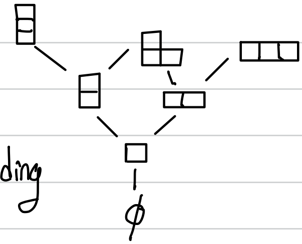
2. Central measures, the Thoma simplex and symmetric functions

We want to study the Plancherel measure $Pl_n(\lambda) = \frac{|ST(\lambda)|^2}{n!}$

1. Central measures on the Young graph

Consider the Young graph $\mathbb{Y}$:



$\lambda \nearrow \Lambda$ if Λ can be obtained from λ by adding one cell at the top-right border.

A standard tableau with shape λ corresponds to a sequence $\emptyset \nearrow \lambda_1 \nearrow \dots \nearrow \lambda_n = \lambda$.
recurrence relation :

$$|ST(\Lambda)| = \sum_{\lambda : \lambda \nearrow \Lambda} |ST(\lambda)|$$

We can therefore define a transition kernel $p_n^{n+1} : \mathbb{Y}(n+1) \times \mathbb{Y}(n) \to \mathbb{R}$

$$\Lambda, \lambda \mapsto \mathbb{1}_{\lambda \nearrow \Lambda} \frac{|ST(\lambda)|}{|ST(\Lambda)|}$$

Given $\lambda \in \mathbb{Y}(n)$ the transition kernels $p_{n-1}^{n}, p_{n-2}^{n-1}, \dots, p_0^{1}$ allow one to choose a uniform standard tableau in $ST(\lambda)$.

<u>Definition</u>: A central measure is a family of probability measures $(\pi_n)_{n \in \mathbb{N}}$ on the levels $\mathbb{Y}(n)$ of the Young graph such that:

$$\forall n, \quad \pi_{n+1} \, p_n^{n+1} = \pi_n .$$

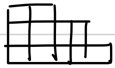
<u>Proposition</u>: the Plancherel measures form a central measure.
<u>Proof</u>: We have to establish:

$$Pl_n(\lambda) = \sum_{\Lambda : \lambda \nearrow \Lambda} Pl_{n+1}(\Lambda) \frac{|ST(\lambda)|}{|ST(\Lambda)|}$$

$$\Leftrightarrow \frac{|ST(\lambda)|}{n!} = \sum_{\Lambda : \lambda \nearrow \Lambda} \frac{|ST(\Lambda)|}{n+1 \, !} \Leftrightarrow (n+1)|ST(\lambda)| = \sum_{\Lambda : \lambda \nearrow \Lambda} |ST(\Lambda)|.$$

Given an integer partition λ, we denote λ' the conjugate partition given by the sizes of the columns of λ.

ex: $(5,4,2)' = (3,3,2,2,1)$.



We claim that both sides of the equation are equal to the number of permutations $\sigma \in \mathfrak{S}(n+1)$ s.t.

$$\begin{array}{|c|c|} \hline \sigma(1) & \\ \hline \sigma(2) & \sigma(\lambda'_1+1) \\ \hline \vdots & \vdots \\ \hline \sigma(\lambda'_1) & \sigma(\lambda'_1+\lambda'_2) \\ \hline \end{array} \quad \cdots \quad \boxed{\sigma(n)}$$

is a semistandard tableau with shape λ

* LHS

indeed, if σ is such a permutation, then:

- $\sigma(n+1)$ can be any integer in $[\![1, n+1]\!]$. ($n+1$ possibilities).
- $P(\sigma_1 \dots \sigma_n)$ is the semi-standard tableau indicated above; knowing $\sigma(n+1)$, there are $|ST(\lambda)|$ possibilities
- $Q(\sigma_1 \dots \sigma_n) = \begin{array}{|c|c|} \hline \lambda'_1 & \\ \hline \vdots & \lambda'_1+\lambda'_2 \\ \hline 2 & \vdots \\ \hline 1 & \lambda'_1+1 \\ \hline \end{array} \; \cdots$ (1 possibility).

