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Techniques d'analyse harmonique et résultats asymptotiques en théorie des probabilités

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Avant-propos

L'objectif de ce mémoire est de présenter les résultats de recherche que j'ai obtenus depuis la thèse de doctorat que j'ai soutenue en 2010. J'ai donc dû trouver un moyen de résumer ces résultats, et initialement cette tâche m'a parue assez difficile. En effet, j'ai travaillé sur des sujets aussi variés que :

- les estimées de type Berry–Esseen pour des suites de variables aléatoires (théorie des probabilités classique) ;
- les mouvements browniens sur des espaces symétriques (théorie des processus aléatoires, géométrie, un peu de théorie des représentations) ;
- les nombres de diviseurs premiers d'entiers aléatoires (théorie analytique des nombres, théorie des probabilités) ;
- les spectres de graphes aléatoires géométriques tracés sur des groupes de Lie (théorie des représentations des groupes de Lie, bases crystallines de Lusztig–Kashiwara).

Un point commun entre ces divers sujets est que j'aime regarder des objets mathématiques *aléatoires*, et étudier leurs *propriétés asymptotiques* (la plupart du temps, lorsque la taille du modèle tend vers l'infini). On pourrait faire l'erreur de penser que c'est le caractère aléatoire de ces modèles qui suscite mon intérêt. En fait, c'est tout l'inverse : je m'intéresse aux *structures mathématiques* cachées derrière ces modèles aléatoires au comportement chaotique, et je recherche la façon dont on peut utiliser ces structures algébriques ou analytiques pour démontrer des résultats asymptotiques.

Ainsi, le fil rouge de mes travaux n'est pas l'aspect aléatoire, mais plutôt l'outil que j'utilise presque toujours pour vaincre l'aléatoire : l'*analyse harmonique*, et les divers avatars de la *transformée de Fourier*. Rappelons que si f appartient à l'espace $\mathscr{L}^1(\mathbb{R})$ des fonctions intégrables sur la droite réelle, sa transformée de Fourier est la fonction continue bornée

$$\xi \mapsto \widehat{f}(\xi) = \int_{\mathbb{R}} f(x)\, \mathrm{e}^{\mathrm{i}\xi x}\, dx.$$

Dans la suite, $\mathscr{C}_{\mathrm{b}}(\mathbb{R})$ désigne l'espace des fonctions continues bornées sur $\mathbb{R}$ et à valeurs complexes. Par densité ou par dualité, on peut étendre la transformée de Fourier à d'autres espaces fonctionnels, par exemple l'espace des fonctions de carré intégrale $\mathscr{L}^2(\mathbb{R})$, ou l'espace des distributions tempérées $\mathscr{S}'(\mathbb{R})$. Si f et $\widehat{f}$ sont toutes les deux intégrables, alors on peut retrouver f à partir de $\widehat{f}$, en utilisant la formule d'inversion de Fourier

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}(\xi)\, \mathrm{e}^{-\mathrm{i}\xi x}\, d\xi. \tag{0.1}$$

D'autre part, la transformée de Fourier est une isométrie de $\mathscr{L}^2(\mathbb{R})$ (à un facteur $\frac{1}{2\pi}$ près) :

$$\forall f, g \in \mathscr{L}^2(\mathbb{R}),\;\; \langle f \mid g \rangle = \frac{1}{2\pi} \left\langle \widehat{f} \,\middle|\, \widehat{g} \right\rangle. \tag{0.2}$$

Finalement, la transformée de Fourier de la convolution de deux fonctions est le produit de leurs transformées de Fourier :

$$\forall f,g \in \mathscr{L}^1(\mathbb{R}),\ \ \widehat{(f*g)}(\xi) = \widehat{f}(\xi)\,\widehat{g}(\xi), \tag{0.3}$$

avec $(f*g)(x) = \int_{\mathbb{R}} f(x-y)\,g(y)\,dy$. Les trois formules (0.1) (*formule d'inversion*), (0.2) (*formule de Parseval*) et (0.3) (*formule de convolution*) sont les propriétés générales que l'on attend de toute extension ou analogue de la transformée de Fourier classique $\mathscr{L}^1(\mathbb{R}) \to \mathscr{C}_{\mathrm{b}}(\mathbb{R})$. L'essentiel de mes travaux repose sur de telles généralisations de la transformée de Fourier, avec $\mathscr{L}^1(\mathbb{R})$ remplacé par :

- l'espace $\mathscr{M}^1(\mathbb{R})$ des mesures de probabilité boréliennes sur $\mathbb{R}$;
- ou, l'algèbre de groupe $\mathbb{C}\mathfrak{S}(n)$ des fonctions sur le groupe symétrique de taille n ;
- ou, l'algèbre de convolution $\mathscr{L}^2(X)$ des fonctions de carré intégrable sur un espace $X = G/K$, où G est un groupe de Lie compact.

▷ **Fluctuations des variables aléatoires et convergence mod-ϕ.** En théorie des probabilités classique, l'espace $\mathscr{L}^1(\mathbb{R})$ est remplacé par l'espace des mesures de probabilité $\mathscr{M}^1(\mathbb{R})$, et on peut définir la transformée de Fourier d'une mesure $\mu \in \mathscr{M}^1(\mathbb{R})$ en utilisant essentiellement la même formule que précédemment :

$$\widehat{\mu}(\xi) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\xi x}\,\mu(dx).$$

La formule de convolution est encore vérifiée, et il en va de même pour la formule de Parseval dans le contexte suivant : si f est une fonction dans $\mathscr{L}^1(\mathbb{R})$ avec $\widehat{f}$ également dans $\mathscr{L}^1(\mathbb{R})$, alors pour toute mesure de probabilité μ sur la droite réelle,

$$\mu(f) = \int_{\mathbb{R}} f(x)\,\mu(dx) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}(\xi)\,\widehat{\mu}(-\xi)\,d\xi.$$

D'autre part, la formule d'inversion de Fourier est différente selon que la distribution μ est discrète supportée par un réseau, ou continue. Par exemple, si μ n'est pas supportée par un réseau et si sa transformée de Fourier $\widehat{\mu}$ est intégrable sur la droite réelle, alors μ est absolument continue par rapport à la mesure de Lebesgue, et sa densité est donnée par la formule d'inversion (0.1). Nous renvoyons à [Fel71, Chapitre XV], [Mal95, Chapitre III] et [Str11, Section 2.3] pour la théorie classique des transformées de Fourier de mesures de probabilité ; un court exposé est également proposé dans [FMN17b, Section 2.2]. La formule d'inversion de Fourier fournit un dictionnaire entre les propriétés d'une mesure de probabilité μ, et les propriétés de sa transformée de Fourier $\widehat{\mu}$. Par exemple, le comportement de $\widehat{\mu}$ au voisinage de 0 impose la taille de la queue de la distribution μ, *via* l'inégalité

$$\mu(\mathbb{R} \setminus [-2C, 2C]) \leq C \int_{-\frac{1}{C}}^{\frac{1}{C}} (1 - \widehat{\mu}(\xi))\,d\xi.$$

Une autre raison pour laquelle les techniques d'analyse harmonique sont très utiles en théorie des probabilités est que la transformée de Fourier échange la convergence en loi dans $\mathscr{M}^1(\mathbb{R})$, et la convergence localement uniforme dans $\mathscr{C}_{\mathrm{b}}(\mathbb{R})$. Étant donnée une suite $(V_n)_{n\in\mathbb{N}}$ de variables aléatoires à valeurs réelles, il existe souvent une renormalisation $(V_n/s_n)_{n\in\mathbb{N}}$ de la suite telle que les transformées de Fourier $\mathbb{E}[\mathrm{e}^{\mathrm{i}\xi V_n/s_n}]$ admettent une limite $F(\xi)$, et donc telle que les lois μ_n des variables aléatoires renormalisées V_n/s_n convergent au sens faible vers la distribution μ dont la transformée de Fourier est $\widehat{\mu} = F$. Lorsqu'on remet à l'échelle les variables aléatoires V_n et lorsqu'on regarde V_n/s_n, on obtient la bonne normalisation pour décrire l'aspect général des fluctuations de V_n, mais on perd également beaucoup d'information concernant le comportement précis des variables V_n. Ce point n'avait pas été mis en lumière jusqu'à récemment, lorsqu'une autre *théorie de la renormalisation* a été inventée afin de compléter la convergence en loi $\frac{V_n}{s_n} \rightharpoonup \mu$ par d'autres résultats plus précis :

- des grandes déviations (comportement au bord de la distribution asymptotique des fluctuations),
- des théorèmes locaux (comportement dans des régions infinitésimales),
- des estimées de vitesse de convergence,

etc. Cette théorie est celle de la *convergence mod-ϕ*, et la première définition de cette notion a été donnée il y a presque dix ans dans un article de Jacod, Kowalski et Nikeghbali [JKN11] (dans le cas spécifique de la convergence mod-gaussienne). Durant mon séjour post-doctoral à Zürich, j'ai eu la chance de commencer à travailler sur cette théorie, et nous l'avons développée depuis avec Valentin Féray et Ashkan Nikeghbali [FMN16; FMN17a; FMN17b; FMN17c; MN15], et plusieurs autres coauteurs [Chh+15; BMN17]. Nous avons ainsi établi de nombreux résultats théoriques reliés à la notion de suites mod-ϕ convergentes, et nous avons aussi étudié de larges classes d'exemples, et montré que ces exemples rentraient dans ce cadre.

Les deux premiers chapitres de ce mémoire sont consacrés à une présentation de ces résultats. Afin de donner un sommaire précis de ces deux chapitres, expliquons informellement la notion de convergence mod-ϕ, en nous concentrant sur le cas gaussien. Supposons donnée une suite $(V_n)_{n\in\mathbb{N}}$ de variables aléatoires telle que $V_n/s_n \rightharpoonup \mathcal{N}_{\mathbb{R}}(0,1)$, les s_n étant des paramètres de renormalisation tendant vers l'infini. En termes de transformées de Fourier, la convergence en loi est équivalente à l'estimée asymptotique

$$\mathbb{E}\left[\mathrm{e}^{\mathrm{i}\xi \frac{V_n}{s_n}}\right] = \mathrm{e}^{-\frac{\xi^2}{2}}\,(1+o(1)). \tag{0.4}$$

Une idée assez naturelle est que si l'on connaît plus de choses sur les transformées de Fourier, alors on obtiendra des résultats asymptotiques plus précis que la convergence en loi. Avec la renormalisation V_n/s_n, le $o(1)$ dans le terme de droite de l'équation (0.4) est en fait une fonction $f(\xi,n)$ telle que $\lim_{n\to\infty} f(\xi,n) = 0$ pour tout ξ. En prenant ces limites, on perd la dépendance en ξ et toutes les informations pertinentes en dehors du fait que la suite $(V_n/s_n)_{n\in\mathbb{N}}$ est asymptotiquement normale. Une façon de surmonter ce problème consiste à chercher une autre renormalisation V_n/r_n telle que, si $\sqrt{t_n} = \frac{s_n}{r_n}$, alors

$$\begin{aligned}\mathbb{E}\left[\mathrm{e}^{\mathrm{i}\xi \frac{V_n}{r_n}}\right] &= \mathbb{E}\left[\mathrm{e}^{\mathrm{i}\sqrt{t_n}\xi \frac{V_n}{s_n}}\right] \\ &= \mathrm{e}^{-\frac{t_n \xi^2}{2}}\,\left(1+f(\sqrt{t_n}\xi,n)\right) \quad \text{avec } \lim_{n\to\infty}\left(1+f(\sqrt{t_n}\xi,n)\right) = \theta(\xi)\end{aligned}$$

pour une certaine fonction non triviale $\theta(\xi)$. Ainsi, au lieu de dire que V_n/s_n converge vers une distribution gaussienne, on dira que V_n/r_n ressemble à une grande variable gaussienne de variance t_n, plus un résidu encodé asymptotiquement par $\theta(\xi)$. Notons que la bonne renormalisation $X_n = \frac{V_n}{r_n}$ n'est pas toujours facile à trouver, et qu'on ne peut pas établir son existence simplement en prenant un terme additionnel dans le développement de Taylor des transformées de Fourier de ces variables. Le résidu $\theta(\xi)$ contient en fait des informations spécifiques au modèle qui produit les variables aléatoires V_n, et la théorie de la convergence mod-ϕ nous a permis d'extraire ces informations et de démontrer de nouvelles estimées asymptotiques des probabilités pour la suite $(V_n)_{n\in\mathbb{N}}$.

Dans le chapitre 1, nous présentons les conséquences théoriques de la notion de convergence mod-ϕ, qui est la généralisation de la définition informelle précédente avec une loi infiniment divisible arbitraire au lieu de la distribution gaussienne standard. Si l'on est capable de démontrer la convergence mod-ϕ d'une suite, alors on obtient d'un seul coup de nombreux résultats asymptotiques :

- un théorème central limite avec une zone de convergence étendue (Théorème 1.21) ;
- des résultats de grandes déviations (Théorèmes 1.12 and 1.13) ;
- des estimées de Berry–Esseen pour la vitesse de convergence (Théorèmes 1.14 et 1.28 dans le cas continu, et Théorème 1.36 dans le cas discret) ;
- et un théorème limite local valable pour une grande plage d'échelles infinitésimales (Théorème 1.49).

Ces divers résultats reposent sur des hypothèses légèrement différentes. Pour les deux premiers (théorème central limite étendu et principes de grandes déviations), on demandera des estimées de la transformée de Laplace complexe au lieu de la transformée de Fourier. Alors, dans le cas particulier où $(X_n)_{n\in\mathbb{N}}$ est une suite de variables aléatoires telle que

$$\mathbb{E}[\mathrm{e}^{zX_n}] = \mathrm{e}^{t_n \frac{z^2}{2}}\, \psi(z)\, (1+o(1))$$

avec $t_n \to +\infty$ (ce qui sera la définition d'une suite convergente au sens mod-Gaussien avec paramètres t_n et limite ψ, voir Définition 1.1), on obtient les résultats suivants :

- Si $Y_n = \frac{X_n}{\sqrt{t_n}}$, alors pour toute suite déterministe $x_n = o(\sqrt{t_n})$,

$$\mathbb{P}[Y_n \geq x_n] = \left(\int_{x_n}^{\infty} \frac{\mathrm{e}^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\,dx\right)(1+o(1)),$$

  donc la zone de normalité de $(Y_n)_{n\in\mathbb{N}}$ est de taille $o(\sqrt{t_n})$.
- Au bord de cette zone, l'estimée gaussienne est corrigée par le résidu ψ, et si $x > 0$, alors

$$\mathbb{P}[Y_n \geq x\sqrt{t_n}] = \frac{\mathrm{e}^{-\frac{t_n x^2}{2}}}{\sqrt{2\pi t_n}\, x}\, \psi(x)\, (1+o(1)).$$

Les deux autres types de résultats asymptotiques nécessitent seulement une estimée des transformées de Fourier, mais sur une zone assez large si l'on veut des résultats optimaux. Dans le cas particulier où $(X_n)_{n\in\mathbb{N}}$ est une suite de variables aléatoires telle que

$$\mathbb{E}[\mathrm{e}^{\mathrm{i}\xi X_n}]\, \mathrm{e}^{t_n \frac{\xi^2}{2}} = \theta_n(\xi) = \theta(\xi)\, (1+o(1))$$

avec $t_n \to +\infty$, si l'on a par exemple une borne supérieure

$$|\theta_n(\xi) - 1| \leq K_1 |\xi|^3 \exp(K_2 |\xi|^3)$$

sur une zone $\xi \in [-K(t_n)^\gamma, K(t_n)^\gamma]$ avec $\gamma \in [0,1]$, alors la distance de Kolmogorov entre $Y_n = X_n/\sqrt{t_n}$ et la distribution gaussienne standard est un $O((t_n)^{\frac{1}{2}+\gamma})$. Par ailleurs, sous les mêmes hypothèses, l'approximation normale de Y_n est valable sur des parties Jordan mesurables B_n de mesure de Lebesgue $L(B_n) \gg (t_n)^{-\frac{1}{2}-\gamma}$.

Dans le chapitre 2, nous expliquons quelles structures mènent à la convergence mod-ϕ. Si les transformées de Fourier ou de Laplace des variables aléatoires V_n étudiées sont explicites, alors on a seulement besoin des techniques usuelles d'analyse (Section 2.1). Néanmoins, cette classe inclut des exemples nouveaux et non triviaux : nombre de zéros d'une série entière aléatoire, nombre de cycles d'une permutation aléatoire choisie suivant une mesure de probabilité donnant un poids différent à chaque taille de cycle, nombre de facteurs irréductibles d'un polynôme aléatoire sur un corps fini, nombre de facteurs premiers (distincts ou comptés avec multiplicité) d'un entier aléatoire, *etc.* Dans chacun de ces cas, la convergence mod-ϕ est établie en étudiant l'asymptotique de la formule explicite pour la transformée de Fourier, celle-ci pouvant être donnée par une intégrale de Cauchy ou de Dirichlet dans le plan complexe.

Dans le cas de variables mod-Poisson, la théorie de la convergence mod-ϕ permet la construction de schémas d'approximation des variables discrètes étudiées par des mesures signées, ces approximations étant bien plus précises que la simple approximation poissonienne. De plus, le formalisme des fonctions symétriques et de leurs spécialisations permet d'encoder ces approximations de façon très concise. Par exemple, supposons que l'on souhaite approximer la loi du nombre aléatoire ω_n de diviseurs premiers distincts d'un entier aléatoire dans $[\![1, n]\!]$. Alors, pour construire une mesure signée ν_n dont la distance en variation totale à la loi de ω_n est un $O((\log \log n)^{-b})$ pour un entier $b \geq 1$, il suffit de manipuler l'alphabet infini

$$\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots\right\}_{n \geq 1} \sqcup \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots, \frac{1}{p}, \ldots\right\}_{p \in \mathbb{P}}$$

et la spécialisation correspondante de certaines fonctions symétriques de degré plus petit que $2b$. Une méthode similaire d'approximation est proposée pour de nombreuses autres suites aléatoires discrètes de nature combinatoire ou arithmétique.

À l'opposée de ces structures très rigides et algébriques, la convergence mod-gaussienne ne nécessite pas un calcul explicite des transformées de Fourier. Ainsi, nous donnons dans le second chapitre un critère général sur les *cumulants* d'une suite de variables aléatoires qui garantit la convergence mod-gaussienne, et qui à son tour est impliqué par certaines structures de dépendance : les graphes de dépendance creux (Théorème 2.19) et les graphes de dépendance pondérés (Théorème 2.35). Ces résultats permettent de montrer que de nombreux modèles de graphes aléatoires, partitions aléatoires, permutations aléatoires, configurations aléatoires de spins, *etc.* produisent des variables aléatoires (les *observables* du modèle) qui sont convergentes mod-gaussiennes. Par exemple, étant donné un graphe aléatoire $G(n, \gamma)$ sur n sommets associé à un graphon γ (classe d'équivalence dans l'espace des fonctions mesurables symétriques $[0, 1]^2 \to [0, 1]$, modulo les Lebesgue-isomorphismes de $[0, 1]$), pour tout motif F (graphe fini fixé), lorsque n tend vers l'infini, le nombre de motifs $\hom(F, G(n, \gamma))$ est convergent au sens mod-gaussien après une renormalisation appropriée, comme conséquence de la méthode des cumulants et des graphes de dépendance. De même, si l'on considère la magnétisation $M_n = \sum_{x \in \Lambda_n} \sigma(x)$ d'une boîte $\Lambda_n \subset \mathbb{Z}^d$ dans le modèle d'Ising à très haute température $T \gg T_{\text{critique}}$ et sans champ extérieur, et si la taille de cette boîte tend vers l'infini, alors une renormalisation appropriée de M_n converge au sens mod-gaussien, comme conséquence de la théorie des graphes de dépendance pondérés.

Le critère sur les cumulants mentionné précédemment est une borne supérieure impliquant trois paramètres A, D_n, N_n :

$$\forall r \geq 1,\ \ \forall n \in \mathbb{N},\ \ \left|r!\,[z^r](\log \mathbb{E}[\mathrm{e}^{zS_n}])\right| = |\kappa^{(r)}(S_n)| \leq r^{r-2}\,(2D_n)^{r-1}\,N_n\,A^r. \tag{0.5}$$

Si cette borne supérieure est satisfaite, si $D_n \ll N_n$ et si les trois premiers cumulants de S_n vérifient certaines hypothèses asymptotiques, alors la suite renormalisée

$$X_n = \frac{S_n - \mathbb{E}[S_n]}{(N_n\,(D_n)^2)^{1/3}}$$

est convergente au sens mod-gaussien. D'autre part, si S_n est une somme de N_n variables bornées par A et admettant un graphe de dépendance ou un graphe de dépendance pondéré de degré maximal plus petit que D_n, alors les cumulants de S_n satisfont l'inégalité (0.5). Ce résultat est à rapprocher de certains résultats d'analyticité en physique mathématique, car il donne des informations sur le rayon de convergence des séries génératrices des variables aléatoires S_n. En plus de la convergence mod-gaussienne, il implique des inégalités de concentration semblables à l'inégalité classique de Hoeffding, mais valables dans un contexte nettement plus général (Proposition 2.33). On retrouve ainsi des inégalités qui jusque-là ne pouvaient être obtenues qu'à l'aide de techniques de martingales, et qui sont fondamentales pour l'étude des motifs dans des modèles de graphes aléatoires ou de permutations aléatoires. La théorie des graphes de dépendance (pondérés) offre une nouvelle approche pour l'étude des fluctuations de sommes de variables aléatoires, et on conjecture qu'elle nous permettra de comprendre précisément le comportement des chaos arithmétiques et des fonctionnelles de systèmes dynamiques ergodiques mélangeants.

Au terme des deux premiers chapitres, il aura été montré que la théorie de la convergence mod-ϕ permet une étude unifiée des diverses échelles de fluctuations d'une suite de variables aléatoires. Dans le cas spécifique mod-gaussien, cette approche unifiée est complétée par des résultats d'universalité. Par exemple, pour une classe très générale de modèles de graphes aléatoires (les modèles de graphons), toute observable de tout modèle est génériquement mod-gaussienne, avec des paramètres de convergence mod-gaussienne qui dépendent continuement des paramètres du modèle. Un phénomène semblable a lieu pour des modèles de permutations aléatoires et des modèles de partitions aléatoires. Nous proposons dans la section 2.3 une notion d'*espace de modules mod-gaussien* qui formalise cette universalité, et qui autorise une étude géométrique des modèles. En particulier, les points singuliers de ces espaces de modules sont des modèles aléatoires avec des symétries additionnelles, et ces symétries se traduisent parfois par une modification des paramètres de convergence mod-gaussienne, et par la modification correspondante des échelles de fluctuations. À notre connaissance, cette approche géométrique de la classification des modèles aléatoires est inédite. Dans ce cadre, nous devrions pouvoir étudier les modèles singuliers en construisant de nouvelles structures de dépendance, qui seraient des dégénérations de la structure générique de graphe de dépendance :

- graphes de dépendance pondérés ;
- graphes de dépendance lacunaires où seulement certains arbres couvrants sont pris en compte dans les bornes sur les cumulants ;
- surfaces de dépendance qui miment les expansions topologiques des moments de certains modèles, mais les adaptent à l'étude des cumulants.

▷ **Probabilités sur des groupes de Lie et sur des espaces symétriques compacts.** Notre troisième chapitre est consacré à l'étude d'un autre type d'objets aléatoires, et à l'utilisation d'un autre avatar de la transformée de Fourier, à savoir, la *transformée de Fourier non commutative* d'un groupe compact ou d'un quotient de groupes compacts. Fixons un groupe compact G, et notons dg sa mesure de Haar, qui est l'unique mesure de probabilité sur G

qui est invariante à gauche et à droite. Pour définir la transformée de Fourier d'une fonction $f \in \mathscr{L}^1(G, dg)$, on doit remplacer la fonction

$$\rho^\xi : x \mapsto e^{ix\xi}$$

par une représentation unitaire de G sur un espace vectoriel complexe de dimension finie :

$$\begin{aligned} \rho^\lambda : G &\to \mathrm{U}(V^\lambda) \\ g &\mapsto \rho^\lambda(g). \end{aligned}$$

Ainsi, la transformée de Fourier non commutative $\widehat{f}$ est une fonction sur l'ensemble $\widehat{G}$ des représentations unitaires irréductibles $\lambda = (V^\lambda, \rho^\lambda)$, avec $\widehat{f}(\lambda) \in \mathrm{End}_{\mathbb{C}}(V^\lambda)$ donné par la formule

$$\widehat{f}(\lambda) = \int_G f(g)\, \rho^\lambda(g)\, dg.$$

Notons qu'on manipule uniquement des représentations de dimension finie, grâce à la compacité de G. Si $f \in \mathscr{L}^2(G, dg)$, alors on obtient des analogues parfaits des trois formules (0.1), (0.2) et (0.3). En particulier, la transformée de Fourier est une isométrie entre $\mathscr{L}^2(G)$ et la somme hilbertienne $\mathscr{L}^2(\widehat{G}) = \bigoplus_{\lambda \in \widehat{G}} \mathrm{End}(V^\lambda)$. Ces propriétés peuvent être utilisées pour étudier une mesure de probabilité μ sur G dont la densité par rapport à la mesure de Haar est de carré intégrable. Considérons par exemple une suite de mesures de probabilités $(\mu_n)_{n \in \mathbb{N}}$ qui converge vers la mesure de Haar $\eta = dg$ lorsque n tend vers l'infini. Dans le monde de Fourier non commutatif, ceci se traduit par :

$$\forall \lambda \in \widehat{G},\ \ \widehat{\mu_n}(\lambda) \to_{n \to \infty} \widehat{\eta}(\lambda) = \begin{cases} 0 & \text{si } V^\lambda \neq \mathbb{C}, \\ 1 & \text{si } V^\lambda = \mathbb{C}, \end{cases}$$

où $\mathbb{C}$ désigne la représentation triviale de dimension 1 de G. En particulier, la théorie des représentations du groupe G fournit une méthode très générale pour étudier des marches aléatoires sur des groupes compacts et leur convergence vers la loi uniforme. Cette idée est apparue pour la première fois dans des travaux de Poincaré au début du vingtième siècle, et elle est devenue assez populaire après plusieurs articles célèbres de Diaconis sur la combinatoire de l'opération de brassage de cartes ; voir [DS81; AD86; BD92; Dia96], et également [CST08, Section 10.7]. Dans [Mél14b], nous avons résolu une conjecture de Saloff-Coste [Sal10] concernant la convergence vers la loi stationnaire des mouvements browniens sur des groupes de Lie compacts tels que $\mathrm{SU}(n)$ ou $\mathrm{SO}(n)$. Ainsi, nous avons démontré que cette convergence avait lieu dans une courte fenêtre autour d'un temps de coupure

$$t_{\text{coupure}} \propto \log n,$$

où n est le rang du groupe (dimension d'un tore maximal). En particulier, plus la dimension du groupe de Lie est grande, et plus la transition de phase vers la stationnarité est étroite (par rapport au temps de coupure). Ces résultats sont présentés dans le chapitre 3 (Théorème 3.23), après un bref survol de la théorie des représentations des groupes de Lie compacts (Section 3.1). Ils sont également vrais sur des *espaces symétriques* compacts, qui sont des généralisations des groupes de Lie compacts, et qui en sont des quotients. La famille des espaces symétriques contient les sphères, les espaces projectifs, les variétés grassmanniennes, *etc.*, et le survol proposé dans la section 3.1 traite également de l'analyse harmonique de ces espaces.

Détaillons un peu plus les idées de la preuve du phénomène de coupure pour les mouvements browniens. Étant donné un espace symétrique X, la transformée de Fourier non commutative fournit une décomposition explicite de la densité de la loi du mouvement brownien au temps t par rapport à la mesure de Haar :

$$\frac{d\mu_t(x)}{d\nu} = \sum_{\lambda} m_\lambda \, \mathrm{e}^{-a_\lambda t} \, f_\lambda(x),$$

où les fonctions f_λ sont orthornormales dans $\mathscr{L}^2(X)$, et où la somme porte sur un ensemble de représentations irréductibles

- de X si X est un groupe de Lie compact,
- de $\mathrm{Isom}(X)$ si X est un espace symétrique de type non-groupe.

Cette décomposition permet d'estimer les normes $\mathscr{L}^p$ de $|d\mu_t/d\nu - 1|$, en utilisant en particulier l'orthonormalité des fonctions f_λ pour le cas $p = 2$. On en déduit que $d\mu_t/d\nu$ est uniformément proche de 1 pour les normes $\mathscr{L}^p$ dès que les quantités $m_\lambda \, \mathrm{e}^{-a_\lambda t}$ deviennent petites ; et ceci se produit à peu près au même moment t_{coupure} pour toutes les représentations irréductibles λ impliquées dans la décomposition de la densité. Ces arguments donnent une borne supérieure pour le temps de coupure, et ils sont très robustes, car ils peuvent être utilisés pour n'importe quel processus aléatoire sur un espace possédant un groupe de symétrie compact. Pour la borne inférieure, il faut montrer à l'inverse que pour tout temps $t < t_{\text{coupure}}$, la loi μ_t charge des événements dont la mesure de Haar est presque nulle. L'idée est alors de trouver des fonctions discriminantes dont les distributions sont très différentes sous μ_t et sous ν ; et les fonctions f_λ sont des candidates naturelles. Le calcul de l'espérance et de la variance de ces fonctions sous une mesure brownienne μ_t est rendu possible par des arguments de calcul stochastique, en utilisant l'équation différentielle stochastique qui définit un mouvement brownien ou plus généralement un processus de Lévy sur un groupe de Lie.

La théorie des représentations d'un groupe G permet d'étudier des objets plus complexes que des points aléatoires pris dans un G-espace homogène. L'idée générale est la suivante : si un objet aléatoire tracé sur un G-espace homogène peut être encodé au moins partiellement dans une représentation du groupe G, et si ce groupe G appartient à une famille classique (groupes symétriques, groupes unitaires, *etc.*), alors les propriétés asymptotiques de cet objet aléatoire (loi des grands nombres et fluctuations lorsque la taille de l'objet tend vers l'infini) peuvent être reliées à la *théorie asymptotique des représentations* de cette famille de groupes :

- soit lorsque la taille du groupe tend vers l'infini,
- ou, lorsque la dimension de la représentation tend vers l'infini.

Un exemple classique de cette approche est la résolution du problème d'Ulam des plus longues sous-suites croissantes d'une permutation aléatoire, qui repose sur l'étude asymptotique des mesures de Plancherel des groupes symétriques (mesures spectrales des représentations régulières). Des généralisations de cet exemple formaient le contenu de ma thèse, voir les articles [Mél10; Mél11; Mél12; FM12; Mél14a] et la monographie [Mél17]. Plus récemment, j'ai commencé à étudier des *graphes aléatoires géométriques* tracés sur des espaces symétriques compacts [Mél18]. Ces graphes sont construits de la façon suivante : on prend N points aléatoires indépendents et uniformes dans un espace symétrique X, et on connecte toutes les paires de points $\{x, y\}$ qui sont suffisamment proches. Plus précisément, on relie x à y si et seulement si $d(x, y) \leq L$, avec un niveau L qui peut dépendre de N. Lorsque N tend vers l'infini, l'analyse harmonique des espaces symétriques permet de décrire le spectre asymptotique de la matrice d'adjacence de tels graphes aléatoires. Les deux dernières sections 3.3 et 3.4 de ce mémoire présentent les résultats obtenus jusqu'ici dans cette direction.

Si $L > 0$ est fixé et si N tend vers l'infini (*régime gaussien* des graphes géométriques), le comportement asymptotique du spectre de la matrice d'adjacence d'un graphe aléatoire géométrique est relié aux valeurs propres d'un opérateur intégral de classe Hilbert–Schmidt sur l'espace métrique compact mesuré X sous-jacent au graphe. Lorsque X est un groupe de Lie compact, une sphère ou un espace projectif (Théorèmes 3.28 et 3.35), ces valeurs propres peuvent être calculées explicitement. Ainsi, on montre que si $A_{\Gamma(N,L)}$ est la matrice d'adjacence, alors les valeurs propres de $A_{\Gamma(N,L)}/N$ convergent presque sûrement vers des quantités déterministes indexées par les représentations irréductibles du groupe ou par les représentations sphériques du groupe d'isométries de la sphère ou de l'espace projectif. Les formules donnant ces quantités limites mettent en jeu des fonctions de Bessel dans le cas des groupes, et des polynômes orthogonaux de Laguerre ou Jacobi dans le cas des sphères et des espaces projectifs. La preuve de ces formules asymptotiques repose :

- sur un résultat général dû à Kolchinskii et Giné, qui assure que certaines grandes matrices aléatoires symétriques ont leurs spectres proches de celui d'un opérateur intégral de classe Hilbert–Schmidt, cet argument pouvant être appliqué aux matrices d'adjacence des graphes aléatoires géométriques.
- pour les espaces de type groupe, sur la formule d'intégration de Weyl, qui permet de calculer les valeurs propres des opérateurs limites des matrices d'adjacence.
- pour les espaces de type non-groupe et de rang 1, sur des formules explicites pour les fonctions zonales sphériques, qui mettent en jeu des polygônes orthogonaux du paramètre zonal.

Le cas le plus intéressant est celui où N tend vers l'infini et $L = L_N$ tend vers 0 de telle sorte qu'un sommet du graphe aléatoire géométrique sur X avec N points et niveau L_N a un nombre de voisins en $O(1)$ (*régime poissonien* des graphes géométriques). Lorsque $X = G$ est un groupe de Lie compact, nous avons une conjecture et de nombreux résultats partiels sur la distribution spectrale asymptotique de la matrice d'adjacence (voir en particulier le Théorème 3.38 et la Conjecture 3.41). Plus précisément, notons

$$\nu_N = \frac{1}{N} \sum_{e \text{ valeur propre de } A_{\Gamma(N,L)}} \delta_e$$

la mesure spectrale aléatoire de la matrice d'adjacence, et $\mu_N = \mathbb{E}[\nu_N]$, l'espérance étant prise dans l'espace vectoriel des mesures signées et donnant une mesure de probabilité déterministe pour tout N. En utilisant des arguments géométriques et la convergence locale au sens Benjamini–Schramm des graphes géométriques aléatoires poissoniens, on peut montrer qu'il existe une mesure limite μ_∞ telle que $\mu_N \rightharpoonup_{N\to\infty} \mu_\infty$. Nous avons démontré ceci en établissant un résultat général reliant la convergence au sens Lipschitz des espaces métriques propres pointés aléatoirement à la convergence Benjamini–Schramm des graphes géométriques dessinés sur ces espaces. Malheureusement, notre preuve de la convergence des espérances de mesures spectrales ne donne aucune information sur la limite μ_∞ (support, atomes, régularité, *etc.*). Néanmoins, nous conjecturons les deux propriétés suivantes :

- Les mesures spectrales μ_N des matrices d'adjacence $A_{\Gamma(N,L)}$ devraient converger en probabilité vers μ_∞ (sans prendre l'espérance).
- Les moments de μ_∞ devraient admettre un développement combinatoire dont les termes correspondent à certains graphes finis (circuits et circuits réduits).

Chacun des termes de ces développements combinatoires est donné par une intégrale de fonctions de Bessel sur un produit de chambres de Weyl, avec une mesure d'intégration qui est une fonction polynomiale par morceaux et dont les domaines de polynomialité sont polyhédraux. Ces mesures d'intégration sont issues de la théorie asymptotique des représentations de G, et leur forme conjecturelle provient de la théorie des bases cristallines et des polytopes en cordes. Cette branche récente de la théorie des représentations des groupes et algèbres de Lie permet d'indexer de façon cohérente des bases de vecteurs de poids des représentations irréductibles d'un groupe de Lie G, et de calculer les coefficients de structure (coefficients de Littlewood–Richardson et leurs généralisations) de l'algèbre des représentations du groupe. Plus précisément, on peut voir la *théorie des cristaux* comme une amélioration importante de la formule des caractères de Weyl

$$\mathrm{ch}^{\lambda} = \frac{\sum_{\sigma \in W} \varepsilon(\sigma)\, \mathrm{e}^{\sigma(\lambda+\rho)}}{\sum_{\sigma \in W} \varepsilon(\sigma)\, \mathrm{e}^{\sigma(\rho)}} = \sum_{\omega} K^{\lambda}_{\omega}\, \mathrm{e}^{\omega}, \tag{0.6}$$

λ étant un poids dominant du réseau des poids d'un groupe de Lie G simple, simplement connexe et compact, et la somme du terme de droite de l'équation (0.6) portant sur tout le réseau des poids, avec des coefficients de Kostka K^{λ}_{ω} entiers positifs. Cette formule permet de calculer la dimension d'un espace de poids $V^{\lambda}(\omega)$, mais elle ne dit pas comment le groupe G et son algèbre de Lie $\mathfrak{g}$ agissent précisément sur les vecteurs de poids d'un tel espace. Ceci rend difficile le calcul de quantités plus complexes que les multiplicités des poids, par exemple les coefficients de Littlewood–Richardson. Le cristal $\mathfrak{C}(\lambda)$ associé à la représentation irréductible de plus haut poids λ est un objet combinatoire qui encode cette action par un graphe pondéré, et qui par ailleurs a des interprétations géométriques qui rendent possibles son calcul et sa manipulation. Ainsi, Littelmann a montré que les éléments du cristal $\mathfrak{C}(\lambda)$ pouvaient être représentés par des chemins dans le réseau des poids, le produit tensoriel s'apparentant alors à la concaténation des chemins, et les poids correspondant aux points terminaux des chemins. Une autre interprétation géométrique a été proposée par Berenstein et Zelevinsky : les éléments du cristal peuvent être vus comme les points entiers d'un certain polytope (le polytope en cordes $\mathscr{P}(\lambda)$), les poids étant alors obtenus à partir de ces points entiers grâce à une application affine (projection du polytope sur un espace vectoriel de dimension égale au rang du groupe). Ces descriptions permettent une compréhension beaucoup plus fine des représentations du groupe G, qui est requise pour l'étude des graphes géométriques poissoniens. En particulier, comme la mesure spectrale des multiplicités des poids peut être décrite comme la projection affine de la mesure uniforme sur les points entiers d'un certain polytope, ceci induit des résultats de polynomialité locale pour ces multiplicités, au moins dans le cadre asymptotique où les plus hauts poids des représentations irréductibles tendent vers l'infini.

À l'aide de cette théorie, j'ai réduit le problème de l'asymptotique du régime poissonien des graphes géométriques à une autre conjecture purement algébrique (Conjecture 3.50) qui concerne certaines fonctionnelles des représentations irréductibles de G, et le lien entre ces fonctionnelles et les polytopes contenant les bases cristallines des représentations. Notons que pour les six premiers moments de la mesure limite μ_∞, les conjectures précédemment évoquées sont des théorèmes démontrés. La résolution complète de la conjecture permettra de déterminer les propriétés caractéristiques de la mesure limite μ_∞. Au moment d'écrire ce mémoire, un premier *preprint* [Mél18] contenant tous les résultats sus-mentionnés et la présentation du problème ouvert sur les cristaux est presque achevé. Bien que je n'ai pas encore de publication sur ce sujet, j'ai décidé d'y consacrer une large part de ce mémoire. En effet, j'ai beaucoup travaillé sur ces graphes aléatoires durant les trois dernières années, et j'ai donné de nombreux exposés sur les résultats partiels que j'ai obtenus. Il sera clair à la lecture des sections 3.3-3.4 que nous sommes proches d'avoir une bonne compréhension de ces objets aléatoires, de nouveau grâce aux outils d'analyse harmonique (non commutative); et je considère que ce thème est une direction très prometteuse pour mes futurs travaux.

Remerciements

Pour clore cet avant-propos du mémoire, je voudrais adresser mes remerciements à tous les mathématiciens avec lesquels j'ai eu le plaisir de travailler durant la dernière décennie. Mes premiers remerciements vont à Philippe Biane, mon directeur de thèse, et à Ashkan Nikeghbali, avec lequel j'ai effectué un postdoctorat à Zürich pendant la période 2011-2013. Je les considère tous les deux comme mes mentors, et j'ai appris de nombreuses nouvelles parties des mathématiques grâce à eux : par exemple, la théorie des représentations des groupes symétriques et le modèle des chemins d'une part, et l'analyse harmonique classique et la théorie probabiliste des nombres d'autre part. J'apprends toujours beaucoup d'eux, et ils ont une influence profonde sur le type de mathématiques que je fais. Des remerciements spéciaux sont dus à Ashkan, qui est impliqué dans la plupart de mes travaux mathématiques récents (et de mes voyages mathématiques !).

J'adresse également ma sincère gratitude à deux de mes principaux collaborateurs, Valentin Féray et Reda Chhaibi, avec lesquels j'ai écrit plusieurs articles, et qui m'ont toujours donné des explications particulièrement éclairantes sur certains sujets (*e.g.*, la théorie des cumulants et des graphes de dépendance, ou celle des cristaux de représentations). C'est avec grand plaisir que je travaillerai de nouveau avec eux à l'avenir.

Je suis également très redevable au département de mathématiques d'Orsay, et particulièrement l'équipe de probabilités et statistiques, dont je suis membre permanent depuis 2013. Je souhaiterais en particulier remercier Maxime Février, avec qui je partage le bureau et qui me vois régulièrement être peut-être un peu trop enthousiaste et bavard à propos de mes recherches. J'ai également beaucoup bénéficié des séminaires qui sont organisés à Orsay et à l'institut Henri Poincaré ; par exemple, mes récents travaux sur les graphes aléatoires géométriques m'ont été inspirés par des exposés donnés lors de ces séminaires.

Il y a quatre ans, Miklós Bóna m'a proposé d'écrire un livre sur les représentations des groupes symétriques [Mél17]. Je voudrais le remercier car cela a été une grande opportunité pour moi, et parce que cela m'a permis de publier des résultats issus de ma thèse et que j'avais laissés de côté. L'écriture de ce livre m'a pris beaucoup de temps, et cela a probablement guéri ma tendance à écrire des papiers trop longs ; le lecteur du présent mémoire appréciera certainement cela.

Je remercie bien sûr Alexei Borodin et Thierry Lévy, qui ont eu la patience de lire ce mémoire et d'écrire les rapports autorisant la soutenance. Je suis très admiratif de leurs travaux, et je suis très honoré qu'ils aient accepté cette tâche. J'ai la même admiration et gratitude envers les six membres du jury qui ont accepté de venir juste avant les vacances d'été pour ma soutenance, et je ferai en sorte de donner le meilleur exposé possible à cette occasion.

Finalement, la fin de ma thèse et le début des travaux que je vais présenter ici coïncident incidemment avec la rencontre de ma femme Véronique. Elle a été un support constant et une source d'inspiration durant toutes ces années, et les plus grands remerciements lui sont évidemment réservés.

Pierre-Loïc Méliot

Foreword

The purpose of this memoir is to present the research results that I have obtained since my Ph.D. thesis in 2010. I therefore tried to find a way to summarise these results, and this task seemed quite difficult in the beginning. Indeed, I worked on subjects as different as:

- estimates of Berry–Esseen type for sequences of random variables (classical probability theory);
- Brownian motions on symmetric spaces (theory of random processes, geometry, a bit of representation theory);
- numbers of prime divisors in random integers (analytic number theory, probability theory);
- spectra of random geometric graphs drawn on Lie groups (representation theory of Lie groups, Lusztig–Kashiwara crystal bases).

A common trend in these various topics is that I like to look at *random* mathematical objects, and to study their *asymptotic properties* (most of the time, when the size of the model grows to infinity). One might therefore be mistaken and think that my interest lies in the randomness of these models. In fact, it is the exact opposite: I am interested in the *mathematical structures* behind these random models whose behavior is chaotic, and I look for a way to use these algebraic or analytic structures in order to obtain some asymptotic results.

Thus, the common thread of my works is not the random aspect, but rather the tool that I almost always use in order to vanquish this randomness, namely, *harmonic analysis* and the many forms of the *Fourier transform*. Recall that if f belongs to the space $\mathscr{L}^1(\mathbb{R})$ of integrable functions on the real line, then its Fourier transform is the bounded continuous function

$$\xi \mapsto \widehat{f}(\xi) = \int_{\mathbb{R}} f(x)\, \mathrm{e}^{\mathrm{i}\xi x}\, dx.$$

In the sequel, we denote $\mathscr{C}_{\mathrm{b}}(\mathbb{R})$ the space of continuous bounded functions on $\mathbb{R}$ with complex values. By density or duality, one can extend the Fourier transform to other functional spaces, for instance the space of square-integrable functions $\mathscr{L}^2(\mathbb{R})$, or the space of tempered distributions $\mathscr{S}'(\mathbb{R})$. If f and $\widehat{f}$ are both integrable, then one can recover f from $\widehat{f}$ by means of the Fourier inversion formula

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}(\xi)\, \mathrm{e}^{-\mathrm{i}\xi x}\, d\xi. \tag{0.1}$$

On the other hand, the Fourier transform is up to a factor $\frac{1}{2\pi}$ an isometry of $\mathscr{L}^2(\mathbb{R})$:

$$\forall f, g \in \mathscr{L}^2(\mathbb{R}),\ \ \langle f \mid g \rangle = \frac{1}{2\pi} \left\langle \widehat{f} \,\middle|\, \widehat{g} \right\rangle. \tag{0.2}$$

Finally, the Fourier transform of the convolution of two functions is the product of their Fourier transforms:

$$\forall f,g \in \mathscr{L}^1(\mathbb{R}),\ \ \widehat{(f*g)}(\xi) = \widehat{f}(\xi)\,\widehat{g}(\xi), \tag{0.3}$$

where $(f*g)(x) = \int_{\mathbb{R}} f(x-y)\,g(y)\,dy$. The three formulas (0.1) (*inversion formula*), (0.2) (*Parseval's formula*) and (0.3) (*convolution formula*) are the generic properties that are expected for any extension or analogue of the classical Fourier transform $\mathscr{L}^1(\mathbb{R}) \to \mathscr{C}_{\mathrm{b}}(\mathbb{R})$. Most of my works rely on such generalisations of the Fourier transform, with $\mathscr{L}^1(\mathbb{R})$ replaced by:

- the space $\mathscr{M}^1(\mathbb{R})$ of Borel probability measures on $\mathbb{R}$;
- or, the group algebra $\mathbb{C}\mathfrak{S}(n)$ of functions on the symmetric group of size n;
- or, the convolution algebra $\mathscr{L}^2(X)$ of square-integrable functions on a space $X = G/K$, where G is a compact Lie group.

▷ **Fluctuations of random variables and mod-ϕ convergence.** In classical probability theory, the space $\mathscr{L}^1(\mathbb{R})$ is replaced by the space of probability measures $\mathscr{M}^1(\mathbb{R})$, and one can define the Fourier transform of $\mu \in \mathscr{M}^1(\mathbb{R})$ by using essentially the same formula as before:

$$\widehat{\mu}(\xi) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\xi x}\,\mu(dx).$$

The convolution formula holds true, and the Parseval formula as well in the following setting: if f is a function in $\mathscr{L}^1(\mathbb{R})$ with $\widehat{f}$ also in $\mathscr{L}^1(\mathbb{R})$, then for any probability measure μ on the real line,

$$\mu(f) = \int_{\mathbb{R}} f(x)\,\mu(dx) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}(\xi)\,\widehat{\mu}(-\xi)\,d\xi.$$

On the other hand, the Fourier inversion formula depends whether the distribution μ is lattice or non-lattice distributed. For instance, if μ is non-lattice distributed and with Fourier transform $\widehat{\mu}$ integrable on the real line, then μ is absolutely continuous with respect to the Lebesgue measure, and its density is given by the inversion formula (0.1). We refer to [Fel71, Chapter XV], [Mal95, Chapter III] and [Str11, Section 2.3] for the classical theory of Fourier transforms of probability measures; a short survey is also proposed in [FMN17b, Section 2.2]. The Fourier inversion formula yields a dictionary between the properties of a probability measure μ, and the properties of its Fourier transform $\widehat{\mu}$. For example, the behavior of $\widehat{\mu}$ around 0 dictates the size of the tail of the distribution μ, *via* the inequality

$$\mu(\mathbb{R} \setminus [-2C,2C]) \leq C \int_{-\frac{1}{C}}^{\frac{1}{C}} (1-\widehat{\mu}(\xi))\,d\xi.$$

Another reason why techniques from harmonic analysis are very useful in probability theory is that the Fourier transform interchanges the convergence in law in $\mathscr{M}^1(\mathbb{R})$ and the local uniform convergence in $\mathscr{C}_{\mathrm{b}}(\mathbb{R})$. Given a sequence $(V_n)_{n\in\mathbb{N}}$ of real-valued random variables, there exists often a renormalisation $(V_n/s_n)_{n\in\mathbb{N}}$ of the sequence such that the Fourier transforms $\mathbb{E}[\mathrm{e}^{\mathrm{i}\xi V_n/s_n}]$ admit a limit $F(\xi)$, and therefore such that the laws μ_n of the rescaled random variables V_n/s_n converge weakly to the distribution μ whose Fourier transform is $\widehat{\mu} = F$. When rescaling the random variables V_n and looking at V_n/s_n, one gets the right normalisation in order to describe the general aspect of the fluctuations of V_n, but as a result one also loses a lot of information about the precise behavior of the variables V_n. This point had not been shed light upon until recently, when another *renormalisation theory* was invented in order to complete the convergence in law $\frac{V_n}{s_n} \rightharpoonup \mu$ by other more precise results:

- large deviations (behavior at the edge of the asymptotic distribution of the fluctuations),
- local limit theorems (behavior in infinitesimal regions),
- speed of convergence estimates,

etc. This theory is the one of *mod-ϕ convergence*, and the first definition of this notion was given almost ten years ago in a paper by Jacod, Kowalski and Nikeghbali [JKN11] (in the specific case of mod-Gaussian convergence). During my postdoctoral stay in Zürich, I had the chance to start working on this theory, and we have developed it since then with Valentin Féray and Ashkan Nikeghbali [FMN16; FMN17a; FMN17b; FMN17c; MN15], and with several other coauthors [Chh+15; BMN17]. Thus, we established numerous theoretical results related to the notion of mod-ϕ convergent sequences, and we also studied large classes of examples and showed that they fitted to this framework.

The two first chapters of this memoir are devoted to a presentation of these results. In order to give a precise outline of these two chapters, let us explain informally the notion of mod-ϕ convergence, by focusing on the Gaussian case. Suppose given a sequence $(V_n)_{n\in\mathbb{N}}$ of random variables such that $V_n/s_n \rightharpoonup \mathcal{N}_{\mathbb{R}}(0,1)$, where the s_n's are some renormalisation parameters growing to infinity. In terms of Fourier transforms, the convergence in law is equivalent to the asymptotic estimate

$$\mathbb{E}\left[\mathrm{e}^{\mathrm{i}\xi\frac{V_n}{s_n}}\right] = \mathrm{e}^{-\frac{\xi^2}{2}}\,(1+o(1)). \tag{0.4}$$

A quite natural idea is that, if one knows more about the Fourier transforms, then one will get more precise asymptotic results than the convergence in law. With the renormalisation V_n/s_n, the $o(1)$ in the right-hand side of Equation (0.4) is actually a function $f(\xi,n)$ such that $\lim_{n\to\infty} f(\xi,n) = 0$ for any ξ. By taking these limits, one loses the dependence in ξ and all the relevant information beyond the fact that $(V_n/s_n)_{n\in\mathbb{N}}$ is asymptotically normal. A way to overcome this problem consists in looking for another renormalisation V_n/r_n such that, if $\sqrt{t_n} = \frac{s_n}{r_n}$, then

$$\begin{aligned}\mathbb{E}\left[\mathrm{e}^{\mathrm{i}\xi\frac{V_n}{r_n}}\right] &= \mathbb{E}\left[\mathrm{e}^{\mathrm{i}\sqrt{t_n}\xi\frac{V_n}{s_n}}\right]\\ &= \mathrm{e}^{-\frac{t_n\xi^2}{2}}\,\left(1+f(\sqrt{t_n}\xi,n)\right) \quad \text{with } \lim_{n\to\infty}\left(1+f(\sqrt{t_n}\xi,n)\right) = \theta(\xi)\end{aligned}$$

for some non-trivial function $\theta(\xi)$. Hence, instead of saying that V_n/s_n converges to a Gaussian distribution, one says that V_n/r_n looks like a large Gaussian variable with variance t_n, plus a residue encoded asymptotically by $\theta(\xi)$. Note that the right renormalisation $X_n = \frac{V_n}{r_n}$ is not always easy to find, and that one cannot establish its existence just by taking an additional term in the Taylor expansion of the Fourier transforms of these random variables. The residue $\theta(\xi)$ actually contains specific information about the model producing the random variables V_n, and the theory of mod-ϕ convergence allowed us to extract this information and to prove some new asymptotic estimates of probabilities for the sequence $(V_n)_{n\in\mathbb{N}}$.

In Chapter 1, we present the theoretical consequences of the notion of mod-ϕ convergence, which is the generalisation of the previous informal definition with an arbitrary infinitely divisible law instead of the standard Gaussian distribution. If one is able to prove the mod-ϕ convergence of a sequence, then one gets at once many asymptotic results:

- a central limit theorem with a extended zone of convergence (Theorem 1.21);
- large deviation results (Theorems 1.12 and 1.13);

- Berry–Esseen estimates of the speed of convergence (Theorems 1.14 and 1.28 in the continuous case, and Theorem 1.36 in the discrete case);
- and a local limit theorem for a large range of infinitesimal scales (Theorem 1.49).

These distinct kinds of results rely on slightly different hypotheses. For the two first kinds (extended central limit theorem and large deviation principles), we shall ask for estimates of the complex Laplace transform instead of the Fourier transform. Then, in the particular case where $(X_n)_{n\in\mathbb{N}}$ is a sequence of random variables such that

$$\mathbb{E}[\mathrm{e}^{zX_n}] = \mathrm{e}^{t_n \frac{z^2}{2}}\, \psi(z)\,(1+o(1))$$

with $t_n \to +\infty$ (this will be the definition of a convergent sequence in the mod-Gaussian sense with parameters t_n and limit ψ, see Definition 1.1), we obtain the following results:

- If $Y_n = \frac{X_n}{\sqrt{t_n}}$, then for any deterministic sequence $x_n = o(\sqrt{t_n})$,

$$\mathbb{P}[Y_n \geq x_n] = \left(\int_{x_n}^{+\infty} \frac{\mathrm{e}^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\,dx\right)(1+o(1)),$$

  so the normality zone of $(Y_n)_{n\in\mathbb{N}}$ has size $o(\sqrt{t_n})$.
- At the edge of this zone, the Gaussian estimate is corrected by the residue ψ, and if $x>0$, then

$$\mathbb{P}[Y_n \geq x\sqrt{t_n}] = \frac{\mathrm{e}^{-\frac{t_n x^2}{2}}}{\sqrt{2\pi t_n}\,x}\,\psi(x)\,(1+o(1)).$$

The two other kinds of asymptotic results only need an estimate of the Fourier transforms, but on a sufficiently large zone if one wants optimal results. In the particular case where $(X_n)_{n\in\mathbb{N}}$ is a sequence of random variables such that

$$\mathbb{E}[\mathrm{e}^{\mathrm{i}\xi X_n}]\,\mathrm{e}^{t_n \frac{\xi^2}{2}} = \theta_n(\xi) = \theta(\xi)\,(1+o(1))$$

with $t_n \to +\infty$, if one has for instance an upper bound

$$|\theta_n(\xi) - 1| \leq K_1|\xi|^3 \exp(K_2|\xi|^3)$$

on a zone $\xi \in [-K(t_n)^\gamma, K(t_n)^\gamma]$ with $\gamma \in [0,1]$, then the Kolmogorov distance between $Y_n = X_n/\sqrt{t_n}$ and the standard Gaussian distribution is a $O((t_n)^{\frac{1}{2}+\gamma})$. Besides, with the same hypotheses, the normal approximation of Y_n is valid for Jordan-measurable subsets B_n with Lebesgue measure $L(B_n) \gg (t_n)^{-\frac{1}{2}-\gamma}$.

In Chapter 2, we explain which structures lead to mod-ϕ convergence. If the Fourier or Laplace transforms of the random variables V_n under study are explicit, then only standard techniques of analysis are required. However, this class includes many examples that are new and non trivial: number of zeroes of a random power series, number of cycles of a random permutation chosen according to a probability measure that gives a distinct weight to each size of cycle, number of irreducible factors of a random polynomial over a finite field, number of prime factors (distinct or counted with multiplicity) of a random integer, *etc.* In each of these cases, the mod-ϕ convergence is proved by means of an asymptotic study of the explicit formula for the Fourier transform, which can be given by a Cauchy or Dirichlet integral in the complex plane.

In the case of mod-Poisson variables, the theory of mod-ϕ convergence enables the construction of approximation schemes of the discrete variables under study by signed measures, these approximations being much more precise than the simple Poissonian approximation. Moreover, the formalism of symmetric functions and of their specialisations allows one to encode these approximations in a very concise way. For instance, suppose that one wants to approximate the distribution of the random number ω_n of distinct prime divisors of a random integer in $[\![1, n]\!]$. Then, in order to construct a signed measure ν_n whose total variation distance to the law of ω_n is a $O((\log\log n)^{-b})$ for some integer $b \geq 1$, one only needs to deal with the infinite alphabet

$$\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots\right\}_{n\geq 1} \sqcup \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots, \frac{1}{p}, \ldots\right\}_{p\in\mathbb{P}}$$

and with the corresponding specialisation of certain symmetric functions of degree smaller than $2b$. A similar method of approximation is proposed for many other random discrete sequences of combinatorial or arithmetic nature.

In contrast to these structures which are very rigid and algebraic, the mod-Gaussian convergence does not require an explicit calculation of the Fourier transforms. Thus, we give in the second chapter a general criterion on the *cumulants* of a sequence of random variables that ensures the mod-Gaussian convergence, and in turn is implied by certain dependency structures: sparse dependency graphs (Theorem 2.19) and weighted dependency graphs (Theorem 2.35). These results allow one to show that many models of random graphs, random partitions, random permutations, random spin configurations, *etc.* yield random variables (the *observables* of the model) that are mod-Gaussian convergent. For example, given a random graph $G(n, \gamma)$ on n vertices that is associated to a graphon γ (equivalence class in the space of measurable symmetric functions $[0, 1]^2 \to [0, 1]$ modulo the Lebesgue-isomorphisms of $[0, 1]$), for every motive F (fixed finite graph), when n goes to infinity, the number of motives $\hom(F, G(n, \gamma))$ is mod-Gaussian convergent after an adequate renormalisation, as a consequence of the method of cumulants and of dependency graphs. Similarly, if one considers the magnetisation $M_n = \sum_{x\in\Lambda_n} \sigma(x)$ of a box $\Lambda_n \subset \mathbb{Z}^d$ in the Ising model with very high temperature $T \gg T_{\text{critical}}$ and without external field, and if the size of this box goes to infinity, then an appropriate renormalisation of M_n is mod-Gaussian convergent, as a consequence of the theory of weighted dependency graphs.

The criterion on cumulants previously mentioned is an upper bound involving three parameters D_n, N_n, A:

$$\forall r \geq 1,\ \ \forall n \in \mathbb{N},\ \ \left|r!\,[z^r](\log \mathbb{E}[\mathrm{e}^{zS_n}])\right| = |\kappa^{(r)}(S_n)| \leq r^{r-2}\,(2D_n)^{r-1}\,N_n\,A^r. \tag{0.5}$$

If this upper bound holds, if $D_n \ll N_n$ and if the three first cumulants of S_n verify certain asymptotic hypotheses, then the renormalised sequence

$$X_n = \frac{S_n - \mathbb{E}[S_n]}{(N_n\,(D_n)^2)^{1/3}}$$

is mod-Gaussian convergent. On the other hand, if S_n is a sum of N_n variables bounded by A and endowed with a dependency graph or a weighted dependency graph with maximal degree smaller than D_n, then the cumulants of S_n satisfy the inequality (0.5). This result should be compared with certain analyticity results in mathematical physics, as it gives information on the radius of convergence of the generating series of the random variables S_n. In addition to the mod-Gaussian convergence, it implies concentration inequalities similar to the classical Hoeffding inequality, but that hold in a much more general setting (Proposition 2.33). We thus recover inequalities which until now could only be obtained by means of martingale techniques, and which are fundamental for the study of the motives in models of random graphs or random permutations. The theory of (weighted) dependency graphs yields a new approach for studying the fluctuations of sums of random variables, and we conjecture that it will enable us to understand precisely the behavior of arithmetic chaoses and of functionals of mixing ergodic dynamical systems.

At the end of the two first chapters, it will have been proved that the theory of mod-ϕ convergence enables a unified study of the various scales of fluctuations of a sequence of random variables. In the specific mod-Gaussian case, this unified approach is completed by universality results. For instance, for a very general class of models of random graphs (the graphon models), any observable of any model is generically mod-Gaussian, with the parameters of mod-Gaussian convergence that depend continuously on the parameters of the model. A similar phenomenon occurs for some models of random permutations and some models of random partitions. We propose in Section 2.3 a notion of *mod-Gaussian moduli space* which formalises this universality, and which enables a geometric study of the models. In particular, the singular points of these moduli spaces are random models with additional symmetries, and these symmetries translate sometimes into a modification of the parameters of mod-Gaussian convergence, and into the corresponding modification of the scales of fluctuations. To the best of our knowledge, this geometric approach of the classification of the random models is unprecedented. In this framework, we should be able to study the singular models by constructing new dependency structures, which would be degenerations of the generic graph dependency structure:

- weighted dependency graphs;
- lacunary dependency graphs where only certain spanning trees appear in the upper bounds on cumulants;
- dependency surfaces which mimic the topological expansions of moments of certain models, but which adapt them to the study of cumulants.

▷ **Probability theory on compact Lie groups and symmetric spaces.** Our third chapter is devoted to the study of another kind of random objects, and to the use of another avatar of the Fourier transform, namely, the *non-commutative Fourier transform* of a compact group or a quotient of compact groups. Fix a compact group G, and denote dg its Haar measure, which is the unique probability measure on G which is left- and right-invariant. To define the Fourier transform of a function $f \in \mathscr{L}^1(G, dg)$, one needs to replace the function

$$\rho^{\xi} : x \mapsto \mathrm{e}^{\mathrm{i}x\xi}$$

by a unitary representation of G on a finite-dimensional complex vector space:

$$\begin{aligned} \rho^{\lambda} : G &\to \mathrm{U}(V^{\lambda}) \\ g &\mapsto \rho^{\lambda}(g). \end{aligned}$$

Thus, the non-commutative Fourier transform $\widehat{f}$ is defined as a function on the set $\widehat{G}$ of irreducible unitary representations $\lambda = (V^{\lambda}, \rho^{\lambda})$, with $\widehat{f}(\lambda) \in \mathrm{End}_{\mathbb{C}}(V^{\lambda})$ given by the formula

$$\widehat{f}(\lambda) = \int_G f(g)\, \rho^{\lambda}(g)\, dg.$$

Notice that one deals only with finite-dimensional representations, thanks to the compactness of G. If $f \in \mathscr{L}^2(G, dg)$, then one obtains exact analogues of the three formulas (0.1), (0.2) and (0.3). In particular, the Fourier transform is an isometry between $\mathscr{L}^2(G)$ and the Hilbert sum

$\mathscr{L}^2(\widehat{G}) = \bigoplus_{\lambda \in \widehat{G}} \mathrm{End}(V^\lambda)$. These properties can be used in order to study a probability measure μ on G whose density with respect to the Haar measure is square-integrable. Consider for instance a sequence of probability measures $(\mu_n)_{n\in\mathbb{N}}$ that converges towards the Haar measure $\eta = dg$ as n goes to infinity. In the non-commutative Fourier world, this translates to:

$$\forall \lambda \in \widehat{G},\ \ \widehat{\mu_n}(\lambda) \to_{n\to\infty} \widehat{\eta}(\lambda) = \begin{cases} 0 & \text{if } V^\lambda \neq \mathbb{C}, \\ 1 & \text{if } V^\lambda = \mathbb{C}, \end{cases}$$

where $\mathbb{C}$ denotes the trivial one-dimensional representation of G. In particular, the representation theory of the group G yields a very general method in order to study random walks on compact groups and their convergence to the uniform measure. This idea appeared for the first time in the works of Poincaré at the beginning of the twentieth century [Poi12], and it became quite popular after several celebrated papers by Diaconis on the combinatorics of the operation of shuffling of cards; see [DS81; AD86; BD92; Dia96], and also [CST08, Section 10.7]. In [Mél14b], we solved a conjecture due to Saloff-Coste [Sal10] regarding the convergence to stationarity of the Brownian motions on compact Lie groups such as $\mathrm{SU}(n)$ or $\mathrm{SO}(n)$. Thus, we proved that this convergence happens in a short window of time around a cut-off time

$$t_{\text{cut-off}} \propto \log n,$$

where n is the rank of the group (dimension of a maximal torus). In particular, the larger the dimension of the Lie group is, and the narrower the phase transition to stationarity gets (in comparison to the cut-off time). These results are presented in Chapter 3 (Theorem 3.23), after a short survey of the representation theory of compact Lie groups (Section 3.1). They also hold true over compact *symmetric spaces*, which are generalisations of compact Lie groups, and which are quotients thereof. The class of symmetric spaces includes the spheres, the projective spaces, the Grassmannian manifolds, *etc.*, and the survey of Section 3.1 also deals with the harmonic analysis of these spaces.

Let us detail a bit more the ideas of the proof of the cut-off phenomenon for the Brownian motions. Given a symmetric space X, the non-commutative Fourier transform yields an explicit expansion of the density of the law of the Brownian motion at time t with respect to the Haar measure:

$$\frac{d\mu_t(x)}{d\nu} = \sum_\lambda m_\lambda\, \mathrm{e}^{-a_\lambda t}\, f_\lambda(x),$$

where the functions f_λ are orthonormal in $\mathscr{L}^2(X)$, and where the sum runs over a set of irreducible representations

- of X if X is a compact Lie group,
- of $\mathrm{Isom}(X)$ if X is a symmetric space of type non-group.

This expansion allows one to estimate the $\mathscr{L}^p$ norms of $|d\mu_t/d\nu - 1|$, by using in particular the orthonormality of the functions f_λ for the case $p = 2$. From this, one deduces that $d\mu_t/d\nu$ is uniformly close to 1 for the $\mathscr{L}^p$ norms as soon as the quantities $m_\lambda\, \mathrm{e}^{-a_\lambda t}$ get small; and this occurs about the same time $t_{\text{cut-off}}$ for all the irreducible representations λ involved in the expansion of the density. These arguments provide us with an upper bound on the cut-off time, and they are very robust, as they can be used for any random process endowed with a compact group of symmetry. For the lower bound, one needs to show conversely that for any time $t < t_{\text{cut-off}}$, the law μ_t charges events whose Haar measure is almost zero. The idea is then to find discriminating functions whose distributions are very different under μ_t and under ν; and the functions f_λ are natural candidates for this. The computation of the expectation and of the variance of these functions under a Brownian motion μ_t is enabled by arguments of stochastic calculus, and by using the stochastic differential equation that defines a Brownian motion or more generally a Lévy process on a Lie group.

The representation theory of a group G allows one to study objects that are more complex than random points taken in a homogeneous G-space. The general idea is the following: if a random object traced on an homogeneous G-space can be encoded at least partially in a representation of the group G, and if this group G belongs to a classical family (symmetric groups, unitary groups, *etc.*), then the asymptotic properties of this random object (law of large numbers and fluctuations as the size of the object goes to infinity) can be related to the *asymptotic representation theory of this family of groups*:

- either, when the size of the group goes to infinity,
- or, when the dimension of the representation goes to infinity.

