① Ising model - Exam solutions 2025

<u>Ex 1</u>: We start with a finite domain $\Lambda \subset\subset \mathbb{Z}^d$, with wired b.c. on $\partial\Lambda$. Then, for $\omega \in \Omega_\Lambda = \{0,1\}^{E(\Lambda)}$,

$\forall e \in E(\Lambda)$, $e = \{u,v\}$,

$$\frac{\phi^1_{\Lambda,p,q}(\omega^{+e})}{\phi^1_{\Lambda,p,q}(\omega^{-e})} = \begin{cases} \frac{p}{1-p} & \text{if } u \overset{\omega^{-e}}{\longleftrightarrow} v \text{, possibly using the boundary to connect them,} \\ \frac{p}{q(1-p)} & \text{otherwise.} \end{cases}$$

$$= \frac{p}{q(1-p)} q^{\mathbb{1}_{u \overset{\omega^{-e}}{\longleftrightarrow} v}}.$$

Let $\omega \leq \omega'$, then $u \overset{\omega^{-e}}{\longleftrightarrow} v$ implies $u \overset{\omega'^{-e}}{\longleftrightarrow} v$, and since $1 \leq q \leq q'$ we get $q^{\mathbb{1}_{u \overset{\omega^{-e}}{\longleftrightarrow} v}} \leq q'^{\mathbb{1}_{u \overset{\omega'^{-e}}{\longleftrightarrow} v}}$.

With this and $\frac{p}{q(1-p)} \leq \frac{p'}{q'(1-p')}$, we get

$$\frac{\phi^1_{\Lambda,p,q}(\omega^{+e})}{\phi^1_{\Lambda,p,q}(\omega^{-e})} \leq \frac{\phi^1_{\Lambda,p',q'}(\omega'^{+e})}{\phi^1_{\Lambda,p',q'}(\omega'^{-e})}.$$

By a criterion, we have $\phi^1_{\Lambda,p,q} \leq_{st} \phi^1_{\Lambda,p',q'}$.

Finally, we take a sequence $\Lambda_n = [\![-n,n]\!]^d$, then for any increasing local function, we know that

② $\Phi^1_{p,q}(f) = \lim_n \nearrow \Phi^1_{\Lambda_n,p,q}(f)$

so we have $\Phi^1_{p,q}(f) \leq \Phi^1_{p',q'}(f)$.

By approximation of $\nearrow$ functions via $\nearrow$ local functions, this gives $\Phi^1_{p,q} \leq_{st} \Phi^1_{p',q'}$

(For instance, if f is $\nearrow$ on $\Omega_{\mathbb{Z}^d} = \{0,1\}^{\mathbb{Z}^d}$, define $f_p(\omega) = f(\tilde{\omega}_p)$ where $\tilde{\omega}_p = \begin{cases} \omega \text{ on } [\![-p,p]\!]^d \\ 0 \text{ elsewhere.} \end{cases}$

then f_p is local, $\nearrow$, and $f_p \xrightarrow[p\to\infty]{} f$ for the local topology...)

Exercise 2 :

1) $$Z = \sum_{\sigma\in\Omega} \exp\Big(\sum_{U\subset V} J_U \sigma_U\Big)$$

$$= \sum_{\sigma\in\Omega} \prod_{U\subset V} \cosh(J_U) + \sigma_U \sinh(J_U)$$

$$= \Big(\prod_{U\subset V} \cosh J_U\Big) \sum_{\sigma\in\Omega} \sum_{\mathcal{E}\subset\mathcal{P}(V)} \Big(\prod_{U\in\mathcal{E}} \tanh J_U \cdot \sigma_U\Big)$$

$$= \Big(\prod_{U\subset V} \cosh J_U\Big) \sum_{\mathcal{E}\subset\mathcal{P}(V)} w(\mathcal{E}).\Big(\sum_{\sigma\in\Omega} \prod_{U\in\mathcal{E}} \sigma_U\Big)$$

where $w(\mathcal{E}) = \prod_{U\in\mathcal{E}} \tanh(J_U)$.

Let $\partial\mathcal{E} = \{x\in V / \sum_{U\in\mathcal{E}} \mathbb{1}_{x\in U} \text{ is odd}\}$.