  $\parallel$

  Q

* RHS

- $Q(\sigma)$ is one of the standard tableau obtained by adding $n+1$ at the top right boundary of Q $\longrightarrow$ one possibility for each Λ with $\lambda \nearrow \Lambda$.
- $P(\sigma)$ is arbitrary in $ST(\Lambda)$, knowing Λ.

$\square$.

Given a central measure $(\pi_n)_{n\in\mathbb{N}}$, one can use the Kolmogorov extension theorem in order to define a random infinite tableau $T = \emptyset \nearrow \lambda_1 \nearrow \dots \nearrow \lambda_n \nearrow \dots$ such that:

1) $\forall n$, $\text{law}(\lambda_n) = \pi_n$.

2) conditionally to $\lambda_n \nearrow \lambda_{n+1} \nearrow \dots$, λ_{n-1} has distribution $p^n_{n-1}(\lambda_n, \cdot)$

Then: $\forall n$, ______________, the subtableau of T formed by the integers $\in [\![1, n]\!]$ has uniform law in $ST(\lambda_n)$.

problem: find all the central measures on infinite standard tableaux

$\iff$ all the families (π_n) of probability measures which satisfy the harmonicity property:

$$\frac{\pi_n(\lambda)}{|ST(\lambda)|} = \sum_{\Lambda : \lambda \nearrow \Lambda} \frac{\pi_{n+1}(\Lambda)}{|ST(\Lambda)|} .$$

The set of such functions $\mathbb{Y} \to \mathbb{R}_+$ is a compact convex set

$\to$ it suffices to describe the extremal central measures. (product topology on $\mathbb{R}_+^{\mathbb{Y}}$).

example: $\pi_n = \delta_{(n)}$

$\pi_n = \delta_{(1^n)}$ (we shall see that all these examples are extremal).

$\pi_n = Pl_n$

Theorem (Thoma, 1964; Kerov-Vershik 1981)

There exists
- an algebra of functions $Sym = \bigoplus_{\lambda \in \mathbb{Y}} \mathbb{R} s_\lambda$
- a compact set Ω
- a way to associate to any $\omega \in \Omega$ a morphism of algebras $Sym \to \mathbb{R}$, $f \mapsto \Psi_\omega(f)$

such that any extremal central measure writes uniquely as $\pi_n(\lambda) = |ST(\lambda)| \, \Psi_\omega(s_\lambda)$ for some $\omega \in \Omega$.

2. Symmetric functions.

We fix an infinite alphabet of variables $X = \{x_1, x_2, \dots, x_n, \dots\}$.

A monomial is a finite product of variables.

ex: $x_1^3 x_2 x_4^2$.

notation: if $I \in \mathbb{N}^{(\mathbb{N}^*)}$ is a sequence of integers eventually equal to 0, we set $x^I = x_1^{i_1} x_2^{i_2} \dots x_s^{i_s}$.

The degree of a monomial is its number of terms. $\deg(x^I) = |I| = \sum_{s\geq 1} i_s$.
A polynomial is a formal linear combination $\sum c_I x^I$ of monomials which is bounded in degree: $c_I = 0$ for all monomials with $\deg(x^I) >$ some $d \in \mathbb{N}$.

ex: $2x_1^2 x_3 - x_2^5$, $x_1 + x_2 + \dots + x_n + \dots$ (not necessarily finite sums!).

We denote $\mathbb{R}[x]$ the real vector space of polynomials. It is graded by

$$\deg\left(\sum c_I x^I\right) = \max(|I|,\ c_I \neq 0)$$

($-\infty$ by convention for the 0 polynomial).