A classical instance of this approach is the solution of Ulam's problem of the longest increasing subsequences of a random permutation, which relies on the asymptotic study of the Plancherel measures of the symmetric groups (spectral measures of the regular representations). Some generalisations of this example were the content of my Ph.D. thesis, see the papers [Mél10; Mél11; Mél12; FM12; Mél14a] and the monograph [Mél17]. More recently, I started to study *random geometric graphs* drawn on compact symmetric spaces [Mél18]. These graphs are constructed as follows: one takes N independent points uniformly at random in a symmetric space X, and one connects all the pairs of points $\{x,y\}$ that are sufficiently close. More precisely, one connects x to y if and only if $d(x,y) \leq L$, with a level L that might depend on N. When N grows to infinity, the harmonic analysis of the symmetric spaces allows one to describe the asymptotic spectrum of the adjacency matrix of such random geometric graphs. The two last Sections 3.3 and 3.4 of this memoir present the results obtained so far in this direction.

If $L > 0$ is fixed and if N goes to infinity (*Gaussian regime* of the geometric graphs), the asymptotic behavior of the spectrum of the adjacency matrix of a random geometric graph is connected to the eigenvalues of an integral operator of Hilbert–Schmidt class on the compact measured metric space X underlying the graph. When X is a compact Lie group, a sphere or a projective space (Theorems 3.28 and 3.35), these eigenvalues can be computed explicitly. Thus, one shows that if $A_{\Gamma(N,L)}$ is the adjacency matrix, then the eigenvalues of $A_{\Gamma(N,L)}/N$ converge almost surely to deterministic quantities which are labeled by the irreducible representations of the group, or by the spherical representations of the isometry group of the sphere or of the projective space. The formulas that give these limiting quantities involve Bessel functions in the case of groups, and Laguerre or Jacobi orthogonal polynomials in the case of spheres and projective spaces. The proof of these asymptotic formulas relies:

- on a general result due to Koltchinskii and Giné, which ensures that certain large symmetric random matrices have their spectra close to the one of an integral operator of Hilbert–Schmidt class, this argument being valid for the adjacency matrices of the random geometric graphs.
- for the spaces with type group, on Weyl's integration formula, which allows one to compute the eigenvalues of the limiting operators of the adjacency matrices.
- for the spaces with non-group type and rank 1, on explicit formulas for the zonal spherical functions, which involve some orthogonal polynomials of the zonal parameter.

The most interesting case is the one where N goes to infinity and $L = L_N$ goes to 0 in such a way that a vertex of the random geometric graph on G with N points and level L_N has a $O(1)$ number of neighbors (*Poissonian regime* of the geometric graphs). When $X = G$ is a compact Lie group, we have a conjecture and many partial results on the asymptotic spectral distribution of the adjacency matrix (see in particular Theorem 3.38 and Conjecture 3.41). More precisely, denote

$$\nu_N = \frac{1}{N} \sum_{e \text{ eigenvalue of } A_{\Gamma(N,L)}} \delta_e$$

the random spectral measure of the adjacency matrix, and $\mu_N = \mathbb{E}[\nu_N]$, the expectation being taken in the vector space of signed measures and yielding a deterministic probability measure for each N. By using geometric arguments and the local Benjamini–Schramm convergence of a Poissonian random geometric graph, one can show that there exists a limiting measure μ_∞ such that $\mu_N \rightharpoonup_{N\to\infty} \mu_\infty$. We proved this by establishing a general result which connects the Lipschitz convergence of random pointed proper metric spaces to the Benjamini–Schramm convergence of the geometric graphs drawn on these spaces. Unfortunately, our proof of the convergence of the expected spectral measures does not give any information on the limit μ_∞ (support, atoms, regularity, *etc.*). However, we conjecture the two following properties:

- The spectral measures ν_N of the adjacency matrices $A_{\Gamma(N,L)}$ should converge in probability towards μ_∞ (without taking the expectation).
- The moments of μ_∞ should admit a combinatorial expansion whose terms correspond to certain finite graphs (circuits and reduced circuits).

Each of the terms of these combinatorial expansions is given by an integral of Bessel functions on a product of Weyl chambers, with an integration measure that is polynomial by parts, with domains of polynomiality that are polyhedral. These integration measures stem from the asymptotic representation theory of G, and their conjectured form comes from the theory of crystal bases and string polytopes. This recent branch of representation theory of Lie groups and algebras allows one to label in a coherent way some bases of weight vectors of the irreducible representations of a Lie group G, and to compute the structure coefficients (Littlewood–Richardson coefficients and their generalisations) of the algebra of representations of the group. More precisely, one can consider *crystal theory* as an important amelioration of Weyl's character formula

$$\mathrm{ch}^\lambda = \frac{\sum_{\sigma\in W} \varepsilon(\sigma)\, \mathrm{e}^{\sigma(\lambda+\rho)}}{\sum_{\sigma\in W} \varepsilon(\sigma)\, \mathrm{e}^{\sigma(\rho)}} = \sum_{\omega} K^\lambda_\omega\, \mathrm{e}^{\omega}, \tag{0.6}$$

λ being a dominant weight of the weight lattice of a simple simply connected compact Lie group, and the sum in the right-hand side of Equation (0.6) running over the whole weight lattice, with the Kostka coefficients K^λ_ω that are non-negative integers. This formula enables the computation of the dimension of a weight space $V^\lambda(\omega)$, but it does not tell us precisely how the the group G and its Lie algebra $\mathfrak{g}$ act on the weight vectors of such a space. This makes it difficult to compute quantities that are more complex than the multiplicities of weights, for instance the Littlewood–Richardson coefficients. The crystal $\mathfrak{C}(\lambda)$ associated to the irreducible representation with highest weight λ is a combinatorial object which encodes this action by a weighted graph, and which besides admits geometric interpretations which enable its computation and manipulation. Thus, Littelmann showed that the elements of the crystal $\mathfrak{C}(\lambda)$ can be represented by paths in the weight lattice, the tensor product being then related to the concatenation of paths, and the weights corresponding to the endpoints of the paths. Another geometric interpretation was proposed by Berenstein and Zelevinsky: the elements of the crystal can be seen as integer points in a certain polytope (the string polytope $\mathscr{P}(\lambda)$), the weights being then obtained from these integer points thanks to an affine map (projection of the polytope on a vector space with dimension equal to the rank of the group). These descriptions enable a much finer understanding of the representations of the group G, which is required for the study of Poissonian geometric graphs. In particular, since the spectral measure of the weight multiplicities can be described as the affine projection of the uniform measure on the integer poins of some polytope, this leads to local polynomiality results for these multiplicities, at least in the asymptotic setting where the highest weights of the irreducible representations go to infinity.

Thanks to this theory, I was able to reduce the problem of the asymptotics of the Poissonian behavior of geometric graphs to another entirely algebraic conjecture (Conjecture 3.50), which regards certain functionals of the irreducible representations of G, and the connection between these functionals and the polytopes that contain the crystal bases of the representations. Note that for the six first moments of the limiting measure μ_∞, the conjectures previoulsy evoked are proven theorems. The complete resolution of the conjecture will allow us to determine the characteristic properties of the limiting measure μ_∞. At the time of writing this memoir, a first preprint [Mél18] including all the aforementioned results and a presentation of the open problem on crystals is almost finished. Although I do not have yet a publication on this subject, I decided to devote to it a large part of my memoir. Indeed, I have worked a lot on these random graphs during the three last years, and I gave numerous talks on the partial results that I have already obtained. It will be clear from the reading of Sections 3.3-3.4 that we are very close to have a good understanding of these random objects, thanks again to tools from (non-commutative) harmonic analysis; and I consider this topic to be a very promising direction for my future works.

Acknowledgements

To close this foreword of the memoir, I would like to address my thanks to all the mathematicians with whom I had the pleasure to work during the last decade. My first thanks go to Philippe Biane, my Ph.D. advisor, and to Ashkan Nikeghbali, who offered me a post-doc position in Zürich for the period 2011-2013. I consider them both as my mentors, and I learned many new parts of mathematics thanks to them: for example, representation theory of symmetric groups and the path model on the one hand, and classical harmonic analysis and probabilistic number theory on the other hand. I still learn a lot from them, and they have a profound influence on the kind of mathematics that I am doing. Special thanks are due to Ashkan who is involved in most of my recent mathematical works (and mathematical journeys!).

I also would like to express my sincere gratitude to two of my main collaborators, Valentin Féray and Reda Chhaibi, with whom I wrote several papers, and who always provided me with particularly enlightening explanations on certain topics (*e.g.*, the theory of cumulants and of dependency graphs, or the theory of crystals of representations). It is with great pleasure that I look forward to working with them again in the future.

I am also greatly indebted to the mathematics department of Orsay, and especially the probability and statistics team, of which I am a permanent member since 2013. I want in particular to thank Maxime Février, with whom I share the office and who regularly sees me being maybe a bit too enthousiastic and talkative about my research. I also benefited a lot from the seminars that are organised in Orsay or at the Institute Henri Poincaré; for instance,

my recent works on random geometric graphs were inspired by some talks given during these seminars.

Four years ago, I was proposed by Miklós Bóna to write a book on the representations of the symmetric groups [Mél17]. I would like to thank him since it was a great opportunity for me, and since it allowed me to publish some results from my Ph.D. thesis that I had left besides. The writing of this book took me a lot of time, and it probably cured my tendency of writing too long papers; the reader of the present memoir will certainly appreciate it.

I thank of course Alexei Borodin and Thierry Lévy, who had the patience to read this memoir and write the reports authorising the defense. I am very appreciative of their works, and I am very honored that they have accepted this task. I have the same admiration and gratitude towards the six members of the jury who have agreed to come just before the summer break for my defense, and I will try to give the best possible presentation on this occasion.

Finally, the end of my Ph.D. thesis and the beginning of the works that I will present here incidentally coincide with the meeting of my wife Véronique. She has been a constant support and source of inspiration during these years, and the greatest thanks are of course reserved for her.

Pierre-Loïc Méliot

Chapter 1

Mod-ϕ convergence and its probabilistic consequences

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables with values in $\mathbb{R}$. When looking at a mathematical system that produces these random variables, probabilists often try to find the *asymptotic distribution* of X_n, possibly after an appropriate renormalisation. The oldest result in this direction is probably De Moivre's law [Moi56]: if X_n is the total number of heads obtained after n tosses of an equilibrated coin, and if n is large, then

$$G_n = \frac{2}{\sqrt{n}}\left(X_n - \frac{n}{2}\right)$$

is distributed almost like a standard Gaussian random variable G:

$$\forall x \in \mathbb{R},\ \lim_{n\to\infty} \mathbb{P}[G_n \leq x] = \lim_{n\to\infty}\left(\sum_{k=0}^{\lfloor \frac{n}{2}+\sqrt{n}x\rfloor} \frac{1}{2^n}\binom{n}{k}\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \mathrm{e}^{-\frac{s^2}{2}}\,ds,$$

see Figure 1.1.

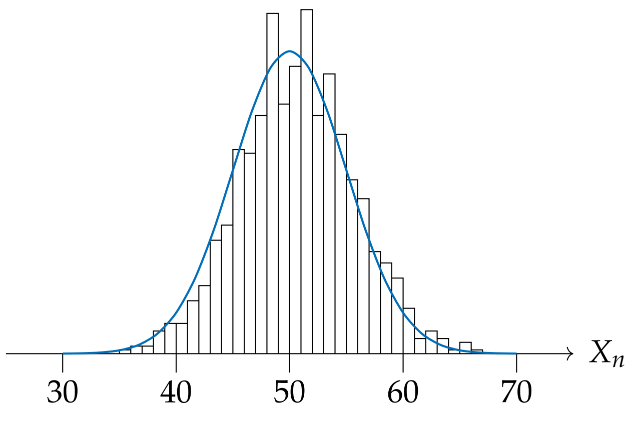


Figure 1.1. De Moivre's law with $n = 100$ and $N = 1000$ tries: the asymptotic distribution of tails and heads in tosses of coins is the Gaussian law.

The notion that captures this asymptotic behavior is the *convergence in law* or in distribution of random variables. Thus, $(X_n)_{n\in\mathbb{N}}$ is said to converge towards a probability measure μ on the real line if and only if, for any continuous bounded function $f : \mathbb{R} \to \mathbb{R}$, $\lim_{n\to\infty} \mathbb{E}[f(X_n)] = \int_{\mathbb{R}} f(x)\,\mu(dx)$. For instance, the central limit theorem (see [Lya00; Lya01; Lin22; Lév25], and [Fis11] for a historical account), which generalises De Moivre's law, states that if $(A_n)_{n\in\mathbb{N}}$ is a sequence of independent and identically distributed random variables with finite mean $\mathbb{E}[A_1] = m$ and finite variance $\mathrm{var}(A_1) = \sigma^2$, then

$$X_n = \frac{1}{\sigma\sqrt{n}}\,(A_1 + A_2 + \cdots + A_n - nm)$$

converges in law towards the standard Gaussian distribution $\mathcal{N}_{\mathbb{R}}(0,1)$.

In this setting, one of the most important result is Lévy's continuity theorem: a sequence of random variables $(X_n)_{n\in\mathbb{N}}$ converges in law towards a probability measure μ if and only if, for any $\xi \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathbb{E}\left[\mathrm{e}^{\mathrm{i}\xi X_n}\right] = \widehat{\mu}(\xi) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}x\xi}\,\mu(dx).$$

Thus, the laws μ_n of the variables X_n converge towards the law μ if and only if the Fourier transforms $\widehat{\mu}_n$ converge pointwise to the Fourier transform $\widehat{\mu}$. It is then natural to expect that, if one has a good estimate on the difference between $\widehat{\mu}_n$ and $\widehat{\mu}$, then one will get precisions on the central limit theorem: large deviation results, bounds on the Kolmogorov distance, *etc.* It turns out that one should not look at the *difference* $\widehat{\mu}_n(\xi) - \widehat{\mu}(\xi)$, but rather at the *ratio* $\widehat{\mu}_n(\xi)/\widehat{\mu}(\xi)$. This is only possible if $\widehat{\mu}(\xi)$ does not vanish, and therefore we shall restrict ourselves to the case where the limiting law is infinitely divisible, this being a sufficient condition for the non-vanishing of the Fourier transform. This leads us to the notion of mod-ϕ convergence, which was invented by Delbaen, Kowalski, Jacod and Nikeghbali; see [JKN11] for the Gaussian case, and [DKN15] for the general case of an infinitely divisible reference law. We also refer to [BKN09; KN10; KN12], which are previous works on the notion of mod-ϕ convergence. The first section of this chapter will recall the definitions introduced in these articles and in [FMN16].

1.1 Infinitely divisible laws and second-order central limit theorems

In the sequel, ϕ is a fixed *infinitely divisible probability measure* on $\mathbb{R}$. This means that for any $n \in \mathbb{N}^*$, one can find a probability measure ϕ_n such that $\phi = (\phi_n)^{*n}$, where $\pi_1 * \pi_2$ denotes the convolution of two probability measures:

$$\text{for any Borel subset } A,\quad (\pi_1 * \pi_2)(A) = \int_{\mathbb{R}^2} 1_{(x+y\in A)}\,\pi_1(dx)\,\pi_2(dy).$$

The classification of infinitely divisible probability measures on $\mathbb{R}$, or even $\mathbb{R}^d$ is well known since the works of Lévy and Khintchine, see for instance [Sat99, Chapters 1 and 2]. Thus, for any infinitely divisible law ϕ on $\mathbb{R}$, there exists a unique triplet (a, σ^2, π) with $a \in \mathbb{R}$, $\sigma^2 \in \mathbb{R}_+$ and π positive Borel measure on $\mathbb{R} \setminus \{0\}$ such that $\int_{\mathbb{R}} \min(1, x^2)\,\pi(dx) < +\infty$, and

$$\widehat{\phi}(\xi) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}x\xi}\,\phi(dx) = \exp\left(\mathrm{i}a\xi - \frac{\sigma^2\xi^2}{2} + \int_{\mathbb{R}} (\mathrm{e}^{\mathrm{i}x\xi} - 1 - 1_{|x|<1}\,\mathrm{i}x\xi)\,\pi(dx)\right).$$

A proof of this Lévy–Khintchine formula is given in [Sat99, Theorem 8.1]. Important examples of infinitely divisible distributions are:

- the Gaussian law $\mathcal{N}_{\mathbb{R}}(a, \sigma^2)$ with density $\frac{1}{\sqrt{2\pi\sigma^2}}\,\mathrm{e}^{-\frac{(x-a)^2}{2\sigma^2}}\,dx$, and with Fourier transform

  $$\widehat{\phi}(\xi) = \mathrm{e}^{\mathrm{i}a\xi - \frac{\sigma^2\xi^2}{2}}$$

  $(a \in \mathbb{R}, \sigma^2 \in \mathbb{R}_+, \pi = 0)$;

- the Poisson law $\mathcal{P}(\lambda)$ with $\lambda > 0$, supported on $\mathbb{N}$ and given by

$$\forall k \in \mathbb{N},\ \ \mathbb{P}[X = k] = \mathrm{e}^{-\lambda}\,\frac{\lambda^k}{k!} \qquad ; \qquad \widehat{\phi}(\xi) = \mathrm{e}^{\lambda(\mathrm{e}^{\mathrm{i}\xi}-1)};$$

  $(a = 0, \sigma^2 = 0, \pi = \lambda\,\delta_1)$;
- the standard Cauchy law $\mathcal{C}$, with density $\frac{1}{\pi(1+x^2)}\,dx$ and with Fourier transform

$$\widehat{\phi}(\xi) = \mathrm{e}^{-|\xi|}$$

  $(a = 0, \sigma^2 = 0, \pi = \frac{1}{\pi x^2}\,dx)$.

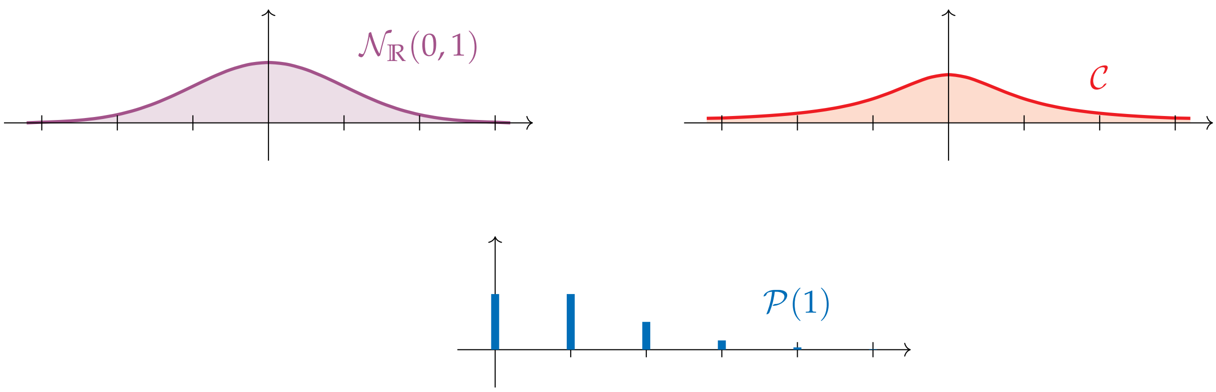


Figure 1.2. The standard Gaussian, Cauchy and Poisson laws.

Note that several of these laws have convergent Laplace transforms $\mathbb{E}[\mathrm{e}^{zX}] = \int_{\mathbb{R}} \mathrm{e}^{zx}\,\mu(dx)$ on a disk around 0 in the complex plane, and even on the whole complex plane $\mathbb{C}$ (Gaussian and Poisson laws). In the following we shall sometimes work with the Laplace transform, and sometimes with the Fourier transform, which is the restriction of the Laplace transform to the imaginary line $\mathrm{i}\mathbb{R}$.

▷ **Mod-ϕ convergent sequences.** We now give the main definition of this chapter, which is the correct way to go beyond convergence in law. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of real-valued random variables, and D be a connected subset of $\mathbb{C}$ that contains 0. We also fix an infinitely divisible law ϕ, and we suppose that $\int_{\mathbb{R}} \mathrm{e}^{zx}\,\phi(dx)$ is well defined for $z \in D$; we denote

$$\eta(z) = az + \frac{\sigma^2 z^2}{2} + \int_{\mathbb{R}} (\mathrm{e}^{zx} - 1 - 1_{|x|<1} zx)\,\pi(dx)$$

the *Lévy–Khintchine exponent* of ϕ. Note that if $D = \mathrm{i}\mathbb{R}$, then the Lévy exponent η is always well defined on D.

Definition 1.1 (Mod-ϕ convergence, [DKN15; FMN16]). *We say that $(X_n)_{n\in\mathbb{N}}$ converges in the mod-ϕ sense on the domain D with parameters $t_n \to +\infty$ and limit ψ if, locally uniformly on D,*

$$\lim_{n\to\infty} \mathbb{E}[\mathrm{e}^{zX_n}]\,\mathrm{e}^{-t_n\eta(z)} = \lim_{n\to\infty} \psi_n(z) = \psi(z),$$

ψ being a continuous function on D such that $\psi(0) = 1$.

If $D = \mathrm{i}\mathbb{R}$, we shall simply speak of mod-ϕ convergence, without specifying the domain. In this case, it is convenient to set $\theta_n(\xi) = \psi_n(\mathrm{i}\xi)$ and $\theta(\xi) = \psi(\mathrm{i}\xi)$, and the mod-$\phi$ convergence on $\mathrm{i}\mathbb{R}$ amounts to the local uniform convergence $\theta_n \to \theta$ on $\mathbb{R}$. The case $D = \mathrm{i}\mathbb{R}$ is the original definition of mod-ϕ convergence, but later we shall need to look at more general domains of convergence, in particular bands $(c,d) + \mathrm{i}\mathbb{R}$; see Section 1.2.

Remark 1.2. Intuitively, a sequence of random variables $(X_n)_{n\in\mathbb{N}}$ converges in the mod-ϕ sense if X_n is equal to a large infinitely divisible distribution of exponent $t_n\eta$, plus some small residue which is encoded asymptotically by the limit ψ. We shall see in a moment that in many situations, this implies that a certain renormalisation Y_n of X_n converges in law towards the infinitely divisible distribution of exponent η (Proposition 1.9). Therefore, mod-ϕ convergence can be seen as a refinement of the notion of convergence in law, and as a second-order central limit theorem. Then, one of the main objective is to obtain precise informations from the residue ψ and from the convergence $\psi_n \to \psi$.

Remark 1.3. We restrict ourselves to reference laws ϕ that are infinitely divisible, but in practice this is not really a restriction. Indeed, many interesting sequences of random variables $(X_n)_{n\in\mathbb{N}}$ inherit the scaling properties of the system that produces them, and in the limit $n \to \infty$ these scaling properties usually lead to infinite divisibility.

Remark 1.4. In the following, when $\phi = \mathcal{N}_{\mathbb{R}}(0,1)$, we shall simply speak of *mod-Gaussian convergence*; and when $\phi = \mathcal{P}(1)$, we shall speak of *mod-Poisson convergence*.

Example 1.5 (Sums of i.i.d. random variables). Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of independent and identically distributed random variables with values in $\mathbb{R}$. We suppose that $\mathbb{E}[A_1] = 0$ and $\mathbb{E}[|A_1|^3] < \infty$. We set $\mathbb{E}[(A_1)^2] = \sigma^2$, $\mathbb{E}[(A_1)^3] = L$ and we define $(X_n)_{n\in\mathbb{N}}$ by:

$$X_n = \frac{1}{n^{1/3}} \sum_{i=1}^{n} A_i.$$

Then, a Taylor expansion of the Fourier transform of A_1 shows that $(X_n)_{n\in\mathbb{N}}$ converges in the mod-Gaussian sense (on $D = \mathrm{i}\mathbb{R}$) with parameters $t_n = n^{1/3}\sigma^2$ and limit $\exp(\frac{L}{6}(\mathrm{i}\xi)^3)$:

$$\lim_{n\to\infty} \mathbb{E}[\mathrm{e}^{\mathrm{i}\xi X_n}]\,\mathrm{e}^{\frac{n^{1/3}\sigma^2\xi^2}{2}} = \exp\left(\frac{L\,(\mathrm{i}\xi)^3}{6}\right) \quad \text{locally uniformly on } \mathbb{R}.$$

As is well known, in this setting, $Y_n = \frac{1}{\sigma n^{1/6}}\,X_n$ converges in law towards a standard Gaussian $\mathcal{N}_{\mathbb{R}}(0,1)$. This result is generalised by Proposition 1.9. On the other hand, the sequence $(Y_n)_{n\in\mathbb{N}}$ also satisfies:

- Cramér's large deviation principle [Cra38],
- Gnedenko's local limit theorem [Gne48],
- Berry–Esseen estimates on $d_{\mathrm{Kol}}(Y_n, \mathcal{N}_{\mathbb{R}}(0,1))$ [Ber41; Ess45],

and various other precisions of the central limit theorem. The purpose of the theory of mod-ϕ convergence is to extend these results to much more general sequences of random variables, stemming for instance from combinatorics, graph theory, analytic number theory, or asymptotic representation theory.

Example 1.6 (Numbers of prime divisors). If $k \geq 1$ is an integer, let $\omega(k)$ denote the number of distinct prime divisors of k; for instance, $\omega(12) = 2$ since $12 = 2^2 \cdot 3$. We consider the random variable ω_n that is the value of the arithmetic function ω on a random integer $k \in [\![1,n]\!]$, k being chosen according to the uniform probability on this integer interval. The Selberg–Delange method yields the following asymptotics for the Laplace transform of ω_n:

$$\mathbb{E}[\mathrm{e}^{z\omega_n}] = \frac{1}{n}\sum_{k=1}^{n} \mathrm{e}^{z\omega(k)} = \mathrm{e}^{(\log\log n)(\mathrm{e}^z-1)}\,\frac{1}{\Gamma(\mathrm{e}^z)}\,\left(\prod_{p\in\mathbb{P}}\left(1+\frac{\mathrm{e}^z-1}{p}\right)\mathrm{e}^{-\frac{\mathrm{e}^z-1}{p}}\right)\,(1+o(1))$$

see [Del71] and [Ten95, Section II.6, Theorem 1]. So, $(\omega_n)_{n\in\mathbb{N}}$ converges in the mod-Poisson sense on the whole complex plane $D = \mathbb{C}$, with parameters $t_n = \log\log n$ and limit $\psi(z) = \frac{1}{\Gamma(\mathrm{e}^z)} \prod_{p\in\mathbb{P}}(1+\frac{\mathrm{e}^z-1}{p})\,\mathrm{e}^{-\frac{\mathrm{e}^z-1}{p}}$.

Since a large Poisson random variable $\mathcal{P}(\lambda)$ can be approximated by a Gaussian random variable $\mathcal{N}_{\mathbb{R}}(\lambda,\lambda)$, there is also mod-Gaussian convergence for an adequate renormalisation of ω_n, namely,

$$\widetilde{\omega}_n = \frac{\omega_n - \log\log n}{(\log\log n)^{1/3}}.$$

Thus,

$$\mathbb{E}\left[\mathrm{e}^{z\widetilde{\omega}_n}\right] = \mathrm{e}^{\frac{(\log\log n)^{1/3}z^2}{2}}\,\widetilde{\psi}(z)\,(1+o(1))$$

with $\widetilde{\psi}(z) = \exp(\frac{z^3}{6})$. Hence, one has mod-Gaussian convergence of $(\widetilde{\omega}_n)_{n\in\mathbb{N}}$, on $D = \mathbb{C}$ and with parameters $t_n = (\log\log n)^{1/3}$. This result allows one to go beyond the celebrated Erdös–Kac central limit theorem [EK40], which states that

$$\frac{\omega_n - \log\log n}{\sqrt{\log\log n}} \rightharpoonup \mathcal{N}_{\mathbb{R}}(0,1).$$

Example 1.7 (Winding numbers). Let $(Z_t)_{t\in\mathbb{R}_+}$ be a standard complex Brownian motion, starting from $Z_0 = 1$. Almost surely, this planar Brownian motion never touches 0, so one has a well-defined polar decomposition $Z_t = R_t\,\mathrm{e}^{\mathrm{i}\varphi_t}$, with $t \mapsto R_t$ and $t \mapsto \varphi_t$ (random) continuous functions, and $\varphi_0 = 0$; see Figure 1.3.

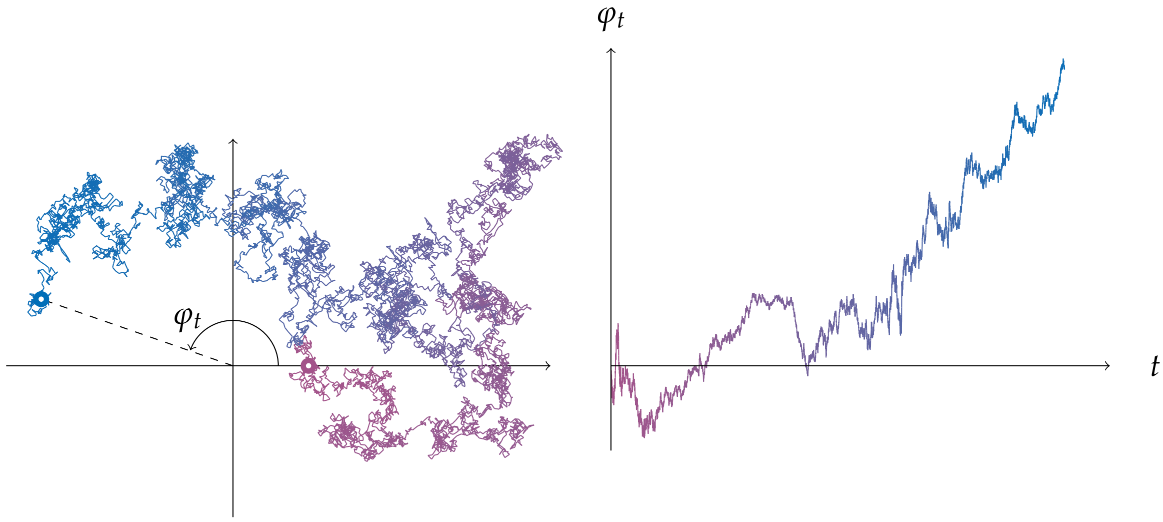


Figure 1.3. Winding number of the planar Brownian motion.

In [Spi58], Spitzer computed the Fourier transform of the *winding number* φ_t:

$$\mathbb{E}[\mathrm{e}^{\mathrm{i}\xi\varphi_t}] = \sqrt{\frac{\pi}{8t}}\,\mathrm{e}^{-\frac{1}{4t}}\left(I_{\frac{|\xi|-1}{2}}\left(\frac{1}{4t}\right) + I_{\frac{|\xi|+1}{2}}\left(\frac{1}{4t}\right)\right),$$

where

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{1}{k!\,\Gamma(\nu+k+1)}\left(\frac{z}{2}\right)^{\nu+2k}$$

is the modified Bessel function of the first kind. An asymptotic study of this formula shows that $(\varphi_t)_{t\in\mathbb{R}_+}$ converges in the mod-Cauchy sense on $D = \mathrm{i}\mathbb{R}$, with parameters $\frac{\log 8t}{2}$ and limit

$$\theta(\xi) = \frac{\sqrt{\pi}}{\Gamma\left(\frac{|\xi|+1}{2}\right)};$$

see [DKN15, Theorem 10].

Example 1.8 (Characteristic polynomials of random matrices). Let M_n be a random matrix of the unitary group $\mathrm{U}(n)$, chosen according to the Haar measure. We set

$$X_n = \mathrm{Re}(\log\det(I_n - M_n)) = \sum_{i=1}^{n} \mathrm{Re}(\log(1-\mathrm{e}^{\mathrm{i}\theta_i}))$$

where $(\mathrm{e}^{\mathrm{i}\theta_1}, \mathrm{e}^{\mathrm{i}\theta_2}, \ldots, \mathrm{e}^{\mathrm{i}\theta_n})$ are the eigenvalues of M_n. Here, the complex logarithm is chosen so that $X_n = X(M_n)$ defines a continuous function on an open dense subset of $\mathrm{U}(n)$. There is an exact formula for the Laplace transform of X_n:

$$\mathbb{E}[\mathrm{e}^{zX_n}] = \prod_{j=1}^{\infty} \frac{\Gamma(j)\,\Gamma(j+z)}{\left(\Gamma(j+\frac{z}{2})\right)^2}$$

if $\mathrm{Re}(z) > -1$. This is an application of Selberg's integral formula, see [Sel44], and [KS00b; KS00a; KN12] for the asymptotics of this formula. From this exact formula, one can deduce the mod-Gaussian convergence of $(X_n)_{n\in\mathbb{N}}$ on $D = \{z \in \mathbb{C} \,|\, \mathrm{Re}(z) > -1\}$, with parameters $t_n = \frac{\log n}{2}$ and limit

$$\psi(z) = \frac{\left(G(1+\frac{z}{2})\right)^2}{G(1+z)}.$$

Here, $G(z)$ is Barnes' G-function, which is the unique entire solution of the functional equation $G(z+1) = \Gamma(z)\,G(z)$, with $G(1) = 1$. This result can be extended to logarithms of characteristic polynomials of other classical compact Lie groups such as $\mathrm{USp}(n)$ or $\mathrm{SO}(2n)$; see [KN12, §4.2 and §4.3].

The previous examples already show the large scope of the theory of mod-ϕ convergence. Starting from Definition 1.1, there are two possible directions:

(1) From the mod-ϕ convergence of a sequence $(X_n)_{n\in\mathbb{N}}$, one can try to deduce the asymptotics of the distribution of X_n, either "in the bulk" (central limit theorem, local limit theorem, speed of convergence) of the distribution, or "at the edge" (normality zones, large deviations).

(2) One can also try to establish the mod-ϕ convergence of sequences of random variables stemming from a probabilistic model: random graphs, random matrices, arithmetic functions of random integers, random permutations, *etc.*

In this first chapter, we shall concentrate on general probabilistic results of the first kind. The second chapter will be devoted to general methods that allow to prove the mod-ϕ convergence of a sequence of random variables: analysis of singularities of generating functions, or the method of moments and cumulants.

▷ **Central limit theorems.** To end this section, let us state a general central limit theorem that follows from mod-ϕ convergence. In the sequel we shall need to manipulate *stable laws*, which are the infinitely divisible laws of exponent

$$\eta(\mathrm{i}\xi) = \eta_{c,\alpha,\beta}(\mathrm{i}\xi) = -|c\xi|^{\alpha}\,(1 - \mathrm{i}\beta\, h(\alpha,\xi)\,\mathrm{sgn}(\xi))$$

with $c \in \mathbb{R}_+^*$ (the *scale* parameter), $\alpha \in (0,2]$ (the *stability* parameter), $\beta \in [-1,1]$ (the *skewness* parameter), and

$$h(\alpha,\xi) = \begin{cases} \tan\left(\frac{\pi\alpha}{2}\right) & \text{if } \alpha \neq 1, \\ -\frac{2}{\pi}\,\log|\xi| & \text{if } \alpha = 1. \end{cases}$$

The stable laws $\phi_{c,\alpha,\beta}$ are involved in a generalised central limit theorem for sums of i.i.d. variables without second moment [GK54]. We recover the Gaussian law for $\alpha = 2$, $\beta = 0$; and the Cauchy law for $\alpha = 1$, $\beta = 0$. We refer to [Sat99, Chapter 3] for a detailed study of the stable laws, which have the following scaling property:

$$t\,\eta_{c,\alpha,\beta}\left(\frac{\mathrm{i}\xi}{t^{1/\alpha}}\right) = \begin{cases} \eta_{c,\alpha,\beta}(\mathrm{i}\xi) & \text{if } \alpha \neq 1, \\ \eta_{c,\alpha,\beta}(\mathrm{i}\xi) - \left(\frac{2c\beta}{\pi}\,\log t\right)\mathrm{i}\xi & \text{if } \alpha = 1. \end{cases}$$

This implies the following proposition (see [FMN17b, Proposition 3]):

Proposition 1.9 (Central limit theorem). *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of real random variables.*

(1) Suppose that $(X_n)_{n\in\mathbb{N}}$ converges in the mod-ϕ sense on $D = \mathrm{i}\mathbb{R}$, with parameters $t_n \to +\infty$ and with $\phi = \phi_{c,\alpha,\beta}$ that is a stable law. Set

$$Y_n = \begin{cases} \frac{X_n}{(t_n)^{1/\alpha}} & \text{if } \alpha \neq 1, \\ \frac{X_n}{t_n} - \frac{2c\beta}{\pi}\log t_n & \text{if } \alpha = 1. \end{cases}$$

The sequence of random variables $(Y_n)_{n\in\mathbb{N}}$ converges in law towards $\phi_{c,\alpha,\beta}$.

(2) Suppose that $(X_n)_{n\in\mathbb{N}}$ converges in the mod-ϕ sense on $D = \mathrm{i}\mathbb{R}$, with parameters $t_n \to +\infty$ and with ϕ that admits a moment of order 3. Then,

$$\widetilde{X}_n = \frac{X_n - t_n\,\eta'(0)}{(t_n)^{1/3}}$$

converges in the mod-Gaussian sense on $\mathrm{i}\mathbb{R}$, with parameters $\widetilde{t}_n = (t_n)^{1/3}\,\eta''(0)$ and limit $\theta(\xi) = \exp(\frac{\eta'''(0)\,(\mathrm{i}\xi)^3}{6})$.

Remark 1.10. An infinitely divisible law ϕ admits a moment of order 3 if and only if its Lévy measure π satisfies $\int_{|x|\geq 1} |x|^3\,\pi(dx) < +\infty$ (see [Sat99, Corollary 25.8]). On the other hand, the Lévy exponent η might only be defined on $\mathrm{i}\mathbb{R}$ in the second item of Proposition 1.9; in this case, by $\eta'(0)$ we mean $-\mathrm{i}\,\eta'_{\mathrm{imag}}(0)$, where η_{imag} is the real function $\xi \mapsto \eta(\mathrm{i}\xi)$; and similarly for the higher derivatives.

The proof of Proposition 1.9 is immediate by a Taylor expansion of the adequate Fourier transforms. By combining the first and the second item, one sees that in most cases, the mod-ϕ convergence of a sequence of random variables $(X_n)_{n\in\mathbb{N}}$ can be considered as an improved or "second-order" central limit theorem for an adequate renormalisation $(Y_n)_{n\in\mathbb{N}}$ of the original sequence. The goal of the next sections is to explain precisely what is gained by means of the mod-ϕ convergence.

1.2 Normality zones and precise large deviations

Since stable laws have integrable Fourier transforms, they also have continuous densities and Proposition 1.9 amounts to the pointwise convergence of the cumulative distribution functions. Thus, if $(X_n)_{n\in\mathbb{N}}$ converges in the mod-ϕ sense with parameters $(t_n)_{n\in\mathbb{N}}$, and if $\phi = \phi_{c,\alpha,\beta}$, then for any $x \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathbb{P}[Y_n \leq x] = \int_{-\infty}^{x} m_{c,\alpha,\beta}(s)\, ds, \tag{1.1}$$

where $m_{c,\alpha,\beta}(s)$ is the density of $\phi_{c,\alpha,\beta}$, *e.g.* $\frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{s^2}{2}}$ in the Gaussian case. In this setting, a first application of the mod-ϕ convergence consists in allowing $x = x_n$ to vary with n, and in computing the asymptotics of probabilities in this setting. More precisely, we shall compute probabilities at the edge of the distribution of X_n, that is with x growing to infinity. If x is not too large, then Equation (1.1) stays true; in the case $\phi = \mathcal{N}_{\mathbb{R}}(0,1)$, we shall then speak of the *normality zone* of Y_n (or X_n). On the other hand, if x is too large, then Equation (1.1) becomes false, but under the assumption of complex mod-ϕ convergence, one can correct it for x belonging to a certain range. This correction involves the limit ψ of the residues $\psi_n(z) = \mathbb{E}[\mathrm{e}^{zX_n}]\,\mathrm{e}^{-t_n\eta(z)}$. Thus, mod-$\phi$ convergence implies *precise large deviation estimates*; this is one of the main results of [FMN16].

▷ **The hypothesis of convergence on a band.** In the remainder of this section, $(X_n)_{n\in\mathbb{N}}$ is a sequence of real-valued random variables that converges in the mod-ϕ sense on a band $\mathcal{S}_{(c,d)} = \{z \in \mathbb{R} \,|\, c < \mathrm{Re}(z) < d\}$, with $c < 0 < d$, and possibly $c = -\infty$ or $d = +\infty$. In particular, we assume that the Laplace transform of $(X_n)_{n\in\mathbb{N}}$ and of ϕ is well defined in a neighborhood of 0; this forces the variables X_n and the reference distribution ϕ to have moments of all order. Note that ϕ has a well-defined Laplace transform on the symmetric band $\mathcal{S}_{(-d,d)}$ if and only if its Lévy measure π satisfies $\int_{|x|\geq 1} \mathrm{e}^{t|x|}\,\pi(dx) < \infty$ for any $t < d$. We set

$$\psi_n(z) = \mathbb{E}[\mathrm{e}^{zX_n}]\,\mathrm{e}^{-t_n\eta(z)} \quad ; \quad \psi(z) = \lim_{n\to\infty} \psi_n(z).$$

The functions ψ_n are holomorphic on $\mathcal{S}_{(c,d)}$, and so is their limit ψ; moreover, the local uniform convergence $\psi_n \to \psi$ also holds true for the complex derivatives up to any order.

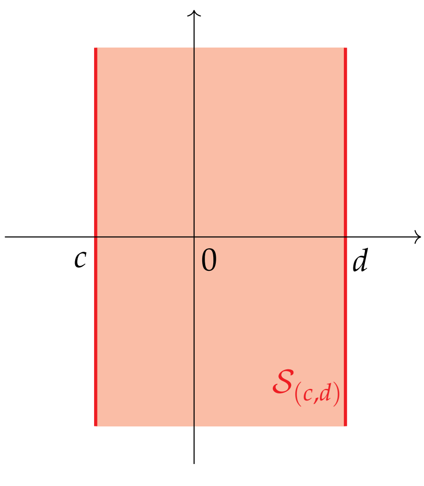


Figure 1.4. Mod-ϕ convergence on a band $\mathcal{S}_{(c,d)}$ of the complex plane yields a normality zone and precise large deviations.

In order to state the results on the zone of normality and the precise deviation estimates that follow from these hypotheses, we need to recall the theory of *Legendre–Fenchel transforms*; see [Roc70], and [DZ98, Section 2.2] for its applications in probability theory. Let η be a convex and lower semicontinuous function on the real line (for any x, $\liminf_{x\to x_0} \eta(x) \geq \eta(x_0)$), with values in the extended real line $\mathbb{R} \sqcup \{+\infty\}$, and not equal everywhere to $+\infty$. The Legendre–Fenchel transform of η is defined by:

$$F(x) = \sup_{h\in\mathbb{R}} (hx - \eta(h)).$$

The Legendre–Fenchel transform is an involution on convex lower semicontinuous functions. It is well known that, if $(A_n)_{n\in\mathbb{N}}$ is a sequence of independent and identically distributed random variables with log-Laplace transform $\eta(x) = \log \mathbb{E}[\mathrm{e}^{xA_1}]$, then the large deviations of $M_n = \frac{1}{n}\sum_{i=1}^n A_i$ are prescribed by the Legendre–Fenchel transform F of η:

$$\limsup_{n\to\infty} \frac{\log \mathbb{P}[M_n \in A]}{n} \leq -\inf_{x\in\overline{A}} F(x);$$
$$\liminf_{n\to\infty} \frac{\log \mathbb{P}[M_n \in A]}{n} \geq -\inf_{x\in A^\circ} F(x).$$

We shall see in a moment that the same function F (the Legendre–Fenchel transform of the logarithm of the Lévy exponent η) is involved in the large deviation estimates coming from mod-ϕ convergence (Theorems 1.12 and 1.13). If $\phi = \mathcal{N}(m,\sigma^2)$, then

$$\eta(x) = mx + \frac{\sigma^2 x^2}{2} \qquad ; \qquad F(x) = \frac{(x-m)^2}{2\sigma^2},$$

and if $\phi = \mathcal{P}(\lambda)$, then

$$\eta(x) = \lambda(\mathrm{e}^x - 1) \qquad ; \qquad F(x) = \begin{cases} x \log \frac{x}{\lambda} - (x-\lambda) & \text{if } x > 0 \\ +\infty & \text{otherwise.} \end{cases}$$

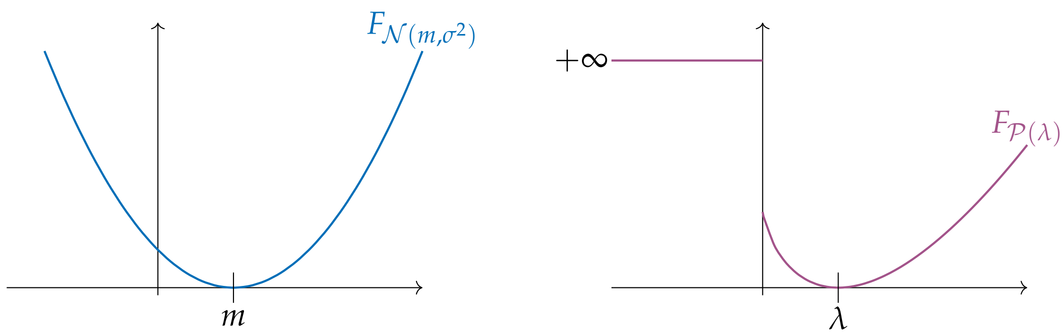


Figure 1.5. The Legendre–Fenchel transforms of the Lévy exponents of a Gaussian law and of a Poisson law.

Another prerequisite for our results is the distinction between lattice distributed and non-lattice distributed infinitely divisible laws. Let ϕ be a non-constant reference infinitely divisible law, and (a, σ^2, π) the triplet of its Lévy–Khintchine representation. The support of ϕ is described by the following classification:

Lemma 1.11 (Support of an infinitely divisible law). *The infinitely divisible law ϕ belongs to one of the following classes:*

- *Suppose that $\sigma^2 = 0$, that π integrates $|x|$ and that the support of π generates a discrete subgroup $\mathbb{Z}[\mathrm{supp}(\pi)] = \lambda\mathbb{Z} \subset \mathbb{R}$, with $\lambda > 0$. Then, the support of ϕ is included in the shifted discrete subgroup $\gamma + \lambda\mathbb{Z}$, where $\gamma = a - \int_{\mathbb{R}} 1_{|x|<1}\, x\, \pi(dx)$. We then say that ϕ is lattice distributed, and the characteristic function*
$$\exp(\eta(\mathrm{i}\xi)) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\xi x}\, \phi(dx)$$
satisfies $|\exp(\eta(\mathrm{i}\xi))| = 1$ if and only if $\xi \in \frac{2\pi}{\lambda}$.
- *Otherwise, we say that ϕ is non-lattice distributed, and $|\exp(\eta(\mathrm{i}\xi))| < 1$ for any $\xi \neq 0$. If $\sigma^2 = 0$ and γ is well-defined, then the support of ϕ is $\gamma + \overline{\mathbb{N}[\mathrm{supp}(\pi)]}$. Otherwise, the support of ϕ is $\mathbb{R}$, or one of the two half-lines $(-\infty, \gamma]$ or $[\gamma, +\infty)$.*

We refer to [SH04, Chapter 4, Theorem 8.4] for a proof of this classification. In the lattice case, replacing Y with law ϕ by $\frac{Y}{\lambda}$, one can assume without loss of generality that $\lambda = 1$; we then say that $\mathbb{Z}$ is the lattice of ϕ, and $|\exp(\eta(\mathrm{i}\xi))| = 1$ if and only if $\xi \in 2\pi\mathbb{Z}$.

▷ **Large deviation estimates.** We can now state our results of large deviations, see [FMN16, Theorems 3.2.2 and 4.2.1]. Recall that $(X_n)_{n\in\mathbb{N}}$ is a sequence that converges mod-ϕ on a band $\mathcal{S}_{(c,d)} = (c, d) + \mathrm{i}\mathbb{R}$ with parameters $(t_n)_{n\in\mathbb{N}}$ and limit ψ. We exclude the trivial case where ϕ is constant. We assume $c < 0 < d$, and if $x \in (c, d)$, we denote $h = h(x)$ the solution of $\eta'(h(x)) = x$. Equivalently, if F is the Legendre–Fenchel transform of the Lévy exponent η of ϕ, then $F(x) = xh(x) - \eta(h(x))$, and $F'(x) = h(x)$. On the other hand, until the end of this section, we assume that ψ does not vanish on the real part of the domain $\mathcal{S}_{(c,d)}$ (by definition, the residues ψ_n do not vanish on (c, d), but their limit might vanish and we exclude this case).

Theorem 1.12 (Large deviations in the lattice case). *We suppose that the reference law is lattice distributed, with lattice $\mathbb{Z}$. If $t_n x \in \mathbb{Z}$ and $x \in (\eta'(c), \eta'(d))$, then*
$$\mathbb{P}[X_n = t_n x] = \frac{\mathrm{e}^{-t_n F(x)}}{\sqrt{2\pi t_n \eta''(h)}}\, \psi(h)\, (1 + o(1)).$$
If x is in the range of $\eta'_{|(0,d)}$, then
$$\mathbb{P}[X_n \geq t_n x] = \frac{\mathrm{e}^{-t_n F(x)}}{\sqrt{2\pi t_n \eta''(h)}}\, \frac{\psi(h)}{1 - \mathrm{e}^{-h}}\, (1 + o(1)).$$
If x is in the range of $\eta'_{|(c,0)}$, then
$$\mathbb{P}[X_n \leq t_n x] = \frac{\mathrm{e}^{-t_n F(x)}}{\sqrt{2\pi t_n \eta''(h)}}\, \frac{\psi(h)}{1 - \mathrm{e}^{h}}\, (1 + o(1)).$$

Theorem 1.13 (Large deviations in the non-lattice case). *We suppose that the reference law ϕ is non-lattice distributed. If x is in the range of $\eta'_{|(0,d)}$, then*
$$\mathbb{P}[X_n \geq t_n x] = \frac{\mathrm{e}^{-t_n F(x)}}{h\, \sqrt{2\pi t_n \eta''(h)}}\, \psi(h)\, (1 + o(1)).$$
If x is in the range of $\eta'_{|(c,0)}$, then
$$\mathbb{P}[X_n \leq t_n x] = \frac{\mathrm{e}^{-t_n F(x)}}{|h|\, \sqrt{2\pi t_n \eta''(h)}}\, \psi(h)\, (1 + o(1)).$$

Both theorems are proved by a careful use of the Parseval formula, either on the circle (lattice case) or on the real line (non-lattice case); and by using the Laplace method to estimate asymptotics of integrals. In the non-lattice case, an important intermediary result is the following general Berry–Esseen estimate [FMN16, Proposition 4.1.1]:

Theorem 1.14 (General Berry–Esseen estimates). *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence that converges mod-ϕ on a band $\mathcal{S}_{(c,d)}$ with $c<0<d$. We suppose that the reference law ϕ is non-lattice distributed, and we consider the renormalisation*

$$Y_n = \frac{X_n - t_n\eta'(0)}{\sqrt{t_n\,\eta''(0)}},$$

which converges in law towards a standard Gaussian distribution by Proposition 1.9. We have more precisely:

$$\sup_{x\in\mathbb{R}}\left|\mathbb{P}[Y_n\leq x] - \int_{-\infty}^{x}\left(1+\frac{\psi'(0)\,s}{\sqrt{t_n\eta''(0)}}+\frac{\eta'''(0)\,(s^3-3s)}{6\sqrt{t_n(\eta''(0))^3}}\right)\frac{\mathrm{e}^{-\frac{s^2}{2}}}{\sqrt{2\pi}}\,ds\right| = o\left(\frac{1}{\sqrt{t_n}}\right).$$

The following remarks will help the understanding of these important asymptotic results.

Remark 1.15. In the lattice case, one can actually give a full expansion in powers of $\frac{1}{t_n}$ of the probabilities $\mathbb{P}[X_n = t_n x]$ and $\mathbb{P}[X_n \geq t_n x]$. Thus, with the same assumptions as in Theorem 1.12, for any $v \geq 0$

$$\mathbb{P}[X_n = t_n x]\,\mathrm{e}^{t_n F(x)}\,\sqrt{2\pi t_n\,\eta''(h)} = a_0 + \frac{a_1}{t_n}+\cdots+\frac{a_v}{(t_n)^v}+o\left(\frac{1}{(t_n)^v}\right),$$

where $a_0=\psi(h)$, and more generally a_k is a rational fraction in the derivatives of η and ψ at h. Similarly, for $x\in(\eta'(0),\eta'(d))$,

$$\mathbb{P}[X_n \geq t_n x]\,\mathrm{e}^{t_n F(x)}\,\sqrt{2\pi t_n\,\eta''(h)}\,(1-\mathrm{e}^{-h}) = b_0 + \frac{b_1}{t_n}+\cdots+\frac{b_v}{(t_n)^v}+o\left(\frac{1}{(t_n)^v}\right)$$

for some other explicit rational fractions b_k in the derivatives of η and ψ at h. As far as we know, it is not possible to have such an expansion in the non-lattice case, but in this setting we have Berry–Esseen estimates (see Theorem 1.14 and Section 1.3).

Remark 1.16. In the non-lattice case, Theorem 1.14 gives an estimate of the Kolmogorov distance between Y_n and a standard Gaussian $G\sim\mathcal{N}_{\mathbb{R}}(0,1)$:

$$\begin{aligned}d_{\mathrm{Kol}}(Y_n,G) &= \sup_{s\in\mathbb{R}}|\mathbb{P}[Y_n\leq s]-\mathbb{P}[G\leq s]| \\ &\lesssim \sup_{x\in\mathbb{R}}\left(\mathrm{e}^{-\frac{x^2}{2}}\left|\psi'(0)+(x^2-1)\,\frac{\eta'''(0)}{6\eta''(0)}\right|\right)\frac{1}{\sqrt{2\pi t_n\,\eta''(0)}}.\end{aligned}$$

In particular, $d_{\mathrm{Kol}}(Y_n,G) = O(\frac{1}{\sqrt{t_n}})$. If $\psi'(0)=\eta'''(0)=0$, then we can get much better estimates, see Section 1.3.

Remark 1.17. Theorems 1.12 and 1.13 are examples of *precise* large deviation results, because they give estimates of probabilities and not of their logarithms. They are reminiscent of a result due to Bahadur and Rao [BR60] that is a precise large deviation estimate for means

$M_n = \frac{1}{n}\sum_{i=1}^n A_i$ of i.i.d. random variables:

$$\forall x > \mathbb{E}[A_1],\ \ \mathbb{P}[M_n \geq x] = \begin{cases} \frac{\mathrm{e}^{-n\,F(x)}}{(1-\mathrm{e}^{-F'(x)})\,\sqrt{2\pi n\,\mathrm{Var}(A_1)}}\,(1+o(1)) & \text{in the } \mathbb{Z}\text{-lattice case,} \\ \frac{\mathrm{e}^{-n\,F(x)}}{F'(x)\,\sqrt{2\pi n\,\mathrm{Var}(A_1)}}\,(1+o(1)) & \text{in the non-lattice case,} \end{cases}$$

where F is the Legendre–Fenchel transform of the log-Laplace transform of A_1. In [FMN16, Section 4.5.1], we explained how to recover these results from our general theory of mod-ϕ convergence.

Example 1.18 (Number of cycles in a random permutation). Let $C_n = c(\sigma_n)$ be the total number of cycles (including cycles of length 1 which are fixed points) of a random permutation σ_n of $\mathfrak{S}(n)$ chosen according to Ewens' measure [Ewe72]

$$\mathbb{P}[\sigma_n = \sigma] = \frac{\theta^{c(\sigma)}}{\theta(\theta+1)(\theta+2)\cdots(\theta+n-1)},$$

with $\theta > 0$. We recover the uniform measure for $\theta = 1$. The identity in the symmetric group algebra $\mathbb{C}(\theta)(\mathfrak{S}(n))$

$$\sum_{\sigma\in\mathfrak{S}(n)} \theta^{c(\sigma)}\,\sigma = \prod_{i=1}^n ((1,i)+\cdots+(i-1,i)+\theta) \tag{1.2}$$

proves that the number of cycles has the same law as a sum of independent random Bernoulli variables:

$$C_n =_{\text{law}} \sum_{i=1}^n \mathcal{B}\left(\frac{\theta}{\theta+i-1}\right).$$

The algebraic formula (1.2) can be seen as one of the many properties of the so-called Jucys–Murphy elements; see the papers [Juc74; Mur81; OV04], and [CST10, §3.2] or [Mél17, Chapter 8] for a survey of their properties. In probability theory, the decomposition of C_n is known as Feller's coupling; see for instance [ABT03, Chapter 4]. A consequence is the formula

$$\mathbb{E}[\mathrm{e}^{zC_n}] = \prod_{i=1}^n \left(1+\frac{\theta(\mathrm{e}^z-1)}{\theta+i-1}\right) = \mathrm{e}^{\left(\sum_{i=1}^n \frac{\theta}{\theta+i-1}\right)(\mathrm{e}^z-1)}\ \prod_{i=1}^n \left(1+\frac{\theta(\mathrm{e}^z-1)}{\theta+i-1}\right)\mathrm{e}^{-\frac{\theta(\mathrm{e}^z-1)}{\theta+i-1}}.$$

The product $\psi_n(z)$ converges to an entire function as n goes to infinity, which one can compute by using the infinite product representation of the Γ function. We shall give the details of the computation in Section 2.1, in the more general situation of probabilities on permutations given by a family of weights $(\theta_k)_{k\geq 1}$ associated to the different lengths of the cycles. So, one has mod-Poisson convergence on the whole complex plane, with parameters $t_n = \theta \log n$ and limit $\psi(z) = \frac{\Gamma(\theta)}{\Gamma(\theta\mathrm{e}^z)}$. It follows then from Theorem 1.12 that for any $x > 1$ such that $x\,\theta\log n \in \mathbb{N}$,

$$\mathbb{P}[C_n \geq x\,\theta\log n] = \frac{\mathrm{e}^{-\theta\log n(x\log x-(x-1))}}{\sqrt{2\pi\log n}}\,\frac{\sqrt{x}}{x-1}\,\frac{\Gamma(\theta)}{\Gamma(\theta x)}\,(1+o(1)),$$

since $F(x) = x\log x - (x-1)$, $h(x) = \log x$ and $\eta''(h(x)) = \mathrm{e}^{h(x)} = x$. In [FMN16, Section 7.4], we gave with a similar analysis the precise large deviations of the number of rises in a uniform random permutation of $\mathfrak{S}(n)$.

Example 1.19 (Number of prime divisors). With essentially the same calculations as in the previous example, we get the asymptotics of the number of distinct prime divisors ω_n of a random integer smaller than n. For instance, for any $x > 1$ such that $x \log\log n \in \mathbb{N}$,

$$\mathbb{P}[\omega_n = x \log\log n] = \frac{\mathrm{e}^{-\log\log n (x \log x - (x-1))}}{\sqrt{2\pi x \log\log n}} \, \frac{1}{\Gamma(x)} \left(\prod_{p \in \mathbb{P}} \left(1 + \frac{x-1}{p}\right) \mathrm{e}^{-\frac{x-1}{p}} \right) (1 + o(1)).$$

This result was also obtained by Radziwiłł in [Rad09]. In [FMN16, Section 7.2], we extended these estimates to a larger class of arithmetic functions of random integers; see Example 2.8.

Example 1.20 (Characteristic polynomials of random matrices). Consider as in Example 1.8 the real part X_n of the logarithm $\log\det(I_n - M_n)$, where $M_n \in \mathrm{U}(n)$ is Haar-distributed. The mod-Gaussian convergence of this quantity implies the following large deviation results:

$$\forall x > 0,\ \mathbb{P}\left[|\det(I_n - M_n)| \geq n^{\frac{x}{2}}\right] = \frac{G(1+\frac{x}{2})^2}{G(1+x)} \, \frac{n^{-\frac{x^2}{4}}}{x\sqrt{\pi \log n}} (1 + o(1));$$

$$\forall x \in (0,1),\ \mathbb{P}\left[|\det(I_n - M_n)| \leq n^{-\frac{x}{2}}\right] = \frac{G(1-\frac{x}{2})^2}{G(1-x)} \, \frac{n^{-\frac{x^2}{4}}}{x\sqrt{\pi \log n}} (1 + o(1)).$$

Besides, if $Y_n = \sqrt{2/\log n}\, X_n$, then $d_{\mathrm{Kol}}(Y_n, \mathcal{N}_{\mathbb{R}}(0,1))$ is asymptotically a $o((\log n)^{-1/2})$, because in this case $\psi'(0) = 0$ and $\eta'''(0) = 0$. The theory of zones of control which we shall develop in Section 1.3 allows one to show that $d_{\mathrm{Kol}}(Y_n, \mathcal{N}_{\mathbb{R}}(0,1))$ is actually a $O((\log n)^{-1})$, and in fact that $d_{\mathrm{Kol}}(\frac{X_n}{\sqrt{\mathrm{var}(X_n)}}, \mathcal{N}_{\mathbb{R}}(0,1)) = O((\log n)^{-3/2})$.

▷ **Normality zones.** The previous theorems regard the behavior of X_n at the scale t_n. On the other hand, in the lattice and in the non-lattice cases, by combining the two items of Proposition 1.9, one sees that under the hypotheses of mod-ϕ convergence made at the beginning of this section,

$$Y_n = \frac{X_n - t_n \eta'(0)}{\sqrt{t_n\, \eta''(0)}}$$

converges in law towards a standard Gaussian distribution $\mathcal{N}_{\mathbb{R}}(0,1)$. In other words, once recentred, at the scale $\sqrt{t_n}$, X_n has a Gaussian behavior. A natural problem is then to describe what happens between these two scales $\sqrt{t_n}$ and t_n. It turns out that one can again use the theory of mod-ϕ convergence to compute the zone of normality of the sequence $(X_n)_{n\in\mathbb{N}}$, which is the largest scale $(t_n)^\alpha$ up to which the asymptotic behavior of the random variables X_n is prescribed by the Gaussian distribution. A small caveat is that in the non-lattice case, one needs to assume that ϕ is absolutely continuous with respect to the Lebesgue measure. This happens as soon as $\sigma^2 > 0$, or if $\sigma^2 = 0$ and the Lévy measure π of ϕ has an absolutely continuous part with infinite mass [SH04, Chapter 4, Theorem 4.20]. We then obtained in [FMN16, Theorems 3.3.1 and 4.3.1]:

Theorem 1.21 (Zone of normality). *We consider as before a sequence $(X_n)_{n\in\mathbb{N}}$ of random variables that converges mod-ϕ on a band $\mathcal{S}_{(c,d)}$ which contains 0 in its interior. Suppose that ϕ is lattice distributed, or non-lattice distributed and absolutely continuous with respect to the Lebesgue measure.*

(1) If $y = o((t_n)^{1/6})$ and $Y_n = \frac{X_n - t_n\eta'(0)}{\sqrt{t_n\, \eta''(0)}}$, then

$$\mathbb{P}[Y_n \geq y] = \mathbb{P}[\mathcal{N}_{\mathbb{R}}(0,1) \geq y]\,(1 + o(1)).$$

In particular, if $1 \ll y \ll (t_n)^{1/6}$, *then*

$$\mathbb{P}[Y_n \geq y] = \frac{e^{-\frac{y^2}{2}}}{y\sqrt{2\pi}}\,(1+o(1)),$$

the right-hand side being the tail of a standard Gaussian distribution.

(2) *Suppose more generally that* $y = o((t_n)^{1/2})$, *and denote* $x = \eta'(0) + \sqrt{\frac{\eta''(0)}{t_n}}\,y$. *Then,*

$$\mathbb{P}[Y_n \geq y] = \mathbb{P}[X_n \geq t_n x] = \frac{e^{-t_n F(x)}}{y\sqrt{2\pi}}\,(1+o(1)),$$

where F is as in Theorems 1.12 and 1.13 the Legendre–Fenchel transform of the log-Laplace transform of ϕ.

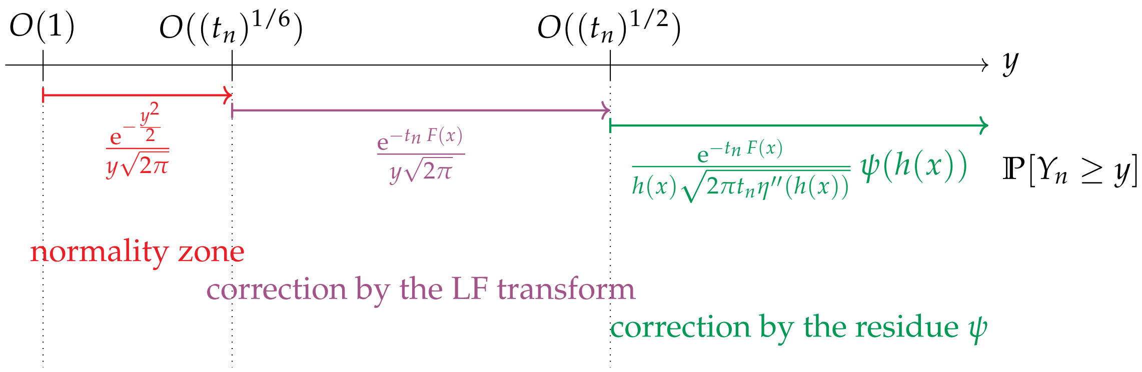


Figure 1.6. Panorama of the fluctuations of a mod-ϕ convergent sequence.

Let us restate Theorem 1.21 in terms of zones of normality. If $y = o((t_n)^{1/6})$, then the y-tail of the distribution of Y_n is asymptotically equivalent to the Gaussian tail. Hence, the generic zone of normality of Y_n is $o((t_n)^{1/6})$. When y becomes larger but stays a $o((t_n)^{1/4})$, the Gaussian tail has to be corrected by an exponential of y^3:

$$\mathbb{P}[Y_n \geq y] = \frac{e^{-\frac{y^2}{2}}}{y\sqrt{2\pi}}\,\exp\left(\frac{\eta'''(0)}{6\,(\eta''(0))^{3/2}}\,\frac{y^3}{\sqrt{t_n}}\right)(1+o(1)).$$

This is a consequence of the second part of Theorem 1.21, by making a Taylor expansion of F around $\eta'(0)$. Between $(t_n)^{1/4}$ and $(t_n)^{3/10}$, one needs to add another term in y^4, and more generally, if $y = o((t_n)^{\frac{1}{2}-\frac{1}{m}})$ with $m \geq 3$, then

$$\mathbb{P}[Y_n \geq y] = \frac{e^{-\frac{y^2}{2}}}{y\sqrt{2\pi}}\,\exp\left(-\sum_{i=3}^{m}\frac{F^{(i)}(\eta'(0))}{i!}\,\frac{(\eta''(0))^{\frac{i}{2}}\,y^i}{(t_n)^{\frac{i-2}{2}}}\right)(1+o(1)).$$

When $y = O((t_n)^{1/2})$, by Theorems 1.12 and 1.13, one has the full Legendre–Fenchel transform F in the asymptotic expansion of $\mathbb{P}[Y_n \geq y]$, plus a new correcting term which involves the limiting residue ψ of mod-ϕ convergence.

Remark 1.22. The size of the zone of normality is $o((t_n)^{1/6})$ unless $\eta'''(0) = 0$, in which case it can be larger. A particular case is when ϕ is a Gaussian law $\mathcal{N}_{\mathbb{R}}(m,\sigma^2)$. Then, all the derivatives of F but the two first vanish, and therefore, the zone of normality is in this case (at least) a $o((t_n)^{1/2})$.

Remark 1.23. Our results on the zone of normality are close to those obtained by Hwang by using the formalism of quasi-powers ([Hwa96; Hwa98] and [FS09, Section IX.5]). One of the main difference is the hypothesis of analyticity that we make on $\int_{\mathbb{R}} \mathrm{e}^{zx}\, \phi(dx) = \mathrm{e}^{\eta(z)}$. It is more restrictive than the framework of quasi-powers, but it allows one to describe a full panorama of fluctuations, from the central limit theorem of Proposition 1.9 to the moderate or large deviations of Theorems 1.12 and 1.13.

Example 1.24 (Zones of normality). The zone of normality of the rescaled number of cycles

$$\frac{C_n - \theta \log n}{\sqrt{\theta \log n}}$$

of a random permutation under Ewens' measure is $o((\log n)^{1/6})$, and the zone of normality of the rescaled number of prime divisors

$$\frac{\omega_n - \log \log n}{\sqrt{\log \log n}}$$

of a random integer smaller than n is $o((\log \log n)^{1/6})$. On the other hand, because one has mod-Gaussian convergence, the zone of normality of $(2/\log n)^{1/2}\,\mathrm{Re}(\log \det(I_n - M_n))$ with M_n Haar-distributed unitary matrix is a $o((\log n)^{1/2})$ (and not just a $o((\log n)^{1/6})$).

