Peeling random planar maps

(Very) preliminary version

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Gabriel Metsu (Dutch Baroque Era Painter, 1629-1667): Woman Peeling an Apple

Contents

Abstract

This is the draft of the lecture notes for the M2 course given at the University of Geneva in April 2016 and for the "cours Peccot" given at Collège de France in May 2016. The subject of the course is to study the geometry of random planar maps by discovering them step-by-step using the so-called peeling process.

The spatial Markov property of random planar maps is one of the most important properties of these random lattices. Roughly speaking, this property says that, after a region of the map has been explored, the law of the remaining part only depends on the perimeter of the discovered region. The spatial Markov property was first used in the physics literature, without a precise justification: Watabiki [60] introduced the so-called "peeling process", which is a growth process discovering the random lattice step by step and used it to derived the so-called "two-point" function of 2D quantum gravity. A rigorous version of the peeling process and its Markovian properties was given by Angel [4] in the case of the Uniform Infinite Planar Triangulation (UIPT), which had been defined by Angel and Schramm [8] as the local limit of uniformly distributed plane triangulations with a fixed size. The peeling process has been used since to derive information about the metric properties of the UIPT [4], about percolation [4, 6, 53] and simple random walk [9] on the UIPT and its generalizations, and more recently about the conformal structure [33] of random planar maps. It also plays a crucial role in the construction of "hyperbolic" random triangulations [7, 32]. In these lecture notes we review and extend all these results via the new and more universal peeling process recently introduced by Budd [23] which enables us to treat all the Boltzmann map models at once.

The references and proper citations to recent results are gathered at the end of every chapter in a bibliographical paragraph in order to lighten the presentation. May the concerned authors forgive this.

Chapter I : Generalities on planar maps

In this chapter we introduce the basic notions about planar maps: several equivalent definitions, local topology, duality. We also gather a few applications of Euler's formula (Platonic solids, isoperimetric inequalities, Fàry theorem, 5 and 6-colors theorem). Finally we present briefly the theory of circle packings which is a means to represent a (simple) planar map faithfully in the plane.

1.1 Definitions

A planar graph is a locally finite (multi-)graph which can be drawn on the plane (or equivalently on the sphere) in such a way that the edges are non-crossing except at the vertices. Such a drawing is called a proper embedding. Notice that a planar graph may have several topologically different proper embeddings and the definition only tells us the existence of such. In particular, the notion of face of the graph is subject to vary with the embedding.

Definition 1. *A finite planar map is a finite connected planar (multi-)graph properly embedded in the plane (or in the sphere) viewed up to homeomorphisms that preserve the orientation.*

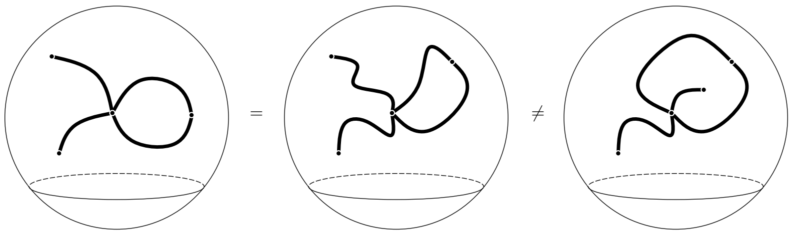


Figure 1.1: The same underlying planar graph can yield different planar maps.

There is an analogous definition of a finite map drawn on the torus or more generally on a compact (orientable say) surface of genus $g \geq 0$, but since we will restrict ourselves to the planar case we sometimes drop the adjective planar and speak of a map instead of a planar map. May the reader forgive this.

If $\mathfrak{m}$ is a planar map, we denote respectively $\mathsf{Edges}(\mathfrak{m}), \mathsf{Vertices}(\mathfrak{m})$ and $\mathsf{Faces}(\mathfrak{m})$ the set of its edges, vertices and faces (the connected components of the complementary of its embedding). Actually, these sets are only defined for one specific embedding of the map, but we make this abuse of notion since the only important thing are the incidence relations between those objects and they do not depend on the precise embedding considered (see also the equivalent definition of a map in terms of oriented graphs below). The degree $\deg(f)$ of a face f (we sometimes also say the perimeter of f) is the number of edges incident to this face with the convention that when an edge is lying completely inside a face it is counted twice in the degree. Similarly the degree $\deg(x)$ of a vertex x is the number of edges adjacent to x, where loops attached to x are counted twice.

According to Definition 1, a finite planar map is thus an equivalence class of embeddings of a finite planar graph. This may seem hard to manipulate at first glance, luckily it admits several equivalent points of view:

- a finite planar map can be seen as a gluing of finitely many polygons (the faces of the map) along their edges so that the manifold produced this way is a topological sphere,
- a finite planar map can also be seen as a finite graph with a system of coherent orientations around each vertex of the graph which correspond to the cyclic ordering of the edges when going clockwise around a vertex in the map.

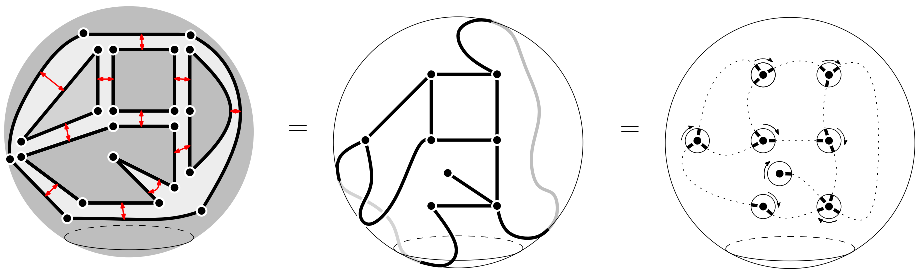


Figure 1.2: The same planar map seen : [left] as a gluing of polygon (notice that two edges of a same polygon could be folded to give a single edge in the map), [center] as an equivalence class of embeddings of a finite planar graph in $\mathbb{S}_2$, [right] as a graph with cyclic orientations around vertices.

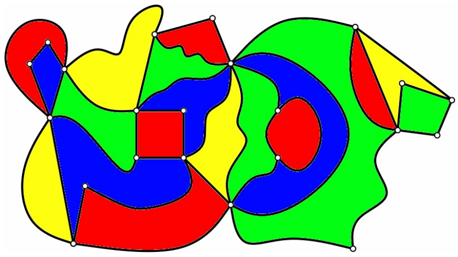
Using any of the the last two definitions it should be clear that the number of planar maps with a given number of edges is finite.

(Planar) maps are more rigid than planar graphs since they are given with an embedding (equivalently a planar orientation) whereas planar graphs only *possess* such an embedding. This rigidity enables us to enumerate planar maps more easily than planar graphs and this is mainly why we will consider maps instead of graphs. For a complete rigidity we will only consider *rooted* maps, that are, maps given with one distinguished oriented edge called the *root edge* which we denote by $\vec{e}$. The origin vertex ρ of $\vec{e}$ is the root vertex (also called origin) and the face incident on the right of $\vec{e}$ is the root face f_r of the map. Once rooted, maps have no non-trivial symmetry. From now on, all the maps considered are rooted. We denote by

$$\mathcal{M} := \text{ the set of all (rooted) finite planar maps},$$

in the following a generic planar map will be denoted by $\mathfrak{m} \in \mathcal{M}$. For technical reason we shall also consider that there exists a unique "vertex map" denoted by $\dagger$ which is made of a unique vertex and no edge nor face. A *simple* map is a map in which multiple edges or loops are forbidden.

A famous theorem on planar maps which looks childish is the 4-colors theorem which proves that 4 colors suffice to color any planar map such that any pair of adjacent faces (i.e. sharing an edge) have different colors. The proof is extremely difficult and requires the help of a computer to check numerous cases, but a version with 6 or even 5 colors is much easier to do (see Exercise 5).



1.2 Local topology and infinite maps

If $\mathfrak{m}$ is a map and $r \in \{0, 1, 2, 3, \dots\}$ we denote by $[\mathfrak{m}]_r$ and call the ball of radius r in $\mathfrak{m}$, the map formed by all the vertices of $\mathfrak{m}$ which are at graph distance less than r from the origin ρ of $\mathfrak{m}$ together with the edges linking them. The map $[\mathfrak{m}]_r$ inherits the planar orientation from $\mathfrak{m}$ and is indeed a planar map rooted at the root edge of $\mathfrak{m}$ as soon as $r \geq 1$. When $r = 0$ we put $[\mathfrak{m}]_0 = \dagger$ the "vertex-map".

Definition 2. *We put a distance (check it!) on the set $\mathcal{M}$ of all finite maps by the formula*

$$\mathrm{d_{loc}}(\mathfrak{m}, \mathfrak{m}') \quad = \quad \Big(1 + \sup\{r \geq 0 : [\mathfrak{m}]_r = [\mathfrak{m}']_r\}\Big)^{-1}. \tag{1.1}$$

Note that the space $(\mathcal{M}, \mathrm{d_{loc}})$ is not complete. The elements in $\overline{\mathcal{M}} \backslash \mathcal{M}$ correspond to infinite maps (in this setup they can be seen as a coherent system of balls of radius r for $r \geq 0$). Once this completion has been done, we have

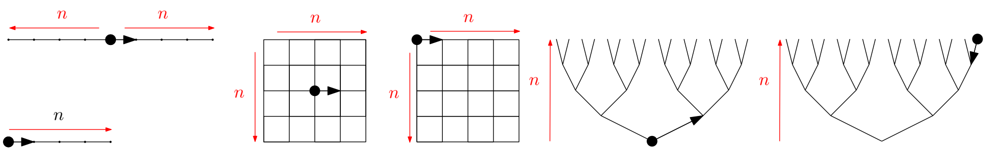
Proposition 1. *The space $(\overline{\mathcal{M}}, \mathrm{d_{loc}})$ is Polish (metric, separable and complete). Furthermore, a subset $A \subset \overline{\mathcal{M}}$ is relatively compact (its closure is compact) if and only if for any $r \geq 0$*

$$\#\{[\mathfrak{m}]_r : \mathfrak{m} \in A\} < \infty \qquad \text{or equivalently} \qquad \sup\{\deg(x) : x \in \mathsf{Vertices}([\mathfrak{m}]_r), \mathfrak{m} \in A\} < \infty.$$

Proof. It is easy to see that $\mathrm{d_{loc}}$ is a distance. The separation is granted since $\mathcal{M}$ is dense in $\overline{\mathcal{M}}$ and countable. **Completeness.** If $(\mathfrak{m}_n)$ is a Cauchy sequence for $\mathrm{d_{loc}}$ then for every r, the ball $[\mathfrak{m}_n]_r$ stabilizes to a certain map $\mathfrak{m}_r^*$. By coherence we have $[\mathfrak{m}_{r'}^*]_r = \mathfrak{m}_r^*$ for any $r' \geq r$ and so we can define a unique possibly infinite map $\mathfrak{m}_\infty \in \mathcal{E}$ such that $\mathfrak{m}_r^* = [\mathfrak{m}_\infty]_r$. It is then clear that

$\mathfrak{m}_n \to \mathfrak{m}_\infty$ for $\mathrm{d_{loc}}$. **Characterization of the compacts.** The condition in the theorem is clearly necessary for A to be relatively compact for otherwise there exists $r_0 \geq 0$ and a sequence $(\mathfrak{m}_n)$ in A whose balls of radius r_0 are all at distance $\varepsilon = \frac{1}{1+r_0}$ from each other. Such a sequence cannot admit a convergent subsequence. Conversely, a subset A satisfying the condition of the theorem is easily seen to be pre-compact for $\mathrm{d_{loc}}$: just cover with balls of radius $1/r$ centered on each element of $\{[\mathfrak{m}]_r : \mathfrak{m} \in A\}$. We leave the equivalent condition to the reader. □

Exercise 1. Compute the limit of the following 6 sequences of planar maps.



For the above definition, it is easy to check that if $(\mathbb{P}_n)_{n\geq 0}$ and $\mathbb{P}_\infty$ are probability measures on $\overline{\mathcal{M}}$ then $\mathbb{P}_n \to \mathbb{P}_\infty$ in distribution for the local distance if and only if for any $r \geq 0$ and any fixed planar map $\mathfrak{m}_0$ we have

$$\mathbb{P}_n([\mathfrak{m}]_r = \mathfrak{m}_0) \xrightarrow[n\to\infty]{} \mathbb{P}_\infty([\mathfrak{m}]_r = \mathfrak{m}_0).$$

In other words, convergence in distribution for the local distance is equivalent to convergence in distribution of the ball of radius r, for any $r \geq 0$. Beware though, the convergence of the probabilities $\mathbb{P}_n([\mathfrak{m}]_{r_0} = \mathfrak{m}_0)$ is not sufficient to imply convergence in distribution because tightness is missing (these probabilities could all converge to 0 for example).

Remark 1. An infinite map $\mathfrak{m}$ may have faces of infinite degrees. For example, the line graph of length $2n$ rooted in the middle converges locally towards the map made of an infinite line separating two faces of infinite degrees.

1.2.1 Infinite maps of the plane and the half-plane

The above definition of infinite maps (as coherent sequences of balls) is equivalent to infinite locally finite graphs given with a system of coherent (i.e. giving rise to a planar structure) cyclic orientations of the edges around each vertex. However, this is not equivalent to Definition 1 for infinite planar maps: the same infinite map may be drawn in two ways on the sphere so that it is impossible to map the first on the second via a homeomorphism (see Fig. 1.3 below).

For one-ended maps however, things are easier. Let us recall the definition of end in a graph:

Definition 3. *Let* g *be a graph and* $\mathrm{k}_1 \subset \mathrm{k}_2 \subset \cdots$ *an increasing sequence of finite subgraphs of* g *which exhausts* g*, that is* $\bigcup_{i\geq 0} \mathrm{k}_i = \mathrm{g}$*. An* end *of* g *is a nested sequence* $\cdots \subset U_3 \subset U_2 \subset U_1$ *where* U_i *is an infinite connected component of* $\mathrm{g}\backslash\mathrm{k}_i$*. A priori, the number of ends may depend on the sequence* $(\mathrm{k}_i)_{i\geq 1}$ *however it is an exercise to see that it does not.*

Exercise 2. Show that: any finite graph has 0 end, any infinite graph has at least one end, $\mathbb{Z}$ has two ends, $\mathbb{Z}^d$ for $d \geq 2$ has one end and that the complete k-ary tree with $k \geq 3$ has uncountably many ends.

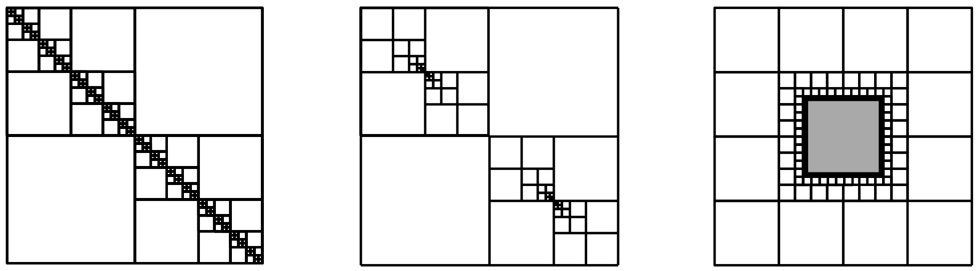


Figure 1.3: Three examples of infinite maps, the left-most one has infinitely many ends, the center one has two ends whereas the right-most one has only one end (the centered region in gray is not a face). In particular, the right-most map can be drawn on the sphere after shrinking the gray region (and the latter can not be obtained by a homeomorphism of the first drawing).

There is also a natural way to put a topology on the space of ends of a graph to turn it into a topological space (see wikipedia "ends of graphs" for example). In particular, the space of ends of the full k-ary tree is homeomorphic to a Cantor set, i.e. a closed set with no interior and no isolated points. Coming back to maps (the number of ends of a map is the number of ends of the graph it induces) then

Proposition 2. *Infinite planar maps with one end can be seen as equivalence classes (for orientation preserving homeomorphisms of the plane) of proper embeddings of infinite planar graphs on the plane $\mathbb{R}^2$ such that every compact of $\mathbb{R}^2$ intersects only finitely many edges of the embedding.*

Proof. It should be clear that an (equivalence class of) embedding such as the one above defines an infinite planar map with only one end since any finite set in the map is contained in a compact set of the embedding and its complement contains at most one unbounded region. Reciprocally, if $\mathfrak{m}$ is an infinite planar map with one end then for every r recall that $[\mathfrak{m}]_r$ is the ball of radius r and write $\overline{[\mathfrak{m}]}_r$ for the hull of the ball of radius r obtained by filling-in all the finite components of $\mathfrak{m}\backslash[\mathfrak{m}]_r$. Denote the vertices which are at distance r from the origin of $\mathfrak{m}$ and located on the boundary of $\overline{[\mathfrak{m}]}_r$ by $\partial\overline{[\mathfrak{m}]}_r$. We then claim that it is possible to draw $\mathfrak{m}$ on $\mathbb{R}^2$ in such a way that the vertices of $\partial\overline{[\mathfrak{m}]}_r$ are drawn on the circle of center $(0,0)$ and radius r, and that all the edges and vertices of $\overline{[\mathfrak{m}]}_r$ are inside that circle. Such an embedding is indeed of the required form. □

Using the last proposition it is legitimate to call infinite planar maps with one end *infinite planar maps of the plane* and we almost do so after splitting this group into two further subclasses: An infinite map with one end can have 0 or 1 face of infinite degree. If it has one such face it can be drawn on $\mathbb{R}\times\mathbb{R}_+$ by saying the face of infinite degree contains the half-plane $\mathbb{R}\times\mathbb{R}_-$.

Definition 4 (Maps of the plane and half-plane)**.** *A map of* the plane *is a rooted infinite planar map with only one end such that all faces are of finite degree. A map of the* half-plane *is a rooted infinite planar map with one end such that the root face is of infinite degree.*

1.3 Euler's formula and applications

The first non-trivial result about planar maps is the famous Euler relation which links the number of faces, of edges and of vertices of any finite planar map.

Theorem 1 (*Euler*)

For any finite planar map $\mathfrak{m}$ we have

$$\#\mathsf{Vertices}(\mathfrak{m}) + \#\mathsf{Faces}(\mathfrak{m}) - \#\mathsf{Edges}(\mathfrak{m}) \quad = \quad 2. \tag{1.2}$$

Proof. The proof is done by induction on the number of edges. The only map with 0 edge has 1 vertex and 1 face so that (1.2) is true. Suppose now that $\#\mathsf{Edges}(\mathfrak{m}) \geq 1$ and erase an arbitrary edge of $\mathfrak{m}$, then two cases may happen:

- either the new map $\mathfrak{m}'$ is still connected and so applying the induction hypothesis we have $\#\mathsf{Vertices}(\mathfrak{m}') + \#\mathsf{Faces}(\mathfrak{m}') - \#\mathsf{Edges}(\mathfrak{m}') = 2$. Also we have $\#\mathsf{Vertices}(\mathfrak{m}) = \#\mathsf{Vertices}(\mathfrak{m}')$ and $\#\mathsf{Edges}(\mathfrak{m}) = \#\mathsf{Edges}(\mathfrak{m}') + 1$ and a careful inspection shows that $\#\mathsf{Faces}(\mathfrak{m}) = \#\mathsf{Faces}(\mathfrak{m}') + 1$. Gathering-up the pieces we find that $\mathfrak{m}$ obeys (1.2).

- or the removal of the edge breaks $\mathfrak{m}$ into two connected maps $\mathfrak{m}_g$ and $\mathfrak{m}_y$. Applying (1.2) to each block we find that $\#\mathsf{Vertices}(\mathfrak{m}_1) + \#\mathsf{Faces}(\mathfrak{m}_1) - \#\mathsf{Edges}(\mathfrak{m}_1) = 2$ as well as $\#\mathsf{Vertices}(\mathfrak{m}_2) + \#\mathsf{Faces}(\mathfrak{m}_2) - \#\mathsf{Edges}(\mathfrak{m}_2) = 2$. Also, we have $\#\mathsf{Vertices}(\mathfrak{m}) = \#\mathsf{Vertices}(\mathfrak{m}_1) + \#\mathsf{Vertices}(\mathfrak{m}_2)$ and $\#\mathsf{Edges}(\mathfrak{m}) = \#\mathsf{Edges}(\mathfrak{m}_1) + \#\mathsf{Edges}(\mathfrak{m}_2) + 1$ and another careful inspection shows that $\#\mathsf{Faces}(\mathfrak{m}) = \#\mathsf{Faces}(\mathfrak{m}_1) + \#\mathsf{Faces}(\mathfrak{m}_2) - 1$, the minus 1 terms stems from the fact that the external face of $\mathfrak{m}_1$ and $\mathfrak{m}_2$ is counted twice otherwise. Putting everything together we indeed verify (1.2).

□

Remark 2. As we already noticed, the notion of face is not well-defined for planar graphs as it may depend on its planar embedding. However, we see from Euler's formula that the number of faces does not depend on the embedding but only on the underlying graph structure.

Exercise 3. Using Euler's formula show that the complete graph K_5 on 5 vertices (with an edge between any pair of vertices) and the graph $K_{3,3}$ made of 3 black vertices and 3 white vertices such that there is an edge between any pair of black and white vertices are not planar graphs.

Remark 3. The converse of the above exercise is also true: By the well-known Kuratowski theorem, a graph is planar if it does not contain K_5 or $K_{3,3}$ as a minor. However, in this course, the planar maps come with their embeddings, and so planarity testing is never an issue.

Euler's formula is particularly useful when we deal with special classes of planar maps:

Definition 5. *Fix $k \in \{3, 4, 5, \dots\}$. A k-angulation is a planar map $\mathfrak{m}$ whose faces all have degree k.*

In the following we will use a lot *triangulations* in the case $k = 3$ and *quadrangulations* in the case $k = 4$. Beware, since we allow multiple edges and loops, a triangle (or a quadrangle) can be folded on itself and look weird at first glance, see Fig. 1.4. The dual (see below) of a k-angulation, that is a planar map where all the vertices have degree k is called a k-valent map.

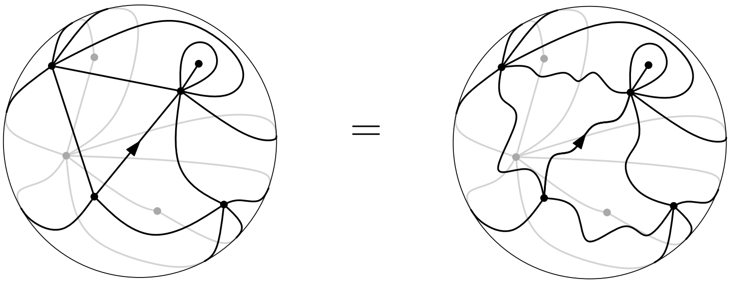


Figure 1.4: A finite triangulation of the sphere. Notice the triangle which is folded on itself and looks like a loop with an inner edge: this is indeed a triangle!