The rule $x^I * x^J = x^{I+J}$ can be extended by linearity to make $\mathbb{R}[x]$ into a graded real algebra.
$\deg PQ = \deg P + \deg Q$.

$$\mathbb{R}[x] = \bigoplus_{n=0}^{\infty} \underbrace{\mathbb{R}_n[x]}_{}$$ homogeneous polynomial with degree n.

remark: For any $N \geq 0$, we have a specialisation morphism

$$\phi_N : \mathbb{R}[x] \longrightarrow \mathbb{R}[x_1, x_2, \dots x_N].$$
$$P(x_1, x_2, \dots) \longmapsto P(x_1, \dots, x_N, x_{N+1} = 0, x_{N+2} = 0, \dots)$$

$$\mathbb{R}[x] = \varprojlim_{N \to +\infty} \mathbb{R}[x_1, x_2, \dots, x_N]$$ with respect to the morphisms

$$\mathbb{R}[x_1, \dots x_{N+1}] \xrightarrow{\phi_N^{N+1}} \mathbb{R}[x_1, \dots, x_N].$$ (in the category of graded real algebras).

Definition A polynomial $P \in \mathbb{R}[x]$ is a symmetric function if, for any permutation $\sigma \in \mathfrak{S}(\infty) = \bigcup_{n=1}^{\infty} \uparrow \mathfrak{S}(n)$,

$$P(x_{\sigma(1)}, x_{\sigma(2)}, \dots) = P(x_1, x_2, \dots)$$

$$\iff \forall N,\ \phi_N(P) \in \mathbb{R}[x_1, \dots, x_N]^{\mathfrak{S}(N)}.$$

$\mathrm{Sym} = \{\text{symmetric functions}\}$ is a graded subalgebra of $\mathbb{R}[x]$.

example 1 : $\forall k \geq 1$, $p_k(X) = \sum_{i=1}^{\infty} (x_i)^k$ k-th power sum.

For $\lambda \in \mathcal{Y} = \bigsqcup_{n \in \mathbb{N}} \mathcal{Y}(n)$, we set $p_\lambda(X) = p_{\lambda_1}(X) p_{\lambda_2}(X) \cdots p_{\lambda_\ell}(X)$.

p_λ is a symmetric function with degree $|\lambda|$.

example 2 : $\forall k \geq 1$, $e_k(X) = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1} x_{i_2} \cdots x_{i_k}$ k-th elementary symmetric function

$e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_\ell}$. $\deg e_\lambda = |\lambda|$

example 3 : $\forall k \geq 1$, $h_k(X) = \sum_{i_1 \leq i_2 \leq \cdots \leq i_k} x_{i_1} x_{i_2} \cdots x_{i_k}$ k-th homogeneous symmetric function.

$h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_\ell}$. $\deg h_\lambda = |\lambda|$.

We have for instance $h_2(X) = e_2(X) + p_2(X)$
$= e_{1^2}(X) - e_2(X)$.

Theorem : We have
$$\mathrm{Sym} = \mathbb{R}[p_1, p_2, \ldots] = \mathbb{R}[e_1, e_2, \ldots] = \mathbb{R}[h_1, h_2, \ldots].$$

Proof : ① First, let us show that the three families generate the same algebra.

$$P(t, X) = \sum_{k=1}^{\infty} \frac{p_k(X)}{k} t^k = \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \frac{x_i^k t^k}{k} = \sum_{i=1} -\log(1 - x_i t)$$
$$= \log\left(\prod_{i=1}^{\infty} \frac{1}{1 - x_i t} \right).$$

However, $\prod_{i=1}^{\infty} \frac{1}{1 - x_i t} = \prod_{i=1}^{\infty} \sum_{k_i \geq 0} (x_i t)^{k_i} = \sum_{k=0}^{\infty} h_k(X) t^k$ (with $h_0 = 1$)
$= H(t, X)$.

So, $\log H = P \Rightarrow$ the two families generate the same algebra.