③ Then we get that for $\partial\mathcal{E} \neq \emptyset$, the sum $\sum_\sigma \ldots$ is 0, and otherwise, it is $|\Omega| = 2^{|V|}$.

So

$$Z = \left(2^{|V|} \prod_{U \subset V} \cosh J_U\right) \cdot \sum_{\substack{\mathcal{E} \subset \mathcal{P}(V) \\ \partial\mathcal{E} = \emptyset}} w(\mathcal{E}).$$

2) Similarly,

$$\langle \sigma_A \rangle = \frac{\sum_{\substack{\mathcal{E} \subset \mathcal{P}(V) \\ \partial\mathcal{E} = A}} w(\mathcal{E})}{\sum_{\substack{\mathcal{E} \subset \mathcal{P}(V) \\ \partial\mathcal{E} = \emptyset}} w(\mathcal{E})}.$$

3) As all weights are positive, we see that $\forall A \subset V$, $\langle \sigma_A \rangle \geq 0$.

Then, let $A, B \subset V$.

$$\langle \sigma_A \sigma_B \rangle - \langle \sigma_A \rangle \langle \sigma_B \rangle = \frac{1}{Z^2} \sum_{\sigma, \sigma' \in \Omega} (\sigma_A \sigma_B - \sigma_A \sigma'_B) \exp\left(\sum_{U \subset V} J_U (\sigma_U + \sigma'_U)\right)$$

$\tau = \sigma\sigma'$

$$= \frac{1}{Z^2} \sum_{\sigma, \sigma'} \sigma_A \sigma_B (1 - \sigma_B \sigma'_B) \exp\left(\sum_{U \subset V} J_U \sigma_U (1 + \sigma_U \sigma'_U)\right)$$

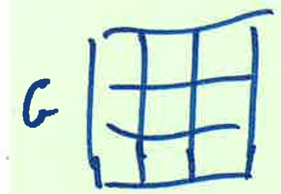
$$= \frac{1}{Z^2} \sum_{\sigma, \tau} (1 - \tau_B)\, \sigma_A \sigma_B \exp\left(\sum_{U \subset V} J_U (1 + \tau_U) \sigma_U\right)$$

$$= \frac{1}{Z^2} \sum_{\tau} (1 - \tau_B) \cdot \underbrace{\left(\sum_{\sigma} \sigma_A \sigma_B \exp\left(\sum_{U \subset V} J_U (1 + \tau_U) \sigma_U\right)\right)}_{\langle \sigma_A \sigma_B \rangle_{J_\tau} \text{ where } (J_\tau)_U = J_U (1 + \tau_U)}$$

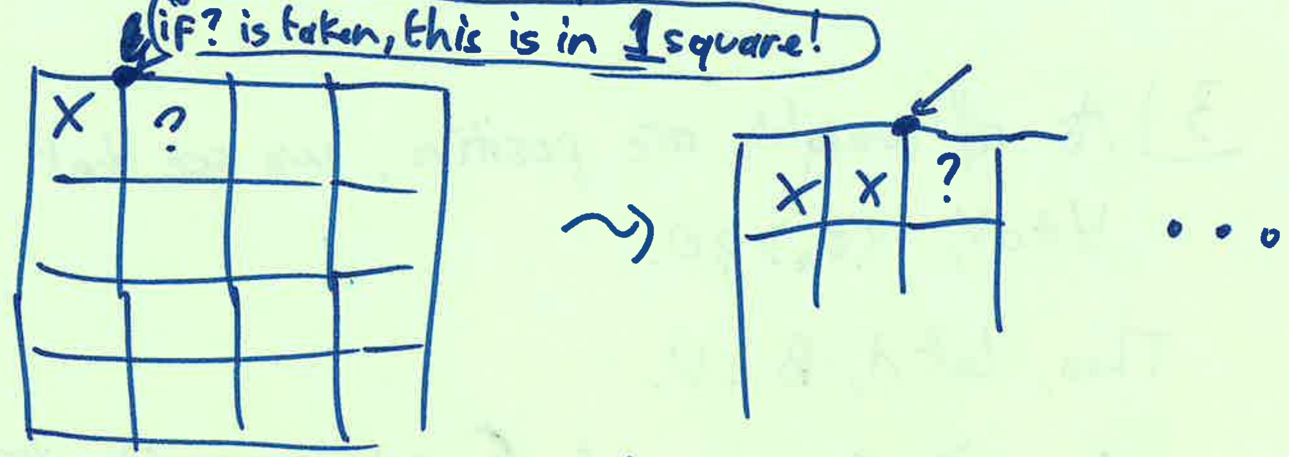
④ By the First ineq, all terms are positive, so

