Ising model - Exam

You may use your handwritten and printed notes. No book is allowed.

9:00 - 12:00

Exercise 1: stochastic domination in FK-percolation

We consider the Gibbs measure of the FK-percolation model on $\mathbb{Z}^d$, for parameters $q \geq 1$ and $p \in (0,1)$, with wired boundary conditions, which we denote by $\Phi^1_{p,q}$. Show that for $\frac{p}{(1-p)q} \leq \frac{p'}{(1-p')q'}$ and $1 \leq q \leq q'$, we have $\Phi^1_{p,q} \leq_{st} \Phi^1_{p',q'}$.

Exercise 2: generalized High Temperature Expansion

Let $G = (V, E)$ be a finite graph. For every **subset** $U \subset V$, we fix a constant $J_U \geq 0$. We consider the measure on $\Omega = \{-1, +1\}^V$:

$$\mu(\sigma) = \frac{1}{Z} \exp\left(\sum_{U \subset V} J_U \sigma_U\right)$$

where, as usual, $\sigma_U = \prod_{v \in U} \sigma_v$, and Z_J is the partition function (which makes this a probability measure).

1. Adapt the High Temperature Expansion to that setting. That is, show that we may write
$$Z = \text{cst} \sum_{\mathcal{C} \subset \mathcal{P}(V) \text{ s.t. } \dots} \text{weight}(\mathcal{C})$$
filling in the gaps and writing down the constant and weight.

2. For $A \subset V$, similarly write down $\langle \sigma_A \rangle$ using the HTE of this model.

3. Show the two Griffith inequalities for this model.

From now on, we consider the case where G is a square region of the square lattice: let $n \geq 1$, we take $V = \{0, \dots, n\}^2$. We also assume that, for some $\beta > 0$, if $U \subset V$ is a unit square[1], then $J_U = \beta$, and otherwise $J_U = 0$.

4. Using question 1, find a simple expression for the partition function $Z_{n,\beta}$ in that case.

5. Deduce the existence and value of the free energy
$$f(\beta) = \lim_{n \to \infty} \frac{1}{n^2} \log(Z_{n,\beta}).$$

6. Give an expression of the quantity $\frac{\partial f}{\partial \beta}$ as an expectation, and compute it.

[1] that is, $U = \{(i,j), (i+1,j), (i+1,j+1), (i,j+1)\}$ for some $0 \leq i, j \leq n-1$

7. Let a, b, c, d be the four corners of G, that is, $a = (0,0), b = (n,0), c = (n,n), d = (0,n)$. Compute $\langle \sigma_a \sigma_b \sigma_c \sigma_d \rangle$.

8. Let u, v be two distinct vertices. Show that $\langle \sigma_u \sigma_v \rangle = 0$.

9. Based on those observations, what do you think of the phase transition of this model?

Exercise 3: Mixing time for the Curie-Weiss model

The aim of this exercise is to find the mixing time for the Glauber dynamics of the Curie-Weiss model. We will use results from the generic theory of mixing times as black boxes, but feel free to think about them (and write about it if you have time).

We consider the complete graph with n vertices, labeled from 0 to $n-1$, with an inverse temperature $\beta > 0$. We consider the Curie-Weiss probability measure:

$$\forall \sigma \in \Omega = \{-1, +1\}^{\{0,\ldots,n-1\}}, \quad \pi(\sigma) = \frac{1}{Z} \exp\left(\frac{\beta}{n} \sum_{\substack{0 \le i \le n-1 \\ 0 \le j \le n-1}} \sigma_i \sigma_j \right)$$

with Z the partition function.

1. Define the Glauber dynamics in that case. Show in particular that the transition matrix is such that for $\sigma \neq \sigma'$,

$$P(\sigma, \sigma') = 0 \text{ or } \frac{1 \pm \tanh\left(\frac{\beta}{n}(\sum_{i=0}^{n-1} \sigma_i \ \pm 1)\right)}{2},$$

saying when each case happens, and quickly justify that π is its (unique) invariant measure.

For the rest of the exercise, let $(X_n)_{n \ge 0}$ be this Markov chain. For two probabilities μ, ν on Ω, we define the *total variation distance*

$$d_{TV}(\mu, \nu) = \max_{A \subset \Omega} |\mu(A) - \nu(A)|.$$

Recall that $P^n(\sigma, \cdot)$ is the distribution of X_n for the deterministic initial measure $X_0 = \sigma$. We are interested in the worst-case distance

$$d(n) = \max_{\sigma \in \Omega} \quad d_{TV}(P^n(\sigma, \cdot), \pi).$$

The *mixing time* of order $\epsilon > 0$ is defined as

$$t_{mix}(\epsilon) = \inf\{n \mid d(n) \le \epsilon\}.$$

A - Low temperature: bottleneck ratio

For $A, B \subset \Omega$, let $Q(A,B)$ be the probability to jump from A to B in one step under π, that is

$$Q(A,B) = \sum_{\sigma \in A, \sigma' \in B} \pi(\sigma) P(\sigma, \sigma').$$

For $S \subset \Omega$, let

$$\Phi(S) = \frac{Q(S, S^c)}{\pi(S)}.$$

A classical result is that if S is such that $\pi(S) \leq \frac{1}{2}$, then

$$t_{mix}\left(\frac{1}{4}\right) \geq \frac{1}{4\ \Phi(S)}.$$

2. For $k \in \{0, \dots, n\}$, let $A_k \subset \Omega$ be the set of spin configurations with exactly k spins equal to $+1$. Write down $\pi(A_k)$.

3. Let $S = \{\sigma \in \Omega \mid \sum_{i=0}^{n-1} \sigma_i < 0\}$. Show that $Q(S, S^c) \leq \pi(A_{\lfloor n/2 \rfloor})$.

4. Using these and the proofs from the course, show that for $\beta > 1$, there exists a constant c (depending on β) such that $\Phi(S) \leq \exp(-cn)$.

5. Find the logical conclusion.

B - High temperature: contraction

For $\sigma, \sigma' \in \Omega$, we define

$$\rho(\sigma, \sigma') = \frac{1}{2} \sum_{i=0}^{n-1} |\sigma_i - \sigma'_i|.$$

Suppose that there exists a constant $0 < \theta < 1$ such that for any σ, σ' **satisfying** $\rho(\sigma, \sigma') = 1$, we can find a coupling (X, Y) of $P^1(\sigma, \cdot)$ and $P^1(\sigma', \cdot)$ such that

$$\mathbb{E}[\rho(X, Y)] \leq \theta.$$

Then a useful result of Bubley and Dyer gives

$$d(n) \leq \theta^n \mathrm{diam}(\Omega)$$

where $\mathrm{diam}(\Omega) = \max_{\sigma, \sigma' \in \Omega} \rho(\sigma, \sigma')$.

6. Apply this strategy to our case, and find a lower bound for $t_{mix}(\epsilon)$ when $\beta < 1$.

To complete the picture, at the critical point $\beta = 1$, the mixing times scales like $n^{3/2}$, but this is beyond the scope of this already elusive exercise.

Exercise 4

Invent a model of statistical mechanics, and say something non-trivial about it.