1.3 Estimates on the speed of convergence

In this section, $(X_n)_{n\in\mathbb{N}}$ is a sequence of random variables that converges mod-ϕ on $D = \mathrm{i}\mathbb{R}$. Another way to precise the central limit theorem that follows from this hypothesis is to estimate the distance between the distribution of X_n and the distribution of an infinitely divisible random variable Z_n of exponent $t_n\eta$, where η is the exponent of ϕ. Concretely, when $\phi = \phi_{c,\alpha,\beta}$ is a stable law, one tries to find an upper bound on the *Kolmogorov distance*

$$\begin{aligned} d_{\mathrm{Kol}}(X_n, Z_n) &= \sup_{x\in\mathbb{R}} |\mathbb{P}[X_n \leq x] - \mathbb{P}[Z_n \leq x]| \\ &= \sup_{x\in\mathbb{R}} |\mathbb{P}[Y_n \leq x] - \mathbb{P}[Z \leq x]|\,, \end{aligned} \tag{1.3}$$

where Y_n is the same renormalisation of X_n as in the first item of Proposition 1.9, and Z is a stable random variable with law $\phi_{c,\alpha,\beta}$. This is the problem of the *speed of convergence* in the central limit theorem, and the objective of this section is to explain how to compute it. Such results were obtained in [FMN17b]. In order to obtain optimal estimates, we shall replace the hypothesis of mod-ϕ convergence by the closely related hypothesis of *zone of control* (Definition 1.25); in most examples, it comes from the same arguments and it is simultaneously satisfied.

When the reference law ϕ is discrete (lattice-distributed with lattice $\mathbb{Z}$), one does not have a rescaled random variable Y_n to compare with ϕ, but one can still look at the first line of Equation (1.3), and compare directly Z_n with the law of exponent $t_n\eta$. We followed this approach in [Chh+15]. In this setting, two other interesting quantities are the *local distance*

$$d_{\mathrm{loc}}(X_n, Z_n) = \sup_{k\in\mathbb{Z}} |\mathbb{P}[X_n = k] - \mathbb{P}[Z_n = k]|$$

and the *total variation distance*

$$d_{\mathrm{TV}}(X_n, Z_n) = \sup_{A\subset\mathbb{Z}} |\mathbb{P}[X_n \in A] - \mathbb{P}[Z_n \in A]| = \frac{1}{2} \sum_{k\in\mathbb{Z}} |\mathbb{P}[X_n = k] - \mathbb{P}[Z_n = k]|\,.$$

We shall see that all these distances are bounded by negative powers of t_n, and that in the discrete case, there exists a general scheme of approximation (Definition 1.35 and Remark 1.41) that allows one to improve these bounds, by replacing the infinitely divisible law of exponent $t_n\eta$ by certain signed measures.

▷ **Zone of control for mod-stable sequences.** We start with the continuous case and we consider a sequence of random variables $(X_n)_{n\in\mathbb{N}}$, and a reference law ϕ that is stable with parameters (c, α, β). We set as in Proposition 1.9

$$Y_n = \begin{cases} \frac{X_n}{(t_n)^{1/\alpha}} & \text{if } \alpha \neq 1, \\ \frac{X_n}{t_n} - \frac{2c\beta}{\pi}\log t_n & \text{if } \alpha = 1. \end{cases}$$

To control the speed of convergence, we shall work with the Fourier transform and with the residues

$$\theta_n(\xi) = \mathbb{E}[\mathrm{e}^{\mathrm{i}\xi X_n}]\,\mathrm{e}^{-t_n\eta_{c,\alpha,\beta}(\mathrm{i}\xi)}.$$

Under the hypothesis of mod-ϕ convergence, $\theta_n(\xi)$ converges locally uniformly towards a function θ which is continuous and satisfies $\theta(\xi) = 1$. To compute the Kolmogorov distance between Y_n and a random variable Y with law $\phi_{c,\alpha,\beta}$, it will be more convenient to have a control on $\theta_n(\xi) - 1$, with a zone of control that might grow with n. The right definition is the following [FMN17b, Definition 5]:

Definition 1.25 (Zone of control). *Fix $v, w > 0$ and $\gamma \in \mathbb{R}$. We say that $(X_n)_{n\in\mathbb{N}}$ has a zone of control $[-K(t_n)^\gamma, K(t_n)^\gamma]$ with exponents v and w and constants K_1 and K_2 if the two following conditions are satisfied:*

(Z1) For any ξ in the zone of control,

$$|\theta_n(\xi) - 1| \leq K_1|\xi|^v \exp(K_2|\xi|^w).$$

(Z2) The parameters of the zone of control satisfy:

$$\alpha \leq w \quad ; \quad -\frac{1}{\alpha} < \gamma \leq \frac{1}{w-\alpha} \quad ; \quad 0 < K \leq \left(\frac{c^\alpha}{2K_2}\right)^{\frac{1}{w-\alpha}}.$$

Example 1.26 (Sums of i.i.d. random variables). Consider a sum $S_n = \sum_{i=1}^n A_i$ of i.i.d. centered random variables with a moment of order 3. We set $\sigma^2 = \mathbb{E}[(A_1)^2]$ and $L = \mathbb{E}[(A_1)^3]$, and we consider the renormalised random variables

$$X_n = \frac{S_n}{\sigma\, n^{1/3}},$$

which converge mod-Gaussian with parameters $t_n = n^{1/3}$ and limit $\theta(\xi) = \exp(\frac{L(\mathrm{i}\xi)^3}{6\sigma^3})$. Here, there is also a zone of control with exponents $v = 3$ and $w = 3$. Indeed, integrating the inequality

$$\left|\mathrm{e}^{\mathrm{i}\xi} - 1 - \mathrm{i}\xi + \frac{\xi^2}{2}\right| \leq \frac{|\xi|^3}{6}$$

against the law of $\frac{A_1}{\sigma n^{1/3}}$ yields

$$\left|\mathbb{E}\left[\mathrm{e}^{\frac{\mathrm{i}\xi A_1}{\sigma n^{1/3}}}\right] - 1 + \frac{\xi^2}{2n^{2/3}}\right| \leq \frac{\rho|\xi|^3}{6n\sigma^3},$$

where $\rho = \mathbb{E}[|A_1|^3]$. Set $K = \frac{\rho}{\sigma^3}$. Combining the previous inequality with $K \leq 1$ and $|a - b|^n \leq n\,(\max(|a|,|b|))^{n-1}\,|a-b|$, one obtains on the zone $[-Kn^{1/3}, Kn^{1/3}]$ the inequality

$$|\theta_n(\xi) - 1| \leq \frac{7\mathrm{e}^{\frac{1}{2}}\rho}{24\sigma^3}\,|\xi|^3\,\mathrm{e}^{\frac{\rho}{6\sigma^3}\,|\xi^3|}.$$

Therefore, there is a zone of control with $v = w = 3$, $\gamma = 1$, $\alpha = 2$, $K = \frac{\rho}{\sigma^3}$, $K_1 = \frac{7\mathrm{e}^{\frac{1}{2}}K}{24}$ and $K_2 = \frac{K}{6}$. The inequalities of the second item of Definition 1.25 are all satisfied.

Example 1.27 (Winding numbers). Consider as in Example 1.7 the winding number φ_t of a complex Brownian motion Z_t starting from $Z_0 = 1$. Analyzing Spitzer's formula, we obtain for any time $t > 0$ and any $\xi \in \mathbb{R}$:

$$|\theta_t(\xi) - 1| = \left|\mathbb{E}[\mathrm{e}^{\mathrm{i}\xi\varphi_t}]\,\mathrm{e}^{\frac{\log 8t}{2}|\xi|} - 1\right| \leq |\xi|,$$

see [FMN17b, §3.2]. Therefore, one has a zone of control with $v = w = 1$, $\alpha = 1$, $\gamma = K = +\infty$, $K_1 = 1$ and $K_2 = 0$.

▷ **Berry–Esseen type estimates.** The notion of zone of control is the right tool in order to compute optimal bounds on the Kolmogorov distance between Y_n and $Y \sim \phi_{c,\alpha,\beta}$. Thus, the main result obtained in [FMN17b, Theorem 20] is the following:

Theorem 1.28 (Bounds on the Kolmogorov distance). *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of real-valued random variables, $(t_n)_{n\in\mathbb{N}}$ a sequence growing to infinity, and $\phi = \phi_{c,\alpha,\beta}$ a reference stable law. We denote Y a random variable with law ϕ, and we suppose that $(X_n)_{n\in\mathbb{N}}$ admits a zone of control $[-K(t_n)^\gamma, K(t_n)^\gamma]$ of exponents (v,w).*

(1) If $(Y_n)_{n\in\mathbb{N}}$ is defined as in the first part of Proposition 1.9, then one has convergence in law $Y_n \rightharpoonup_{n\to\infty} Y$.

(2) Suppose $\gamma \leq \frac{v-1}{\alpha}$. Then,

$$d_{\mathrm{Kol}}(Y_n, Y) \leq C(c,\alpha,v,K,K_1)\,\frac{1}{(t_n)^{\gamma+\frac{1}{\alpha}}}.$$

The constant $C(c,\alpha,v,K,K_1)$ can be taken equal to

$$\inf_{\lambda>0}\left(\frac{1+\lambda}{\alpha\pi c}\left(\frac{2^{\frac{v}{\alpha}}\,\Gamma\!\left(\frac{v}{\alpha}\right)K_1}{c^{v-1}} + \frac{\Gamma\!\left(\frac{1}{\alpha}\right)}{\sqrt[3]{\pi}\,K}\left(4\sqrt[3]{1+\frac{1}{\lambda}} + 3\sqrt[3]{3}\right)\right)\right).$$

Before giving applications of this result, let us detail a bit its proof. The fastest way to prove Theorem 1.28 is by means of an inequality already involved in the proof of the classical Berry–Esseen bounds, see [Fel71, Lemma XVI.3.2]: if X and Y are two random variables with the density of Y bounded by m, then for any $T > 0$,

$$d_{\mathrm{Kol}}(X,Y) \leq \frac{1}{\pi}\int_{-T}^{T}\left|\frac{\mathbb{E}[\mathrm{e}^{\mathrm{i}\xi X}] - \mathbb{E}[\mathrm{e}^{\mathrm{i}\xi Y}]}{\xi}\right|d\xi + \frac{24m}{\pi T}.$$

Using this inequality with $T = K(t_n)^\gamma$ yields Theorem 1.28, though not with the same constant. In [FMN17b, Proposition 16], we proved precise bounds on test functions:

$$\mathbb{E}[f_n(Y_n) - f_n(Y)] \leq C(c,\alpha,v,K_1,f_n)\,\frac{1}{(t_n)^{\frac{v}{\alpha}}}.$$

These bounds hold true for any smooth function f_n whose Fourier transform has its support included in $[-K(t_n)^{\gamma+\frac{1}{\alpha}}, K(t_n)^{\gamma+\frac{1}{\alpha}}]$. They are also true for certain tempered distributions, and this allows one to use smoothing techniques in order to replace $f_n(y)$ by an Heaviside function $1_{y\leq x}$. Thus, Theorem 1.28 can be seen as a particular case of a general bound on test functions.

Remark 1.29. Another extremely popular method to get optimal bounds on Kolmogorov distances is *Stein's method*, see the survey [Ros11] or the two monographs [BC05; CGS11]. When both methods can be applied to the same model, they usually give the same bound up to a constant. One of the advantage of our method is that it only relies on Fourier analysis. On the other hand, it yields without additional work moderate deviations and local limit theorems. Finally, there are many examples where one can apply Theorem 1.28 and where Stein's method is not known to yield an estimate of the speed of convergence. To the best of our knowledge, this is for instance the case for the fluctuations of the models from Section 2.3, and for the magnetisation of the Ising model (Section 2.4).

Remark 1.30. In the mod-Gaussian case, let us compare the general Berry–Esseen estimate of Theorem 1.14 with the bound from Theorem 1.28. Given a mod-Gaussian convergent sequence $(X_n)_{n\in\mathbb{N}}$, we saw in Remark 1.16 that $d_{\mathrm{Kol}}(Y_n, G)$ is:

- asymptotically equivalent to $\frac{|\theta'(0)|}{\sqrt{2\pi t_n}}$ if $\theta'(0) \neq 0$;
- a $o(\frac{1}{\sqrt{t_n}})$ if $\theta'(0) = 0$.

In the next chapter, we shall explain which kinds of models yield mod-Gaussian convergent sequences. Many of these models yield a residue $\theta(\xi) = \exp(L(\mathrm{i}\xi)^3)$, where L is a certain constant. Therefore, the second situation is more frequent, and Theorem 1.14 is not optimal in this case. On the other hand, the form of the limiting residue $\theta(\xi) = \exp(L(\mathrm{i}\xi)^3)$ implies most of the time that one has a zone of control of size $O(t_n)$, with $v = w = 3$. The Kolmogorov distance $d_{\mathrm{Kol}}(Y_n, G)$ is in this case a $O(\frac{1}{(t_n)^{3/2}})$, and Theorem 1.28 provides an optimal bound (up to a multiplicative constant). This comparison explains why one had to introduce the notion of zone of control.

Example 1.31 (Classical Berry–Esseen estimates). If one applies Theorem 1.28 to sums of i.i.d. centered random variables, one recovers the classical Berry–Esseen estimates from [Ber41; Ess45]:

$$d_{\mathrm{Kol}}\left(\frac{1}{\sigma\sqrt{n}}\sum_{i=1}^{n} A_i \,,\, \mathcal{N}_{\mathbb{R}}(0,1)\right) \leq \frac{C\rho}{\sigma^3\sqrt{n}},$$

with a constant $C = 4.815$. This is better than the original constant $C = 7.59$ of Esseen, but worse than the best known constant today, which is $C = 0.4748$. On the other hand, Theorem 1.28 applies to a much larger set of examples.

Example 1.32 (Winding numbers). If φ_t is the winding number of a complex Brownian motion, then

$$d_{\mathrm{Kol}}\left(\frac{2\varphi_t}{\log 8t}\,,\, \mathcal{C}\right) \leq \frac{4}{\log 8t},$$

where $\mathcal{C}$ is the standard Cauchy distribution.

Example 1.33 (Compound Poisson laws). Consider any stable law $\phi = \phi_{c,\alpha,\beta}$, and denote Y a random variable with law ϕ, and Y_n a compound Poisson process of intensity $n\,\mu_n$, where

μ_n is the unique stable law such that $(\mu_n)^{*n} = \phi$. The Fourier transform of Y_n is

$$\mathbb{E}[e^{i\xi Y_n}] = \exp\left(n\left(e^{\frac{\eta_{c,\alpha,\beta}(i\xi)}{n}} - 1\right)\right) \to_{n\to\infty} e^{\eta_{c,\alpha,\beta}(i\xi)} = \mathbb{E}[e^{i\xi Y}],$$

so one has convergence in law $Y_n \rightharpoonup Y$ (*cf.* [Sat99, Chapter 2, §8]). By using a zone of control, one can control the Kolmogorov distance:

$$d_{\mathrm{Kol}}(Y_n, Y) = \begin{cases} O\left(n^{-\frac{1}{\alpha}}\right) & \text{if } \alpha \in (1,2]; \\ O((\log n)^2\, n^{-1}) & \text{if } (\alpha = 1, \beta \neq 0); \\ O(n^{-1}) & \text{if } \alpha \in (0,1) \text{ or } (\alpha = 1, \beta = 0); \end{cases}$$

see [FMN17b, §3.3].

In the mod-Gaussian case, numerous other examples stemming from Markov chains, random graphs, models from statistical mechanics, *etc.* will be given in Chapter 2.

▷ **Approximation of discrete mod-ϕ convergent random variables.** Consider now a sequence of $\mathbb{Z}$-valued random variables $(X_n)_{n\in\mathbb{N}}$, and a reference infinitely divisible law ϕ supported on this lattice. We denote μ_n the law of X_n, and ν_n the infinitely divisible law with Lévy exponent $t_n\eta$, where η is the exponent of ϕ, and $(t_n)_{n\in\mathbb{N}}$ is a sequence growing to infinity. If $(X_n)_{n\in\mathbb{N}}$ converges mod-ϕ with parameters $(t_n)_{n\in\mathbb{N}}$, then we can try as before to bound

$$d_{\mathrm{Kol}}(\mu_n, \nu_n) = \sup_{l\in\mathbb{Z}} \left|\sum_{k=-\infty}^{l} (\mu_n(k) - \nu_n(k))\right|.$$

To measure how close the distribution of X_n is from ν_n, we can also look at the *local distance*

$$d_{\mathrm{loc}}(\mu_n, \nu_n) = \sup_{k\in\mathbb{Z}} |\mu_n(k) - \nu_n(k)|$$

and at the *total variation distance*

$$d_{\mathrm{TV}}(\mu_n, \nu_n) = \sup_{A\subset\mathbb{Z}} |\mu_n(A) - \nu_n(A)| = \frac{1}{2} \sum_{k\in\mathbb{Z}} |\mu_n(k) - \nu_n(k)|\,.$$

These distances are related by the inequalities

$$\frac{1}{2}\, d_{\mathrm{loc}}(\mu, \nu) \leq d_{\mathrm{Kol}}(\mu, \nu) \leq d_{\mathrm{TV}}(\mu, \nu).$$

Example 1.34 (Poisson approximation). In the aforementioned framework, the toy-model is the Poisson approximation of a sum $X_n = \sum_{i=1}^n B_i$ of independent Bernoulli random variables, with $\mathbb{P}[B_i = 1] = p_i$ and $\mathbb{P}[B_i = 0] = 1 - p_i$. Set $t_n = \sum_{i=1}^n p_i$, and suppose that $\lim_{n\to\infty} t_n = +\infty$. If $\sum_{i=1}^\infty (p_i)^2 < +\infty$, then one has mod-Poisson convergence:

$$\mathbb{E}[e^{i\xi X_n}]\, e^{-t_n(e^{i\xi}-1)} = \prod_{i=1}^n \left(1 + p_i(e^{i\xi} - 1)\right) e^{-p_i(e^{i\xi}-1)} \to_{n\to\infty} \prod_{i=1}^\infty \left(1 + p_i(e^{i\xi} - 1)\right) e^{-p_i(e^{i\xi}-1)}.$$

On the other hand, Le Cam's inequality allows one to bound the total variation distance between X_n and the Poisson law $\mathcal{P}(t_n)$:

$$d_{\mathrm{TV}}(X_n, \mathcal{P}(t_n)) \leq \sum_{i=1}^n (p_i)^2,$$

see [Cam60; Ker64]. Using Stein's method, Chen and Steele improved this bound:

$$d_{\mathrm{TV}}(X_n, \mathcal{P}(t_n)) \leq \left(1 - \mathrm{e}^{-\sum_{i=1}^n p_i}\right) \frac{\sum_{i=1}^n (p_i)^2}{\sum_{i=1}^n p_i},$$

see [Che74; Che75; Ste94], and [AGG89] for an extension to dependent random variables. We shall see in this paragraph that the theory of mod-ϕ convergence leads to a similar inequality; besides, the inequality can be improved by an adequate modification of the Poisson approximation.

The Poisson approximation of a sum of independent Bernoulli random variables yields a distribution which is close to X_n up to a $O((t_n)^{-1})$ for the total variation distance. If one wants a better approximation, say up to a $O((t_n)^{-2})$, it is convenient to allow *signed* measures. This is possible in the following setting, which we called general approximation scheme in [Chh+15, Definition 8]:

Definition 1.35 (Approximation scheme). *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of $\mathbb{Z}$-valued random variables that converges mod-ϕ with parameters $(t_n)_{n\in\mathbb{N}}$, ϕ being an infinitely divisible law with lattice $\mathbb{Z}$ and Lévy exponent η. An approximation scheme for the laws μ_n of the variables X_n is a sequence of signed measures $(\nu_n)_{n\in\mathbb{N}}$ on $\mathbb{Z}$, such that*

$$\widehat{\mu}_n(\xi) = \mathrm{e}^{t_n \eta(\mathrm{i}\xi)}\, \theta_n(\xi);$$
$$\widehat{\nu}_n(\xi) = \mathrm{e}^{t_n \eta(\mathrm{i}\xi)}\, \chi_n(\xi)$$

with $\lim_{n\to\infty} \theta_n = \theta$ and $\lim_{n\to\infty} \chi_n = \chi$. The residues θ_n, θ, χ_n and χ are functions on the circle $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ which are equal to 1 at $\xi = 0$ (hence, $\nu_n(\mathbb{Z}) = 1$).

In this definition, we did not precise the nature of the convergence of the residues ψ_n and χ_n; typically, we want it to happen in a space of functions $\mathscr{C}^r(\mathbb{T})$, with respect to the topology of uniform convergence of all the derivatives up to order r. Before stating the exact hypotheses of convergence that are required, let us present the *standard approximation schemes* for a mod-ϕ convergent sequence. Let $(X_n)_{n\in\mathbb{N}}$ be a $\mathbb{Z}$-valued mod-ϕ convergent sequence, and $(\theta_n)_{n\in\mathbb{N}}$ be the sequence of residues. We suppose that these residues θ_n can be represented as convergent power series on the circle:

$$\theta_n(\xi) = 1 + \sum_{k=1}^{\infty} b_{n,k}\,(\mathrm{e}^{\mathrm{i}\xi} - 1)^k + \sum_{k=1}^{\infty} c_{n,k}\,(\mathrm{e}^{-\mathrm{i}\xi} - 1)^k$$

The coefficients $b_{n,k}$ and $c_{n,k}$ can be expressed in terms of the coefficients of the Fourier series of $\theta_n(\xi) = \sum_{k\in\mathbb{Z}} a_{n,k}\,\mathrm{e}^{\mathrm{i}k\xi}$ if θ_n is sufficiently smooth. The standard approximation scheme of order $r \geq 0$ of the sequence $(X_n)_{n\in\mathbb{N}}$ is the sequence of signed measures $(\nu_n^{(r)})_{n\in\mathbb{N}}$ defined by

$$\widehat{\nu}_n^{(r)}(\xi) = \mathrm{e}^{t_n \eta(\mathrm{i}\xi)} \left(1 + \sum_{k=1}^{r} b_{n,k}\,(\mathrm{e}^{\mathrm{i}\xi} - 1)^k + \sum_{k=1}^{r} c_{n,k}\,(\mathrm{e}^{-\mathrm{i}\xi} - 1)^k\right).$$

In other words, one keeps the Taylor expansion up to order r of the residue θ_n to define $\chi_n^{(r)}$, so that $\widehat{\nu}_n^{(r)}(\xi) = \mathrm{e}^{t_n\eta(\mathrm{i}\xi)}\,\chi_n^{(r)}(\xi)$. The calculation of the residues $\chi_n^{(r)}$ will be performed in the next chapter, see Section 2.1.

Let us now detail the hypotheses required in order to have bounds on the distances between the distributions of the variables X_n and their approximation schemes. Until the end of this section, we assume that the reference infinitely divisible law ϕ with lattice $\mathbb{Z}$ has a moment of order 2. This amounts to the following:

- the Lévy measure π of ϕ is supported on $\mathbb{Z}$ and satisfies $\sum_{k\neq 0} \pi(k)\, k^2 < +\infty$.
- if (a, σ^2, π) is the Lévy–Khintchine triplet of ϕ, then $\sigma^2 = 0$ and $a \in \mathbb{Z}$.

We fix an approximation scheme $(\nu_n)_{n\in\mathbb{N}}$ of $(\mu_n)_{n\in\mathbb{N}}$, and we consider the following hypotheses:

(H1) We have

$$\forall n \in \mathbb{N},\ \theta_n(\xi) - \chi_n(\xi) = \beta_n\,(\mathrm{i}\xi)^{r+1}\,(1 + o_\xi(1))$$
$$\text{and }\ \theta(\xi) - \chi(\xi) = \beta\,(\mathrm{i}\xi)^{r+1}\,(1 + o_\xi(1))$$

with $\lim_{n\to\infty} \beta_n = \beta$.

(H2) The residues θ_n, θ, χ_n and χ are in $\mathscr{C}^1(\mathbb{T})$, and we have

$$\forall n \in \mathbb{N},\ \theta_n'(\xi) - \chi_n'(\xi) = \mathrm{i}(r+1)\,\beta_n\,(\mathrm{i}\xi)^r + \mathrm{i}(r+2)\,\gamma_n\,(\mathrm{i}\xi)^{r+1}\,(1 + o_\xi(1))$$
$$\text{and }\ \theta'(\xi) - \chi'(\xi) = \mathrm{i}(r+1)\,\beta\,(\mathrm{i}\xi)^r + \mathrm{i}(r+2)\,\gamma\,(\mathrm{i}\xi)^{r+1}\,(1 + o_\xi(1))$$

with $\lim_{n\to\infty} \beta_n = \beta$ and $\lim_{n\to\infty} \gamma_n = \gamma$.

Since $\theta_n(0) = \chi_n(0) = \theta(0) = \chi(0) = 1$, integrating Condition (H2) gives estimates on $\theta_n - \chi_n$ and $\theta - \chi$, so (H2) is stronger than (H1). On the other hand, if θ_n converges in $\mathscr{C}^{r+1}(\mathbb{T})$ (respectively, in $\mathscr{C}^{r+2}(\mathbb{T})$) to θ and if $\chi_n = \chi_n^{(r)}$ is the residue of the standard approximation scheme of order r, then (H1) (respectively, (H2)) is satisfied. Thus, one easily produces approximation schemes that satisfy the hypotheses of the following result, up to an arbitrary order r (Theorems 3.4, 3.7 and 3.12 in [Chh+15]).

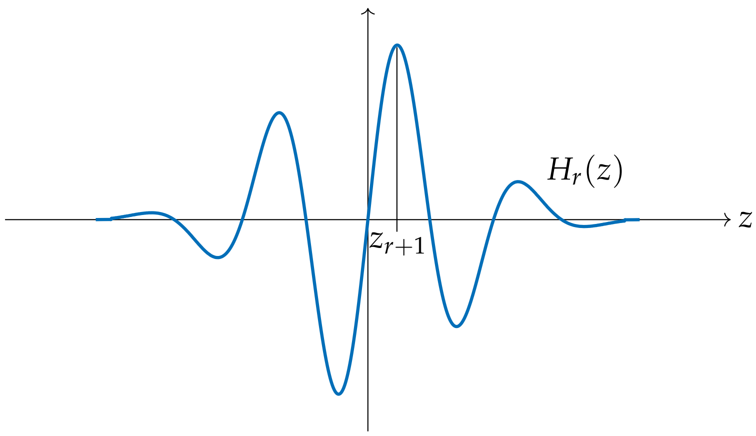


Figure 1.7. The Hermite function H_9, which attains its global extremas at the smallest zeroes of H_{10}.

Theorem 1.36 (Bounds on the local, Kolmogorov and total variation distances). *In the previous framework, consider an approximation scheme $(\nu_n)_{n\in\mathbb{N}}$ of $(X_n)_{n\in\mathbb{N}}$, with Fourier transforms $\widehat{\nu}_n(\xi) = \mathrm{e}^{t_n \eta(\mathrm{i}\xi)}\,\chi_n(\xi)$. We denote $\eta''(0)$ the variance of ϕ.*

(1) Under the hypothesis (H1),

$$d_{\mathrm{loc}}(\mu_n, \nu_n) = \frac{|\beta|\,|H_{r+1}(z_{r+2})|}{\sqrt{2\pi}\,(\eta''(0)\,t_n)^{\frac{r}{2}+1}} + o\left(\frac{1}{(t_n)^{\frac{r}{2}+1}}\right),$$

where $H_r(z) = \frac{\partial^r}{\partial z^r}\left(\mathrm{e}^{-\frac{z^2}{2}}\right)$ *is the r-th Hermite function , and* z_{r+1} *is the smallest zero in absolute value of* H_{r+1}*, and a global extrema of* $|H_r|$ *(see Figure 1.7).*

(2) Under the same hypothesis (H1),

$$d_{\mathrm{Kol}}(\mu_n,\nu_n) = \frac{|\beta|\,|H_r(z_{r+1})|}{\sqrt{2\pi}\,(\eta''(0)\,t_n)^{\frac{r+1}{2}}} + o\left(\frac{1}{(t_n)^{\frac{r+1}{2}}}\right).$$

(3) Under the stronger hypothesis (H2), if ϕ *has a third moment, then*

$$d_{\mathrm{TV}}(\mu_n,\nu_n) = \frac{|\beta|}{2\sqrt{2\pi}\,(\eta''(0)\,t_n)^{\frac{r+1}{2}}}\left(\int_{\mathbb{R}} |H_{r+1}(z)|\,dz\right) + o\left(\frac{1}{(t_n)^{\frac{r+1}{2}}}\right).$$

Remark 1.37. Our results are exact asymptotics of the distances, but they are not unconditional bounds. Still, one can easily adapt the proofs to get such unconditional bounds; see in particular [BKN09, Propositions 2.1, 2.2 and 2.4]. The two first inequalities of Theorem 1.36 rely mainly on the Laplace method and on a careful use of the hypothesis (H1). On the other hand, to obtain the inequality on the total variation distance, we used various formulas and approximation techniques in the *Wiener algebra* $\mathscr{A}(\mathbb{T})$, which is the convolution algebra of absolutely convergent Fourier series $f(\xi) = \sum_{k\in\mathbb{Z}} a_k\,\mathrm{e}^{\mathrm{i}k\xi}$ endowed with the norm $\|f\|_{\mathscr{A}(\mathbb{T})} = \sum_{k\in\mathbb{Z}} |a_k|$. For instance, an important tool in the proof of the bounds on distances is the inequality

$$\|f\|_{\mathscr{A}(\mathbb{T})} \leq |a_0(f)| + \frac{\pi}{\sqrt{3}}\,\|f'\|_{\mathscr{L}^2(\mathbb{T})},$$

see [Kat04, §6.2]. If μ and ν are two (signed) measures, then their total variation distance is nothing else than the norm in $\mathscr{A}(\mathbb{T})$ of $\frac{1}{2}(\widehat{\mu}-\widehat{\nu})$; this explains why the Wiener algebra is the adequate setting for these computations. Thus, we were able to reduce the problem of approximation of discrete laws to harmonic analysis on the circle.

Remark 1.38. The first values of $M_r = |H_r(z_{r+1})|$ are

$$M_0 = 1,\;\; M_1 = \mathrm{e}^{-\frac{1}{2}},\;\; M_2 = 1,\;\; M_3 = \mathrm{e}^{-\frac{3-\sqrt{6}}{2}}(3\sqrt{6}-6),\;\; M_4 = 3,$$

and we have in fact $M_{2r} = (2r-1)!! = (2r-1)(2r-3)\cdots 3\cdot 1$ for any $r \geq 0$. On the other hand, the first values of $V_r = \int_{\mathbb{R}} |H_r(z)|\,dz$ are

$$V_0 = 2,\;\; V_1 = 4\mathrm{e}^{-\frac{1}{2}},\;\; V_2 = 2\left(1+4\mathrm{e}^{-\frac{3}{2}}\right).$$

▷ **Basic scheme of approximation and derived scheme of approximation.** A particular case is when the scheme of approximation $(\nu_n)_{n\in\mathbb{N}}$ is given by the infinitely divisible laws, that is with residues $\chi_n(\xi) = \chi(\xi) = 1$. We then speak of the *basic* scheme of approximation. Let us analyze it for the classical Poisson approximation of a sum $X_n = \sum_{i=1}^n \mathcal{B}(p_i)$ of independent Bernoulli random variables. We then have $\eta(\mathrm{i}\xi) = \mathrm{e}^{\mathrm{i}\xi}-1$, and

$$\begin{aligned}\theta_n(\xi) &= \prod_{i=1}^n \left(1+p_i(\mathrm{e}^{\mathrm{i}\xi}-1)\right)\mathrm{e}^{-p_i(\mathrm{e}^{\mathrm{i}\xi}-1)} = \exp\left(\sum_{i=1}^n \log(1+p_i(\mathrm{e}^{\mathrm{i}\xi}-1)) - p_i(\mathrm{e}^{\mathrm{i}\xi}-1)\right)\\ &= \exp\left(\sum_{i=1}^n\sum_{k=2}^\infty \frac{(-1)^{k-1}}{k}\,(p_i(\mathrm{e}^{\mathrm{i}\xi}-1))^k\right) = \exp\left(\sum_{k=1}^\infty \frac{(-1)^{k-1}\,\mathfrak{p}_{k,n}}{k}\,(\mathrm{e}^{\mathrm{i}\xi}-1)^k\right)\end{aligned}$$

where the parameters $\mathfrak{p}_{k,n}$ are defined by

$$\mathfrak{p}_{1,n} = 0 \qquad ; \qquad \mathfrak{p}_{k\geq 2,n} = \sum_{i=1}^{n} (p_i)^k.$$

The expansion of the exponential can be performed by using the algebra of symmetric functions Sym, see [Mac95, Chapter 1] or [Mél17, Chapter 2]. We shall explain this expansion in a general setting in Section 2.1; for the moment we only detail what is needed for the classical Poisson approximation. Let $\mathfrak{Y}$ be the set of integer partitions, that are non-increasing sequences $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{\ell(\lambda)})$ of positive integers. The size of an integer partition λ is the sum $|\lambda|$ of its parts, and we denote $\mathfrak{Y}(k) \subset \mathfrak{Y}$ the set of integer partitions of size k. On the other hand, if λ is an integer partition, we can write it multiplicatively $\lambda = 1^{m_1(\lambda)}2^{m_2(\lambda)}\ldots s^{m_s(\lambda)}$; for instance $\lambda = (4,4,2,2,2,1)$, which is of size 15, writes also as $\lambda = 1\,2^3\,4^2$. Set $z_\lambda = \prod_{s\geq 1} s^{m_s(\lambda)}\,(m_s(\lambda))!$; hence, $\lambda = (4,4,2,2,2,1)$ gives $z_\lambda = 1536$. The combinatorial coefficient z_λ is the size of the centralizer of a permutation with cycle type λ in $\mathfrak{S}(|\lambda|)$. We now define for any $k \geq 1$:

$$\mathfrak{e}_{k,n} = \sum_{\lambda \in \mathfrak{Y}(k)} (-1)^{|\lambda|-\ell(\lambda)}\, \frac{\mathfrak{p}_{\lambda,n}}{z_\lambda},$$

where $\mathfrak{p}_{\lambda,n} = \prod_{i=1}^{\ell(\lambda)} \mathfrak{p}_{\lambda_i,n}$. These coefficients are those of the expansion in power series of θ_n:

$$\theta_n(\xi) = \sum_{k=0}^{\infty} \mathfrak{e}_{k,n}\,(\mathrm{e}^{\mathrm{i}\xi}-1)^k$$

with by convention $\mathfrak{e}_{0,n} = 1$. We have:

$$\mathfrak{e}_{1,n} = 0 \quad ; \quad \mathfrak{e}_{2,n} = -\frac{\mathfrak{p}_{2,n}}{2} \quad ; \quad \mathfrak{e}_{2,n} = -\frac{\mathfrak{p}_{3,n}}{3}.$$

Since $\mathfrak{e}_{1,n} = 0$, the basic scheme of approximation for a sum of independent Bernoulli random variables is the standard scheme of order $r = 1$. It follows from Theorem 1.36 that, if $\mathfrak{p}_2 = \sum_{i=1}^{\infty}(p_i)^2 < \infty$ and if ν_n is the Poisson law of parameter $t_n = \sum_{i=1}^{n} p_i$, then

$$d_{\mathrm{loc}}(\mu_n,\nu_n) \simeq \frac{\mathfrak{p}_2}{2\sqrt{2\pi}\,(t_n)^{\frac{3}{2}}};$$
$$d_{\mathrm{Kol}}(\mu_n,\nu_n) \simeq \frac{\mathfrak{p}_2}{2\sqrt{2\pi\mathrm{e}}\,t_n};$$
$$d_{\mathrm{TV}}(\mu_n,\nu_n) \simeq \frac{\mathfrak{p}_2}{\sqrt{2\pi\mathrm{e}}\,t_n}.$$

This agrees with the classical results coming from the Chen–Stein method.

When dealing with the standard schemes of higher order, it is convenient to introduce the notion of *derived scheme* of an approximation scheme. Suppose given an approximation scheme $(\nu_n)_{n\in\mathbb{N}}$ of the sequence of discrete laws $(\mu_n)_{n\in\mathbb{N}}$; their Fourier transforms are

$$\widehat{\nu}_n(\xi) = \mathrm{e}^{t_n\eta(\mathrm{i}\xi)}\,\chi_n(\xi).$$

Instead of working with the approximating residues $\chi_n(\xi)$ which vary with n, one can look at the signed measures σ_n defined by

$$\widehat{\sigma}_n(\xi) = \mathrm{e}^{t_n\eta(\mathrm{i}\xi)}\,\chi(\xi), \quad \text{with } \chi = \lim_{n\to\infty}\chi_n.$$

One then says that $(\sigma_n)_{n\in\mathbb{N}}$ is the derived scheme of approximation of $(\nu_n)_{n\in\mathbb{N}}$. Dealing with a fixed approximating residue is more practical for applications, and on the other hand, in many cases, the convergences $\theta_n \to \theta$ and $\chi_n \to \chi$ happen at a speed faster than any power of t_n. Therefore, one does not lose much by working with derived schemes. A precise statement is the following (*cf.* [Chh+15, Theorem 3.17]):

Theorem 1.39 (Derived scheme of approximation). *Let $(\nu_n)_{n\in\mathbb{N}}$ be a general approximation scheme of $(\mu_n)_{n\in\mathbb{N}}$, and $(\sigma_n)_{n\in\mathbb{N}}$ be the derived scheme of $(\nu_n)_{n\in\mathbb{N}}$. Suppose that $\|\chi_n - \chi\|_\infty = o((t_n)^{-\frac{r}{2}-1})$ and that $\|\chi_n' - \chi'\|_\infty = o((t_n)^{-\frac{r}{2}-\frac{5}{4}})$. Then, under the hypotheses of Theorem 1.36 (Conditions (H1) and (H2)), one has the same estimates as before with σ_n instead of ν_n, so for instance*

$$d_{\mathrm{TV}}(\mu_n, \sigma_n) = \frac{|\beta|}{2\sqrt{2\pi}\,(\eta''(0)\,t_n)^{\frac{r+1}{2}}} \left(\int_{\mathbb{R}} |H_{r+1}(z)|\,dz\right) + o\left(\frac{1}{(t_n)^{\frac{r+1}{2}}}\right)$$

if ϕ has a third moment and if Condition (H2) is satisfied.

Example 1.40 (Derived schemes for the classical Poisson approximation). In the classical Poisson approximation with $\sum_{i=1}^\infty p_i = +\infty$ and $\sum_{i=1}^\infty (p_i)^2 < +\infty$, the limiting residue $\theta(\xi) = \prod_{i=1}^\infty (1 + p_i(\mathrm{e}^{\mathrm{i}\xi} - 1))\,\mathrm{e}^{-p_i(\mathrm{e}^{\mathrm{i}\xi}-1)}$ admits for expansion $\theta(\xi) = \sum_{k=0}^\infty \mathfrak{e}_k\,(\mathrm{e}^{\mathrm{i}\xi} - 1)^k$, where $\mathfrak{e}_k = \sum_{\lambda\in\mathfrak{Y}(k)} (-1)^{|\lambda|-\ell(\lambda)}\,\frac{\mathfrak{p}_\lambda}{z_\lambda}$, $\mathfrak{p}_1 = 0$ and $\mathfrak{p}_{k\geq 2} = \sum_{i=1}^\infty (p_i)^k$.

The derived scheme of the standard scheme of approximation of order $r \geq 1$ is the sequence of signed measures $(\sigma_n^{(r)})_{n\in\mathbb{N}}$ defined by:

$$\widehat{\sigma}_n^{(r)}(\xi) = \mathrm{e}^{t_n(\mathrm{e}^{\mathrm{i}\xi}-1)} \left(\sum_{k=0}^r \mathfrak{e}_k\,(\mathrm{e}^{\mathrm{i}\xi} - 1)^k\right)$$

with $t_n = \sum_{i=1}^n p_i$. When $r = 1$, we recover the basic scheme of approximation. When $r = 2$, we obtain

$$\widehat{\sigma}_n^{(2)}(\xi) = \mathrm{e}^{t_n(\mathrm{e}^{\mathrm{i}\xi}-1)} \left(1 - \frac{\mathfrak{p}_2}{2}\,(\mathrm{e}^{\mathrm{i}\xi} - 1)^2\right);$$
$$\sigma_n^{(2)}(k) = \frac{\mathrm{e}^{-t_n}\,(t_n)^k}{k!} \left(1 - \frac{\mathfrak{p}_2}{2}\left(1 - \frac{2k}{t_n} + \frac{k(k-1)}{(t_n)^2}\right)\right).$$

It is a much better approximation of X_n than the basic scheme, since

$$d_{\mathrm{loc}}(X_n, \sigma_n^{(2)}) = \frac{(\sqrt{6}-2)\,\mathfrak{p}_3}{\sqrt{2\pi\,\mathrm{e}^{3-\sqrt{6}}}\,(t_n)^2} + o\left(\frac{1}{(t_n)^2}\right);$$
$$d_{\mathrm{Kol}}(X_n, \sigma_n^{(2)}) = \frac{\mathfrak{p}_3}{3\sqrt{2\pi}\,(t_n)^{\frac{3}{2}}} + o\left(\frac{1}{(t_n)^{\frac{3}{2}}}\right);$$
$$d_{\mathrm{TV}}(X_n, \sigma_n^{(2)}) = \frac{(1 + 4\mathrm{e}^{-\frac{3}{2}})\,\mathfrak{p}_3}{3\sqrt{2\pi}\,(t_n)^{\frac{3}{2}}} + o\left(\frac{1}{(t_n)^{\frac{3}{2}}}\right)$$

as soon as the convergent series $\sum_{i=1}^\infty (p_i)^2$ has a sufficiently small tail (in order to apply Theorem 1.39, we need $\sum_{i=n+1}^\infty (p_i)^2 = o((t_n)^{-\frac{9}{4}})$).

Remark 1.41. Let us summarize the general technique of approximation of the laws of a sequence $(X_n)_{n\in\mathbb{N}}$ of discrete random variables:

(1) We find an infinitely divisible reference law ϕ with exponent η, such that X_n looks like a large random variable of exponent $t_n\eta$, in the sense of mod-ϕ convergence.

(2) We fix an order $r \geq 0$ sufficiently large, depending on the order of approximation $O((t_n)^{-\alpha})$ that is needed for the distances. For instance, if we want a total variation distance of order $O((t_n)^{-2})$, then we take $r = 3$.

(3) We consider the derived scheme $(\sigma_n^{(r)})_{n\in\mathbb{N}}$ of the standard approximation scheme $(\nu_n^{(r)})_{n\in\mathbb{N}}$ of order r. It is defined by

$$\widehat{\sigma}_n^{(r)}(\xi) = \mathrm{e}^{t_n\eta(\mathrm{i}\xi)}\, P^{(r)}(\xi),$$

where $P^{(r)}$ is a Laurent polynomial of degree r in $\mathrm{e}^{\mathrm{i}\xi}$, with the same Taylor expansion at $\xi = 0$ up to order r than the residue $\theta(\xi) = \lim_{n\to\infty} \mathbb{E}[\mathrm{e}^{\mathrm{i}\xi X_n}]\, \mathrm{e}^{-t_n\eta(\mathrm{i}\xi)}$.

Then, assuming that certain technical conditions are satisfied, $\sigma_n^{(r)}$ is the required approximation of the law μ_n of X_n. In Section 2.1, we shall apply this program to a large set of examples: number of cycles in a random permutation, number of prime divisors of a random integer, number of prime factors of a random polynomial over $\mathbb{F}_q$, *etc.* The use of symmetric functions and formal alphabets will shed light on the structure of discrete mod-ϕ convergent random variables, and it will enable the explicit approximation of their laws up to any order $r \geq 0$.

1.4 Local limit theorems and multi-dimensional extensions

The two previous Sections 1.2 and 1.3 have shown that the theory of mod-ϕ convergence allows an excellent understanding of the speed of convergence in a central limit theorem, and of the behavior of the random variables at the edge of their limiting distribution. In this last section of the chapter, we shall discuss some asymptotic results that we have not yet explored as thoroughly as the previous ones but that fall in the same framework of mod-ϕ convergence. They correspond to the preprint [FMN17c], and to an ongoing project with Ashkan Nikeghbali and his student Martina Dal Borgo on local limit theorems [BMN17].

▷ **Multi-dimensional mod-Gaussian convergence.** We start with the multi-dimensional generalisation of the previous results. Let $d \geq 2$ be a fixed positive integer, and $(\mathbf{X}_n)_{n\in\mathbb{N}}$ be a sequence of random vectors in $\mathbb{R}^d$. Though the theory of infinitely divisible distributions can be extended without difficulty to finite-dimensional vector spaces, here we shall only consider *mod-Gaussian* convergent sequences when $d \geq 2$. For non-lattice distributed reference laws, this is the only case where we were able to prove significant results.

Definition 1.42 (Mod-Gaussian convergence in dimension $d \geq 2$). *Let K be a positive-definite symmetric matrix of size $d \times d$, and $(t_n)_{n\in\mathbb{N}}$ a sequence growing to infinity. We say that $(\mathbf{X}_n)_{n\in\mathbb{N}}$ is mod-Gaussian convergent with parameters t_nK and limit $\psi(\mathbf{z})$ on a domain $D \subset \mathbb{C}^d$ if, locally uniformly on this domain,*

$$\lim_{n\to\infty} \mathbb{E}[\mathrm{e}^{\langle \mathbf{z}\,|\,\mathbf{X}_n\rangle}]\, \mathrm{e}^{-t_n\frac{\mathbf{z}^tK\mathbf{z}}{2}} = \lim_{n\to\infty} \psi_n(\mathbf{z}) = \psi(\mathbf{z}),$$

where $\langle \mathbf{z} \mid \mathbf{X}_n\rangle = \mathbf{z}^t\mathbf{X}_n$, and ψ is a continuous function on D.

As before, the domain D is always assumed to contain $\mathbf{0}$, and it is usually taken equal to $(\mathrm{i}\mathbb{R})^d$, or $\mathbb{C}^d$, or possibly a multi-band $\prod_{i=1}^d \mathcal{S}_{(c_i,d_i)}$. Assuming that D contains $(\mathrm{i}\mathbb{R})^d$, the mod-Gaussian convergence of $(\mathbf{X}_n)_{n\in\mathbb{N}}$ implies as in Proposition 1.9 a central limit theorem:

$$\mathbf{Y}_n = \frac{\mathbf{X}_n}{\sqrt{t_n}} \rightharpoonup \mathcal{N}_{\mathbb{R}^d}(\mathbf{0}, K).$$

Here, $\mathcal{N}_{\mathbb{R}^d}(\mathbf{0},K)$ denotes the multi-dimensional Gaussian distribution with covariance matrix K; its Fourier transform is $\mathbb{E}[\mathrm{e}^{\mathrm{i}\langle \xi \,|\, \mathbf{X}\rangle}] = \exp(-\frac{\xi^t K \xi}{2})$.

Suppose that $(\mathbf{X}_n)_{n\in\mathbb{N}}$ is a mod-Gaussian convergent sequence of random vectors, with parameters $t_n K$ and limit ψ on $\mathbb{C}^d$. To state the multi-dimensional analogue of Theorem 1.13, one can assume without loss of generality that $K = I_d$: indeed, if $(\mathbf{X}_n)_{n\in\mathbb{N}}$ is mod-Gaussian convergent with parameters $t_n K$ and limit $\psi(\mathbf{z})$, then $\widetilde{\mathbf{X}}_n = K^{-1/2}\mathbf{X}_n$ is mod-Gaussian convergent with parameters $t_n I_d$ and limit $\psi(K^{-1/2}\mathbf{z})$. Now, we proved in [FMN17c, Proposition 27 and Theorem 28]:

Theorem 1.43 (Large deviations in dimension $d \geq 2$). *Let $(\mathbf{X}_n)_{n\in\mathbb{N}}$ be a mod-Gaussian convergent sequence of random vectors in $\mathbb{R}^d$, with parameters $t_n I_d$ and limit ψ on $\mathbb{C}^d$. We suppose that ψ does not vanish on $\mathbb{R}^d$. We fix a subset $S \subset \mathbb{S}^{d-1}$ of the unit sphere that is Jordan measurable, and with non-zero surface measure. If $B = S \times [b,+\infty)$ is the spherical sector with basis bS ($b>0$), then*

$$\mathbb{P}[\mathbf{X}_n \in t_n B] = \left(\frac{t_n}{2\pi}\right)^{\frac{d}{2}} \frac{\mathrm{e}^{-\frac{t_n b^2}{2}}}{t_n b} \left(\int_{bS} \psi(\mathbf{s})\, \mu_{surface}(d\mathbf{s})\right)(1+o(1)).$$

In dimension $d=1$, one recovers the results of Theorem 1.13, since the surface measure on the 0-dimensional sphere $\mathbb{S}^0$ is the counting measure. The proof of Theorem 1.43 relies on a general study of the difference on convex Borel subsets $C \subset \mathbb{R}^d$ between the distribution of $\mathbf{Y}_n = \mathbf{X}_n/\sqrt{t_n}$ and the Gaussian distribution $\mathcal{N}_{\mathbb{R}^d}(\mathbf{0},K)$. Indeed, one can prove an analogue of Theorem 1.14 in dimension $d \geq 2$ if one replaces the Kolmogorov distance by the *convex distance*

$$d_{\mathrm{convex}}(\mu,\nu) = \sup_{\substack{C \text{ convex Borel}\\ \text{subset of } \mathbb{R}^d}} |\mu(C) - \nu(C)|.$$

We refer to [BR10] for a study of the relation between convergence in convex distance, and weak convergence of probability measures. The two notions are not always equivalent, but they become equivalent if the limiting distribution is regular with respect to convex sets [BR10, Theorem 2.11].

Theorem 1.44 (Berry–Esseen estimates in dimension $d \geq 2$). *Let $(\mathbf{X}_n)_{n\in\mathbb{N}}$ be a sequence of random vectors in $\mathbb{R}^d$ that is mod-Gaussian convergent on $D = \mathbb{C}^d$, with parameters $t_n K$ and limit ψ. We set $\mathbf{Y}_n = \frac{\mathbf{X}_n}{\sqrt{t_n}}$, and we denote μ_n the law of $\mathbf{Y}_n$, and $\nu = \mathcal{N}_{\mathbb{R}^d}(0,K)$. Then,*

$$d_{\mathrm{convex}}(\mu_n,\nu) = O\left(\frac{1}{\sqrt{t_n}}\right),$$

see [FMN17c, Theorem 18]. More precisely, if ν_n is the deformation of ν defined as the signed measure with density

$$\nu_n(d\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^d \det K}}\, \mathrm{e}^{-\frac{\mathbf{x}^t K^{-1}\mathbf{x}}{2}} \left(1 + \frac{\mathbf{x}^t K^{-1}(\nabla\psi)(\mathbf{0})}{\sqrt{t_n}}\right) d\mathbf{x}$$

and with Fourier transform

$$\widehat{\nu}_n(\boldsymbol{\xi}) = \mathrm{e}^{-\frac{\xi^t K \xi}{2}} \left(1 + \mathrm{i}\, \frac{\boldsymbol{\xi}^t(\nabla\psi)(\mathbf{0})}{\sqrt{t_n}}\right),$$

then $d_{\mathrm{convex}}(\mu_n,\nu_n) = o\left(\frac{1}{\sqrt{t_n}}\right)$ [FMN17c, Theorem 22].

Using the second part of Theorem 1.44 and an argument of tilting (exponential change of measure), one can compute estimates of the probability of $\mathbf{X}_n$ being in a cone $t_n C$, and then by approximation the probability of $\mathbf{X}_n$ being in any spherical sector $t_n B$ with Jordan measurable basis; this leads to Theorem 1.43.

Remark 1.45. In dimension $d \geq 2$, one can define zones of control similarly to Definition 1.25, and they yield much better estimates of the convex distance. However, the only examples that we have in mind rely on the method of cumulants, so we postpone the presentation of these results to Section 2.2.

Example 1.46 (Characteristic polynomials of random unitary matrices). We consider as in Example 1.8 a unitary matrix M_n taken according to the Haar measure of the unitary group, and we set $\mathbf{X}_n = \log\det(I_n - M_n)$, Thus, we consider the complex logarithm of the characteristic polynomial of M_n, instead of its real part. The random variable $\mathbf{X}_n = X_n^{(1)} + \mathrm{i}X_n^{(2)}$ converges in the mod-Gaussian sense on $\mathbb{C} = \mathbb{R}^2$, with parameters $\frac{\log n}{2}\, I_2$ and limiting residue

$$\psi(\mathbf{z}) = \psi(z^{(1)}, z^{(2)}) = \frac{G\left(1 + \frac{z^{(1)}+\mathrm{i}z^{(2)}}{2}\right) G\left(1 + \frac{z^{(1)}-\mathrm{i}z^{(2)}}{2}\right)}{G(1+z^{(1)})};$$

cf. [KN12, Equation (24)]. The domain of convergence is $D = \mathcal{S}_{(-1,+\infty)} \times \mathbb{C}$. Since $(\nabla\psi)(\mathbf{0}) = \mathbf{0}$, the second part of Theorem 1.44 guarantees that

$$d_{\text{convex}}\left(\sqrt{\frac{2}{\log n}}\,\mathbf{X}_n\,,\,\mathcal{N}_{\mathbb{R}^2}(\mathbf{0}, I_2)\right) = o\left(\frac{1}{\sqrt{\log n}}\right).$$

On the other hand, by using Theorem 1.43 and considering the spherical sector $C(r, \theta_1, \theta_2) = \{z = R\mathrm{e}^{\mathrm{i}\theta} \,|\, R \geq r,\ \theta \in (\theta_1, \theta_2)\}$, one sees that for any $r < 1$,

$$\mathbb{P}\left[\mathbf{X}_n \in \frac{\log n}{2}\, C(r, \theta_1, \theta_2)\right] = n^{-\frac{r^2}{4}} \left(\int_{\theta_1}^{\theta_2} \frac{G(1+\frac{r\mathrm{e}^{\mathrm{i}\theta}}{2})\, G(1-\frac{r\mathrm{e}^{\mathrm{i}\theta}}{2})}{G(1+r\cos\theta)}\, \frac{d\theta}{2\pi}\right)(1+o(1)).$$

The restriction on r is due to the fact that the domain of convergence is smaller than $\mathbb{C}^2$. With r fixed, the function

$$G_r(\theta) = \frac{G(1+\frac{r\mathrm{e}^{\mathrm{i}\theta}}{2})\, G(1-\frac{r\mathrm{e}^{\mathrm{i}\theta}}{2})}{G(1+r\cos\theta)}$$

takes higher values for θ close to π, and smaller values for θ close to 0 (Figure 1.8). So, conditioned to be of size $O(\log n)$, $\mathbf{X}_n$ has a higher probability to have its argument close to π.

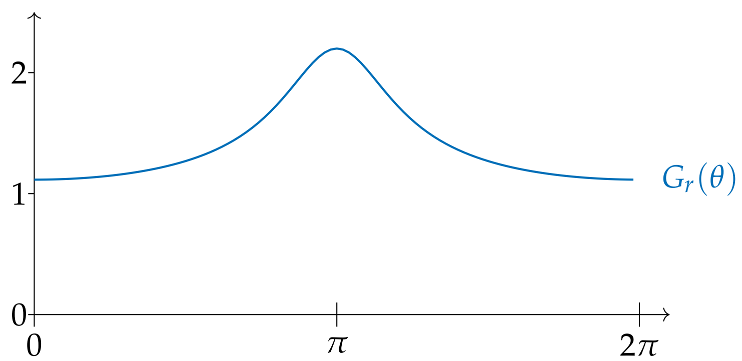


Figure 1.8. Breaking of symmetry of the logarithm of the characteristic polynomial $\mathbf{X}_n$, when conditioned to be of size $O(\log n)$ (here, $r = 0.7$).

Example 1.47 (Random walk on the lattice $\mathbb{Z}^2$). Consider a random walk $\mathbf{S}_n = \sum_{i=1}^n \mathbf{A}_i$, where the steps $\mathbf{A}_i$ are independent and identically distributed on $\mathbb{Z}^2$, with

$$\mathbb{P}[\mathbf{A}_i = (1,0)] = \mathbb{P}[\mathbf{A}_i = (-1,0)] = \mathbb{P}[\mathbf{A}_i = (0,1)] = \mathbb{P}[\mathbf{A}_i = (0,-1)] = \frac{1}{4}.$$

If $\mathbf{X}_n = \frac{\mathbf{S}_n}{n^{1/4}}$, then $(\mathbf{X}_n)_{n\in\mathbb{N}}$ converges in the mod-Gaussian sense with parameters $\frac{n^{1/2}}{2} I_2$ and limit

$$\psi(\mathbf{z}) = \exp\left(-\frac{(z^{(1)})^4 + (z^{(2)})^4 + 6(z^{(1)}z^{(2)})^2}{96}\right).$$

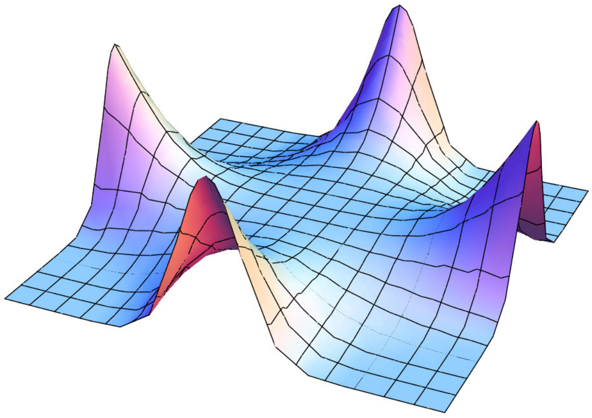


Figure 1.9. The function $F(r,\theta)$ measuring the breaking of symmetry of a two-dimensional random walk $\mathbf{S}_n$ conditioned to be of size $O(n^{3/4})$.

Our general Berry–Esseen estimate 1.44 is not very good in this case, but we shall describe in the next chapter a technique to get an optimal bound on the convex distance. On the other hand, the large deviation result 1.43 leads in this setting to the following estimate:

$$\mathbb{P}\left[\mathbf{S}_n \in n^{3/4}\, C(r,\theta_1,\theta_2) \mid \|\mathbf{S}_n\| \geq n^{3/4} r\right] = \left(\int_{\theta_1}^{\theta_2} F(r,\theta)\, d\theta\right)\,(1+o(1))$$

where $C(r,\theta_1,\theta_2)$ denotes the same spherical sector as in the previous example, and

$$F(r,\theta) = \frac{\exp(-\frac{r^4 (\sin 2\theta)^2}{6})}{\int_{\theta=0}^{2\pi} \exp(-\frac{r^4(\sin 2\theta)^2}{6})\, d\theta}.$$

The function $F(r,\theta)$ drawn in Figure 1.9 is larger when θ is close to $\{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$. Thus, in order to be larger than its expected size $O(n^{1/2})$, a random walk on the lattice $\mathbb{Z}^2$ needs to stay close to one of the four cardinal axes, and mod-Gaussian convergence allows one to make this statement quantitative. This result generalizes readily to arbitrary dimensions $d \geq 2$, see [FMN17c, Section 4.1].

Remark 1.48. In [Chh+15, Section 5], mod-ϕ convergent sequences with values in $\mathbb{Z}^d$ and with reference law ϕ supported on this lattice are studied. In this setting, one can define schemes of approximation as in Definition 1.35, and prove bounds on the local distance and the total variation distance similar to those of Theorem 1.36. This allows one to study for instance the asymptotic distribution of conjugacy classes of random coloured permutations in $\mathfrak{S}(n) \wr (\mathbb{Z}/d\mathbb{Z})$, or the number of prime divisors of a random integer that fall in a fixed residue class in $(\mathbb{Z}/a\mathbb{Z})^*$. Thus, we also have interesting results related to mod-ϕ convergence in the multi-dimensional discrete setting.

▷ **Mod-ϕ convergence and local limit theorems.** Another direction that one can explore in the framework of mod-ϕ convergence is local limit theorems (the first part of Theorem 1.12 is already a kind of local limit). A general principle in real harmonic analysis is that, if one has a control on the Fourier transform $\widehat{\mu}(\xi)$ of a probability measure on a large interval, then it gives information on the regularity of the distribution μ on a small interval. Therefore, mod-ϕ convergence should lead to local limit theorems, and mod-ϕ convergence with a zone of control should lead to better local limit theorems. Let us state the following theorem (see [BMN17, Theorem 9]), which confirms this intuition and is an amelioration of earlier results from [KN12; DKN15]:

Theorem 1.49 (Local limit theorem with a zone of control). *Let $\phi = \phi_{c,\alpha,\beta}$ be a reference stable law. We denote $m_{c,\alpha,\beta}(x)$ its density with respect to the Lebesgue measure, and we consider a sequence $(X_n)_{n\in\mathbb{N}}$ that admits a zone of control $[-K(t_n)^\gamma, K(t_n)^\gamma]$ with exponents (v,w). Let Y_n be the usual renormalisation of X_n (Proposition 1.9), such that $Y_n \rightharpoonup \phi$. For every $\delta \in (0, \frac{1}{\alpha}+\gamma)$, any $x \in \mathbb{R}$ and any bounded Jordan measurable subset $B \subset \mathbb{R}$ with Lebesgue measure $L(B) > 0$,*

$$\lim_{n\to\infty} (t_n)^\delta \; \mathbb{P}\left[Y_n - x \in \frac{1}{(t_n)^\delta}\, B\right] = m_{c,\alpha,\beta}(x)\, L(B).$$

Thus, the size of the zone of control dictates up to which scale the approximation given by the central limit theorem can be used, when looking at small scales. On the other hand, if ϕ is the Gaussian distribution, then there is an immediate generalisation of Theorem 1.49 to a multi-dimensional setting, see [BMN17, Proposition 18] and our Proposition 2.17.

Remark 1.50. If one only assumes mod-ϕ convergence of $(X_n)_{n\in\mathbb{N}}$ in Theorem 1.49, then one has a zone of control with index $\gamma = 0$. This case corresponds to [KN12, Theorem 4], which was stated in the multi-dimensional mod-Gaussian setting.

Example 1.51 (Characteristic polynomials of random unitary matrices). With M_n Haar-distributed in $\mathrm{U}(n)$, consider as before $X_n = \mathrm{Re}(\log\det(I_n - M_n))$. One can show that the mod-Gaussian convergence of $(X_n)_{n\in\mathbb{N}}$, which occurs with parameters $t_n = \frac{\log n}{2}$, can be given a zone of control of size $O(t_n)$ ($\gamma = 1$, $v = 2$, $w = 3$). Therefore, for any $\delta \in (0, \frac{3}{2})$ and any $x \in \mathbb{R}$,

$$\lim_{n\to\infty} (\log n)^\delta \; \mathbb{P}\left[\sqrt{\frac{2}{\log n}}\, X_n - x \in \frac{B}{(\log n)^\delta}\right] = \frac{\mathrm{e}^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\, L(B).$$

Similar results can be stated for characteristic polynomials of random matrices in the following classical ensembles: GUE, β-Laguerre, β-Jacobi, β-Gram and circular β-Jacobi, see [BMN17, Section 5].

Perspectives

This chapter focused on the theoretical consequences of the notion of mod-ϕ convergence; the most interesting examples will be presented in the next chapter. We consider the results of Sections 1.2-1.3 (large deviations and speed of convergence) to be quite optimal, and we do not expect to be able to get better results in these directions. On the opposite, the results of Section 1.4 are not always optimal. For instance, the local limit theorem 1.49 yields the asymptotics of the probabilities of Y_n being in a small interval up to size $\gg (t_n)^{-(\gamma+\frac{1}{\alpha})}$. Unfortunately, when Y_n is the renormalisation of a discrete random variable (*e.g.*, a statistic of a combinatorial model), this exponent $\gamma + \frac{1}{\alpha}$ is not always the one such that $(t_n)^{\delta}\, Y_n \in \mathbb{Z}$. Thus, one can have $\gamma + \frac{1}{\alpha} < \delta$, and then one cannot say anything precise on the probabilities

$$\mathbb{P}[Y_n \in (t_n)^{-\varepsilon}(a,b)]$$

with $\gamma + \frac{1}{\alpha} \leq \varepsilon < \delta$. So, we still have to understand what happens at these scales, in particular when the mod-Gaussian convergence follows from the method of cumulants (to be presented in Section 2.2).

On the other hand, the multi-dimensional large deviation principle of Theorem 1.43 only concerns spherical sectors $t_n C$, but one can also be interested in other growing sets $t_n D$ with D not spherical. If D touches $D_{\min} = \{\mathbf{x} \in D \,|\, \mathbf{x}^t K \mathbf{x} \text{ is minimal}\}$ along a hypersurface of dimension $d-1$, then Theorem 1.43 usually suffices. Otherwise, it needs to be extended to more general situations, *e.g.* when $D_{\min}$ is of dimension $< d-1$. In this setting we expect to obtain similar asymptotics, but with a different power of t_n. Another extension of Theorem 1.43 that we shall look for is with respect to other reference infinitely divisible laws, in particular multi-dimensional self-decomposable laws [Sat99, Chapter 3, Section 15]. These self-decomposable laws are naturally approached by marginales of Ornstein–Uhlenbeck type processes (Section 17 in *loc. cit.*), and in this framework one has mod-ϕ convergence. It would be an interesting example to look at, for which Theorems 1.43 and 1.44 need to be generalised.

Another important perspective is the transposition of all the results from this chapter to the framework of *free probability* (see [NS06] or [Mél17, Section 9.1] for an introduction to this non-commutative analogue of probability theory). Roughly speaking, free probability theory is the study of random variables that stem:

- from algebras of (infinite-dimensional) operators,
- and also from large random matrices, which in many situations can be considered as approximations of infinite-dimensional operators.

In this framework, the notion of independence of random variables is replaced by the notion of freeness, and the Gaussian distribution, which is the limit of sums of independent random variables, gets replaced by Wigner's semicircle distribution [Voi91]. One can develop a theory of free-infinitely divisible distributions [Maa92; BV93; BP99; Ben05], so it is natural to try to develop a theory of free-mod-ϕ convergence. The convergence in law in free probability is usually obtained by looking at the Cauchy–Stieltjes transform

$$C_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-s}\, \mu(ds)$$

and at various other related functions on the upper-half plane, in particular the R-transform. Thus, in a free analogue of the mod-ϕ theory, the Fourier transform needs to be replaced by the R-transform. Additionally, the cumulants of random variables which play a crucial role in many proofs of mod-Gaussian convergence have free analogues (the so-called free cumulants). So, one can hope to also extend the main techniques of proof of mod-Gaussian convergence to the free setting.

Chapter 2

Structure of mod-ϕ convergent sequences

We now have a good understanding of the theory of mod-ϕ convergence, and the main goal is then to establish the mod-ϕ convergence for large classes of models and random variables. This second chapter is devoted to this objective, and we shall describe some general structures and constructions that ensure a mod-Poisson or a mod-Gaussian convergence. If one has an exact formula for the generating function $\mathbb{E}[e^{zX_n}]$, possibly as a contour integral of a function of two variables, then standard techniques from complex analysis (singularity analysis, Laplace method, stationary phase or steepest descent method) yield the mod-ϕ convergence of $(X_n)_{n\in\mathbb{N}}$. This is already a large source of examples, which we present in Section 2.1. A particularly interesting class is provided by random combinatorial objects whose generating series have algebraico-logarithmic singularities. In this framework, a quite fascinating fact is that one can use the formalism of symmetric functions and formal alphabets in order to give a totally explicit description of the approximation schemes of order $r \geq 0$ of the sequence $(X_n)_{n\in\mathbb{N}}$ (Theorem 2.9).