In particular, in a finite k-angulation $\mathfrak{m}$, we have $k \cdot \#\mathsf{Faces}(\mathfrak{m}) = 2 \cdot \#\mathsf{Edges}(\mathfrak{m})$, because each edges is counted by two faces (or twice by the same face). This combined with Euler's formula gives an affine relation between the number of vertices, edges and faces of $\mathfrak{m}$ (depending on k, there are congruence constraints).

Definition 6. *A planar map is bipartite if one can color its vertices in two colors (black and white say) so that two neighbor vertices do not share the same color.*

We introduce the above definition here because it is easy to see that a planar map is bipartite if and only if all its faces have even degree (exercise). For example, quadrangulations are bipartite. As we will see later on, bipartite planar maps are in a sense more regular than general maps and their enumeration formulas are nicer.

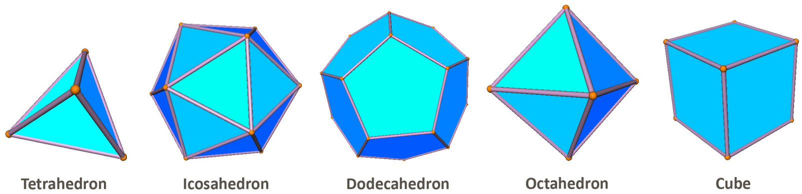
1.3.1 Platonic solids

A well-known application of Euler's formula is the classification of all regular polyhedrons or Platonic solids. Indeed, a regular polyhedron can be seen as a finite map such that the degrees of the vertices and faces are constant. If $\alpha \geq 3$ and $\gamma \geq 3$ denote respectively the common degree of the vertices and faces of the map $\mathfrak{m}$ with v vertices f faces and e edges then we have

$$\alpha v = 2e, \qquad \gamma f = 2e, \qquad \text{and} \qquad v + f - e = 2.$$

It is easy to see that there are only 5 solutions to these equations giving rise to 5 regular polyhedrons described below. Notice the symmetry between the number of vertices and faces which play the same role once exchanged. This is explained below by the duality operation.

Name	α	γ	e
Tetrahedron	3	3	6
Cube	3	4	12
Octahedron	4	3	12
Dodecahedron	3	5	30
Icosahedron	5	3	30



Exercise 4. Show that the above Platonic solids do exist that is, can be constructed in three dimensions by gluing identical flat regular polygons together.

1.3.2 Fàry theorem

By definition, planar maps can always be drawn on the plane in a proper way. One can wonder whether it is possible to do this with straight lines. Obvious obstructions are multiple edges or loops, but once these have been forbidden the answer is yes!

Theorem 2 (***Fàry 1948***)

Any simple (without multiple edges nor loops) finite planar map can be properly drawn in the plane with straight edges.

Proof. The proof is done by induction on the number of vertices of the map $\mathfrak{m}$. If $\mathfrak{m}$ has only one vertex then (since loops are forbidden) it corresponds to the vertex map which can be drawn on the plane with no line (in particular straight). If $\mathfrak{m}$ has more than 2 vertices and is simple then we use the following lemma:

Lemma 3. *Any finite simple planar map* $\mathfrak{m}$ *possesses a vertex of degree less than or equal to* 5.

Proof of the lemma. By Euler's formula we have $v + f - e = 2$ with obvious notations, whereas the edge count gives

$$2e = f_1 + 2f_2 + 3f_3 + \cdots,$$

where f_i is the number of faces of degree i in the map $\mathfrak{m}$. Since $\mathfrak{m}$ is simple, there are no faces of degree 1 or 2 and it follows that $2e \geq 3f$. Combining with Euler's formula we get that $3v - e \geq 6$ or equivalently $\frac{2e}{v} \leq 6 - \frac{12}{v}$. Since $\frac{2e}{v}$ represents the mean degree of a vertex in the map $\mathfrak{m}$, the last inequality implies the existence of a vertex of degree less than 5. □

Coming back to the proof of the theorem, we take a vertex v of degree less than 5 in the map $\mathfrak{m}$. We can suppose, wlog, that $\mathfrak{m}$ is a (simple) triangulation since adding edges makes the drawing with straight lines even more difficult. Now erase v from the map (as well as its incident edges). By the induction hypothesis, the rest of the map can be drawn on the plane with straight lines. Consider now the vertices to which the vertices v should have been linked. They form a polygonal face of degree less than 5. By the art gallery theorem (see wikipedia) we can place back v inside this face in such a way that it can be linked by straight lines to all its neighbor vertices. □

Exercise 5. Deduce from Lemma 3 that any planar map can be properly colored with 6 colors. Harder: prove that 5 colors actually suffices.

1.3.3 Duality

If $\mathfrak{m}$ is a finite or infinite planar map such that all the faces of $\mathfrak{m}$ are of finite degree, one can define the dual map $\mathfrak{m}^\dagger$ obtained informally speaking by placing inside each face of $\mathfrak{m}$ a vertex of $\mathfrak{m}^\dagger$ and linking two vertices of $\mathfrak{m}^\dagger$ by an edge if the corresponding faces in $\mathfrak{m}$ share an edge. The root edge of the dual map is dual to the root edge of the primal map and crosses it from left to right. The duality mapping is clearly an involution on the set of all planar maps with finite face degrees and exchanges the roles of vertices and faces.

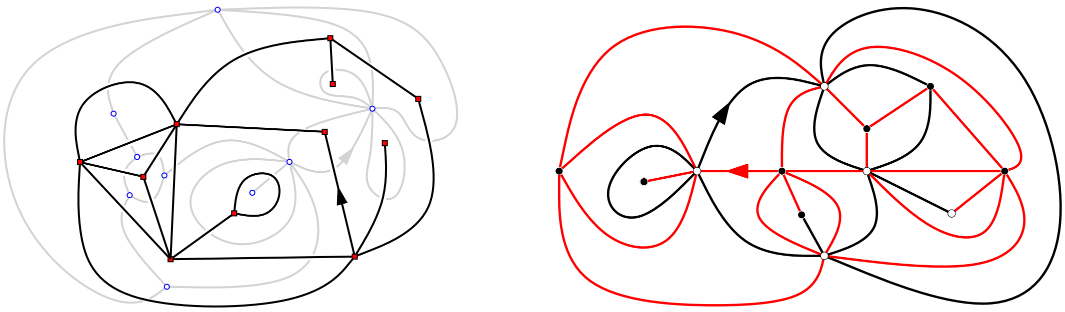


Figure 1.5: Duality between planar maps (left) and between quadrangulations and planar maps (right).

There is also another bijection between, on the one hand, the set of all quadrangulations with n faces, and on the other hand, the set of all planar maps with n edges. The one-to-one correspondence is given as follows: If $\mathfrak{m}$ is a planar map with n edges, then in each face of $\mathfrak{m}$ we put an extra point that we link to all (corners of) the vertices adjacent to this face, see Fig. 1.5. We then erase all the edges of $\mathfrak{m}$ and are left with a quadrangulation $\mathfrak{q}$ with n faces (which is clearly bipartite!). The root edge is transferred from $\mathfrak{m}$ to $\mathfrak{q}$ as depicted on Fig. 1.5.

As a consequence of the last bijection, the number of planar maps with n edges and m vertices is the same as that of planar maps with n edges and m faces. Also, the number of planar maps with n edges is the same as the number of quadrangulations with n faces which will turn out to be relatively simple, see next chapter.

1.4 Curvature and isoperimetric inequalities

In this section we consider triangulations only. Let us start with a warmup. If $\mathfrak{t}$ is a finite triangulation with $v > 0$ vertices, f faces and e edges then combining Euler's formula with the relation $3f = 2e$ we get

$$3v = 6 + e \qquad \Longleftrightarrow \qquad \frac{2e}{v} = 6 - \frac{12}{v}.$$

The quantity $2e/v$ represents the mean degree of the triangulation and we see that it goes to 6 as $v \to \infty$. In conformal geometry it represents the "average curvature": if the mean degree is equal to 6 the surface is flat, if it is larger than or equal to 7 the surface is negatively curved and if it is smaller than 6, it is positively curved. Everybody knows the standard 6-regular triangulation, which is flat, known as the honey-comb lattice. However, it is easy to see that there exist infinite triangulations whose vertex degrees are bounded from below by 7 say (e.g. the 7-regular triangulation) but they grow very rapidly. This can be encoded in the so-called isoperimetric profile:

Theorem 3 (*Degrees and isoperimetric profile*)

Let $\mathfrak{t}$ be a triangulation with a boundary of length p, that is a planar map whose faces are all triangles except for one face, called the external face which is of degree p. We denote by n the number of inner vertices of $\mathfrak{t}$

- *If all the inner vertex degrees are larger than or equal to 7 then for some $c > 0$*

$$p \geq c \cdot n.$$

- *If all the inner vertex degrees are larger than or equal to 6 then*

$$p \geq \sqrt{12n}.$$

Proof. Let $\mathfrak{t}$ be a triangulation with n inner vertices and whose minimal inner vertex degree is $d \in \{6, 7\}$. We may choose $\mathfrak{t}$ so that p is the smallest possible. Notice that this forces the boundary $\partial\mathfrak{t}$ to be a simple cycle since otherwise if there are pinch points by a simple surgical operation we can glue two edges and diminish the perimeter while keeping the number of inner vertices and their degrees unchanged. We write f for the number of faces of $\mathfrak{t}$ and e its the number of edges. Counting the edges from the face point of view gives $2e = 3(f-1) + p$ and Euler's formula writes $(n+p) + f - e = 2$ which once combined give

$$3n + 2p = 3 + e. \tag{1.3}$$

On the other hand counting edges from the vertex point of view yields

$$2e = \sum_{u \in \mathrm{Vertices}(\mathfrak{t})} \deg(u). \tag{1.4}$$

From the above display we deduce that $2e \geq dn$ where d is the minimal inner vertex degree and combining this with (1.3) already yields the first point of the theorem.

In the case $d = 6$ we must do better and we adapt here the proof of [5]. We introduce Σ the edges incident to both a vertex on the boundary $\partial\mathfrak{t}$ of $\mathfrak{t}$ and an inner vertex of $\mathfrak{t}$ and Δ the edges linking two vertices of $\partial\mathfrak{t}$. Coming back to (1.4) more carefully we get $2e = 6n + 2p + \#\Sigma + 2\#\Delta$ which together with with (1.3) yields

$$\#\Sigma + 2\#\Delta \leq 2p - 6. \tag{1.5}$$

We already deduce that $p \geq 4$ unless $\mathfrak{t}$ is made of a single triangle. We now assume that the triangulation has been chosen so that the ratio $c = p^2/n$ is the smallest possible among all triangulations with boundary so that $n \leq N$ where N is fixed (if there are several choices, pick on with minimal p). Let us examine a bit more the structure of such a minimal triangulation. Recall that the boundary is necessarily simple, and let us now rule-out the possibility of a non-boundary edge linking two vertices of $\partial\mathfrak{t}$. Indeed, if there was such an edge it would split the map into two triangulations with boundary of perimeter $p_1 + 1$ and $p_2 + 1$ with n_1 and n_2 inner vertices respectively such that $n_1 + n_2 = n$ and $p_1 + p_2 = p$.

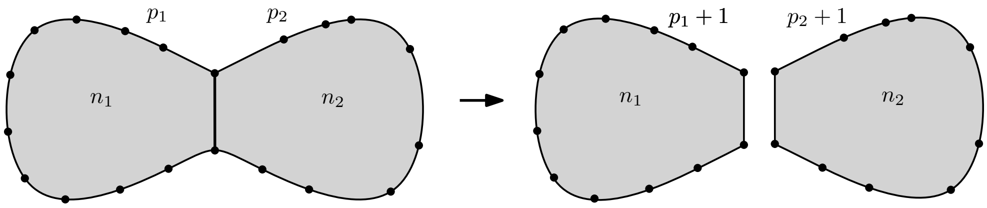


Figure 1.6: One cannot split $\mathfrak{t}$ into two parts by the minimality assumption.

Notice then that necessarily $p > p_1 \geq 2$ and $p > p_2 \geq 2$ and $(p_1 - 1)(p_1 - 1) > 1$ otherwise $\mathfrak{t}$ is made of two triangles glued together and n would be equal to 0. By our minimality assumption we must have $(p_1 + 1)^2 \geq cn_1 + 1$ and $(p_2 + 1)^2 \geq cn_2 + 1$. Using the fact that $(p_1 - 1)(p_2 - 1) > 1$ it follows that

$$(p_1 + p_2)^2 > (p_1 + 1)^2 + (p_2 + 1)^2 - 2 \geq c(n_1 + n_2),$$

which is absurd. Hence we have with the above notation $\Delta = \varnothing$. Now if we consider the set of all inner vertices adjacent to the boundary of $\mathfrak{t}$, by our deduction on $\mathfrak{t}$ they form a connected subset which encloses a triangulation $\mathfrak{t}'$ with a boundary of perimeter p'. It is easy to see on the figure that $\#\Sigma = p + p'$ and so using (1.5) we have that $p \geq p' + 6$. Using our minimality

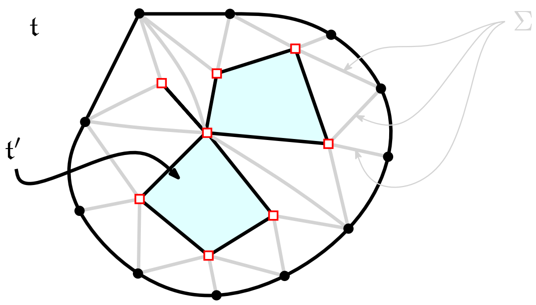


Figure 1.7: Induction hypothesis: passing from $\mathfrak{t}$ to $\mathfrak{t}'$.

assumption again we deduce that $(p')^2 \geq cv' + 1$ where v' is the number of inner vertices of $\mathfrak{t}'$. Since we obviously have $v \leq v' + p'$ (with equality if $\mathfrak{t}'$ has a simple boundary) we deduce that

$$c(v' + p') \geq cv = p^2 \geq (p' + 6)^2 \geq (p')^2 + 12p' \geq cv' + 12p',$$

and this can only work if $c \geq 12$. As N was arbitrary, this proves the second statement of the theorem. □

Remark 4. The isoperimetric constant $\sqrt{12}$ is achieved in the case of balls of large radius in the standard infinite 6-regular triangulation.

Exercise 6. Show that there is no infinite triangulation of the plane whose vertex degrees are bounded by 5. Is there an infinite triangulation of the plane with degrees only in $\{5, 6\}$ apart the six-regular triangulation?

1.5 Circle packings

A planar map does not a priori have any canonical representation in the plane (or the sphere) since even in the finite case, it is given as an equivalence class of embeddings. Still one can ask if we can make sense of a "faithful" representation of a map.

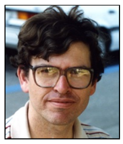
As in Section 1.3.2 we focus on the case of simple maps where multiple edges and loops have been forbidden (since in any representation with straight lines, the latter are squashed). We say that a simple map $\mathfrak{m}$ is represented by a circle packing if there is a collection $(\mathcal{C}_v : v \in \mathsf{Vertices}(\mathfrak{m}))$ of non overlapping disks in the plane $\mathbb{R}^2$ such that $\mathcal{C}_v$ is tangent to $\mathcal{C}_u$ if and only if u and v are neighbors in $\mathfrak{m}$. Recall that the completed plane $\hat{\mathbb{C}} = \mathbb{R}^2 \cup \{\infty\}$ can be identified with the Riemann sphere $\mathbb{S}^2$ by the stereographic projection from the north pole. This projection transforms circles and lines in $\hat{\mathbb{C}}$ into circles on the Riemann sphere. Recall also that the Möbius group

$$\left\{ z \in \hat{\mathbb{C}} \mapsto \frac{az+b}{cz+d} \right\}$$

acts triply transitively on the Riemann sphere (i.e. we can map any triplet of points to any other triplet of points) and preserves circles.

Theorem 4 (*Finite circle packing theorem. Koebe, Andreev–Thurston*)

Any finite simple map $\mathfrak{m}$ admits a circle packing representation on the Riemann sphere. Furthermore if $\mathfrak{m}$ is a simple triangulation then the circle packing is unique up to Möbius transformations.



Remark 5. Fàry's theorem is a trivial corollary of the above theorem!

Sketch of the proof. First, it is easy to see that it suffices to prove the theorem for simple triangulations because we can embed any simple planar map inside a simple triangulation by further triangulating inside each face. Fix a triangulation $\mathfrak{t}$ and pick a face $f \in \mathsf{Faces}(\mathfrak{t})$ that we will see as the exterior face. We will prove that we can construct a circle packing of $\mathfrak{t}$ such that the three circles corresponding to this outer face are three mutually tangent circles of radius 1, or equivalently that the three vertices of the triangles form an equilateral triangle. The rest of the circles are in-between these three circles. We start with the uniqueness statement.

Uniqueness. Since the Möbius group of the Riemann sphere acts triply transitively we can

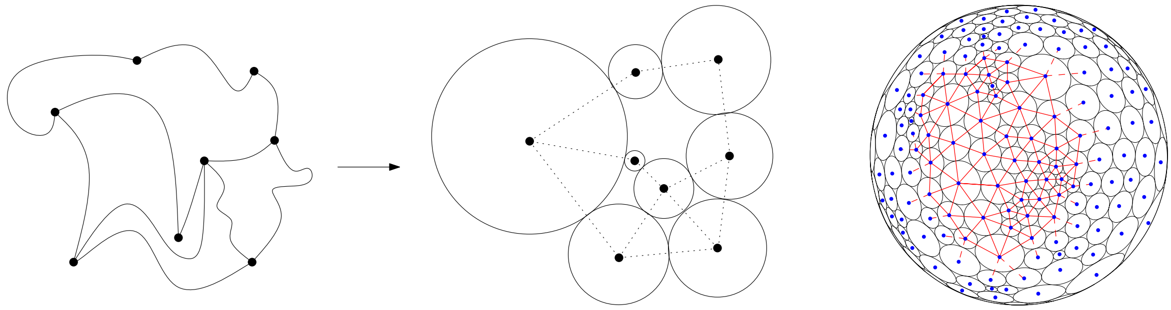


Figure 1.8: On the left, a finite circle packing of a planar map. On the right a circle packing of a triangulation seen on the sphere $\mathbb{S}_2$.

transform any circle packing into a packing of the above form (with the marked face forming an equilateral triangle and the rest of the vertices inside). Imagine that we are given two packings $\mathcal{P}$ and $\mathcal{P}'$ of the above form, in particular the three exterior circles are of radius 1. We then choose an interior vertex v of the triangulation such that the ratio of the corresponding circles in the packing is maximal i.e.

$$\lambda(v) = \frac{r_{\mathcal{P}}(v)}{r_{\mathcal{P}'}(v)} \quad \text{is maximal.}$$

We then examine the structure of the packing around this circle in $\mathcal{P}$. By dividing all the distances by $\lambda(v)$ we end up with a circle of radius $\mathrm{r}_{\mathcal{P}'}(v)$ and such that all the neighboring circles have a radius which is less than the corresponding radius in $\mathcal{P}'$. By an obvious monotonicity property of the angles around a circle we deduce that these new radii must coincide with those in $\mathcal{P}'$ i.e.

$$\lambda(u) = \frac{r_{\mathcal{P}}(u)}{r_{\mathcal{P}'}(u)} = \lambda(v),$$

for all u neighbors of v (for otherwise if the inequality were strict, the neighboring circles would not surround the circle associated to v). Since the graph is connected we deduce step by step that $\lambda(\cdot)$ is constant and must be equal to 1 by the assumption on the exterior circles. Hence $\mathcal{P} = \mathcal{P}'$.

Existence. We will not prove the existence but just describe the algorithm that can be used (even in practice!) to construct the packing. The idea is to first find all the radii of the circles. Once these radii are found, one can reconstruct the packing step by step by starting from the external face and deploying the circles one by one around the circles already explored (notice that given the radii and the combinatorial layout we can determine the angles, and for this we crucially use the fact that the underlying map is a triangulation). To find the radii we start with an arbitrary assignment of radii to the vertices of the triangulations except the three vertices of the marked face which have their radii fixed for ever to 1. We then examine all the internal vertices in a cyclic order and repeat forever the following adjustment : see Fig. 1.9.

Repeatedly applying this updating rule, it can be proved (but it is not trivial) that this algorithm indeed converges towards the unique fixed point for the right values of the radii for

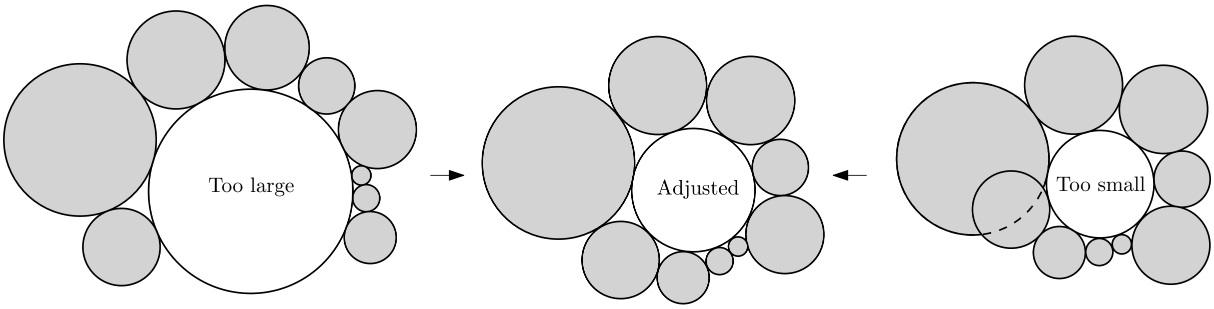


Figure 1.9: Adjustment rule : For an internal vertex v with radius r_v, we examine the radii of the neighbors of v. Using these radii, one can see whether or not placing the circles of the corresponding radii around a circle of radius r_v would close exactly. For most of the time it will not. But by a simple monotonicity property, one can always update the radius r_v so that the latter property holds true.

the circles (with the outer three circles normalized) and that these values give rise to a non-degenerate (all the radii are positive) circle packing for the triangulation $\mathfrak{t}$. This can be shown by understanding the monotonicity properties of the "flow of angles" on the graph when looping this rule, see [30]. □

1.5.1 Applications

We will use the theory of circles packings in Chapter 8 when studying random walks on random planar maps. In the mean time, although we will not need it, we state some applications of the theory of circle packings without giving the proofs. First they lead to a beautiful proof [55] of the following well-known theorem:

Theorem 5 (*Lipton-Tarjan [49]*)

If $\mathfrak{m}$ is a planar map with n vertices, then there is a simple closed path made of less than $\sqrt{8n}$ vertices which separates the maps into two components containing less than $2n/3$ vertices each.

Second, there is a wonderful link between circle packings and the Riemann mapping theorem (which was the initial motivation for Thurston to study circle packings). Imagine that we have a circle packing of a region Ω, with the hexagonal packing say, and that the same combinatorial triangulation structure is circle-packed in the disk (we can always do so by the finite circle packing theorem). We suppose also that the center $z \in \Omega$ is mapped to $0 \in \mathbb{D}$. Then as the maximal radius of the circles goes to 0, the mapping induced by the circle packings approximates a conformal bijection $\Omega \to \mathbb{D}$ such that z is mapped to 0.