We also have $\frac{1}{H(-t, X)} = \prod_{i=1}^{\infty} (1 + x_i t) = \sum_{k=0}^{\infty} e_k(X) t^k$ (with $e_0 = 1$)

So, $\frac{1}{H(-t)} = E(t) \Rightarrow$ the two families generate the same algebra.

(2) . If $x^I = x_{i_1}^{m_1} \cdots x_{i_\ell}^{m_\ell}$, denote $\lambda = t(x^I)$ the type of x^I, which is the integer partition obtained by reordering the integers $m_1 \dots m_\ell$.

$P \in Sym \iff$ the coefficient of x^I only depends of $t(x^I)$.

Therefore, a linear basis of Sym consists in the monomial functions $m_\lambda(x) = \sum_{t(x^I)=\lambda} x^I$.

examples : $m_2(x) = \sum_i x_i^2$

$m_{2,1}(x) = \sum_{i \neq j} x_i^2 x_j$

$m_{1,1}(x) = \sum_{i<j} x_i x_j$.

Denote $\lambda < \mu$ if μ can be obtained from λ by adding certain parts, and reordering

ex: $(5,3,3,1) < (6,5,1)$.

Then : $\forall \lambda$, $p_\lambda(x) = \left(\prod_{s \geq 1} m_s(\lambda)!\right) m_\lambda(x) + \sum_{\mu > \lambda} * \, m_\mu(x)$

$\longrightarrow$ triangular relation, and $(p_\lambda)_{\lambda \in \mathcal{Y}}$ is a linear basis of Sym.

example : $p_{5,3,3,1} = \sum_{i,j,k,\ell} x_i^5 x_j^3 x_k^3 x_\ell$

$= \sum_{i \neq j \neq k = \ell} \underline{\qquad}$ + terms with equalities

$= 2 m_{5,3,3,1} + 2 m_{8,3,1} + 2 m_{6,3,3} + m_{8,5,1} + 2 m_{5,4,3}$

$+ 2 m_{5,4,1} + m_{11,1} + 2 m_{9,3} + m_{7,5} + m_{12}$.

$+ 2 m_{6,6} + 2 m_{8,4}$ $\square$.

A last interesting basis of Sym consists in the Schur functions.

3. Schur functions

An antisymmetric polynomial in N variables is a polynomial $P \in \mathbb{R}[x_1 \dots x_N]$ such that

$\forall \sigma \in \mathfrak{S}(N)$, $P(x_{\sigma(1)}, \dots, x_{\sigma(N)}) = \varepsilon(\sigma)\, P(x_1, \dots, x_N)$.

example : $\Delta(x_1, \dots x_N) = \det\left(x_i^{N-j}\right)_{1 \le i,j \le N}$ = Vandermonde determinant $= \prod_{1 \le i < j \le N} (x_i - x_j)$.

<u>lemma 1</u> : if $P \in \mathrm{Antisym}_N$ and x^I appears with a non zero coefficient in P, then $t(x^I)$ is an integer partition with its parts all distinct.

$\longrightarrow$ a linear basis of $\mathrm{Antisym}_N$ consists in the functions $a_{\lambda+\rho_N} = \sum_{\sigma \in \mathfrak{S}(N)} \varepsilon(\sigma)\, x^{\sigma(\lambda+\rho_N)}$ with $\rho_N = (N-1, N-2, \dots, 0)$ and $\lambda \in \mathcal{Y}_{\text{length} \le N}$.

<u>lemma 2</u> : if $P \in \mathrm{Antisym}_N$, then $P = \Delta Q$ with $Q \in \mathrm{Sym}_N = \mathbb{R}[x_1 \dots x_N]^{\mathfrak{S}(N)}$.

the map $\psi_N : \begin{array}{ccc} \mathrm{Sym}_N & \to & \mathrm{Antisym}_N \\ f & \mapsto & f\Delta \end{array}$ is a $+\frac{N(N-1)}{2}$ graded isomorphism.