$$\langle \sigma_A \sigma_B \rangle - \langle \sigma_A \rangle \langle \sigma_B \rangle \geq 0.$$

4) We are looking for sets of squares s.t. every vertex belongs to an even number of squares. We claim that only the empty set works.



Indeed, the corners squares cannot be present, otherwise the corner vertices belong to just one square. But then, the rest of the ~~edges~~ "boundary" of G cannot be present either, by the same argument on subsequent vertices:



We get that the whole "boundary" of squares is excluded, and by induction, we see that no square can be present.

Therefore, only $\mathcal{E} = \emptyset$ contributes to Z:

$$Z_{n,\beta} = 2^{(n+1)^2} \coth \beta^{n^2}.$$

⑤ 5) $\frac{1}{n^2}\log(Z_{n,\beta}) = \frac{(n+1)^2}{n^2}\log 2 + \log\cosh\beta$

$\xrightarrow[n\to\infty]{} f(\beta) = \log 2 + \log\cosh(\beta).$

6) For all $n \geq 1$,

$$\frac{\partial}{\partial\beta}\left(\log(Z_{n,\beta})\right) = \frac{\frac{\partial}{\partial\beta}\left(\sum_{\sigma\in\Omega}\exp\left(\sum_{U \text{ unit square}}\beta\sigma_U\right)\right)}{Z_{n,\beta}}$$

$$= \sum_{\sigma\in\Omega}\left(\sum_{U \text{ unit square}}\sigma_U\right)\frac{\exp\left(\sum_{U}\beta\sigma_U\right)}{Z_{n,\beta}}$$

($\frac{\exp(\sum_U \beta\sigma_U)}{Z_{n,\beta}} = \mu(\sigma)$)

$$= \left\langle \sum_{U \text{ unit square}} \sigma_U \right\rangle_{n,\beta}.$$

$$\frac{\partial}{\partial\beta}\left(\log(Z_{n,\beta})\right) = n^2\frac{\sinh\beta}{\cosh\beta} = n^2\tanh\beta$$

So $\frac{\partial f}{\partial\beta} = \tanh\beta$ is also equal to $\left\langle \frac{1}{n^2}\sum_{U \text{ unit square}}\sigma_U \right\rangle_{n,\beta}$

for all n.

IF there are N squares s.t. $\sigma_U = +1$, then

$\tanh\beta = \left\langle \frac{1}{n^2}\sum_U \sigma_U \right\rangle_{n,\beta} = \left\langle \frac{1}{n^2}\left(N - (n^2 - N)\right)\right\rangle_{n,\beta}$

so $\left\langle \frac{1}{n^2}N \right\rangle_{n,\beta} = \frac{1+\tanh\beta}{2}$.

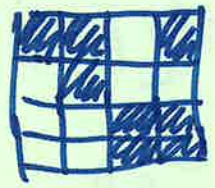
7) By a similar argument, we see that the only $\mathcal{C}$ set of squares s.t. $\partial\mathcal{C} = \{a,b,c,d\}$ is the set of all squares. So

$$\langle \sigma_a \sigma_b \sigma_c \sigma_d \rangle_{n,\beta} = (\operatorname{th} \beta)^{n^2}.$$

In particular, $\forall \beta > 0$ this goes to 0 as $n \to \infty$

8) There is no set of squares with boundary $\{u, v\}$.

Indeed, for a set of squares $\mathcal{C}$, let's color it:



Then the boundary is exactly the set of vertices that look like  or  (up to rotations).