In the special case where $\phi = \mathcal{N}_{\mathbb{R}}(0,1)$ is the Gaussian distribution, it turns out that one can prove mod-Gaussian convergence *without computing precisely the Laplace transforms* $\mathbb{E}[e^{zX_n}]$. Indeed, it suffices to prove adequate upper bounds on the coefficients of the series $\log(\mathbb{E}[e^{zX_n}])$, which are called the *cumulants* of the random variable X_n. One of the most important result from [FMN16; FMN17b] is that, if X_n is a renormalisation of a sum S_n of dependent random variables *with a sparse dependency graph*, then one can bound accordingly the cumulants of X_n, thereby proving the mod-Gaussian convergence of the sequence, and even with a large zone of control (Proposition 2.13 and Theorems 2.14 and 2.19). Section 2.2 is devoted to the presentation of this theory of dependency graphs. In Section 2.3, we describe spaces of models of

- random graphs,
- random permutations,
- random integer partitions,

whose statistics are always mod-Gaussian [FMN17a]. Indeed, for all these models, the fluctuations of the statistics have underlying sparse dependency graphs. A general conjecture is that many random combinatorial objects behave similarly.

If a random variable S_n does not admit an underlying sparse dependency graph, in certain cases, one can still prove upper bounds on the cumulants of S_n that look exactly the same as if there were a dependency graph. A typical example of this theory is provided by the empirical measures of finite Markov chains. We study these examples in Section 2.4, also examining certain models from statistical mechanics [MN15; FMN17b]. The upper bounds on cumulants obtained for these models involve *weighted* dependency graphs, which generalise the standard dependency graphs from Sections 2.2-2.3.

2.1 Analysis of generating functions and formal alphabets

In the first chapter, the mod-ϕ convergent sequences that we studied had explicit Laplace transforms $\mathbb{E}[\mathrm{e}^{zX_n}]$. Let us give two more important examples with this property:

Example 2.1 (Zeroes of a random analytic series). Set $f(z) = \sum_{n=0}^{\infty} A_n\, z^n$, where $(A_n)_{n\in\mathbb{N}}$ is a sequence of independent complex Gaussian variables, with

$$\mathbb{E}[A_1] = 0 \quad ; \quad \mathbb{E}[(\mathrm{Re}(A_1))^2] = \mathbb{E}[(\mathrm{Im}(A_1))^2] = \frac{1}{2} \quad ; \quad \mathbb{E}[(\mathrm{Re}(A_1))(\mathrm{Im}(A_1))] = 0.$$

The radius of convergence of the random series $f(z)$ is almost surely equal to 1, and the set of zeroes $\mathcal{Z}(f) = \{z \in \mathbb{C} \,|\, |z| < 1,\ f(z) = 0\}$ is a determinantal point process with kernel

$$K(z,w) = \frac{1}{\pi(1-w\overline{z})^2};$$

see [PV05] and [Hou+09, Theorem 5.1.1]. Now, a general property of determinantal point processes [Hou+09, Theorem 4.5.3] is that, if C is a compact subset of the open disk of radius 1, then the number of points of $\mathcal{Z}(f)$ that fall in C has the law of a sum of independent Bernoulli variables:

$$\mathrm{card}(\mathcal{Z}(f)\cap C) =_{\mathrm{law}} \sum_{k=1}^{\infty} \mathcal{B}(\lambda_k),$$

where the λ_k's are the eigenvalues of the integral operator $K_{|C} : \mathscr{L}^2(C) \to \mathscr{L}^2(C)$. When $C = \overline{D}_{(0,r)}$ is the disk of radius $r < 1$, these eigenvalues can be computed, and one obtains the identity in law

$$N_r = \mathrm{card}\{z \in \mathcal{Z}(f) \,|\, |z| \leq r\} =_{\mathrm{law}} \sum_{k=1}^{\infty} \mathcal{B}(r^{2k}).$$

Hence, the Laplace transform of N_r is explicit, given by the infinite product:

$$\mathbb{E}[\mathrm{e}^{zN_r}] = \prod_{k=1}^{\infty}(1 + r^{2k}(\mathrm{e}^z - 1)).$$

Let $h = \frac{4\pi r^2}{1-r^2}$ be the hyperbolic area of $\overline{D}_{(0,r)}$. As r goes to 1, h goes to $+\infty$ and $N_r = N^h$ admits a renormalisation that is mod-Gaussian convergent [FMN16, Section 7.1]:

$$\left(\frac{N^h - \frac{h}{4\pi}}{h^{1/3}}\right)_{h\geq 0} \text{ is mod-Gaussian with parameters } \frac{h^{1/3}}{8\pi} \text{ and limit } \psi(z) = \exp\left(\frac{z^3}{144\pi}\right).$$

More generally, given a determinantal point process with trace-class kernel K on a domain D, if $(C_n)_{n\in\mathbb{N}}$ is a growing sequence of compact subsets of D, then the number of points S_n of the determinantal point process that fall in C_n is given by a series of independent Bernoulli variables. Therefore, if the eigenvalues of $K_{|C_n}$ satisfy certain limiting conditions, then one has mod-Gaussian convergence of a certain renormalisation of $(S_n)_{n\in\mathbb{N}}$. Unfortunately, it is usually not possible to compute explicitly the eigenvalues of $K_{|C_n}$.

Example 2.2 (Magnetisation of the one-dimensional Ising model). Fix an inverse temperature $\beta > 0$. The Ising model on the discrete torus $\mathbb{Z}/n\mathbb{Z}$ is the probability measure on configurations $\sigma : \mathbb{Z}/n\mathbb{Z} \to \{\pm 1\}$ that is proportional to

$$\exp\left(\frac{\beta}{2}\sum_{i=1}^{n}\sigma(i)\sigma(i+1)\right).$$

We refer to [Bax82; FV17] for a mathematical study of this model, which is a toy-model in a large class of models stemming from statistical mechanics. If $M_n = \sum_{i=1}^n \sigma(i)$ is the total magnetisation of the system, then the transfer matrix method yields

$$\mathbb{E}[\mathrm{e}^{zM_n}] = \frac{\mathrm{tr}(T(z))^n}{\mathrm{tr}(T(0))^n}, \quad \text{where } T(z) = \begin{pmatrix} \mathrm{e}^{-z} & \mathrm{e}^{z-\beta} \\ \mathrm{e}^{-z-\beta} & \mathrm{e}^{z} \end{pmatrix} \text{ is the transfer matrix.}$$

The two eigenvalues of $T(z)$ are $\cosh z \pm \sqrt{(\sinh z)^2 + \mathrm{e}^{-2\beta}}$, so M_n has an explicit Laplace transform. Its asymptotic analysis shows that $\frac{M_n}{n^{1/4}}$ is mod-Gaussian convergent with parameters $n^{1/2}\mathrm{e}^{\beta}$ and limit

$$\psi(z) = \exp\left(-\frac{(3\mathrm{e}^{3\beta} - \mathrm{e}^{\beta})\,z^4}{24}\right).$$

This model and the Curie–Weiss model were studied in [MN15]. The analogue result in dimension $d \geq 2$ will be presented in Section 2.4. On the other hand, if one considers the Ising model on an interval $[\![1, n]\!]$ instead of the torus $\mathbb{Z}/n\mathbb{Z}$, then M_n is a linear statistic of a Markov chain, and we shall also see in Section 2.4 (Example 2.36) that these functionals are generically mod-Gaussian (after an appropriate renormalisation).

▷ **Double generating series and their asymptotic analysis.** A more complex situation is when $X_n = X(\omega_n)$ is a statistic of a random object ω_n of size n, such that one has an explicit formula for the *double generating series* of $(X_n)_{n\in\mathbb{N}}$. Consider a combinatorial class $\mathfrak{C} = (\mathfrak{C}_n)_{n\in\mathbb{N}}$, that is a sequence of finite sets [FS09, Section I.1]. We endow each set $\mathfrak{C}_n$ with a probability measure $\mathbb{P}_n$ and with a statistic $X : \mathfrak{C}_n \to \mathbb{N}$. For example, if one considers the combinatorial class $\mathfrak{S} = (\mathfrak{S}(n))_{n\in\mathbb{N}}$ of permutations, then $\mathbb{P}_n$ can be Ewens' measure (defined in Example 1.18), and $X(\sigma)$ can be the number of cycles of a permutation σ. We also fix a sequence $(\alpha_n)_{n\in\mathbb{N}}$ of positive renormalisation parameters. The double or bivariate generating series of the random variables $X_n = X(\sigma \in \mathfrak{C}_n)$ is defined by:

$$F(t,w) = \sum_{n=0}^{\infty} \alpha_n t^n\, \mathbb{E}_n[w^{X_n}].$$

Usually, the renormalisation parameters are taken in one of the following sequences:

$$\alpha_n = 1 \quad ; \quad \alpha_n = \frac{1}{n!} \quad ; \quad \alpha_n = |\mathfrak{C}_n| \quad ; \quad \alpha_n = \frac{|\mathfrak{C}_n|}{n!},$$

where $|\mathfrak{C}_n|$ denotes the cardinality of the finite set $\mathfrak{C}_n$. Given a random combinatorial model, the choice of the double generating series is made so that $F(t,w)$ satisfies the following hypotheses:

Definition 2.3 (Algebraico-logarithmic singularities, [FO90]). *Consider an annular domain $\mathcal{A}_{(a,b)} = \{w \in \mathbb{C} \mid \exp(a) < |w| < \exp(b)\} = \exp(\mathcal{S}_{(a,b)})$, with $a < 0 < b$. We say that $F(t,w)$ is a generating series with algebraico-logarithmic singularities if the following assertions are satisfied:*

(1) For any $w \in \mathcal{A}_{(a,b)}$, the map $t \mapsto F(t,w)$ is holomorphic on a domain $\Delta_0(r(w), R(w), \phi)$, where $0 < r(w) < R(w)$ and $0 < \phi < \frac{\pi}{2}$, and

$$\Delta_0(r, R, \phi) = \{z \in \mathbb{C} \,|\, |z| < R,\, z \neq r,\, |\arg(z-r)| > \phi\}.$$

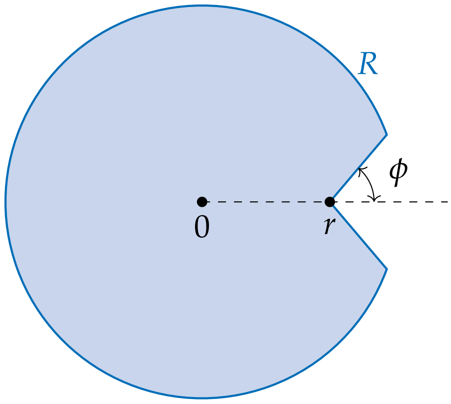


Figure 2.1. Domain of analyticity $\Delta_0(r, R, \phi)$.

(2) As t converges to $r(w)$, $F(t,w)$ admits a singularity of algebraico-logarithmic type:

$$F(t,w) = K(w) \left(\frac{1}{1-\frac{z}{r(w)}}\right)^{\alpha(w)} \left(\log\left(\frac{1}{1-\frac{z}{r(w)}}\right)\right)^{\beta(w)} (1+o(1)).$$

A well-known principle in analytic combinatorics is that the singularities of a generating series dictate the asymptotics of its coefficients; see [FS09, Chapter VI]. This follows from an asymptotic analysis of the Cauchy formula

$$\alpha_n\, \mathbb{E}[w^{X_n}] = [t^n]\,(F(t,w)) = \frac{1}{2\mathrm{i}\pi} \oint \frac{F(t,w)}{t^{n+1}}\, dt,$$

where the integral is taken over any contour that makes a single loop around 0 (for the asymptotic analysis, one chooses a contour that follows closely the boundary of the domain $\Delta_0(r,R,\phi)$). Hence, one has the following transfer theorem:

Theorem 2.4 (Transfer theorem, [FO90]). *Under the assumptions of Definition 2.3, if $w \in \mathcal{A}_{(a,b)}$, then*

$$\begin{aligned}\mathbb{E}[w^{X_n}] &= \frac{[t^n](F(t,w))}{[t^n](F(t,1))} \\ &= \frac{K(w)}{K(1)}\,\frac{\Gamma(\alpha(1))}{\Gamma(\alpha(w))}\,\left(\frac{r(1)}{r(w)}\right)^n\, n^{\alpha(w)-\alpha(1)}\,(\log n)^{\beta(w)-\beta(1)}\,\left(1+O\left(\frac{1}{\log n}\right)\right).\end{aligned}$$

The remainder $O(\frac{1}{\log n})$ can be taken uniform if w stays in a sufficiently small compact subset of $\mathcal{A}_{(a,b)}$. If $\beta(w) = \beta(1)$ for all w, then the remainder is in fact a $O(\frac{1}{n})$.

Setting $w = \mathrm{e}^z$, the transfer theorem leads to many results of complex mod-Poisson convergence, which we list hereafter; they come from [NZ13], [FMN16, Section 7.3] and [Chh+15, Section 4].

Example 2.5 (Random permutations under a generalised weighted measure). We fix a sequence of non-negative parameters $\Theta = (\theta_k)_{k\geq 1}$, and we consider the combinatorial class of permutations $\mathfrak{S}$. We endow each symmetric group $\mathfrak{S}(n)$ with the probability measure

$$\mathbb{P}_{n,\Theta}[\sigma] = \frac{1}{n!\,\mathfrak{h}_n(\Theta)} \prod_{k\geq 1} (\theta_k)^{m_k(\sigma)},$$

where $m_k(\sigma)$ is the number of cycles of length k in σ; and $\mathfrak{h}_n(\Theta) = \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}(n)} \prod_{k\geq 1}(\theta_k)^{m_k(\sigma)}$ is the normalisation constant such that $\mathbb{P}_{n,\Theta}$ is a probability measure. This model appears in the study of the quantum Bose gas in statistical mechanics, see [BU09; BU11; BUV11; EU12]. One recovers the Ewens measures when $\Theta = (\theta)_{k\geq 1}$ is a constant sequence. As in Example 1.18, we are interested in the number of cycles of σ under $\mathbb{P}_{n,\Theta}$; $\ell_n = X(\sigma) = \sum_{k\geq 1} m_k(\sigma)$. Consider the following double generating series:

$$\begin{aligned} F(t,w) &= \sum_{n=0}^{\infty} t^n\, \mathfrak{h}_n(\Theta)\, \mathbb{E}_{n,\Theta}[w^{\ell_n}] = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{\sigma\in\mathfrak{S}(n)} \prod_{k\geq 1} (w\theta_k)^{m_k(\sigma)} \\ &= \sum_{n=0}^{\infty} t^n\, \mathfrak{h}_n(w\Theta) = \exp\left(\sum_{k=1}^{\infty} \frac{w\theta_k}{k}\, t^k\right). \end{aligned} \tag{2.1}$$

For the last identity in Equation (2.1), one considers the specialisation of the algebra of symmetric functions Sym that sends the k-th power sum $\mathfrak{p}_k$ to $\mathfrak{p}_k(w\Theta) = w\theta_k$, see [Mac95, Chapter 1] or [Mél17, Chapter 2]; and one uses the identity of symmetric functions

$$\mathfrak{h}_n = \sum_{\lambda\in\mathfrak{Y}(n)} \frac{\mathfrak{p}_\lambda}{z_\lambda} = \frac{1}{n!} \sum_{\lambda\in\mathfrak{Y}(n)} |C_\lambda|\, \mathfrak{p}_\lambda,$$

where $|C_\lambda|$ is the number of permutations with cycle type λ in $\mathfrak{S}(n)$. Set $g_\Theta(t) = \sum_{k=1}^{\infty} \frac{\theta_k t^k}{k}$, and suppose that g_Θ is a power series that is convergent on a domain $\Delta_0(r,R,\phi)$ and with logarithmic singularity:

$$g_\Theta(t) = \theta\, \log\left(\frac{1}{1-\frac{t}{r}}\right) + L + O(|t-r|).$$

Then, $F(t,w) = \exp(w g_\Theta(t))$ satisfies the hypotheses of Theorem 2.4 with $r(w) = r$, $R(w) = R$, $K(w) = \mathrm{e}^{wL}$, $\alpha(w) = w\theta$ and $\beta(w) = 0$. Therefore, one has the mod-Poisson convergence

$$\mathbb{E}[\mathrm{e}^{z\ell_n}] = \mathrm{e}^{(\theta\log n + L)(\mathrm{e}^z - 1)}\, \frac{\Gamma(\theta)}{\Gamma(\theta \mathrm{e}^z)}\, (1 + O(n^{-1}))$$

with parameters $t_n = \theta \log n + L$ and limit $\psi(z) = \frac{\Gamma(\theta)}{\Gamma(\theta\mathrm{e}^z)}$. For Ewens measures, $g_\Theta(t) = g_\theta(t) = -\theta\, \log(1-t)$ and $L = 0$.

Example 2.6 (Connected components in functional graphs). Let $\mathfrak{F}(n)$ be the set of maps from $[\![1,n]\!]$ to $[\![1,n]\!]$; it contains $\mathfrak{S}(n)$. If $f \in \mathfrak{F}(n)$, its functional graph is the directed graph with vertex set $V = [\![1,n]\!]$, and with edge set $E = \{(k, f(k)) \,|\, k \in [\![1,n]\!]\}$. The functional graph $G(f)$ is a disjoint union of a finite number $X(f)$ of connected components, these connected components being cycles on which trees are grafted (Figure 2.2). We endow the sets $\mathfrak{F}(n)$ with their uniform probability measures. The double generating series with $\alpha_n = \frac{|\mathfrak{F}(n)|}{n!} = \frac{n^n}{n!}$ is

$$F(t,w) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{f\in\mathfrak{F}(n)} w^{X(f)} = \exp(-w\, \log(1-T(t))),$$

where

$$T(t) = \sum_{n=1}^{\infty} \frac{t^n\,|\mathfrak{T}(n)|}{n!}$$

is the generating series of unordered rooted labeled trees, and is the entire solution of $T(t) = t\,\mathrm{e}^{T(t)}$. The hypotheses of Theorem 2.4 are satisfied with $r(w) = \frac{1}{\mathrm{e}}$, $K(w) = 2^{-\frac{w}{2}}$, $\alpha(w) = \frac{w}{2}$ and $\beta(w) = 0$. Hence, one has the complex mod-Poisson convergence

$$\mathbb{E}_n[\mathrm{e}^{zX(f)}] = \mathrm{e}^{(\frac{1}{2}\log\frac{n}{2})(\mathrm{e}^z-1)}\,\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{\mathrm{e}^z}{2})}\,(1+O(n^{-1})),$$

with parameters $t_n = \frac{1}{2}\log\frac{n}{2}$ and limit $\psi(z) = \frac{\Gamma(1/2)}{\Gamma(\mathrm{e}^z/2)}$.

$$\begin{tikzcd}
& & & 2 \arrow[d] & \\
& & & 8 \arrow[dl] & 6 \arrow[l] \\
& 3 \arrow[r] & 4 \arrow[d] & & \\
& 1 \arrow[u] & 7 \arrow[l] & & \\
5 \arrow[ur] & & & &
\end{tikzcd}$$

Figure 2.2. The functional graph associated to the function $f \in \mathfrak{F}(8)$ with word 38471814.

Example 2.7 (Number of irreducible factors in a polynomial over $\mathbb{F}_q$). Let $\mathfrak{P}(n,\mathbb{F}_q)$ be the set of monic polynomials of degree n over $\mathbb{F}_q$, and $\mathfrak{I}(n,\mathbb{F}_q)$ be the set of monic irreducible polynomials of degree n over $\mathbb{F}_q$. We are interested in:

- the number of distinct irreducible factors $X_n = X(P)$ of a random polynomial $P \in \mathfrak{P}(n,\mathbb{F}_q)$, with P taken according to the uniform probability measure;
- and the number of irreducible factors $Y_n = Y(P)$, this time counted with multiplicities.

Their generating series are

$$F_X(t,w) = \sum_{n=0}^{\infty} t^n \sum_{P\in\mathfrak{P}(n,\mathbb{F}_q)} w^{X(P)} = \exp\left(\sum_{k=1}^{\infty} \frac{I(t^k)}{k}\,(1-(1-w)^k)\right);$$

$$F_Y(t,w) = \sum_{n=0}^{\infty} t^n \sum_{P\in\mathfrak{P}(n,\mathbb{F}_q)} w^{Y(P)} = \exp\left(\sum_{k=1}^{\infty} \frac{I(t^k)}{k}\,w^k\right).$$

Here, $I(t)$ is the generating series of the irreducible polynomials over $\mathbb{F}_q$, given by

$$I(t) = \sum_{n=1}^{\infty} t^n\,|\mathfrak{I}(n,\mathbb{F}_q)| = \sum_{k=1}^{\infty} \frac{\mu(k)}{k}\,\log\left(\frac{1}{1-qt^k}\right).$$

This leads to the complex mod-Poisson convergences:

$$\mathbb{E}[e^{zX_n}] = e^{(\log n + R(q^{-1}))(e^z-1)} \frac{1}{\Gamma(e^z)} \exp\left(\sum_{k=2}^{\infty} \frac{(-1)^{k-1}\, I(q^{-k})}{k} (e^z-1)^k\right) (1+O(n^{-1}));$$

$$\mathbb{E}[e^{zY_n}] = e^{(\log n + R(q^{-1}))(e^z-1)} \frac{1}{\Gamma(e^z)} \exp\left(\sum_{k=2}^{\infty} \frac{I(q^{-k})}{k} (e^{kz}-1)\right) (1+O(n^{-1})),$$

where $R(t) = \sum_{k=2}^{\infty} \frac{\mu(k)}{k} \log\left(\frac{1}{1-qt^k}\right)$.

Similar techniques can be used in analytic number theory, leading to the mod-Poisson convergence of certain statistics of random integers. More precisely, let $f : \mathbb{N} \to \mathbb{Z}$ be an arithmetic function, and $L_f(s,w)$ be its *double Dirichlet series*

$$L_f(s,w) = \sum_{n=0}^{\infty} \frac{w^{f(n)}}{n^s}.$$

If the parameter w is fixed, then the Dirichlet series usually converges absolutely on a half-plane $\{s \in \mathbb{C} \,|\, \mathrm{Re}(s) > r(w)\}$. The analogue of Cauchy's formula in this setting is Perron's summation formula [Ten95, Chapter II.2]: if f_n is the random variable $f(k)$ with k random integer in $[\![1,n]\!]$, then

$$\mathbb{E}[e^{zf_n}] = \int_{r-i\infty}^{r+i\infty} L_f(s,e^z)\, \frac{n^{s-1}}{s}\, ds + \frac{e^{zf(n)}}{2n}$$

for any $r > \max(0, r(e^z))$. A careful analysis of this formula relates the behavior of $L_f(s,w)$ near its pole $r(w)$, and the asymptotics of $\mathbb{E}[e^{zf_n}]$, in a fashion similar to Theorem 2.4. We refer to [Ten95, Section II.5, Theorem 3] for the details.

Example 2.8 (Additive arithmetic functions with linear growth). Let $f : \mathbb{N} \to \mathbb{Z}$ be an arithmetic function with the three following properties:

(1) f is additive: if $m \wedge n = 1$, then $f(mn) = f(m) + f(n)$;

(2) $f(p) = 1$ for any prime number p;

(3) $|f(p^k)| \leq Ck + B$ for some positive constants B, C that do not depend on the prime number p.

The double Dirichlet series has then an infinite Euler product:

$$L_f(w,s) = \prod_{p\in\mathbb{P}} \left(1 + \frac{w^{f(p)}}{p^s} + \frac{w^{f(p^2)}}{p^{2s}} + \cdots\right).$$

In this framework, one has complex mod-Poisson convergence of $(f_n)_{n\in\mathbb{N}}$, with parameters $t_n = \log\log n$, domain $\mathcal{S}_{(-\frac{\log 2}{C}, \frac{\log 2}{C})}$ and limit

$$\psi(z) = \frac{1}{\Gamma(e^z)} \prod_{p\in\mathbb{P}} \left(1 + \frac{e^{zf(p)}}{p} + \frac{e^{zf(p^2)}}{p^2} + \cdots\right) e^{e^z \log(1-\frac{1}{p})};$$

see [FMN16, Proposition 7.2.11].

▷ **Approximation of lattice-valued random variables.** In the previous paragraph, we saw that if $(X_n)_{n\in\mathbb{N}}$ is a sequence of random variables whose double generating series or double Dirichlet series admits an explicit formula with known singularities, then this formula leads in many situations to the mod-Poisson convergence of the sequence $(X_n)_{n\in\mathbb{N}}$. There is more to this story: it turns out that in all the previous examples, one can describe precisely the derived approximation schemes of the standard approximation schemes of order $r \geq 0$ [Chh+15, Section 4]. To this purpose, let us recall briefly some facts from the theory of symmetric functions. The algebra of *symmetric functions* Sym is the projective limit in the category of graded algebras of the algebras $\mathbb{C}[x_1, x_2, \ldots, x_n]^{\mathfrak{S}(n)}$ of symmetric polynomials. So, Sym is the algebra of power series in an infinity of (commutative) variables $x_1, x_2, \ldots, x_n, \ldots$ that are:

- invariant under any finite permutation of the variables,
- with bounded degree.

A transcendence basis of Sym over $\mathbb{C}$ consists in the *power sums*

$$\mathfrak{p}_{k\geq 1} = \sum_{i=1}^{\infty} (x_i)^k.$$

Two other transcendence bases are the *homogeneous symmetric functions*

$$\mathfrak{h}_{k\geq 1} = \sum_{1\leq i_1\leq i_2\leq \cdots \leq i_k} x_{i_1} x_{i_2} \cdots x_{i_k}$$

and the *elementary symmetric functions*

$$\mathfrak{e}_{k\geq 1} = \sum_{1< i_1< i_2< \cdots < i_k} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

The connection between these three bases is encoded in the relations $H(t)\,E(-t) = 1$ and $H(t) = \exp(P(t))$, where

$$H(t) = 1 + \sum_{k=1}^{\infty} \mathfrak{h}_{k\geq 1}\, t^k;$$
$$E(t) = 1 + \sum_{k=1}^{\infty} \mathfrak{e}_{k\geq 1}\, t^k;$$
$$P(t) = \sum_{k=1}^{\infty} \frac{\mathfrak{p}_k}{k}\, t^k.$$

A *formal alphabet* is a specialisation of Sym, that is a morphism of algebras Sym $\to \mathbb{C}$. It is convenient to denote such morphisms $f \mapsto f(A)$. Among the formal alphabets, the true alphabets are the summable sequences $A = (a_1, a_2, \ldots a_n, \ldots)$, which yield the morphisms

$$\mathfrak{p}_k \mapsto \mathfrak{p}_k(A) = \sum_{i=1}^{\infty} (a_i)^k.$$

Given a formal alphabet A, the conjugate alphabet $\varepsilon(A)$ is the formal alphabet defined by $\mathfrak{p}_k(\varepsilon(A)) = (-1)^{k-1}\, \mathfrak{p}_k(A)$. On the other hand, given two formal alphabets A and B, their sum $A + B$ is the formal alphabet defined by $\mathfrak{p}_k(A+B) = \mathfrak{p}_k(A) + \mathfrak{p}_k(B)$. These two operations on alphabets are related to the Hopf algebra structure of Sym, see [Mél17, Section 2.3]. Finally, consider a sequence $A = (a_1, a_2, \ldots, a_n, \ldots)$ that is not summable, but with $\sum_{i=1}^{\infty} |a_i|^2 < +\infty$.

We then convene that A is associated to the specialisation

$$\mathfrak{p}_1(A) = 0 \qquad ; \qquad \mathfrak{p}_{k\geq 2}(A) = \sum_{i=1}^{\infty} (a_i)^k.$$

The results of [Chh+15, Section 4] are summarised by the following:

Theorem 2.9 (Approximation schemes and formal alphabets). *Consider the following random sequences $(X_n)_{n\in\mathbb{N}}$, associated to sequences of parameters $(t_n)_{n\in\mathbb{N}}$ and to formal alphabets A:*

X_n	t_n	*formal alphabet A*
sum of independent Bernoulli variables with parameters p_j	$\sum_{i=1}^n p_i$	$\{p_1, p_2, p_3 \ldots\}$
number of cycles of a random uniform permutation	$\log n + \gamma$	$\{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$
number of cycles of a random Ewens permutation with parameter θ	$\theta(\log n + \gamma)$	$\{1, \frac{\theta}{\theta+1}, \frac{\theta}{\theta+2}, \ldots\}$
number of connected components of a uniform random map	$\frac{1}{2}(\log 2n + \gamma)$	$\{1, \frac{1}{3}, \frac{1}{5}, \ldots\}$
number of distinct irreducible factors of a random monic polynomial	$\log n + R(q^{-1}) + \gamma$	$\{1, \frac{1}{2}, \frac{1}{3}, \ldots\} + \left\{\frac{1}{q^{\deg P}} \mid P \in \mathfrak{I}(\mathbb{F}_q)\right\}$
number of irreducible factors of a random monic polynomial	$\log n + S(q^{-1}) + \gamma$	$\{1, \frac{1}{2}, \frac{1}{3}, \ldots\} + \varepsilon\left(\left\{\frac{1}{q^{\deg P}-1} \mid P \in \mathfrak{I}(\mathbb{F}_q)\right\}\right)$
number of distinct prime divisors of a random integer	$\log\log n + \gamma$	$\{1, \frac{1}{2}, \frac{1}{3}, \ldots\} + \{\frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots\}$

Here, n is the size (or degree) of the random object whose statistic is X_n; γ is the Euler–Mascheroni constant; $\mathfrak{I}(\mathbb{F}_q)$ is the set of irreducible polynomials over $\mathbb{F}_q$; and

$$R(q^{-1}) = \sum_{k=2}^{\infty} \frac{\mu(k)}{k} \log\left(\frac{1}{1-q^{1-k}}\right);$$
$$S(q^{-1}) = \sum_{k=2}^{\infty} \frac{\varphi(k)}{k} \log\left(\frac{1}{1-q^{1-k}}\right)$$

where μ and φ are Möbius' and Euler's arithmetic functions.

(1) In all the previous examples, one has mod-Poisson convergence on $D = \mathrm{i}\mathbb{R}$ of $(X_n)_{n\in\mathbb{N}}$, with parameters $(t_n)_{n\in\mathbb{N}}$ and limit

$$\theta(\xi) = 1 + \sum_{k=1}^{\infty} \mathfrak{e}_k(A)\,(\mathrm{e}^{\mathrm{i}\xi} - 1)^k.$$

(2) The derived scheme $(\sigma_n^{(r)})_{n\in\mathbb{N}}$ of the standard approximation scheme $(\nu_n^{(r)})_{n\in\mathbb{N}}$ of order $r \geq 0$ of the law μ_n of X_n is defined by

$$\widehat{\sigma}_n^{(r)}(\xi) = \mathrm{e}^{t_n(\mathrm{e}^{\mathrm{i}\xi}-1)} \left(1 + \sum_{k=1}^{r} \mathfrak{e}_k(A)\,(\mathrm{e}^{\mathrm{i}\xi} - 1)^k\right),$$

or equivalently by

$$\sigma_n^{(r)}(m) = \frac{\mathrm{e}^{-t_n}(t_n)^m}{m!}\left(1+\sum_{k=1}^{r}\mathfrak{e}_k(A)\,c(k,m,t_n)\right),$$

where $c(k,m,t_n) = \sum_{l=0}^{\min(k,m)}(-1)^{k-l}\binom{k}{l}\frac{(t_n)^{-l}m!}{(m-l)!}$ *is a Poisson–Charlier polynomial.*

(3) *For any* $r \geq 1$*, the scheme of approximation* $(\sigma_n^{(r)})_{n\in\mathbb{N}}$ *satisfies in the previous examples:*

$$d_{\mathrm{loc}}(\mu_n,\sigma_n^{(r)}) = \frac{|\mathfrak{e}_{r+1}(A)\,G_{r+1}(z_{r+2})|}{\sqrt{2\pi}(t_n)^{\frac{r}{2}+1}} + o\left(\frac{1}{(t_n)^{\frac{r}{2}+1}}\right);$$

$$d_{\mathrm{Kol}}(\mu_n,\sigma_n^{(r)}) = \frac{|\mathfrak{e}_{r+1}(A)\,G_{r}(z_{r+1})|}{\sqrt{2\pi}(t_n)^{\frac{r+1}{2}}} + o\left(\frac{1}{(t_n)^{\frac{r+1}{2}}}\right);$$

$$d_{\mathrm{TV}}(\mu_n,\sigma_n^{(r)}) = \frac{|\mathfrak{e}_{r+1}(A)|\,\int_{\mathbb{R}}|G_{r+1}(z)|\,dz}{2\sqrt{2\pi}(t_n)^{\frac{r+1}{2}}} + o\left(\frac{1}{(t_n)^{\frac{r+1}{2}}}\right).$$

Thus, the approximation of the lattice-valued random variables introduced in this section is reduced by Theorem 2.9 to a simple algorithm depending on a set of parameters A (the formal alphabet), which is naturally connected to the structure of the model. For instance, when studying the numbers of prime divisors in random integers, the corresponding alphabet involves the inverses of the integers and the inverses of the prime numbers. So far, we do not have a clear explanation of why we obtain very simple alphabets in all the aforementioned cases.

2.2 Joint cumulants and dependency graphs

In the previous section, we proved the mod-Gaussian or mod-Poisson convergence of sequences of random variables $(X_n)_{n\in\mathbb{N}}$ that had explicit generating series, possibly given by contour integrals. However, the complete knowledge of $\mathbb{E}[\mathrm{e}^{zX_n}]$ is not required to establish the mod-Gaussian convergence of the sequence, and in many situations it suffices to prove estimates on the coefficients of this series (or its logarithm). This idea leads to the powerful *method of cumulants*.

▷ **Cumulants, joint cumulants and mod-Gaussian convergence.** Let X be a random variable that admits a convergent Laplace transform $\mathbb{E}[\mathrm{e}^{zX}]$ in a disc around the origin; in particular, X has moments of all order.

Definition 2.10 (Cumulant). *The cumulants of* X *are the coefficients* $\kappa^{(r)}(X)$ *of*

$$\log\mathbb{E}[\mathrm{e}^{zX}] = \sum_{r=1}^{\infty}\frac{\kappa^{(r)}(X)}{r!}\,z^r.$$

In other words, $\kappa^{(r)}(X) = \frac{d^r}{dz^r}(\log\mathbb{E}[\mathrm{e}^{zX}])_{|z=0}$.

The first cumulants of a random variable X are

$$\kappa^{(1)}(X) = \mathbb{E}[X] \quad ; \quad \kappa^{(2)}(X) = \mathbb{E}[X^2]-\mathbb{E}[X]^2 = \mathrm{var}(X) \quad ;$$
$$\kappa^{(3)}(X) = \mathbb{E}[X^3]-3\,\mathbb{E}[X^2]\mathbb{E}[X]+2\,(\mathbb{E}[X])^3.$$

In general, the r-th cumulant of X is a homogeneous polynomial of degree r in the moments $\mathbb{E}[X^k]$. The relation between cumulants and moments is better understood by introducing the notion of joint cumulants [LS59], which generalises Definition 2.10:

Definition 2.11 (Joint cumulant). *Let $X_1, X_2, \ldots, X_r$ be real random variables with moments of all order. The joint cumulant of $X_1, \ldots, X_r$ is*

$$\kappa(X_1, X_2, \ldots, X_r) = \frac{\partial^r}{\partial z_1 \partial z_2 \cdots \partial z_r} \left(\log \mathbb{E}[\mathrm{e}^{z_1 X_1 + z_2 X_2 + \cdots + z_r X_r}] \right) \Big|_{z_1 = z_2 = \cdots = z_r = 0}.$$

Proposition 2.12 (Properties of joint cumulants). *Denote $\mathfrak{P}(r)$ the set of set partitions of $[\![1, r]\!]$, and if $\pi = \pi_1 \sqcup \pi_2 \sqcup \cdots \sqcup \pi_{\ell(\pi)}$ belongs to $\mathfrak{P}(r)$, set $\mu(\pi) = (-1)^{\ell(\pi)-1} (\ell(\pi) - 1)!$ (Möbius function of the lattice of set partitions). The joint cumulant of the random variables $X_1, \ldots, X_r$ is given by the following formula:*

$$\kappa(X_1, \ldots, X_r) = \sum_{\pi \in \mathfrak{P}(r)} \mu(\pi) \prod_{i=1}^{\ell(\pi)} \mathbb{E}\left[\prod_{j \in \pi_i} X_j \right].$$

Moreover, the joint cumulants have the following properties:

(1) Multilinearity and invariance by permutation.

(2) If X is a random variable with moments of all order, then

$$\kappa^{(r)}(X) = \kappa(\underbrace{X, X, \ldots, X}_{r \text{ occurrences}}).$$

(3) If $\{X_1, \ldots, X_r\}$ can be split into two non-empty families $\{X_{i_1}, \ldots, X_{i_s}\}$ and $\{X_{j_1}, \ldots, X_{j_t}\}$ that are independent, then $\kappa(X_1, \ldots, X_r) = 0$.

The Gaussian distributions are characterised by the vanishing of their cumulants of order $r \geq 3$. On the other hand, a sequence of random variables is mod-Gaussian if its Fourier or Laplace transform is close to the one of a Gaussian. Therefore, it is natural to characterise mod-Gaussian convergence by means of cumulants. The right way to do it is the following ([FMN16, Chapter 5] and [FMN17b, Definition 28 and Lemma 29]):

Proposition 2.13 (Method of cumulants). *Let $(S_n)_{n \in \mathbb{N}}$ be a sequence of real-valued random variables. We say that $(S_n)_{n \in \mathbb{N}}$ satisfies the hypotheses of the method of cumulants with parameters (D_n, N_n, A) if:*

(MC1) The sequences of real numbers $(D_n)_{n \in \mathbb{N}}$ and $(N_n)_{n \in \mathbb{N}}$ satisfy $N_n \to +\infty$ and $\frac{D_n}{N_n} \to 0$.

(MC2) The first cumulants of S_n satisfy:

$$\kappa^{(1)}(S_n) = 0;$$
$$\kappa^{(2)}(S_n) = (\sigma_n)^2 \, N_n D_n;$$
$$\kappa^{(3)}(S_n) = L_n \, N_n (D_n)^2$$

with $\lim_{n \to \infty} (\sigma_n)^2 = \sigma^2 > 0$, and $\lim_{n \to \infty} L_n = L$.

(MC3) The cumulants of S_n are bounded as follows:

$$\forall r \geq 1, \ |\kappa^{(r)}(S_n)| \leq A^r \, r^{r-2} \, N_n \, (2D_n)^{r-1}.$$

Set $X_n = \frac{S_n}{(N_n)^{1/3}(D_n)^{2/3}}$. *Under the assumptions (MC1)-(MC3),* $(X_n)_{n\in\mathbb{N}}$ *is mod-Gaussian convergent on the complex plane, with*

$$t_n = (\sigma_n)^2 \left(\frac{N_n}{D_n}\right)^{1/3} \qquad ; \qquad \psi(z) = \exp\left(\frac{Lz^3}{6}\right).$$

Moreover, one has a zone of control $[-Kt_n, Kt_n]$ *with exponents* $(3,3)$ *and*

$$K = \frac{1}{(8+4\mathrm{e})A^3} \qquad ; \qquad K_1 = K_2 = (2+\mathrm{e})A^3.$$

Theorem 2.14 (Fluctuations of random variables with the method of cumulants). *Consider a sequence* $(S_n)_{n\in\mathbb{N}}$ *that satisfies the hypotheses of the method of cumulants, and* $Y_n = \frac{S_n}{\sqrt{\mathrm{var}(S_n)}}$.

(1) Central limit theorem: $Y_n \rightharpoonup \mathcal{N}_{\mathbb{R}}(0,1)$.

(2) Normality zones and moderate deviations [FMN16, Proposition 5.2.1]: if $y = o((\frac{N_n}{D_n})^{1/6})$*, then*

$$\mathbb{P}[Y_n \geq y] = \mathbb{P}[\mathcal{N}_{\mathbb{R}}(0,1) \geq y]\,(1+o(1)).$$

If $y = o((\frac{N_n}{D_n})^{1/4})$ *and* $y \gg 1$*, then*

$$\mathbb{P}[Y_n \geq y] = \frac{\mathrm{e}^{-\frac{y^2}{2}}}{y\sqrt{2\pi}} \exp\left(\frac{Ly^3}{6\sigma^3}\sqrt{\frac{D_n}{N_n}}\right)\,(1+o(1)).$$

(3) Speed of convergence [FMN17b, Corollary 30 and Remark 31]:

$$d_{\mathrm{Kol}}(Y_n, \mathcal{N}_{\mathbb{R}}(0,1)) \leq \frac{76.36\,A^3}{(\sigma_n)^3}\sqrt{\frac{D_n}{N_n}}.$$

(4) Local limit theorem: for any $\varepsilon \in (0, \frac{1}{2})$*, and any bounded Jordan measurable subset* B *with Lebesgue measure* $L(B)$*,*

$$\lim_{n\to\infty} \left(\frac{N_n}{D_n}\right)^{\varepsilon} \mathbb{P}\left[Y_n - y \in \left(\frac{D_n}{N_n}\right)^{\varepsilon} B\right] = \frac{L(B)}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{y^2}{2}}.$$

Remark 2.15. In the method of cumulants, if $\lim_{n\to\infty}\sigma_n = 0$, then the asymptotic normality and the bound on the speed of convergence still hold, assuming only that

$$t_n = (\sigma_n)^2\left(\frac{N_n}{D_n}\right)^{1/3} \to +\infty.$$

More generally, if there exists $\varepsilon \in (0,1)$ such that

$$(\sigma_n)^2\left(\frac{N_n}{D_n}\right)^{\varepsilon} \to +\infty,$$

then one still has the asymptotic normality $Y_n \rightharpoonup \mathcal{N}_{\mathbb{R}}(0,1)$ (without estimates of the speed of convergence when $\varepsilon > \frac{1}{3}$).

Remark 2.16. If $L_n = 0$, which happens for instance when S_n is a symmetric random variable, then one gets improved normality zones and moderate deviation estimates, depending on the first index $r \geq 3$ such that $\lim_{n\to\infty}\frac{\kappa^{(r)}(S_n)}{N_n(D_n)^{r-1}} \neq 0$.

In a moment, we shall see that given a sum $S_n = \sum_{i=1}^{N_n} A_{i,n}$ of random variables, the bound (MC3) of the method of cumulants can be proven under certain natural hypotheses on the dependency structure of the variables $A_{i,n}$. This is the reason why the method of cumulants is such a powerful tool in the theory of mod-Gaussian convergence. Before that, let us evoke the multi-dimensional generalisation of this method [FMN17c, Theorems 32 and 33].

Proposition 2.17 (Multi-dimensional method of cumulants). *Let $(\mathbf{S}_n)_{n\in\mathbb{N}}$ be a sequence of random vectors in $\mathbb{R}^d$. We say that $(\mathbf{S}_n)_{n\in\mathbb{N}}$ satisfies the hypotheses of the multi-dimensional method of cumulants with parameters (D_n, N_n, A) if (MC1) is satisfied, as well as:*

(MC2-d) The first cumulants of the coordinates of S_n satisfy:

$$\kappa(S_n^{(i)}) = 0;$$

$$\kappa(S_n^{(i)}, S_n^{(j)}) = K_{ij}\, N_n D_n \left(1 + O\left(\sqrt{\frac{D_n}{N_n}}\right)\right);$$

$$\kappa(S_n^{(i)}, S_n^{(j)}, S_n^{(k)}) = L_{n,(i,j,k)}\, N_n (D_n)^2$$

with K_{ij} positive-definite symmetric matrix, and $\lim_{n\to\infty} L_{n,(i,j,k)} = L_{i,j,k}$.

(MC3-d) The cumulants of the coordinates of S_n are bounded as follows:

$$\forall r \geq 1,\ |\kappa(S_n^{(i_1)}, \ldots, S_n^{(i_r)})| \leq A^r\, r^{r-2}\, N_n\, (2D_n)^{r-1}.$$

Then, $\mathbf{X}_n = \mathbf{S}_n/((N_n)^{1/3}(D_n)^{2/3})$ is mod-Gaussian convergent on $\mathbb{C}^d$, with parameters $(\frac{N_n}{D_n})^{1/3} K$. This implies in particular the convergence in law of $\mathbf{Y}_n = \frac{\mathbf{S}_n}{(N_n D_n)^{1/2}}$ to $\mathcal{N}_{\mathbb{R}^d}(\mathbf{0}, K)$, and more precisely,

$$d_{\text{convex}}(\mathbf{Y}_n\,,\, \mathcal{N}_{\mathbb{R}^d}(\mathbf{0}, K)) = O\left(\sqrt{\frac{D_n}{N_n}}\right).$$

One has also the following local limit theorem: assuming to simplify that $K = I_d$, for any $\varepsilon \in (0, \frac{1}{2})$, and any bounded Jordan measurable subset B with d-dimensional Lebesgue measure $L(B)$,

$$\lim_{n\to\infty} \left(\frac{N_n}{D_n}\right)^{d\varepsilon} \mathbb{P}\left[\mathbf{Y}_n - \mathbf{y} \in \left(\frac{D_n}{N_n}\right)^{\varepsilon} B\right] = \frac{\mathrm{e}^{-\frac{\|\mathbf{y}\|^2}{2}}}{(2\pi)^{d/2}}\, L(B).$$

Note that in this setting, the method of cumulants yields better estimates of speed of convergence than the general estimates from Theorem 1.44.

▷ **Sums of random variables with a sparse dependency graph.** We now explain how to obtain the bounds (MC3) that enable the method of cumulants. Consider a sum $S = \sum_{v\in V} A_v$ of bounded random variables, with $\|A_v\|_\infty \leq A$.

Definition 2.18 (Dependency graph). *A dependency graph for a family of random variables $\{A_v\}_{v\in V}$ is a graph $G = (V, E)$ such that, if $\{A_v\}_{v\in V_1}$ and $\{A_v\}_{v\in V_2}$ are two disjoint subsets of variables without edges $(v_1, v_2) \in E$ with $v_1 \in V_1$ and $v_2 \in V_2$, then these two families $\{A_v\}_{v\in V_1}$ and $\{A_v\}_{v\in V_2}$ are independent.*

The definition is better understood on an example. Suppose that one has a family of random variables with dependency graph drawn in Figure 2.3. The random vectors $(A_1, A_2, \ldots, A_5)$ and (A_6, A_7) are independent, because these vectors belong to two distinct connected components of G. However, the vectors (A_1, A_2, A_3) and A_5 are also independent, because there is no *direct* edge between the corresponding vertices of G.

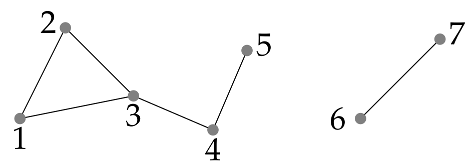


Figure 2.3. A dependency graph for a family of random variables.

The parameters of a dependency graph for a family of bounded random variables $(A_v)_{v\in V}$ are:

$$D = 1 + \max_{v\in V} \deg(v);$$
$$N = \mathrm{card}(V);$$
$$A = \sup_{v\in V} \|A_v\|_\infty.$$

The dependency graphs and the cumulants of random variables have been used to prove asymptotic normality in [PL83; Jan88; BR89a; BR89b; AB93; CS04; PY05; DE13]. In [FMN16, Theorem 9.1.7], we obtained the following strong improvement of these previous results:

Theorem 2.19 (Dependency graphs and bounds on cumulants). *Let $S = \sum_{v\in V} A_v$ be a sum of bounded random variables admitting a dependency graph G with parameters (D, N, A). Then, for any $r \geq 1$,*

$$|\kappa^{(r)}(S)| \leq A^r\, r^{r-2}\, N\, (2D)^{r-1}.$$

Corollary 2.20 (Sums of random variables with a sparse dependency graph). *Let $(S_n)_{n\in\mathbb{N}}$ be a sequence of sums $S_n = \sum_{i=1}^{N_n} A_{i,n}$ of random variables. We fix for each family $(A_{i,n})_{1\leq i\leq N_n}$ a dependency graph G_n with parameters (D_n, N_n, A). Then, the bound (MC3) of the method of cumulants holds true, so assuming that the graphs G_n are sparse ($\frac{D_n}{N_n} \to 0$) and that the first cumulants satisfy (MC2), we get all the results from Theorem 2.14.*

In particular, one gets asymptotic normality and an estimate of the speed of convergence for sums of random variables with a sparse dependency graph; these estimates were also obtained by Rinott, who used Stein's method [Rin94]. Before examining applications of Theorem 2.19, let us give a few details on its proof. The main idea is to develop by multilinearity

$$\kappa^{(r)}(S) = \sum_{v_1,\ldots,v_r\in V} \kappa(A_{v_1}, A_{v_2}, \ldots, A_{v_r}),$$

and to prove a bound on each joint cumulant $\kappa(A_{v_1}, A_{v_2}, \ldots, A_{v_r})$. We already know that if $G = (V, E)$ is a dependency graph for $\{A_v\}_{v\in V}$, then the sum can be restricted to families $\{A_{v_1}, \ldots, A_{v_r}\}$ such that the induced mulitgraph $H = G[v_1, v_2, \ldots, v_r]$ is connected (otherwise, the joint cumulant vanishes). Actually, one has the more precise bound:

$$|\kappa(A_{v_1}, \ldots, A_{v_r})| \leq 2^{r-1} A^r\, \mathrm{ST}_H,$$

where ST_H denotes the number of spanning trees of the multigraph H. The bound on $\kappa^{(r)}(S)$ follows by summing these inequalities, and using the fact that the number of Cayley trees on r vertices is r^{r-2}. On the other hand, the apparition of spanning trees brings light to certain combinatorial properties of the joint cumulants, and it will lead us to a generalisation of the notion of dependency graph, see Section 2.4.

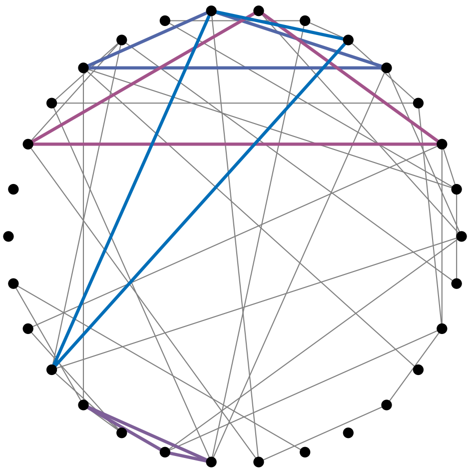


Figure 2.4. Count of triangles in a random Erdös–Rényi graph with parameters $n = 30$ and $p = 0.1$. Here, there are $|\hom(K_3, G(n,p))| = 4 \times 3! = 24$ ways to embed a triangle in the graph.

Example 2.21 (Subgraph counts in Erdös–Rényi random graphs). As a first application of Theorem 2.19, consider a random *Erdös–Rényi graph* $G(n,p)$, that is a random graph on n vertices with each edge $\{i,j\}$ that appears with probability p, independently of all the other edges [ER59; ER60]. We are interested in the random number of copies of a fixed graph H in $G = G(n,p)$. It is defined as the number $|\hom(H,G)|$ of maps $\psi : V_H \to V_G$ such that, if $\{h_1,h_2\} \in E_H$, then $\{\psi(h_1),\psi(h_2)\} \in E_G$ (one could also look at *injective* maps, with similar results). Set $S_n = |\hom(H,G(n,p))|$, H and $p \in (0,1)$ being fixed. The random variable S_n is a sum of dependent random Bernoulli variables:

$$S_n = \sum_{\psi:[\![1,k]\!]\to[\![1,n]\!]} \left(\prod_{\{i,j\}\in E_H} 1_{\{\psi(i),\psi(j)\}\in E_{G(n,p)}} \right)$$

where $k = \mathrm{card}(H)$, and V_H is identified with $[\![1,k]\!]$. The random variables $B_{\psi,n}$ indexed by maps $\psi : [\![1,k]\!] \to [\![1,n]\!]$ admit a dependency graph with parameters

$$D_n = 2\binom{k}{2}^2 n^{k-2} \quad ; \quad N_n = n^k \quad ; \quad A = 1,$$

Indeed, if ψ and ϕ are two maps that do not share at least two values $\psi(i) = \phi(j)$ and $\psi(k) = \phi(l)$, then the corresponding random variables $B_{\psi,n}$ and $B_{\phi,n}$ do not share any edge and therefore are independent. One can check that the other hypotheses of the method of cumulants are satisfied, with

$$\sigma^2 = \left(\frac{h}{\binom{k}{2}}\right)^2 p^{2h-1}(1-p) \quad ; \quad L = \left(\frac{h}{\binom{k}{2}}\right)^4 p^{3h-2}(1-p)\left(3(1-p) + \frac{p-2}{h}\right),$$

where k and h are the numbers of vertices and of edges of H. Therefore, if one considers the renormalised random variable

$$Y_n = \frac{|\hom(H,G(n,p))| - n^k p^h}{\sqrt{\mathrm{var}(|\hom(H,G(n,p))|)}},$$

then $Y_n \rightharpoonup \mathcal{N}_{\mathbb{R}}(0,1)$; this result already appeared in [Ruc88; JŁR00]. However, we also get without additional work

$$d_{\mathrm{Kol}}(Y_n, \mathcal{N}_{\mathbb{R}}(0,1)) = O\left(\frac{k^4(k-1)^4}{p^{3h}(p^{-1}-1)^{3/2}h^3}\,\frac{1}{n}\right)$$

or the moderate deviation estimate

$$\mathbb{P}[Y_n \geq n^{1/3}x] = \frac{\mathrm{e}^{-\frac{n^{2/3}x^2}{2}}}{n^{1/3}x\sqrt{2\pi}}\,\exp\left(\frac{x^3\,(3h(1-p)+p-2)}{6}\sqrt{\frac{2p}{1-p}}\right)\,(1+o(1)) \quad \text{for } x > 0.$$

2.3 Mod-Gaussian moduli spaces

In [FMN17b, Example 36] and [FMN17c, Section 4.3], we applied Theorem 2.19 to a model of random walks with positively correlated steps, in one or several dimensions. In this section, we present three models which have more structure, and which illustrate the following informal result: each time one tries to approximate a continuous object by a random discrete one, some observables of the corresponding random model are mod-Gaussian convergent and satisfy the hypotheses of the method of cumulants. The results of this section correspond to an ongoing work with Valentin Féray and Ashkan Nikeghbali [FMN17a].

▷ **The space of graphons, the space of permutons and the Thoma simplex.** We start by introducing three spaces $\mathscr{G}$, $\mathscr{P}$ and $\mathscr{T}$ which are parameter spaces for models of random graphs, of random permutations and of random integer partitions.

→ *Graphons.* Let us start with random graphs. The following definition is due to Lovász and Szegedy [LS06]; see also [Bor+06; LS07; Bor+08].

Definition 2.22 (Graphon). *A graph function is a measurable function $g : [0,1]^2 \to [0,1]$ that is symmetric: $g(x,y) = g(y,x)$ almost everywhere (with respect to the Lebesgue measure). A graphon is an equivalence class of graph functions for the relation*

$$g \sim h \quad \iff \quad \exists \sigma \text{ Lebesgue isomorphism of } [0,1] \text{ such that } h(x,y) = g(\sigma(x),\sigma(y)).$$

We denote $\mathscr{F}$ the space of graph functions, and $\mathscr{G} = \mathscr{F}/\sim$ the space of graph functions.

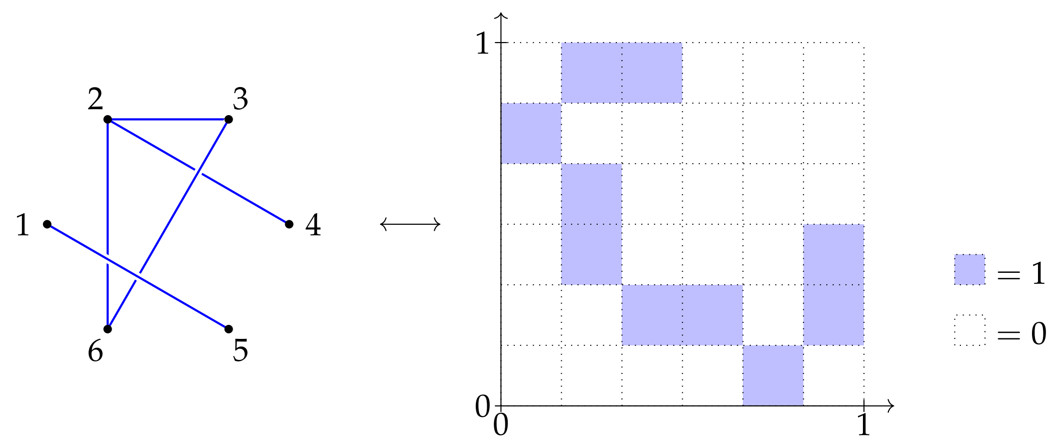


Figure 2.5. The graph function g_G associated to a graph G.

The space of graphons is a universal object for the parametrisation of models of dense random graphs. If $G = (V, E)$ is a simple graph on n vertices, identifying V with $[\![1, n]\!]$, one can associate to it a canonical graph function $g = g_G$: it is the function on the square that takes its values in $\{0, 1\}$, and is such that

$$g(x,y) = 1 \text{ if } x \in \left(\frac{i-1}{n}, \frac{i}{n}\right],\ y \in \left(\frac{j-1}{n}, \frac{j}{n}\right] \text{ and } \{i,j\} \in E_G,$$

and 0 otherwise. The function g_G is essentially the adjacency matrix of G, drawn as a function on the square $[0,1]^2$. Therefore, any graph G yields a canonical graphon $\gamma_G = [g_G]$. Conversely, starting from a graphon γ, one can produce for any n a random graph $G(n, \gamma)$:

(1) one picks at random n independent uniform random variables $X_1, \ldots, X_n$ in $[0, 1]$;

(2) if $\gamma = [g]$, then one connects i to j in $G(n, \gamma)$ according to a Bernoulli random variable B_{ij} with parameter $g(X_i, X_j)$, these random variables being independent conditionally to $X_1, \ldots, X_n$ (in practice, one can define $B_{ij} = 1_{U_{ij} \leq g(X_i, X_j)}$, where $(U_{i,j})_{1 \leq i < j \leq n}$ is a new family of independent uniform random variables in $[0, 1]$).

The law of $G(n, \gamma)$ does not depend on the choice of a representative g of the equivalence class γ. We refer to Figure 2.6 for two examples of random graphs associated to two graphons, in size $n = 20$. When $\gamma = p$ is a constant function, one recovers the Erdös–Rényi random graphs from Example 2.21. We shall see in this section that the random graphs $G(n, \gamma)$ are associated to random graphons $\gamma_{G(n,\gamma)} = \Gamma(n, \gamma)$ that converge back to γ (in a sense to be precised later), and that the fluctuations of these models are mod-Gaussian.

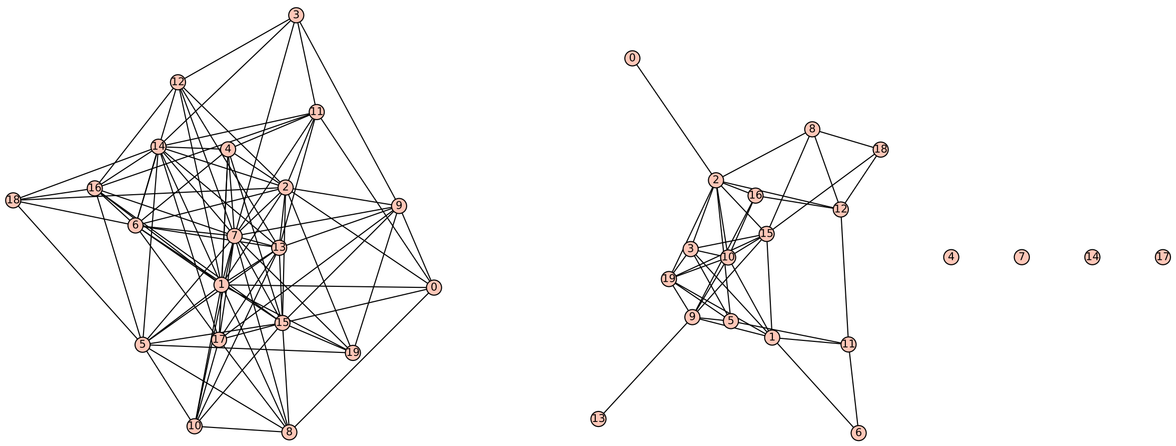


Figure 2.6. Two random graphs of size $n = 20$ associated to the graph functions $g(x, y) = \frac{x+y}{2}$ and $g'(x, y) = xy$.

$\to$ *Permutons.* A construction similar to the space of graphons allows one to study models of random permutations [Hop+11; Hop+13].

Definition 2.23 (Permuton). *A permuton is a Borel probability measure π on the square $[0, 1]^2$ whose marginale laws are uniform:*

$$(p_1)_* \pi = (p_2)_* \pi = \text{ uniform measure on } [0, 1],$$

where $p_1(x, y) = x$, $p_2(x, y) = y$, and $(p_1)_*, (p_2)_* : \mathscr{M}^1([0, 1]^2) \to \mathscr{M}^1([0, 1])$ are the corresponding maps between sets of probability measures. We denote $\mathscr{P}$ the space of permutons.

Again, the space of permutons is a universal object for the parametrisation of models of random permutations. Denote $\mathfrak{S}(n)$ the symmetric group of order n, and consider a permutation $\sigma \in \mathfrak{S}(n)$. One can associate to it a canonical permuton π_σ, which is the probability measure on $[0,1]$ with density

$$\pi_\sigma(dx\,dy) = n\,1_{\sigma(\lceil nx\rceil)=\lceil ny\rceil}\;dx\,dy.$$

We refer to Figure 2.7 for an example; the permuton π_σ is essentially the graph of the permutation σ, drawn inside the square $[0,1]^2$.

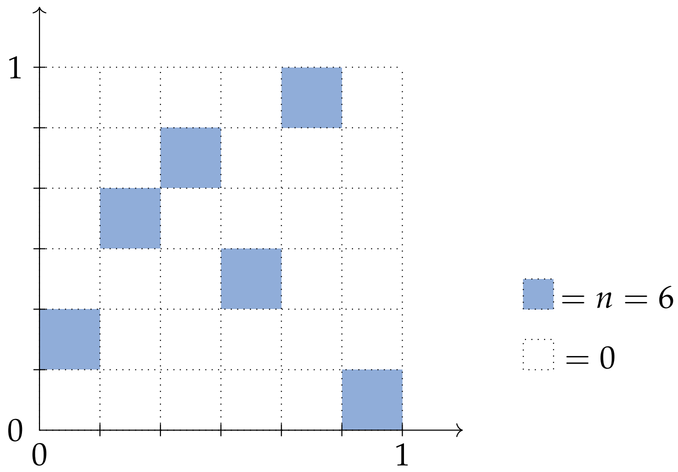


Figure 2.7. The density of the permuton π_σ associated to the permutation $\sigma = 245361$.

Conversely, starting from a permuton π, one can produce for any $n \in \mathbb{N}$ a random permutation $\sigma(n,\pi)$. If $(x_1,y_1),\ldots,(x_n,y_n)$ is a family of points in the square $[0,1]^2$, we say that these points are in a general configuration if all the x_i's are distinct, and if all the y_i's are also distinct. To a general family of k points, we can associate a unique permutation $\sigma \in \mathfrak{S}(n)$ with the following property: if $\psi_1 : \{x_1,\ldots,x_n\} \to [\![1,n]\!]$ and $\psi_2 : \{y_1,\ldots,y_n\} \to [\![1,n]\!]$ are increasing bijections, then

$$\sigma(\psi_1(x_i)) = \psi_2(y_i)$$

for any $i \in [\![1,n]\!]$. We then say that σ is the *configuration* of the set of points; and we denote $\sigma = \mathrm{conf}((x_1,y_1),\ldots,(x_n,y_n))$. What this means is that the graph of σ is the same as the graph of the points $(x_1,y_1),\ldots,(x_n,y_n)$, up to translations of these points that do not change their order (horizontally or vertically). Given a permuton π, if $(X_1,Y_1),\ldots,(X_n,Y_n)$ are random points of $[0,1]^2$ chosen independently and according to the probability measure π, then they are with probability 1 in a general configuration, which allows one to define

$$\sigma(n,\pi) = \mathrm{conf}((X_1,Y_1),\ldots,(X_n,Y_n)).$$

When π is the uniform measure on $[0,1]^2$, one simply obtains the uniform random permutations in $\mathfrak{S}(n)$. We shall see that the random permutations $\sigma(n,\pi)$ are associated to random permutons $\pi_{\sigma(n,\pi)} = \Pi(n,\pi)$ that converge back to π in the sense of weak convergence of probability measures; and that the fluctuations of these models are mod-Gaussian.

$\to$ *Partitions.* Finally, there is a third parameter space that allows one to construct random integer partitions:

Definition 2.24 (Thoma simplex). *The Thoma simplex is the set $\mathscr{T}$ of pairs $\omega = (\alpha, \beta)$ of infinite non-negative and non-increasing sequences*

$$\alpha = (\alpha_1 \geq \alpha_2 \geq \cdots \geq 0)\,,\ \beta = (\beta_1 \geq \beta_2 \geq \cdots \geq 0)$$

that satisfy

$$\sum_{i=1}^{\infty} \alpha_i + \sum_{i=1}^{\infty} \beta_i = 1 - \gamma \leq 1.$$

Recall that an integer partition of size n is a sequence $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r)$ of non-negative integers with $|\lambda| = \sum_{i=1}^{r} \lambda_i = n$; the set of integer partitions of size n is denoted $\mathfrak{Y}(n)$. Given such a partition, its *Young diagram* is the diagram with n boxes and λ_1 boxes on the first row, λ_2 boxes on the second row, *etc.* The *Frobenius coordinates* $A(\lambda)$ and $B(\lambda)$ of a

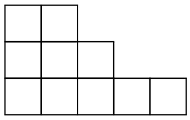


Figure 2.8. The Young diagram of the integer partition $(5, 3, 2)$ of size 10.

partition λ are then defined as follows. Denote $d \geq 0$ the number of boxes of the diagram that are on the principal diagonal. We set $A(\lambda) = (a_1, a_2, \ldots, a_d)$ and $B(\lambda) = (b_1, b_2, \ldots, b_d)$, where $a_i - \frac{1}{2}$ is the number of boxes on the right of the i-th diagonal box, and $b_i - \frac{1}{2}$ is the number of boxes on top of the i-th diagonal box. Equivalently, the a_i's and the b_i's can be described as the lengths of the segments connecting the center of the diagonal boxes to the borders of the Young diagram; see Figure 2.9. With the previous example $\lambda = (5, 3, 2)$, we

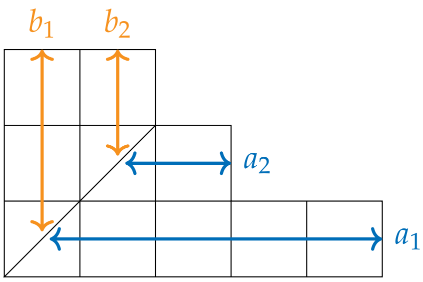


Figure 2.9. Frobenius coordinates of the Young diagram $\lambda = (5, 3, 2)$.

obtain $A = (\frac{9}{2}, \frac{3}{2})$ and $B = (\frac{5}{2}, \frac{3}{2})$. Note that the sum of the Frobenius coordinates is equal to the size of λ. The Frobenius coordinates enable us to associate to any integer partition λ an element ω_λ of the Thoma simplex:

$$\omega_\lambda = \left(\left(\frac{a_1}{n}, \frac{a_2}{n}, \ldots, \frac{a_d}{n}, 0, \ldots \right), \left(\frac{b_1}{n}, \frac{b_2}{n}, \ldots, \frac{b_d}{n}, 0, \ldots \right) \right),$$

where $n = |\lambda|$. The parameter ω_λ encodes the row and column frequencies of the Young diagram of λ.