Bibliographical notes. Most of this chapter can be found in textbooks on planar graphs/maps. The local distance has been introduced in the context of random planar maps by Benjamini &

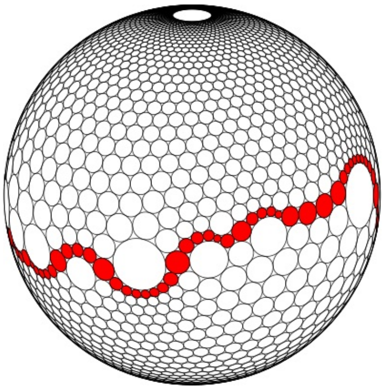


Figure 1.10: A small balanced separator of a planar graph represented by a circle packing on the sphere $\mathbb{S}_2$. Image of Kenneth Stephenson

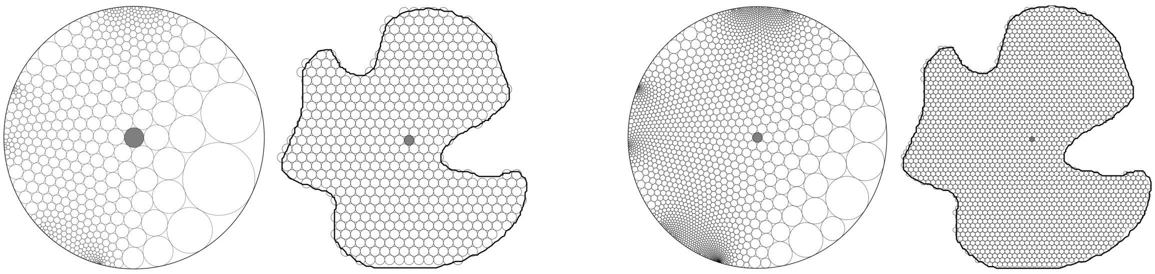


Figure 1.11: Thurston conjecture (Rodin–Sullivan/Schramm theorem) : Circle packings can be used to approximate conformal mappings. Images of Kenneth Stephenson.

Schramm [11]. Part II of Theorem 3 is due to Angel, Benjamini and Horesh [5]. We refer to the wonderful book [58] for many more applications of the beautiful theory of circle packings. The proof of uniqueness in the circle packing theorem via maximum principle is taken from wikipedia and is due to O. Schramm.

Chapter II : Tutte's equation

Our goal now is to enumerate planar maps, quadrangulations say. The basic idea, which goes back to Tutte, is to find a recurrence relation between quadrangulations of different sizes. The natural idea is to erase the root edge to diminish the size of the map. Unfortunately when doing so the remaining map is generally not a quadrangulation anymore. The key idea is then to generalize the model and consider quadrangulation with a boundary which are now stable under erasing the root edge. We start by solving Tutte's equation explicitly in the case of quadrangulations because this yields to remarkably close formulas. Then, in order to present the results in a unified form, we will discuss a general model called Boltzmann planar maps which includes $2k$-angulations as specific cases.

Tutte's equation is very important not only because it leads to exact and asymptotic enumeration of planar maps but also because it is the true spirit of the peeling process. In fact, as we will see later on, the peeling process is *in fine* just a probabilistic way to re-interpret Tutte's equation.

2.1 Maps with a boundary.

Recall that if $\mathfrak{m}$ is planar map, the face incident on the right of the root edge is called the root face or sometimes the external face. This enables us to see a planar map as a map with a boundary: the boundary is just the contour of the root face, the degree of the root face is in this case also called the perimeter of the map. Notice that this boundary usually contains pinch points, if it does not, the boundary is said to be simple. For $k \geq 3$, a k-angulation with a boundary is a planar map whose faces are all of degree k except the root face which can be of arbitrary degree (subject to parity constraints).

Remark 6 (Maps with external face of degree 2)*.* It is easy to see that there is a bijection between the set of all planar maps (different from the vertex map $\dagger$) and maps with a boundary of perimeter 2: just see the root edge as a zipper that we zip or unzip. We shall sometimes implicitly use this identification. Notice that in the case of bipartite $\mathfrak{m}$ the root edge cannot be a loop and hence there is a bijection between the set of all bipartite maps (minus the vertex map) and bipartite maps with a *simple* root face of degree 2.

Recall our convention that $\dagger$ is the only map with a unique vertex and no face, which will be seen as the unique map with a boundary of perimeter 0.

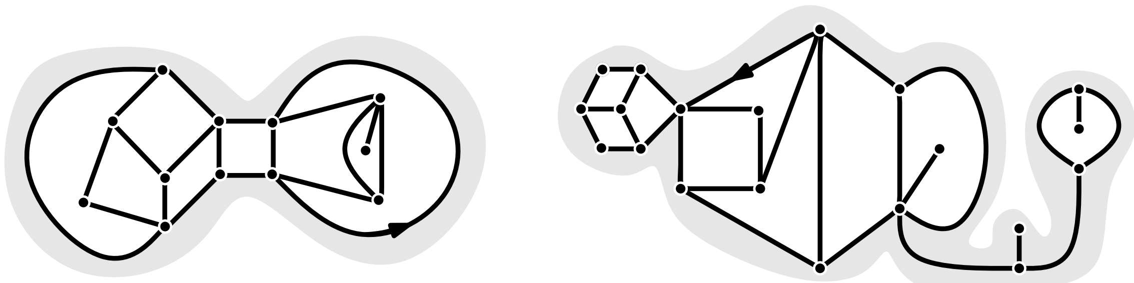


Figure 2.1: A quadrangulation with a simple boundary (on the left) and a quadrangulation with a general boundary (on the right). The root (or external) face is light gray.

2.2 Tutte's equation in the case of quadrangulations

By bipartiteness a quadrangulation with a boundary necessarily has an even perimeter and to ease notation in the following we generically denote its half-perimeter by $\ell \geq 1$, more precisely for $n \geq 0$ and $\ell \geq 0$ write $\mathcal{Q}_n^{(\ell)}$ for the set of all (rooted) quadrangulation with n inner faces and with a root face of degree 2ℓ. Recall that when $n = p = 0$ the set $\mathcal{Q}_0^{(0)} = \{\dagger\}$ only contains the vertex map. The idea of Tutte's equation is to write a relation between $\mathcal{Q}_n^{(\ell)}$ by erasing the root edge. Let us first describe this decomposition via a figure.

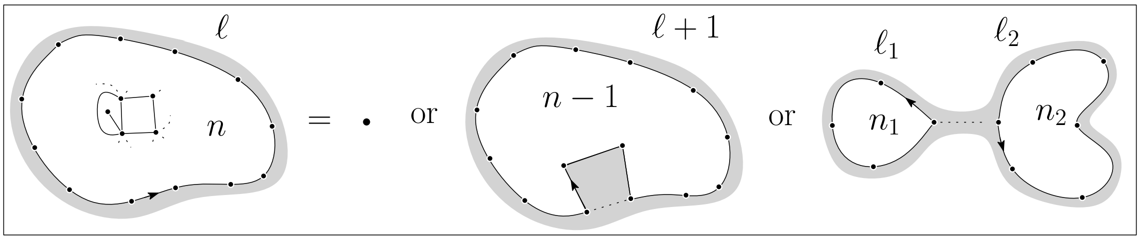


Figure 2.2: Tutte's decomposition.

In words, this decomposition says that a map of $\mathcal{Q}_n^{(\ell)}$ is either the vertex map (if both $n = \ell = 0$) or we have the following alternative: if after erasing the root edge, the map stays connected then we can associate with it an element of $\mathcal{Q}_{n-1}^{(\ell+1)}$ –necessarily $\ell + 1 \geq 2$–, if it does not then erasing the root edge splits the map into two elements of $\mathcal{Q}_{n_1}^{(\ell_1)}$ and $\mathcal{Q}_{n_2}^{(\ell_2)}$ respectively with $n_1 + n_2 = n$ and $\ell_1 + \ell_2 = \ell - 1$. After introducing the generating function (which can be first seen as a formal power series)

$$Q(g,z) = \sum_{n \geq 0, \ell \geq 0} g^n z^\ell \# \mathcal{Q}_n^{(\ell)},$$

the former equation becomes Tutte's equation

$$Q(g,z) = 1 + \frac{g}{z}\Big(Q(g,z) - [z^0]Q(g,z) - [z^1]Q(g,z)\Big) + z\Big(Q(g,z)\Big)^2. \tag{2.1}$$

Notice the terms $[z^0]Q(g,z)$ and $[z^1]Q(g,z)$, which represent the terms in z^0 and z^1 in Q, must be subtracted since the quadrangulations for which the erasure of the root edge leaves a connected part must be of half-perimeter at least 2 (without these terms, the equation would be very easy to solve!). Tutte has developed the so-called quadratic method to solve such equations. We will not enter the details of this techniques and rather directly provide the reader with the answer.

Exercise 7. Prove that the above equation characterizes Q as a formal power series in g and z, in other words, for any $n, \ell \geq 0$ fixed, Equation (2.1) enables us to compute in finite time the number of quadrangulations with a boundary of perimeter ℓ and n inner faces.

2.2.1 The solution

For $g, z > 0$ let $R(g)$ and $Q_0(g,z)$ be the positive solutions to

$$R(g) = 1 + 3g\,R^2(g) \quad \text{and} \quad Q_0(g,z) = 1 + zR(g)Q_0^2(g,z)$$

then we have

$$Q(g,z) = Q_0(g,z) \cdot (1 - gR^2(g)(Q_0(g,z) - 1)).$$

The anxious reader may develop explicitly the above calculations to give a closed but rather bad-looking formula for Q and check that it indeed satisfies (2.1). In particular one sees that the formal generating series is actually finite for $g \leq 1/12$ and $z \leq 1/8$; one may also (for example using Lagrange inversion formula or via a direct tedious calculation) extract the exact values for the number of quadrangulations with a boundary: for $(n, \ell) \neq (0,0)$ we have

$$\#\mathcal{Q}_n^{(\ell)} = \frac{(2\ell)!}{\ell!(\ell-1)!} 3^n \frac{(2n+\ell-1)!}{(n+\ell+1)!n!} \quad \text{and} \quad \#\mathcal{Q}_0^{(0)} = 1. \tag{2.2}$$

Corollary 4. *The number of planar maps with n edges is equal to* $3^n \cdot \dfrac{2 \cdot (2n)!}{(n+2)!n!}$.

Proof. Specify the last display with $\ell = 1$ and use the bijection presented in Section 1.3.3. □

Rather than the above exact formulas, one should remember the following asymptotic results for which the critical polynomial exponents and the following functions are universal among many classes of planar maps:

Definition 7 (The functions $h^\uparrow$ and $h^\downarrow$). *For $\ell \geq 0$ put*

$$h^\uparrow(\ell) = 2\ell\, 2^{-2\ell}\binom{2\ell}{\ell} \quad \textit{and} \quad h^\downarrow(\ell) = h^\uparrow(\ell+1) - h^\uparrow(\ell) = 2^{-2\ell}\binom{2\ell}{\ell}, \qquad \ell \geq 0,$$

and $h^\uparrow(\ell) = h^\downarrow(\ell) = 0$ for $\ell \leq -1$.

Then using the above exact formula and Stirling formula we get

$$\begin{aligned} \#\mathcal{Q}_n^{(\ell)} &\underset{n\to\infty}{\sim} C(\ell)\,\kappa^n n^{-5/2}, \\ C(\ell) &= c_0 \alpha^\ell h^\uparrow(\ell), \end{aligned} \tag{2.3}$$

where $\kappa = 12$, $\alpha = 8$ and $c_0 = (4\sqrt{\pi})^{-1}$ are non-universal factors (they depend on the model of maps considered such triangulations, quadrangulations or more generally regular critical Boltzmann maps, see below). Using these asymptotic one sees that at the critical point $g_c = \frac{1}{\kappa}$ the numbers $[z^\ell]Q(g_c, z)$ are finite and can easily be computed, we shall use a slight variant of them: recall that by Euler's formula $n + \ell + 1$ is the number of vertices in a quadrangulation of $\mathcal{Q}_n^{(\ell)}$ and introduce for $\ell \geq 1$

$$W^{(\ell)} = \sum_{n\geq 0} \#\mathcal{Q}_n^{(\ell)} \kappa^{-n} = \alpha^\ell \cdot \frac{h^\uparrow(\ell)}{\ell(\ell+1)(\ell+2)}, \tag{2.4}$$

$$W_\bullet^{(\ell)} = \sum_{n\geq 0} (n+\ell+1)\#\mathcal{Q}_n^{(\ell)} \kappa^{-n} = \alpha^\ell \cdot h^\downarrow(\ell), \tag{2.5}$$

and $W^{(0)} = W_\bullet^{(0)} = 1$. The numbers $W^{(\ell)}$ and $W_\bullet^{(\ell)}$ will be interpreted as the partition function for the following measures. The Boltzmann distribution $\mathbb{P}^{(\ell)}$ on quadrangulations with a boundary of perimeter 2ℓ is the distribution on $\mathcal{Q}^{(\ell)} = \cup_{n\geq 0} \mathcal{Q}_n^{(\ell)}$ such that any element $\mathfrak{q} \in \mathcal{Q}^{(\ell)}$ gets a weight

$$\mathbb{P}^{(\ell)}(\mathfrak{q}) = \frac{\kappa^{-\#\mathsf{InnerFaces}(\mathfrak{q})}}{W^{(\ell)}}.$$

Similarly the Boltzmann distribution $\mathbb{P}_\bullet^{(\ell)}$ on pointed quadrangulations with a boundary of perimeter 2ℓ is the distribution on $\mathcal{Q}_\bullet^{(\ell)}$ giving weight

$$\mathbb{P}_\bullet^{(\ell)}(\mathfrak{q}_\bullet) = \frac{\kappa^{-\#\mathsf{InnerFaces}(\mathfrak{q}_\bullet)}}{W_\bullet^{(\ell)}},$$

to each quadrangulation $\mathfrak{q}_\bullet = (\mathfrak{q}, v)$ equipped with a distinguished vertex $v \in \mathsf{Vertices}(\mathfrak{q})$. The above calculations may seem miraculous (and in some sense they are) but in fact these forms are universal among a large class of bipartite planar maps as we shall now see.

> The reader only interested in applications to random quadrangulations may skip the next sections and directly jump to the desired chapter and plug the last formulas for the numbers $W^{(\ell)}$ and $W_\bullet^{(\ell)}$ as well as $q_k = 12^{-1}\delta_2(k)$ and $c_\mathbf{q} = 8$ when necessary.

In the rest of these lecture notes we only deal with bipartite planar maps.

2.3 Tutte's equation for Boltzmann maps

In the rest of these lecture notes we focus on *bipartite* planar maps because their enumeration is simpler than that of general planar maps. However, most of the material developed can be adapted to the general case and in particular to triangulations to the cost of additional technicalities and heavier notation. Recall from Definition 7 the functions $h^\uparrow(\ell)$ and $h^\downarrow(\ell)$.

2.3.1 Boltzmann planar maps.

Let $\mathbf{q} = (q_k)_{k\geq 1}$ be a non-zero sequence of non-negative reals which will be called the *weight sequence* in these pages. We use this sequence to define a σ-finite measure on the set of all planar maps by the formula

$$w_{\mathbf{q}}(\mathfrak{m}) \quad = \prod_{f\in \mathsf{Faces}(\mathfrak{m})\setminus\{f_r\}} q_{\deg(f)/2}. \tag{2.6}$$

Notice that the root face do not appear in the above product. In particular, if we take the weight sequence $\mathbf{q} = (\delta_{k_0}(k))_{k\geq 1}$ then $w_{\mathbf{q}}(\mathfrak{m})$ is simply 1 if all the faces apart from the root face are of degree $2k_0$. For $\ell \geq 0$, we denote by $\mathcal{M}^{(\ell)}$ the set of all bipartite planar maps such that the root face has degree 2ℓ (in particular $\mathcal{M}^{(0)} = \{\dagger\}$). We insist on the fact that the root face may not be simple, that is, may contain pinch points. We sometimes call a map $\mathfrak{m} \in \mathcal{M}^{(\ell)}$ a map with a boundary of half-perimeter ℓ. For reasons that will become clearer later on, we will also need the notion of a *pointed* planar map. A pointed planar map $\mathfrak{m}_\bullet = (\mathfrak{m}, v)$ is just a planar map $\mathfrak{m}$ given with a distinguished vertex $v \in \mathsf{Vertices}(\mathfrak{m})$. The weight $w_{\mathbf{q}}(\mathfrak{m}_\bullet)$ is simply the weight of $\mathfrak{m}$. We denote accordingly $\mathcal{M}_\bullet^{(\ell)}$ the set of all (bipartite) pointed planar maps with a boundary of perimeter 2ℓ and put

$$W^{(\ell)}(\mathbf{q}) = w_{\mathbf{q}}(\mathcal{M}^{(\ell)}) \quad \text{and} \quad W_\bullet^{(\ell)}(\mathbf{q}) = w_{\mathbf{q}}(\mathcal{M}_\bullet^{(\ell)}), \tag{2.7}$$

in particular $W^{(0)}(\mathbf{q}) = W_\bullet^{(0)}(\mathbf{q}) = 1$. We write simply $W^{(\ell)}$ and $W_\bullet^{(\ell)}$ when the weight sequence $\mathbf{q}$ is implicit. As we saw above, enumerating maps of $\mathcal{M}^{(\ell)}$ is a more general problem than enumerating planar maps since there is a bijection between maps (different from the vertex map $\dagger$) and maps with a boundary of perimeter 2. It is easy to convince ourselves that the above definitions do not always make sense since $W_\bullet^{(\ell)}(\mathbf{q})$ and $W^{(\ell)}(\mathbf{q})$ may be infinite.

Definition 8. *The weight sequence* $\mathbf{q}$ *is called* admissible *when* $W_\bullet^{(1)}(\mathbf{q})$ *is finite*[a] *(and hence one easily checks that* $W_\bullet^{(\ell)}(\mathbf{q})$ *as well as* $W^{(\ell)}(\mathbf{q})$ *is finite for all* $\ell \geq 1$*).*

[a] The reader may wonder at this point whether the weaker assumption of finiteness of $W^{(\ell)}(\mathbf{q})$ is equivalent to admissibility. It is indeed the case (see Corollary 23) but the above definition with using pointed maps is more useful thanks to the following Theorem 6.

For example if one takes the weight sequence $q_k = g \cdot \delta_2(k)$ corresponding to counting quadrangulations with a weight g per inner face, the corresponding measure $w_{\mathbf{q}}$ is finite if and only if $g \leq \frac{1}{12}$ by (2.3). We recall the admissibility condition for the sequence $\mathbf{q}$ proved in [51], see Appendix A. For $k \geq 1$ we put

$$\tilde{q}_k = q_k \binom{2k-1}{k-1} \quad \text{and} \quad f_{\mathbf{q}}(x) = \sum_{k=1}^{\infty} \tilde{q}_k x^{k-1}.$$

Proposition 5 (Admissibility criterion)**.** *The sequence* $\mathbf{q}$ *is admissible if and only if the following equation has a positive solution*

$$f_{\mathbf{q}}(x) = 1 - \frac{1}{x}.$$

This admissibility criterion is easily proved using the Bouttier–Di Francesco–Guitter [21] bijection between bipartite planar maps and mobiles (labeled trees) but we shall admit it in these lecture notes. In the case when $\mathbf{q}$ is admissible, we shall denote by $Z_{\mathbf{q}}$ the smallest solution to the equation $f_{\mathbf{q}}(x) = 1 - \frac{1}{x}$, and put $c_{\mathbf{q}} = 4Z_{\mathbf{q}}$. As in the case of quadrangulations, when $\mathbf{q}$ is admissible the numbers $W^{(\ell)}$ and $W_\bullet^{(\ell)}$ can be interpreted as the partition functions of Boltzmann distributions: we shall denote by $\mathbb{P}_{\mathbf{q}}^{(\ell)}$ the $\mathbf{q}$-Boltzmann distribution on $\mathcal{M}^{(\ell)}$ giving a weight

$$\mathbb{P}_{\mathbf{q}}^{(\ell)}(\mathfrak{m}) = \frac{w_{\mathbf{q}}(\mathfrak{m})}{W^{(\ell)}(\mathbf{q})},$$

and similarly the pointed $\mathbf{q}$-Boltzmann distributed on $\mathcal{M}_\bullet^{(\ell)}$ given weight

$$\mathbb{P}_{\bullet,\mathbf{q}}^{(\ell)}(\mathfrak{m}_\bullet) = \frac{w_{\mathbf{q}}(\mathfrak{m}_\bullet)}{W_\bullet^{(\ell)}(\mathbf{q})}.$$

As usual, we shall drop the dependence in $\mathbf{q}$ when it is implicit. Notice a simple but important interpretation of the pointed partition function $W_\bullet^{(\ell)}$. If $\mathfrak{m}$ is a map we denote by $|\mathfrak{m}|$ the number of vertices of $\mathfrak{m}$. It follows readily from the above definitions that the expected volume of a $\mathbf{q}$-Boltzmann map is given by

$$\int \mathbb{P}^{(\ell)}(d\mathfrak{m})|\mathfrak{m}| = \frac{1}{W^{(\ell)}} \sum_{\mathfrak{m} \in \mathcal{M}^{(\ell)}} w(\mathfrak{m})|\mathfrak{m}| = \frac{W_\bullet^{(\ell)}}{W^{(\ell)}}. \tag{2.8}$$

The ideal goal now is to compute exactly $W^{(\ell)}(\mathbf{q})$ and $W_\bullet^{(\ell)}(\mathbf{q})$ as a function of ℓ and of the weight sequence $\mathbf{q}$. The starting point is the same as in the last section: the deletion of the root edge give us the following recursive equation (Tutte's equation)

$$W^{(\ell)} = \sum_{k=1}^{\infty} q_k W^{(\ell+k-1)} + \sum_{\ell_1+\ell_2=\ell-1} W^{(\ell_1)} W^{(\ell_2)}, \tag{2.9}$$

or in terms of the pointed versions

$$W_\bullet^{(\ell)} = \sum_{k=1}^{\infty} q_k W_\bullet^{(\ell+k-1)} + 2\sum_{\ell_1=0}^{\ell-1} W_\bullet^{(\ell_1)} W^{(\ell-\ell_2-1)}. \tag{2.10}$$

As in the last section, the above identities completely characterize $W^{(\ell)}(\mathbf{q})$ and $W_\bullet^{(\ell)}(\mathbf{q})$ which may be seen as formal series in $q_1, q_2, q_3, \ldots$.