If we look at the interface between ▨ and ◯, this means that the interface has a "corner":

┐ or ┘ or └ or ┌ .

But if we consider a connected component of the set of black squares, it must have at least 4 corners on its outer boundary.

So $|\partial\mathcal{C}| \geq 4$ (unless $\mathcal{C} = \emptyset$).

As a result, $\langle \sigma_u \sigma_v \rangle_{n,\beta} = 0$.

9) No quantity studied here had any discontinuity or even lack of analyticity, for $\beta > 0$.

Also $\langle \sigma_u \sigma_v \rangle = 0$ means that we cannot expect Ising order at any temperature. So it seems that there is no phase transition (or if you prefer it happens at $\beta = \infty$), and this is similar to Ising in dimension 1.

Ex 3 :

1) At every step, we select a vertex $v \in \{0, .., n-1\}$ uniformly at random, and we resample its spin using the conditional distribution of π given all the other spins. That is, for any $\sigma \in \Omega$, if σ, σ' differ at exactly one spin v,

$$P(\sigma, \sigma') = \frac{1}{n} \frac{\pi(\sigma')}{\pi(\sigma) + \pi(\sigma')} = \frac{1}{n} \frac{\exp\left(\frac{2\beta}{n} \sigma'_v \sum_{i \neq v} \sigma_i\right)}{\exp\left(\frac{2\beta}{n} \sigma_v \left(\sum_{i \neq v} \sigma_i\right)\right) + \exp\left(\frac{2\beta}{n} \sigma'_v \sum_{i \neq v} \sigma_i\right)}$$

other mistake in the subject, sorry! there is a factor 2 to take into account both $i = v$ and $j = v$

$$= \frac{1 - \sigma_v \tanh\left(\frac{2\beta}{n}\left(\sum_{i=0}^{n-1} \sigma_i - \sigma_v\right)\right)}{2n}.$$

In all other cases, $P(\sigma, \sigma') = 0$, except for $P(\sigma, \sigma)$ which we don't have to compute.

This defines a Markov chain on Ω, and it is clearly irreducible, so it is recurrent positive.

(8) Moreover, it is aperiodic

$\left(P(\sigma,\sigma) \geq \frac{1}{n}\left(1 - P(\sigma,\sigma')\right) > 0 \right.$ by taking any σ' differing from σ at just one site).

So it has a unique invariant measure, and converges to it in distribution. Finally, π is reversible, so it is the invariant measure:

$\forall \sigma, \sigma'$, if they differ at exactly one spin,

$$\pi(\sigma) P(\sigma,\sigma') = \pi(\sigma) \frac{\pi(\sigma')}{\pi(\sigma)+\pi(\sigma')} \cdot \frac{1}{n}$$

$$= \pi(\sigma') P(\sigma',\sigma).$$

In all other cases this equality is clear.

A-2) A_k contains $\binom{n}{k}$ spin configs, and each one is such that

$$\pi(\sigma) = \frac{1}{Z} \exp\left(\frac{\beta}{n}\left(\sum_{i=0}^{n-1} \sigma_i\right)^2\right)$$

$$= \frac{1}{Z} \exp\left(\frac{\beta}{n}\left(k-(n-k)\right)^2\right)$$

$$= \frac{1}{Z} \exp\left(\frac{\beta}{n}\left(2k-n\right)^2\right)$$

So $\pi(A_k) = \frac{1}{Z} \binom{n}{k} \exp\left(\frac{\beta}{n}(2k-n)^2\right)$

(also, $Z = \sum_{k=0}^{n} \binom{n}{k} \exp\left(\frac{\beta}{n}(2k-n)^2\right)$.)

(9) 3) We suppose first that n is even. Then, denoting $\mathbb{P}_\pi$ the distribution of the Markov chain started at $X_0 \sim \pi$, we have

$$Q(S,S^c) = \mathbb{P}_\pi(X_0 \in S,\ X_1 \notin S).$$

In that case, since we change one spin at a time, we must have X_0 with $\frac{n}{2}-1$ positive spins, and change a -1 into a $+1$ to get X_1. So $X_1 \in A_{\frac{n}{2}}$. Therefore,

$$Q(S,S^c) \leq \mathbb{P}_\pi(X_1 \in A_{\frac{n}{2}}) = \pi(A_{\frac{n}{2}})$$

since π is invariant.