As for graphons and permutons, there is conversely a way to associate to any parameter $\omega \in \mathscr{T}$ a random integer partition $\lambda(n, \omega) \in \mathfrak{Y}(n)$. However, it is much more complicated than before, since it relies on the combinatorics of the RSK algorithm (see for instance [Ful97, Chapter 4] and [Mél17, Section 3.2]) and of the operation of shuffling. First, let us associate to ω a random permutation $\sigma(n, \omega) \in \mathfrak{S}(n)$; this construction appeared first in [Ful02], and it was reinterpreted in the setting of the Hopf algebra of free quasi-symmetric functions in [Mél12] and [Mél17, Chapter 12].

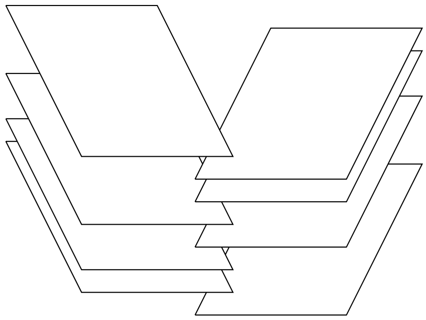


Figure 2.10. Shuffling two blocks of cards.

The parameter ω being fixed, we start from a deck of ordered cards $12\ldots n$, and we perform the following operations:

(GRS1) We cut the deck in blocks of sizes $a_1 + a_2 + \cdots + b_1 + b_2 + \cdots + c = n$ chosen randomly according to the probability measure

$$\mathbb{P}[a,b,c] = \frac{n!}{(\prod_{i=1}^{\infty} a_i!)\,(\prod_{i=1}^{\infty} b_i!)\,c!} \prod_{i=1}^{\infty} (\alpha_i)^{a_i} \prod_{i=1}^{\infty} (\beta_i)^{b_i}\,\gamma^c.$$

The blocks of sizes $a_1, a_2, \ldots$ will be called type A, the blocks of sizes $b_1, b_2, \ldots$ will be called type B, and the last block of size c will be called type C.

(GRS2) We reverse each block of cards $(k+1)(k+2)\ldots(k+b_i)$ of type B in order to obtain a block $(k+b_i)\ldots(k+2)(k+1)$, and we randomize the last block of type C, replacing $(n-c+1)(n-c+2)\ldots n$ by a uniform random permutation of these c letters.

(GRS3) We shuffle randomly all the blocks, the random shuffle of words $w_1, \ldots, w_r$ of lengths $\ell_1, \ldots, \ell_r$ being the random word of length $\ell_1 + \cdots + \ell_r$ that is chosen uniformly among the $\binom{\ell_1+\ell_2+\cdots+\ell_r}{\ell_1,\ell_2,\ldots,\ell_r}$ words w that can be decomposed into disjoint subwords $w_1, \ldots, w_r$.

This operation is called *generalised riffle shuffle*, and the arrangement of the cards in the resulting deck is a random permutation $\sigma(n,\omega) \in \mathfrak{S}(n)$. The shuffle operation is really intuitive in terms of decks of cards, see Figure 2.10.

Example 2.25 (Generalised riffle shuffle). Set $n = 15$ and $\omega = ((\frac{1}{6}, \frac{1}{4}, 0, \ldots), (\frac{1}{3}, 0, \ldots))$. We start from the deck of cards

123456789ABCDEF.

(1) The probability measure on sequences (a,b,c) that is associated to the parameter ω is supported by sequences $((a_1,a_2),b_1,c)$, all the other numbers $a_{i\geq 3}$ and $b_{i\geq 2}$ being equal to 0. One possibility is $a_1 = 3$, $a_2 = 3$, $b_1 = 6$, and $c = 3$. We then cut the deck into the four blocks

123 | 456 | 789ABC | DEF.

(2) For the second step, we reverse the third block which is of type B, and we randomize the last block, one possibility being FDE. Thus, we get

123 | 456 | CBA987 | FDE.

(3) Finally, we shuffle back together these four blocks, and one possibility is

$$4CB1A25F9387D6E.$$

Thus, one possible result of the generalized riffle shuffle with parameter ω and with $n = 15$ is the permutation $\sigma(n, \omega) = 4CB1A25F9387D6E$.

A standard tableau of shape $\lambda \in \mathfrak{Y}(n)$ is a numbering of the cells of the Young diagram of λ by the numbers $1, 2, \ldots, n$, such that the rows and the columns are strictly increasing. For instance,

7	10			
2	5	8		
1	3	4	6	9

is a Young tableau of shape $(5, 3, 2)$. If $\sigma \in \mathfrak{S}(n)$ is a permutation, one can associate to it two standard tableaux $P(\sigma)$ and $Q(\sigma)$ with the same shape $\lambda(\sigma) \in \mathfrak{Y}(n)$. This operation is bijective and known as the Robinson–Schensted–Knuth correspondence; see [Rob38; Sch61] or the aforementioned chapter of the book by Fulton [Ful97]. The most important property of $\lambda = \lambda(\sigma)$ is that it encodes the length of the longest increasing and decreasing subwords of σ: for any $r \geq 1$

$$\lambda_1 + \cdots + \lambda_r = \max\{\ell(w_1) + \cdots + \ell(w_r),\ w_1, \ldots, w_r \text{ disjoint increasing subwords of } \sigma\},$$

and

$$\lambda'_1 + \cdots + \lambda'_r = \max\{\ell(w_1) + \cdots + \ell(w_r),\ w_1, \ldots, w_r \text{ disjoint decreasing subwords of } \sigma\},$$

where λ'_i denotes the length of the i-th column of the Young diagram of λ. Using the RSK algorithm, we can thus associate to any parameter $\omega \in \mathscr{T}$ a random integer partition $\lambda(n, \omega) = \lambda(\sigma(n, \omega))$.

Proposition 2.26 (Fulman, Méliot). *The law of $\lambda(n, \omega)$ is given by*

$$\mathbb{P}[\lambda(n, \omega)] = (\dim \lambda)\, \mathfrak{s}_\lambda(\omega),$$

where:

- $\dim \lambda$ *is the number of standard tableaux of shape λ, and is equal to the dimension of the irreducible Specht representation S^λ of $\mathfrak{S}(n)$ with Young diagram λ.*
- $\mathfrak{s}_\lambda$ *is the Schur function of label λ, that is the symmetric function in* Sym *defined as the projective limit of the symmetric polynomials*

$$\mathfrak{s}_\lambda(x_1, \ldots, x_n) = \frac{\det((x_i^{\lambda_j + n - j})_{1 \leq i,j \leq n})}{\det((x_i^{n-j})_{1 \leq i,j \leq n})}.$$

- $\mathfrak{s}_\lambda(\omega)$ *is the specialisation of the function $\mathfrak{s}_\lambda \in$* Sym *obtained from:*

$$\mathfrak{p}_1(\omega) = 1 \qquad ; \qquad \mathfrak{p}_{k \geq 2}(\omega) = \sum_{i=1}^{\infty} (\alpha_i)^k + (-1)^{k-1} \sum_{i=1}^{\infty} (\beta_i)^k.$$

In particular, the law of $\lambda(n, \omega)$ is equal to the central measure on partitions associated to the extremal character of label ω of the infinite symmetric group $\mathfrak{S}(\infty)$ [Tho64; KV81; KV86].

Remark 2.27. One can also give a formula for the law of $\sigma(n, \omega)$, by using specialisations of the Hopf algebra QSym of quasi-symmetric functions; see [Mél17, Theorem 12.17].

Remark 2.28. If $\omega = ((0,0,\ldots),(0,0,\ldots))$, then $\sigma(n,\omega)$ is the uniform random permutation in $\mathfrak{S}(n)$, and $\lambda(n,\omega)$ is the random integer partition under the well-known *Plancherel measure* $\mathbb{P}_n$ of $\mathfrak{S}(n)$, whose asymptotics have been studied for instance in [LS77; KV77]. These probability measures are connected to the problem of the longest increasing subsequence [Ula61; Ham72; AD95] and to random matrix theory [BDJ99; BOO00; Oko00; Joh01].

If we consider the random parameter $\omega_{\lambda(n,\omega)} = \Omega(n,\omega)$, then it has been proved by Kerov and Vershik that for any fixed parameter ω, $\Omega(n,\omega) \to \omega$ in probability, the convergence in the Thoma simplex $\mathscr{T}$ being coordinate-wise. Again, we are going to explain that this convergence is mod-Gaussian.

▷ **Topology and observables of the mod-Gaussian moduli spaces.** We have now three spaces of parameters (or *moduli spaces*) $\mathscr{G}$, $\mathscr{P}$ and $\mathscr{T}$ which correspond to models of random graphs, random permutations and random integer partitions. Moreover, these combinatorial objects can be considered as elements of the spaces of parameters. To study the mod-Gaussian convergence of these models, we need to introduce observables in all these cases. These functionals will actually control the topology of the moduli spaces.

→ *Graphons.* If g is a graph function and $F = (V_F, E_F)$ is a finite graph, the *density* of F in g is the real number

$$t(F,g) = \int_{\mathbb{R}^k} \left(\prod_{\{i,j\}\in E_F} g(x_i,x_j) \right) dx_1\,dx_2\,\cdots\,dx_k$$

where k is the number of vertices of F, and V_F is identified with $[\![1,k]\!]$. The density of F in g is invariant by Lebesgue isomorphisms of $[0,1]$, so $t(F,\cdot)$ is well-defined on the space of graphons $\mathscr{G}$. On the other hand, if $g = g_G$ is the graph function associated to a graph G, then

$$t(F,g_G) = t(F,G) = \frac{|\,\mathrm{hom}(F,G)|}{|V_G|^{|V_F|}},$$

where as in Example 2.21, $\mathrm{hom}(F,G)$ denotes the set of morphisms from F to G.

We endow the space $\mathscr{F}$ with the cut-metric $d(g_1,g_2) = \inf_\sigma \|(g_1)^\sigma - g_2\|_\Box$, where the infimum runs over Lebesgue isomorphisms $\sigma : [0,1] \to [0,1]$, and where

$$\|g\|_\Box = \sup_{S,T \text{ measurable subsets of } [0,1]} \left| \int_{S\times T} g(x,y)\,dx\,dy \right|.$$

The cut metric $d(g_1,g_2)$ depends only on the equivalence classes $[g_1]$ and $[g_2]$ in $\mathscr{G}$, and it yields a map $\delta : \mathscr{G} \times \mathscr{G} \to \mathbb{R}_+$ which makes $\mathscr{G}$ into a compact metric space [Bor+08, Proposition 3.6]. Let $\mathscr{O}_G$ be the algebra of finite graphs, defined over $\mathbb{R}$ and with product $F_1 \times F_2 = F_1 \sqcup F_2$ if F_1, F_2 are two graphs. We evaluate a formal sum of graphs $f = \sum_F c_F\, F$ on a graphon by the rule

$$f(\gamma) = \sum_F c_F\, t(F,\gamma).$$

In [Bor+08, Theorem 3.8], it is shown that this rule yields a map $\mathscr{O}_G \to \mathscr{C}(\mathscr{G})$ whose image is a dense subalgebra of the algebra of continuous functions. Hence, a sequence of graphons $(\gamma_n)_{n\in\mathbb{N}}$ converges in $\mathscr{G}$ if and only if all the observables $f(\gamma_n)$ with $f \in \mathscr{O}_G$ converge. Moreover, for any graphon γ and any graph F, one can show that

$$\mathbb{E}[t(F,\Gamma(n,\gamma))] = t(F,\gamma) \qquad ; \qquad \mathrm{var}(t(F,\Gamma(n,\gamma))) = O\left(\frac{1}{n}\right).$$

Therefore, $(\Gamma(n,\gamma))_{n\in\mathbb{N}}$ converges in probability to γ.

$\to$ *Permutons.* A similar approach can be followed with permutons. If π is a permuton and τ is a permutation of size k, the density of τ in π is defined by

$$t(\tau,\pi) = \int_{\mathbb{R}^{2k}} 1_{\mathrm{conf}((x_1,y_1),\ldots,(x_k,y_k))=\tau}\, d\pi^{\otimes k}(x_1,y_1,\ldots,x_k,y_k).$$

If $\tau \in \mathfrak{S}(k)$ and $\pi = \pi_\sigma$ is the permuton associated to a permutation $\sigma \in \mathfrak{S}(n)$, then one can relate $t(\tau,\pi_\sigma)$ to the density

$$t(\tau,\sigma) = \frac{\mathrm{card}\{S \subset [\![1,n]\!] \mid |S| = k \text{ and } \mathrm{conf}(\{(i,\sigma(i)),\ i \in S\}) = \tau\}}{\binom{n}{k}}$$

of the motive τ in σ. Hence,

$$|t(\tau,\pi_\sigma) - t(\tau,\sigma)| \leq \binom{k}{2}\frac{1}{n},$$

see [Hop+13, Lemma 3.5]. Therefore, the density of τ in permutons and in permutations is almost the same notion.

Let $\mathscr{O}_P$ be the algebra of all permutations, which is defined over $\mathbb{R}$ and endowed with the following product. If $\tau_1 \in \mathfrak{S}(n)$ and $\tau_2 \in \mathfrak{S}(m)$, then

$$\tau_1 \times \tau_2 = \frac{n!\,m!}{(n+m)!}\sum_\rho \rho,$$

where the sum runs over the multiset of $(\frac{(n+m)!}{n!\,m!})^2$ permutations $\rho \in \mathfrak{S}(n+m)$ whose graphs $\{(i,\rho(i)) \mid i \in [\![1,n+m]\!]\}$ can be split into two disjoint subsets $S_1 = \{(i,\rho(i)) \mid i \in I_1\}$ and $S_2 = \{(i,\rho(i)) \mid i \in I_2\}$ of size n and m and with respective configurations τ_1 and τ_2. Here we count the permutations ρ according to the sets S_1 and S_2, so a permutation ρ can be counted several times. This *graphical shuffle product* of permutations is compatible with the densities: if $\tau_1 \times \tau_2 = \frac{n!\,m!}{(n+m)!}\sum_{\rho\in R} \rho$, then for any permuton $\pi \in \mathscr{P}$,

$$t(\tau_1,\pi)\,t(\tau_2,\pi) = \frac{n!\,m!}{(n+m)!}\sum_{\rho\in R} t(\rho,\pi).$$

Therefore, one has a natural map $\mathscr{O}_P \to \mathscr{C}(\mathscr{P})$, the space of permutons $\mathscr{P} \subset \mathscr{M}^1([0,1]^2)$ being endowed with the topology of weak convergence of probability measures. Again, one can show that the image of this map is a dense subalgebra of the algebra of continuous functions, see [Hop+13]. Therefore, a sequence of permutons $(\pi_n)_{n\in\mathbb{N}}$ converges in $\mathscr{P}$ if and only if all the observables $f(\pi_n)$ with $f \in \mathscr{O}_P$ converge. Moreover, for any permuton π and any permutation τ, one can show that

$$\mathbb{E}[t(\tau,\sigma_n(\pi))] = t(\tau,\pi) \qquad ; \qquad \mathrm{var}(t(\tau,\sigma_n(\pi))) = O\left(\frac{1}{n}\right).$$

Therefore, $(\Pi(n,\pi))_{n\in\mathbb{N}}$ converges in probability to π.

$\to$ *Partitions.* If $\omega = (\alpha,\beta)$ is a parameter of the Thoma simplex, we associate to it the probability measure

$$\pi_\omega = \sum_{i=1}^\infty \alpha_i\,\delta_{\alpha_i} + \sum_{i=1}^\infty \beta_i\,\delta_{-\beta_i} + \gamma\,\delta_0,$$

where $\gamma = 1 - \sum_{i=1}^\infty \alpha_i - \sum_{i=1}^\infty \beta_i$. This rule enables us to embed $\mathscr{T}$ into the space of probability measures $\mathscr{M}^1([-1,1])$, and $\mathscr{T}$ is a compact metrisable subset, where convergence in law is equivalent to the pointwise convergence of all the coordinates α_i and β_i. For $k \geq 1$, we define the observable

$$p_k(\omega) = \int_{-1}^{1} t^{k-1}\, \pi_\omega(dt) = \begin{cases} \sum_{i=1}^\infty (\alpha_i)^k + (-1)^{k-1} \sum_{i=1}^\infty (\beta_i)^k & \text{if } k \geq 2, \\ 1 & \text{if } k = 1. \end{cases}$$

If λ is an integer partition and ω_λ is the associated parameter of the Thoma simplex, then $p_k(\omega(\lambda))$ is a renormalisation of a power sum of the Frobenius coordinates of λ:

$$p_k(\omega_\lambda) = \frac{1}{|\lambda|^k} \left(\sum_{i=1}^d (a_i)^k + (-1)^{k-1} \sum_{i=1}^d (b_i)^k \right).$$

One can show that if $n = |\lambda|$ and if $\chi^\lambda(k)$ is the value on a cycle of length k of the normalised irreducible character $\chi^\lambda = \frac{\operatorname{tr} \rho^\lambda(\cdot)}{\operatorname{tr} \rho^\lambda(1)}$ of the Specht representation S^λ of $\mathfrak{S}(n)$, then

$$p_k(\omega_\lambda) = \chi^\lambda(k) + O\left(\frac{1}{n}\right).$$

Thus, the functions p_k belong to a large algebra of observables which relate the geometry of the integer partitions λ to the algebraic properties of the corresponding irreducible representations S^λ. We refer to [KO94; Ker98; IK99; IO02] and [Mél17, Chapter 7] for details on this essential tool of the asymptotic representation theory of the symmetric groups; it played a major role in the papers written during the Ph.D. thesis of the author [Mél10; Mél11; Mél12; FM12; Mél14a].

The algebra of partitions $\mathscr{O}_T$ is the algebra defined over $\mathbb{R}$, with combinatorial basis formed by the integer partitions of arbitrary size, and with product $\mu \times \nu = \mu \sqcup \nu$. The rule

$$t(\mu, \omega) = \prod_{i=1}^{\ell(\mu)} p_{\mu_i}(\omega)$$

yields a natural map $\mathscr{O}_T \to \mathscr{C}(\mathscr{T})$ which generates a dense subalgebra of the algebra of continuous functions. Therefore, a sequence of parameters $(\omega_n)_{n\in\mathbb{N}}$ converges in $\mathscr{T}$ if and only if all the observables $f(\omega_n)$ with $f \in \mathscr{O}_T$ converge. For any integer partition μ and any parameter $\omega \in \mathscr{T}$, if $n \geq |\mu|$, then

$$\mathbb{E}[\chi^{\lambda_n(\omega)}(\mu)] = t(\mu, \omega) \qquad ; \qquad \operatorname{var}(\chi^{\lambda_n(\omega)}(\mu)) = O\left(\frac{1}{n}\right).$$

Therefore, $(\Omega(n, \omega))_{n\in\mathbb{N}}$ converges in probability to ω; this result was first obtained by Kerov and Vershik in [KV81].

▷ **Mod-Gaussian convergence for graphons, permutons and Thoma parameters.** We can now state a result of mod-Gaussian convergence for the random models $(\Gamma(n, \gamma))_{n\in\mathbb{N}}$, $(\Pi(n, \pi))_{n\in\mathbb{N}}$ and $(\Omega(n, \omega))$; they correspond to [FMN17a, Theorems 21, 26 and 29].

Theorem 2.29 (Féray–Méliot–Nikeghbali, 2017). *There exists two linear maps*

$$\kappa_{2,G} : \mathscr{O}_G \otimes \mathscr{O}_G \to \mathscr{O}_G \quad \textit{and} \quad \kappa_{3,G} : \mathscr{O}_G \otimes \mathscr{O}_G \otimes \mathscr{O}_G \to \mathscr{O}_G$$

such that, if $f \in \mathscr{O}_G$ is an observable of graphons of degree k, $\gamma \in \mathscr{G}$ is a graphon, and $S_n(f, \gamma) = n^k\,(f(\Gamma(n, \gamma)) - f(\gamma))$, then $(S_n(f, \gamma))_{n\in\mathbb{N}}$ satisfies the hypotheses of the method of cumulants with parameters $D_n = k^2\, n^{k-1}$, $N_n = n^k$, $A = \|f\|_{\mathscr{O}_G}$ and

$$\sigma^2 = \kappa_{2,G}(f, f)(\gamma) \qquad ; \qquad L = \kappa_{3,G}(f, f, f)(\gamma).$$

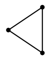
Therefore, if $\kappa_{2,G}(f,f)(\gamma) \neq 0$, then $Y_n(f,\gamma) = S_n(f,\gamma)/\sqrt{\mathrm{var}(S_n(f,\gamma))}$ satisfies all the limiting results listed in Theorem 2.14.

Here, by observable of degree k we mean a linear combination $f = \sum_F c_F\, F$ of graphs with k vertices, and the norm $\|f\|_{\mathscr{O}_G}$ is then $\sum_F |c_F|$. On the other hand, the linear maps κ_2 and κ_3 that enable the computation of the limiting variances and third cumulants are related to the operation of junction of graphs. For instance,

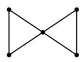
$$\kappa_{2,G}(F_1,F_2) = \frac{1}{k^2} \sum_{1\leq a,b\leq k} (F_1 \bowtie F_2)(a,b) - F_1 \times F_2,$$

where $(F_1 \bowtie F_2)(a,b)$ is the graph on $2k-1$ vertices obtained by identifying the vertex a in F_1 with the vertex b in F_2. A similar but more complicated description can be given for $\kappa_{3,G}(F_1,F_2,F_3)$.

Example 2.30 (Number of triangles in a graphon model). We fix a graphon γ, and we consider the subgraph counts in $G(n,\gamma)$ of the triangle

$$F = K_3 =$$  .

Let

$$H =$$  ;

this is the only isomorphism class of graphs that can be written as $(F \bowtie F)(a,b)$. If $t(H,\gamma) \neq (t(K_3,\gamma))^2$, then

$$Y_n = \frac{t(K_3,\Gamma(n,\gamma)) - t(K_3,\gamma)}{\sqrt{\mathrm{var}(t(K_3,\Gamma(n,\gamma)))}} = \sqrt{n}\,\frac{t(K_3,\Gamma(n,\gamma)) - t(K_3,\gamma)}{3\sqrt{t(H,\gamma)-(t(K_3,\gamma))^2}}\left(1+O\left(\frac{1}{n}\right)\right)$$

converges to a standard Gaussian distribution $\mathcal{N}_{\mathbb{R}}(0,1)$. Moreover, $d_{\mathrm{Kol}}(Y_n,\mathcal{N}_{\mathbb{R}}(0,1)) = O(\frac{1}{\sqrt{n}})$, and the Gaussian approximation of Y_n can be used between the scales $n^{-1/2}$ and $n^{1/6}$.

We also have the mod-Gaussian convergence of observables for the permuton models and the models of random integer partitions:

Theorem 2.31 (Féray–Méliot–Nikeghbali, 2017). *There exists two linear maps*

$$\kappa_{2,P} : \mathscr{O}_P \otimes \mathscr{O}_P \to \mathscr{O}_P \quad \text{and} \quad \kappa_{3,P} : \mathscr{O}_P \otimes \mathscr{O}_P \otimes \mathscr{O}_P \to \mathscr{O}_P$$

such that, if $f \in \mathscr{O}_P$ is an observable of permutons of degree k, $\pi \in \mathscr{P}$ is a permuton, and $S_n(f,\pi) = n^k\,(f(\Pi(n,\pi)) - \mathbb{E}[f(\Pi(n,\pi))])$, then $(S_n(f,\pi))_{n\in\mathbb{N}}$ satisfies the hypotheses of the method of cumulants with parameters $D_n = k^2\,n^{k-1}$, $N_n = n^k$, $A = \|f\|_{\mathscr{O}_P}$ and

$$\sigma^2 = \kappa_{2,P}(f,f)(\pi) \qquad ; \qquad L = \kappa_{3,P}(f,f,f)(\pi).$$

Similarly, there exists two linear maps

$$\kappa_{2,T} : \mathscr{O}_T \otimes \mathscr{O}_T \to \mathscr{O}_T \quad \text{and} \quad \kappa_{3,T} : \mathscr{O}_T \otimes \mathscr{O}_T \otimes \mathscr{O}_T \to \mathscr{O}_T$$

such that, if $f \in \mathscr{O}_T$ is an observable of degree k, $\omega \in \mathscr{T}$ is a Thoma parameter, and $S_n(f,\omega) = n^k\,(f(\Omega(n,\omega)) - \mathbb{E}[f(\Omega(n,\omega))])$, then $(S_n(f,\omega))_{n\in\mathbb{N}}$ satisfies the hypotheses of the method of cumulants with parameters $D_n = k^2\,n^{k-1}$, $N_n = n^k$, $A = \|f\|_{\mathscr{O}_T}$ and

$$\sigma^2 = \kappa_{2,T}(f,f)(\omega) \qquad ; \qquad L = \kappa_{3,T}(f,f,f)(\omega).$$

The linear maps $\kappa_{i,P}$ and $\kappa_{i,T}$ can be described by means of the operations of junction of two or three permutations, or of junction of two or three integer partitions; these combinatorial operations are quite similar to those defined for graphs. On the other hand, the proofs of Theorems 2.29 and 2.31 rely on Theorem 2.19 and on the construction of adequate dependency graphs for the random variables considered.

Remark 2.32. In the same setting, one can show that:

(1) For any permuton $\pi \in \mathscr{P}$ and any permutation τ of size k, the random variables $\binom{n}{k}\, t(\tau, \sigma_n(\pi))$ satisfy the hypotheses of the method of cumulants with parameters $D_n = k\binom{n}{k-1}$, $N_n = \binom{n}{k}$ and $A = 1$.

(2) For any Thoma parameter $\omega \in \mathscr{T}$ and any integer partition μ of size k, the random variables $n^{\downarrow k} \chi^{\lambda_n(\omega)}(\mu)$ satisfy the hypotheses of the method of cumulants with parameters $D_n = k^2 n^{\downarrow k-1}$, $N_n = n^{\downarrow k}$ and $A = 1$. Here, $n^{\downarrow k} = n(n-1)(n-2)\cdots(n-k+1)$.

Hence, the motive subcounts in random permutations, and the random character values are also generically mod-Gaussian convergent (as soon as the limiting variance does not vanish).

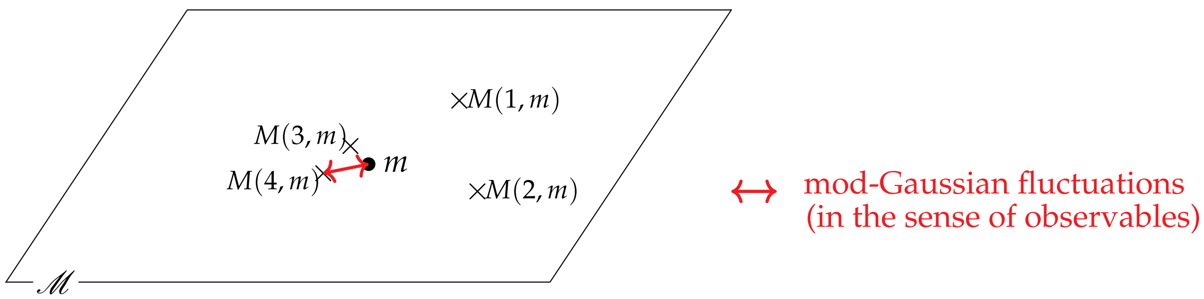


Figure 2.11. Mod-Gaussian moduli spaces as fields of fluctuations.

We have thus shown that generically, an observable

- of a random graph in a graphon model,
- or of a random permutation in a permuton model,
- or of a random integer partition under a central measure

is mod-Gaussian. The last case was essentially dealt with in [FMN16, Chapter 11], by adapting the theory of dependency graphs to a setting of variables in a non-commutative probability space. We call these classes of models *mod-Gaussian moduli spaces*. Each time, we have:

- a compact topological space $\mathscr{M}$;
- a graded algebra $\mathscr{O}_M$ endowed with a natural morphism $\mathscr{O}_M \to \mathscr{C}(\mathscr{M})$ with dense image, and therefore which determines the topology of $\mathscr{M}$;
- and a way to construct for any parameter $m \in \mathscr{M}$ a sequence of random discrete objects $M(n,m)$ which can be considered as elements of $\mathscr{M}$, and which:
  - converge in probability back to the parameter m as n goes to infinity.
  - provide for any homogeneous observable $f \in \mathscr{O}_M$ a sequence $(f(M(n,m)))_{n\in\mathbb{N}}$ of random variables whose fluctuations are mod-Gaussian (unless f and m yield a limiting variance $\kappa_2(f,f)(m)$ equal to zero).

There are several advantages to this geometric study of the random models. First, one is able to prove generic central limit theorems, local limit theorems, *etc.* which are valid simultaneously for almost any model associated to a parameter of the moduli space. On the other hand, the cases where $\kappa_2(f,f)(m) = 0$ and where one does not have *a priori* mod-Gaussian convergence can be considered as *singularities* of the moduli space. The singular parameters correspond most of the time to models with additional symmetries (this is one of the reasons why the terminology of moduli spaces is relevant). For these singularities, one can sometimes find another (non-generic) normalisation of the random observables that yields mod-Gaussian sequences. For instance, the constant graph functions $g = p$ correspond to the Erdös–Rényi random graphs, and the corresponding graphons yield generic limiting variances $\kappa_2(f,f)(p) = 0$ for any observable. As detailed in Example 2.21, for these Erdös–Rényi random graphs, one needs to renormalise the densities $t(F,\Gamma(n,p))$ in a different way: indeed, when looking at the subgraph count for a graph with k vertices, the parameters of the method of cumulants in this setting are $D_n = O(n^{k-2})$ and $N_n = O(n^k)$, instead of $D_n = O(n^{k-1})$ and $N_n = O(n^k)$ in the generic case. A similar phenomenon occurs in the Thoma simplex, when looking at the parameter $\omega = 0 = ((0,0,\ldots),(0,0,\ldots))$ which corresponds to the Plancherel measures on partitions:

$$\mathbb{P}[\lambda_n(0) = \lambda] = \mathbb{P}_n[\lambda] = \frac{(\dim S^\lambda)^2}{n!}, \qquad \text{where } S^\lambda \text{ is the Specht module of label } \lambda \text{ for } \mathfrak{S}(n).$$

Kerov's central limit theorem [Ker93; IO02] shows that the fluctuations of certain observables of the Plancherel model are asymptotically normal, but not with the same renormalisation as in the generic case (we also do not know whether the fluctuations are *mod*-Gaussian). In this setting, one can also try to develop a differential calculus in the neighborhood of the singular point $m \in \mathscr{M}$, and study the fluctuations of observables $f(M(n,m_n))$ when the driving parameter m_n of the models converges to m at an adequate speed and with a certain asymptotic direction. For parameters of the Thoma simplex close to the point 0, this is essentially what was done in [Bia01; Śni06a; Śni06b; Mél11], and central limit theorems were proved for instance for Schur–Weyl measures (the mod-Gaussian framework did not exist at that time).

To conclude this section, let us mention an additional result which is implied by the method of cumulants and the theory of dependency graphs (see [FMN17a, Propositions 6 and 7]).

Proposition 2.33 (Concentration inequality). *Let $(S_n)_{n\in\mathbb{N}}$ be a sequence of random variables whose cumulants satisfy the bound (MC3) in the method of cumulants, with parameters (D_n, N_n, A). We assume moreover that $|S_n| \leq N_n A$ almost surely; this is always the case if the bound (MC3) comes from the existence of dependency graphs with parameters (D_n, N_n, A). Then, for any $\varepsilon \geq 0$,*

$$\mathbb{P}[|S_n| \geq \varepsilon] \leq 2\,\exp\left(-\frac{\varepsilon^2}{9D_nN_nA^2}\right).$$

For instance, if F is a graph with k vertices and if γ is a graphon, then

$$\mathbb{P}[|t(F,G_n(\gamma)) - t(F,\gamma)| \geq \varepsilon] \leq 2\,\exp\left(-\frac{n\varepsilon^2}{9k^2}\right)$$

for any $\varepsilon > 0$. These concentration inequalities were previously obtained by means of martingale techniques, and they played an essential role in the study of the topology of the spaces $\mathscr{G}$ and $\mathscr{P}$. Here, we obtain them immediately as a consequence of the existence of adequate dependency graphs, and similar inequalities hold true for the observables of permuton models and the observables of the models of random partitions.

2.4 Weighted dependency graphs

In the two previous sections, we studied random variables that satisfied the hypotheses of the method of cumulants, because they wrote as sums of random variables with a sparse dependency graph. It turns out that one can also establish the validity of these hypotheses for sums of random variables that do not admit a sparse dependency graph, but that have a weak dependency structure, for instance Markovian. The key notion that enables this method is:

Definition 2.34 (Weighted dependency graph). *A (uniform) weighted dependency graph for a family of random variables $\{A_v\}_{v\in V}$ is a graph $G = (V, E)$ endowed with a weight map $\mathrm{wt} : E \to \mathbb{R}_+$ such that, for any variables $A_{v_1}, \ldots, A_{v_r}$, the joint cumulant of these variables satisfies:*

$$|\kappa(A_{v_1}, \ldots, A_{v_r})| \leq A^r \sum_{\substack{T \text{ spanning tree} \\ \text{of } G[v_1,v_2,\ldots,v_r]}} \left(\prod_{e\in E_T} \mathrm{wt}(e) \right)$$

for some positive constant A.

This notion was introduced in [Fér16] in a slightly more general setting, and in [FMN17b, Section 5], it was used to establish the mod-Gaussian convergence of certain random variables. Since the proof of the method of cumulants for sums with sparse dependency graphs involved spanning trees (Theorem 2.19), Definition 2.34 is a natural generalisation of the discussion of Section 2.2. In particular, a dependency graph in the sense of Definition 2.18 is a weighted dependency graph with a weight $\mathrm{wt}(e) = 2$ for all its vertices.

Theorem 2.35 (Weighted dependency graphs and bounds on cumulants). *Let $S = \sum_{v\in V} A_v$ be a sum of random variables endowed with a weighted dependency graph $(G = (V, E), A, \mathrm{wt})$. We set*

$$N = \mathrm{card}\, V \qquad ; \qquad D = \frac{1}{2}\left(1 + \max_{v\in V}\left(\sum_{w\neq v} \mathrm{wt}(\{v, w\})\right)\right),$$

being understood that $\mathrm{wt}(\{v, w\}) = 0$ if $\{v, w\} \notin E$. Then, for any $r \geq 1$,

$$|\kappa^{(r)}(S)| \leq A^r\, r^{r-2}\, N\, (2D)^{r-1}.$$

As a consequence, given a sequence $(S_n)_{n\in\mathbb{N}}$ of sums of random variables, if one can construct weighted dependency graphs for the S_n's whose weights are summable, or at least such that

$$\max_{v\in V_n}\left(\sum_{w\neq v} \mathrm{wt}(\{v, w\})\right) \ll \mathrm{card}(V_n),$$

then one usually obtains the mod-Gaussian convergence of the sequence.

Example 2.36 (Linear functional of a Markov chain). Consider an ergodic Markov chain (irreducible and aperiodic) $(X_n)_{n\in\mathbb{N}}$ on a finite state space $\mathfrak{X} = [\![1, M]\!]$, with transition matrix P and stationary measure π. We assume that X_0 is distributed as π, and we fix a state $a \in \mathfrak{X}$. The number of visits

$$N_{a,n} = \sum_{i=1}^{n} 1_{X_i=a}$$

is a sum of random variables that admit a weighted dependency graph with constant $A = 1$ and weight map

$$\mathrm{wt}(\{i, j\}) = 2\, (\theta_P)^{|j-i|},$$

where θ_P is some constant strictly smaller than 1 and determined by P. More precisely, let us introduce the time reversal

$$\widetilde{P}(x,y) = \frac{\pi(y)\,P(y,x)}{\pi(x)}$$

of the transition matrix P, and the multiplicative reversiblisation $M(P) = P\widetilde{P}$ [Fil91]. The matrix $M(P)$ is a reversible stochastic matrix, and we define

$$(\theta_P)^2 = \max\{|z| \mid z \neq 1 \text{ and } z \text{ is an eigenvalue of } M(P)\}.$$

As a consequence of the existence of a weighted dependency graph, we get that the sequence $(N_{a,n} - n\pi(a))_{n\in\mathbb{N}}$ satisfies the hypotheses of the method of cumulants with parameters $D_n = \frac{1+\theta_P}{1-\theta_P}$, $N_n = n$ and $A = 1$. Therefore, the sequence

$$Y_{a,n} = \frac{N_{a,n} - n\pi(a)}{\sqrt{\mathrm{var}(N_{a,n})}}$$

satisfies all the limiting results of Theorem 2.14. In particular, denoting T_a the first return time to a, since $\mathrm{var}(N_{a,n}) \simeq n\,(\pi(a))^3\,\mathrm{var}(T_a)$, for n large enough, one obtains the Berry–Esseen estimate

$$d_{\mathrm{Kol}}(Y_{a,n}\,,\,\mathcal{N}_{\mathbb{R}}(0,1)) = O\left(\left(\frac{1+\theta_P}{1-\theta_P}\right)^2 \frac{1}{(\pi(a))^{9/2}\,(\mathrm{var}_a(T_a))^{3/2}}\,\frac{1}{\sqrt{n}}\right),$$

with a universal constant in the $O(\cdot)$. The result extends to the fluctuations of any linear functional $\sum_{i=1}^n f(X_i)$ with $f : \mathfrak{X} \to \mathbb{R}$; then, the constant A in the method of cumulants depends on f and on the size M of the space $\mathfrak{X}$ [FMN17b, Sections 5.4-5.5]. Similar estimates of the speed of convergence had previously been obtained by using different techniques in [Bol80; Lez96; Man96]; the advantage of our approach is that it gives without additional work normality zones and large deviation estimates (besides, it only relies on Fourier analysis).

Example 2.37 (Magnetisation of the Ising model in dimension $d \geq 2$). Consider the Ising model in dimension $d \geq 2$: in a box $[-C,C]^d$, it is the unique probability measure $\mathbb{P}_{\beta,h,[-C,C]^d}$ on spin configurations $\sigma : [-C,C]^d \to \{\pm 1\}$ such that $\mathbb{P}_{\beta,h,[-C,C]^d}[\sigma]$ is proportional to

$$\exp\left(\beta \sum_{\{x,y\}\,|\,x\sim y} \sigma(x)\sigma(y) + h\sum_x \sigma(x)\right).$$

In this expression, the sums are restricted to a box $[-C,C]^d$, $\beta > 0$ is the inverse temperature of the system, and h is the value of the external magnetic field. We refer to Figure 2.12 for an example in dimension 2 and at high temperature, without external field. If $h \neq 0$, or if $h = 0$ and $\beta < \beta_c(d)$ (high temperature), then the probability measures $\mathbb{P}_{\beta,h,[-C,C]^d}$ have a unique limiting distribution (Gibbs measure) $\mathbb{P}_{\beta,h,\mathbb{Z}^d}$ as C goes to infinity. This allows us to deal with infinite spin configurations on $\mathbb{Z}^d$, see for instance [FV17, Theorem 3.41]. With β and h fixed in this domain of uniqueness, let us consider a growing sequence of boxes $(\Lambda_n)_{n\in\mathbb{N}}$ with $\bigcup_{n\in\mathbb{N}} \Lambda_n = \mathbb{Z}^d$, and the total magnetisations

$$M_n = \sum_{x\in\Lambda_n} \sigma(x)$$

of these boxes. One can show that if $h \neq 0$ or if $h = 0$ and $\beta < \beta_b(d) < \beta_c(d)$ (very high temperature), then under the Gibbs measure $\mathbb{P}_{\beta,h,\mathbb{Z}^d}$, the family of spins $(\sigma(x))_{x\in\mathbb{Z}^d}$ admits a weighted dependency graph for a certain constant $A = A(d,\beta,h)$ and for the weight

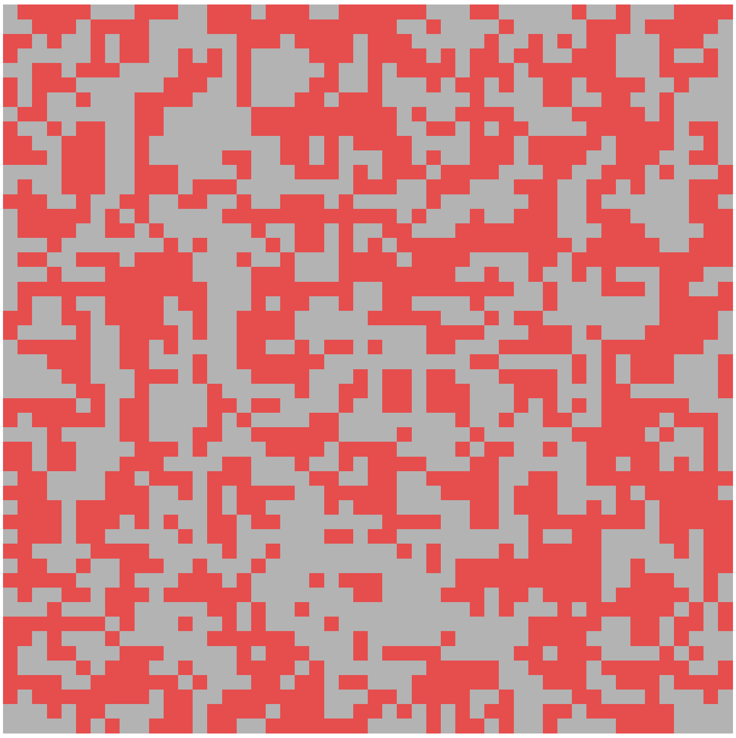


Figure 2.12. A configuration of the Ising model when $d = 2$, $\beta = 0.2$ and $h = 0$.

$\mathrm{wt}(x, y) = \varepsilon^{\|x-y\|_1}$ with $\varepsilon < 1$. This result was obtained by Duneau, Iagolnitzer and Souillard in the seventies [DIS73; DIS74], in the framework of *strong cluster properties*. As a consequence, $(M_n - \mathbb{E}[M_n])_{n\in\mathbb{N}}$ satisfies the hypotheses of the method of cumulants with parameters $D_n = O(1)$, $N_n = |\Lambda_n|$ and $A = O(1)$. We thus obtain a Berry–Esseen estimate

$$d_{\mathrm{Kol}}\left(\frac{M_n - \mathbb{E}_{\beta,h,\mathbb{Z}^d}[M_n]}{\sqrt{\mathrm{var}(M_n)}}, \mathcal{N}_{\mathbb{R}}(0,1)\right) = O\left(\frac{1}{\sqrt{|\Lambda_n|}}\right),$$

or the concentration inequality:

$$\mathbb{P}_{\beta,h,\mathbb{Z}^d}\left[\left|M_n - \mathbb{E}_{\beta,h,\mathbb{Z}^d}[M_n]\right| \geq |\Lambda_n|\varepsilon\right] \leq 2\,\exp(-K|\Lambda_n|\varepsilon^2)$$

for some $K = K(d, \beta, h) > 0$. These probabilistic estimates are true when the magnetic field is non-zero, or at very high temperature [FMN17b, Section 5.3].

Perspectives

In this chapter, we have presented various mathematical structures leading to mod-ϕ convergence: generating series with algebraico-logarithmic singularities, dependency graphs, weighted dependency graphs, *etc.* An important goal of future works will consist in:

- identifying new structures that also imply mod-ϕ convergence,
- using the corresponding probabilistic estimates in order to prove new results on a large variety of models.

In the following, we detail several research directions fulfilling this objective.

$\to$ *Moment conjecture and Ramachandra's conjecture.* The theory of mod-ϕ convergence initially found its source in several conjectures on Riemann's ζ function, see [KS00b; KS00a; Kea05; Con+05] and [JKN11, Section 4.1]. Recall that the ζ function on the critical line satisfies Selberg's central limit theorem [Sel46; Sel92; Gho83; BH95; RS15]: if $t \in [T, 2T]$ is chosen at random and uniformly, then as T goes to infinity, the random variables

$$Y_T = \frac{\log \zeta(\frac{1}{2} + \mathrm{i}t)}{\sqrt{\log \log T}}$$

converge in distribution to a complex Gaussian variable $\mathcal{N}_{\mathbb{C}} = \mathcal{N}_{\mathbb{R}}(0, \frac{1}{\sqrt{2}}) + \mathrm{i}\,\mathcal{N}_{\mathbb{R}}(0, \frac{1}{\sqrt{2}})$. The Keating–Snaith *moment conjecture* is a much stronger statement: for any $z \in \mathbb{C}$ with $\mathrm{Re}(z) > -2$,

$$\lim_{T\to\infty} \frac{1}{T\,(\log T)^{\frac{z^2}{4}}} \int_0^T \left|\zeta\left(\frac{1}{2} + \mathrm{i}t\right)\right|^z dt = M(z)\, A(z),$$

where $M(z) = \frac{(G(1+\frac{z}{2}))^2}{G(1+z)}$ is the limiting residue of mod-Gaussian convergence for the characteristic polynomials of random unitary matrices (Example 1.8), and $A(z)$ is an arithmetic factor:

$$A(z) = \prod_{p\in\mathbb{P}} \left(\left(1 - \frac{1}{p}\right)^{\frac{z^2}{4}} \left(\sum_{n=0}^{\infty} \left(\frac{\Gamma(n+\frac{z}{2})}{n!\,\Gamma(\frac{z}{2})}\right)^2 \frac{1}{p^n}\right)\right).$$

In other words, the random variables $X_T = \mathrm{Re}(\log \zeta(\frac{1}{2} + \mathrm{i}t))$ with $t \sim \mathcal{U}([0, T])$ are conjectured to be mod-Gaussian convergent on $\mathcal{S}_{(-2,+\infty)}$, with parameters $t_T = (\log \log T)/2$ and limit $\psi(z) = M(z)\,A(z)$. There exist analogue conjectures for random values of L-functions, and for the complex random variables $\log \zeta(\frac{1}{2} + \mathrm{i}t)$. Besides, the moment conjecture for Riemann's ζ function is known to hold true for $z = 2$ and $z = 4$.

Though the moment conjecture seems out of reach with the tools developed so far, a consequence of its validity might be easier to deal with, namely, *Ramachandra's conjecture*: the values of ζ on the critical line form a dense subset of $\mathbb{C}$. We refer to Figure 2.13 for a representation of the function $t \mapsto \zeta(\frac{1}{2} + \mathrm{i}t)$ with $t \in [0, 100]$; Ramachandra's conjecture ensures that this curve eventually fills the whole complex plane. This conjecture was first stated at the conference *Recent progress in analytic number theory* held in Durham in 1979; it is for instance mentioned in [Tit86, Section 11.13]. In probabilistic terms, the density of the values of $t \mapsto \zeta(\frac{1}{2} + \mathrm{i}t)$ can be reinterpreted as a local limit theorem for the random variables Y_T. Consequently, in order to prove this result, instead of attacking directly the moment conjecture, one can:

- try to compare the random values of the ζ function on $[0, T]$ with those of the Euler products $\prod_{p\leq N} \frac{1}{1-p^{-\frac{1}{2}-\mathrm{i}t}}$, and then compare the Euler products with the random zeta functions
$$Z_N = \prod_{p\leq N} \frac{1}{1 - \frac{X_p}{\sqrt{p}}},$$
where the X_p's are independent and uniformly distributed on the unit circle.
- use the mod-Gaussian convergence of the variables $\log Z_N$ and the corresponding local limit theorem.

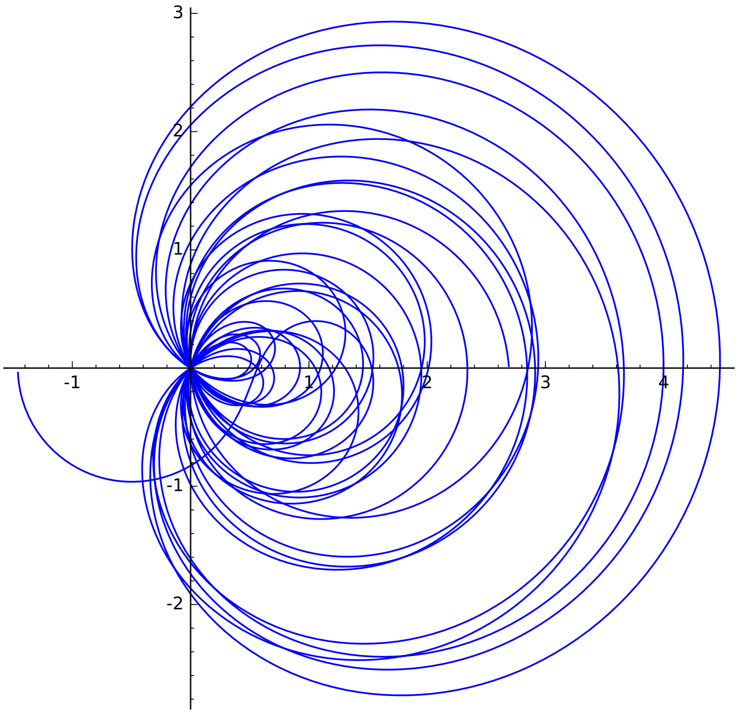


Figure 2.13. Values of $\zeta(\frac{1}{2} + \mathrm{i}t)$ for $t \in [0, 100]$.

Notice that the first item is a natural approach in order to understand the behavior of the ζ function on large intervals; see for instance [LLR14] for results outside the critical line. Ramachandra's conjecture is one of the main incentive to studying local limit theorems in the framework of mod-ϕ convergence, and it would also be its most spectacular application.

Easier cases or analogues of Ramachandra's conjecture should also be considered in the framework of mod-ϕ convergence of complex-valued random variables. More precisely:

- The Bohr–Jessen central limit theorem ensures that if $\sigma > \frac{1}{2}$, then the random variables
$$Y_{T,\sigma} = \log \zeta(\sigma + \mathrm{i}t)$$
with t uniform in $[T, 2T]$ converge as T goes to infinity towards limiting distributions BJ_σ, which are supported by the whole complex plane if $\sigma \in (\frac{1}{2}, 1]$; see [BJ30; BJ32; JW35; BJ48]. Note that the convergence in distribution occurs here without renormalisation. The article [LLR14] yields strong estimates of the speed of convergence in these central limit theorems, and the theory of mod-ϕ convergence should enable one to recover or improve on these results.
- On the other hand, it is known since the works of Montgomery that
  - the Riemann $\zeta(\sigma + \mathrm{i}t)$ function on its critical line $\sigma = \frac{1}{2}$,
  - and the characteristic polynomial $\det(I_n - zM_n)$ with z on the unit circle and M_n unitary Haar-distributed random matrix of size n

  share many asymptotic properties, at least conjecturally. We refer in particular to [KS00a; HKO01] for the central limit theorems; to [Mon73; RS96; CNN17] for the correlations of zeroes or eigenvalues; to [KS00b; Kea05; Con+05] for the asymptotics of moments; and to [Arg+16; ABB17; CMN16] for the asymptotics of the extreme values. In particular, the analogue of Selberg's central limit theorem is the following result from [HKO01]: if U is uniformly distributed on the circle, then

$$\frac{\log \det(I_n - U\,M_n)}{\sqrt{\log n}}$$

is a random variable whose law converges in probability towards the law of a complex Gaussian variable $\mathcal{N}_{\mathbb{C}}$. A local limit theorem in this framework would prove that the characteristic polynomial $P_n(z) = \det(I_n - zM_n)$ is a random curve which fills a large part of the complex plane with high probability as n goes to infinity; see Figure 2.14. This is a unitary analogue of Ramachandra's conjecture, which is related to the computation of bounds on the cumulants of the random variables $(\mathrm{tr}(M_n)^r)$ (the traces of powers of random unitary matrices are known to converge towards independent complex Gaussian variables, see [DS94; Joh97], and [Sos00a; Sos00b; Sos01] for partial results on the cumulants).

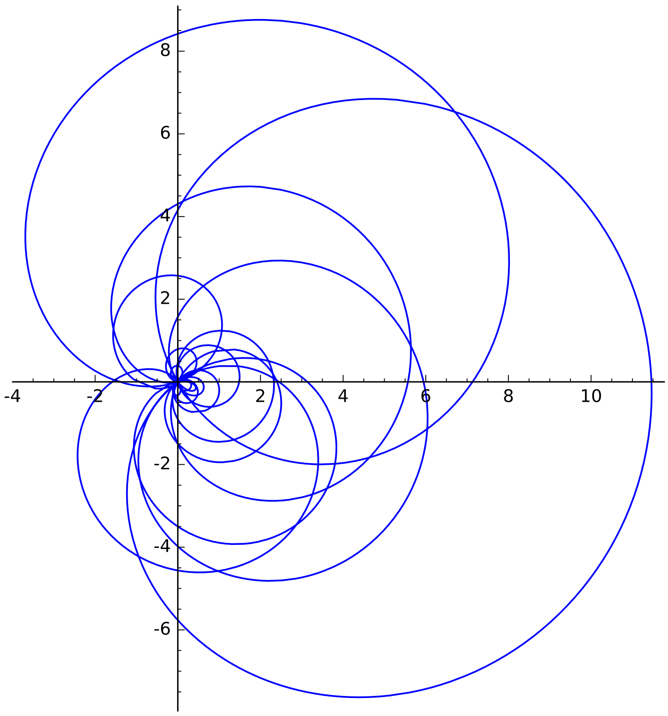


Figure 2.14. The characteristic polynomial P_n of a random unitary matrix of size $n = 20$, viewed as a function on the unit circle.

$\to$ *Sums of random arithmetic functions.* The connection between Ramachandra's conjecture and the theory of mod-ϕ convergence is an instance of a general method in number theory: in order to predict the behavior of some arithmetic functions, one can replace them by appropriate random variables which are easier to deal with. For instance, it is well known that the Riemann hypothesis is equivalent to the statement

$$\forall \varepsilon > 0, \quad \sum_{k \leq n} \mu(k) = O\left(n^{\frac{1}{2}+\varepsilon}\right),$$

where $\mu(n)$ is the arithmetic Möbius function defined by $\mu(p_1 p_2 \cdots p_k) = (-1)^k$, and $\mu(n) = 0$ if n has a square factor. To understand the behavior of the Mertens function $M(n) =$

$\sum_{k\leq n}\mu(k)$, one can try to replace the arithmetic function $\mu(n)$ by random variables

$$X_n = \begin{cases} \prod_{i=1}^k \varepsilon_{p_i} & \text{if } n = p_1 p_2 \cdots p_k, \\ 0 & \text{if } n \text{ has a square factor,} \end{cases}$$

where the ε_p's are independent random variables labeled by the prime numbers $p \in \mathbb{P}$, and uniformly distributed over $\{\pm 1\}$ or over the unit circle. Then, $\widetilde{M}(n) = \sum_{k\leq n} X_k$ satisfies $\widetilde{M}(n) = O(n^{1/2+\varepsilon})$ almost surely [Hal83]. Moreover, if one restricts the sum to small intervals or to integers with a small number of prime factors:

$$\widetilde{M}(n, n+y) = \sum_{\substack{n\leq k\leq n+y \\ y=O(\frac{n}{\log n})}} X_k \qquad ; \qquad \widetilde{M}^{(L)}(n) = \sum_{\substack{k\leq n,\ \omega(k)\leq L \\ L\ll \log\log n}} X_k$$

then these sums are asymptotically normal; see [CS12; Har13]. In this setting, we would like to show that the method of cumulants can be used, by exhibiting an adequate weighted dependency graph. Note that the classical theory of sparse dependency graphs can be applied here. However, it does not yield mod-Gaussian convergence: the random variables X_n are pairwise independent, and as a consequence, with the notations of Proposition 2.13 and with a natural choice for a dependency graph, $\sigma^2 = 0$. This is why we need to look for a *weighted* dependency graph, whose structure would shed light on the arithmetic properties of the prime numbers. This problem might also require the construction of new dependency structures which would be adapted to the study of sums of symmetric and pairwise independent random variables. Similar questions can be asked when the ring $\mathbb{Z}$ and the prime numbers are replaced by a finite function field $\mathbb{F}_q[X]$ and by the monic irreducible polynomials over $\mathbb{F}_q$. The moment conjecture for the zeta function of a random irreducible polynomial is then a theorem (see for instance [JKN11, Proposition 4.8]), and one could study the random arithmetic sums corresponding to these function fields, as a first step in the understanding of the case of number fields.

$\to$ *Mixing dynamical systems.* In Example 2.36, we explained that under the appropriate hypotheses, a linear functional of a Markov chain is mod-Gaussian convergent (after an adequate renormalisation). A more general framework where the cumulant method could be used is the theory of mixing ergodic dynamical systems. Let X be a compact metric space, and $T : X \to X$ be a measurable map. We recall that T is called *ergodic* with respect to some probability measure μ on X if T preserves μ, and if the only T-invariant measurable subsets of X have μ-measure 0 or 1 [KH95, Definition 4.1.6]. The map T is called (strongly) *mixing* if, for any measurable subsets A and B, $\lim_{n\to\infty}\mu(T^{-n}(A)\cap B) = \mu(A)\,\mu(B)$ [KH95, Definition 4.2.8]; this is stronger than the ergodicity. Classical examples of mixing maps are:

- the expanding maps $E_m : x \in \mathbb{R}/\mathbb{Z} \mapsto mx \in \mathbb{R}/\mathbb{Z}$ with m integer greater than 2 and with respect to the Lebesgue measure;
- the hyperbolic automorphisms of the torus $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$ with respect to the Lebesgue measure;
- the Gauss map $G : x \in \mathbb{R}/\mathbb{Z} \mapsto \frac{1}{x} \in \mathbb{R}/\mathbb{Z}$ with respect to the Gauss measure $\frac{1}{\log 2}\frac{dx}{1+x}$ (see Figure 2.15);
- the Bernoulli shifts $\sigma : [\![1, M]\!]^{\mathbb{N}} \to [\![1, M]\!]^{\mathbb{N}}$ with respect to any probability measure on sequences associated to an irreducible Markov chain.

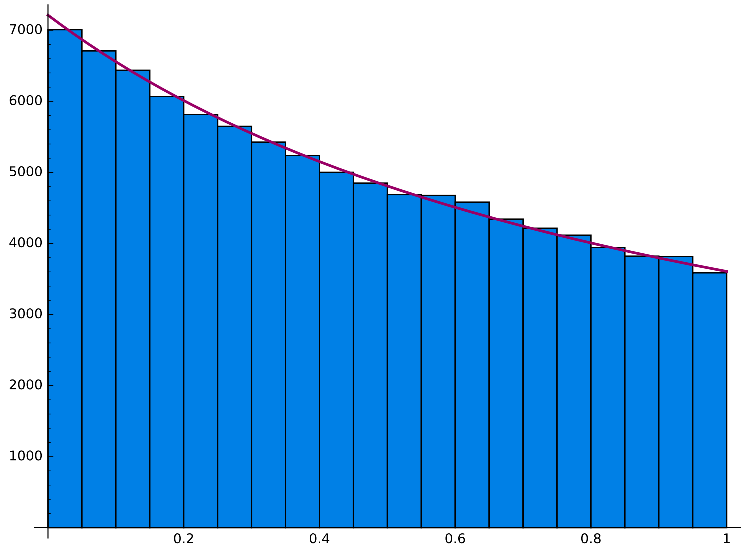


Figure 2.15. Distribution of the 10^5 first iterates of the Gauss map, starting from a random point in $[0,1]$.

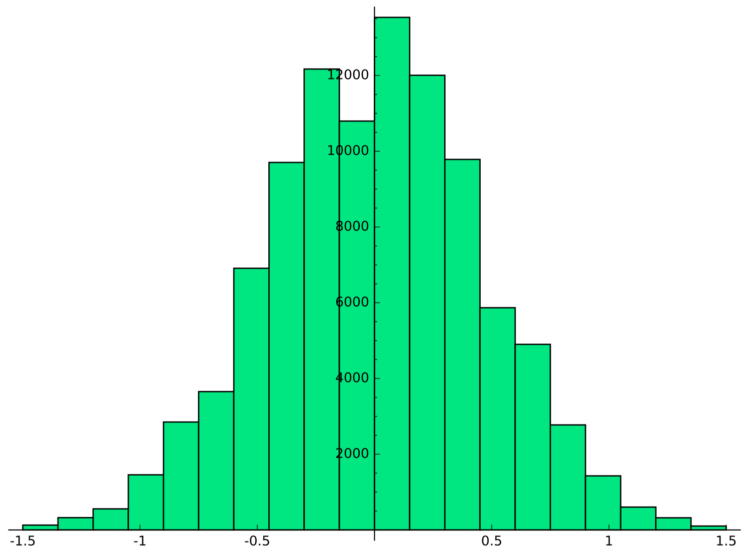


Figure 2.16. Fluctuations $Y_n(T,f) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(f(T^n(x)) - \mu(f))$, when T is the Gauss map, $f = 1_{[0,\frac{1}{2}]}$ and $n = 1000$, over $N = 10^5$ tries.

Given an ergodic map $T : (X,\mu) \to (X,\mu)$, for μ-almost any $x \in X$,

$$\frac{1}{n}\sum_{i=1}^{n} f(T^n(x)) \to \int_X f(y)\,\mu(dy)$$

for any function $f \in \mathscr{L}^1(X,\mu)$ (Birkhoff ergodic theorem). The mixing condition can then be used to prove a central limit theorem for

$$Y_n(T,f) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(f(T^n(x)) - \mu(f)),$$

x being chosen at random according to μ. We refer to [Ros56; Ibr62; Ibr75; Bra81; Pel96], and [Bra05] for a survey of the various possible mixing conditions. For instance, the empirical distribution of these fluctuations when T is the Gauss map and $f = 1_{[0,\frac{1}{2}]}$ is drawn in Figure 2.16, and one has indeed asymptotic normality. To the best of our knowledge, the cumulants of such random variables have never been considered, and their study would be an interesting contribution to the theory of mixing processes. Notice that when the state space is the torus $\mathbb{T}^d$, the ergodic theory of the corresponding dynamical systems is often attained by means of Fourier analysis (see *e.g.* [KH95, Section 4.2]). In this setting, a first step would therefore consist in estimating the cumulants of $Y_n(T,f)$ with $f(t) = \mathrm{e}^{\mathrm{i}\langle m\,|\,t\rangle}$ character of the torus.

$\to$ *Statistics of random processes.* Many examples of mod-ϕ convergent random variables can probably be found in the theory of random processes. We already mentioned in Example 1.7 the winding number of a planar Brownian motion, which is mod-Cauchy convergent. Other interesting statistics stemming from the planar Brownian motion are the occupation densities

$$D_V(t) = \int_0^t 1_V(B_s)\,ds,$$

where $(B_s)_{s\geq 0}$ is a standard 2-dimensional Brownian motion started from 0, and V is a bounded measurable domain with positive area. The Kallianpur–Robbins theorem [KR53] ensures that these densities are distributed asymptotically like exponential random variables:

$$\frac{2\pi}{(\log t)}\,D_V(t) \rightharpoonup_{t\to+\infty} \mathscr{E}\left(\frac{1}{\mathrm{area}(V)}\right)$$

where $\mathscr{E}(\frac{1}{\lambda})$ denotes the exponential distribution with density $\frac{1_{x\geq 0}}{\lambda}\,\mathrm{e}^{-\frac{x}{\lambda}}$, and with mean λ. It would be interesting to restate this result in terms of mod-exponential convergence, as this would imply local limit theorems and speed of convergence estimates.

Another domain where mod-ϕ convergence is conjectured to occur is the theory of self-similar processes. For example, let us detail the case of the *random clocks* associated to these processes. If $\alpha > 0$, a positive self-similar Markov process of index α is a strong Markov process with càdlàg paths on $\mathbb{R}_+$, which satisfies the identities in law

$$((bX_{b^{-\alpha}t})_{t\geq 0}, \mathbb{P}_a) =_{\mathrm{law}} ((X_t)_{t\geq 0}, \mathbb{P}_{ba})$$

for any $a, b > 0$. Here, $\mathbb{P}_a$ is the law of the Markov process starting from a. Lamperti's representation theorem [Lam62; Lam72] establishes a bijection between positive self-similar Markov processes and Lévy processes starting from 0. Hence, let $(X, (\mathbb{P}_a)_{a>0})$ be a pssmp of index α, and

$$T^{(X)}(t) = \int_0^t \frac{1}{(X_s)^\alpha}\,ds \qquad ; \qquad A^{(X)}(t) = \inf\{u \geq 0 \,|\, T^{(X)}(u) \geq t\}.$$

The process $(T^{(X)}(t))_{t\geq 0}$ is called the clock of the pssmp, and $(A^{(X)}(t))_{t\geq 0}$ is the inverse of this increasing process. If

$$Y_t = \log\left(\frac{X_{A^{(X)}(t)}}{X_0}\right),$$

then $(Y_t)_{t\geq 0}$ is under $\mathbb{P}_a$ a Lévy process started from 0. Moreover, one can reconstruct entirely $(X, (\mathbb{P}_a)_{a>0})$ from $(Y_t)_{t\geq 0}$; in particular, every pssmp is the exponential of a time-changed Lévy process. In the recent paper [DRZ15], the large deviations of the clocks $T^{(X)}(t)$ of pssmp are studied (as t goes to $+\infty$), by means of a careful estimation of the Laplace transforms of these random variables. A central limit theorem is also conjectured, see Equation (30) in *loc. cit.*, and the article [YZ01] in the case where X is a Bessel process. In this framework, one expects that an adequate renormalisation of $(T^{(X)}(t))_{t\geq 0}$ is mod-Gaussian convergent.

$\to$ *Random measured metric spaces.* In Section 2.3, we introduced the notion of mod-Gaussian moduli space, which formalises the idea that for some classes of random models (graphs, permutations, integer partitions), a generic observable of a generic model is mod-Gaussian convergent. Hence, the mod-Gaussian convergence is a universal result for certain classes of models. A natural objective is then to find new classes of random models with this property. A space of parameters where we expected this to be true was the space $\mathscr{M}$ of separable complete metric spaces (X, d_X) equipped with a probability measure $\mathbb{P}_X$. This space was introduced in [GPW09], and it is polish (complete metrisable separable) for the topology induced by the Gromov–Hausdorff–Prohorov metric:

$$\begin{aligned}
&d_{\mathrm{GHP}}\left((X,d_X,\mathbb{P}_X),(Y,d_Y,\mathbb{P}_Y)\right)\\
&= \inf_{\substack{\psi_X : X\to Z\\ \psi_Y : Y \to Z}} \left(\inf\{\varepsilon>0 \,|\, \forall C \text{ closed subset of } Z,\ (\psi_X)_*\mathbb{P}_X(C) \leq (\psi_Y)_*\mathbb{P}_Y(C^\varepsilon)+\varepsilon\}\right).
\end{aligned}$$

Here,

- the infimum is taken over all isometric embeddings $\psi_X : X \to Z$ and $\psi_Y : Y \to Z$ of (X, d_X) and (Y, d_Y) into a common metric space (Z, d_Z);
- $(\psi_X)_*\mathbb{P}_X$ and $(\psi_Y)_*\mathbb{P}_Y$ are the images of the probability measures $\mathbb{P}_X$ and $\mathbb{P}_Y$ by ψ_X and ψ_Y, and thus are probability measures on Z;
- if $C \subset Z$, then $C^\varepsilon = \{z \in Z \,|\, d_Z(z, C) < \varepsilon\}$.

There is an algebra $\mathscr{O}_M$ of observables corresponding to this topology, see [GPW09, Theorem 5]. If $\phi : (\mathbb{R}_+)^{\binom{n}{2}} \to \mathbb{R}$ is a bounded continuous function, denote

$$t(\phi,(X,d_X,\mathbb{P}_X)) = \int_{X^n} \phi((d(x_i,x_j))_{1\leq i<j\leq n})\,(\mathbb{P}_X)^{\otimes n}(dx_1\,dx_2\cdots dx_n).$$

The convergence with respect to the Gromov–Hausdorff–Prohorov topology is equivalent to the convergence of all the observables $t(\phi,\cdot)$. On the other hand, given a separable complete metric and measured space $v = (X, d_X, \mathbb{P}_X)$, one can associate to it random discrete spaces:

- $X_n(v) = \{x_1, \ldots, x_n\}$ is a finite set of points taken independently in X and according to the probability measure $\mathbb{P}_X$.
- $d_n(v)$ is the restriction of d_X to $X_n(v)$.
- finally, $\mathbb{P}_n(v)$ is the uniform probability measure $\frac{1}{n}\sum_{i=1}^n \delta_{x_i}$ on $X_n(v)$.