2.3.2 The universal solution

The wonderful magic of enumeration of (bipartite) maps is that these equations admit a universal solution regardless of the weight sequence $\mathbf{q}$. Recall from above that $c_{\mathbf{q}}/4$ is the smallest positive solution to the equation $f_{\mathbf{q}}(x) = 1 - \frac{1}{x}$, then we have:

Theorem 6 (*Universal form for the resolvent of pointed bipartite Boltzmann maps*)
For any admissible weight sequence $\mathbf{q}$, *there exists* $c_{\mathbf{q}} > 4$ *such that for* $z > c_{\mathbf{q}}$ *we have*

$$\sum_{\ell=0}^{\infty} z^{-\ell} W_{\bullet}^{(\ell)}(\mathbf{q}) = \left(1 - \frac{c_{\mathbf{q}}}{z}\right)^{-1/2} \quad \text{or equivalently} \quad W_{\bullet}^{(\ell)}(\mathbf{q}) = c_{\mathbf{q}}^{\ell} h^{\downarrow}(\ell), \quad \ell \geq 0.$$

For example when $\mathbf{q} = \frac{1}{12}\delta_2$, which corresponds to the model of critical quadrangulations then it is easy to use the exact enumeration formulas of the last section to deduce that the above theorem holds with $c_{\mathbf{q}} = \alpha = 8$. The above theorem gives a universal formula for $W_{\bullet}^{(\ell)}$ depending on a single parameter. The easiest explanation for this universal formula goes through the use of the Bouttier–Di Francesco–Guitter bijection between bipartite planar maps and mobiles, see Appendix A. However at some point we will need an access to the asymptotic behavior of $W^{(\ell)}$ which is a bit more complicated and less universal than that of its pointed analog (see Chapter 5). For the first few chapters we will only need the following fact which we prove in Chapter 5

Lemma 6. *If* $\mathbf{q}$ *is an admissible weight sequence then we have*

$$\lim_{\ell\to\infty} \frac{W^{(\ell+1)}}{W^{(\ell)}} = c_{\mathbf{q}}. \tag{2.11}$$

2.3.3 Critical and subcriticial weight sequences

Recall that the weight sequence $\mathbf{q}$ is admissible if the equation

$$f_{\mathbf{q}}(x) = 1 - \frac{1}{x}$$

has a positive solution. It is easy to see that the last equation has at most two positive solutions and recall that $c_{\mathbf{q}}$ is four times the smallest of such solutions denoted by $Z_{\mathbf{q}}$. We further distinguish the weight sequence according to whether the graphs of the function $x \mapsto f_{\mathbf{q}}(x)$ and $x \mapsto 1 - \frac{1}{x}$ are tangent at $Z_{\mathbf{q}}$ or not. More precisely, in the case when $\mathbf{q}$ is admissible we must have $Z_{\mathbf{q}}^2 f_{\mathbf{q}}'(Z_{\mathbf{q}}) \leq 1$.

Definition 9. *The admissible weight sequence* $\mathbf{q}$ *is called* subcritical[a] *if* $Z_{\mathbf{q}}^2 f_{\mathbf{q}}'(Z_{\mathbf{q}}) < 1$ *and critical if* $Z_{\mathbf{q}}^2 f_{\mathbf{q}}'(Z_{\mathbf{q}}) = 1$, *see Fig. 2.3 below.*

[a] The names subcritical and critical come from the coding of random planar maps by random labeled trees via the BDG bijection: the weight sequence $\mathbf{q}$ is (sub)critical when the distribution it induces on trees is a (sub)critical multi-type Galton–Watson distribution.

For further use let us write explicitly the two equations defining an admissible and critical weight sequence

$$\sum_{k\geq 0} q_k \binom{2k-1}{k-1} Z_{\mathbf{q}}^{k-1} = 1 - \frac{1}{Z_{\mathbf{q}}}, \qquad \text{(admissible)} \tag{2.12}$$

$$\sum_{k\geq 0} q_k \binom{2k-1}{k-1} Z_{\mathbf{q}}^{k-2}(k-1) = \frac{1}{Z_{\mathbf{q}}^2}, \qquad \text{(critical).} \tag{2.13}$$

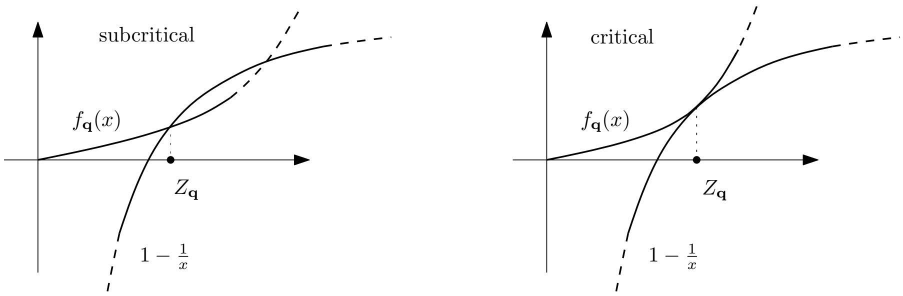


Figure 2.3: Graphs of the function $x \mapsto f_{\mathbf{q}}(x)$ and $x \mapsto 1-\frac{1}{x}$ in the case of a subcritical (left) and critical (right) weight sequence.

We will see in Chapter 5 that the asymptotic for $W^{(\ell)}$ are very different in the subcritical and in the critical case. This, in turns, change dramatically the geometry of the underlying Boltzmann random maps.

Bibliographical notes. Although provable using Tutte's equation the enumeration results presented in this section are more efficiently proved (and understood) using bijections between maps and labeled trees (Schaeffer's bijection [57] in the case of quadrangulations and its extension by Bouttier–Di Francesco–Guitter [21] in the case of bipartite maps). The beginning of this chapter is adapted from [24]. Tutte's equation is naturally due to Tutte [59] (first in the context of triangulations). The solution for quadrangulations here is taken from [22]. The formalism for Boltzmann planar maps is taken from [51] where the fundamental Proposition 5 is proved using the BDG bijection [21]. Although easy to derive from [51], Theorem 6 is taken under this form from [23]. The concept of critical weight sequence is due to [51] later re-interpreted in [23]. We also use notation of [23].

Chapter III : Peeling of finite Boltzmann maps

In this chapter we define precisely what we mean by peeling a deterministic (bipartite) planar map. We then compute the law of the Markov chains given by peeling explorations of finite **q**-Boltzmann planar maps and show that a natural random walk hides behind such explorations.

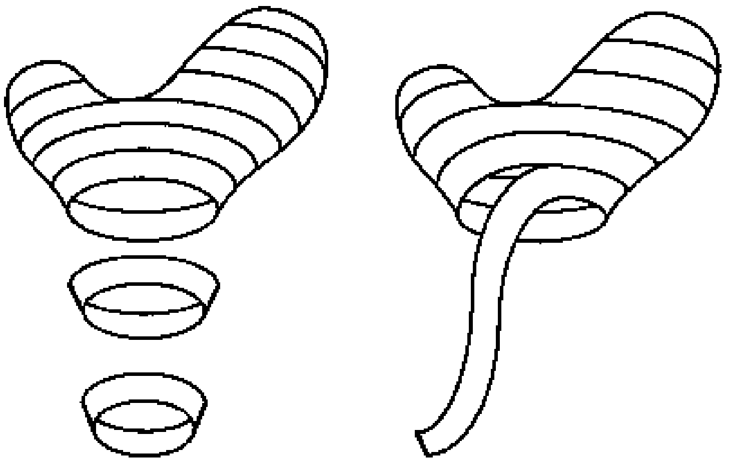


Figure 3.1: Original figures from the paper of Watabiki.

> The reader eager to manipulate planar maps in order to get a better understanding of the notions developed in this chapter is warmly encouraged to play with the wonderful (and free!) software developed by Timothy Budd available here:
>
> *http://www.nbi.dk/~budd/planarmap/examples/editor.html*

3.1 Peeling processes

3.1.1 Gluing maps with a boundary.

Let $\mathfrak{m}$ be a (rooted bipartite) planar map and recall that $\mathfrak{m}^\dagger$ stands for its dual map whose vertices are the faces of $\mathfrak{m}$ and whose edges are dual to those of $\mathfrak{m}$. The origin of $\mathfrak{m}^\dagger$ is the root face f_r of $\mathfrak{m}$. Let $\mathfrak{e}^\circ$ be a finite connected subset of edges of $\mathfrak{m}^\dagger$ such that the origin of $\mathfrak{m}^\dagger$ is incident to $\mathfrak{e}^\circ$ (the letter "$\mathfrak{e}$" stands for *explored*). We associate with $\mathfrak{e}^\circ$ a planar map $\mathfrak{e}$ which, roughly speaking, is obtained by gluing the faces of $\mathfrak{m}$ corresponding to the vertices adjacent to $\mathfrak{e}^\circ$ along the (dual) edges of $\mathfrak{e}^\circ$, see Fig. 3.2. The resulting map, rooted at the root edge of $\mathfrak{m}$, is a finite (rooted bipartite) planar map with several distinguished faces $h_1, \dots, h_k \in \mathsf{Faces}(\mathfrak{e})$ that correspond to the connected components of $\mathfrak{m}^\dagger \backslash \mathfrak{e}^\circ$. These distinguished faces are called the holes of $\mathfrak{e}$. Notice that the holes are simple, meaning that there is no pinch-point on their boundaries, and these boundaries also do not have vertices in common. Such an object will be called a planar map with holes. See Fig. 3.2 below.

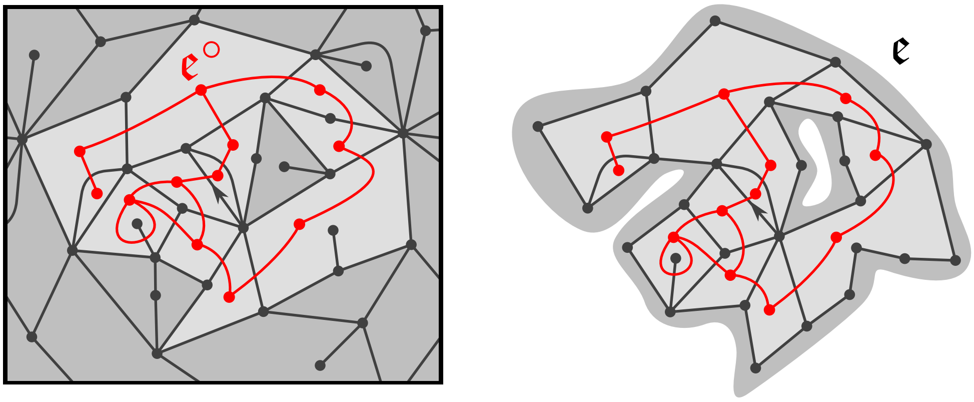


Figure 3.2: Illustration of the duality between connected subsets of edges on the dual map and their associated submaps on the primal lattice.

We say that $\mathfrak{e}$ is a submap of $\mathfrak{m}$ and write

$$\mathfrak{e} \subset \mathfrak{m}$$

since $\mathfrak{m}$ can be obtained back from $\mathfrak{e}$ by gluing inside each hole h_i of $\mathfrak{e}$ a (uniquely defined) bipartite planar map $\mathfrak{u}_i$ of perimeter $\deg(h_i)$ (the letter $\mathfrak{u}$ stands for *unexplored*). To perform this gluing operation, we implicitly assume that an oriented edge is distinguished on the boundary of each hole h_i of $\mathfrak{e}$, on which we glue the root edge of $\mathfrak{u}_i$. We will not mention this further, since these edges can be arbitrarily chosen using a deterministic procedure given $\mathfrak{e}$. Notice that after this gluing operation, it might happen that several edges on the boundary of a given hole of $\mathfrak{e}$ get identified because the boundary of $\mathfrak{u}_i$ may not be simple, see Fig. 3.3 below. We will alternatively speak of "gluing" or "filling-in the hole".

It is easy to see that this operation is rigid (see [8, Definition 4.7]) in the sense that if $\mathfrak{e} \subset \mathfrak{m}$, then the maps $(\mathfrak{u}_i)_{1 \le i \le k}$ are uniquely defined (in other words, if one glues different maps inside a given planar map with holes, one gets different maps after the gluing procedure). This definition even makes sense when $\mathfrak{e}$ is a finite map and $\mathfrak{m}$ is an infinite map. Conversely, if $\mathfrak{e} \subset \mathfrak{m}$, one can recover $\mathfrak{e}^\circ$ in a unique way as the set of all the dual edges between faces of $\mathfrak{e}$ which are not holes.

Exercise 8. Prove the above rigidity statement.

This discussion shows that there are two points of view on submaps of $\mathfrak{m}$ which are equivalent: either submaps can be seen as objets of the type of $\mathfrak{e}^\circ$ (which are connected components of edges

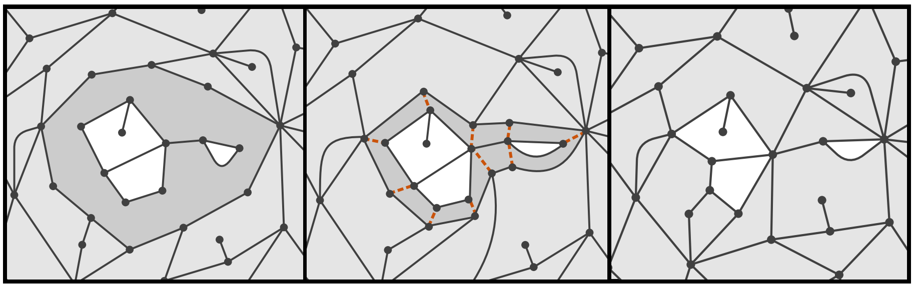


Figure 3.3: The gluing operation is illustrated below.

containing the origin in $\mathfrak{m}^\dagger$), or as planar maps $\mathfrak{e} \subset \mathfrak{m}$ with holes (possibly none) which may be filled-in to obtain $\mathfrak{m}$. In this paper, we will mostly work with the second point of view.

In the case when $\mathfrak{e}^\circ$ contains only one point (the root face) then we abuse notation and say that $\mathfrak{e}$ is equal to the root face f_r. In fact, when doing so we see the root face of $\mathfrak{m}$ as a map with hole made of one simple face of degree $\deg(f_\mathrm{r})$ and the corresponding hole of the same perimeter.

Recall from Section 1.2 that $[\mathfrak{m}]_r$, the ball of radius r in a map $\mathfrak{m}$ was composed of all the vertices and edges at distance less than r from the origin of $\mathfrak{m}$. Notice that with this definition we may not have $[\mathfrak{m}]_r \subset \mathfrak{m}$ which is a bit annoying (in particular the "holes" of $[\mathfrak{m}]_r$ may not be simple). We thus introduce another notion:

Definition 10 (The ball of radius r, new version)**.** *If $\mathfrak{m}$ is a map, for $r \geq 0$ we denote by*

$$\mathrm{Ball}_r(\mathfrak{m}) \subset \mathfrak{m},$$

the map with holes obtained by cutting all edges in $\mathfrak{m}$ that have both endpoints at distance larger than or equal to r from the origin of the root edge and taking the component of the root. Equivalently, $(\mathrm{Ball}_r(\mathfrak{m}))^\circ$ is given by those edges of $\mathfrak{m}^\dagger$ whose dual edges have at least one endpoint at distance strictly less than r from the origin. We also put $\mathrm{Ball}_0(\mathfrak{m}) = f_\mathrm{r}$ seen as a map with hole.

It is easy to see that $\mathrm{Ball}_r(\mathfrak{m})$ contains all the faces of $\mathfrak{m}$ which are adjacent to at least one vertex at distance strictly less than r from the origin of the map and that $[\mathfrak{m}]_r$ can be recovered from $\mathrm{Ball}_r(\mathfrak{m})$. But remark that it is important not to define the local distance with this notion for otherwise we would not have convergence of the polygon with k sides towards $\mathbb{Z}$.

3.1.2 Peeling process

A peeling exploration is a means to explore a planar map $\mathfrak{m}$ edge after edge. If $\mathfrak{e} \subset \mathfrak{m}$ is a planar map with holes, the *active boundary*[1] of $\mathfrak{e}$ is given by the union of all the edges adjacent to the holes of $\mathfrak{e}$. We denote it by $\mathsf{Active}(\mathfrak{e})$. Formally, a peeling exploration depends on a function

[1]Contrary to [14], the active boundary of $\mathfrak{e}$ cannot be seen as the union of self-avoiding loops on the original map $\mathfrak{m}$ since they are closed paths which may visit twice the same edge, they are called frontiers in [23].

$\mathcal{A}$, called the *peeling algorithm*, which associates with any planar map with holes $\mathfrak{e}$ an edge of $\mathsf{Active}(\mathfrak{e}) \cup \{\dagger\}$, where $\dagger$ is a cemetery point which we interpret as the will to end the exploration. In particular, if $\mathfrak{e}$ has no holes, we must have $\mathcal{A}(\mathfrak{e}) = \dagger$. We say that this peeling algorithm is deterministic since no randomness is involved in the definition of $\mathcal{A}$.

Intuitively speaking, given the peeling algorithm $\mathcal{A}$, the peeling process of a (bipartite) planar map $\mathfrak{m}$ is a way to iteratively explore $\mathfrak{m}$ by starting from its boundary and by discovering a piece of $\mathfrak{m}$ by *peeling an edge* determined by the algorithm $\mathcal{A}$. If $\mathfrak{e} \subset \mathfrak{m}$ is a planar map with holes and e is an edge of $\mathsf{Active}(\mathfrak{e})$ or $e = \dagger$, the planar map with holes $\mathsf{Peel}(\mathfrak{e}, e, \mathfrak{m})$ obtained by peeling e is defined as follows. Let F_e be the face of $\mathfrak{m}$ adjacent to e (provided that $e \neq \dagger$) and located on the other side of e with respect to $\mathfrak{e}$. Then there are three possibilities, see Fig. 3.4:

- Either $e = \dagger$ and $\mathsf{Peel}(\mathfrak{e}, \dagger, \mathfrak{m}) = \mathfrak{e}$.
- *Event* C_k*:* the face F_e is not a face of $\mathfrak{e}$ and has degree $2k$. Then $\mathsf{Peel}(\mathfrak{e}, e, \mathfrak{m})$ is obtained by gluing F_e on e without performing the possible identifications of its other edges inside $\mathfrak{m}$.
- *Event* G_{k_1,k_2}*:* the face F_e is actually a face of $\mathfrak{e}$. In this case, the edge e is identified in $\mathfrak{m}$ with another edge e' on the boundary of the same hole where k_1 (resp. k_2) is half of the number of edges on the boundary of the hole strictly between e and e' when turning in clockwise order around the hole, and $\mathsf{Peel}(\mathfrak{e}, e, \mathfrak{m})$ is the map after this identification in $\mathfrak{e}$.

When $k_1 > 0$ and $k_2 > 0$, note that the event G_{k_1,k_2} results in the splitting of a hole into two holes. If $k_1 = 0$ or $k_2 = 0$ the corresponding hole of perimeter 0 is actually a vertex of the map since by our convention the vertex map is the only one map of perimeter 0. In particular the event $\mathsf{G}_{0,0}$ results in the disappearance of a hole.

Definition 11 (Peeling exploration)**.** *If $\mathfrak{m}$ is a (finite or infinite) planar map, the peeling exploration of $\mathfrak{m}$ with algorithm $\mathcal{A}$ is the sequence of planar maps with holes*

$$\mathfrak{e}_0 \subset \mathfrak{e}_1 \subset \cdots \subset \mathfrak{e}_n \subset \cdots \subset \mathfrak{m},$$

such that the map $\mathfrak{e}_0$ is the root face f_{r} seen as a map with hole and for every $i \geq 0$

$$\mathfrak{e}_{i+1} = \mathsf{Peel}(\mathfrak{e}, \mathcal{A}(\mathfrak{e}_i), \mathfrak{m}).$$

In particular, observe that if $\mathfrak{e}_i \neq \mathfrak{e}_{i-1}$ with $i \geq 1$, then $\mathfrak{e}_i$ has exactly i internal edges. If $i \geq 0$, the map with holes $\mathfrak{e}_i(\mathfrak{m})$ is obviously a (deterministic) function of $\mathfrak{m}$. But note that $(\mathfrak{e}_j(\mathfrak{m}); 0 \leq j \leq i)$ is also a (deterministic) function of $\mathfrak{e}_i(\mathfrak{m})$. Finally, to simplify notation, we will write $\mathfrak{e}_i$ instead of $\mathfrak{e}_i(\mathfrak{m})$. Notice also that although not visible in our notation, the sequence of explored maps $(\mathfrak{e}_i)$ depends obviously on the underlying map, but also on the peeling algorithm $\mathcal{A}$. It should be clear in the following which are the statements valid for all peeling explorations and those for specific ones.

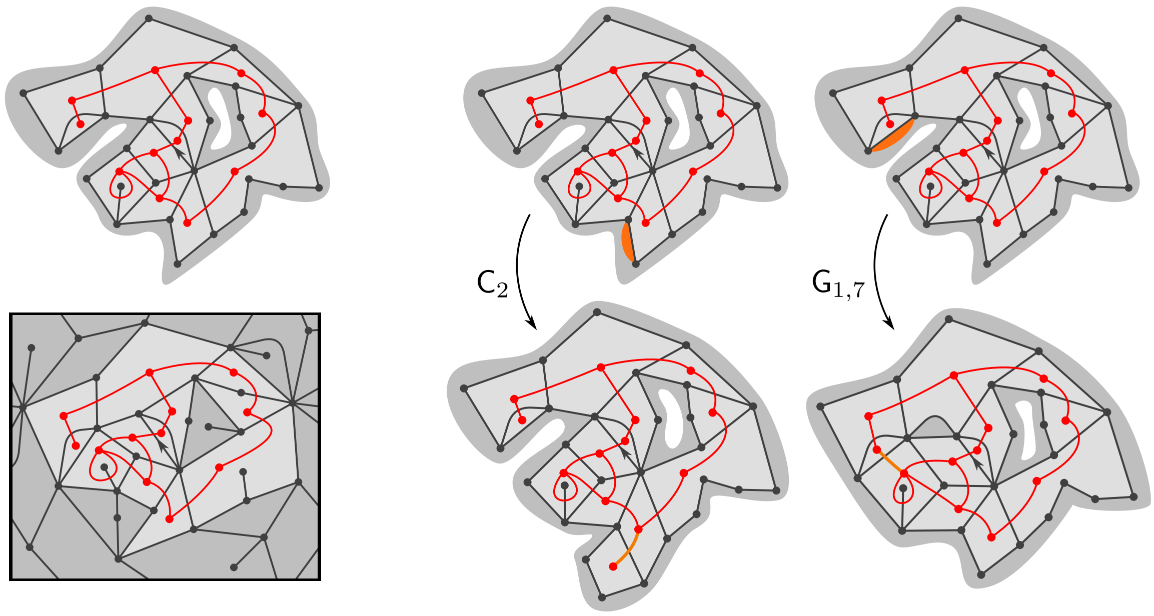


Figure 3.4: Illustration of the different peeling events. On the left column is represented the submap $\mathfrak{e} \subset \mathfrak{m}$ as well as its associated dual version $\mathfrak{e}^\circ$ in red. The center and right columns represent two different peeling events, the edge to be peeled is in thick orange. The event C_2 occurs in the center column, whereas event $\mathsf{G}_{1,7}$ occurs in the right column.

