Y}$. The random set of descents $D(\lambda)$ forms a determinantal point process on $\mathbb{Z}'$, meaning that:

$\forall \{x_1, \dots, x_n\} \subset \mathbb{Z}'$,

$$\mathbb{P}[\{x_1, \dots, x_n\} \subset D_\lambda] = \det(K_{X,Y}(x_i, x_j))_{1 \le i,j \le n}.$$

The kernel $K_{X,Y}$ has an explicit generating function:

$$\mathcal{K}(z,w) = \sum_{x,y \in \mathbb{Z}'} K_{X,Y}(x,y)\, z^x w^{-y} \underset{|z|>|w|}{=} \frac{\sqrt{zw}}{z-w} \exp\left(\sum_{k=1}^{\infty} \frac{p_k(X)}{k}(z^k - w^k) - \sum_{k=1}^{\infty} \frac{p_k(Y)}{k}(z^{-k} - w^{-k})\right)$$

<u>example</u>: For the Poissonised Plancherel measure $\mathbb{P}\mathrm{Pl}_\theta$, $X = Y = \sqrt{\theta}\, E$

$$\mathcal{K}(z,w) = \frac{\sqrt{zw}}{z-w} \exp\left(\sqrt{\theta}\left((z-w) - (z^{-1} - w^{-1})\right)\right).$$