One can then study the convergence of $\mathrm{Y}(n, v) = (X_n(v), d_n(v), \mathbb{P}_n(v))$ to v, and the fluctuations of the observables $t(\phi, \mathrm{Y}(n, v))$.

My first Ph. D. student Jacques de Catelan is currently working on this subject, and he proved recently the mod-Gaussian convergence for a generic observable and a generic model v. Hence, the pair $(\mathscr{M}, \mathscr{O}_M)$ forms a (non-compact) mod-Gaussian moduli space. Moreover, it turns out that the singularities of this mod-Gaussian moduli space are precisely the compact homogeneous spaces $X = G/K$, X being endowed with a G-invariant distance d and with the projection of the Haar measure of G. We now conjecture that these singularities still yield mod-Gaussian observables, but with a non-generic renormalisation and a non-generic underlying dependency structure. On the other hand, as an application of these results, we hope to obtain asymptotic estimates on random metric trees, which are important random combinatorial models.

$\to$ *Random partitions under Plancherel measure and random matrices.* To conclude this list of sequences of random variables that we conjecture to be mod-ϕ convergent, let us evoke the random character values under the Plancherel measures of the symmetric groups. We thus consider a random integer partition $\lambda \in \mathfrak{Y}(n)$ under the Plancherel measure $\mathbb{P}_n[\lambda] = (\dim \lambda)^2/n!$, and we look at the random variable

$$X_n(k) = \chi^\lambda(k) = \frac{\operatorname{tr} \rho^\lambda(c_k)}{\operatorname{tr} \rho^\lambda(1)}$$

where c_k is a k-cycle, $\rho^\lambda : \mathfrak{S}(n) \to \mathrm{GL}(S^\lambda)$ is the defining morphism of the irreducible representation S^λ, and λ is chosen at random according to $\mathbb{P}_n$. By Kerov's central limit theorem, for any $k \geq 2$,

$$\frac{n^{\frac{k}{2}}\, X_n(k)}{\sqrt{k}} \rightharpoonup_{n\to\infty} \mathcal{N}_{\mathbb{R}}(0,1).$$

Quite surprisingly, this was one of the first example that we tried to study five years ago, and this led eventually to the theory of sparse dependency graphs and to the method of cumulants. Unfortunately, the parameter $\omega = 0$ corresponding to the Plancherel measures in the Thoma simplex is a singularity of this mod-Gaussian moduli space, and the generic argument does not apply. Therefore, we were not able to prove the mod-Gaussian convergence of $(X_n(k))_{n\in\mathbb{N}}$. The more recent theory of weighted dependency graphs, or a new 2-dimensional theory of dependency surfaces might remove the problems that we encountered in our first approach. On the other hand, it is well known that the Poissonised Plancherel measures

$$\mathbb{P}_\theta = \sum_{n=0}^{\infty} \frac{\mathrm{e}^{-\theta}\,\theta^n}{n!}\,\mathbb{P}_n$$

give rise to determinantal point processes [BOO00, Theorem 1], so the suspected connection between determinantal correlations and mod-Gaussian asymptotics (see the discussion in Example 2.1) might bring a new approach to the estimation of the fluctuations of the random character values under Plancherel measures.

Another domain of mathematics that produces mod-Gaussian convergent sequences and that we already evoked for the unitary Ramachandra conjecture is random matrix theory; see the recent papers [BHR17a; BHR17b]. The linear statistics of the eigenvalues of the classical ensembles from random matrix theory satisfy a central limit theorem (see for instance [DS94; Joh97; Joh98; Sos00b; Sos01; Cab01; LP09]), and several techniques might be used in order to establish their mod-Gaussian convergence: cumulant method, determinantal structure, Szegö's asymptotics of Toeplitz determinants, formulas of topological recursion, *etc.* In this setting, another problem is the identification of the class of functions that yield mod-Gaussian linear statistics. Thus, with the terminology of mod-Gaussian moduli spaces, a difficult question will be to find an appropriate algebra of observables (subalgebra of the algebra of continuous functions on spectral measures).

Chapter 3

Random objects on symmetric spaces

In the two previous chapters, we explained how to use classical harmonic analysis on the real line in order to study the fluctuations of numerous random models. If the observables of a model are not real-valued and if they take their values in another kind of parameter space X, then one can still use harmonic analysis in order to study them, but it is then required to deal with the Fourier transform of X (instead of the classical Fourier transform $\mathscr{L}^2(\mathbb{R}) \to \mathscr{L}^2(\mathbb{R})$). A case that is particularly interesting is when X is a compact Lie group, or more generally a compact symmetric space. This class of parameter spaces includes spheres, projective spaces, Grassmannians, and orthogonal or unitary groups. By using classical results from the representation theory of Lie groups (due to Cartan, Weyl, *etc.*), we were able to study several kinds of random objects drawn on X:

(1) random processes $(Y_t)_{t\geq 0}$ with values in X, and which converge to the uniform measure [Mél14b];

(2) random geometric graphs drawn on X, and the spectra of the associated random adjacency matrices ([Mél18], work in progress).

The second topic actually led us to new conjectures in the representation theory of compact Lie groups, so the connection between representation theory and random objects on Lie groups goes both ways: one can prove new results on the random objects by using harmonic analysis, but one can also discover new properties of the representations of Lie groups by means of probabilistic models.

This last chapter is organised as follows. In Section 3.1, we propose a survey of the harmonic analysis on compact symmetric spaces.

- It will considerably ease the understanding of the later sections, particularly for readers with a probabilistic background.
- It allows us to fix notations and to introduce all the relevant objects, including those on which we shall make conjectures and which correspond to recent developments in representation theory (crystals and string polytopes).
- Except for the theory of crystal bases, everything that we shall say about the harmonic analysis on a compact symmetric space can be found in the celebrated books by Helgason [Hel78; Hel84]. However, it is not always easy to extract the relevant information from these 1000+ pages, in particular because the hypotheses are sometimes stated for the isometry group of the symmetric space, and sometimes for the corresponding Lie algebra, or complexification thereof. For one particular result due

to Cartan and Helgason (Theorem 3.11), a concise statement of its exact hypotheses (H1) and (H2) is especially difficult to find in the literature (to the best of our knowledge, if one wants to avoid to have to search several articles or book chapters, then one can only find them in [GW09, Section 12.3]).

This is why we decided to recall the main results of this theory.

In Section 3.2, we show that a Brownian motion drawn on a compact symmetric space converges in law towards the uniform measure with a cut-off, meaning that the total variation distance to the stationary measure decreases abruptly after a certain time $t_{\text{cut-off}}$ which one can compute explicitly for any symmetric space. These results are presented in such a way that one can easily adapt the method of proof to other random processes on groups, *e.g.* the mixing of cards and the corresponding random walks on the symmetric groups.

In Section 3.3, we introduce the random geometric graphs on symmetric spaces, which have two interesting limiting regimes as their size grows to infinity: the Gaussian regime (large dense graphs) and the Poissonian regime (large sparse graphs). We explain how to compute the limiting spectrum of a random geometric graph in the Gaussian regime, when the underlying graph X is a simple simply connected compact Lie group, or a compact symmetric space of rank 1. Last, in Section 3.4, we present conjectures and partial results on the Poissonian regime of random geometric graphs drawn on Lie groups. The result that one expects (Conjecture 3.41) is a convergence in probability of the spectral measures of these graphs, with the moments of the limit that can be computed by using techniques from asymptotic representation theory. This theorem depends on a conjecture on certain functionals of the dominant weights of the compact Lie groups (Conjecture 3.50). These functionals should admit a combinatorial description as numbers of integers points in polytopes determined by certain linear equations.

3.1 Symmetric spaces and non-commutative Fourier transforms

In this first section, we recall the main results of harmonic analysis on a compact Lie group or symmetric space:

- Peter–Weyl's decomposition of the space of square-integrable functions (Theorem 3.2);
- Cartan's classification of simple simply connected compact Lie groups and symmetric spaces (Theorems 3.6 and 3.8);
- Weyl's highest weight theorem and the Cartan–Helgason classification of spherical representations (Theorems 3.9 and 3.11);
- finally, Lusztig–Kashiwara theory of crystal bases, for which we provide Littelmann's interpretation using paths in the weight lattice (Theorem 3.15), and the Berenstein–Zelevinsky description using string polytopes (Propositions 3.19 and 3.20).

This survey does not contain any new result, but its content is required in order to have a full understanding of the later sections.

▷ **Harmonic analysis on a compact group.** Let G be a compact topological group. Recall that there exists a unique Borel probability measure on G called the *Haar measure* and denoted $\mathrm{Haar}(dg)$ or simply dg, which is invariant on the left and on the right: if $L_h : g \mapsto hg$ and $R_h : g \mapsto gh$, then $(L_h)_*\mathrm{Haar} = (R_h)_*\mathrm{Haar} = \mathrm{Haar}$ for any $h \in G$ (see [Lan93, Chapter

XII]). The harmonic analysis on G relies on a Fourier transform of the space $\mathscr{L}^2(G, dg)$ of (complex-valued) measurable functions on G that are square-integrable with respect to the Haar measure. This space $\mathscr{L}^2(G)$ is an Hilbert space for the scalar product

$$\langle f_1 \mid f_2 \rangle = \int_G \overline{f_1(g)}\, f_2(g)\, dg$$

and a Banach algebra for the convolution product

$$(f_1 * f_2)(g) = \int_G f_1(gh^{-1})\, f_2(h)\, dh.$$

On the other hand, recall that a linear (complex) *representation* of G is given by a finite-dimensional complex vector space V and by a continuous morphism of groups $\rho : G \to \mathrm{GL}(V)$ (*cf.* [Ser77, Chapter 4]). Then, one can always find an Hermitian scalar product $\langle \cdot \mid \cdot \rangle_V$ on V such that ρ takes its values in the unitary group $\mathrm{U}(V)$ with respect to this scalar product. The action of G on vectors $v \in V$ is denoted $(\rho(g))(v)$, or simply $g \cdot v$. A linear representation (V, ρ) is called *irreducible* if one cannot find a vector subspace W with $0 \subsetneq W \subsetneq V$ and that is G-stable, *i.e.*, $g \cdot w \in W$ for any $g \in G$ and any $w \in W$. If V is a linear representation of G, then one can always split it into irreducible components:

$$V = \bigoplus_{\lambda \in \widehat{G}} m_\lambda V^\lambda,$$

where $\widehat{G}$ is the countable set of isomorphism classes of irreducible representations $\lambda = (V^\lambda, \rho^\lambda)$ of G, and the coefficients m_λ are non-negative integers. Moreover, this decomposition is unique, so the Grothendieck group of isomorphism classes of representations of G is

$$\mathrm{R}(G) = \mathbb{Z}\widehat{G} = \bigoplus_{\lambda \in \widehat{G}} \mathbb{Z} V^\lambda.$$

Let us now introduce the non-commutative Fourier transform of G:

Definition 3.1 (Non-commutative Fourier transform). *Let G be a compact group, and $f \in \mathscr{L}^1(G, dg)$. The Fourier transform of f is the function $\widehat{f}$ on $\widehat{G}$ with values in $\bigsqcup_{\lambda \in \widehat{G}} \mathrm{End}_{\mathbb{C}}(V^\lambda)$ and that is defined by:*

$$\widehat{f}(\lambda) = \int_G f(g)\, \rho^\lambda(g)\, dg.$$

If $f \in \mathscr{L}^2(G)$, then it is convenient to see the Fourier transform $\widehat{f}$ as an element of

$$\mathscr{L}^2(\widehat{G}) = \bigoplus_{\lambda \in \widehat{G}}^{\perp} \mathrm{End}_{\mathbb{C}}(V^\lambda),$$

where the direct sum is orthogonal and completed to get an Hilbert space. Here, each endomorphism space $\mathrm{End}_{\mathbb{C}}(V^\lambda)$ is endowed with the scalar product

$$\langle u \mid v \rangle_{\mathrm{End}(V^\lambda)} = d_\lambda \,\mathrm{tr}(u^* v),$$

where $d_\lambda = \dim_{\mathbb{C}}(V^\lambda)$, and where the adjoint u^* of u is taken with respect to a G-invariant scalar product on V^λ.

Theorem 3.2 (Peter–Weyl, [PW27]). *The map $f \in \mathscr{L}^2(G) \mapsto \widehat{f} \in \mathscr{L}^2(\widehat{G})$ is an isometry of Hilbert spaces, and an isomorphism of algebras. It is also compatible with the action of G on the left or the right of $\mathscr{L}^2(G)$ and $\mathscr{L}^2(\widehat{G})$.*

Here, $\mathscr{L}^2(G)$ and the spaces $\mathrm{End}_{\mathbb{C}}(V^\lambda)$ are endowed with their natural structures of (G,G)-bimodules:

$$((g_1,g_2)\cdot f)(h) = f(g_1hg_2) \qquad ; \qquad (g_1,g_2)\cdot u = (\rho^\lambda(g_1))\circ u \circ (\rho^\lambda(g_2)).$$

Theorem 3.2 amounts to the properties (0.2) and (0.3) listed in the preface (Parseval's formula and convolution formula). The inversion formula (Equation (0.1)) reads in this setting:

$$f(g) = \sum_{\lambda\in\widehat{G}} d_\lambda \,\mathrm{tr}\left(\widehat{f_-}(\lambda)\,\rho^\lambda(g)\right),$$

where $f_-(g) = f(g^{-1})$, and the sum on the right-hand side converges in the Hilbert space $\mathscr{L}^2(G)$. This formula is a generalisation to arbitrary compact groups of the Fourier expansion of square-integrable functions on the circle:

$$f(\theta) = \sum_{n\in\mathbb{Z}} c_n(f)\,\mathrm{e}^{in\theta}, \quad \text{with } c_n(f) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,\mathrm{e}^{-in\theta}\,d\theta.$$

There is an extension of Theorem 3.2 to quotients $X = G/K$ of a compact topological group by a closed subgroup. The most convenient setting for this extension is the one of compact *Gelfand pairs* (*cf.* [CST08]):

Definition 3.3 (Gelfand pair). *If G is a compact topological group and $K \subset G$ is a closed subgroup, one says that (G,K) is a compact Gelfand pair if, for any irreducible representation V^λ of G, the space of fixed vectors*

$$V^{\lambda,K} = \{v \in V^\lambda \,|\, \forall k \in K,\ k\cdot v = v\}$$

has dimension 0 *or* 1.

Given a compact Gelfand pair (G,K), an irreducible representation V^λ is called *spherical* if $\dim_{\mathbb{C}}(V^{\lambda,K}) = 1$. We denote $\widehat{G}^K$ the set of spherical representations of the pair (G,K), and if $\lambda \in \widehat{G}^K$, we denote e^λ a unit vector in $V^{\lambda,K}$ (with respect to a G-invariant scalar product); it is unique up to multiplication by a scalar of modulus 1.

Definition 3.4 (Spherical Fourier transform). *If (G,K) is a compact Gelfand pair, the spherical Fourier transform of a function $f \in \mathscr{L}^1(G/K)$ is the function $\widetilde{f}$ from $\widehat{G}^K$ to $\bigsqcup_{\lambda\in\widehat{G}^K} V^\lambda$ that is defined by*

$$\widetilde{f}(\lambda) = \sqrt{d_\lambda}\,\left(\widehat{f}(\lambda)\right)(e^\lambda) = \sqrt{d_\lambda}\int_G f(g)\,\left(\rho^\lambda(g)\right)(e^\lambda)\,dg.$$

If $f \in \mathscr{L}^2(G/K)$, then it is convenient to see $\widetilde{f}$ as an element of $\mathscr{L}^2(\widehat{G}^K) = \bigoplus_{\lambda\in\widehat{G}^K} V^\lambda$. We endow each spherical representation V^λ with the product

$$a\cdot b = \frac{1}{\sqrt{d_\lambda}}\left\langle e^\lambda \,\middle|\, b\right\rangle_{V^\lambda} a.$$

We then have an analogue of the Peter–Weyl theorem 3.2 for compact Gelfand pairs (see [Hel84, Chapter IV]):

Theorem 3.5 (Cartan). *The map $f \in \mathscr{L}^2(G/K) \mapsto \widetilde{f} \in \mathscr{L}^2(\widehat{G}^K)$ is an isometry of Hilbert spaces, and an isomorphism of algebras. It is compatible with the left action of G on these spaces, and one has the spherical Fourier inversion formula:*

$$f(x=gK) = \sum_{\lambda\in\widehat{G}^K} \sqrt{d_\lambda}\left\langle \widetilde{\widetilde{f}}(\lambda) \,\middle|\, (\rho^\lambda(g))(e^\lambda)\right\rangle_{V^\lambda}.$$

As a particular case of Theorems 3.2 and 3.5, one gets the Fourier expansions of conjugacy-invariant functions on G and of bi-K-invariant functions on G:

- The *irreducible characters*
$$\mathrm{ch}^\lambda(g) = \mathrm{tr}\, \rho^\lambda(g)$$
form an orthonormal basis of the Hilbert space $\mathscr{L}^2(G)^G$ of square-integrable functions on G that are invariant by conjugation: $f(ghg^{-1}) = f(h)$ for any $g, h \in G$. Moreover,
$$\mathrm{ch}^\lambda * \mathrm{ch}^\mu = \frac{\delta_{\lambda,\mu}}{d_\lambda}\, \mathrm{ch}^\lambda.$$
- Given a compact Gelfand pair (G, K), the *zonal spherical functions*
$$\mathrm{zon}^\lambda(g) = \sqrt{d_\lambda}\, \left\langle e^\lambda \,\middle|\, (\rho^\lambda(g))(e^\lambda) \right\rangle_{V^\lambda}$$
form an orthonormal basis of the Hilbert space $\mathscr{L}^2(K\backslash G/K)$ of bi-K-invariant functions: $f(k_1 g k_2) = f(g)$ for any $k_1, k_2 \in K$. Moreover,
$$\mathrm{zon}^\lambda * \mathrm{zon}^\mu = \frac{\delta_{\lambda,\mu}}{\sqrt{d_\lambda}}\, \mathrm{zon}^\lambda.$$

▷ **Compact Lie groups.** In the remainder of this section, we shall make the previous results explicit in the case where G is a (semi)simple compact Lie group. Recall that a real *Lie group* is a smooth manifold G endowed with a product $\nabla : G \times G \to G$ and with an inversion $i : G \to G$ that are smooth maps, and that make G into a group. One associates to a Lie group G its *Lie algebra* $\mathfrak{g} = T_eG$, which is the tangent space of G at the neutral element $e = e_G$. It is endowed with the Lie bracket $[X, Y] = (\mathrm{ad}\, X)(Y)$, where
$$\begin{aligned} \mathrm{Ad} : G &\to \mathrm{End}_{\mathbb{R}}(T_eG) \\ g &\mapsto D_e(c_g) \end{aligned} \qquad ; \qquad \begin{aligned} \mathrm{ad} : T_eG &\to \mathrm{End}_{\mathbb{R}}(T_eG) \\ X &\mapsto (D_e(\mathrm{Ad}))(X) \end{aligned}$$
$c_g : G \to G$ being the conjugation by g, and D_e the differential at e. There is an equivalence of categories between real Lie algebras and simply connected real Lie groups. Thus, any real Lie algebra $\mathfrak{g}$ is the Lie algebra of a connected Lie group G, which is unique up to isomorphism if one assumes G simply connected; see for instance [Kna02, Chapter I, Section 10]. Moreover, a simply connected Lie group G is compact if and only if the Killing form
$$B(X, Y) = \mathrm{tr}((\mathrm{ad}\, X) \circ (\mathrm{ad}\, Y))$$
of its Lie algebra $\mathfrak{g}$ is negative definite. Cartan's criterion for semisimplicity [Kna02, Theorem 1.45] implies then that a simply connected compact Lie group can always be written as a direct product $G = G_1 \times G_2 \times \cdots \times G_r$, where the G_i's are *simple* simply connected compact Lie group (in short sscc Lie group). Here, by simple we mean a Lie group G whose Lie algebra $\mathfrak{g}$ is simple, that is nonabelian and without non-trivial ideal. A classification of all the simple complex Lie algebras has been given by Cartan [Car33], and it was later related to the classification of root systems by Dynkin [Dyn47]. For a modern treatment, we refer to [Hel78, Chapter X] or [Kna02, Chapter II]; other useful references are [BD85; FH91; GW09; Bum13]. In order to present this classification, it is useful to introduce the notions of weights and roots of G. Given a compact Lie group G, we fix a maximal torus $T \subset G$, that is a connected abelian subgroup of maximal dimension. A *weight* of a representation of G on a complex vector space V is a character $\chi : T \to \mathrm{U}(1)$ such that
$$V(\chi) = \{v \in V \mid \forall t \in T,\ t \cdot v = \chi(t)\, v\} \neq \{0\}.$$

Let $\mathfrak{g}$ be the Lie algebra of G, $\mathfrak{t}$ be the Lie algebra of T, and $\mathfrak{g}_\mathbb{C}$ and $\mathfrak{t}_\mathbb{C}$ be the corresponding complex Lie algebras obtained by tensoring by $\mathbb{C}$. The differential at e of a weight χ is a linear function $\omega : \mathfrak{t} \to \mathrm{i}\mathbb{R}$, so it can be seen as an element of $(\mathfrak{t}_\mathbb{C})^*$. With this interpretation, the weights form a discrete additive subgroup of $(\mathfrak{t}_\mathbb{C})^*$ (the *weight lattice* $\mathbb{Z}\Omega$), and the evaluation of a weight ω writes multiplicatively:

$$\chi(\mathrm{e}^X) = \mathrm{e}^{\omega(X)}$$

if $X \in \mathfrak{t}$ and $\omega = D_e\chi$. Any complex linear representation V of G splits into its weight spaces [Kna02, Proposition 5.4]:

$$V = \bigoplus_{\substack{\omega \in \mathbb{Z}\Omega \\ V(\omega) \neq \{0\}}} V(\omega).$$

In the following, we denote $\mathbb{R}\Omega = \mathbb{R} \otimes_\mathbb{Z} \mathbb{Z}\Omega$ the real vector subspace of $(\mathfrak{t}_\mathbb{C})^*$ spanned by the weight lattice. We endow $\mathfrak{t}_\mathbb{C}$ with the restriction of the Killing form of $\mathfrak{g}_\mathbb{C}$, $(\mathfrak{t}_\mathbb{C})^*$ with the dual of this non-degenerate bilinear form, and $\mathbb{R}\Omega \subset (\mathfrak{t}_\mathbb{C})^*$ with the restriction of this dual form. One can show that this restriction is a positive-definite scalar product on $\mathbb{R}\Omega$; in the following we always use this particular scalar product on elements of the weight space. The *Weyl group* $W = \mathrm{Norm}(T)/T$ of G is a finite group, and its action on T induces an action by isometries on $\mathbb{R}\Omega$. For any linear representation V of G, the weights of V and their multiplicities are invariant by action of the Weyl group. We call *root* of G a non-zero weight of the adjoint representation of G on its Lie algebra $\mathfrak{g}$. All these roots have multiplicity one in the adjoint representation, and one has the decomposition $\mathfrak{g}_\mathbb{C} = (\bigoplus_{\alpha\in\Phi} \mathfrak{g}_\mathbb{C}(\alpha)) \oplus \mathfrak{t}_\mathbb{C}$, where Φ denotes the set of roots. This set Φ is a (reduced, crystallographic) *root system* in $\mathbb{R}\Omega$ [Kna02, Theorem 2.42], meaning that:

(RS1) The roots span linearly $\mathbb{R}\Omega$.

(RS2) If α is a root, the the roots β colinear to α are $\beta = \pm\alpha$.

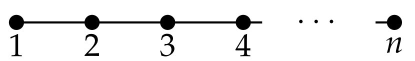
(RS3) If $\alpha \in \Phi$, then the reflection $s_\alpha : x \mapsto x - 2\frac{\langle x \,|\, \alpha\rangle}{\langle \alpha \,|\, \alpha\rangle}\,\alpha$ yields a permutation of the roots.

(RS4) If $\alpha, \beta \in \Phi$, then $2\,\frac{\langle \alpha \,|\, \beta\rangle}{\langle \alpha \,|\, \alpha\rangle}$ is an integer.

The second condition allows one to split Φ into two sets Φ_+ and $\Phi_- = -\Phi_+$. A positive root $\alpha \in \Phi_+$ is called simple if it cannot be written as a sum of two other positive roots. The Weyl group W is a Coxeter group generated by the reflections s_α with α simple root [Bum13, Theorem 18.3]. The geometry of the root system Φ is classically encoded by the Dynkin diagram of the simple roots (see [Hel78, Section X.3.3]), and one has the following classification:

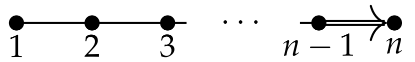
Theorem 3.6 (Cartan's classification of sscc Lie groups). *If G is a sscc Lie group, then G falls into one of the following infinite families:*

- *type* $\mathrm{A}_{n\geq 1}$*: special unitary group* $\mathrm{SU}(n+1)$*, with Dynkin diagram*



;

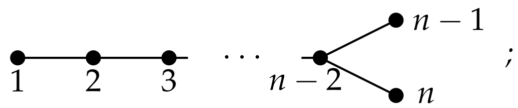
- *type* $\mathrm{B}_{n\geq 2}$*: odd spin group* $\mathrm{Spin}(2n+1)$*, with Dynkin diagram*



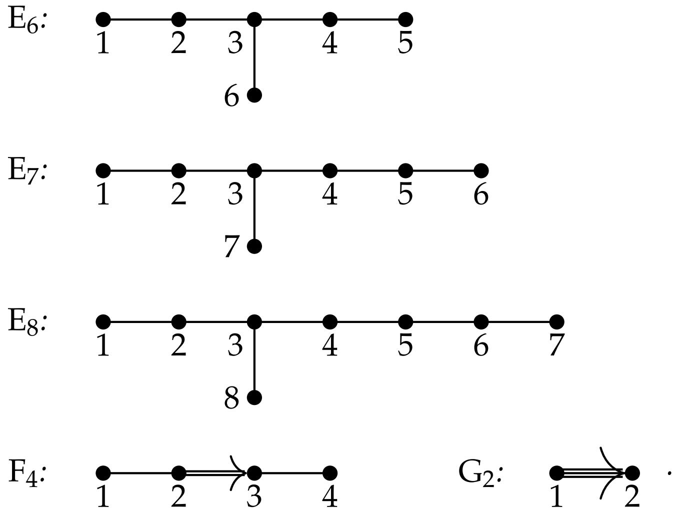
;

- *type* $\mathrm{C}_{n\geq 3}$*: compact symplectic group* $\mathrm{USp}(n)$*, with Dynkin diagram*

- *type* $\mathrm{D}_{n\geq 4}$*: even spin group* $\mathrm{Spin}(2n)$*, with Dynkin diagram*



or, it is one of the five exceptional compact Lie groups associated to the root systems E_6*,* E_7*,* E_8*,* F_4 *or* G_2*:*



Given a sscc Lie group, since the Killing form $B(\cdot,\cdot)$ on $\mathfrak{g}$ is negative definite, its opposite defines a scalar product on T_eG, which can be transported to all the other tangent spaces T_gG by using the action of the group. We thus obtain a left G-invariant Riemannian structure on G, which is also right G-invariant since B is $(\mathrm{Ad}\,G)$-invariant. This is the standard structure of Riemannian manifold on a compact Lie group, and in the classical cases A_n, B_n, C_n, and D_n, the opposite of the Killing form is a multiple of

$$\langle X \mid Y \rangle_{\mathfrak{g}} = -\mathrm{Re}(\mathrm{tr}(XY)),$$

the real part being only needed in type C_n. Depending on the applications, it is convenient to use certain renormalisations of this scalar product, and the corresponding Riemannian structures. For instance, in random matrix theory, a common choice is

$$\langle X \mid Y \rangle_{\mathfrak{g}} = -\frac{\beta n}{2}\,\mathrm{Re}(\mathrm{tr}(XY)), \tag{3.1}$$

where n is the size of the matrices considered, and $\beta = 1$, 2 or 4 for real, complex or quaternionic matrices.

▷ **Symmetric spaces.** The sscc Lie groups belong to a larger class of compact Riemannian manifolds called *symmetric spaces*:

Definition 3.7 (Symmetric space). *A (globally) symmetric space is a complete Riemannian manifold X with the following property: for any $x \in X$, there exists a (unique) isometry $i_x : X \to X$ such that, if $t \mapsto \gamma(t)$ is a geodesic with constant speed and with $\gamma(0) = x$, then $i_x(\gamma(t)) = \gamma(-t)$.*

Intuitively, a symmetric space is a Riemannian manifold where geodesics are nicely arranged in a symmetric way around any point, see Figure 3.1. Given a compact symmetric

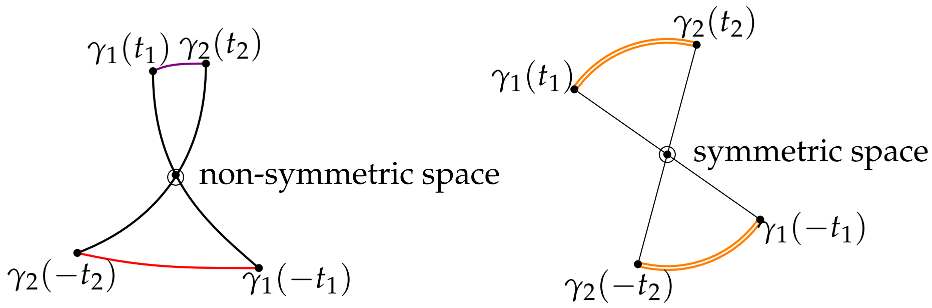


Figure 3.1. Geodesics on a symmetric space.

space X, its universal cover $\widetilde{X}$ is again a symmetric space, and the cover map $\pi : \widetilde{X} \to X$ has finite degree. On the other hand, any simply connected compact symmetric space is isometric to a product of simple simply connected compact symmetric spaces (in short ssccss), which cannot be split further. A classification of all the ssccss has been proposed by Cartan [Car27]:

Theorem 3.8 (Cartan's classification of ssccss). *Let X be a ssccss. The connected component G of the identity in $\mathrm{Isom}(X)$ is a compact Lie group, and there exists an involutive automorphism $\sigma : G \to G$ such that $X = G/K$ and*

$$G^{\sigma,0} \subset K \subset G^{\sigma},$$

where $G^{\sigma} = \{g \in G \,|\, \sigma(g) = g\}$ and $G^{\sigma,0}$ is the connected component of the identity in this subgroup of G. Moreover, (G, K) forms a compact Gelfand pair, and X is one of the following objects:

(1) group type: $X = K$ is a sscc Lie group classified by Theorem 3.6, $G = K \times K$ and $\sigma(k_1, k_2) = (k_2, k_1)$ (and X is the diagonal subgroup).

(2) non-group type: X falls into one of the following infinite families:
- *real Grassmannians:* $\mathrm{SO}(p+q)/(\mathrm{SO}(p) \times \mathrm{SO}(q)$;
- *complex Grassmannians:* $\mathrm{SU}(p+q)/(\mathrm{S}(\mathrm{U}(p) \times \mathrm{U}(q)))$;
- *quaternionic Grassmannians:* $\mathrm{USp}(p+q)/(\mathrm{USp}(p) \times \mathrm{USp}(q))$;
- *space of real structures on a complex space:* $\mathrm{SU}(n)/\mathrm{SO}(n)$;
- *space of quaternionic structures on an even complex space:* $\mathrm{SU}(2n)/\mathrm{USp}(n)$;
- *space of complex structures on a quaternionic space:* $\mathrm{USp}(n)/\mathrm{U}(n)$;
- *space of complex structures on an even real space:* $\mathrm{SO}(2n)/\mathrm{U}(n)$;

or, it is one of the twelve exceptional sscc symmetric spaces that write as quotients of one of the five exceptional compact Lie groups (all of them being related to the geometry of octonions, see [Bae02]).

Given a symmetric quotient $X = G/K$ as in Theorem 3.8, the tangent space of X at $e_G K$ can be identified with the orthogonal $\mathfrak{p}$ of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Killing form. The restriction of the opposite Killing form to $\mathfrak{p}$ can then be transported to any other tangent space $T_x X$ by using the action of G, and one thus gets a G-invariant Riemannian structure on X, which is up to a scalar multiple its structure of Riemannian symmetric space. As in the group case, if X falls into one of the seven infinite families of symmetric spaces of non-group type, then it is sometimes convenient to rescale the distances and work with another renormalisation of the scalar product on $\mathfrak{p}$, for instance the one given by Equation (3.1). As explained in the introduction of the chapter, a general objective consists in studying random objects drawn on a ssccss, by using the non-commutative Fourier transform if $X = G$ is a Lie group, and the spherical Fourier transform if $X = G/K$ is a symmetric space of type non-group.

▷ **Irreducible and spherical representations.** Given a ssccss, we now make explicit the set $\widehat{G}$ in the group case, and the set $\widehat{G}^K$ in the non-group case. The positive roots $\alpha \in \Phi_+$ determine in $\mathbb{R}\Omega$ a cone

$$C = \{x \in \mathbb{R}\Omega \mid \forall \alpha \in \Phi_+,\ \langle \alpha \mid x \rangle \geq 0\}$$

called the fundamental *Weyl chamber*, and which is a fundamental domain for the action of the Weyl group W. This cone determines a partial order on the set of weights:

$$\omega_1 \leq \omega_2 \iff \omega_2 - \omega_1 \in C.$$

A weight is called *dominant* if it belongs to C.

Theorem 3.9 (Weyl's highest weight theorem, [Wey25; Wey26; Wey32]). *The representations of a sscc Lie group G are completely reducible, and there is an equivalence of categories between the representations of G and those of the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$. Given an irreducible complex linear representation V of G, it admits a unique highest weight λ, which is dominant. This highest weight determines the isomorphism class of V, so*

$$\widehat{G} = C \cap \mathbb{Z}\Omega$$

is in bijection with the set of dominant weights.

Example 3.10 (Representation theory of $\mathrm{SU}(3)$). Consider the group $G = \mathrm{SU}(3)$. A maximal torus is $T = \{\mathrm{diag}(z_1, z_2, z_3) \mid |z_i| = 1 \text{ and } z_1 z_2 z_3 = 1\}$, and a basis of the weight lattice consists in the two fundamental weights ω_1 and ω_2 with $e^{\omega_1}(z) = z_1$ and $e^{\omega_2}(z) = (z_3)^{-1}$. The positive roots are $\alpha_1 = 2\omega_1 - \omega_2$, $\alpha_2 = 2\omega_2 - \omega_1$ and $\alpha_1 + \alpha_2$. The dominant weights, which label the irreducible representations of $\mathrm{SU}(3)$, are the linear combinations $n_1\omega_1 + n_2\omega_2$ with $n_1, n_2 \in \mathbb{N}$, see Figure 3.2. The Weyl group is $W = \mathfrak{S}(3)$.

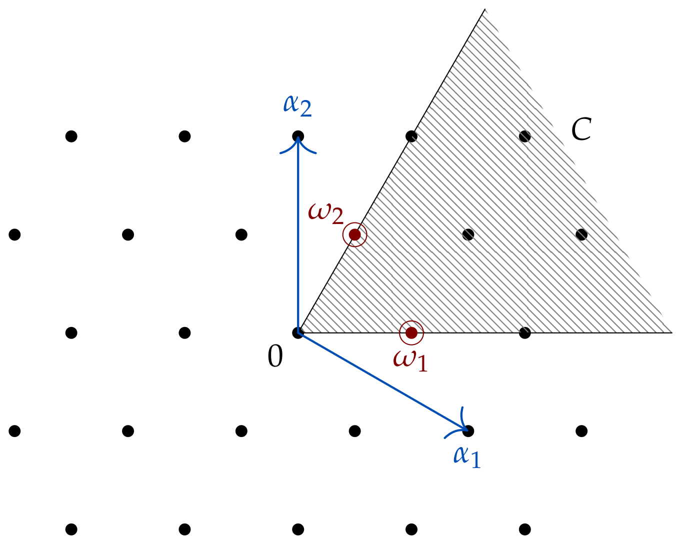


Figure 3.2. The weight lattice of the group $\mathrm{SU}(3)$.

The highest weight theorem is completed by a formula for the character ch^{λ} of the module V^{λ} with highest weight λ. Every element $g \in G$ is conjugated to an element t of the maximal torus, which is unique up to action of the Weyl group W; hence, it suffices to give a formula for $\mathrm{ch}^\lambda(t)$ with $t \in T$. This formula is

$$\mathrm{ch}^\lambda(t) = \frac{\sum_{w\in W} \varepsilon(w)\,\mathrm{e}^{\lambda+\rho}(w(t))}{\sum_{w\in W} \varepsilon(w)\,\mathrm{e}^{\rho}(w(t))},$$

where $\rho = \frac{1}{2}\sum_{\alpha\in\Phi_+} \alpha$ [Bum13, Chapter 22]. Specializing to $t = e$, one gets Weyl's dimension formula:

$$\dim_{\mathbb{C}}\left(V^\lambda\right) = \frac{\prod_{\alpha\in\Phi_+} \langle \lambda+\rho \mid \alpha\rangle}{\prod_{\alpha\in\Phi_+} \langle \rho \mid \alpha\rangle}.$$

Consider now a symmetric space of non-group type $X = G/K$. The identification of the spherical irreducible representations of the pair (G,K) is due to Cartan; the first rigorous proofs of this rule are due to Helgason and Satake, see [Sat60; Sug62; Hel70; Hel84]. The precise hypotheses (H1) and (H2) of this rule are quite subtle, and as mentioned in the introduction of the chapter, it is a bit difficult to find them written in a concise way in the literature. Let σ be the Cartan automorphism associated to a ssccss $X = G/K$, such that K is an open subgroup of G^σ; we denote $P = \{g \in G \mid \sigma(g) = g^{-1}\}$.

(H1) The maximal torus T in G is chosen so that $A = T \cap P$ is a maximal torus inside P: it is always possible to do so, up to conjugation of T (or, up to conjugation of K and of the Cartan automorphism σ).

We then set $S = T \cap K$; it is always a product of a subtorus of T by an elementary abelian 2-group. At the level of Lie algebras, the hypothesis (H1) is equivalent to the fact that $\mathfrak{a} = \mathfrak{t} \cap \mathfrak{p}$ is a maximal abelian subalgebra of $\mathfrak{p}$.

(H2) The decomposition of the set of roots Φ into positive and negative roots is such that, for $a \in \mathfrak{a}$, the set of inequalities

$$\forall \alpha \in \Phi_+,\ \alpha(\mathrm{i}a) \geq 0$$

determine a non-degenerate cone in $\mathrm{i}\mathfrak{a}$ (the $\mathbb{R}_+$-span of a linear basis of $\mathrm{i}\mathfrak{a}$).

The maximal torus T being chosen according to (H1), there always exists a specific choice of positive and negative roots that ensures that (H2) is also satisfied; see [GW09, Section 12.3] for details.

Theorem 3.11 (Cartan–Helgason). *Consider a ssccss $X = G/K$ of type non-group. Under the hypotheses (H1) and (H2), a dominant weight $\lambda \in \widehat{G}$ is spherical (hence, in $\widehat{G}^K$) if and only if, viewed as a morphism $T \to \mathrm{U}(1)$, it is trivial on S:*

$$\forall s \in S,\ (\mathrm{e}^\lambda)(s) = 1.$$

Moreover, one can always find a sublattice $L \subset \mathbb{Z}\Omega$ such that $\widehat{G}^K = C \cap L$, and the rank of this sublattice corresponds to the geometric rank of the symmetric space (maximal dimension of a flat totally geodesic submanifold).

Example 3.12 (Spherical representations for the complex projective plane). Consider the complex projective plane $\mathbb{CP}^2 = G/K$, with $G = \mathrm{SU}(3)$ and $K = \mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(1))$. A Cartan automorphism that determines a subgroup G^σ isomorphic to K is

$$\sigma(g) = NgN \quad \text{with } N = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then,

$$T \cap P = \{\mathrm{diag}(z_1, 1, (z_1)^{-1}) \mid |z_1| = 1\};$$
$$T \cap K = \{\mathrm{diag}(z_1, (z_1)^{-2}, z_1) \mid |z_1| = 1\}.$$

One can check that Hypotheses (H1) and (H2) are satisfied. The dominant weights that are trivial on S are the multiples of the spherical weight $\mathrm{e}^{\rho}(t) = \mathrm{e}^{\omega_1+\omega_2}(t) = t_1(t_3)^{-1}$, so by Theorem 3.11, $\widehat{G}^K = \mathbb{N}\rho$, see Figure 3.3.

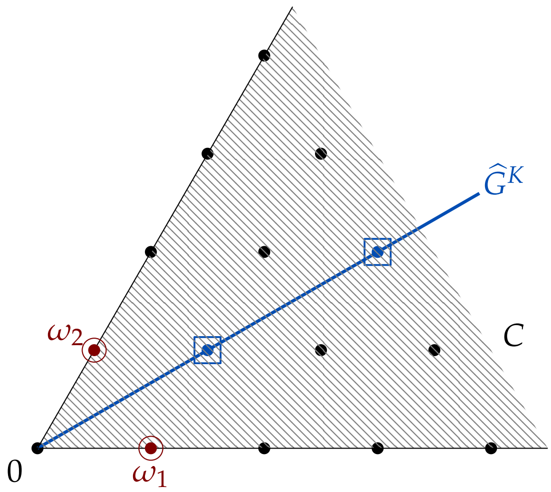


Figure 3.3. The spherical weights for the symmetric space $\mathbb{CP}^2$.

▷ **Crystals, Littelmann paths and string polytopes.** The harmonic analysis of the spaces $\mathscr{L}^2(G)^G$ and $\mathscr{L}^2(K\backslash G/K)$ is entirely solved by the previous results. Hence, these results enable the study of random objects drawn on a compact symmetric space, under an additional symmetry condition (invariance by conjugation in the group case, and bi-K-invariance in the non-group case). Unfortunately, this additional symmetry is easily broken, and for many calculations, a better understanding of the weight distribution of a representation of G is required. The modern point of view on this problem is the theory of crystal bases; we summarise it hereafter, trying to make it as concrete as possible. We shall see in Section 3.4 that the study of random geometric graphs on Lie groups leads to some conjectures related to the crystals of representations. Until the end of this section, G is a fixed sscc Lie group, $\mathfrak{g}$ is its Lie algebra, and d is the rank of G (dimension of the maximal torus). The set of simple roots of G is denoted $(\alpha_i)_{i\in[\![1,d]\!]}$. This is a linear basis of $\mathbb{R}\Omega$, and we denote $(\alpha_i^\vee)_{i\in[\![1,d]\!]}$ the basis of simple coroots in $(\mathbb{R}\Omega)^*$, defined by the relations

$$\forall i,j \in [\![1,d]\!]\,,\ \ \alpha_i(\alpha_j^\vee) = 2\,\frac{\langle \alpha_i \mid \alpha_j\rangle}{\langle \alpha_j \mid \alpha_j\rangle}.$$

The elements of the dual basis $(\omega_i)_{i\in[\![1,d]\!]}$ of the basis of coroots $(\alpha_i^\vee)_{i\in[\![1,d]\!]}$ are the fundamental weights, such that $\mathbb{Z}\Omega = \mathrm{Span}_{\mathbb{Z}}(\omega_1,\ldots,\omega_d)$.

→ *Crystals.* Let $U_q(\mathfrak{g}_{\mathbb{C}})$ be the quantum group of the complexification $\mathfrak{g}_{\mathbb{C}}$ of the Lie algebra $\mathfrak{g}$; it is a deformation with a complex parameter q of the universal enveloping algebra $U(\mathfrak{g}_{\mathbb{C}})$, see [Jim85; Jim86]. There is a corresponding deformation V_q^λ of the irreducible module V^λ, and a notion of weight vectors in V_q^λ, such that if

$$V_q^\lambda = \bigoplus_{\substack{\omega \in \mathbb{Z}\Omega \\ V_q^\lambda(\omega) \neq 0}} V_q^\lambda(\omega),$$

then the weights and the multiplicities are the same for V^λ and for V_q^λ:

$$\forall \omega \in \mathbb{Z}\Omega,\ \dim_{\mathbb{C}}\left(V_q^\lambda(\omega)\right) = \dim_{\mathbb{C}}\left(V^\lambda(\omega)\right).$$

This is the Lusztig–Rosso correspondence, see the original papers [Lus88; Ros88; Ros90], and [Mél17, Chapter 5] for a detailed exposition of the case $\mathfrak{g} = \mathfrak{gl}(n)$. The correspondence holds for any q that is not 0 or a root of unity.

Definition 3.13 (Crystal basis and crystal of a representation). *A crystal basis of the irreducible representation V_q^λ is a linear basis $\mathfrak{C}(\lambda)$ of V_q^λ that consists of weight vectors, and such that if $(e_i, f_i, q^{h_i})_{i \in [\![1,d]\!]}$ are the Chevalley generators of $U_q(\mathfrak{g}_{\mathbb{C}})$, then for any vector v of the crystal basis, $e_i \cdot v$ is either 0 or another vector v' of the crystal basis; and similarly for $f_i \cdot v$. The crystal of V_q^λ is the weighted labeled oriented graph:*

- *with vertices $v \in \mathfrak{C}(\lambda)$,*
- *with labeled oriented edges $v \to_{f_i} v'$ if $v' = f_i \cdot v$,*
- *with a weight map $\mathrm{wt}(\cdot)$ that associates to $v \in \mathfrak{C}(\lambda)$ the corresponding weight in $\mathbb{Z}\Omega$.*

Theorem 3.14 (Lusztig–Kashiwara, [Lus90; Kas90]). *Let $\mathfrak{g}_{\mathbb{C}}$ be a semisimple complex Lie algebra, and λ a dominant weight. There exists a crystal basis $\mathfrak{C}(\lambda)$ of the irreducible module V^λ, and the combinatorial structure of the corresponding weighted labeled oriented graph does not depend on q or on the choice of a crystal basis.*

If one knows the crystal $\mathfrak{C}(\lambda)$of an irreducible representation V^λ, then one recovers immediately the highest weight of the representation, and all the multiplicities of the weights: for any $\omega \in \mathbb{Z}\Omega$,

$$\dim_{\mathbb{C}}\left(V^\lambda(\omega)\right) = \mathrm{card}\{v \in \mathfrak{C}(\lambda) \,|\, v \text{ has weight } \omega\}.$$

$\to$ *Paths.* There is a concrete description of the crystal $\mathfrak{C}(\lambda)$ due to Littelmann, see [Lit95; Lit98b], and [BBO05; BBO09] for applications of this path model in probability. We call *path* on the weight space $\mathbb{R}\Omega$ a piecewise linear map $\pi : [0,1] \to \mathbb{R}\Omega$ starting at 0, and such that $\pi(1)$ belongs to the weight lattice $\mathbb{Z}\Omega$. We identify two paths if they differ by a continuous reparametrisation. The set of paths is a semigroup for the operation of concatenation:

$$(\pi_1 * \pi_2)(t) = \begin{cases} \pi_1(2t) & \text{if } t \in [0, \frac{1}{2}], \\ \pi_1(1) + \pi_2(2t-1) & \text{if } t \in [\frac{1}{2}, 1]. \end{cases}$$

Fix a simple root α. Given a path π, we set $g_\alpha(\pi, t) = \pi(t)(\alpha^\vee) = 2\,\frac{\langle \pi(t) \,|\, \alpha \rangle}{\langle \alpha \,|\, \alpha \rangle}$; this is a piecewise linear function on $[0,1]$. Suppose

$$m_\alpha = \min_{t \in [0,1]} g_\alpha(\pi, t) \leq -1.$$

We cut the path π in parts $\pi_0, \pi_1, \ldots, \pi_\ell, \pi_{\ell+1}$ such that $\pi = \pi_0 * \pi_1 * \cdots * \pi_\ell * \pi_{\ell+1}$ and:

(0) $\pi_0 * \pi_1 * \cdots * \pi_\ell$ ends at a point with a value of g_α minimal, and it is the smallest part of the whole path π with this property; and then π_0 is the largest part of the path $\pi_0 * \pi_1 * \cdots * \pi_\ell$ that ends with a value of g_α equal to $m_\alpha + 1$.

(1) either $g_\alpha(\pi)$ is strictly decreasing on the interval $[t_{i-1}, t_i]$ corresponding to the part π_i, and $g_\alpha(\pi, s) \geq g_\alpha(\pi, t_{i-1})$ for $s \leq t_{i-1}$; in other words, $g_\alpha(\pi)$ is minimal on the segment $[t_{i-1}, t_i]$.

(2) or, $g_\alpha(\pi, t_{i-1}) = g_\alpha(\pi, t_i)$ and $g_\alpha(\pi, s) \geq g_\alpha(\pi, t_{i-1})$ for $s \in [t_{i-1}, t_i]$.

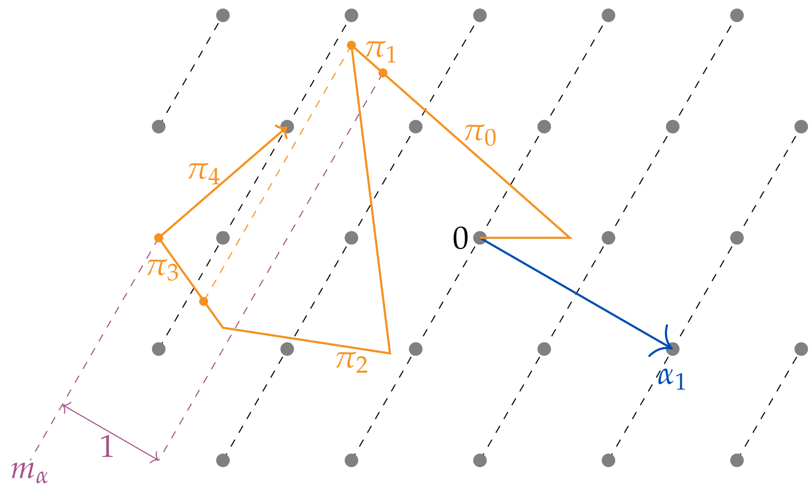


Figure 3.4. The decomposition of a path in the weight lattice of $\mathrm{SU}(3)$.

This decomposition is better understood in a picture, see Figure 3.4 for an example on the weight lattice of type A_2. Denote s_α the reflection with respect to the root α, that is the map $x \mapsto x - 2\,\frac{\langle x \,|\, \alpha\rangle}{\langle \alpha \,|\, \alpha\rangle}\,\alpha$. For $j \in [\![1, \ell]\!]$, we define

$$\pi'_j = \begin{cases} s_\alpha(\pi_j) & \text{if } \pi_j \text{ is of type } (1), \\ \pi_j & \text{if } \pi_j \text{ is of type } (2). \end{cases}$$

We then set

$$e_\alpha(\pi) = \begin{cases} \emptyset & \text{if } m_\alpha > -1, \\ \pi_0 * (\pi'_1 * \pi'_2 * \cdots * \pi'_\ell) * \pi_{\ell+1} & \text{if } m_\alpha \leq -1, \end{cases}$$

and $f_\alpha(\pi) = c \circ e_\alpha \circ c(\pi)$, where c is the map on paths defined by $(c(\pi))(t) = \pi(1-t) - \pi(1)$. Here, $\emptyset$ is a "ghost" path. An example of action of a root operator e_α is drawn in Figure 3.5.

Theorem 3.15 (Littelmann). *Let λ be a dominant weight in $\widehat{G}$, and π_λ be the segment that connects 0 to λ, considered as a path. We introduce the weighted labeled oriented graph:*

- *with vertices the paths $\pi = f_{\alpha_{i_1}} f_{\alpha_{i_2}} \cdots f_{\alpha_{i_r}}(\pi_\lambda)$ that are not the ghost path $\emptyset$, and that are obtained from π_λ by applying operators f_{α_i};*
- *with an oriented labeled edge $\pi \to_{f_i} \pi'$ if $\pi' = f_{\alpha_i}(\pi)$;*
- *with the weight map $\mathrm{wt} : \pi \mapsto \pi(1)$.*

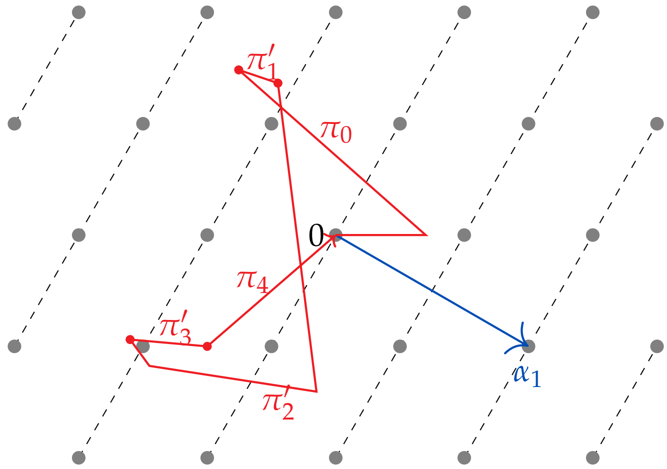


Figure 3.5. The action of $e_1 = e_{\alpha_1}$ on the path of Figure 3.4.

The crystal $\mathfrak{C}(\pi_\lambda)$ that one obtains is finite, and it is isomorphic to the crystal $\mathfrak{C}(\lambda)$ of the irreducible representation V^λ. In particular,

$$\dim_{\mathbb{C}}\left(V^\lambda(\omega)\right) = \mathrm{card}\{\textit{paths in the crystal } \mathfrak{C}(\pi_\lambda) \textit{ with endpoint } \omega\},$$

and the character ch^λ is given by the formula $\mathrm{ch}^\lambda = \sum_{\pi \in \mathfrak{C}(\pi_\lambda)} \mathrm{e}^{\mathrm{wt}(\pi)}$. Moreover, one can take instead of π_λ any path from 0 to λ that stays in the Weyl chamber: all these paths span the same crystal $\mathfrak{C}(\lambda)$.

Example 3.16 (Crystal of the adjoint representation of $\mathrm{SU}(3)$). Consider the adjoint representation of $\mathrm{SU}(3)$ on $\mathfrak{su}(3)$, which has dimension 8. The highest weight of this representation is $\lambda = \rho = \omega_1 + \omega_2$, and the crystal $\mathfrak{C}(\lambda)$ is drawn in Figure 3.6.

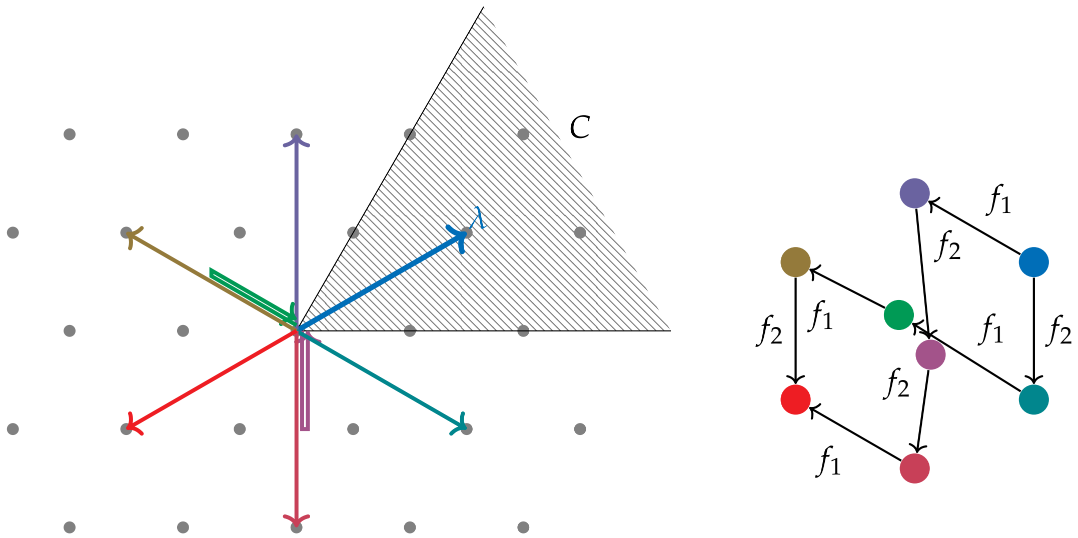


Figure 3.6. The crystal of the adjoint representation of $\mathrm{SU}(3)$, viewed as a set of paths.

Suppose that one needs to compute the moments of some random variables stemming from random objects drawn on G, and that one uses to this purpose harmonic analysis on G. One then often needs to calculate tensor products of irreducible representations, and the path model is particularly useful in this setting [Lit98b, Proposition 2 and Corollary 1]:

Proposition 3.17 (Tensor product of crystals). *Fix two dominant weights λ and μ, and consider the tensor product*

$$V^\lambda \otimes V^\mu = \sum_{\nu \in \widehat{G}} c_\nu^{\lambda,\mu} \, V^\nu.$$

The concatenation product of crystals $\mathfrak{C}(\pi_\lambda) * \mathfrak{C}(\pi_\mu)$ is a set of paths such that the action of the root operators on these paths generate a crystal whose connected components are isomorphic to the elements of the multiset

$$\left\{ (\mathfrak{C}(\pi_\nu))^{c_\nu^{\lambda,\mu}},\ \nu \in \widehat{G} \right\}.$$

In particular, the Littlewood–Richardson coefficient $c_\nu^{\lambda,\mu}$ is equal to the number of paths $\pi \in \mathfrak{C}(\pi_\mu)$ such that $\pi_\lambda * \pi$ always stays in the Weyl chamber C, and $\pi_\lambda * \pi$ ends at the dominant weight ν.

$\to$ *Polytopes.* The previous paragraph has shown how to interpret the Kostka numbers $K_\omega^\lambda = \dim_{\mathbb{C}}(V^\lambda(\omega))$ and the Littlewood–Richardson coefficients $c_\nu^{\lambda,\mu}$ as certain numbers of paths in the weight lattice. If one is interested in the asymptotics of these quantities as the weights $\lambda, \mu, \ldots$ grow to infinity, instead of the path description of the crystal $\mathfrak{C}(\lambda)$, it is more convenient to deal with a polyhedral description, which first appeared in works of Berenstein, Littelmann, Nakashima and Zelevinsky [BZ93; BZ96; NZ97; Lit98a; BZ01]. In this paragraph, we fix a dominant weight λ, and a decomposition of the longest element w_0 of the Weyl group as a product of reflections s_{α_i} along the walls of the Weyl chamber C:

$$w_0 = s_{\alpha_{i_1}} s_{\alpha_{i_2}} \cdots s_{\alpha_{i_l}}, \qquad \text{with } l = \operatorname{card} \Phi_+.$$

If v is an element of the crystal $\mathfrak{C}(\lambda)$, we call *string parametrisation* of v the vector of integers $(n_1, n_2, \ldots, n_l) \in \mathbb{N}^r$ such that:

- n_1 is the maximal integer such that $e_{\alpha_{i_1}}^{n_1}(v) \neq 0$;
- n_2 is the maximal integer such that $e_{\alpha_{i_2}}^{n_2} e_{\alpha_{i_1}}^{n_1}(v) \neq 0$;
- if $n_1, \ldots, n_{s-1}$ are known, then n_s is the maximal integer such that $e_{\alpha_{i_s}}^{n_s} \cdots e_{\alpha_{i_1}}^{n_1}(v) \neq 0$.

Example 3.18 (String parametrisation of a SU(3)-crystal). For SU(3), we fix the decomposition $s_1 s_2 s_1$ of the longest element w_0 of the Weyl group $W = \mathfrak{S}(3)$. Then, the string parametrisation of the crystal graph of the adjoint representation appears in Figure 3.7.

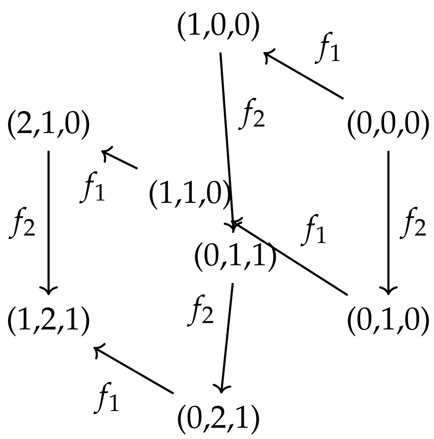


Figure 3.7. String parametrisation of the elements of the crystal of the adjoint representation of SU(3).

Given a vertex $v \in \mathfrak{C}(\lambda)$ with string parametrisation $(n_1, \ldots, n_l)$, one has

$$v = f_{\alpha_{i_1}}^{n_1} f_{\alpha_{i_2}}^{n_2} \cdots f_{\alpha_{i_l}}^{n_l}(v_\lambda),$$

where v_λ is the unique element of the crystal with weight λ. As a consequence,

$$\mathrm{wt}(v) = \lambda - \sum_{j=1}^{l} n_j \, \alpha_{i_j}.$$

We denote $\mathscr{S}(\lambda)$ the set of all string parametrisations of elements of the crystal $\mathfrak{C}(\lambda)$, and $\mathscr{S}(G) = \bigcup_{\lambda \in \widehat{G}} \mathscr{S}(\lambda)$. We also denote $\mathscr{SC}(G)$ the *string cone* of G, that is the real cone (set of non-negative linear combinations) spanned by the elements of $\mathscr{S}(G)$. Finally, for $\lambda \in \widehat{G}$, let $\mathscr{P}(\lambda)$ be the *string polytope* of λ: it is the set of elements $(u_1, \ldots, u_l)$ in the string cone and such that

$$\begin{aligned}
u_l &\leq \lambda(\alpha_{i_l}^\vee);\\
u_{l-1} &\leq (\lambda - u_l \alpha_{i_l})(\alpha_{i_{l-1}}^\vee);\\
u_{l-2} &\leq (\lambda - u_l \alpha_{i_l} - u_{l-1}\alpha_{i_{l-1}})(\alpha_{i_{l-2}}^\vee);\\
&\vdots \qquad\quad \vdots\\
u_1 &\leq (\lambda - u_l \alpha_{i_l} - \cdots - u_2 \alpha_{i_2})(\alpha_{i_1}^\vee).
\end{aligned}$$

Proposition 3.19 (Littelmann). *The string cone $\mathscr{SC}(G)$ is a rational convex cone delimited by a finite number of hyperplanes in $\mathbb{R}^l$. The string parametrisations in $\mathscr{S}(G)$ are the integer points of the string cone $\mathscr{SC}(G)$, and the string parametrisations in $\mathscr{S}(\lambda)$ are the integer points of the string polytope $\mathscr{P}(\lambda)$.*

The string polytope $\mathscr{P}(\lambda)$ has maximal dimension l as long as λ does not belong to the walls of the Weyl chamber. On the other hand, an explicit description of the string cone is given in [BZ01, Theorem 3.10]. A *trail* from a weight ϕ to another weight π of an (irreducible) representation V of $\mathfrak{g}_\mathbb{C}$ is a sequence of weights $\phi = \phi_0, \phi_1, \ldots, \phi_l = \pi$ of V such that:

(1) $\phi_{j-1} - \phi_j = k_j \, \alpha_{i_j}$ for any $j \in [\![1, l]\!]$, with the k_j's non-negative integers;

(2) $e_{i_1}^{k_1} e_{i_2}^{k_2} \cdots e_{i_l}^{k_l}$ is a non-negative linear map from $V(\pi)$ to $V(\phi)$.

The trails are simply the images by the weight map of the directed paths on the crystal graph.

Proposition 3.20 (Berenstein–Zelevinsky). *Let ${}^{\mathrm{L}}\mathfrak{g}_\mathbb{C}$ be the dual Langlands Lie algebra of the complexified Lie algebra $\mathfrak{g}_\mathbb{C}$ of a sscc Lie group G; it is the Lie algebra obtained from $\mathfrak{g}_\mathbb{C}$ by exchanging roots and coroots, respectively weights and coweights. The string cone $\mathscr{SC}(G)$ consists in all the sequences $(x_1, x_2, \ldots, x_l) \in (\mathbb{R}_+)^l$ such that, for any $i \in [\![1, d]\!]$ and any trail $(\phi_0^\vee, \phi_1^\vee, \ldots, \phi_l^\vee)$ from $\omega_i^\vee$ to $w_0 s_i(\omega_i^\vee)$ in the fundamental representation $V^{\omega_i^\vee}$ of ${}^{\mathrm{L}}\mathfrak{g}_\mathbb{C}$,*

$$\sum_{j=1}^{l} x_j \, \alpha_{i_j}\left(\frac{\phi_{j-1}^\vee + \phi_j^\vee}{2}\right) \geq 0.$$

Example 3.21 (String cone and string polytope of a representation of SU(3)). For $G = \mathrm{SU}(3)$, the string cone is the set of triples $(x, y, z) \in (\mathbb{R}_+)^3$ such that $y \geq z$; see [Lit98a, Corollary 2]. The string polytope of the adjoint representation with highest weight $\lambda = \omega_1 +$

ω_2 is then the subset of the string cone:

$$\mathscr{P}(\omega_1+\omega_2) = \{(x,y,z) \in (\mathbb{R}_+)^3 \,|\, z \leq 1,\ z \leq y \leq 1+z,\ x \leq 1-2z+y\}.$$

This polytope is drawn in Figure 3.8, and one can check that it contains eight integer points.

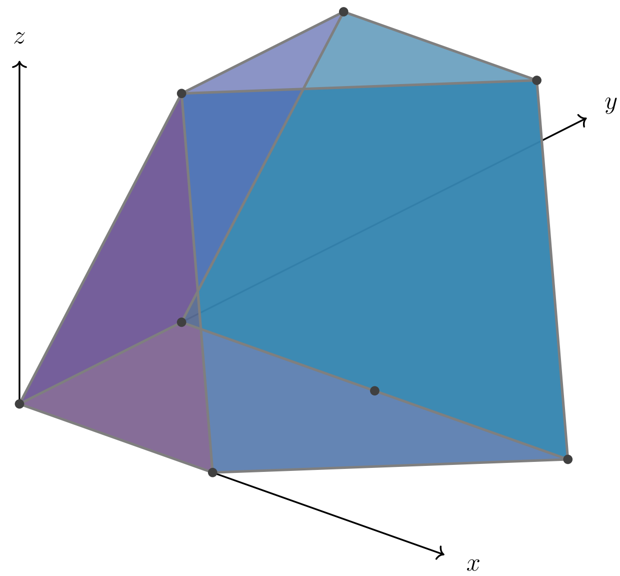


Figure 3.8. The string polytope of the adjoint representation of SU(3).

Propositions 3.19 and 3.20 show that the combinatorics of the representations of a sscc Lie group can often be restated in terms of numbers of integer points in a polyhedral domain. For instance, if λ is a dominant weight and if ω is a weight, then the Kostka number K_ω^λ is the number of integer points $(x_1, x_2, \ldots, x_l)$ in the string polytope $\mathscr{P}(\lambda)$ such that

$$\lambda - \sum_{j=1}^{l} x_j\, \alpha_{i_j} = \omega.$$

Moreover, the equations determining the string polytope $\mathscr{P}(\lambda)$ are homogeneous in λ, which allows one to prove asymptotic results. For instance, by using the fact that the image of the Lebesgue measure on a compact polytope by an affine map is a piecewise polynomial measure which is also compactly supported by a polytope, one shows readily (see [BBO09, §5.3] and [Mél18, Proposition 5.18]):

Proposition 3.22 (Asymptotics of Kostka numbers). *Fix a direction x in the Weyl chamber $C \subset \mathbb{R}\Omega$, and a continuous bounded function f on $\mathbb{R}\Omega$. We assume that x does not belong to the walls of the Weyl chamber. Then, there exists a probability measure $m_x(y)\,dy$ on $\mathbb{R}\Omega$ that is supported by* $\mathrm{Conv}(\{w(x)\,|\,w \in W\})$*, that has a piecewise polynomial density m_x, and such that*

$$\lim_{\substack{t\to\infty \\ tx \in \widehat{G}}} \left(\frac{\prod_{\alpha\in\Phi_+} \langle \rho \,|\, \alpha\rangle}{t^l \prod_{\alpha\in\Phi_+} \langle x \,|\, \alpha\rangle} \sum_{\omega\in\mathbb{Z}\Omega} K_\omega^{tx}\, f\left(\frac{\omega}{t}\right)\right) = \int_{\mathrm{Conv}(W(x))} f(y)\, m_x(y)\, dy.$$

The local degree of $y \mapsto m_x(y)$ is bounded by $l - d = |\Phi_+| - \mathrm{rank}(G)$, and one has the scaling property

$$m_{\gamma x}(\gamma y) = \frac{m_x(y)}{\gamma^d}.$$

The probability measure $m_x(y)\,dy$ is a version of the *Duistermaat–Heckman measure*. Here, dy is the Lebesgue measure with respect to the scalar product on $\mathbb{R}\Omega$ associated to the Killing form. Informally, Proposition 3.22 can be restated as

$$K_{ty}^{tx} \simeq t^{l-d} \left(\prod_{\alpha \in \Phi_+} \frac{\langle x \mid \alpha \rangle}{\langle \rho \mid \alpha \rangle} \right) m_x(y) \sqrt{\det R_G}\, 1_{t(x-y) \in R}, \tag{3.2}$$

where $R_G = (\langle \alpha_i \mid \alpha_j \rangle)_{1 \le i,j \le d}$, and $\sqrt{\det R_G}$ is the volume of a fundamental domain of the root lattice $R = \mathrm{Span}_{\mathbb{Z}}(\alpha_1, \ldots, \alpha_d)$ of G. There are similar results for the Littlewood–Richardson coefficients, which we shall give in Section 3.4 (see Proposition 3.48). Moreover, a polyhedral description of more general structure coefficients of the representations of G is conjectured, and conditionally to this conjecture, we shall be able to describe the asymptotic behavior of Poissonian random geometric graphs, see Conjectures 3.41 and 3.50 and the corresponding discussion.