Remark 7. One can alternatively represent a peeling exploration $\mathfrak{e}_0 \subset \mathfrak{e}_1 \subset \cdots \subset \mathfrak{e}_n \subset \cdots \subset \mathfrak{m}$ as the associated sequence of growing connected subset of edges $(\mathfrak{e}_i^\circ)_{i\geq 0}$ of the dual map $\mathfrak{m}^\dagger$, such that $\mathfrak{e}_{i+1}^\circ$ is obtained from $\mathfrak{e}_i^\circ$ by adding one edge of $\mathfrak{m}^\dagger$ (unless the exploration has stopped), see Fig. 3.4. We will however mostly use the first point of view.

Remark 8. The reader may also compare the above presentation with that of [14, Sec. 2.3]. In the peeling process considered in [14, Sec. 2.3] and which has its origins in [4], the sequence $\mathfrak{e}_0 \subset \cdots \subset \mathfrak{e}_n \subset \cdots \subset \mathfrak{m}$ is again a sequence of maps with simple holes (with the slight difference that in this case the holes can share vertices but not edges) but (unless the peeling has stopped), $\mathfrak{e}_{i+1}$ is obtained from $\mathfrak{e}_i$ by the addition of a new *face*. Furthermore, in this peeling process, $\mathfrak{m}$ is obtained from $\mathfrak{e}_i$ by the filling-in the holes of $\mathfrak{e}_i$ with maps having *simple* boundary. In other words, the peeling process of [14, Sec. 2.3] is "face"-peeling, while in the present work we have an "edge"-peeling.

3.1.3 Peeling process with a target

In what follows we will sometimes explore maps with a distinguished target: if $\mathfrak{m}_\bullet = (\mathfrak{m}, v)$ is a pointed map and if $\mathfrak{e} \subset \mathfrak{m}$ is a submap of $\mathfrak{m}$ we write $\mathfrak{e}^*$ for the submap $\mathfrak{e} \subset \mathfrak{m}$ together with the knowledge of the hole of $\mathfrak{e}$ which contains the distinguished vertex $v \in \mathsf{Vertices}(\mathfrak{m})$. If v is already an inner vertex of $\mathfrak{e}$ (an inner vertex of a map with holes is a vertex that is not adjacent to the active boundary) then $\mathfrak{e}^*$ is given by $\mathfrak{e}$ together with the knowledge of the distinguished vertex v. In that case we thus speak of

$$\mathfrak{e}_0^* \subset \mathfrak{e}_1^* \subset \cdots \subset \mathfrak{e}_n^* \subset \cdots \subset \mathfrak{m}$$

as the peeling exploration with target of $(\mathfrak{m}, v)$. Actually in this case, the peeling algorithm $\mathcal{A}$ may even depend on the knowledge of the distinguished hole. If $\mathfrak{m}$ does not have a distinguished point but is infinite and one-ended, then we may consider that there is a distinguished point "at infinity" and we can define a peeling exploration targeting ∞ so that the distinguished hole of $\mathfrak{e}_n^*$ is the only hole containing an infinite part in $\mathfrak{m}$.

3.1.4 Filled-in explorations

In the case of a peeling exploration with a target one can define what we call the respective *filled-in* peeling exploration (with a target). If $\mathfrak{e}^* \subset \mathfrak{m}$ is a submap of $\mathfrak{m}$ with a distinguished hole, we call the *hull* of $\mathfrak{e}^*$ in $(\mathfrak{m}, v)$ the submap $\bar{\mathfrak{e}}$ obtained by filling-in all the holes of $\mathfrak{e}^*$ with the respective map it contains inside $\mathfrak{m}$ except for the distinguished hole. Hence $\bar{\mathfrak{e}}$ is a submap with a unique hole (unless $\mathfrak{e}^*$ had no distinguished hole, but a distinguished vertex in which case $\bar{\mathfrak{e}}$ is $\mathfrak{m}$ together with the distinguished point). If $\mathfrak{e}_0^* \subset \mathfrak{e}_1^* \subset \cdots \subset \mathfrak{e}_n^* \subset \cdots \subset \mathfrak{m}$ is a peeling with target of $(\mathfrak{m}, v)$ then

$$\bar{\mathfrak{e}}_0 \subset \bar{\mathfrak{e}}_1 \subset \cdots \subset \mathfrak{m}$$

is called the filled-in exploration with target of $(\mathfrak{m}, v)$. Notice that this sequence may contain repetitions but this can be avoided if the peeling algorithm $\mathcal{A}$ always peels an edge on the distinguished hole of $\mathfrak{e}_n^*$ which we will implicitly suppose when dealing with filled-in explorations. Such an exploration process then proceeds by peeling an edge and immediately filling-in the possible hole not containing the target.

When $\mathfrak{m}$ does not have a distinguished target (possibly at infinity) and that there is a priori no distinguished hole in $\mathfrak{e}_n$, we can still decide to track a particular family of holes which we call the *locally largest* one and consider the corresponding filled-in exploration. More precisely if $\bar{\bar{\mathfrak{e}}}_n \subset \mathfrak{m}$ is a submap of $\mathfrak{m}$ with at most one hole, then the map $\bar{\bar{\mathfrak{e}}}_{n+1}$ is obtained by first peeling the edge $\mathcal{A}(\bar{\bar{\mathfrak{e}}}_n)$: If the peeling of that edge results in a split of the hole into two holes then we immediately fill-in the smallest of those two (if there is a tie, then choose according to a deterministic rule) with its respective map inside $\mathfrak{m}$. The exploration

$$\bar{\bar{\mathfrak{e}}}_0 \subset \cdots \subset \bar{\bar{\mathfrak{e}}}_n \subset \cdots \subset \mathfrak{m},$$

is then called the locally largest filled-in exploration in $\mathfrak{m}$. Notice that when $\mathfrak{m}$ has a distinguished target we generally do NOT have $(\bar{\mathfrak{e}}_n)_{n\geq 0} = (\bar{\bar{\mathfrak{e}}}_n)_{n\geq 0}$.

3.2 Law of the peeling under the Boltzmann measures

We suppose in this section that $\mathbf{q}$ is an admissible weight sequence so that we can speak of $\mathbf{q}$-Boltzmann planar maps (possibly pointed). When dealing with random planar maps, we work on the canonical space Ω of all the (rooted bipartite, possibly infinite) random maps with holes, possibly pointed and possibly given with a distinguished hole. This space is equipped with the Borel σ-field for the (extended) local topology. The notation

$$\mathbb{P}^{(\ell)}, \mathbb{E}^{(\ell)}, \quad \text{resp. } \mathbb{P}^{(\ell)}_{\bullet}, \mathbb{E}^{(\ell)}_{\bullet}$$

is used for the probability and expectation on Ω relative to the law of a $\mathbf{q}$-Boltzmann map with perimeter 2ℓ, resp. a pointed $\mathbf{q}$-Boltzmann map with perimeter 2ℓ. A generic element of the canonical space will be either denoted by $\mathfrak{m}$ or by $\mathfrak{m}_{\bullet} = (\mathfrak{m}, v)$ if it is pointed. In this section, we fix a peeling algorithm $\mathcal{A}$ and first compute the law of the peeling exploration $\mathfrak{e}_0 \subset \mathfrak{e}_1 \subset \cdots \subset \mathfrak{m}$ under $\mathbb{P}^{(\ell)}$ and $\mathbb{P}^{(\ell)}_{\bullet}$. We denote by $\mathcal{F}_n$ the σ-algebra on Ω generated by the exploration (i.e. the functions $(\mathfrak{e}_i)_{0\le i\le n}$ or $(\mathfrak{e}^*_i)_{0\le i\le n}$ or $(\bar{\mathfrak{e}}_i)_{0\le i\le n}$ or $(\bar{\bar{\mathfrak{e}}}_i)_{0\le i\le n}$ depending on the type of exploration we consider).

3.2.1 q-Boltzmann maps

Let us first define the following probability transitions:

Definition 12 (Transition probabilities in the Boltzmann case)**.** *For any $\ell \ge 1, k \ge 0$ and $k_1, k_2 \ge 0$ such that $k_1 + k_2 + 1 = \ell$ we put*

$$b^{(\ell)}(k) = q_k \frac{W^{(\ell+k-1)}}{W^{(\ell)}}, \qquad b^{(\ell)}(k_1, k_2) = \frac{W^{(k_1)}W^{(k_2)}}{W^{(\ell)}}.$$

The fact that $b^{(\ell)}$ defines probability transitions, that is for all $\ell \ge 1$

$$1 = \sum_{k\ge 0} b^{(\ell)}(k) + \sum_{k_1+k_2+1=\ell} b^{(\ell)}(k_1, k_2)$$

is equivalent to Tutte's equation (2.9).

Proposition 7 (Law of the peeling process under the Boltzmann distribution)**.** *Fix $\ell \ge 1$ and a peeling algorithm $\mathcal{A}$. Then under $\mathbb{P}^{(\ell)}$ the peeling exploration $\mathfrak{e}_0 \subset \mathfrak{e}_1 \subset \cdots \subset \mathfrak{e}_n \subset \cdots \subset \mathfrak{m}$ with algorithm $\mathcal{A}$ is a Markov chain whose probability transitions are as follows: Conditionally on $\mathcal{F}_n$ and provided that $\mathcal{A}(\mathfrak{e}_n) \neq \dagger$ (which belongs to $\mathcal{F}_n$), if we denote by $\mathcal{L}_n$ the half-perimeter of the hole on which $\mathcal{A}(\mathfrak{e}_n)$ is selected then the events C_k and G_{k_1,k_2} (where $k \ge 1$ and $k_1 + k_2 + 1 = \mathcal{L}_n$ with $k_1, k_2 \ge 0$) occur respectively with probabilities*

$$b^{(\mathcal{L}_n)}(k) \quad \text{and} \quad b^{(\mathcal{L}_n)}(k_1, k_2).$$

Proof. Let us first consider the first step of the peeling process assuming that $\mathcal{A}(\mathfrak{e}_0) \neq \dagger$. By the rigidity of the gluing operation, the event C_k happens if and only if the map $\mathfrak{m}$ is obtained from the gluing of a map of perimeter $2\ell + 2k - 2$ onto the map $\mathfrak{e}_0$ to which we glued a face of degree $2k$ on $\mathcal{A}(\mathfrak{e}_0)$, hence we have

$$\mathbb{P}^{(\ell)}(\mathsf{C}_k \mid \mathcal{F}_0) = \frac{1}{W^{(\ell)}} \cdot q_k \sum_{\mathfrak{m}_1 \in \mathcal{M}^{(\ell+k-1)}} w(\mathfrak{m}_1) = q_k \frac{W^{(\ell+k-1)}}{W^{(\ell)}} = b^{(\ell)}(k).$$

Furthermore, the above calculation shows that conditionally on the above event, the map $\mathfrak{m}_1$ filling-in the hole of $\mathfrak{e}_1$ is distributed according to $\mathbb{P}^{(\ell+k-1)}$. Similarly and again by rigidity, the event G_{k_1,k_2} where $k_1+k_2+1=\ell$ happens if and only if the map $\mathfrak{m}$ is obtained by first identifying the edge $\mathcal{A}(\mathfrak{e}_0)$ with the edge of the same hole located $2k_1$ steps on its left and then gluing two maps of respective perimeters $2k_1$ and $2k_2$ into the two holes created (recall that when $k_i=0$ then we just glue a vertex in). The same calculation shows that

$$\mathbb{P}^{(\ell)}(\mathsf{G}_{k_1,k_2} \mid \mathcal{F}_0) = \frac{1}{W^{(\ell)}} \sum_{\mathfrak{m}_1 \in \mathcal{M}^{(k_1)}} w(\mathfrak{m}_1) \sum_{\mathfrak{m}_2 \in \mathcal{M}^{(k_2)}} w(\mathfrak{m}_2) = b^{(\ell)}(k_1,k_2).$$

Again, conditionally on the above event, an easy extension of the last calculation shows that the maps $\mathfrak{m}_1$ and $\mathfrak{m}_2$ filling-in the two holes of $\mathfrak{e}_1$ are independent and respectively distributed according to $\mathbb{P}^{(k_1)}$ and $\mathbb{P}^{(k_2)}$. This proves that the transitions are correct for the first step of the chain. But this calculation easily propagates to later steps of the chain because we have seen that after the first step (provided that the peeling has not stopped) the holes of $\mathfrak{e}_1$ are filled-in with independent Boltzmann planar maps having the proper perimeter. □

From the last proposition we easily deduce that under $\mathbb{P}^{(\ell)}$ the locally largest filled-in exploration process $\bar{\bar{\mathfrak{e}}}_0 \subset \cdots \subset \bar{\bar{\mathfrak{e}}}_n \subset \cdots \subset \mathfrak{m}$ is also a Markov chain: we first use the above transition probabilities and then fill-in the smallest of the two holes (if there are two) with a $\mathbf{q}$-Boltzmann planar map of the proper perimeter sampled independently of the past exploration.

3.2.2 pointed $\mathbf{q}$-Boltzmann maps

We now proceed to similar calculations in the case of pointed Boltzmann planar maps where we use a peeling exploration with target. Recall the definition of the events C_k and G_{k_1,k_2}. In the case of the peeling of a distinguished hole of $\mathfrak{e}^*_n$ we shall write G_{*,k_2} for the event G_{k_1,k_2} where the hole on the left of the peeled edge becomes the next distinguished hole of $\mathfrak{e}^*_{n+1}$ and similarly for $\mathsf{G}_{k_1,*}$. Notice that if the perimeter of the hole on which we peel is ℓ then on the events $\mathsf{G}_{*,\ell-1}$ and $\mathsf{G}_{\ell-1,*}$ the new distinguished hole becomes of perimeter 0: it's a vertex of the underlying map.

Definition 13 (Transitions probabilities for the distinguished hole in the pointed Boltzmann case)**.** *For any $\ell \geq 1, k \geq 0$ and $k_1, k_2 \geq 0$ such that $k_1+k_2+1=\ell$ we put*

$$b_\bullet^{(\ell)}(k) = q_k \frac{W_\bullet^{(\ell+k-1)}}{W_\bullet^{(\ell)}}, \qquad b_\bullet^{(\ell)}(*,k_2) = \frac{W_\bullet^{(k_1)} W^{(k_2)}}{W_\bullet^{(\ell)}}, \qquad \text{and } b_\bullet^{(\ell)}(k_1,*) = \frac{W^{(k_1)} W_\bullet^{(k_2)}}{W_\bullet^{(\ell)}}.$$

Again the fact that $b_\bullet^{(\ell)}$ indeed define probability transitions for all $\ell \geq 1$ is equivalent to Tutte's equation in the pointed case (2.10).

Proposition 8 (Law of the peeling process with target under the pointed Boltzmann distribution)**.** *Fix $\ell \geq 1$ and a peeling algorithm $\mathcal{A}$. Then under $\mathbb{P}_\bullet^{(\ell)}$ the peeling exploration with target $\mathfrak{e}^*_0 \subset \mathfrak{e}^*_1 \subset$*

$\cdots \subset \mathfrak{e}_n^* \subset \cdots \subset \mathfrak{m}$ is a Markov chain whose probability transitions are as follows. Conditionally on $\mathcal{F}_n$ and provided that $\mathcal{A}(\mathfrak{e}_n^*) \neq \dagger$ we denote by $\mathcal{L}_n$ the half-perimeter of the hole on which $\mathcal{A}(\mathfrak{e}_n^*)$ is selected. On the event where $\mathcal{A}(\mathfrak{e}_n^*)$ belongs to the distinguished hole of $\mathfrak{e}_n^*$ the events C_k, $\mathsf{G}_{k,*}$ and $\mathsf{G}_{*,k}$ (where $0 \le k \le \mathcal{L}_n - 1$) occur respectively with probabilities

$$b_\bullet^{(\mathcal{L}_n)}(k), \qquad b_\bullet^{(\mathcal{L}_n)}(k,*) \quad \text{and} \quad b_\bullet^{(\mathcal{L}_n)}(*,k).$$

Otherwise if $\mathcal{A}(\mathfrak{e}_n^*)$ is not on the distinguished hole of $\mathfrak{e}_n^*$ then the probability transitions are those described in Proposition 7.

Proof. This is mutatis mutandis the same proof as for Proposition 7 and one just needs to check the appearance of the transitions $b_\bullet^{(\ell)}$ for the first step of the Markov chain starting from the distinguished hole of $\mathfrak{e}_0^*$ and to establish the fact that the (possibly) two holes of $\mathfrak{e}_1^*$ are filled-in with independent maps, the one in the distinguished hole being a pointed **q**-Botlzmann map and the one in the non-distinguished hole being a **q**-Boltzmann map with the proper perimeter. □

From the last proposition we easily deduce that under $\mathbb{P}_\bullet^{(\ell)}$ the filled-in exploration process with target $\bar{\mathfrak{e}}_0 \subset \cdots \bar{\mathfrak{e}}_n \subset \cdots \subset \mathfrak{m}$, is also a Markov chain: we first use the transitions probabilities $b_\bullet^{(\cdot)}$ to see how to make the distinguished hole evolve and then fill-in the non-distinguished hole (if there is a splitting event) with a **q**-Boltzmann planar map of the proper perimeter sampled independently from the past exploration. For later uses, remark that the chain $|\partial\bar{\mathfrak{e}}_n|$ made by the half-perimeter of the hole of $\bar{\mathfrak{e}}_n$ is itself a Markov chain whose probability transitions are expressed as

$$\mathbb{P}_\bullet^{(\ell)}(|\partial\bar{\mathfrak{e}}_{n+1}| = m + k \mid |\partial\bar{\mathfrak{e}}_n| = m) = b_\bullet^{(m)}(k+1)\mathbf{1}_{k\ge 0} + 2b_\bullet^{(m)}(*,-k-1)\mathbf{1}_{k\le -1}. \tag{3.1}$$

3.3 A random walk

In this section we study the peeling process under $\mathbb{P}^{(\ell)}$ and $\mathbb{P}_\bullet^{(\ell)}$ as $\ell \to \infty$. This makes a limit random walk appear which will turn out to be the key to many results on the peeling process.

3.3.1 The step distribution ν

We fix a peeling algorithm $\mathcal{A}$ and make it run on a Boltzmann planar map with a very large boundary. Recall from Propositions 7 and 8 the probability transitions of the peeling process once an edge has been selected on the boundary of a hole containing either a Boltzmann map or a pointed Boltzmann map. Using Theorem 6 in the pointed case and (2.11) in the non-pointed case, as $\ell \to \infty$, we see that these transitions admit a limit:

$$\lim_{\ell\to\infty} b^{(\ell)}(k) = \lim_{\ell\to\infty} b_\bullet^{(\ell)}(k) \quad = \quad q_k\, c_{\mathbf{q}}^{k-1} \tag{3.2}$$

and

$$\begin{aligned}\lim_{\ell\to\infty} b^{(\ell)}(k,\ell-k-1) = \lim_{\ell\to\infty} b_\bullet^{(\ell)}(k,*) \quad &= \quad W^{(k)}\, c_{\mathbf{q}}^{-k-1}, \\ \lim_{\ell\to\infty} b^{(\ell)}(\ell-k-1,k) = \lim_{\ell\to\infty} b_\bullet^{(\ell)}(*,k) \quad &= \quad W^{(k)}\, c_{\mathbf{q}}^{-k-1}.\end{aligned} \tag{3.3}$$

This leads us to introduce the two related objects:

Definition 14 (Transitions probabilities for infinite holes)**.** *For any* $\ell \geq 1, k \geq 0$ *we put*

$$b^{(\infty)}(k) = q_k c_{\mathbf{q}}^{k-1}, \qquad b^{(\infty)}(\infty, k) = b^{(\infty)}(k, \infty) = W^{(k)} c_{\mathbf{q}}^{-k-1}.$$

Definition 15. *Let* ν *be the measure on* $\mathbb{Z}$ *defined by*

$$\nu(k) = \begin{cases} q_{k+1} c_{\mathbf{q}}^{k} & \text{for } k \geq 0 \\ 2W^{(-1-k)} c_{\mathbf{q}}^{k} & \text{for } k \leq -1, \end{cases}$$

or equivalently

$$\nu(k) = \begin{cases} b^{(\infty)}(k+1) & \text{for } k \geq 0 \\ b^{(\infty)}(-k-1, \infty) + b^{(\infty)}(\infty, -k-1) & \text{for } k \leq -1. \end{cases}$$

A priori, these two measures are sub-probability measures as limit of probability measures. By the one-step peeling transitions of the peeling with target in the pointed Boltzmann case (Eq 3.1) and Theorem 6 we deduce that the half-perimeter of the distinguished hole (when the edge selected is on that hole) in a filled-in exploration evolves as a Markov chain whose probability transitions can be rewritten as

$$p(m, m+k) = \nu(k) \frac{h^{\downarrow}(m+k)}{h^{\downarrow}(m)}, \quad k \in \mathbb{Z} \tag{3.4}$$

where we recall from Definition 7 that $h^{\downarrow}(\ell) = 2^{-2\ell}\binom{2\ell}{\ell}$ if $\ell \geq 0$ and $h^{\downarrow}(\ell) = 0$ if $\ell < 0$. We will use this to prove:

Lemma 9. *The measure* ν *is a probability measure (and* $b^{(\infty)}$ *defines probability transitions).*

Proof. So far we can only deduce (by Fatou) that ν is of mass less than or equal to 1 since it is a limit of transition probabilities. From (3.4) we deduce that for any $\ell \geq 1$

$$h^{\downarrow}(\ell) = \sum_{k \in \mathbb{Z}} \nu(k)\, h^{\downarrow}(\ell + k). \tag{3.5}$$

Summing-up these equalities for $\ell = 1, \ldots, n$ and using Definition 7 we deduce that

$$\left(h^{\uparrow}(n+1) - h^{\uparrow}(1)\right) = \sum_{k \in \mathbb{Z}} \nu(k) \left(h^{\uparrow}(n+k+1) - h^{\uparrow}(k+1)\right)$$

and so

$$1 = \sum_{k \in \mathbb{Z}} \nu(k) \frac{h^{\uparrow}(n+k+1) - h^{\uparrow}(k+1)}{(h^{\uparrow}(n+1) - h^{\uparrow}(1))}.$$

Now, using the form of the function $h^{\uparrow}$ we see that the ratio involving $h^{\uparrow}$ in the right display is bounded by some constant $C > 0$ independent of n and k and goes to 1 for fixed k as $n \to \infty$. We can then apply the bounded convergence theorem and deduce that $\sum_{k \in \mathbb{Z}} \nu(k) = 1$, and so ν is a probability measure. □

3.3.2 $h^{\downarrow}$-transform

In the rest of these lecture notes we shall denote by $S_n = X_1 + \cdots + X_n$ a random walk with independent increments distributed according to ν. We assume that under the probability $\mathbf{P}_\ell$ this walk starts from $\ell \geq 0$. It follows from (3.5) that the function $h^{\downarrow}$ is harmonic at the points $\{1, 2, \ldots\}$ for the random walk (S). Notice that $h^{\downarrow}(0) = 1$. Furthermore, the form of the transition (3.4) shows that the half-perimeter of the unique hole in a filled-in exploration under $\mathbb{P}_{\bullet}^{(\ell)}$ is obtained as the Doob $h^{\downarrow}$-transform of the Markov chain (S) started from ℓ and killed at first entrance in $\mathbb{Z}_{\leq 0} = \{\ldots, -2, -1, 0\}$ for the harmonic function $h^{\downarrow}$.