We set
$$s_\lambda(x_1, \dots, x_N) = \psi_N^{-1}(a_{\lambda+\rho_N}) = \frac{a_{\lambda+\rho_N}(x_1 \dots x_N)}{a_\lambda(x_1, \dots, x_N)} = \frac{\det\left(x_i^{\lambda_j + N - j}\right)}{\det\left(x_i^{N-j}\right)}.$$

<u>lemma 3</u> : We have $s_\lambda(x_1, \dots, x_N, x_{N+1} = 0) = s_\lambda(x_1, \dots, x_N)$ if $\ell(\lambda) \le N$.

$$\frac{\det\begin{pmatrix} x_i^{\lambda_j + N+1-j} & \begin{matrix} * \\ \vdots \\ * \end{matrix} \\ 0 \;\text{——}\; 0 & 1 \end{pmatrix}}{\det\begin{pmatrix} x_i^{N+1-j} & \begin{matrix} * \\ \vdots \\ * \end{matrix} \\ 0 \;\text{——}\; 0 & 1 \end{pmatrix}} = \frac{x_1 \cdots x_N}{x_1 \cdots x_N} \; \frac{a_{\lambda+\rho_N}(x_1 \cdots x_N)}{a_{\rho_N}(x_1 \cdots x_N)}.$$

$\Rightarrow$ There exists symmetric functions $s_{\lambda \in \mathcal{Y}}$ with $\deg s_\lambda = |\lambda|$
$$\mathrm{Sym} = \bigoplus_{\lambda \in \mathcal{Y}} \mathbb{R} s_\lambda .$$

new statement of the Thoma / Kerov-Vershik theorem

$\Omega = \{(\alpha, \beta) : \alpha, \beta$ nonincreasing non-negative sequence with $\sum_{i=1}^{\infty} \alpha_i + \beta_i \leq 1\}$

Thoma simplex.

Set $p_1(\alpha, \beta) = 1$ and $p_{k\geq 2}(\alpha, \beta) = \sum_{i=1}^{\infty} \alpha_i^k + (-1)^{k-1} \beta_i^k$.

As $\mathrm{Sym} = \mathbb{R}[p_1, p_2, \dots]$, this defines a morphism of algebras $\mathrm{Sym} \to \mathbb{R}$.

1. $\forall \lambda \in \mathbb{Y}$, $s_\lambda(\alpha, \beta) \geq 0$.
2. $\pi_n(\lambda) = |ST(\lambda)|\, s_\lambda(\alpha, \beta)$ is an extremal central measure
3. the corresponding infinite tableau $T = \emptyset \nearrow \lambda_1 \nearrow \lambda_2 \dots \nearrow \lambda_n \nearrow \dots$ satisfies:

   $\forall k \geq 1$, $\frac{\lambda_{n,k}}{n} \xrightarrow[p.s.]{} \alpha_k$; $\frac{\lambda'_{n,k}}{n} \xrightarrow[p.s.]{} \beta_k$.
4. All extremal central measures are of this kind.

4. Cauchy determinants and several properties of Schur functions.

Consider for $N \geq 1$ the Cauchy determinant:

$$C_N = \det\left(\frac{1}{1-x_i y_j}\right)_{1 \leq i,j \leq N}.$$

We have
$$C_N = \begin{vmatrix} \frac{1}{1-x_1y_1} & \cdots & \frac{1}{1-x_1y_N} \\ \vdots & & \vdots \\ \frac{1}{1-x_Ny_1} & \cdots & \frac{1}{1-x_Ny_N} \end{vmatrix} = \begin{vmatrix} \frac{(y_1-y_N)x_1}{1-x_1y_1} & \cdots & \frac{(y_{N-1}-y_N)x_1}{1-x_1y_{N-1}} & 1 \\ \vdots & & \vdots & \vdots \\ \frac{(y_1-y_N)x_N}{1-x_Ny_1} & \cdots & \frac{(y_{N-1}-y_N)x_N}{1-x_Ny_{N-1}} & 1 \end{vmatrix} \cdot \prod_{i=1}^{N} \frac{1}{1-x_iy_N}$$