3.2 Cut-off phenomenon for Brownian motions

Let X be a fixed sscc symmetric space that is either a classical compact Lie group in one of the four infinite families of Theorem 3.6, or a quotient in one the seven infinite families described in Theorem 3.8. Thus, we exclude the case of the five exceptional compact Lie groups and of the twelve exceptional quotients. In the case of the spin groups, we shall actually state our results for the special orthogonal groups, which are covered twice by the corresponding spin groups; it is easy to deduce one case from the other case, see Remark 3.32. If $X = G$ is of group type, then we endow X with its Riemannian structure corresponding to the normalisation (3.1). If $X = G/K$ is of non-group type, then we endow X with the Riemannian structure corresponding to the restriction to $\mathfrak{p} = \mathfrak{k}^\perp$ of the scalar product on $\mathfrak{g}$ given again by Equation (3.1).

▷ **Brownian motions and their asymptotic behavior.** We are interested in the behavior of a *Brownian motion* $(B_t)_{t \ge 0}$ drawn on X and started from e_G in the group case, and from $e_G K$ in the non-group case. This is the continuous Feller process on X whose infinitesimal generator restricted to the space of twice differentiable functions is $\frac{1}{2}\Delta$, where Δ is the Laplace–Beltrami operator. This random process is unique in law, and for any $t > 0$, the random variable B_t has a smooth density $p_t(x)\,dx$ with respect to the Haar measure of the ssccss. We refer to [Lia04b] for details on the construction of these processes; it suffices to deal with the group case, because the projection on $X = G/K$ of a Brownian motion on the sscc Lie group G is a Brownian motion on X. The Brownian motions on compact matrix groups can be written as solutions of certain stochastic differential equations, and also as translation-invariant continuous Lévy processes. As t goes to infinity, the law μ_t of B_t converges to the uniform probability measure on X, and the density $p_t(x) = d\mu_t(x)/dx$ converges uniformly to 1; this is actually true for more general hypo-elliptic diffusion processes, see [Lia04a] and [Lia04b, Chapter 4]. This raises the question of the speed of convergence, which can be measured thanks to the total variation distance:

$$d_{\mathrm{TV}}(\mu_t, \mathrm{Haar}) = \sup_{A \subset X} |\mathbb{P}[B_t \in A] - \mathrm{Haar}(A)| = \frac{1}{2} \int_X |p_t(x) - 1|\, dx.$$

It was well known since the works of Diaconis [DS81; AD86; BD92; Dia96] that random walks on groups can exhibit a *cut-off phenomenon*, meaning that the total variation distance between the marginal law and the stationary measure goes from 1 to 0 in a very short window of time. In [Mél14b, Theorem 6], we proved an analogue result for Brownian motions on ssccss:

Theorem 3.23 (Cut-off times of Brownian motions). *Let $(\mu_t)_{t\geq 0}$ be the family of marginal laws of the Brownian motion traced on a ssccss X of size n. There exist positive constants $\alpha, \gamma_a, \gamma_b, c, C$ such that*

$$\forall \varepsilon \in (0, 1/4),\ \ d_{\mathrm{TV}}(\mu_t, \mathrm{Haar}) \geq 1 - \frac{c}{n^{\gamma_b \varepsilon}} \quad \text{if } t = \alpha\,(1-\varepsilon)\,\log n; \tag{3.3}$$

$$\forall \varepsilon \in (0, \infty),\ \ d_{\mathrm{TV}}(\mu_t, \mathrm{Haar}) \leq \frac{C}{n^{\gamma_a \varepsilon/4}} \quad \text{if } t = \alpha\,(1+\varepsilon)\,\log n. \tag{3.4}$$

These constants only depend on the type of the ssccss that is considered. Assuming $n \geq 10$, we can take the following constants:

X	β	α	γ_b	γ_a	c	C
$\mathrm{SO}(n)$	1	2	2	2	36	6
$\mathrm{SU}(n)$	2	2	2	4	8	10
$\mathrm{USp}(n)$	4	2	2	2	5	3
$\mathrm{Gr}(n, q, \mathbb{R})$	1	1	1	1	32	2
$\mathrm{Gr}(n, q, \mathbb{C})$	2	1	1	2	32	2
$\mathrm{Gr}(n, q, \mathbb{H})$	4	1	1	1	16	2
$\mathrm{SO}(2n)/\mathrm{U}(n)$	1	1	2	1	8	2
$\mathrm{SU}(n)/\mathrm{SO}(n)$	2	1	2	2	24	8
$\mathrm{SU}(2n)/\mathrm{USp}(n)$	2	1	2	2	22	8
$\mathrm{USp}(n)/\mathrm{U}(n)$	4	1	2	1	17	2

where $\mathrm{Gr}(n, q, \mathbb{F})$ is the Grassmannian of q-dimensional vector subspaces in $\mathbb{F}^n$, with $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$.

Thus, the convergence to the stationary measure occurs at a very specific time $\alpha \log n$, with $\alpha = 2$ in the group case and $\alpha = 1$ in the non-group case; see Figure 3.9. One can also show that the other $\mathscr{L}^p$-distances $\|p_t - 1\|_{\mathscr{L}^p(X)}$ with $p \in (1, +\infty)$ have a cut-off at the same time as the $\mathscr{L}^1$-distance, and that the $\mathscr{L}^\infty$-distance has a cut-off at twice this time [Mél14b, Theorem 7]; this extension relied on previous results of Chen and Saloff-Coste [Sal94; Sal04; CS08].

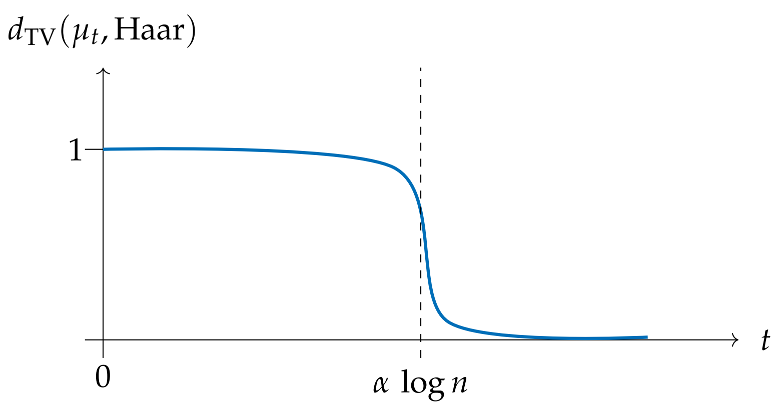


Figure 3.9. Total variation distance between μ_t and the Haar distribution on a ssccss of size n.

▷ **Upper bound and eigenvalues of the kernel.** The method of proof of Theorem 3.23 can theoretically be adapted to many other cases, so let us detail it a bit. To obtain the upper bound (3.4), one can use the Cauchy–Schwarz inequality and the isometric properties of the non-commutative Fourier transform (Theorems 3.2 and 3.5). Thus, in the group case,

$$\begin{aligned}(d_{\mathrm{TV}}(\mu_t,\mathrm{Haar}))^2 &= \frac{1}{4}\left(\|p_t-1\|_{\mathscr{L}^1(G)}\right)^2 \leq \frac{1}{4}\left(\|p_t-1\|_{\mathscr{L}^2(G)}\right)^2 \\ &\leq \frac{1}{4}\sum_{\substack{\lambda\in\widehat{G}\\ \lambda\neq 0}} d_\lambda \,\mathrm{tr}((\widehat{p_t}(\lambda))^*(\widehat{p_t}(\lambda))) = \frac{1}{4}\sum_{\substack{\lambda\in\widehat{G}\\ \lambda\neq 0}} \left|\int_G \mathrm{ch}^\lambda(g)\,p_t(g)\,dg\right|^2.\end{aligned}$$

Similarly, in the non-group case, one has the inequality

$$(d_{\mathrm{TV}}(\mu_t,\mathrm{Haar}))^2 \leq \frac{1}{4}\sum_{\substack{\lambda\in\widehat{G}^K\\ \lambda\neq 0}} \left|\int_X \mathrm{zon}^\lambda(x)\,p_t(x)\,dx\right|^2,$$

where zon^λ is the zonal spherical function of label λ. It turns out that one can compute exactly the terms of these series:

$$f(\lambda) = \begin{cases}(d_\lambda)^2\,\mathrm{e}^{-a_\lambda t} & \text{in the group case,}\\ d_\lambda\,\mathrm{e}^{-a_\lambda t} & \text{in the non-group case}\end{cases}$$

with positive constants a_λ which are polynomials of degree 2 in the coordinates of the dominant weights λ. Moreover, at time $t = t_{\text{cut-off}} = \alpha \log n$, each individual term $f(\lambda)$ is uniformly bounded (independently of λ and of n). When $t = \alpha\,(1+\varepsilon)\,\log n$, one can then compare the total variation distance with what is essentially the generating series of non-empty integer partitions:

$$\sum_{\substack{\lambda\in\mathfrak{Y}\\ \lambda\neq 0}} n^{-\varepsilon|\lambda|} = O(n^{-\varepsilon}).$$

This explains the decay of the total variation distance after cut-off time, and this approach is general in the following sense. Suppose that one wants to compute the speed of convergence of a Markov process on a space $\mathfrak{X}$, and that the kernel of this process is associated to compact operators on the space $\mathscr{L}^2(\mathfrak{X},\mu)$, μ being a reference measure (for instance, the stationary measure of the Markov process). Then, a careful estimation of the eigenvalues of these operators usually provides the time to stationarity, or at least an upper bound. Thus, if the eigenvalues of the convolution by the law μ_t of the random process are $\{e_\lambda(t),\ \lambda\in\Lambda\}$, with each eigenvalue $e_\lambda(t)$ that has multiplicity m_λ, then the time to stationarity is usually obtained by solving the equation

$$m_\lambda\, e_\lambda(t) \approx 1$$

for any $\lambda \in \Lambda$.

▷ **Lower bound and discriminating functions.** Before cut-off time, in order to prove the lower bound (3.3), the idea is to find *discriminating functions* whose values are close to 0 on an Haar distributed element of X, and close to a different value when evaluated on a Brownian motion at time $t < t_{\text{cut-off}}$. These discriminating functions are the irreducible characters or the zonal spherical functions associated to the dominant weights $\lambda = \lambda_{\min}$ that yield the smallest constants a_λ:

X	discriminating representation $V^{\lambda_{\min}}$
$\mathrm{SO}(n)$	$\mathbb{C}^n$
$\mathrm{SU}(n)$	$\mathbb{C}^n$
$\mathrm{USp}(n)$	$\mathbb{C}^{2n}$

X	discriminating representation $V^{\lambda_{\min}}$
$\mathrm{Gr}(n,q,\mathbb{R})$	$\mathfrak{so}^{\perp}(n,\mathbb{C}) \subset \mathfrak{sl}(n,\mathbb{C})$
$\mathrm{Gr}(n,q,\mathbb{C})$	$\mathfrak{sl}(n,\mathbb{C})$
$\mathrm{Gr}(n,q,\mathbb{H})$	$\mathfrak{sp}^{\perp}(2n,\mathbb{C}) \subset \mathfrak{sl}(2n,\mathbb{C})$
$\mathrm{SO}(2n)/\mathrm{U}(n)$	$\mathcal{A}^2(\mathbb{C}^{2n})$
$\mathrm{SU}(n)/\mathrm{SO}(n)$	$\mathcal{S}^2(\mathbb{C}^{n})$
$\mathrm{SU}(2n)/\mathrm{USp}(n)$	$\mathcal{A}^2(\mathbb{C}^{2n})$
$\mathrm{USp}(n)/\mathrm{U}(n)$	$\mathcal{S}^2(\mathbb{C}^{2n})$

For instance, when $X = \mathrm{SU}(n)$, the discriminating function is just the trace of a random matrix. Under the Haar measure, it is a complex random variable with mean 0 and $\mathbb{E}[|\mathrm{tr}\, M|^2] = 1$. On the other hand, under the Brownian measure μ_t, it is a complex random variable with mean

$$n\,\mathrm{e}^{-\frac{t}{2}\left(1-\frac{1}{n^2}\right)}$$

and variance smaller than 1. Therefore, as long as $t \leq 2\log n$, the trace of a unitary Brownian random matrix stays with high probability far from 0, whereas the trace of a Haar distributed unitary random matrix stays close to 0 (Figure 3.10).

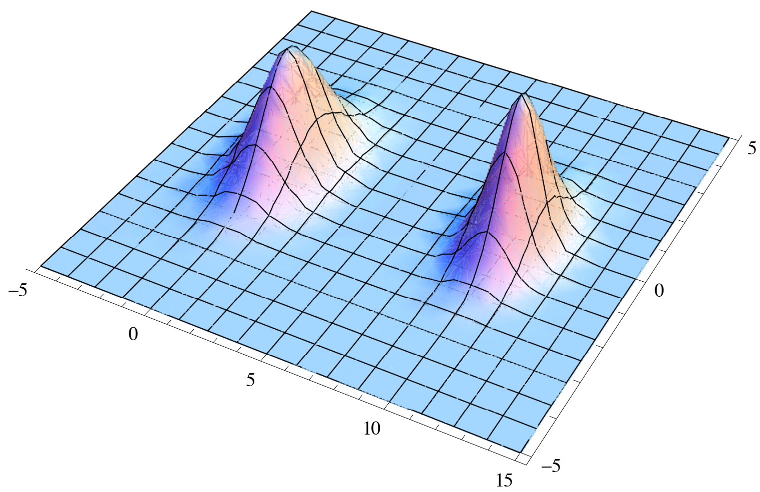


Figure 3.10. Densities of the trace of a random unitary matrix under the Haar measure (left peak) and under the Brownian measure before cut-off time (right peak).

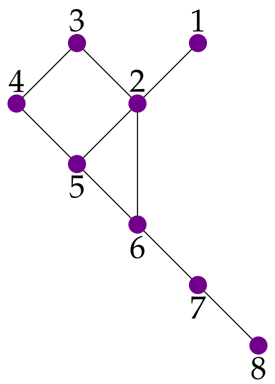
This can be made rigorous by using the Bienaymé–Chebyshev inequality and by computing the two first moments of the discriminating function. In the non-group case, this requires to compute squares of zonal spherical functions, and this was done by relating this calculation to certain stochastic differential equations, and by a brute-force computation. Again, the use of discriminating functions is a general approach if one wants to find a lower bound for the time to stationarity of a random process.

3.3 Spectra of random geometric graphs

In the two last sections of this chapter, we study random graphs drawn on a fixed ssccss. By graph, we mean here a simple unoriented graph without loop or multiple edge, hence a pair $\Gamma = (V, E)$ with E subset of the set of pairs $\{v, w\}$ with $v \neq w$ and $v, w \in V$. The *size* of a graph Γ is the cardinality $N = |V|$ of its vertex set, and the *adjacency matrix* is the matrix A_Γ of size $N \times N$, with rows and columns labeled by the vertices of Γ, and with coefficients

$$A_\Gamma(v, w) = \begin{cases} 1 & \text{if } \{v, w\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, the diagonal coefficients of the adjacency matrix of a simple graph are equal to zero.



$$A_\Gamma = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

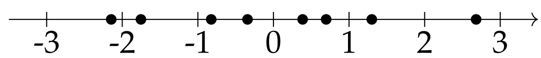


Figure 3.11. A (simple, unoriented) graph, its adjacency matrix and its spectrum.

The *spectrum* $\mathrm{Spec}(\Gamma)$ of a graph Γ consists of the N real eigenvalues $e_1 \geq e_2 \geq \cdots \geq e_N$ of the symmetric matrix A_Γ (see Figure 3.11). The knowledge of the spectrum yields many informations on the geometry of the graph [Chu97; GR01]: mean and maximal number of neighbors of a vertex; chromatic number; number of edges, triangles, spanning trees; expansion properties, Cheeger constant; *etc.* The purpose of the last sections of this memoir is to study the asymptotic properties of the spectra of certain random graphs defined by a geometric condition on a compact symmetric space. Most of the results hereafter appear with more details in the preprint [Mél18].

▷ **Random geometric graphs.** Let (X, d, μ) be a compact metric space endowed with a Borel probability measure μ. We fix a level $L > 0$ and an integer $N \geq 1$.

Definition 3.24 (Random geometric graph). *The random geometric graph on X with N points and level L is the random graph $\Gamma(N, L)$ obtained*

- *by taking at random N independent points $v_1, \ldots, v_N$ on X according to the measure $\mu^{\otimes N}$,*
- *and by connecting v_i to v_j if and only if $d(v_i, v_j) \leq L$.*

Example 3.25 (Random geometric graph on the sphere). In Figure 3.12, we have drawn in stereographic projection a random geometric graph on the real sphere $\mathbb{RS}^2$, with μ equal to Lebesgue's spherical measure, $N = 100$ points, and $L = \frac{\pi}{8}$ (one eighth of the diameter of the space).

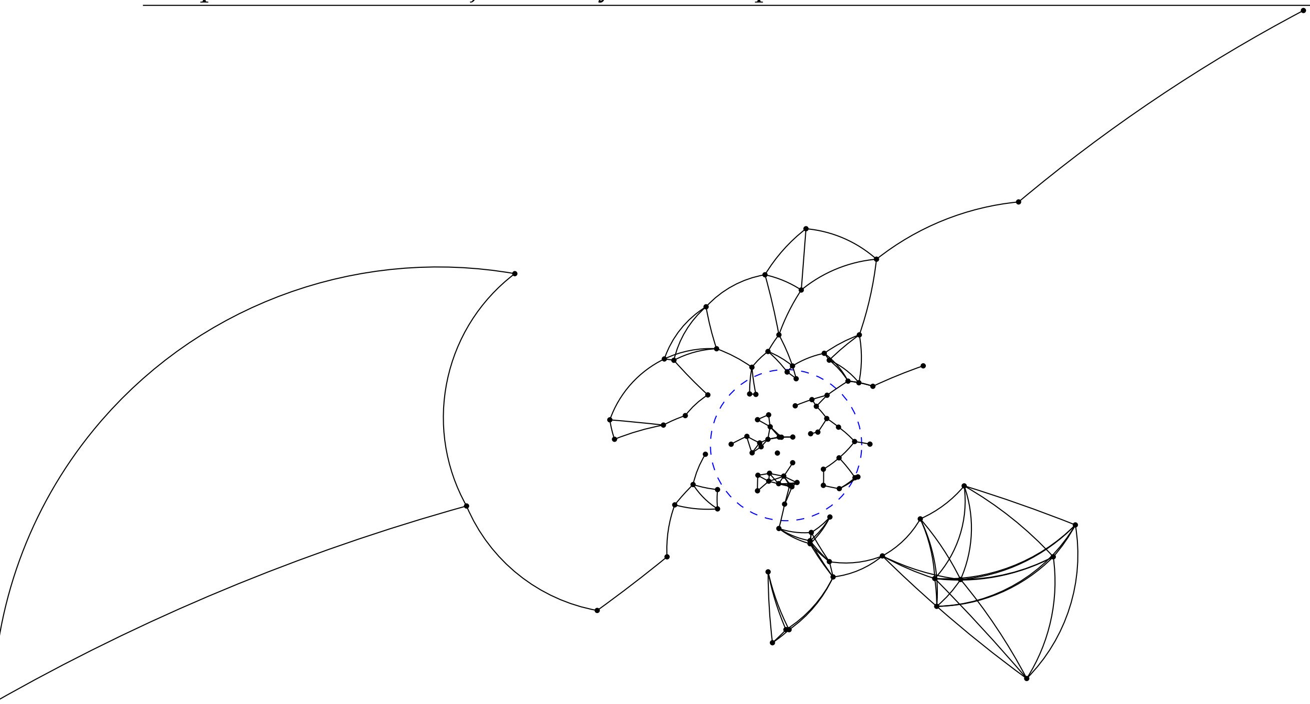


Figure 3.12. A random geometric graph on $\mathbb{RS}^2$, with $N = 100$ and $L = \frac{\pi}{8}$ (the blue circle is the equator).

A particular case is when X is a sscc symmetric space, endowed with the unique probability measure μ that is invariant by action of the isometry group $\mathrm{Isom}(X)$; μ is a scalar multiple of the volume form. In this setting, there are two interesting regimes that one can consider:

(1) the *Gaussian regime*, where L is fixed but N goes to infinity; in this setting the adjacency matrix A_Γ is dense.

(2) the *Poissonian regime*, where $L = L_N$ decreases to zero in such a way that each vertex of $\Gamma(N, L_N)$ has a $O(1)$ number of vertices; in this setting the adjacency matrix A_Γ is sparse. To be more precise, one takes

$$L_N = \left(\frac{\ell}{N}\right)^{\frac{1}{\dim X}},$$

so that the number of neighbors of a fixed vertex v_i in $\Gamma(N, L_N)$ follows a law close to the Poisson law of parameter

$$\lambda = \ell\,\frac{\mathrm{vol}(B_{\mathbb{R}^{\dim X}}(0,1))}{\mathrm{vol}(X)}.$$

Remark 3.26. The spectrum of the adjacency matrix of a random graph is a classical object in random matrix theory; see for instance [Erd+12; Erd+13] for the more recent results in the case of Erdös–Rényi graphs. For the spectral analysis of random geometric graphs in the Euclidean space or on a torus, we refer to [Bor08; BEJ06; DGK17]. Quite surprisingly, the more symmetric case that we present here has not yet been studied. On the other hand, the model of random geometric graphs mimics certain (but not all) properties of large social networks, and the spectral analysis of these networks and of the random graphs approximating these networks has become recently an important research field [Fri93; New03; Vu08; TVW13; NN13].

The two regimes lead to very different asymptotic behaviors for $\mathrm{Spec}(A_{\Gamma(N,L)})$. In the Gaussian regime, the asymptotics are discrete (see the next paragraph): the eigenvalues of $A_{\Gamma(N,L)}$ with the largest modules are of size $O(N)$, and after being rescaled, they converge in probability to specific values. On the opposite, in the Poissonian regime, the asymptotics are continuous (cf. Section 3.4). Hence, the random spectral measure

$$\nu_N = \frac{1}{N} \sum_{e_i \text{ eigenvalue of } A_{\Gamma(N,L_N)}} \delta_{e_i} \tag{3.5}$$

is conjectured to have a limit in probability, which is a probability measure ν_∞ on $\mathbb{R}$ whose moments can be explicitly computed, and which is related to the asymptotic representation theory of the group underlying the symmetric space.

▷ **Approximation of integral operators and asymptotics of the Gaussian regime.** Until the end of this section, X is a fixed ssccss and $L > 0$ is a fixed level, and we are interested in the spectrum of $\Gamma(N,L)$ as N grows to infinity. Denote $h(x,y) = 1_{d(x,y) \leq L}$; the adjacency matrix $A_{\Gamma(N,L_N)}$ is equal to

$$((1-\delta_{ij})\, h(v_i,v_j))_{1\leq i,j\leq N}, \quad \text{with the } v_i\text{'s i.i.d. random variables.}$$

A natural idea from random matrix theory is that after appropriate rescaling, the spectrum of $A_{\Gamma(N,L_N)}$ should approximate the spectrum of the integral operator by convolution by h. More precisely, consider a real symmetric function h on X such that $\iint_{X^2} (h(x,y))^2\, dx\, dy < +\infty$. The convolution by h induces an auto-adjoint integral operator T_h on $\mathscr{L}^2(X,dx)$:

$$\begin{aligned} T_h : \mathscr{L}^2(X,dx) &\to \mathscr{L}^2(X,dx) \\ f &\mapsto \left(T_h(f) : x \mapsto \int_X h(x,y)\, f(y)\, dy\right). \end{aligned}$$

This operator is compact, and even of Hilbert–Schmidt class: if $(e_i)_{i\in I}$ is an orthonormal basis of $\mathscr{L}^2(X,dx)$, then

$$\|T_h\|_{\mathrm{HS}} = \sqrt{\sum_{i\in I} \left(\|T_h(e_i)\|_{\mathscr{L}^2(G)}\right)^2} = \|h\|_{\mathscr{L}^2(X^2)}.$$

We label the discrete real spectrum of T_h by integers:

$$\mathrm{Spec}(T_h) = (e_{-1} \leq e_{-2} \leq \cdots \leq 0 \leq \cdots \leq e_2 \leq e_1 \leq e_0),$$

with $\lim_{|k|\to\infty} e_k = 0$ (here, we add an infinity of zeroes to the sequence $(e_k)_{k\in\mathbb{Z}}$ if needed, for instance when T_h is of finite rank). The Hilbert–Schmidt class ensures that $\sum_{k\in\mathbb{Z}} (e_k)^2 < +\infty$, and a general result due to Giné and Kolchinskii (see [GK00, Theorem 3.1]) ensures that the random matrices

$$T_h(N) = \frac{1}{N}\, ((1-\delta_{ij})\, h(v_i,v_j))_{1\leq i,j\leq N} \tag{3.6}$$

have their spectrum close to the one of T_h. Thus:

Theorem 3.27 (Giné–Koltchinskii). *Let (X,dx) be a probability space, h a square-integrable kernel on X, and T_h the associated Hilbert–Schmidt operator. We consider the random matrices $T_h(N)$ given by Equation (3.6), with the v_i's independent and distributing according to dx. We denote the spectrum of $T_h(N)$*

$$\mathrm{Spec}(T_h(N)) = (e_{-1}(N) \leq e_{-2}(N) \leq \cdots \leq 0 \leq \cdots \leq e_2(N) \leq e_1(N) \leq e_0(N)).$$

Under these assumptions,

$$\delta\left(\mathrm{Spec}(T_h(N)), \mathrm{Spec}(T_h)\right) = \sqrt{\sum_{k\in\mathbb{Z}} (e_k(N) - e_k)^2} \to 0 \quad \textit{almost surely.}$$

It turns out that one can use the non-commutative Fourier transform of compact Lie groups and symmetric spaces in order to compute the spectrum of the operators T_h with $h(x,y) = 1_{d(x,y)\leq L}$. In the sequel, we assume that $X = G$ is a fixed sscc Lie group, with distances normalised in the following way: the scalar product on $\mathfrak{g}$ is taken equal to

$$\langle X \mid Y \rangle_{\mathfrak{g}} = -B(X,Y) = -\mathrm{tr}(\mathrm{ad}\, X \circ \mathrm{ad}\, Y), \tag{3.7}$$

that is the opposite Killing form. Beware that this is not the same normalisation as in Equation (3.1) and in Section 3.2. For $g \in G$, set $Z_L(g) = h(e_G, g) = 1_{d(e_G,g)\leq L}$. We then have

$$(T_h(f))(g_1) = \int_G h(g_1,g_2)\, f(g_2)\, dg_2 = \int_G h(e, g_1 g_2^{-1})\, f(g_2)\, dg_2 = (Z_L * f)(g_1)$$

so T_h is the convolution on the left by the function Z_L. By Theorem 3.2, it is conjugated by isometries to the diagonal operator on $\mathscr{L}^2(\widehat{G})$ that acts on each endomorphism space $\mathrm{End}_{\mathbb{C}}(V^\lambda)$ by multiplication by

$$c_\lambda = \frac{1}{d_\lambda} \int_G Z_L(g)\, \mathrm{ch}^\lambda(g)\, dg.$$

Moreover, one can use Weyl's integration formula [Bum13, Chapter 17] to compute these eigenvalues c_λ: the integral on the compact group G becomes an integral over a maximal torus T, and one uses the fact that the Fourier transform of a Euclidean ball is given by a Bessel function of the first kind. One obtains [Mél18, Theorem 3.1]:

Theorem 3.28 (Spectral asymptotics of a random geometric graph, Gaussian regime). *Let G be a sscc Lie group and $\Gamma(N,L)$ be a random geometric graph of level L and with N points on G. We suppose $L \in (0,\pi)$, and we denote $d = \mathrm{rank}(G)$. As N goes to $+\infty$, the almost sure limits of the set of eigenvalues of $\frac{1}{N} A_{\Gamma(N,L)}$ are given by the infinite multiset*

$$\bigsqcup_{\lambda\in\widehat{G}} \left\{ \underbrace{c_\lambda, c_\lambda, \ldots, c_\lambda}_{(d_\lambda)^2 \text{ occurrences}} \right\},$$

with

$$c_\lambda = \frac{1}{d_\lambda\, \mathrm{vol}(\mathfrak{t}/\mathfrak{t}_{\mathbb{Z}})} \sum_{w\in W} \varepsilon(w) \left(\frac{L}{2\pi\, \|\lambda + \rho - w(\rho)\|} \right)^{\frac{d}{2}} J_{\frac{d}{2}}(L\, \|\lambda + \rho - w(\rho)\|).$$

In this formula:

- *The sum runs over elements of the Weyl group W of G, and $\varepsilon(w)$ is the parity of the number of reflections s_α with α simple root that are needed to write w.*
- *The weight $\rho = \frac{1}{2}\sum_{\alpha\in\Phi_+} \alpha$ is the half-sum of positive roots, and the norm $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}\Omega$ coming from the scalar product (3.7) on $\mathfrak{g}$.*

- *The lattice $2\pi\mathfrak{t}_{\mathbb{Z}}$ is the kernel of the exponential map* $\exp : \mathfrak{t} \to T$, *and the volume of a fundamental domain of the lattice $\mathfrak{t}_{\mathbb{Z}} \subset \mathfrak{t}$ is given in the classical cases by:*

$$G = \mathrm{SU}(n): \quad \mathrm{vol}(\mathfrak{t}/\mathfrak{t}_{\mathbb{Z}}) = 2^{\frac{n-1}{2}}\, n^{\frac{n}{2}};$$
$$G = \mathrm{USp}(n): \quad \mathrm{vol}(\mathfrak{t}/\mathfrak{t}_{\mathbb{Z}}) = 2^{n}\,(n+1)^{\frac{n}{2}};$$
$$G = \mathrm{Spin}(2n+1): \quad \mathrm{vol}(\mathfrak{t}/\mathfrak{t}_{\mathbb{Z}}) = 2^{\frac{n}{2}+1}\,(2n-1)^{\frac{n}{2}};$$
$$G = \mathrm{Spin}(2n): \quad \mathrm{vol}(\mathfrak{t}/\mathfrak{t}_{\mathbb{Z}}) = 2^{n+1}\,(n-1)^{\frac{n}{2}}.$$

- *The function $J_\beta(z)$ is the Bessel function given by the convergent power series*

$$J_\beta(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\beta+1)} \left(\frac{z}{2}\right)^{2m+\beta},$$

see Figure 3.13 for the case $\beta = 1$.

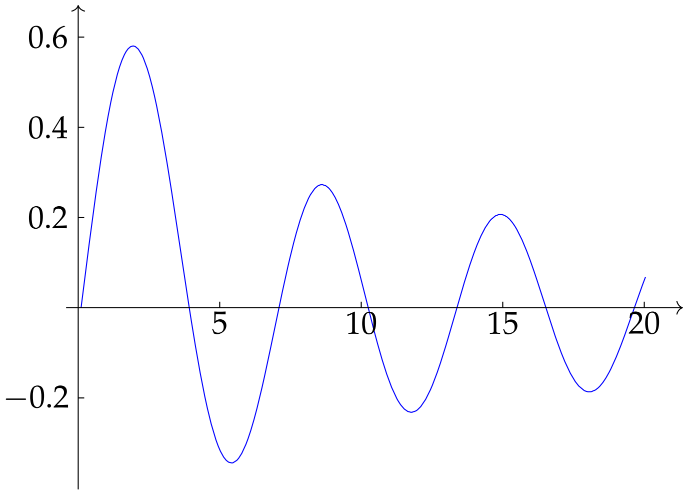


Figure 3.13. The Bessel function $J_1(x)$.

Corollary 3.29 (Spectral radius of a random geometric graph, Gaussian regime). *Given a sscc Lie group G and a level $L \in (0,\pi)$, the spectral radius of $\Gamma(N,L)$ is asymptotically equivalent to*

$$\frac{N}{\mathrm{vol}(\mathfrak{t}/\mathfrak{t}_{\mathbb{Z}})} \left(\sum_{w\in W} \varepsilon(w) \left(\frac{L}{2\pi\,\|\rho - w(\rho)\|}\right)^{\frac{d}{2}} J_{\frac{d}{2}}(L\,\|\rho - w(\rho)\|)\right).$$

Example 3.30 (Random geometric graphs on SU(2) and SU(3)). Suppose $G = \mathrm{SU}(2)$. Then, there is one limiting eigenvalue c_k of $\frac{1}{N} A_{\Gamma(N,L)}$ for each integer $k \geq 0$, with multiplicity $(k+1)^2$. We have

$$c_{k\geq 1} = \frac{1}{\pi\,(k+1)} \left(\frac{1}{k} \sin\!\left(\frac{k\,L}{2\sqrt{2}}\right) - \frac{1}{k+2} \sin\!\left(\frac{(k+2)\,L}{2\sqrt{2}}\right)\right);$$
$$c_0 = \frac{1}{2\pi} \left(\frac{L}{\sqrt{2}} - \sin\!\left(\frac{L}{\sqrt{2}}\right)\right).$$

Suppose now $G = \mathrm{SU}(3)$. The formula for c_λ with $\lambda = n_1\omega_1 + n_2\omega_2$ dominant weight in the Weyl chamber involves 6 weights close to λ, namely, the weights $\lambda + \mu$ with

$$\mu \in \{0, 3\omega_1, 3\omega_2, 2\omega_2 - \omega_1, 2\omega_1 - \omega_2, 2\omega_1 + 2\omega_2\},$$

see Figure 3.14. Thus,

$$c_{n_1,n_2} = \frac{L}{6\pi\sqrt{3}(n_1+1)(n_2+1)(n_1+n_2+2)} \left(\sum_{w\in\mathfrak{S}(3)} \varepsilon(w)\,\frac{J_1(L\,\|\lambda+\rho-w(\rho)\|)}{\|\lambda+\rho-w(\rho)\|}\right)$$

and each eigenvalue c_{n_1,n_2} has multiplicity $(\frac{(n_1+1)(n_2+1)(n_1+n_2+2)}{2})^2$. In this formula, the norm of a weight $k_1\omega_1+k_2\omega_2$ is $\|k_1\omega_1+k_2\omega_2\| = \frac{1}{3}\sqrt{(k_1)^2+k_1k_2+(k_2)^2}$.

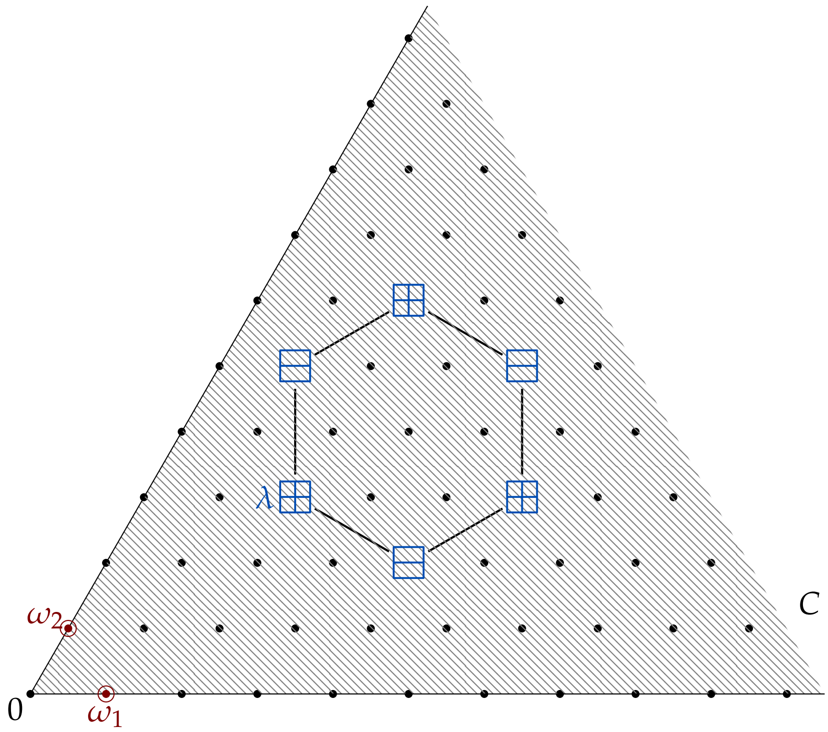


Figure 3.14. The weights involved in the computation of c_λ for $G = \mathrm{SU}(3)$.

Remark 3.31. The formula in Theorem 3.28 involves a kind of discrete derivative of the analytic function

$$\widetilde{J}_{\mathbb{R}\Omega} : \mathbb{R}\Omega \to \mathbb{R}$$

$$x \mapsto \|x\|^{-\frac{\mathrm{rank}(G)}{2}}\, J_{\frac{\mathrm{rank}(G)}{2}}(\|x\|) = \frac{1}{2^{\frac{\mathrm{rank}(G)}{2}}} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\frac{\mathrm{rank}(G)}{2}+1)} \left(\frac{\|x\|}{2}\right)^{2m}. \tag{3.8}$$

In the Poissonian regime, these discrete derivatives will degenerate into true derivatives of the function $\widetilde{J}_{\mathbb{R}\Omega}$.

Remark 3.32. The restriction to *simply connected* compact Lie groups is made in order to be simplify a bit the discussion, and to use Weyl's highest weight theorem and the corresponding description of $\widehat{G}$ (Theorem 3.9). When working with a semisimple connected compact Lie group, the set $\widehat{G}$ of irreducible representations of G can be identified with a subset of the set $\mathbb{Z}\Omega \cap C$ of dominant weights of the Lie algebra $\mathfrak{g}$ of G. Moreover, the lattice Λ spanned by $\widehat{G}$ in $\mathbb{Z}\Omega$ has finite index, and $\mathbb{Z}\Omega/\Lambda$ is isomorphic to the finite fundamental group $\pi_1(G)$ (see for instance [Bum13, Theorem 23.2]). Thus, the whole discussion of this section or of the previous section on Brownian motions still holds for semisimple connected compact Lie groups, and the only difference is that one has to work with a smaller set of dominant weights. In particular, the case of special orthogonal groups can be dealt with without additional difficulty, although $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}/2\mathbb{Z}$ for $n \geq 3$.

▷ **Compact symmetric spaces of rank 1.** The same analysis as before can be performed for random geometric graphs on a ssccss of non-group type. The only additional difficulty is the manipulation of zonal spherical functions. Hence, in the Gaussian regime with L fixed and N going to infinity, the limit of the spectrum of $\frac{1}{N} A_{\Gamma(N,L)}$ with $\Gamma(N,L)$ random geometric graph on a sscc symmetric space $X = G/K$ is the multiset $\bigsqcup_{\lambda \in \widehat{G}} \{c_\lambda^{\otimes d_\lambda}\}$, where

$$c_\lambda = \frac{1}{\sqrt{d_\lambda}} \int_X 1_{d(eK,x) \leq L} \,\mathrm{zon}^\lambda(x)\, dx = \int_G 1_{d(eK,gK)\leq L} \left\langle e^\lambda \,\middle|\, (\rho^\lambda(g))(e^\lambda) \right\rangle_{V^\lambda} dg.$$

Suppose that $X = G/K$ has rank one, meaning that the only flat totally geodesic submanifolds of X are one-dimensional. These compact symmetric spaces are important examples in the study of random geometric graphs, because of the following characterisation (see [Wol67, Chapter 8]): a compact connected Riemannian manifold X is a compact symmetric space of rank one if and only if it is 2-point homogeneous, meaning that given two pairs of points (x_1, x_2) and (y_1, y_2) such that $d(x_1, x_2) = d(y_1, y_2)$, there is an isometry $i : X \to X$ with $i(x_1) = y_1$ and $i(x_2) = y_2$. The second assertion is a natural condition to look at if one wants random geometric graphs that are as symmetric as possible. On the other hand, one has the following classification of the compact symmetric spaces of rank one and of the corresponding spherical representations:

Proposition 3.33 (Compact symmetric spaces of rank one). *The compact symmetric spaces of rank one are:*

- *the real spheres* $\mathbb{RS}^{n\geq 1} = \mathrm{SO}(n+1)/\mathrm{SO}(n)$,
- *the real projective spaces* $\mathbb{RP}^{n\geq 2} = \mathrm{SO}(n+1)/\mathrm{O}(n)$,
- *the complex projective spaces* $\mathbb{CP}^{n\geq 2} = \mathrm{SU}(n+1)/\mathrm{U}(n)$,
- *the quaternionic projective spaces* $\mathbb{HP}^{n\geq 2} = \mathrm{USp}(n+1)/(\mathrm{USp}(n) \times \mathrm{USp}(1))$,
- *the octonionic projective plane* $\mathbb{OP}^2 = \mathrm{F}_4/\mathrm{Spin}(9)$.

All of them are simply connected but the real projective spaces, which have a twofold universal cover for $n \geq 2$. *The set of spherical representations* $\widehat{G}^K$ *is each time a half-line* $\mathbb{N}\omega_0$ *in a one-dimensional sublattice of* $\widehat{G}$. *The representation* V^{ω_0} *and the dimensions* $\dim_{\mathbb{C}}(V^{k\omega_0})$ *with* $k \geq 0$ *are given by the following table:*

X	V^{ω_0}	$\dim_{\mathbb{C}}(V^{k\omega_0})$
$\mathbb{RS}^n$	*geometric representation on* $\mathbb{C}^{n+1}$	$\frac{2k+n-1}{k+n-1}\binom{k+n-1}{n-1}$
$\mathbb{RP}^n$	$\mathfrak{so}^\perp(n+1,\mathbb{C}) \subset \mathfrak{sl}(n+1,\mathbb{C})$	$\frac{4k+n-1}{2k+n-1}\binom{2k+n-1}{n-1}$
$\mathbb{CP}^n$	*adjoint representation* $\mathfrak{sl}(n+1,\mathbb{C})$	$\frac{2k+n}{n}\binom{k+n-1}{n-1}^2$
$\mathbb{HP}^n$	$\mathfrak{sp}^\perp(2n+2,\mathbb{C}) \subset \mathfrak{sl}(2n+2,\mathbb{C})$	$\frac{2k+2n+1}{(2n+1)(k+1)}\binom{k+2n}{2n}\binom{k+2n-1}{2n-1}$
$\mathbb{OP}^2$	*tracefree part of the 27-dimensional Albert algebra* $\mathrm{A}(3,\mathbb{O})$	$\frac{2k+11}{385}\binom{k+7}{4}\binom{k+10}{10}$

Let us now describe the zonal spherical functions of these spaces, and then compute the eigenvalues c_λ (their multiplicities d_λ are given by the previous table). We refer to [Gri83; VV09; AH10], where it is shown that the *Legendre polynomials* are the zonal spherical functions of the real spheres, and that the *Jacobi polynomials* are the zonal spherical functions of the projective spaces (with parameters depending on the base field). Hence, if the real sphere $\mathbb{RS}^n$ is endowed with coordinates $(x_1, x_2, \ldots, x_{n+1})$ such that $\sum_{i=1}^{n+1}(x_i)^2 = 1$, and if the projective space $\mathbb{KP}^n$ is endowed with projective coordinates $[x_1 : x_2 : \cdots : x_{n+1}]$ with the x_i's in $\mathbb{K}$, then the zonal spherical functions of these spaces are given by the following table:

X	spherical coordinate	law of the spherical coordinate	normalised zonal spherical function
$\mathbb{RS}^n$	$x = x_{n+1}$	$\frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{1}{2})\,\Gamma(\frac{n}{2})}\,(1-x^2)^{\frac{n-2}{2}}\,1_{x\in[-1,1]}\,dx$	$P^{n,k}(x)$
$\mathbb{RP}^n$	$s = \frac{\lvert x_{n+1}\rvert^2}{\lvert x_1\rvert^2+\cdots+\lvert x_{n+1}\rvert^2}$	$\frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{1}{2})\,\Gamma(\frac{n}{2})}\,(1-s)^{\frac{n-2}{2}}\,s^{-\frac{1}{2}}\,1_{s\in[0,1]}\,ds$	$J^{(\frac{n}{2},\frac{1}{2}),k}(s)$
$\mathbb{CP}^n$	$t = \frac{\lvert z_{n+1}\rvert^2}{\lvert z_1\rvert^2+\cdots+\lvert z_{n+1}\rvert^2}$	$n\,(1-t)^{n-1}\,1_{t\in[0,1]}\,dt$	$J^{(n,1),k}(t)$
$\mathbb{HP}^n$	$u = \frac{\lvert h_{n+1}\rvert^2}{\lvert h_1\rvert^2+\cdots+\lvert h_{n+1}\rvert^2}$	$2n(2n+1)\,u(1-u)^{2n-1}\,1_{u\in[0,1]}\,du$	$J^{(2n,2),k}(u)$
$\mathbb{OP}^2$	$\xi = \frac{\lvert\theta_3\rvert^2}{\lvert\theta_1\rvert^2+\lvert\theta_2\rvert^2+\lvert\theta_3\rvert^2}$	$1320\,\xi^3(1-\xi)^7\,1_{\xi\in[0,1]}\,d\xi$	$J^{(8,4),k}(\xi)$

In this table, the zonal spherical functions are renormalised so that $f(e_G K) = 1$, so they differ from the functions zon^λ by a factor $\sqrt{d_\lambda}$. The $P^{n,k}$'s are the (real) Legendre polynomials given by Rodrigues' formula

$$P^{n,k}(x) = \frac{(-1)^k}{2^k\,(\frac{n-2}{2}+k)^{\downarrow k}}\,\frac{1}{(1-x^2)^{\frac{n-2}{2}}}\,\frac{d^k}{dx^k}(1-x^2)^{\frac{n-2}{2}+k},$$

and the $J^{(a,b),k}$'s are the Jacobi polynomials given by Rodrigues' formula

$$J^{(a,b),k}(x) = \frac{1}{(a+k-1)^{\downarrow k}}\,\frac{1}{(x-1)^{a-1}x^{b-1}}\,\frac{d^k}{dx^k}((x-1)^{a+k-1}x^{b+k-1}).$$

The Legendre polynomials are the orthogonal polynomials for the law of the last coordinate of a point on the sphere, and the Jacobi polynomials of index (a,b) are the orthogonal polynomials for the β-distribution

$$\beta^{(a,b)}(dx) = \frac{\Gamma(a+b)}{\Gamma(a)\,\Gamma(b)}\,(1-x)^{a-1}\,x^{b-1}\,1_{x\in[0,1]}\,dx.$$

Remark 3.34. The manipulation of the coordinates of the exceptional compact symmetric space of rank one $\mathrm{F}_4/\mathrm{Spin}(9)$ is a bit complicated: this space can be considered as the octonionic projective plane, but since octonions form a non-associative algebra, the definition of a projective plane is not as straightforward as in the other cases. The right way to construct $\mathbb{OP}^2$ is by gluing affine charts, and it is also convenient to relate this space to the geometry of the exceptional Albert algebra $\mathrm{A}(3,\mathbb{O})$ of dimension 27; see [Joh76; Ada96; Bae02].

On the sphere $\mathbb{RS}^n$, the distance from $x = (x_1, x_2, \ldots, x_{n+1})$ to the base point $(0, 0, \ldots, 1)$ is (proportional to) $\arccos x_{n+1}$. Similarly, on the projective space $\mathbb{KP}^n$, the distance from $x = [x_1 : x_2 : \cdots : x_{n+1}]$ to the base point $[0 : 0 : \cdots : 1]$ is $\arccos(|x_{n+1}|/\sqrt{|x_1|^2 + \cdots + |x_{n+1}|^2})$.

In the sequel, we choose the normalisation of the distance so that the distance is equal to the arccosine (not just proportional). Using these formulas and Rodrigues' formula for the orthogonal polynomials, we obtain:

Theorem 3.35 (Random geometric graphs on rank one compact symmetric spaces, Gaussian regime). *Let X be a compact symmetric space of rank one, and $\Gamma(N,L)$ be a random geometric graph of level L and with N points on X. We suppose $L \in (0, \frac{\pi}{2})$. As N goes to $+\infty$, the almost sure limits of the set of eigenvalues of $\frac{1}{N} A_{\Gamma(N,L)}$ are given by the infinite multiset*

$$\bigsqcup_{k=0}^{\infty} \left\{ \underbrace{c_k, c_k, \ldots, c_k}_{d_k \text{ occurrences}} \right\},$$

with $d_k = \dim_{\mathbb{C}}(V^{k\omega_0})$ given by the table of Proposition 3.33, and with c_k given by the following table:

X	*eigenvalue* c_0	*eigenvalue* $c_{k\geq 1}$
$\mathbb{RS}^n$	$\frac{\int_0^L \sin^{n-1}\theta\, d\theta}{\int_0^\pi \sin^{n-1}\theta\, d\theta}$	$\frac{(\sin L)^n}{n\int_0^\pi \sin^{n-1}\theta\, d\theta}\, P^{n+2,k-1}(\cos L)$
$\mathbb{RP}^n$	$\frac{\int_0^L \sin^{n-1}\theta\, d\theta}{\int_0^{\frac{\pi}{2}} \sin^{n-1}\theta\, d\theta}$	$\frac{(\sin L)^n \cos L}{n\int_0^{\frac{\pi}{2}} \sin^{n-1}\theta\, d\theta}\, J^{(\frac{n+2}{2},\frac{3}{2}),k-1}(\cos^2 L)$
$\mathbb{CP}^n$	$(\sin L)^{2n}$	$(\sin L)^{2n}\,(\cos L)^2\, J^{(n+1,2),k-1}(\cos^2 L)$
$\mathbb{HP}^n$	$(\sin L)^{4n}\,(1+2n\cos^2 L)$	$(2n+1)\,(\sin L)^{4n}\,(\cos L)^4\, J^{(2n+1,3),k-1}(\cos^2 L)$
$\mathbb{OP}^2$	$(\sin L)^{16}\,\left(\substack{1+8\cos^2 L\\ +36\cos^4 L+120\cos^6 L}\right)$	$165\,(\sin L)^{16}\,(\cos L)^8\, J^{(9,5),k-1}(\cos^2 L)$

Example 3.36 (Random geometric graphs on the real sphere $\mathbb{RS}^2$). For the real sphere of dimension 2, the spherical representation $V^{k\omega_0}$ has dimension $2k+1$. The corresponding zonal spherical function is proportional to the Legendre polynomial $P^{2,k}(x)$. The limiting eigenvalues of the rescaled adjacency matrix of a random geometric graph of level L on $\mathbb{RS}^2$ are

$$c_0 = \frac{\sin^2 L}{4}\,\frac{2}{1+\cos L};$$
$$c_{k\geq 1} = \frac{\sin^2 L}{4}\,P^{4,k-1}(\cos L)$$

with c_k that has multiplicity $2k+1$, and where

$$P^{4,k-1}(x) = \frac{1}{2^{k-1}\,k!}\,\frac{1}{(x^2-1)}\,\frac{d^{k-1}}{dx^{k-1}}(x^2-1)^k.$$

Thus, up to the multiplicative factor $\frac{\sin^2 L}{4}$, all the limiting eigenvalues of the random geometric graph of level L can be obtained by looking at the values at $x = \cos L$ of the family of functions

$$\left\{ f(x) = \frac{2}{1+x} \right\} \sqcup \{P^{4,k}(x),\ k \geq 0\}$$

see Figure 3.15.

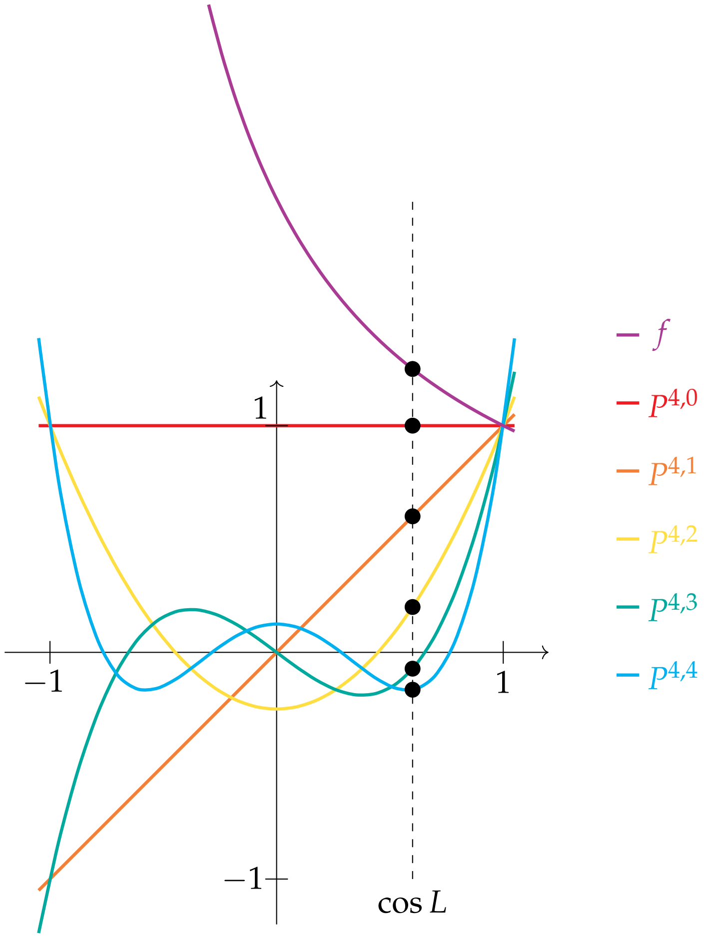


Figure 3.15. The limiting eigenvalues of a random geometric graph on the 2-dimensional real sphere, in the Gaussian regime.

There is no theoretical obstruction to the generalisation of these results to symmetric spaces of non-group type and with higher rank. However, the corresponding zonal spherical functions are much more difficult to deal with (and sometimes not very well known), and we do not know yet if there exist closed formulas for the limiting eigenvalues in the Gaussian regime of random geometric graphs on general Grassmannian manifolds, or on Lagragian spaces of structures.

3.4 Poissonian regime and asymptotic representation theory

In this last section, we fix a sscc Lie group G, and we consider random geometric graphs $\Gamma(N, L_N)$ on G with levels L_N that vary with N as follows:

$$L_N = \left(\frac{\ell}{N}\right)^{\frac{1}{\dim G}}.$$

The normalisation of the distances on G is chosen according to Equation (3.7). In this Poissonian regime, we are interested in the asymptotics of the random spectral measure ν_N given by Equation (3.5). As in Section 3.1, we set $d = \mathrm{rank}(G)$ and $l = |\Phi_+|$.

▷ **Local convergence and conjecture on the limiting spectral distribution.** For any fixed positive real number R and any fixed $g \in G$,

- the subgraph of the random geometric graph of level L_N that is included in the ball of radius RL_N around g in G,
- and the random geometric graph of level 1 obtained from a Poissonian cloud of intensity $\frac{\ell}{\mathrm{vol}(G)}\,dx$ in a Euclidean ball of radius R in $\mathbb{R}^{\dim G}$,

look essentially the same. Indeed, as N goes to infinity, a small ball in G of radius RL_N is almost isometric to its Euclidean approximation in the tangent space to G, and the points falling in this small ball are asymptotically described by a Poisson point process. As a consequence, a limit ν_∞ of the spectral measures ν_N can be considered as "the spectral measure of a Poissonian random geometric graph in $\mathbb{R}^{\dim G}$", whence the terminology of Poissonian regime. This can be made more precise by means of the following notion of local convergence of rooted graphs:

Definition 3.37 (Benjamini–Schramm local convergence). *A sequence of random rooted (possibly infinite) graphs $(\Gamma_n, r_n)_{n\in\mathbb{N}}$ converges to a random rooted graph (Γ, r) if and only if, for any radius $R \in \mathbb{N}$ and any rooted graph (H, s) whose vertices are all at distance smaller than R from s,*

$$\lim_{n\to\infty} \mathbb{P}[\Gamma_n \cap B(r_n, R) = (H, s)] = \mathbb{P}[\Gamma \cap B(r, R) = (H, s)].$$

Here the balls $B(r, R)$ are taken with respect to the graph distance. We refer to [BS01; AL07; Abé+11] for details on this notion of convergence of graphs; beware that it is not the same as the convergence in the graphon sense (Section 2.3). One of the main results from [Mél18, Theorem 4.2] is:

Theorem 3.38 (Local convergence of Poissonian random geometric graphs). *Consider a random geometric graph $\Gamma(N, L_N)$ with $L_N = (\frac{\ell}{N})^{\frac{1}{\dim G}}$, and with a root r_N chosen randomly uniformly among the N vertices of this graph. The pair $(\Gamma(N, L_N), r_N)_{N\in\mathbb{N}}$ converges in the local Benjamini–Schramm sense towards the following random rooted graph $(\Gamma_\infty, 0)$:*

- *One takes random points in $\mathbb{R}^{\dim G}$ according to a Poisson point process with intensity $\frac{\ell}{\mathrm{vol}(G)}\,dX$, dX being the Lebesgue measure; and one adds the point 0.*
- *One connects these vertices when they are at Euclidean distance smaller than 1, and the root of the resulting random graph is 0.*

Remark 3.39. It should be noted that although quite intuitive, this result of local convergence is not at all easy to prove, for the following reason. As explained above, a neighborhood of size RL_N of a point $g \in G$ is almost isometric to the corresponding ball in $\mathbb{R}^{\dim G}$, but since this is only an almost isometry, the projection in $\mathbb{R}^{\dim G}$ of a geometric graph with level L_N in the ball on G *is not* a geometric graph with level L_N in $\mathbb{R}^{\dim G}$. This important problem can be solved by using the regularity with respect to the Benjamini–Schramm topology of the limiting random geometric graph $(\Gamma_\infty, 0)$ on $\mathbb{R}^{\dim G}$ with respect to the parameter ℓ. We actually developed a general theory relating convergence in the Lipschitz sense of random pointed proper metric spaces and convergence of random geometric graphs in the Benjamini–Schramm sense; see [Mél18, Theorem 4.6].

Given a sequence of finite random rooted graphs (Γ_N, r_N) with a uniform bound D on the degrees of the vertices, a Benjamini–Schramm convergence $(\Gamma_N, r_N) \to (\Gamma_\infty, r_\infty)$ towards a (possibly infinite) random rooted graph implies the weak convergence of the expected spectral measures $\mu_N = \mathbb{E}[\nu_N] \in \mathscr{M}^1(\mathbb{R})$, where ν_N is as before the mean of the Dirac distributions δ_e at the eigenvalues of A_{Γ_N}. This result appears in [Abé+11, Theorem 4], and under the assumption of a uniform bound on the degrees, one also have convergence of the atoms. For the random geometric graphs $\Gamma(N, L_N)$, one does not have a uniform bound on the degrees, but one can still adapt the method of proof to establish:

Corollary 3.40 (Convergence of the expected spectral measure of a Poissonian random geometric graph). *Consider a random geometric graph $\Gamma(N, L_N)$ on a sscc Lie group G with $L_N = (\frac{\ell}{N})^{\frac{1}{\dim G}}$, there exists a probability distribution μ_∞ on $\mathbb{R}$ such that*

$$\mathbb{E}[\nu_N] = \mu_N \rightharpoonup_{N\to\infty} \mu_\infty.$$

We refer to [Mél18, Theorem 4.16] for a proof of this result. Unfortunately, this proof does not give any information on the limit μ_∞. However, another approach to the proof of the convergence in distribution $\mu_N \rightharpoonup \mu_\infty$ led us to the following precise conjecture:

Conjecture 3.41 (Spectral asymptotics of a random geometric graph, Poissonian regime). *Let $(\Gamma(N, L_N))_{N\in\mathbb{N}}$ be the sequence of random geometric graphs on a sscc Lie group G, with levels $L_N = (\frac{\ell}{N})^{\frac{1}{\dim G}}$, $\ell > 0$.*

(1) The limit $\mu_\infty = \lim_{N\to\infty} \mu_N$ of the expected spectral measures is in fact the limit in probability $\lim_{N\to\infty} \nu_N$.

(2) For any $r \geq 1$, the r-th moment $M_r = \int_{\mathbb{R}} x^r \, \mu_\infty(dx)$ of the limiting spectral distribution is an explicit polynomial of degree $(r-1)$ in ℓ, and its coefficients admit combinatorial expansions involving:

- *certain labeled graphs* (reduced circuits)*,*
- *certain integrals over products of Weyl chambers C^h of products of partial derivatives of the Bessel function $\widetilde{J}_{\mathbb{R}\Omega}$ defined by Equation* (3.8)*,*
- *and certain measures on products of Weyl chambers C^h which are piecewise polynomial, and obtained by affine projection of uniform measures on certain string polytopes.*

The most important and difficult part of Conjecture 3.41 is its second item, and the remainder of this section is devoted to explaining why it should be true. We shall actually detail the computation of the six first limiting moments M_r, and rely the previous conjecture to another conjecture on certain functionals of the irreducible representations of G (Conjecture 3.50).

▷ **Circuit expansion.** In the sequel, we set $M_{r,N} = \int_{\mathbb{R}} x^r \, \mu_N(dx)$, and $M_r = \lim_{N\to\infty} M_{r,N}$. The first step in the computation of the asymptotic moments M_r is an expansion of $M_{r,N}$ as a sum over certain labeled graphs. This combinatorial technique is classical in random matrix theory, as it is involved in elementary proofs of the Wigner law of large numbers. If $v_1, \ldots, v_N$ are independent Haar distributed points in G, then

$$M_{r,N} = \frac{1}{N} \sum_{i_1,\ldots,i_r} \mathbb{E}\left[h_N(v_{i_1}, v_{i_2})\, h_N(v_{i_2}, v_{i_3}) \cdots h_N(v_{i_r}, v_{i_1})\right]$$

with $h_N(x,y) = 1_{d(x,y)\leq L_N}$, and the sum runs over indices $i_j \in [\![1, N]\!]$ such that two consecutive indices i_j and i_{j+1} are not equal. An expectation $E_{i_1,i_2,\ldots,i_r} = \mathbb{E}[h_N(v_{i_1}, v_{i_2}) \cdots h_N(v_{i_r}, v_{i_1})]$ only depends on the possible equalities of indices. For instance, when computing $M_{4,N}$, we have:

$$\begin{aligned} M_{4,N} &= (N-1)(N-2)(N-3)\, \mathbb{E}[h_N(v_1, v_2)\, h_N(v_2, v_3)\, h_N(v_3, v_4)\, h_N(v_4, v_1)] \\ &\quad + 2\,(N-1)(N-2)\, \mathbb{E}[h_N(v_1, v_2)\, h_N(v_2, v_1)\, h_N(v_1, v_3)\, h_N(v_3, v_1)] \\ &\quad + (N-1)\, \mathbb{E}[h_N(v_1, v_2)\, h_N(v_2, v_1)\, h_N(v_1, v_2)\, h_N(v_2, v_1)]\,. \end{aligned}$$

The first term corresponds to the case where all the indices i_1, i_2, i_3, i_4 are distinct; the second term corresponds to the identities $i_1 = i_3$ or $i_2 = i_4$; and the last term is when $i_1 = i_3$ and $i_2 = i_4$ simultaneously. We associate to these four cases the *circuits* of Figure 3.16. By circuit,

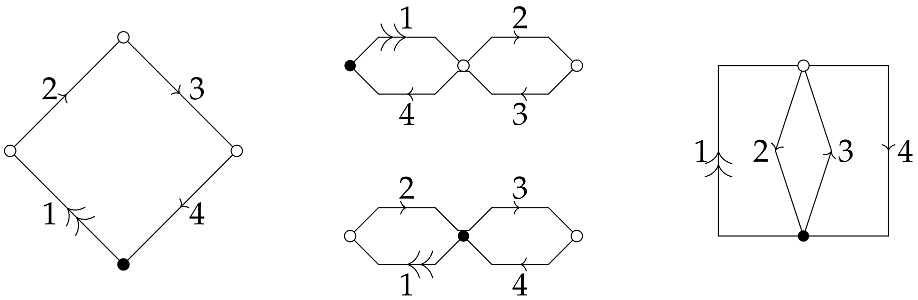


Figure 3.16. The circuits for the calculation of $M_{4,N}$.

we mean a directed graph H, possibly with multiple edges but without loops, endowed with a distinguished traversal T that goes through each directed edge exactly once, and that is cyclic (the starting point is the same as the end point of the traversal). We identify two circuits (H_1, T_1) and (H_2, T_2) if there exists a graph isomorphism $\psi : H_1 \to H_2$ that is compatible with the traversals: $\psi(T_1) = T_2$. A larger example of size 12 and associated to the identities of indices $i_2 = i_5 = i_7$, $i_3 = i_{11}$ and $i_6 = i_{12}$ is drawn in Figure 3.17.

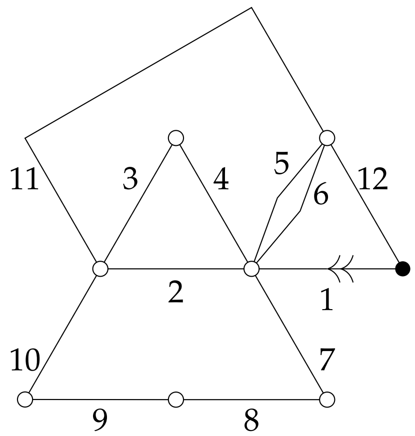


Figure 3.17. Circuit of length $r = 12$ and with $k = 8$ vertices, associated the set of identities $\{i_2 = i_5 = i_7, i_3 = i_{11}, i_6 = i_{12}\}$.

Given a circuit (H, T) with r edges and $k \leq r$ vertices, we associate to it the expectation of a function of k independent points $v_1, \ldots, v_k$ on G: $E_{H,T,N} = \mathbb{E}[\prod_{(i,j)\in T} h_N(v_i, v_j)]$. Notice that $E_{H,T,N}$ only depends on H, and not on the particular traversal T.

Proposition 3.42 (Circuit expansion). *For any $r \geq 0$, we have*

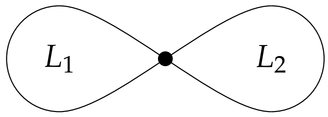
$$M_{r,N} = \sum_{(H,T)} (N-1)\cdots(N-|H|+1)\, E_{H,T,N},$$

where the sum runs over the finite set of circuits with r edges, and $k = |H|$ denotes the number of vertices in H.