For n odd, similarly, we must have that X_0 contains $\lfloor \frac{n}{2} \rfloor$ positive spins (and X_1 has $\lfloor \frac{n}{2} \rfloor + 1$)

so $Q(S,S^c) \leq \mathbb{P}_\pi(X_0 \in A_{\lfloor \frac{n}{2} \rfloor}) = \pi(A_{\lfloor \frac{n}{2} \rfloor})$.

4) Therefore,

$$\phi(S) \leq \frac{\pi(A_{\lfloor \frac{n}{2} \rfloor})}{\pi(S)} = \frac{\pi(A_{\lfloor n/2 \rfloor})}{\sum_{k < \frac{n}{2}} \pi(A_k)}$$

$$= \frac{\binom{n}{\lfloor n/2 \rfloor} \exp\left(\frac{\beta}{n}(2\lfloor \frac{n}{2} \rfloor - n)^2\right)}{\sum_{k \leq \frac{n}{2}} \binom{n}{k} \exp\left(\frac{\beta}{n}(2k-n)^2\right)}$$

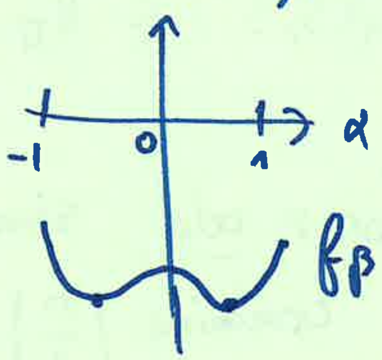
In the proof of the C-W phase transition, we found the asymptotics of this kind of sum. In particular, we saw that

$$-\frac{1}{n}\log\Big(\sum_{k<\frac{n}{2}}\binom{n}{k}\exp\Big(\frac{\beta}{n}(2k-n)^2\Big)\Big)\xrightarrow[n\to\infty]{}\inf_{\alpha<0}F_\beta(\alpha)$$

where $F_\beta(\alpha)=\frac{1+\alpha}{2}\log\frac{1+\alpha}{2}+\frac{1-\alpha}{2}\log\frac{1-\alpha}{2}-\beta\alpha^2$.

Also, $-\frac{1}{n}\log\Big(\binom{n}{\lfloor\frac{n}{2}\rfloor}\exp\Big(\frac{\beta}{n}\big(2\lfloor\frac{n}{2}\rfloor-n\big)^2\Big)\Big)\xrightarrow[n\to\infty]{}F_\beta(0)$.

For $\beta>\frac{1}{2}$, we saw



so $\inf_{\alpha<0}F_\beta(\alpha)<F_\beta(0)$.

Let $c_0:=F_\beta(0)-\inf_{\alpha<0}F_\beta(\alpha)$, then

$\frac{1}{n}\log\phi(S)\xrightarrow[n\to\infty]{}-c_0$ and so $\forall n,\ \phi(S)\leq e^{-cn}$ for a certain c depending only on β.

5) Also, $\pi(S)\leq\frac{1}{2}$

(indeed, by symmetry, $\pi(S)=\pi(\{\sigma/\sum\sigma_i>0\})$

so $2\pi(S)=\pi(\{\sigma/\sum\sigma_i\neq 0\})\leq 1$).

(11) Applying the given result, we find

$$t_{mix}\left(\frac{1}{4}\right) \geq \frac{1}{4}\exp(cn).$$

We see that at low temperature $\left(\beta > \beta_c = \frac{1}{2}\right)$, it takes an exponential time (in n) to get close to the invariant measure π.

B-6) Let σ, σ' be two configs st $\rho(\sigma, \sigma') = 1$, that is, $\exists v \in \{0, .., n-1\}$ s.t. σ and σ' differ only at v. Suppose w.l.o.g. that $\sigma_v = +1$, $\sigma'_v = -1$.

Let us describe a coupling (X, Y) between $P(\sigma, \cdot)$ and $P(\sigma', \cdot)$. First we choose a vertex $w \in \{0, .., n-1\}$ uniformly at random.