Reminder on h-transform of Markov chains. Recall that if $p(x, y)$ are probability transitions for a discrete Markov chain and if h is a non-negative harmonic function for the chain then

$$q(x, y) = \frac{h(y)}{h(x)} p(x, y),$$

defines new probability transitions of a Markov chain called the Doob h-transform of p which is defined for every x, y for which $h(x) > 0$ and $h(y) > 0$. It is assumed that the Markov chain starts from a value x_0 for which $h(x_0) > 0$ and then the form of the transitions shows that it will never reach a value y for which $h(y) = 0$. It is easy to see that the probability of any given path $x_0, x_1, \ldots, x_n$ with $h(x_i) > 0$ for the chain X with transitions p is transformed for the chain Y with transitions q into

$$\mathbb{P}(Y_0 = x_0, \cdots, Y_n = x_n) = \frac{h(x_n)}{h(x_0)} \mathbb{P}(X_0 = x_0, \cdots, X_n = x_n). \tag{3.6}$$

In our case, the underlying Markov chain is the random walk S killed when entering $\mathbb{Z}_{\leq 0}$. We sometimes make an abuse of terminology and speak of the $h^{\downarrow}$-transform of (S) instead of saying the $h^{\downarrow}$-transform of (S) killed when entering $\mathbb{Z}_{\leq 0}$. We assume that under $\mathbf{P}_\ell$ the process $S^{\downarrow}$ is started from ℓ and has the law of the $h^{\downarrow}$-transform of S. In our case, this process has an elegant probabilistic interpretation:

Proposition 10. *Under $\mathbf{P}_\ell$ with $\ell \geq 1$, the Markov chain $(S^{\downarrow})$ has the law of the random walk (S) conditioned on first hitting in a finite time 0 before $\mathbb{Z}_{\leq 0}$ and killed at 0, in particular*

$$h^{\downarrow}(\ell) = \mathbf{P}_\ell(\tau_{\{0\}} = \tau_{\mathbb{Z}_{\leq 0}} < \infty).$$

Proof. If $A \subset \mathbb{Z}$ let τ_A be the first time the random walk (S) started from ℓ enters $A \subset \mathbb{Z}$ and let us compute first $\mathbf{P}_\ell(\tau_{\{0\}} \leq \tau_{\mathbb{Z}_{\leq 0}})$ where $\mathbb{Z}_{\leq 0} = \{\ldots, -2, -1, 0\}$. Since $h^{\downarrow}(S_{n \wedge \tau_{\mathbb{Z}_{\leq 0}}})$ is a bounded martingale (recall that $h^{\downarrow}(k)$ decreases like $1/\sqrt{k}$ when $k \to \infty$) we have by the optional sampling theorem:

$$h^{\downarrow}(\ell) = \mathbf{E}_\ell\left[h^{\downarrow}(S_{n \wedge \tau_{\mathbb{Z}_{\leq 0}}})\right] \quad \xrightarrow[n \to \infty]{} \quad \underbrace{h^{\downarrow}(0)}_{=1} \cdot \mathbf{P}_\ell(\tau_{\{0\}} = \tau_{\mathbb{Z}_{\leq 0}} < \infty).$$

Now Equation (3.6) shows that for any possible particular trajectory Traj starting from ℓ and ending at 0 without touching $\mathbb{Z}_{\le 0}\backslash\{0\}$ we have

$$\mathbf{E}_\ell\left[\left(S_n^\downarrow\right)_{0\le n\le \tau_{\{0\}}} = \mathsf{Traj}\right] = \frac{h^\downarrow(0)}{h^\downarrow(\ell)}\cdot \mathbf{E}_\ell\left[(S_n)_{0\le n\le \tau_{\{0\}}} = \mathsf{Traj}\right],$$

and the conclusion follows from the two displays above. □

3.3.3 $h^\uparrow$-transform

The function $h^\uparrow$ (which is the discrete primitive of $h^\downarrow$) will also play a role in connection with the random walk (S). First, the criticality condition of Definition 9 can be restated in terms of harmonicity of the function $h^\uparrow$.

Lemma 11. *If* $\mathbf{q}$ *is an admissible weight sequence then* $h^\uparrow$ *is super harmonic on* $\mathbb{N}^* = \{1,2,3,\dots\}$ *for the walk* (S). *Furthermore* $\mathbf{q}$ *is critical if and only if* $h^\uparrow$ *is harmonic at those points.*

Proof. Summing equations (2.12) and $Z_\mathbf{q}\times$(2.13) (with a $\le$ instead of $=$ in the general case), using $c_\mathbf{q} = 4Z_\mathbf{q}$ and the definitions of $h^\uparrow$ and of ν, we immediately get after a few easy manipulations that

$$\sum_{k\ge 0}\nu(k)h^\uparrow(k+1)\le 1 = h^\uparrow(1),$$

and with an equality when $\mathbf{q}$ is critical. This is the desired (super)harmonicity of the function $h^\uparrow$ at the point 1. To transfert it to the others points we use the harmonicity of $h^\downarrow$ at point 1:

$$\sum_{k\in\mathbb{Z}}\nu(k)h^\downarrow(k+1) = h^\downarrow(1).$$

Summing the last two displays and using $h^\uparrow(\ell+1)-h^\uparrow(\ell) = h^\downarrow(\ell)$ yields

$$\sum_{k\in\mathbb{Z}}\nu(k)h^\uparrow(k+2)\le h^\uparrow(2),$$

and with equality in the case of a critical weight sequence. This proves (super-)harmonicity of $h^\uparrow$ at point 2. We then iterate to get the same statement at points $\{3,4,\dots\}$. □

Proposition 12. *Coming back to Proposition 10, one can remove the assumption that* $\tau_{\{0\}}$ *is finite since we have* $\tau_{\mathbb{Z}_{\le 0}} < \infty$ *under* $\mathbf{P}_\ell$ *almost surely for any* $\ell\ge 1$.

Proof. By Lemma 11, since the function $h^\uparrow$ is super-harmonic, under $\mathbf{P}_\ell$ the process $h^\uparrow(S_n)$ is a super-martingale. Applying the optimal theorem we have

$$h^\uparrow(\ell)\ge \mathbf{E}_\ell[h^\uparrow(S_{n\wedge\tau_{\mathbb{Z}_{\le 0}}})] = 0\times\mathbf{P}_\ell(\tau_{\mathbb{Z}_{\le 0}}\le n) + \mathbf{E}_\ell[h^\uparrow(S_n)\mathbf{1}_{\tau_{\mathbb{Z}_{\le 0}}>n}].$$

It is easy to see that if the event $\{\tau_{\mathbb{Z}_{\le 0}} = \infty\}$ has positive probability, conditionally on it we must have $S_n\to\infty$. Since $h^\uparrow(n)\to\infty$ as $n\to\infty$, we would reach a contradiction in the last display. Hence $\mathbf{P}_\ell(\tau_{\mathbb{Z}_{\le 0}} = \infty) = 0$ as desired. □

When $h^\uparrow$ is harmonic (on $\mathbb{Z}_{>0}$) one can define the Markov chain $(S^\uparrow)$ via Doob $h^\uparrow$-transform of the walk (S) killed when entering $\mathbb{Z}_{\leq 0}$. We will assume that under the probability measure $\mathbf{P}_\ell$ this chain starts from $\ell \geq 1$. Notice a slight but important difference between $h^\downarrow$ and $h^\uparrow$: although both harmonic for the walk on $\{1, 2, 3, \ldots\}$ we have

$$h^\uparrow(0) = 0 \quad \text{whereas} \quad h^\downarrow(0) = 1.$$

This changes the behavior of those h-transform: on the one hand $S^\downarrow$ we eventually hit 0, whereas on the other hand $S^\uparrow$ will always stay positive. In particular it follows from [38] that for a random walk killed at first entrance of $\mathbb{Z}_{\leq 0}$ there is a unique (up to multiplicative constant) harmonic function on $\mathbb{N}^*$ which is null on $\mathbb{Z}_{\leq 0}$ and we know from [15] that the process $(S^\uparrow)$ can be interpreted as the limit in distribution of the random walk (S) conditioned on staying positive up to time n as $n \to \infty$.

Proposition 13 (Transience of the $h^\uparrow$-transform)**.** *The Markov chain $(S_i^\uparrow)_{i\geq 0}$ is transient.*

Proof. Let us consider the Markov chain $S^\uparrow$ started from ℓ and consider the first time $\tau_{<\ell}$ it reaches a value strictly lower than ℓ. By above properties of the h-transform we can write

$$\mathbf{P}_\ell(\tau_{<\ell}(S^\uparrow) < \infty) = \frac{1}{h^\uparrow(\ell)} \mathbf{E}_\ell[h^\uparrow\left(S_{\tau_{<\ell}}\right)\mathbf{1}_{\tau_\ell < \infty}] \leq \frac{\sup_{\ell' < \ell} h^\uparrow(\ell')}{h^\uparrow(\ell)} < 1.$$

It follows easily that $S^\uparrow$ is transient. □

3.3.4 Rough behavior of (S)

Recall that a random walk (X) with independent increments is said to drift to infinity if $\lim_{n\to\infty} X_n \to +\infty$ almost surely, drifts to $-\infty$ if $\lim_{n\to\infty} X_n \to -\infty$. Otherwise the walk must oscillate meaning that $\limsup_{n\to\infty} X_n = +\infty$ and $\liminf_{n\to\infty} X_n = -\infty$. Then we have:

Proposition 14. *Let $\mathbf{q}$ be an admissible subcritical weight sequence, then the walk (S) drifts towards $-\infty$. If $\mathbf{q}$ is critical then (S) oscillates (in particular if ν has a first moment i.e. $\sum_{\mathbb{Z}} \nu(k)|k| < \infty$ then necessarily ν is centered i.e. $\sum_{\mathbb{Z}} \nu(k)k = 0$).*

Proof. We denote by $(H_i)_{i\geq 0}$ the strict increasing ladder heights of the random walk $-S$, that is $H_0 = 0$ and $-H_{i+1}$ is the first value that $-S$ takes strictly below $-H_i$ for all $i \geq 0$ (with the convention that $H_{i+1} = \infty$ if there is no such value). Then using Proposition 10 we see that

$$h^\downarrow(\ell) = \sum_{i\geq 0} \mathbf{P}_0(H_i = \ell),$$

and so $V(\ell) = \sum_{i\geq 0} \mathbf{P}_0(H_i \leq \ell)$ is equal to $h^\uparrow(\ell + 1)$. The function $V(\ell)$ is known as the *renewal function* of the walk. In particular it is known (see in particular [**?**, Appendix B] and also [15, Section 2]) that

- if S drifts to ∞ then V is bounded (which is not the case here)

- if S drifts to $-\infty$ then V is superharmonic (and not harmonic) on $\{0, 1, 2, \ldots\}$
- if S oscillates then V is harmonic on $\{0, 1, 2, \ldots\}$.

The proposition follows from these results. □

Bibliographical notes. The peeling process was first used in the physics literature by Watabiki [60] without a precise justification. A rigorous version of the peeling process and its Markovian properties was given by Angel [4] in the case of the Uniform Infinite Planar Triangulation (UIPT). The peeling process used by Angel consists roughly speaking in discovering one face at a time. It is well designed to study planar maps with a degree constraint on the faces (such as triangulations or quadrangulations). The peeling process we consider here and which was recently introduced in [23] is different: it discovers one *edge* at a time. The advantage of this "edge-peeling" process over the "face-peeling" process is that it can be treated in a unified fashion for all models of Boltzmann planar maps. The presentation of the beginning of the section is adapted from [13]. The role of an underlying random walk in the peeling process is already present in the works of Angel. In [35] the connection with h-transform was made explicit. The results of Section 3.3 with the above peeling process and in this generality are almost all due to Budd [23].

Chapter IV : Infinite Boltzmann maps

In this chapter we introduce two random infinite planar maps of the plane and of the half-plane (in the sense of Definition 4) which are obtained as limits of conditioned Boltzmann maps. These objects are keys in the theory since the peeling process takes a particularly simple form on them.

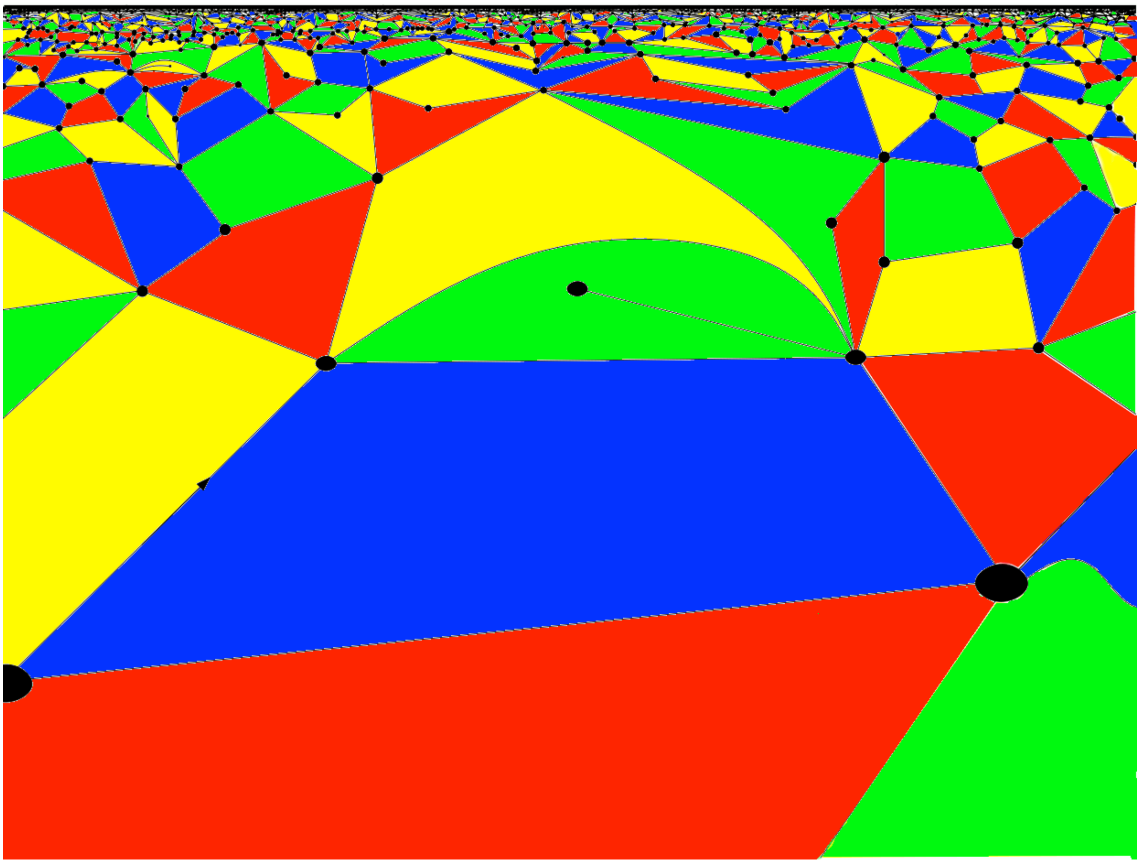


Figure 4.1: An artistic representation of the UIPQ, which is a random map distributed according to the measure $\mathbb{P}_\infty^{(1)}$ when $\mathbf{q} = (12^{-1}\delta_{k=2})$ corresponding to the model of critical random quadrangulations.

4.1 The half-planar Boltzmann map

In this section we suppose that $\mathbf{q}$ is an admissible weight sequence. We study the geometry of random maps sampled according to $\mathbb{P}^{(\ell)}$ and prove that they converge as $\ell \to \infty$ towards a limiting infinite random map with one end and with an infinite (non simple) boundary. Here is the main result of the section:

Theorem 7 (*Half-planar Boltzmann map*)

Let $\mathbf{q}$ be an admissible weight sequence, then we have the following convergence in distribution for the local topology

$$\mathbb{P}^{(\ell)} \xrightarrow[\ell\to\infty]{(d)} \mathbb{P}^{(\infty)}$$

where $\mathbb{P}^{(\infty)}$ is a distribution supported by infinite bipartite planar maps of the half-plane (see Definition 4) that we call the half-planar $\mathbf{q}$-Boltzmann distribution.

Remark 9 (UIHPQ). When $\mathbf{q} = (12^{-1}\delta_2(k))_{k\geq 1}$ corresponds to critical Boltzmann quadrangulations, the measure $\mathbb{P}^{(\infty)}$ is the law of the uniform infinite half-planar quadrangulation (UIHPQ) with a general boundary already considered in [37, 25].

4.1.1 Convergence of the peeling processes

Our main tool is the convergence of the probability transitions proved in Section 3.3. The reader should be convinced that this should imply the convergence of the respective peeling processes under $\mathbb{P}^{(\ell)}$. However this is false for a trivial/obvious reason: under $\mathbb{P}^{(\ell)}$ the peeling process starts with $\mathfrak{e}_0$ which is made of the root face (seen as a map with one hole) of perimeter 2ℓ and this cannot converge as $\ell \to \infty$. To cope with this problem we introduce a slight variation of the definition of submaps.

Let $\mathfrak{e} \subset \mathfrak{m}$ be a submap of $\mathfrak{m} \in \mathcal{M}^{(\ell)}$ where we think of ℓ as large. We suppose that the largest hole of $\mathfrak{e}$ is of perimeter roughly 2ℓ and that it shares a connected component of its boundary of length roughly 2ℓ with the external face. We then introduce $\tilde{\mathfrak{e}}$ the map with holes obtained from $\mathfrak{e}$ after transforming both the giant hole of $\mathfrak{e}$ and its external face into infinite degree faces. The result is then a map with a unique infinite hole and an external face of infinite degree, see Fig. 4.2. When peeling an edge on such a map we can still speak of the events C_k and when we peel an edge on the infinite hole the events G_{k_1,k_2} now become events $\mathsf{G}_{\infty,k}$ or $\mathsf{G}_{k,\infty}$ (we identify the current peeled edge with an edge located $2k+1$ on the left for the event $\mathsf{G}_{k,\infty}$ or on the right for the event $\mathsf{G}_{\infty,k}$ thus creating a new hole of perimeter $2k$).

Recall that a peeling algorithm is a function $\mathcal{A}$ which associates to any map with holes $\mathfrak{e}$ an edge $\mathcal{A}(\mathfrak{e})$ on its active boundary $\mathsf{Active}(\mathfrak{e})$. We suppose in what follows that we fixed an algorithm $\mathcal{A}$ that is "local" in the sense that it does not depend on ℓ. More precisely, we fix $\mathcal{A}$ such that $\mathcal{A}(\mathfrak{e})$ only depends on $\tilde{\mathfrak{e}}$ (for example the algorithm that first peels the edge at distance ℓ from the root edge on $\mathfrak{e}_0$ is not permitted). Once this definition is understood the following should be clear:

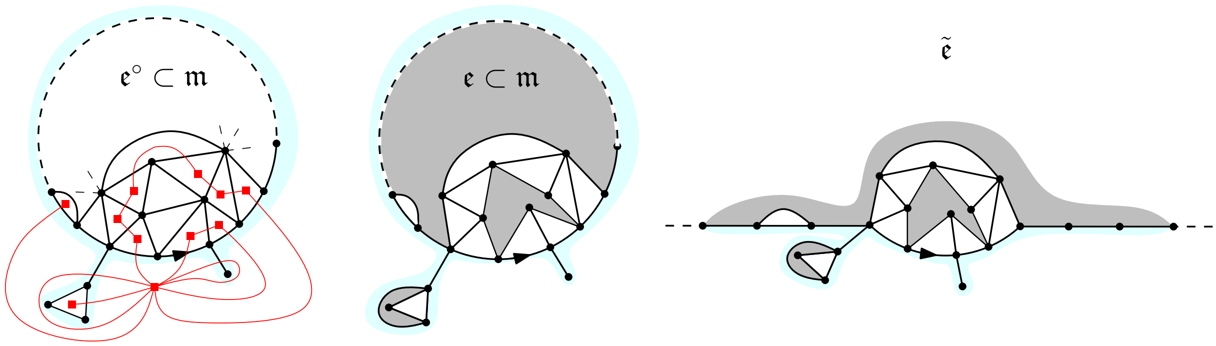


Figure 4.2: Illustration of the definition of $\tilde{\mathfrak{e}}$ (in the non bipartite case). The holes are displayed in gray whereas the external face is in light blue.

Proposition 15. *Consider the peeling process* $(\mathfrak{e}_n)_{n\geq 0}$ *with a peeling algorithm* $\mathcal{A}$ *satisfying the above condition. Then we have the following convergence under* $\mathbb{P}^{(\ell)}$*:*

$$(\tilde{\mathfrak{e}}_n)_{n\geq 0} \quad \xrightarrow[\ell\to\infty]{(d)} \quad (\tilde{\mathfrak{e}}_n)_{n\geq 0},$$

where $(\tilde{\mathfrak{e}}_n)_{n\geq 0}$ *is a Markov chain whose probability transitions are as follows. Conditionally on* $\{\tilde{\mathfrak{e}}_i : 0 \leq i \leq n\}$ *and provided that* $\mathcal{A}(\tilde{\mathfrak{e}}_n) \neq \dagger$*, we denote by* $\mathcal{L}_n \in \{1,2,3,\dots\} \cup \{\infty\}$ *the half-perimeter of the hole on which* $\mathcal{A}(\tilde{\mathfrak{e}}_n)$ *is located. If* $\mathcal{L}_n < \infty$ *then the events* C_k*,* G_{k_1,k_2} *occur with the transitions described in Proposition 7. Otherwise if* $\mathcal{L}_n = \infty$ *the events* C_k*,* $\mathsf{G}_{\infty,k}$ *and* $\mathsf{G}_{k,\infty}$ *occur respectively with probability (see Definition 14)*

$$b^{(\infty)}(k), \quad b^{(\infty)}(\infty,k) \quad \text{and} \quad b^{(\infty)}(k,\infty).$$

Proof. This is merely a consequence of Section 3.3. More precisely, for any $n_0 \geq 0$ the sequence $(\mathfrak{e}_i)_{0\leq i\leq n_0}$ is described by the information of the events of type $\mathsf{C}.$ and $\mathsf{G}.$ together with the initial condition $\mathfrak{e}_0$ which depends on the value of $\ell \geq 1$. The point is that, under our assumption on $\mathcal{A}$, the sequence $(\tilde{\mathfrak{e}}_i)_{0\leq i\leq n_0}$ can be described by the information of the appearance of the events $\mathsf{C}.$ or $\mathsf{G}.$ only, without requiring the value of ℓ, at least as long as the largest hole in the first n_0 steps remains a macroscopic hole of perimeter roughly 2ℓ sharing a huge boundary with the external face. Since the last event occurs with a probability tending to one, the result of the proposition is implied by the convergence of the transition probabilities $b^{(\ell)}$ towards the transitions $b^{(\infty)}$ established in Section 3.3. □

4.1.2 Proof of Theorem 7: a first peeling algorithm

In order to prove Theorem 7 we will apply Proposition 15 with a particular peeling algorithm which explores step after step the local structure around the origin in a map. The most obvious of such algorithms is the following:

Algorithm $\mathcal{A}_{\text{metric}}$: If $\mathfrak{e} \subset \mathfrak{m}$ we associate $\mathcal{A}_{\text{metric}}(\mathfrak{e})$ the edge immediately on the left of the vertex x which is the closet to the origin of the map among all vertices of the active boundary $\mathsf{Active}(\mathfrak{e})$. If there are several choices for x decide between them using a deterministic procedure fixed *a priori*.