$$= \prod_{i=1}^{N} \frac{1}{1-x_iy_N} \prod_{i=1}^{N-1} y_i - y_N \prod_{j=1}^{N-1} \frac{1}{1-x_Ny_j} \begin{vmatrix} \frac{(x_1-x_N)}{1-x_1y_1} & \cdots & \frac{(x_1-x_N)}{1-x_1y_{N-1}} & 0 \\ \frac{(x_{N-1}-x_N)}{1-x_{N-1}y_1} & \vdots & \frac{(x_{N-1}-x_N)}{1-x_{N-1}y_1} & 0 \\ * & & * & 1 \end{vmatrix}$$

$$= \prod_{i=1}^{N-1} (y_N - y_i)(x_N - x_i) \prod_{i=1}^{N} \frac{1}{1-x_i y_N} \prod_{j=1}^{N-1} \frac{1}{1-x_N y_j} \quad C_{N-1}.$$

$$\Longrightarrow \quad C_N = \frac{\Delta(x_1, \dots, x_N)\ \Delta(y_1 \dots y_N)}{\prod_{i,j} (1 - x_i y_j)}.$$

Corollaries. ① the Cauchy formula : given two commuting alphabets of variables, we have

$$\sum_{\lambda \in \mathcal{Y}} s_\lambda(X)\, s_\lambda(Y) = \prod_{i,j \geq 1} \frac{1}{1-x_i y_j}.$$

Indeed, it suffices to show the formula with finite alphabets $(x_1, \dots, x_N)$ $(y_1, \dots, y_N)$.

$$\Longleftrightarrow \quad C_N = \underbrace{\sum_{\lambda \in \mathcal{Y}} a_{\lambda + e_N}(X)\, a_{\lambda+e_N}(Y)}.$$

$$= \sum_{\lambda \in \mathcal{Y}} \Big(\sum_{\sigma \in \mathfrak{S}(N)} \varepsilon(\sigma)\, x^{\sigma(\lambda+e_N)} \Big)\Big(\sum_{\tau \in \mathfrak{S}(N)} \varepsilon(\tau)\, x^{\tau(\lambda+e_N)} \Big)$$

$$= \sum_{I \in \mathbb{N}^N} \sum_{e \in \mathfrak{S}(N)} \varepsilon(e)\, x^I\, y^{e(I)} = \sum_{e \in \mathfrak{S}(N)} \varepsilon(e) \prod_{i=1}^{N} \frac{1}{1-x_i y_{e(i)}}.$$

$$= C_N.$$

② the Jacobi-Trudy formula :

$$s_\lambda(X) = \det\left(h_{\lambda_i + j - i}\right)_{1 \leq i,j \leq \ell(\lambda)} \quad \text{with } h_0 = 1, \quad h_{k<0} = 0.$$

Indeed,

$$s_\lambda(x_1, \dots, x_N) = [y^{\lambda+e_N}] \Big(\sum_{\lambda \in \mathcal{Y}} s_\lambda(x_1 \dots x_N)\, a_{\lambda+e_N}(y_1 \dots y_N) \Big)$$

$$= [y^{\lambda+e_N}] \Big(\prod_{i<j} y_j - y_i \prod_{i,j} \frac{1}{1-x_i y_j} \Big)$$

$$= [y^{\lambda+e_N}] \sum_{\sigma \in \mathfrak{S}(N)} \varepsilon(\sigma) y^{\sigma(e_N)} \sum_{I \in \mathbb{N}^N} y^I h_I(x)$$

$$= \sum_{\sigma \in \mathfrak{S}(N)} \varepsilon(\sigma) \prod_{i=1}^{N} h_{\lambda_i - i + \sigma(i)}$$

$$= \det\left(h_{\lambda_i + j - i} \right) .$$