- If $w = v$, the conditional distribution obtained by resampling at w is the same for σ and for σ'. We sample S with this distribution and we set $X = Y = S$.
- If $w \neq v$, from σ we want to resample σ_w such that it becomes -1 with proba $\frac{1 - \tanh\left(\frac{2\beta}{n}\left(\sum \sigma_i - \sigma_w\right)\right)}{2}$

  $+1$ with proba $\frac{1 + \tanh\left(\frac{2\beta}{n}\left(\sum \sigma_i - \sigma_w\right)\right)}{2}$.

(12) From σ', we want the new spin to be

$+1$ with proba $\frac{1+\tanh\left(\frac{2\beta}{n}\left(\sum\sigma'_i-\sigma'_w\right)\right)}{2}$

-1 with proba $\frac{1-\tanh\left(\frac{2\beta}{n}\left(\sum\sigma'_i-\sigma'_w\right)\right)}{2}$.

Let $S=\sum_{i=0}^{n-1}\sigma_i - \sigma_w$.

Then $\sum_{i=0}^{n-1}\sigma'_i - \sigma'_w = S+\sigma'_v-\sigma_v = S-2$.

We get that σ_w should become $+1$ with proba

$$\frac{1+\tanh\left(\frac{2\beta}{n}S\right)}{2},$$

while for σ'_w it is $\frac{1+\tanh\left(\frac{2\beta}{n}(S-2)\right)}{2}$

(which is smaller, which makes sense: the -1 at v in σ' favors -1 a bit).

We sample a single uniform r.v. $U\sim\mathcal{U}([0,1])$.

If $U\leq\frac{1+\tanh\left(\frac{2\beta}{n}(S-2)\right)}{2}$, we set both σ_w and σ'_w to $+1$.

If $\frac{1+\tanh\left(\frac{2\beta}{n}(S-2)\right)}{2}<U\leq\frac{1+\tanh\left(\frac{2\beta}{n}S\right)}{2}$, we set σ_w to $+1$ and σ'_w to -1.

If $U>\frac{1+\tanh\left(\frac{2\beta}{n}S\right)}{2}$, we set both to -1.

Thus we get a coupling (X,Y) between $P(\sigma,\cdot)$ and $P(\sigma',\cdot)$, in such a way that we try to minimize the distance $\rho(X,Y)$.

(13)

And indeed, whenever we had $w = v$, we ensured
$$\rho(X,Y) = 0$$
(this happens with proba $\frac{1}{n}$).

When $w \neq v$, most of the time we put the same spin at w, so that $\rho(X,Y) = 1$.

However, we can get $\rho(X,Y) = 2$, with proba

$$\frac{1 + th\left(\frac{2\beta}{n} S\right)}{2} - \frac{1 + th\left(\frac{2\beta}{n}(S-2)\right)}{2} = \frac{th\left(\frac{2\beta}{n} S\right) - th\left(\frac{2\beta}{n}(S-2)\right)}{2}$$

$$\leq \frac{2\beta}{n}$$ using that th is 1-lipschitz.

Therefore,

$$\mathbb{E}\left[\rho(X,Y)\right] \leq \frac{1}{n} \cdot 0 + \left(1 - \frac{1}{n}\right)\left(1 + \frac{2\beta}{n}\right)$$

$$\leq 1 - \frac{1-2\beta}{n}.$$

For $\theta = \frac{1-2\beta}{n}$ $\left(\in (0,1) \text{ when } \beta < \frac{1}{2} \right)$, we get the desired hypothesis. So $d(t) \leq \theta^t \operatorname{diam}(\Omega)$
$$\leq n\theta^t.$$

For $\beta < \frac{1}{2}$

So, $\forall \varepsilon > 0$, $t_{mix}(\varepsilon) \leq -\frac{\ln n}{\ln\left(1 - \frac{2\beta}{n}\right)} + \frac{\ln \varepsilon}{\ln\left(1 - \frac{2\beta}{n}\right)}$

$$= \frac{n}{2\beta}\left(\ln n - \ln \varepsilon\right) + O\left(\log n\right)$$