A dubious point in the definition of the algorithm $\mathcal{A}_{\text{metric}}$ is that we have not made precise whether x has minimal distance to ρ for the graph distance in $\mathfrak{m}$ or in the submap $\mathfrak{e}$. Actually, if we want $\mathcal{A}_{\text{metric}}(\mathfrak{e})$ to depend on $\mathfrak{e}$ only, the second choice is the only reasonable one. But it turns out the distance in $\mathfrak{e}_n$ are related to those of $\mathfrak{m}$:

Lemma 16. *If $\mathfrak{e}_0 \subset \mathfrak{e}_1 \subset \cdots$ is the peeling of the map $\mathfrak{m}$ with algorithm $\mathcal{A}_{\text{metric}}$ and if τ_r is the first time $n \geq 0$ such that $\min\{\mathrm{d}_{\mathrm{gr}}^{\mathfrak{e}_n}(x,\rho) : x \in \mathsf{Active}(\mathfrak{e}_n)\} \geq r$ then recalling Definition 10 we have*

$$\mathfrak{e}_{\tau_r} = \mathrm{Ball}_r(\mathfrak{m}).$$

Proof. Let us prove the lemma by induction on r. For $r = 0$ the statement is clear thanks to our definition of $\mathrm{Ball}_0(\mathfrak{m})$. Suppose now that $\mathfrak{e}_{\tau_r} = \mathrm{Ball}_r(\mathfrak{m})$. Let us measure the distances within $\mathfrak{e}_{\tau_r}$ of the vertices of the active boundary of $\mathfrak{e}_{\tau_r}$ to the origin of the map. These distances are by definition larger than $r \geq 0$ but it is easy to see that there are some vertices at distance exactly r. Using the fact that $\mathfrak{e}_{\tau_r} = \mathrm{Ball}_r(\mathfrak{m})$ we deduce that those vertices are also at distance r from the origin within the full map $\mathfrak{m}$ and that the others are at distance strictly larger than r from the origin. Now between time τ_r and τ_{r+1} we always peel edges which are incident to those vertices and all these edges are then part of $\mathrm{Ball}_{r+1}(\mathfrak{m})$ hence

$$\mathfrak{e}_{\tau_{r+1}} \subset \mathrm{Ball}_{r+1}(\mathfrak{m}).$$

For the reverse inclusion we just remark that any edge which is not an inner edge of $\mathfrak{e}_{\tau_{r+1}}$ must have both endpoints are distance larger than $r+1$ from the origin. □

Proof of Theorem 7. Clearly, the above peeling algorithm satisfies the assumption required in Proposition 15 and we thus have convergence of the peeling exploration $(\tilde{\mathfrak{e}}_n)_{n\geq 0}$ with algorithm $\mathcal{A}_{\text{metric}}$ under $\mathbb{P}^{(\ell)}$ towards the peeling exploration $(\tilde{\mathfrak{e}}_n)_{n\geq 0}$ described in the last section. Our first goal is to prove that the increasing sequence $\tilde{\mathfrak{e}}_n$ indeed defines a locally finite planar map of the half-plane. This is implied by the following lemma:

Lemma 17. *For any $r \geq 0$, if $\theta_r = \min\{\mathrm{d}_{\mathrm{gr}}^{\tilde{\mathfrak{e}}_n}(x,\rho) : x \in \mathsf{Active}(\tilde{\mathfrak{e}}_n)\} \geq r$ then we have $\theta_r < \infty$.*

Proof of the lemma. If for some $r \geq 0$ we have $\theta_r = \infty$, this means that when constructing the sequence $(\tilde{\mathfrak{e}}_n)_{n\geq 0}$ using algorithm $\mathcal{A}_{\text{metric}}$ following the transitions of Proposition 15 then $\cup_{n\geq 0}\tilde{\mathfrak{e}}_n$ does not create a locally finite map. The problem comes from the fact that some vertex may stay forever on the boundary of $\tilde{\mathfrak{e}}_n$ and may never be swallowed by the peeling process (i.e. becomes an inner vertex of $\tilde{\mathfrak{e}}_n$). If this happens, it is easy to see that there exists such a vertex x which is on $\mathsf{Active}(\tilde{\mathfrak{e}}_n)$ eventually and such that infinitely many peeling steps are occurring just on the left of x. Such a situation cannot happen with positive probability since each time we peel the edge incident to the left of x an event of type $\mathsf{G}_{\infty,0}$ of $\mathsf{G}_{k-1,0}$ happens with a probability bounded from below by some positive constant and on this event the vertex x becomes an interior vertex of $\tilde{\mathfrak{e}}_{n+1}$. □

Hence the last lemma enables to make the following definition:

Definition 16. *We define the law $\mathbb{P}^{(\infty)}$ (whose associate expectation is denoted by $\mathbb{E}^{(\infty)}$) of an infinite random planar map with an infinite boundary called the half-plane infinite $\mathbf{q}$-Boltzmann map as the law of*

$$\bigcup_{n\geq 0} \tilde{\mathfrak{e}}_n,$$

where $(\tilde{\mathfrak{e}}_n)$ is the Markov chain obtained in Proposition 15 used with algorithm $\mathcal{A}_{\mathrm{metric}}$.

Notice that almost surely under $\mathbb{P}^{(\infty)}(\mathrm{d}\mathfrak{m})$ the map $\mathfrak{m}$ is indeed a map of the half-plane and in particular is one-ended: since the transition probabilities in the finite holes of $\tilde{\mathfrak{e}}_n$ are those of finite Boltzmann maps all the finite holes of $\tilde{\mathfrak{e}}_n$ eventually contain a finite map. So the unique end of $\mathfrak{m}$ is at each step contained in the unique hole with an infinite boundary.

Coming back to the proof of the theorem, the convergence of the peeling explorations established in Proposition 15 together with the last lemma entail $\tilde{\mathfrak{e}}_{\tau_r}$ under $\mathbb{P}^{(\ell)}$ converge towards $\tilde{\mathfrak{e}}_{\theta_r}$. Using Lemma 16 it is an easy exercise to see that this implies the convergence of $[\mathfrak{m}]_r$ under $\mathbb{P}^{(\ell)}$ converges towards $[\mathfrak{m}]_r$ under $\mathbb{P}^{(\infty)}$. This shows the desired result (notice that here it is crucial to use $[\mathfrak{m}]_r$ instead of $\mathrm{Ball}_r(\mathfrak{m})$). □

Exercise 9. Prove that we also have the following convergence $\mathbb{P}^{(\ell)}_{\bullet} \xrightarrow[\ell\to\infty]{(d)} \mathbb{P}^{(\infty)}$ in distribution for the local topology.

4.1.3 Peeling explorations under $\mathbb{P}^{(\infty)}$

In the last section we have *defined* the law $\mathbb{P}^{(\infty)}$ as the law of the lattice produced by the limiting Markov chain appearing in peeling explorations with algorithm $\mathcal{A}_{\mathrm{metric}}$. We now explicitly give the law of the peeling explorations under $\mathbb{P}^{(\infty)}$.

Proposition 18 (Law of the peeling process under $\mathbb{P}^{(\infty)}$)**.** *Fix a peeling algorithm $\mathcal{A}$. Then under $\mathbb{P}^{(\infty)}$ the peeling exploration $\mathfrak{e}_0 \subset \mathfrak{e}_1 \subset \cdots \subset \mathfrak{e}_n \subset \cdots \subset \mathfrak{m}$ is a Markov chain whose probability transitions are as follows. Conditionally on $\mathcal{F}_n$ and provided that $\mathcal{A}(\mathfrak{e}_n) \neq \dagger$ we denote by $\mathcal{L}_n \in \{1,2,3,\ldots\} \cup \{\infty\}$ the half-perimeter of the hole on which $\mathcal{A}(\mathfrak{e}_n)$ is selected. If $\mathcal{L}_n = \infty$ the events C_k, $\mathsf{G}_{k,\infty}$ and $\mathsf{G}_{\infty,k}$ for $k \geq 0$ occur with respective probabilities*

$$b^{(\infty)}(k), \qquad b^{(\infty)}(k,\infty) \quad \textit{and} \quad b^{(\infty)}(\infty,k).$$

Otherwise if $\mathcal{L}_n < \infty$ then the probability transitions are those described in Proposition 7.

Proof. This can be seen in many different ways. Probably the simplest is to proceed as in the initial proof of Proposition 7 and prove that when we peel one edge of $\mathfrak{m}$ under $\mathbb{P}^{(\infty)}$ then we indeed have the above probability transitions and that conditionally on each of those cases the possibly two holes of $\mathfrak{e}_1$ are filled in with independent maps, the one with the infinite perimeter being of law $\mathbb{P}^{(\infty)}$ and the other one being a finite $\mathbf{q}$-Boltzmann map of the proper perimeter. $\square$

We will also consider filled-in explorations $(\bar{\mathfrak{e}}_n)_{n\geq 0}$ under $\mathbb{P}^{(\infty)}$ which are naturally obtained by filling-in the finite holes during a peeling exploration and only keeping the infinite one active. This is again a Markov chain whose probability transitions are deduced from the last proposition. In this case the half-perimeter $|\partial\bar{\mathfrak{e}}_n|$ of the unique hole of $\bar{\mathfrak{e}}_n$ has a priori no meaning since it is infinite. But in this context we denote by $|\partial\bar{\mathfrak{e}}_n|$ the algebraic variation of the half-perimeter of that hole with respect to its initial state (indeed at each step the hole of $\bar{\mathfrak{e}}_n$ is only a local modification of the initial hole of $\bar{\mathfrak{e}}_0$). Recall from Section 3.3 the definition of the random walk (S) with independent increments distributed according to ν. Then using the above probability transitions we deduce that

$$(|\partial\bar{\mathfrak{e}}_n|)_{n\geq 0} \text{ under } \mathbb{P}^{(\infty)} \quad \overset{(d)}{=} \quad (S_n)_{n\geq 0} \text{ under } \mathbf{P}_0. \tag{4.1}$$

Exercise 10 (Translation invariance). Show that the law $\mathbb{P}^{(\infty)}$ is not affected by translating the root edge of $\mathfrak{m}$ by one unit to the left or to the right along the infinite boundary.

4.2 Infinite Boltzmann maps of the plane

In this section on top of being admissible, we suppose that the weight sequence $\mathbf{q}$ is critical (see Definition 9). For $n \geq 0$ we denote by $\mathbb{P}^{(\ell)}_n$ the $\mathbf{q}$-Boltzmann distribution $\mathbb{P}^{(\ell)}$ conditioned on maps with exactly n vertices. Depending on the weight sequence $\mathbf{q}$ there might be parity constraint on n for the $w_{\mathbf{q}}$-weight of bipartite maps with perimeter 2ℓ and n vertices to be non zero and we shall henceforth restrict on those values of n. Recall that one can see a planar map with an external face of degree 2 as a planar map of the sphere and so $\mathbb{P}^{(1)}_n$ will be seen as a distribution on planar maps of the sphere with n vertices. The main result is then:

Theorem 8 (*Convergence towards the infinite Boltzmann planar map*)

Let $\mathbf{q}$ be an admissible and critical weight sequence. For any $\ell \geq 1$ we have the following convergence in distribution for the local topology

$$\mathbb{P}^{(\ell)}_n \xrightarrow[n\to\infty]{(d)} \mathbb{P}^{(\ell)}_\infty,$$

(along the values of n for which $\mathbb{P}^{(\ell)}_n$ has a meaning) where $\mathbb{P}^{(\ell)}_\infty$ is a distribution supported by infinite bipartite planar maps of the plane such that the external face is of degree 2ℓ. This is the so-called infinite $\mathbf{q}$-Boltzmann distribution (with external face of degree 2ℓ).

Remark 10 (UIPQ). When $\mathbf{q} = (12^{-1}\delta_2(k))_{k\geq 1}$ corresponds to critical Boltzmann quadrangulations, the measure $\mathbb{P}^{(1)}_\infty$ induced on quadrangulations of the plane is the law of the Uniform Infinite Planar Quadrangulation defined in [27, 45, 52].

The proof of the above theorem is similar to that of Theorem 7: we prove convergence of the peeling process with algorithm $\mathcal{A}_{\text{metric}}$ under the law $\mathbb{P}_n^{(\ell)}$ as $n \to \infty$ towards an explicit Markov chain, and then define the law $\mathbb{P}_\infty^{(\ell)}$ using the last Markov chain.

4.2.1 Convergence of the peeling transitions under $\mathbb{P}_n^{(\ell)}$

The key to prove Theorem 8 is the following enumeration result. Let $W_n^{(\ell)}$ be the $w_{\mathbf{q}}$ weight of all (bipartite) planar map with external face of degree 2ℓ and n inner vertices (there might be parity constraint on n).

Lemma 19. *Let $\mathbf{q}$ be an admissible and critical weight sequence. For any $\ell, \ell' \geq 1$ and for any $n_0 \geq 0$ we have*

$$\frac{W_n^{(\ell)}}{W_{n-n_0}^{(\ell')}} \xrightarrow[n\to\infty]{} \frac{h^\uparrow(\ell) c_{\mathbf{q}}^\ell}{h^\uparrow(\ell') c_{\mathbf{q}}^{\ell'}},$$

where the limit is taken along the integers n for which both $W_n^{(\ell)}$ and $W_{n-n_0}^{(\ell')}$ are non zero.

Proof. This is proved in [23, Corollary 2]. [Include a direct derivation.] □

Definition 17 (Probability transitions for the distinguished hole under $\mathbb{P}_\infty^{(\cdot)}$)**.** *For any $\ell \geq 1$ and $k \geq 0$ we put*

$$b_\infty^{(\ell)}(k) = c_{\mathbf{q}}^{k-1} q_k \frac{h^\uparrow(\ell+k-1)}{h^\uparrow(\ell)}, \qquad \textit{and} \qquad b_\infty^{(\ell)}(*,k) = b_\infty^{(\ell)}(k,*) = c_{\mathbf{q}}^{-k-1} W^{(k)} \frac{h^\uparrow(\ell-k-1)}{h^\uparrow(\ell)}.$$

Lemma 20. *The probability transitions defining $b_\infty^{(\ell)}$ sum-up to 1, that is for any $\ell \geq 1$*

$$\sum_{k\geq 0} b_\infty^{(\ell)}(k) + \sum_{k\geq 0} b_\infty^{(\ell)}(*,k) + \sum_{k\geq 0} b_\infty^{(\ell)}(k,*) = 1.$$

Proof. Rewriting the identity in terms of ν the above equality is equivalent to the fact that the function $h^\uparrow$ is harmonic for the walk (S) at all points $\{1,2,3,\ldots\}$. This is indeed the case when $\mathbf{q}$ is critical by Lemma 11. □

These numbers arise as limits in the probability transitions for the peeling under $\mathbb{P}_n^{(\cdot)}$ as we shall now see. Consider a peeling step under $\mathbb{P}_n^{(\ell)}$, as usual there are two cases:

- *Event* C_k*:* The probability that the edge to peel is adjacent to a new face of degree $2k$ is simply equal to (again we restrict on the values of n on which the quantities $W_n^{(\ell)}$ are non-zero)

$$q_k \frac{W_n^{(\ell+k-1)}}{W_n^{(\ell)}} \xrightarrow[n\to\infty]{\text{Lem.19}} b_\infty^{(\ell)}(k).$$

- *Event* $\mathrm{G}_{(k_1,n_1),(k_2,n_2)}$: Otherwise the edge to peel is identified with another edge on the same hole splitting the hole into two new holes of half-perimeter k_1 (for the one on the left of the peeled edge) and k_2 with as usual $k_1+k_2-1=\ell$. We also denote by $n_1+n_2=n$ the respective number of vertices inside each of these holes. The probability of event $\mathsf{G}_{(k_1,n_1),(k_2,n_2)}$ is then

$$\frac{W_{n_1}^{(k_1)}W_{n_2}^{(k_2)}}{W_n^{(\ell)}}.$$

If we put $n_1=n-N$ where N is fixed, we deduce using Lemma 19 that

$$\frac{W_{n-N}^{(\ell-k-1)}W_N^{(k)}}{W_n^{(\ell)}}\xrightarrow[n\to\infty]{\text{Lem.}19} c_{\mathbf{q}}^{-k-1}W_N^{(k)}\frac{h^{\uparrow}(\ell-k-1)}{h^{\uparrow}(\ell)}.$$

Taking the sum over all $N\geq 0$ we have

$$\sum_{N\geq 0} c_{\mathbf{q}}^{-k-1}W_N^{(k)}\frac{h^{\uparrow}(\ell-k-1)}{h^{\uparrow}(\ell)}=c_{\mathbf{q}}^{-k-1}W^{(k)}\frac{h^{\uparrow}(\ell-k-1)}{h^{\uparrow}(\ell)}=b_\infty^{(\ell)}(*,k).$$

As a consequence of Lemma 20 we deduce that asymptotically when $n\to\infty$ the probability of an event $\mathsf{G}_{(k_1,n_1),(k_2,n_2)}$ when both n_1 and n_2 are large is negligible as $n\to\infty$. Indeed, by Fatou's lemma we have:

$$\begin{aligned}
\limsup_{n\to\infty}\sum_{\substack{n_1\geq A,n_2\geq A\\ n_1+n_2=n}}\frac{W_{n_1}^{(\ell-k-1)}W_{n_2}^{(k)}}{W_n^{(\ell)}} &= 1-\liminf_{n\to\infty}\left(\sum_{k\geq 0}q_k\frac{W_n^{(\ell+k-1)}}{W_n^{(\ell)}}+\sum_{\substack{n_1<A,\ \text{or}\ n_2<A\\ n_1+n_2=n}}\frac{W_{n_1}^{(\ell-k-1)}W_{n_2}^{(k)}}{W_n^{(\ell)}}\right)\\
&\leq 1-\sum_{k\geq 0}b_\infty^{(\ell)}(k)-\sum_{k<A}b_\infty^{(\ell)}(*,k)-\sum_{k<A}b_\infty^{(\ell)}(k,*)\\
&\xrightarrow[A\to\infty]{} 0,
\end{aligned}$$

by Lemma 20. In particular, when n is large, if we perform one peeling step under $\mathbb{P}_n^{(\ell)}$ then with overwhelming probability there will be only one hole of $\mathfrak{e}_1$ which contains a macroscopic fraction of the mass of $\mathfrak{m}$ whereas the possible other hole is filled-in with a finite map. Using the above computations it is easy to see that the map filling-in the "finite" hole of $\mathfrak{e}_1$ is asymptotically Boltzmann distributed and that the other map is uniform over the constraints imposed on its perimeter and size (in particular its size tend to ∞ as $n\to\infty$). Iterating the above argument we deduce the following rigorous statement:

Proposition 21 (Convergence of the peeling process under $\mathbb{P}_n^{(\ell)}$)**.** *Let $\mathfrak{e}_0\subset\mathfrak{e}_1\subset\cdots\subset\mathfrak{m}$ be a peeling process with algorithm $\mathcal{A}$. Then as $n\to\infty$ the law of $(\mathfrak{e}_i)_{i\geq 0}$ under $\mathbb{P}_n^{(\ell)}$ converges towards the Markov chain*[1] *$(\mathfrak{e}_i^*)_{i\geq 0}$ whose transitions are as follows. Conditionally on $\mathcal{F}_n$ and provided that $\mathcal{A}(\mathfrak{e}_n^*)\neq\dagger$, we denote by $\mathcal{L}_n$ the half-perimeter of the hole on which $\mathcal{A}(\mathfrak{e}_n^*)$ is located. If $\mathcal{A}(\mathfrak{e}_n^*)$ belongs to a non-distinguished hole then the events C_k, G_{k_1,k_2} occur with the transitions described in Proposition 7. Otherwise the events C_k, $\mathsf{G}_{*,k}$ and $\mathsf{G}_{k,*}$ occur respectively with probability (see Definition 17)*

$$b_\infty^{(\mathcal{L}_n)}(k),\quad b_\infty^{(\mathcal{L}_n)}(*,k)\quad\text{and}\quad b_\infty^{(\mathcal{L}_n)}(k,*).$$

[1] after we have forgotten the distinguished cycles in the second chain

4.2.2 Proof of Theorem 8

It follows the same lines as the proof of Theorem 7. More precisely we consider the peeling process $(\mathfrak{e}_i)_{i\geq 0}$ under $\mathbb{P}_n^{(\ell)}$ with algorithm $\mathcal{A}_{\mathrm{metric}}$. By the above proposition, this sequence converges in law towards the chain $(\mathfrak{e}_i^*)_{i\geq 0}$ driven by algorithm $\mathcal{A}_{\mathrm{metric}}$ (after we have forgotten the distinguished holes). As in the preparation of the proof of Theorem 7 we check that $\bigcup_{n\geq 0}\mathfrak{e}_n^*$ defines a infinite map of the plane. This is proved exactly as in Lemma 17 and Definition 16 and we put:

Definition 18. *We define the law $\mathbb{P}_\infty^{(\ell)}$ (whose relative expectation is denoted by $\mathbb{E}_\infty^{(\ell)}$) of an infinite random planar map of the plane whose external face has degree 2ℓ as the law of*

$$\bigcup_{n\geq 0}\mathfrak{e}_n^*,$$

where $(\mathfrak{e}_n^*)$ is the Markov chain obtained in Proposition 21 started from a single face and hole of degree 2ℓ and driven by the algorithm $\mathcal{A}_{\mathrm{metric}}$.

As in the last section and using Proposition 21 we deduce the convergence of $\mathfrak{e}_{\tau_r}$ under $\mathbb{P}_n^{(\ell)}$ towards $\mathfrak{e}_{\theta_r}^*$. By Lemma 16 this entails that $\mathrm{Ball}_r(\mathfrak{m})$ under $\mathbb{P}_n^{(\ell)}$ converges towards $\mathrm{Ball}_r(\mathfrak{m})$ under $\mathbb{P}_\infty^{(\ell)}$ as $n\to\infty$. From this we infer the same convergence for the balls $[\mathfrak{m}]_r$ and this proves our result. $\square$

4.2.3 Law of peeling explorations with target under $\mathbb{P}_\infty^{(\ell)}$

Let us now give the law of peeling explorations under $\mathbb{P}_\infty^{(\ell)}$.