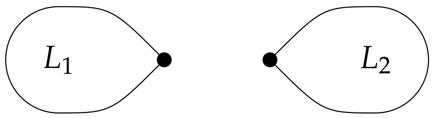
In the combinatorial expansion of $M_{r,N}$, the circuits contain in fact too much information; one can remove these redondancies by means of the operation of *reduction of circuit*. Let

(H, T) be a circuit of length r. Its reduction is the labeled undirected graph which is allowed to be disconnected and to have loops, and which is obtained by performing the following operations:

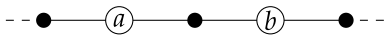
- forgetting the orientation of the edges of H;
- replacing any multiple edge by a single edge;
- putting a label 1 on each of the (single) edges;
- cutting the graph at each of its cut vertices (also called articulation points), replacing a configuration



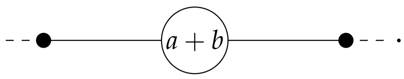
with $L_1 \neq \emptyset$ and $L_2 \neq \emptyset$ by

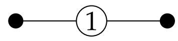
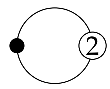


- in the resulting connected components, removing recursively each vertex of degree 2, replacing a configuration



by



- finally, replacing the connected components  by loops  .

Example 3.43 (Reduction of circuits). The reduced circuits corresponding to the four circuits of length 4 are drawn in Figure 3.18, with the middle one that has multiplicity 2.

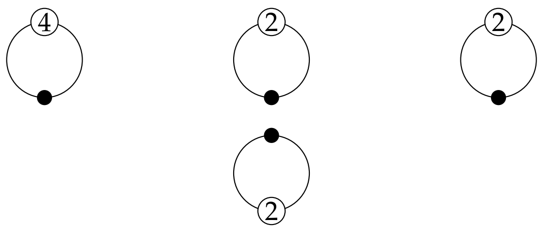


Figure 3.18. Reduction of the circuits of length 4.

The reduction of the circuit of Figure 3.17 is drawn in Figure 3.19.

Note that the operation of reduction can send many distinct circuits to the same reduction; and that it produces two kinds of connected reduced circuits:

- reduced circuits with a single vertex and a single loop based on it.
- reduced circuits with at least two vertices, and where all the vertices have at least degree 3.

It is easily seen that an expectation $E_{H,T,N}$ only depends on the reduction $R(H, T)$ of the circuit (H, T). Thus:

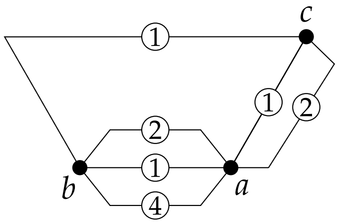


Figure 3.19. Reduction of the circuit of Figure 3.17.

Partial result 3.44 (Reformulation of Conjecture 3.41 in terms of reduced circuits). *For any $r \geq 0$, we have the combinatorial expansion*

$$M_{r,N} = \sum_{(H,T)} (N-1)^{\downarrow |H|-1}\, E_{R(H,T),N},$$

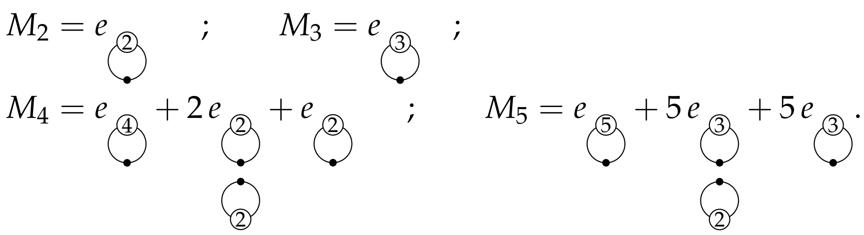
where the sum runs over circuits (H,T) with r edges, and where $E_{R(H,T),N} = E_{H,T,N}$ only depends on the reduced circuit $R(H,T)$. Moreover, for any reduced circuit, it is conjectured that there exists a limit $e_R = \lim_{N\to\infty} N^{|R|-1}\, E_{R,N}$. The number of vertices $|R| = |R(H,T)| = |H|$ is given by the formula

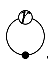
$$|R| = k = k' - (c-1) + \sum_{e \text{ labeled edge of } R} (f_e - 1),$$

where k' is the number of vertices of R, c its number of connected components, and f_e is the label of an edge e in R. Then,

$$M_r = \sum_{(H,T) \text{ circuit with } r \text{ edges}} e_{R(H,T)}.$$

▷ **The five first moments.** The previous discussion corresponds to the first part of the second item in Conjecture 3.41. To explain the second part and the integrals of partial derivatives of $\widetilde{J}_{\mathbb{R}\Omega}$, we shall look at the five first moments M_r; the case $r = 1$ is trivial since $M_1 = M_{1,N} = 0$. We have the following circuit expansions:



Let us explain how to compute e_R when R is a single labeled loop . We have:

$$\begin{aligned}
E_{R,N} &= \mathbb{E}[h_N(v_1,v_2)\, h_N(v_2,v_3) \cdots h_N(v_r,v_1)] \\
&= \mathbb{E}\Big[Z_{L_N}(v_1(v_2)^{-1})\, Z_{L_N}(v_2(v_3)^{-1}) \cdots Z_{L_N}(v_{r-1}(v_r)^{-1})\, Z_{L_N}(v_r(v_1)^{-1})\Big] \\
&= \mathbb{E}\Big[(Z_{L_N})^{*(r-1)}(v_1(v_r)^{-1})\, Z_{L_N}(v_r(v_1)^{-1})\Big] = \Big\langle (Z_{L_N})^{*(r-1)} \,\Big|\, Z_{L_N} \Big\rangle_{\mathscr{L}^2(G,dg)} \\
&= \sum_{\lambda \in \widehat{G}} \frac{\left(\left\langle \mathrm{ch}^\lambda \,\middle|\, Z_{L_N} \right\rangle\right)^r}{(d_\lambda)^{r-2}} = \sum_{\lambda \in \widehat{G}} \frac{(C_{\lambda,N})^r}{(d_\lambda)^{r-2}}
\end{aligned} \tag{3.9}$$

where $C_{\lambda,N} = \int_G \mathrm{ch}^\lambda(g)\, Z_{L_N}(g)\, dg$. The value of $C_{\lambda,N} = d_\lambda\, c_\lambda$ is given by Theorem 3.28:

$$C_{\lambda,N} = \frac{1}{\mathrm{vol}(\mathfrak{t}/\mathfrak{t}_{\mathbb{Z}})} \left(\frac{L_N}{\sqrt{2\pi}}\right)^d \sum_{w\in W} \varepsilon(w)\, \widetilde{J}_{\mathbb{R}\Omega}(L_N\, \|\lambda+\rho-w(\rho)\|).$$

Let $x = L_N\,(\lambda+\rho)$; this is a point in the Weyl chamber C, and we assume that $x = O(1)$. For any root α and any smooth function f on $\mathbb{R}\Omega$, we define the partial derivative

$$(\partial_\alpha f)(x) = \lim_{\eta\to 0}\left(\frac{f(x+\eta\alpha)-f(x)}{\eta}\right) = \lim_{\eta\to 0}\left(\frac{f(x+\frac{\eta\alpha}{2})-f(x-\frac{\eta\alpha}{2})}{\eta}\right).$$

By [Bum13, Proposition 22.7], $\prod_{\alpha\in\Phi_+}(\mathrm{e}^{\frac{\alpha}{2}}-\mathrm{e}^{-\frac{\alpha}{2}}) = \sum_{w\in W}\varepsilon(w)\,\mathrm{e}^{w(\rho)}$ in the group algebra of the space of weights $\mathbb{R}\Omega$, therefore, setting $\partial_{\Phi_-} = \prod_{\alpha\in\Phi_-}(\partial_\alpha) = \prod_{\alpha\in\Phi_+}(-\partial_\alpha)$ and $\delta(x) = \prod_{\alpha\in\Phi_+}\frac{\langle x\,|\,\alpha\rangle}{\langle\rho\,|\,\alpha\rangle}$, we have

$$C_{\lambda,N} \simeq \frac{(L_N)^{(d+l)}}{\mathrm{vol}(\mathfrak{t}/\mathfrak{t}_{\mathbb{Z}})\,(2\pi)^{\frac{d}{2}}}\,(\partial_{\Phi_-}\widetilde{J}_{\mathbb{R}\Omega})(x);$$
$$\frac{N^{r-1}(C_{\lambda,N})^r}{(d_\lambda)^{r-2}} \simeq (L_N)^d\,\frac{\ell^{r-1}}{(\mathrm{vol}(\mathfrak{t}/\mathfrak{t}_{\mathbb{Z}}))^r(\delta(x))^{r-2}}\left(\frac{(\partial_{\Phi_-}\widetilde{J}_{\mathbb{R}\Omega})(x)}{(2\pi)^{\frac{d}{2}}}\right)^r$$

as $L_N\to 0$, since $\dim_{\mathbb{R}}(G) = \dim_{\mathbb{R}}(\mathfrak{g}) = \mathrm{rank}(G)+2|\Phi_+|$. The formula (3.9) for $N^{r-1}\,E_{R,N}$ becomes then a Riemann sum over the lattice $L_N(C\cap\mathbb{Z}\Omega)$. The points of this lattice correspond to domains of volume

$$\frac{(L_N)^d}{\mathrm{vol}(\mathfrak{t}/\mathfrak{t}_{\mathbb{Z}})},$$

so we obtain (see [Mél18, Theorem 5.8]):

Partial result 3.45 (Conjecture 3.41 for the single loop contributions). *Suppose (H,T) is a simple cycle of length $r\geq 2$. Then,*

$$e = \lim_{N\to\infty} N^{r-1}\,E_{H,T,N} = \left(\frac{\ell}{\mathrm{vol}(\mathfrak{t}/\mathfrak{t}_{\mathbb{Z}})}\right)^{r-1}\int_C \frac{1}{(\delta(x))^{r-2}}\left(\frac{(\partial_{\Phi_-}\widetilde{J}_{\mathbb{R}\Omega})(x)}{(2\pi)^{\frac{d}{2}}}\right)^r dx,$$

where dx is the Lebesgue measure on $\mathbb{R}\Omega$ associated to the scalar product of weights coming from Equation (3.7).

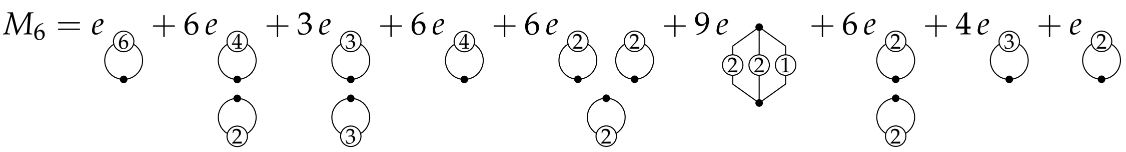
Example 3.46 (Asymptotics of the five first moments). Set $I_r = \int_C \frac{((\partial_{\Phi_-}\widetilde{J}_{\mathbb{R}\Omega})(x))^r}{(\delta(x))^{r-2}\,(2\pi)^{rd/2}}\,dx$ and $\ell' = \frac{\ell}{\mathrm{vol}(\mathfrak{t}/\mathfrak{t}_{\mathbb{Z}})}$. We then have:

$$M_2 = I_2\,\ell';$$
$$M_3 = I_3\,(\ell')^2;$$
$$M_4 = I_4\,(\ell')^3 + 2\,(I_2)^2\,(\ell')^2 + I_2\,(\ell');$$
$$M_5 = I_5\,(\ell')^4 + 5\,I_3\,I_2\,(\ell')^3 + 5\,I_3\,(\ell')^2.$$

For instance, for $G = \mathrm{SU}(2)$, we get

$$\begin{aligned}
M_2 &= \frac{1}{3}\left(\frac{\ell}{8\pi\sqrt{2}}\right);\\
M_3 &= \frac{5}{96}\left(\frac{\ell}{8\pi\sqrt{2}}\right)^2;\\
M_4 &= \frac{34}{2835}\left(\frac{\ell}{8\pi\sqrt{2}}\right)^3+\frac{2}{9}\left(\frac{\ell}{8\pi\sqrt{2}}\right)^2+\frac{1}{3}\left(\frac{\ell}{8\pi\sqrt{2}}\right);\\
M_5 &= \frac{40949}{13934592}\left(\frac{\ell}{8\pi\sqrt{2}}\right)^4+\frac{25}{288}\left(\frac{\ell}{8\pi\sqrt{2}}\right)^3+\frac{25}{96}\left(\frac{\ell}{8\pi\sqrt{2}}\right)^2.
\end{aligned}$$

▷ **The sixth moment.** Starting with the sixth moment, the circuit expansion of M_r involves reduced circuits with more than one vertex:



These new reduced circuits (see Figure 3.20) correspond to integrals over products of Weyl chambers C^h with a measure that is not uniform anymore, and that stems from the crystal theory of the group G (Section 3.1).

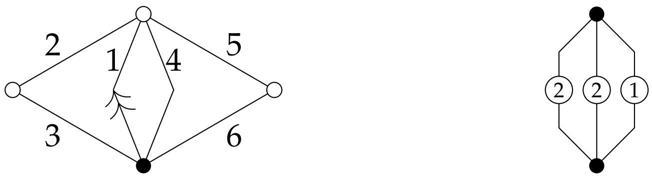


Figure 3.20. The circuit and the reduced circuit in size $r = 6$ and corresponding to the identities $i_1 = i_4$ and $i_2 = i_5$.

Let us explain how to compute E_R with R as in Figure 3.20. We have:

$$\begin{aligned}
E_{R,N} &= \mathbb{E}[h_N(v_1,v_2)\,h_N(v_2,v_3)\,h_N(v_1,v_3)\,h_N(v_1,v_4)\,h_N(v_4,v_1)]\\
&= \mathbb{E}\left[(Z_{L_N}^{*2}(v_1(v_3)^{-1}))^2\,Z_{L_N}(v_3(v_1)^{-1})\right] = N^3\,\left\langle ((Z_{L_N})^{*2})^2 \,\middle|\, Z_{L_N}\right\rangle_{\mathscr{L}^2(G,dg)}\\
&= \sum_{\lambda,\mu,\nu\in\widehat{G}} \frac{(C_{\lambda,N})^2\,(C_N(\mu))^2\,C_N(\nu)}{d_\lambda\,d_\mu}\,\left\langle \mathrm{ch}^\lambda \times \mathrm{ch}^\mu \,\middle|\, \mathrm{ch}^\nu\right\rangle. \qquad (3.10)
\end{aligned}$$

In comparison to Equation (3.9), the novelty is that one needs to understand the asymptotics of the Littlewood–Richardson coefficients

$$c_\nu^{\lambda,\mu} = \left\langle \mathrm{ch}^\lambda \times \mathrm{ch}^\mu \,\middle|\, \mathrm{ch}^\nu\right\rangle_{\mathscr{L}^2(G)}$$

when $\lambda = \frac{x}{L_N}$, $\mu = \frac{y}{L_N}$ and $L_N \to 0$. To this purpose, one can introduce the *relative string polytope* $\mathscr{P}(\lambda,\mu) \subset \mathscr{P}(\mu)$, which is the subset of the string polytope $\mathscr{P}(\mu) \subset \mathbb{R}^l$ that consists in sequences $(x_1,x_2,\ldots,x_l) \in (\mathbb{R}_+)^l$ such that, for any $i \in [\![1,d]\!]$ and any trail $(\phi_0^\vee,\phi_1^\vee,\ldots,\phi_l^\vee)$

from $s_i(\omega_i^\vee)$ to $w_0(\omega_i^\vee)$ in the fundamental representation $V^{\omega_i^\vee}$ of ${}^{\mathrm{L}}\mathfrak{g}_{\mathbb{C}}$,

$$\sum_{j=1}^{l} x_j\, \alpha_{i_j} \left(\frac{\phi_{j-1}^\vee + \phi_j^\vee}{2} \right) \geq -\lambda(\alpha_i^\vee).$$

Berenstein and Zelevinsky proved in [BZ01, Theorem 2.3]:

Proposition 3.47 (Berenstein–Zelevinsky). *For any dominant weights λ, μ, ν, the Littlewood–Richardson coefficient $c_\nu^{\lambda,\mu}$ is the number of integer points in the relative string polytope $\mathscr{P}(\lambda,\mu)$ that have weight $\nu - \lambda$, the weight of a sequence $(n_1, n_2, \ldots, n_l) \in \mathscr{P}(\mu)$ being*

$$\mu - \sum_{j=1}^{l} n_j\, \alpha_{i_j}.$$

We then have the analogue of Proposition 3.22 for Littlewood–Richardson coefficients:

Proposition 3.48 (Asymptotics of Littlewood–Richardson coefficients). *Fix two directions x and y in the interior C' of the Weyl chamber $C \subset \mathbb{R}\Omega$. There exists a function $q_{x,y}(z)$ that is*

- *a compactly supported piecewise polynomial function in z, of total integral over C smaller than $\min(\delta(x), \delta(y))$,*
- *symmetric in x and y,*
- *such that for any bounded continuous function f on C,*

$$\lim_{\substack{t\to\infty \\ tx,ty \in \widehat{G}}} \left(\frac{1}{t^l} \sum_{\nu \in \widehat{G}} c_\nu^{tx,ty}\, f\left(\frac{\nu}{t}\right) \right) = \int_C f(z)\, q_{x,y}(z)\, dz.$$

One has the scaling property $q_{\gamma x, \gamma y}(\gamma z) = \gamma^{l-d}\, q_{x,y}(z)$. More precisely, the function of three variables $(x,y,z) \in (C')^3 \mapsto q_{x,y}(z) \in \mathbb{R}_+$ is:

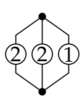
- *piecewise polynomial and locally homogeneous of total degree $l-d$ in (x,y,z),*
- *with domains of polynomiality that are polyhedral cones in $(C')^3$ (subsets that are stable by $(x,y,z) \mapsto (\gamma x, \gamma y, \gamma z)$, and that are bounded by a finite number of affine hyperplanes).*

Example 3.49 (Tensor product of two large representations of SU(3)). With $G = \mathrm{SU}(3)$, $\lambda = 10(\omega_1 + \omega_2)$ and $\mu = 10(2\omega_1 + \omega_2)$, we have drawn in Figure 3.21 the corresponding Littlewood–Richardson coefficients. It is easily seen that they are approximated by a piecewise affine map ($l = 3$, $d = 2$) supported by a polytope.

The positive measures $q_{x,y}(z)\, dz$ appeared in [BBO09, Section 5.6]. Informally, Proposition 3.48 can be restated as

$$c_{tz}^{tx,ty} \simeq t^{l-d}\, q_{x,y}(z)\, \sqrt{\det R_G}\, 1_{t(x+y-z)\in R}, \tag{3.11}$$

with a piecewise polynomial function of z that is compactly supported by a polytope. Proposition 3.48 allows one to consider the series in Equation (3.10) as a Riemann sum, and one obtains:

e 

$$= \left(\frac{\ell}{\mathrm{vol}(\mathfrak{t}/\mathfrak{t}_{\mathbb{Z}})} \right)^3 \int_{C^3} \frac{(\partial_{\Phi_-} \widetilde{J}_{\mathbb{R}\Omega})^2(x)\, (\partial_{\Phi_-} \widetilde{J}_{\mathbb{R}\Omega})^2(y)\, (\partial_{\Phi_-} \widetilde{J}_{\mathbb{R}\Omega})(z)}{(2\pi)^{\frac{5d}{2}}\, \delta(x)\, \delta(y)}\, q_{x,y}(z)\, dx\, dy\, dz.$$

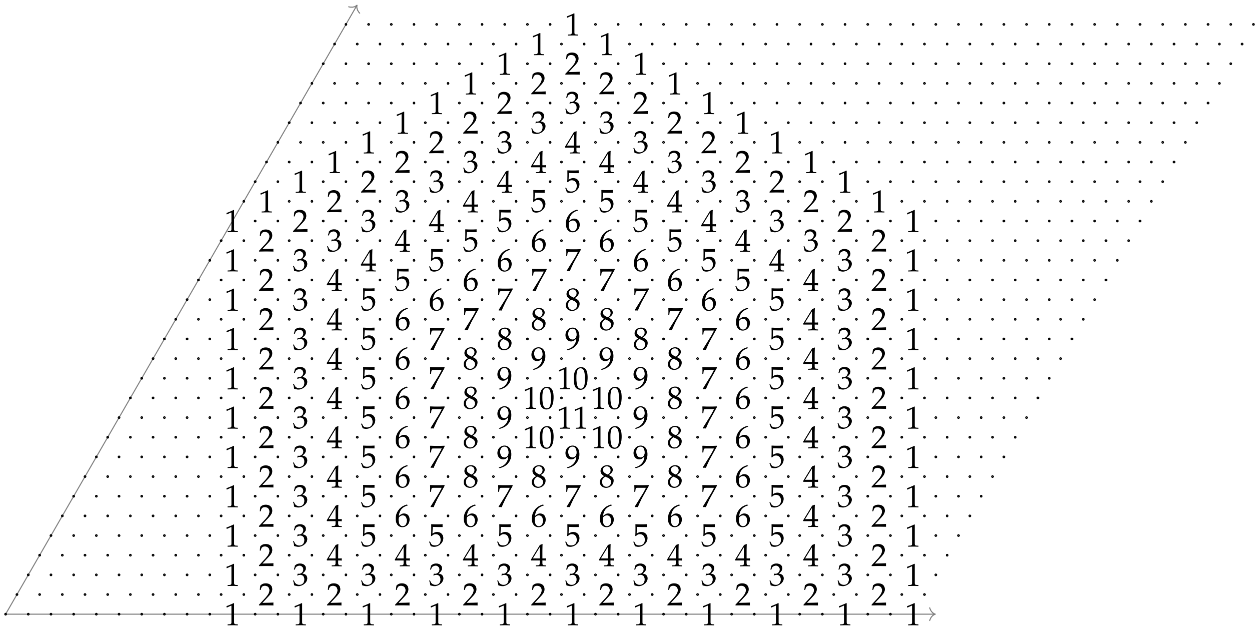


Figure 3.21. The multiplicities $c_\nu^{\lambda,\mu}$ with $\lambda = 10(\omega_1+\omega_2)$ and $\mu = 10(2\omega_1+\omega_2)$.

Therefore, denoting $I_{2,2,1}$ this integral without the factor $(\ell')^3$, we have the formula:

$$\begin{aligned}M_6 &= I_6\,(\ell')^5 + (6\,I_4 I_2 + 3\,(I_3)^2)\,(\ell')^4 + (6\,I_4 + 6\,(I_2)^3 + 9\,I_{2,2,1})\,(\ell')^3 \\ &\quad + (6\,(I_2)^2 + 4\,I_3)\,(\ell')^2 + I_2\,\ell'.\end{aligned}$$

For instance, when $G = \mathrm{SU}(2)$, one obtains

$$\begin{aligned}M_6 &= \frac{92377}{121621500}\left(\frac{\ell}{8\pi\sqrt{2}}\right)^5 + \frac{93257}{2903040}\left(\frac{\ell}{8\pi\sqrt{2}}\right)^4 + \frac{12887}{34560}\left(\frac{\ell}{8\pi\sqrt{2}}\right)^3 \\ &\quad + \frac{7}{8}\left(\frac{\ell}{8\pi\sqrt{2}}\right)^2 + \frac{1}{3}\left(\frac{\ell}{8\pi\sqrt{2}}\right).\end{aligned}$$

The technique presented above allows one to compute e_R for any connected reduced circuit with 2 vertices and an arbitrary number $h \geq 3$ of edges connecting them; see Theorem 5.28 in [Mél18]. Unfortunately, the moments M_r with $r \geq 7$ involve connected reduced circuits on more than 3 vertices, and for these reduced circuits we can for the moment only make conjectures.

▷ **A conjecture on crystals.** Let us detail how we want to deal with the asymptotics of a general circuit contribution $N^{k-1}\,E_{H,T,N}$, where H is some circuit with k vertices and r edges. We denote $R = R(H,T)$ the corresponding reduced circuit, and we fix an arbitrary indexation of the k' vertices of R. Without loss of generality, we can assume R connected, since the contribution $N^{k-1}\,E_{H,T,N}$ factorises over the connected components of R. As a consequence, $k - k' = \sum_e f_e - 1$. If $e = (\{a,b\}, f_e)$ is a labeled edge of R with $a < b$, then we convene to orientate e in the direction $a \to b$. The contribution $E_{H,T,N} = E_{R,N}$ is then

$$E_{R,N} = \mathbb{E}\left[\prod_{e=((a,b),f_e)\text{ edge of }R} (Z_{L_N})^{*f_e}(g_a g_b^{-1})\right],$$

where the random variables $g_1, \ldots, g_{k'}$ are independent and Haar distributed. In the sequel we denote $LE(R)$ the set of labeled edges of the reduced circuit R, and $h = |LE(R)|$ the number of edges of R. Using the character expansion of Z_{L_N}, we obtain

$$E_{R,N} = \sum_{(\lambda_e)_{e\in LE(R)}} \left(\prod_{e\in LE(R)} \frac{(C_{\lambda_e,N})^{f_e}}{(d_{\lambda_e})^{f_e-1}}\right) \mathbb{E}\left[\prod_{e=(a,b)\in LE(R)} \mathrm{ch}^{\lambda_e}(g_a g_b^{-1})\right],$$

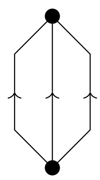
where the sum runs over $\widehat{G}^{LE(R)}$. If $x_e = L_N(\lambda_e + \rho)$, then

$$\frac{N^{k-1}\, E_{R,N}}{(\ell')^{k-k'}\, N^{k'-1} (L_N)^{lh}}$$
$$\simeq \sum_{\substack{x_e \in L_N(\widehat{G}+\rho)\\ e\in LE(R)}} \left(\prod_{e\in LE(R)} \frac{(L_N)^d}{\mathrm{vol}(\mathfrak{t}/\mathfrak{t}_{\mathbb{Z}})} \left(\frac{(\partial_{\Phi_-} \widetilde{J}_{\mathbb{R}\Omega})(x_e)}{(2\pi)^{d/2}}\right)^{f_e} \frac{1}{(\delta(x_e))^{f_e-1}}\right) F(R, (\lambda_e)_{e\in LE(R)}), \quad (3.12)$$

where $\ell' = \frac{\ell}{\mathrm{vol}(\mathfrak{t}/\mathfrak{t}_{\mathbb{Z}})}$ and

$$F(R, (\lambda_e)_{e\in LE(R)}) = \int_{G^{k'}} \left(\prod_{e\in LE(R)} \mathrm{ch}^{\lambda_e}(g_a g_b^{-1})\right) dg_1\, dg_2 \cdots dg_k'.$$

Note that the functional of dominant weights $F(R, (\lambda_e)_{e\in LE(R)})$ and the renormalisation factor $N^{k'-1}\,(L_N)^{lh}$ only depend on the *unlabeled directed graph S* underlying R. The functional F can be considered as a generalisation of the Littlewood–Richardson coefficients; indeed, if

$S =$  ,

and if ν^* is the highest weight associated to the conjugate of the irreducible representation of G with highest weight ν, then $F(R, (\lambda, \mu, \nu^*)) = \int_G \mathrm{ch}^{\lambda}(g)\, \mathrm{ch}^{\mu}(g)\, \overline{\mathrm{ch}^{\nu}(g)}\, dg = c_{\nu}^{\lambda,\mu}$. We call $F(R)$ a *graph functional* of the irreducible representations of G.

Conjecture 3.50 (Graph functional of representations and crystal theory). *Let G be a simple simply connected compact Lie group.*

(1) For any connected reduced circuit R and any set of dominant weights $(\lambda_e)_{e\in LE(R)}$, the graph functional $F(R, (\lambda_e)_{e\in LE(R)})$ is an integer, and it is the number of integer points of a compact polytope $\mathscr{P}((\lambda_e)_{e\in LE(R)})$ lying in the string cone $\mathscr{SC}(G^h) \subset \mathbb{R}^{lh}$ of G^h, with $l = |\Phi_+|$ and h equal to the number of edges of R.

(2) The inequalities that define the polytope $\mathscr{P}((\lambda_e)_{e\in LE(R)})$ are affine functions of the dominant weights λ_e, and the generic dimension of $\mathscr{P}((\lambda_e)_{e\in LE(R)})$ (for dominant weights in the interior of the Weyl chamber) is equal to

$$(\mathrm{card}\, \Phi_+)\, h - (\dim G)(k'-1) = lh - (2l+d)(k'-1),$$

where h is the number of edges of R, and k' is the number of vertices of R.

Therefore, as t grows to infinity, if $(x_e)_{e\in LE(R)}$ is a family of directions in the interior of the Weyl chamber C, then we have the asymptotics

$$F(R,(tx_e)_{e\in LE(R)}) \simeq_{t\to+\infty} t^{(\mathrm{card}\,\Phi_+)\,h-(\dim G)(k'-1)}\, q_R((x_e)_{e\in LE(R)})\, 1_{(tx_e)_{e\in LE(R)}\in I},$$

where the indicator function checks an integrality condition as in Equations (3.2) *and* (3.11), *and where q_R is a piecewise polynomial function that is locally homogeneous of degree $lh-(2l+d)(k'-1)$ in its coordinates, and with domains of polynomiality that are polyhedral cones in C^h.*

Assuming that Conjecture 3.50 holds true, one would obtain without too much additional difficulty a formula for e_R, by interpreting Formula (3.12) as a Riemann sum over C^h:

$$\begin{aligned} e_R &= \lim_{N\to\infty} N^{k-1}\, E_{R,N} \\ &= \ell'^{k-1}\, I_G \int_{C^h} \left(\prod_{e\in LE(R)} \left(\frac{(\partial_{\Phi_-}\widetilde{J}_{\mathbb{R}\Omega})(x_e)}{(2\pi)^{\frac{\mathrm{rank}(G)}{2}}}\right)^{f_e} \frac{1}{(\delta(x_e))^{f_e-1}}\right) q_R((x_e)_{e\in LE(R)}) \prod_{e\in LE(R)} dx_e, \end{aligned}$$

where I_G is some explicit real number that is related to the integrality condition that we did not detail in Conjecture 3.50. A proof of this conjecture should make the polynomials q_R explicit, and thus end the computation of the limiting moments M_r by using the partial result 3.44. In particular, we should then be able to compute bounds on these moments M_r, and thus get some information on the support of the limiting distribution μ_∞. Indeed, an important question which is not solved by the known proof of the convergence $\mu_N \rightharpoonup \mu_\infty$ is whether μ_∞ is compactly supported.

Remark 3.51. The formula $(\mathrm{card}\,\Phi_+)\,h-(\dim G)(k'-1)$ does not give a non-negative number for any connected reduced circuit R. Therefore, the corresponding polytope should be empty if this number is negative, and our conjecture should imply some vanishing results, which can be stated informally as follows: if one takes a graph functional of irreducible representations with too many Haar distributed random variables $g_1,\ldots,g_{k'}$ in comparison to the number h of characters appearing, then this integral vanishes. This is not very surprising since $\int_G \mathrm{ch}^\lambda(g)\,dg=0$ for any non trivial representation, but our conjecture would make this much more precise.

Perspectives

In this chapter, we explained how to use the non-commutative Fourier transform in order to study random objects on sscc symmetric spaces. In this last section, we present two research directions for which we plan to use the same tools.

$\to$ *Convergence to stationarity of hypo-elliptic Brownian motions.* In Section 3.2, we studied the convergence to the stationary law of the Brownian motions drawn on compact symmetric spaces. Assume to simplify that the underlying space is a sscc Lie group G. The Brownian motion on G is the diffusion process associated to the bi-invariant differential operator Δ (the Laplace–Beltrami operator), which corresponds to the element $\sum_{i=1}^{\dim G} X_i\otimes X_i$ of the tensor algebra of $\mathfrak{g}$, $(X_i)_{1\le i\le \dim G}$ being an orthonormal basis of $\mathfrak{g}$. More generally, one can consider left-invariant differential operators L on G which are associated to *hypo-elliptic* elements $\sum_{i=1}^r Y_i\otimes Y_i$, where $(Y_i)_{1\le i\le r}$ is a family of orthogonal vectors such that the Lie algebra spanned by these elements is $\mathfrak{g}$. One can show that given a hypo-elliptic left-invariant differential operator on G, the corresponding continuous Lévy process $(g_t)_{t\ge 0}$ started from e_G

has smooth marginal laws μ_t, and that $d_{\mathrm{TV}}(\mu_t, \mathrm{Haar}) \to 0$ as t goes to infinity [Lia04b, Theorem 4.2]. In this setting, it is natural to try to compute the speed of convergence, and to see whether a cut-off phenomenon still occurs. Intuitively, a hypo-elliptic diffusion is less diffusive than the standard Brownian motion, hence, the time to stationarity should be larger than in the elliptic case. The main problem is then to compute the eigenvalues of the operator L in each endomorphism space $\mathrm{End}_{\mathbb{C}}(V^\lambda)$ (one has several distinct eigenvalues in each such space, since L is not a bi-invariant operator).

Once the stationarity is attained, a (hypo-elliptic) Brownian motion $(x_t)_{t\geq 0}$ on a fixed ssccss X has Gaussian fluctuations, in the following sense. For any sufficiently smooth function $f : X \to \mathbb{R}$,

$$\frac{1}{\sqrt{t}}\left(\int_0^t f(x_t)\,dt - t\int_X f(x)\,dx\right) \rightharpoonup_{t\to\infty} \mathcal{N}_{\mathbb{R}}(0, \sigma^2(f)), \tag{3.13}$$

where $\sigma^2(f)$ is some limiting variance depending on f; see [Lan03] for a general survey of central limit theorems for statistics of Markov processes. This central limit theorem is a continuous analogue of the results on Markov chains presented in Example 2.36 and related to the theory of mod-Gaussian convergence. Thus, Brownian motions and more generally diffusions on ssccss provide an interesting example where one can try to understand the mod-Gaussian convergence of linear statistics of Markov processes. An important point is that the quality of the mod-Gaussian convergence in (3.13) should depend on the regularity of f; in particular, the speed of convergence should also depend on this regularity.

$\to$ *Macdonald polynomials, harmonic analysis on the (q,t)-Young graph and alcove walks.* Another important problem that we want to attack and that sits right between probability theory and representation theory of groups is the harmonic analysis of the (q,t)-*Young graph*. This problem is deeply related to the combinatorics of *Macdonald polynomials* and to their positive specialisations, and it is motivated by the recent theory of Macdonald processes. In [Oko01], Okounkov introduced the Schur measures, which are generalisations of the Plancherel measures and which involve certain specialisations of Schur functions. Given two summable sequences $A = (a_1, a_2, \ldots)$ and $B = (b_1, b_2, \ldots)$ of real numbers in $[0,1)$, the Schur measure with parameters A and B is the probability measure on integer partitions defined by

$$\mathfrak{M}_{A,B}(\lambda) = \prod_{i,j\geq 1}(1 - a_i b_j)\, \mathfrak{s}_\lambda(A)\, \mathfrak{s}_\lambda(B),$$

assuming that the Cauchy product $Z^{-1} = \prod_{i,j\geq 1}(1 - a_i b_j)$ is convergent. In terms of symmetric functions,

$$Z = \exp\left(\sum_{k=1}^{\infty} \frac{\mathfrak{p}_k(A)\,\mathfrak{p}_k(B)}{k}\right),$$

and this allows one to define more generally Schur measures associated to specialisations $\mathfrak{s}_\lambda \mapsto \mathfrak{s}_\lambda(A)$ of Sym that are non-negative on the set of Schur functions. In particular, if $A = B$ is the specialisation defined by

$$\mathfrak{p}_1(A) = \sqrt{\theta} \qquad ; \qquad \mathfrak{p}_{k\geq 2}(A) = 0,$$

then

$$\mathfrak{M}_{A,B}(\lambda) = \frac{\mathrm{e}^{-\theta}\,\theta^{|\lambda|}}{|\lambda|!}\,\frac{(\dim\lambda)^2}{|\lambda|!}$$

is the Poissonised Plancherel measure of parameter θ (law of a random partition where the size n is taken according to the Poisson law $\mathcal{P}(\theta)$, and then the partition λ is taken according to the Plancherel measure $\mathbb{P}_n$ on partitions of size n). The Schur measures were later generalised to Schur processes [OR03; OR07; Bor11], which are random processes on integer partitions whose transition probabilities involve non-negative specialisations of skew Schur functions. These Schur processes enabled the study of various particle systems and random interfaces, see the aforementioned papers or [Joh03; Joh05]. The classification of all Schur processes is related to the classification of all the specialisations of Sym which are non-negative on the set of Schur functions. Assuming the normalisation $\mathfrak{p}_1(X) = 1$, a specialisation X is non-negative if and only if there exists a parameter $\omega = (\alpha, \beta)$ of the Thoma simplex (Section 2.3) such that

$$\mathfrak{p}_1(X) = 1 \qquad ; \qquad \mathfrak{p}_{k\geq 2}(X) = \mathfrak{p}_k(\alpha + \varepsilon(\beta)) = \sum_{i=1}^{\infty} (\alpha_i)^k + (-1)^{k-1} \sum_{i=1}^{\infty} (\beta_i)^k;$$

see [Ais+51; Edr52; Edr53; Tho64; Oko97] and [Mél17, Chapter 11].

In [BC14], Borodin and Corwin proposed an important generalisation of the Schur processes called the *Macdonald processes*, and which involve the (skew) Macdonald polynomials defined in [Mac88]. The correlations of these processes are no longer determinantal, but one can still study them and prove Kardar–Parisi–Zhang asymptotics for the related random particle systems. The Macdonald polynomials $P_\lambda(q, t)$ are elements of the algebra $\mathbb{C}(q, t) \otimes_{\mathbb{C}}$ Sym of symmetric functions with two additional parameters, and one recovers the Schur functions $\mathfrak{s}_\lambda$ by setting $q = t$; we refer to [Hai02; GR05] for a survey of their properties. A classification of all the Macdonald processes rely on a classification of all the non-negative specialisations of the Macdonald polynomials, the parameters q and t being fixed in $[0, 1]$. In the Schur case, given a Thoma parameter $\omega = (\alpha, \beta)$, one has the following formula for the generating series of the ω-specialisation of the homogeneous symmetric functions:

$$\sum_{n=0}^{\infty} \mathfrak{h}_n(\omega)\, z^n = \exp\left(\sum_{k=1}^{\infty} \frac{\mathfrak{p}_k(\omega)\, z^k}{k} \right) = \mathrm{e}^{\gamma z} \prod_{i=1}^{\infty} \frac{1 + \beta_i z}{1 - \alpha_i z}$$

where $\gamma = 1 - \sum_{i=1}^{\infty} \alpha_i - \sum_{i=1}^{\infty} \beta_i$. In the Macdonald case, it has been conjectured by Kerov [Ker03] that with q, t fixed in $(0, 1)$, the normalised specialisations of Sym that are non-negative on the basis $(P_\lambda(q, t))_\lambda$ are again indexed by the Thoma simplex, and that they correspond to the following generating series:

$$\sum_{n=0}^{\infty} \frac{(q; q)_n}{(t; q)_n}\, P_n(q, t; \omega)\, z^n = \mathrm{e}^{\gamma z} \prod_{i=1}^{\infty} \frac{(\alpha_i t z; q)_\infty}{(\alpha_i z; q)_\infty}\, (1 + \beta_i z),$$

where $(a; q)_n = \prod_{k=0}^{n-1} (1 - a q^k)$ and $(a; q)_\infty = \prod_{k=0}^{\infty} (1 - a q^k)$.

Kerov's conjecture has been solved very recently by Matveev in [Mat17]. However, this result and its proof does not answer the following question: do these positive specialisations of the Macdonald polynomials correspond to the extremal points of the convex set of normalised non-negative harmonic functions on the (q, t)-Young graph? The (q, t)-Young graph is the infinite oriented graph:

- whose vertices are the integer partitions of all sizes;
- whose edges are the pairs $(\lambda \nearrow \Lambda)$ such that the Young diagram of Λ is obtained from the Young diagram of λ by adding one box;
- with on each edge a label $c(\lambda \nearrow \Lambda; q, t)$ which is the coefficient of $P_\Lambda(q, t)$ in the Pieri product $\mathfrak{p}_1\, P_\lambda(q, t)$ (this weight is given by an explicit formula involving the arm lengths and the leg lengths of the cells of the Young diagrams).

The Young graph without its labels is drawn in Figure 3.22. When $q = t$, the labels are all equal to 1, and the extremal points of the convex set of non-negative functions h on the Young graph which satisfy $h(\emptyset) = 1$ and the harmonicity condition

$$h(\lambda) = \sum_{\lambda \nearrow \Lambda} h(\Lambda)$$

are indeed in bijection with the normalised non-negative specialisations of the Schur functions. This correspondence relies on the Kerov–Vershik ring theorem (see [GO06, Section

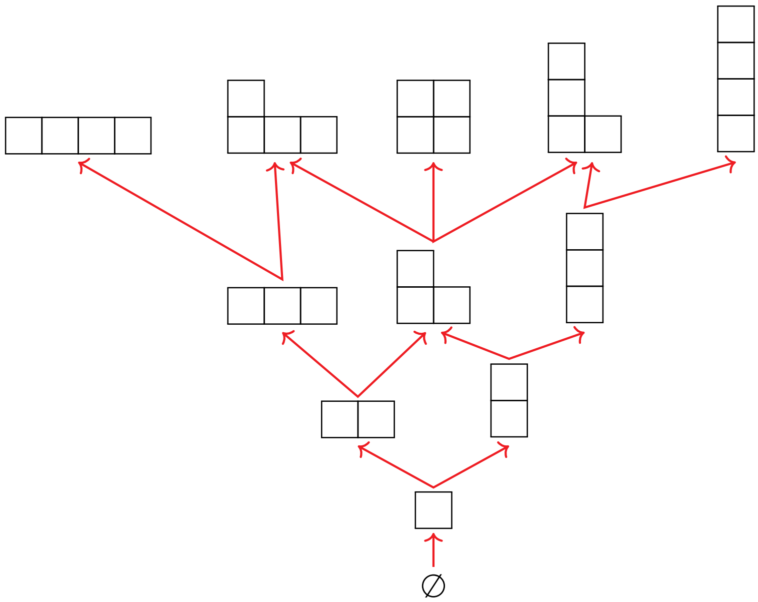


Figure 3.22. The four first levels of the Young graph.

8.7] or [Mél17, Theorem 11.5]), and on general arguments of harmonic analysis (*cf.* [Mar41; Doo59; Car72a; Car72b], and some of the arguments of the recent paper [Tar15]). The same correspondence holds true for the deformation of the Young graph corresponding to Jack polynomials [KOO98].

For Macdonald polynomials and the (q,t)-Young graph, the ring theorem could be applied if one were able to prove the following: with q and t fixed in $(0,1)$, the product

$$P_\lambda(q,t)\, P_\mu(q,t) = \sum_\nu c_\nu^{\lambda,\mu}(q,t)\, P_\nu(q,t) \tag{3.14}$$

involves structure coefficients $c_\nu^{\lambda,\mu}(q,t)$ which are all non-negative. This positivity result is the problem that we want to tackle, and the most promising approach relies on a formula due to Ram and Yip for the (q,t)-Littlewood–Richardson coefficients [Ram06; RY11; Yip12], and which is the analogue of the expansion on Littelmann paths of the classical Littlewood–Richardson coefficients (Proposition 3.17). Fix a root system of rank n and the corresponding weight lattice $\mathbb{Z}\Omega$ and Weyl group W, for instance in type A_n. The *extended affine Weyl group* W_{aff} is the semi-direct product of the group of translations $\{X_\omega,\ \omega \in \mathbb{Z}\Omega\}$ by the Weyl group W. Call *alcove* a pair $\mathfrak{A} = (\omega, A)$ that consists in a weight ω and a fundamental domain A of the weight lattice $\mathbb{Z}\Omega$ such that $\omega \in \partial A$. The fundamental alcove is the pair $\mathfrak{A}_{\mathrm{fund}} = (0, A_0)$ with A_0 unique fundamental domain that contains 0 and is in the Weyl chamber C. The action of the extended affine Weyl group on the fundamental alcove yields a bijection between alcoves and elements of W_{aff}. On the other hand, any element of W_{aff} can be written as $w = \pi^\vee s_{i_1}^\vee s_{i_2}^\vee \cdots s_{i_\ell}^\vee$, with ℓ minimal and where:

- the $s_i^\vee$'s with $i \in [\![0,n]\!]$ are either the reflections $s_1^\vee, \ldots, s_n^\vee$ that span the Weyl group W, or an additional allowed reflection $s_0^\vee$ associated to the maximal root and to the unique wall of A_0 that is not a hyperplane containing the zero weight;
- $\pi^\vee$ is one of the $|R/\mathbb{Z}\Omega|$ elements of length 0 in W_{aff}, which are associated to alcoves (ω, A_0), and which can be considered as changes of sheet in a covering of the space $\mathbb{R}\Omega$.

Given an element $w = \pi^\vee s_{i_1}^\vee s_{i_2}^\vee \cdots s_{i_\ell}^\vee$ in W, an *alcove walk* with type w is a sequence of contiguous alcoves $(\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_{\ell+1})$ where the two first alcoves differ by the change of sheet induced by $\pi^\vee$, and where the alcoves $\mathfrak{A}_{j\geq 1}$ and $\mathfrak{A}_{j+1}$ share a wall associated to the reflection $s_{i_j}^\vee$ (two consecutive alcoves can be equal).

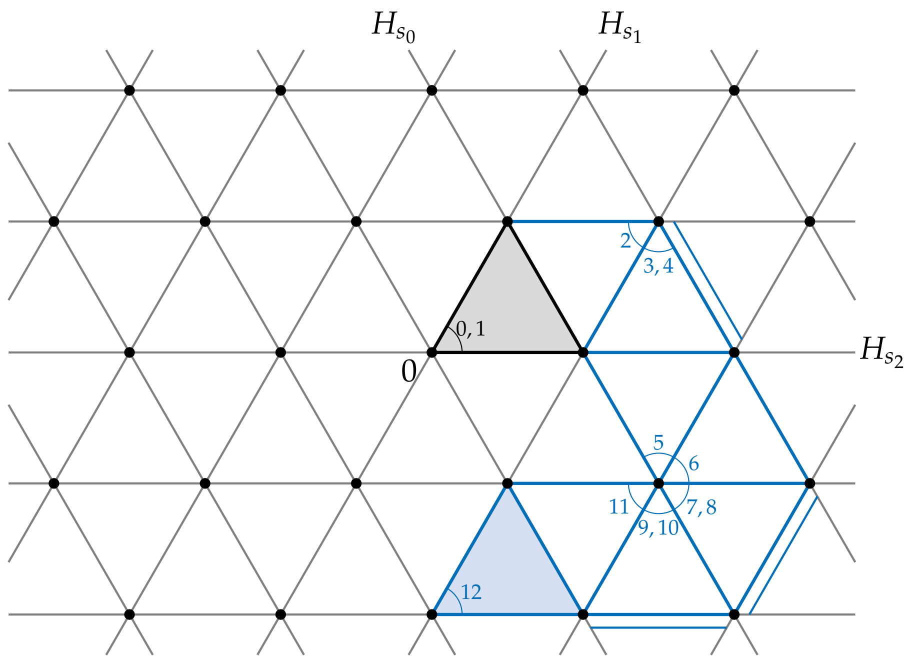


Figure 3.23. An alcove walk in type A_2 starting from the fundamental alcove, and with type $s_0^\vee s_2^\vee s_1^\vee s_0^\vee s_1^\vee s_2^\vee s_0^\vee s_1^\vee s_0^\vee s_2^\vee s_0^\vee$. The doubled walls indicate an alcove fold $(\mathfrak{A}_j, \mathfrak{A}_{j+1})$ with $\mathfrak{A}_j = \mathfrak{A}_{j+1}$.

Recall the expansion of the character $\mathrm{ch}^\lambda = \mathfrak{s}_\lambda$ as a sum over Littlemann paths in the crystal $\mathfrak{C}(\lambda)$ (Theorem 3.15). There exists a similar expansion of the Macdonald polynomial $P_\lambda(q,t)$ as a sum over certain alcove walks [RY11, Theorem 3.4]. There is also a formula involving alcove walks for the (q,t)-Littlewood–Richardson coefficients $c_\nu^{\lambda,\mu}(q,t)$, see [Yip12, Theorem 4.4]. Moreover, this product formula can probably be reinterpreted in a setting of quantum crystals; see [Ram06, Section 4.4] for the case of Hall–Littlewood polynomials, which are the specialisations of the Macdonald polynomials with $q = 0$. There are signs ± 1 appearing in Yip's formula, but there might be a way to gather the alcove walks of this expression in order to make only appear positive terms.

List of Symbols

Combinatorics and symmetric functions.

$\mathfrak{e}_k$	k-th elementary symmetric function, p. 38.
$\mathfrak{F}(n)$	set of functions $[\![1,n]\!] \to [\![1,n]\!]$, p. 35.
$\mathfrak{h}_k$	k-th homogeneous symmetric function, p. 38.
$\mathfrak{I}(n,\mathbb{F}_q)$	set of monic irreducible polynomials of degree n over $\mathbb{F}_q$, p. 36.
$\mathfrak{M}_{A,B}$	Schur measure associated to two alphabets A and B, p. 111.
$\mathfrak{P}(n,\mathbb{F}_q)$	set of monic polynomials of degree n over $\mathbb{F}_q$, p. 36.
$\mathfrak{P}(r)$	set of set partitions of size r, p. 41.
$\mathfrak{p}_k$	k-th power sum, p. 38.
$\mathfrak{s}_\lambda$	Schur function with label λ, p. 51.
$\mathfrak{S}(n)$	group of permutations of size n, p. 12.
$\mathfrak{T}(n)$	set of unordered rooted labeled trees on n vertices, p. 36.
$\mathfrak{Y}$	set of integer partitions (Young lattice), p. 23.
$\mathfrak{Y}(n)$	set of integer partitions of size n, p. 23.
$\varepsilon(A)$	conjugate of a formal alphabet A, p. 38.
γ_G	graphon associated to a graph G, p. 47.
$\mu(\pi)$	Möbius function of a set partition π, p. 41.
π_σ	permuton associated to a permutation σ, p. 48.
ω_λ	parameter of the Thoma simplex that encodes the row and column frequences of an integer partition λ, p. 49.
A_Γ	adjacency matrix of a graph Γ, p. 90.
$c_\nu^{\lambda,\mu}(q,t)$	(q,t)-Littlewood–Richardson coefficients, p. 113.
C_λ	conjugacy class of a permutation with cycle type λ, p. 35.
$\mathscr{F}$	space of graph functions, p. 46.
$\mathscr{G}$	space of graphons, p. 46.
$\hom(H,G)$	set of morphisms from a graph H to a graph G, p. 45.
$m_k(\lambda)$	number of parts of size k in an integer partition λ, p. 23.
$n^{\downarrow k}$	falling factorial $n(n-1)(n-2)\cdots(n-k+1)$, p. 56.
$\mathscr{O}_G$	algebra of finite graphs, p. 52.
$\mathscr{O}_P$	algebra of finite permutations, p. 53.
$\mathscr{O}_T$	algebra of partitions, p. 54.
$\mathbb{P}_n$	Plancherel measure on the set of integer partitions $\mathfrak{Y}(n)$, p. 68.
$\mathscr{P}$	space of permutons, p. 47.
$P_\lambda(q,t)$	Macdonald polynomial with label λ and parameters q and t, p. 112.
$p_k(\omega)$	observable p_k of a parameter ω of the Thoma simplex, p. 54.
$\mathrm{Spec}(\Gamma)$	spectrum of a graph Γ, p. 90.
ST_H	number of spanning trees of a multigraph H, p. 44.

Sym	algebra of symmetric functions (over the complex numbers), p. 38.
$\mathscr{T}$	Thoma simplex, p. 49.
$t(\tau,\cdot)$	density of a permutation motive τ in another permutation or permuton, p. 53.
$t(F,\cdot)$	density of a graph F in another graph or graph function or graphon, p. 52.
z_λ	size of the centralizer of a permutation with cycle type λ, p. 23.
(α,β)	a parameter of the Thoma simplex (also simply denoted ω), p. 49.
$(A(\lambda),B(\lambda))$	Frobenius coordinates of an integer partition λ, p. 49.
$(a;q)_n$	Pochhammer symbol $(1-a)(1-aq)\cdots(1-aq^{n-1})$, p. 112.
(V,E)	sets of vertices and of edges of a graph G, p. 43.

Probability theory and mod-ϕ convergence.

$\Delta_0(r,R,\phi)$	holomorphy domain of a series with an algebraico-logarithmic singularity, p. 34.
$\eta(z)$	Lévy–Khintchine exponent of an infinitely divisible distribution, p. 3.
$\eta_{c,\alpha,\beta}(z)$	Lévy–Khintchine exponent of the distribution $\phi_{c,\alpha,\beta}$, p. 7.
ϕ	an infinitely divisible distribution, p. 2.
$\phi_{c,\alpha,\beta}$	stable law with stability α, skewness β and scaling c, p. 7.
φ_t	winding number of the planar Brownian motion at time t, p. 5.
$\Gamma(n,\gamma)$	random model associated to a graphon γ, p. 47.
$\Gamma(N,L)$	random geometric graph with N points and level L, p. 90.
$\kappa(X_1,\dots,X_r)$	joint cumulant of a family of random variables $X_1,\dots,X_r$, p. 41.
$\kappa^{(r)}(X)$	r-th cumulant of a random variable X, p. 40.
κ_2	function on the algebra of observables of a mod-Gaussian moduli space that enables the computation of the limiting variances, p. 54.
κ_3	function on the algebra of observables of a mod-Gaussian moduli space that enables the computation of the limiting third cumulants, p. 54.
$\lambda(n,\omega)$	random partition of size n associated to a parameter ω of the Thoma simplex, p. 51.
$\mu(n)$	Möbius function of an integer n, p. 39.
μ_t	marginal law of the Brownian motion at time t, p. 86.
$\widehat{\mu}(\xi)$	Fourier transform of a probability measure μ, p. xx.
$\Pi(n,\pi)$	random model associated to a permuton π, p. 48.
$\sigma(n,\omega)$	random permutation of size n associated to a parameter ω of the Thoma simplex, p. 49.
$\sigma(n,\pi)$	random permutation of size n associated to a permuton π, p. 48.
$\theta(\xi)$	limit of the Fourier residues $\theta_n(\xi)$ of a mod-ϕ convergent sequence, p. 3.
θ_P	mixing constant associated to an ergodic transition matrix P, p. 59.
$\Omega(n,\omega)$	random model associated to a parameter ω of the Thoma simplex, p. 52.
ω_n	number of distinct prime divisors of a random integer smaller than n, p. 4.
$\psi(z)$	limit of the Laplace residues $\psi_n(z)$ of a mod-ϕ convergent sequence, p. 3.
$\zeta(s)$	Riemann's zeta function, p. 61.
$\mathcal{A}_{(a,b)}$	annular domain that consists in numbers z with $\mathrm{e}^a < \lvert z\rvert < \mathrm{e}^b$, p. 33.
$\mathscr{A}(\mathbb{T})$	Wiener algebra of absolutely convergent Fourier series, p. 22.
$\mathcal{C}$	standard Cauchy distribution, p. 3.
$d_{\mathrm{convex}}(\mu,\nu)$	convex distance between two measures μ and ν on $\mathbb{R}^d$, p. 26.
$d_{\mathrm{Kol}}(\mu,\nu)$	Kolmogorov distance between two measures μ and ν on the real line, p. 11.
$d_{\mathrm{loc}}(\mu,\nu)$	local distance between two distributions on $\mathbb{Z}$, p. 19.
$d_{\mathrm{TV}}(\mu,\nu)$	total variation distance between two distributions, p. 19.
$d_{\mathrm{GHP}}(X,Y)$	Gromov–Hausdorff–Prohorov metric between two measured metric spaces, p. 67.

$\widehat{f}(\xi)$	Fourier transform of a function f on the real line, p. xix.
$F(t,w)$	double generating series of a random combinatorial statistic, p. 33.
$F(x)$	Legendre–Fenchel transform of a l.s.c.c. function $\eta(h)$, p. 9.
$G(n,\gamma)$	random graph on n vertices associated to a graphon γ, p. 47.
$G(n,p)$	random Erdös–Rényi graph on n vertices and with parameter p, p. 45.
$G(z)$	Barnes' G-function, p. 6.
$H_r(z)$	r-th Hermite function, p. 22.
$L_f(s,w)$	double Dirichlet series of an arithmetic function, p. 37.
L_N	varying level of a random geometric graph in the Poisson regime, p. 91.
$\mathscr{M}^1(X)$	polish space of Borel probability measures on a space X, p. xx.
$m_{c,\alpha,\beta}(s)\,ds$	density of the distribution $\phi_{c,\alpha,\beta}$, p. 8.
$\mathcal{N}_{\mathbb{R}^d}(\mathbf{m},K)$	d-dimensional Gaussian law with mean $\mathbf{m}$ and covariance matrix K, p. 25.
$\mathcal{N}_{\mathbb{R}}(a,\sigma^2)$	Gaussian distribution with mean a and variance σ^2, p. 2.
$\mathcal{P}(\lambda)$	Poisson distribution with parameter λ, p. 3.
$\mathbb{P}_{\beta,h,\mathbb{Z}^d}$	probability measure of the infinite volume Ising configuration with inverse temperature β and external magnetic field h, p. 59.
$\mathbb{P}_{n,\Theta}$	generalised weighted measure with sequence of parameters Θ on the set of permutations of size n, p. 35.
$p_t(g)\,dg$	density of the Brownian motion at time t, p. 86.
$\mathcal{S}_{(c,d)}$	vertical band that consists in numbers z with $c < \mathrm{Re}(z) < d$, p. 8.
wt	the weight function of a weighted dependency graph, p. 58.
$(\chi_n)_{n\in\mathbb{N}}$	sequence of residues of an approximation scheme $(\nu_n)_{n\in\mathbb{N}}$, p. 20.
$(\nu_n)_{n\in\mathbb{N}}$	an approximation scheme of a sequence of discrete laws $(\mu_n)_{n\in\mathbb{N}}$, p. 20.
$(\nu_n^{(r)})_{n\in\mathbb{N}}$	standard approximation scheme of order r of a sequence of laws $(\mu_n)_{n\in\mathbb{N}}$, p. 20.
$(\sigma_n^2,L_n)_{n\in\mathbb{N}}$	renormalised variances and third cumulants for a sequence $(S_n)_{n\in\mathbb{N}}$ that satisfies the hypotheses of the method of cumulants, p. 41.
$(\sigma_n^{(r)})_{n\in\mathbb{N}}$	derived approximation scheme of the standard scheme of order r, p. 25.
(a,σ^2,π)	Lévy–Khintchine triplet of an infinitely divisible distribution, p. 3.
$(B_t)_{t\geq 0}$	Brownian motion (on a real vector space or a compact symmetric space), p. 86.
(D,N,A)	parameters of a dependency graph of a family $(A_v)_{v\in V}$, p. 44.
(D_n,N_n,A)	parameters of the method of cumulants for a sequence $(S_n)_{n\in\mathbb{N}}$, p. 41.
(K_1,K_2)	constants of the zone of control of a mod-stable convergent sequence, p. 16.
$[\pm K(t_n)^\gamma]$	zone of control of a mod-stable convergent sequence, p. 16.
$(S_n)_{n\in\mathbb{N}}$	a sequence of random variables that satisfies the hypotheses of the method of cumulants, p. 41.
$(t_n)_{n\in\mathbb{N}}$	parameters of a mod-ϕ convergent sequence of random variables, p. 3.
(v,w)	exponents of the zone of control of a mod-stable convergent sequence, p. 16.
$(X_n)_{n\in\mathbb{N}}$	a sequence of random variables that converges in the mod-ϕ sense, p. 3.
$(Y_n)_{n\in\mathbb{N}}$	a sequence of random variables that is obtained by renormalisation of a mod-ϕ convergent sequence $(X_n)_{n\in\mathbb{N}}$ and that converges in law, p. 7.

Representation theory and symmetric spaces.

$\mathfrak{C}(\lambda)$	crystal of the irreducible representation with highest weight λ, p. 80.
$\mathfrak{g}$	Lie algebra T_eG of a real Lie group G, p. 73.
$\mathfrak{g}_{\mathbb{C}}$	complexification $\mathbb{C}\otimes_{\mathbb{R}}\mathfrak{g}$ of a real Lie algebra, p. 77.
$\mathfrak{t}$	Lie algebra corresponding to a maximal torus $T\subset G$, p. 93.
$\mathfrak{t}_{\mathbb{Z}}$	lattice in $\mathfrak{t}$ such that $2\pi\mathfrak{t}_{\mathbb{Z}}$ is the kernel of the exponential map, p. 93.

χ^λ	normalised irreducible character of the Specht representation S^λ, p. 54.
Δ	Laplace–Beltrami operator, p. 86.
$\delta(x)$	normalised product of all the scalar products of x with a positive root, p. 105.
Φ	root system of a compact Lie group, p. 74.
$\Phi_\pm$	positive and negative roots of a compact Lie group, p. 74.
λ	an irreducible representation or a dominant weight, p. 71.
∂_{Φ_-}	partial derivative w.r.t. all the negative roots, p. 105.
π_λ	path associated to a dominant weight λ, p. 81.
ρ	half-sum of the positive roots, p. 78.
σ	Cartan automorphism associated to a symmetric space G/K, p. 76.
a_λ	eigenvalue of the Laplace–Beltrami operator associated to the irreducible representation with highest weight λ, p. 88.
$B(X,Y)$	Killing form evaluated on two vectors X and Y of the Lie algebra $\mathfrak{g}$, p. 73.
ch^λ	character of the irreducible representation with label λ, p. 73.
C	Weyl chamber in the space of weights $\mathbb{R}\Omega$, p. 77.
$c_\nu^{\lambda,\mu}$	Littlewood–Richardson coefficient (multiplicity of V^ν in $V^\lambda \otimes V^\mu$), p. 83.
c_λ	eigenvalue of a random geometric graph associated to the irreducible representation with highest weight λ, p. 93.
$C_{\lambda,N}$	$d_\lambda\, c_\lambda$, where c_λ is an eigenvalue of a geometric graph with level L_N, p. 105.
$\dim \lambda$	dimension of the representation S^λ of $\mathfrak{S}(n)$, p. 51.
d	rank of a compact Lie group (dimension of a maximal torus), p. 79.
d_λ	dimension of an irreducible representation λ, p. 71.
e^λ	normalised spherical vector in a spherical representation V^λ, p. 72.
$E_{H,T,N}$	contribution of a circuit (H,T) to the moment $M_{r,N}$, p. 102.
$\mathbb{FP}^n$	projective space of dimension n over the field $\mathbb{F}$, p. 96.
$\widehat{f}(\lambda)$	endomorphism in $\mathrm{End}(V^\lambda)$ defined by the integral $\int_G f(g)\, \rho^\lambda(g)\, dg$ (non-commutative Fourier transform), p. 71.
$\widetilde{f}(\lambda)$	element of V^λ defined by the integral $\sqrt{d_\lambda} \int_G f(g)\, \left(\rho^\lambda(g)\right)(e^\lambda)\, dg$ (spherical Fourier transform), p. 72.
$F(R,(\lambda_e)_e)$	graph functional associated to a reduced circuit R and evaluated on the dominant weights λ_e, p. 109.
$\mathrm{Gr}(n,q,\mathbb{F})$	Grassmannian manifold of q-dimensional vector subspaces in $\mathbb{F}^n$, p. 87.
$\widehat{G}$	set of isomorphism classes of irreducible representations of G, p. 71.
$\widehat{G}^K$	set of spherical representations of a compact Gelfand pair (G,K), p. 72.
G	a compact (Lie) group, p. 70.
$h_N(x,y)$	indicator of the condition $d(x,y) \leq L_N$ for points of a Poissonian random geometric graph, p. 101.
$\widetilde{J}_{\mathbb{R}\Omega}$	modified Bessel function on the weight space of a compact Lie group, p. 95.
$J^{(a,b),k}$	Jacobi polynomial with degree k and parameters (a,b), p. 97.
J_β	Bessel function of the first kind with index β, p. 94.
K	open subgroup of the subgroup of fixed points of a Cartan involution, p. 76.
K_ω^λ	Kostka number (multiplicity of the weight ω in V^λ), p. 83.
$\mathscr{L}^2(\widehat{G})$	dual Hilbert space of $\mathscr{L}^2(G)$, p. 71.
$\mathscr{L}^2(\widehat{G}^K)$	dual Hilbert space of $\mathscr{L}^2(G/K)$, p. 72.
$\mathscr{L}^2(G)$	space of functions that are square-integrable w.r.t. the Haar measure, p. 71.
l	number of positive roots of a compact Lie group, p. 83.
$LE(R)$	set of labeled edges of a reduced circuit R, p. 109.

$M_{r,N}$	expected moment of the spectral measure of a Poissonian random geometric graph, p. 101.
$\mathscr{P}(\lambda)$	string polytope of the irreducible representation with highest weight λ, p. 84.
$P^{n,k}$	Legendre polynomial with degree k and dimension n, p. 97.
$\mathbb{RS}^n$	real sphere of dimension n, p. 96.
$\mathbb{R}\Omega$	real vector space spanned by the weights of a compact Lie group, p. 74.
R	root lattice of a compact Lie group, p. 86.
$R(H,T)$	reduction of a circuit with graph H and traversal T, p. 104.
R_G	Gram matrix of the simple roots of a compact Lie group G, p. 86.
$\mathrm{Spin}(n)$	spin group of order n (universal cover of $\mathrm{SO}(n)$), p. 74.
$\mathscr{S}(\lambda)$	set of string parametrisations of the representation with highest weight λ, p. 84.
$\mathscr{SC}(G)$	string cone of a compact Lie group, p. 84.
$\mathrm{SO}(n)$	special orthogonal group $\{M \in \mathrm{M}(n,\mathbb{R}),\ M^t = M^{-1} \text{ and } \det M = 1\}$, p. 74.
$\mathrm{SU}(n)$	special unitary group $\{M \in \mathrm{M}(n,\mathbb{C}),\ M^* = M^{-1} \text{ and } \det M = 1\}$, p. 74.
S^λ	Specht module of $\mathfrak{S}(n)$ with label $\lambda \in \mathfrak{Y}(n)$, p. 51.
s_α	reflection associated to a simple root α, p. 74.
T	maximal torus in a compact Lie group G, p. 73.
$T_x X$	tangent space of a manifold X at the point x, p. 73.
$\mathrm{USp}(n)$	compact symplectic group $\{M \in \mathrm{M}(n,\mathbb{H}),\ M^* = M^{-1}\}$, p. 75.
$U_q(\mathfrak{g}_\mathbb{C})$	quantum group of a complex Lie algebra, p. 79.
$\mathrm{vol}(\mathfrak{t}/\mathfrak{t}_\mathbb{Z})$	volume of a fundamental domain of the lattice $\mathfrak{t}_\mathbb{Z}$ in $\mathfrak{t}$, p. 93.
$V(\omega)$	weight space in a representation V associated to the weight ω, p. 73.
V^λ	representation space of the irreducible representation with label λ, p. 71.
$V^{\lambda,K}$	subspace of K-fixed vectors in the representation space V^λ, p. 72.
W	Weyl group of a compact Lie group, p. 74.
X	a quotient G/K of a compact (Lie) group, p. 72.
$\mathbb{Z}\Omega$	weight lattice of a compact Lie group, p. 74.
zon^λ	zonal spherical function of the spherical representation with label λ, p. 73.
$Z_L(g)$	indicator of the condition $d(e_G, g) \leq L$ for a point $g \in G$, p. 93.
$(\alpha_i^\vee)_{i\in[\![1,d]\!]}$	basis of coroots for a compact Lie group, p. 79.
$(\omega_i)_{i\in[\![1,d]\!]}$	basis of fundamental weights for a compact Lie group, p. 79.
(G,K)	compact Gelfand pair (associated to a symmetric space), p. 72.
(H,T)	a circuit with graph H and traversal T, p. 102.
(k',h)	number of vertices and number of edges of a reduced circuit, p. 104.
(k,r)	number of vertices and number of edges of a circuit, p. 102.
$\langle u \mid v\rangle_{\mathrm{End}(V^\lambda)}$	standard scalar product on the endomorphism space of an irreducible representation, p. 71.
(V,ρ)	a linear representation of a group with space V and underlying morphism $\rho : G \to \mathrm{GL}(V)$, p. 71.
$\langle X \mid Y\rangle_\mathfrak{g}$	scalar product on a semisimple Lie algebra that is propotional to the opposite of the Killing form, p. 75.

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