Proposition 22 (Law of the peeling process with target under $\mathbb{P}_\infty^{(\ell)}$)**.** *Fix $\ell\geq 1$ and a peeling algorithm $\mathcal{A}$. Then under $\mathbb{P}_\infty^{(\ell)}$ the peeling exploration with target $(\mathfrak{e}_i^*)_{i\geq 0}$ is a Markov chain whose probability transitions are as follows. Conditionally on $\mathcal{F}_n$ and provided that $\mathcal{A}(\mathfrak{e}_n^*)\neq\dagger$ we denote by $\mathcal{L}_n\in\{1,2,3,\dots\}$ the half-perimeter of the hole on which $\mathcal{A}(\mathfrak{e}_n^*)$ is selected. On the event where $\mathcal{A}(\mathfrak{e}_n^*)$ belongs to the distinguished hole of $\mathfrak{e}_n^*$ the events C_k, $\mathrm{G}_{k,*}$ and $\mathrm{G}_{*,k}$ for $k\geq 0$ occur with the respective probabilities*

$$b_\infty^{(\mathcal{L}_n)}(k),\qquad b_\infty^{(\mathcal{L}_n)}(k,*)\quad\text{ and }\quad b_\infty^{(\mathcal{L}_n)}(*,k).$$

Otherwise if $\mathcal{A}(\mathfrak{e}_n^*)$ is on a non-distinguished hole of $\mathfrak{e}_n^*$ then the probability transitions are those described in Proposition 7.

Proof. One can proceed as in the initial proof of Proposition 7 and prove that when we peel one edge of $\mathfrak{m}$ under $\mathbb{P}_\infty^{(\ell)}$ then we have the above probability transitions and that conditionally on each of those cases the possibly two holes of $\mathfrak{e}_1$ are filled in with independent maps, the one within the distinguished hole being of law $\mathbb{P}_\infty^{(\cdot)}$ and the other one of law $\mathbb{P}^{(\cdot)}$. $\square$

As usual it follows from the last display that the filled-in explorations with target $(\bar{\mathfrak{e}}_n)_{n\geq 0}$ under $\mathbb{P}_\infty^{(\ell)}$ are also Markov chains. Let us simply notice that the half-perimeter $|\partial\bar{\mathfrak{e}}_n|$ of the unique hole is itself a Markov chain whose probability transitions can be written under the form

$$p(x, x+k) = \nu(k)\frac{h^{\uparrow}(x+k)}{h^{\uparrow}(x)}. \tag{4.2}$$

In other words, the half-perimeter process of $\bar{\mathfrak{e}}_n$ evolves as $S^{\uparrow}$, the random walk S started from ℓ and conditioned to stay positive, see Section 3.3.3.

Bibliographical notes. Although the results of the first section of this chapter are new in the present generality, they have been known in several particular cases. In particular Angel [3] defined first the uniform infinite half-planar triangulation (UIHPT) by taking the limit of the peeling process in large (in fact infinite) triangulations, see also [54, Section 5]. In fact this object is slightly different from those studied in this work (except from the trivial fact that it is a triangulation and so not bipartite) because the object defined in [3] has a *simple* boundary (no pinch point). See [37] for the definition of the related uniform infinite half-planar quadrangulation with a general boundary as here via a Schaeffer-type construction (this object is studied in depth in [25]). This object has the law $\mathbb{P}^{(\infty)}$ when $\mathbf{q} = (12^{-1}\delta_{k=2})$. The convergence towards infinite maps of the plane has first been proved in the case of triangulations by Angel & Schramm [8] and later by Krikun [46] in the case of quadrangulations. Later Schaeffer-type constructions of the Uniform Infinite Planar Quadrangulation (whose law "is" that of $\mathbb{P}_{\infty}^{(1)}$ when $\mathbf{q} = (12^{-1}\delta_{k=2})$) were given [27, 52, 36] and this opened the door to the more general convergence of Theorem 8 which was proved in this generality in [18] using the Bouttier–Di Francesco–Guitter bijection. The proof given here, only based on the peeling process and the properties of the function $h^{\uparrow}$, is new.

Chapter V : Scaling limit for the peeling process

In this chapter we give the scaling limit for the evolution of the perimeter and volume processes during filled-in explorations under $\mathbb{P}^{(\ell)}, \mathbb{P}_{\bullet}^{(\ell)}$ and $\mathbb{P}^{(\infty)}$. However, to be able to get nice scaling limits for these processes we need regularity assumptions on the weight sequence $\mathbf{q}$ so that the step distribution ν (Definition 15) is in the domain of attraction of a stable law.

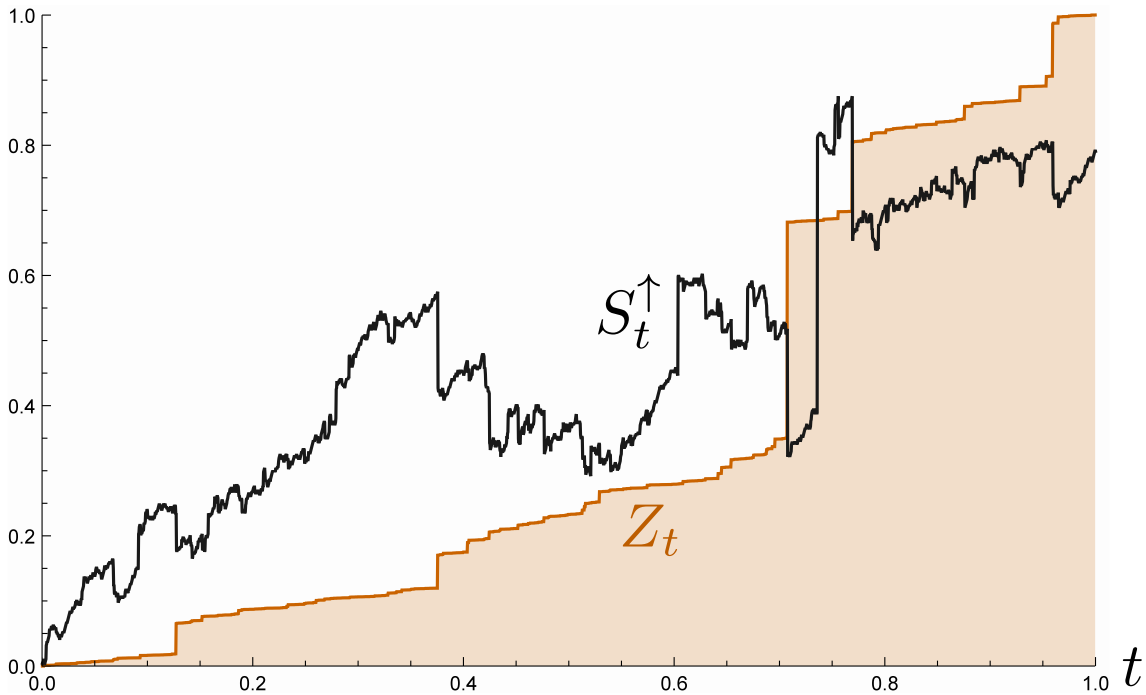


Figure 5.1: Simulation of the processes $\Upsilon_a^{\uparrow}$ and $\mathcal{V}(\Upsilon_a^{\uparrow})$ when $a = 2.3$ [change legend]. These processes describe the scaling limits of the perimeter and volume growth in filled-in explorations under $\mathbb{P}_{\infty}^{(\ell)}$ when $\mathbf{q}$ is of type $a = 2.3$.

5.1 Classification of critical weight sequences

Recall the admissibility and criticality criteria for $\mathbf{q}$ presented in Section 2.3. In the following we will always suppose that $\mathbf{q}$ is admissible and recall the notation $\mathbf{c_q}/4 = Z_{\mathbf{q}}$ for the smallest solution to $f_{\mathbf{q}}(x) = 1 - \frac{1}{x}$.

We have seen in Theorem 6 that, provided $\mathbf{q}$ is admissible, the behavior of the pointed partition function $W_\bullet^{(\ell)}$ is universal and only depends on $c_\mathbf{q}$. To get access to $W^{(\ell)}$ the idea is to vary the weight sequence $\mathbf{q}$. For $g \in (0,1)$ we introduce the weight sequence $\mathbf{q}_g$ defined by $(q_g)_k = q_k g^{k-1}$. If $\mathfrak{m} \in \mathcal{M}^{(\ell)}$ using Euler's formula together with $\sum_{f \in \mathsf{Faces}(\mathfrak{m}) \setminus \{f_r\}} \deg(f)/2 + \ell = \#\mathsf{Edges}(\mathfrak{m})$ we can write that

$$w_{\mathbf{q}_g}(\mathfrak{m}) = g^{-1-\ell} g^{|\mathfrak{m}|} w_\mathbf{q}(\mathfrak{m}). \tag{5.1}$$

Clearly the weight sequence $\mathbf{q}_g$ is admissible (we diminish the weight) and so we put $c_{\mathbf{q}_g}$ so that $c_{\mathbf{q}_g}/4$ is the smallest solution to $f_{\mathbf{q}_g}(x) = 1 - \frac{1}{x}$ or equivalently $f_\mathbf{q}(gx) = 1 - \frac{1}{x}$. The last display together with Theorem 6 shows that

$$\sum_{\mathfrak{m} \in \mathcal{M}^{(\ell)}} |\mathfrak{m}| w_\mathbf{q}(\mathfrak{m}) g^{|\mathfrak{m}|-1} = \left(g c_{\mathbf{q}_g}\right)^\ell h^\downarrow(\ell), \tag{5.2}$$

which by integration yields an explicit expression for $W^{(\ell)}$:

$$W^{(\ell)} = \sum_{\mathfrak{m} \in \mathcal{M}^{(\ell)}} w_\mathbf{q}(\mathfrak{m}) = h^\downarrow(\ell) \cdot \int_0^1 \mathrm{d}g \left(g c_{\mathbf{q}_g}\right)^\ell. \tag{5.3}$$

Before going further, let us deduce a corollary of the last displays which says that the admissibility definition could have equivalently be based on finiteness of $W^{(\ell)}$ rather than finiteness of $W_\bullet^{(\ell)}$.

Corollary 23. *Let $\mathbf{q}$ be a weight sequence such that $W^{(\ell)}(\mathbf{q})$ is finite for some $\ell \geq 1$ (as opposed to the stronger condition $W_\bullet^{(\ell)}(\mathbf{q}) < \infty$). Then $\mathbf{q}$ is admissible in the sense of Definition 8.*

Proof. Let $\mathbf{q}$ be as in the statement and $g \in (0,1)$. By summing (5.1) over all $\mathfrak{m}_\bullet \in \mathcal{M}_\bullet^{(\ell)}$ and using the fact that $k \cdot g^k$ is bounded above by some constant (depending on g) we deduce that

$$W_\bullet^{(\ell)}(\mathbf{q}_g) = g^{-1-\ell} \sum_{\mathfrak{m} \in \mathcal{M}^{(\ell)}} |\mathfrak{m}| w_\mathbf{q}(\mathfrak{m}) g^{|\mathfrak{m}|-1} \leq C_g W^{(\ell)}(\mathbf{q}) \underset{\text{hyp.}}{<} \infty.$$

Hence, for every $g \in (0,1)$ the weight sequence $\mathbf{q}_g$ is admissible in the sense of Definition 8 and by Proposition 5 we deduce that $f_{\mathbf{q}_g}(x) = 1 - \frac{1}{x}$ or equivalently $f_\mathbf{q}(gx) = 1 - \frac{1}{x}$ admits a solution which we denote by $c_{\mathbf{q}_g}/4$ as above. Putting $c = \sup_{g \uparrow 1} c_g$ we easily deduce by monotone convergence that $c/4 < \infty$ is solution to $f_\mathbf{q}(x) = 1 - \frac{1}{x}$ hence $\mathbf{q}$ is indeed admissible by Proposition 5. □

Coming back to (5.3) we see that the function $g \mapsto g c_{\mathbf{q}_g}$ is continuous and increasing with limit $c_\mathbf{q}$ when $g \to 1^-$. By Laplace method we deduce that $(W^{(\ell)})^{1/\ell} \to c_\mathbf{q}$ as $\ell \to 1$. More precisely we can now prove (2.11). We form the ratio:

$$\frac{W^{(\ell+1)}}{W^{(\ell)}} = \frac{h^\downarrow(\ell+1)}{h^\downarrow(\ell)} \frac{\int_0^1 \mathrm{d}g \left(g c_{\mathbf{q}_g}\right)^{\ell+1}}{\int_0^1 \mathrm{d}g \left(g c_{\mathbf{q}_g}\right)^{\ell}}.$$

Since $h^{\downarrow}(\ell+1)/h^{\downarrow}(\ell) \to 1$ and because $gc_{\mathbf{q}_g} \le c_{\mathbf{q}}$ for $g < 1$ we deduce that asymptotically the ratio is less than $c_{\mathbf{q}}$. On the other hand, by Laplace method, the main contribution to the integrals in the last display are for g close to 1, that is for any $\varepsilon > 0$ we have

$$\liminf_{\ell\to\infty} \frac{W^{(\ell+1)}}{W^{(\ell)}} \ge \frac{\int_{1-\varepsilon}^{1} \mathrm{d}g \left(gc_{\mathbf{q}_g}\right)^{\ell+1}}{\int_{1-\varepsilon}^{1} \mathrm{d}g \left(gc_{\mathbf{q}_g}\right)^{\ell}} \ge (1-\varepsilon)c_{\mathbf{q}_{(1-\varepsilon)}} \xrightarrow[\varepsilon\to 0]{} c_{\mathbf{q}},$$

and the lower bound is granted. More generally, the Laplace method tells us that the fine asymptotic of $\int_0^1 f(x)\mathrm{d}x$ where f is non-decreasing is related to the behavior of f near 1. In our case, we thus need to classify the possible behaviors of $c_{\mathbf{q}_g}$ near $g = 1^-$ which can have a few different behaviors as we shall now see. Before classifying those behaviors, let us introduce the notion of *regularity*:

Definition 19. *The admissible weight sequence* $\mathbf{q}$ *is regular if one has* $f_{\mathbf{q}}(Z_q + \varepsilon) < \infty$ *for some* $\varepsilon > 0$ *or equivalently if the probability measure* ν *has an exponential tail on the positive side.*

In particular if the weight sequence is supported by finitely many values $k \ge 1$ then as soon as it is admissible, it is automatically regular.

5.1.1 Regular subcritical weight sequences

To later extend the reasoning of the forthcoming calculation we put $a = \frac{3}{2}$ in this subsection. Suppose that $\mathbf{q}$ is (admissible) and subcritical. That means that the graphs of $x \mapsto f_{\mathbf{q}}(x)$ and $x \mapsto 1 - \frac{1}{x}$ cross at $x = Z_{\mathbf{q}}$ with different tangents $f'_{\mathbf{q}}(Z_{\mathbf{q}}) < 1/Z_{\mathbf{q}}^2$. Then using the notation of the last section, it is easy to see that for $g \in (0,1)$ we have

$$gc_{\mathbf{q}_g} = c_q(1 - \kappa(1-g)^{1/(a-1/2)}) + o(1-g)^{1/(a-1/2)} \quad \text{as } g \to 1^-,$$

for some constant $\kappa > 0$ where above we have written $1 = 1/(a - 1/2)$ in a complicated fashion. Plugging this back into (5.3) and using Laplace method together with the fact that $h^{\downarrow}(\ell) \sim 1/\sqrt{\pi\ell}$ we find that if we put

$$\mathrm{p}_{\mathbf{q}} := \frac{2\kappa^{\frac{1}{2}-a}}{c_{\mathbf{q}}\sqrt{\pi}}\Gamma(a + \frac{1}{2}), \tag{5.4}$$

then we have

$$W^{(\ell)} \underset{\ell\to\infty}{\sim} \frac{\mathrm{p}_{\mathbf{q}}}{2} \cdot c_{\mathbf{q}}^{\ell+1} \cdot \ell^{-a}. \tag{5.5}$$

Combining this with the above regularity assumption we deduce that: In the case of a REGULAR AND SUBCRITICAL SEQUENCE we have for some $C > 0$ and $\lambda \in (0,1)$

$$\nu(-k) \underset{k\to\infty}{\sim} \mathrm{p}_{\mathbf{q}}\, k^{-a} \quad \text{and} \quad \nu(k) \le C\lambda^k \quad \text{for } k \ge 0. \tag{5.6}$$

In particular since $a = \frac{3}{2}$ we see that ν has no first moment because of the left tail.

5.1.2 Regular critical weight sequences

In the case when $\mathbf{q}$ is regular and critical, at the point $Z_{\mathbf{q}}$ the graphs of $x \mapsto f_{\mathbf{q}}(x)$ and of $x \mapsto 1 - \frac{1}{x}$ are tangent. Since $\mathbf{q}$ is regular, the second derivative $f''_{\mathbf{q}}(Z_{\mathbf{q}})$ is finite. Then using again the notation of the last section, a local study shows that for $g \in (0,1)$ we have

$$gc_{\mathbf{q}_g} = c_q(1-(1-g)^{1/(a-1/2)}) + o(1-g)^{1/(a-1/2)} \qquad \text{as } g \to 1^-,$$

for some constant $\kappa > 0$ where now $a = 5/2$ so that $1/(a-1/2)$ is just equal to $1/2$. Plugging this back into (5.3) and using Laplace method, we find that with the same definition of $\mathsf{p}_{\mathbf{q}}$ as that of (5.4) we have

$$W^{(\ell)} \underset{\ell\to\infty}{\sim} \frac{\mathsf{p}_{\mathbf{q}}}{2} \cdot c_{\mathbf{q}}^{\ell+1} \cdot \ell^{-a}, \tag{5.7}$$

still with $a = \frac{5}{2}$. In terms of the step distribution ν we deduce that in the case of a REGULAR AND CRITICAL SEQUENCE we have for some $C > 0$ and $\lambda \in (0,1)$

$$\nu(-k) \underset{k\to\infty}{\sim} \mathsf{p}_{\mathbf{q}}\, k^{-a} \qquad \text{and} \qquad \nu(k) \leq C\lambda^k \qquad \text{for } k \geq 0. \tag{5.8}$$

In particular since $a = \frac{5}{2}$ we see that ν has a first moment. By Proposition 14 it is furthermore centered. This type of behavior is the "generic one" when one deals with natural classes of random maps.

5.1.3 Critical and non-generic

If we want to ensure other behaviors for $c_{\mathbf{q}_g}$ near $g = 1^-$ one must drop the regularity condition. We thus want a weight sequence $\mathbf{q}$ so that $f_{\mathbf{q}}$ is not defined anymore at $Z_{\mathbf{q}}+$ and has an infinite second derivative at $Z_{\mathbf{q}}$. The natural way to do that is to start from a weight sequence $\hat{q}_k$ asymptotic to k^{-a} for $a \in (3/2; 5/2)$ when $k \to \infty$ and then modifying it to achieve the desired criticality. More precisely:

Lemma 24. *We put*

$$c = \frac{4}{4f_{\hat{\mathbf{q}}}(1/4) + f'_{\hat{\mathbf{q}}}(1/4)} \qquad \textit{and} \qquad \beta = \frac{f'_{\hat{\mathbf{q}}}(1/4)}{4f_{\hat{\mathbf{q}}}(1/4) + f'_{\hat{\mathbf{q}}}(1/4)},$$

then the weight sequence $\mathbf{q}$ defined by $q_k = c(\beta/4)^{k-1}\hat{q}_k$ is admissible, critical and furthermore, with the above notation we have

$$gc_{\mathbf{q}_g} = c_q(1-\kappa(1-g)^{\frac{1}{a-1/2}}) + o(1-g)^{1/(a-1/2)} \qquad \textit{as } g \to 1^-.$$

In particular we see that the range of a is constrained by the fact that we want $f'_{\mathbf{q}}(Z_q)$ to exists at $Z_{\mathbf{q}}$ which implies $a > 3/2$ and its second derivative to explode which implies $a < 5/2$. Also we find a posteriori that the β in the above lemma is just $c_{\mathbf{q}}$. We call such a weight sequence a critical non-generic weight sequence with exponent $a \in (3/2; 5/2)$. Using again the Laplace method we find that with the same definition of $p_{\mathbf{q}}$ given in (5.4) we have

$$W^{(\ell)} \underset{\ell \to \infty}{\sim} \frac{p_{\mathbf{q}}}{2} \cdot c_{\mathbf{q}}^{\ell+1} \cdot \ell^{-a}. \tag{5.9}$$

In particular the relation of $p_{\mathbf{q}}$ with the constant c of the last lemma is of uttermost importance: after some easy but tedious analysis we find that

$$p_{\mathbf{q}} = \frac{c}{\cos(a\,\pi)}$$

so that together with (5.9) we have

$$q_k \underset{k \to \infty}{\sim} p_{\mathbf{q}} \cdot \cos(a\,\pi) \cdot c_{\mathbf{q}}^{-k+1}\, k^{-a}. \tag{5.10}$$

These asymptotics translate into asymptotics for the probability measure ν: In the case of a CRITICAL NON GENERIC weight sequence with exponent $a \in (3/2; 5/2)$ we have

$$\nu(k) \underset{k \to \infty}{\sim} p_{\mathbf{q}} \cdot \cos(a\,\pi) \cdot |k|^{-a} \quad \text{and} \quad \nu(-k) \underset{k \to \infty}{\sim} p_{\mathbf{q}} \cdot |k|^{-a}, \qquad \text{as } k \to \infty. \tag{5.11}$$

In particular ν has a first moment and is centered by Proposition 14 when $a > 2$ and has no first moment otherwise. The anxious reader may also consider the following very convenient particular weight sequences:

$$q_k = c \cdot c_{\mathbf{q}}^{-k+1} \frac{\Gamma(\frac{1}{2} - a + k)}{\Gamma(\frac{1}{2} + k)} \mathbf{1}_{k \geq 2}, \qquad c_{\mathbf{q}} = 4a - 2, \qquad c = \frac{-\sqrt{\pi}}{2\,\Gamma(3/2 - a)}, \tag{5.12}$$

for which the probability measure ν takes a simple form:

$$\nu(k) = c\frac{\Gamma(3/2 - a + k)}{\Gamma(3/2 + k)} \mathbf{1}_{k \neq 0}. \qquad (k \in \mathbb{Z}) \tag{5.13}$$

See [24, Section 5] for a proof that this weight sequence is indeed critical and non-generic with exponent $a \in (3/2; 5/2)$. Notice that this weight sequence is term-wise continuous as $a \to 5/2$ taking the value $q_k = \frac{1}{12}\delta_2(k)$, which corresponds exactly to critical quadrangulations.

Notation: In order to state the results in a unified fashion we will say that a weight sequence $\mathbf{q}$ *is of type "*$5/2$*" if it is admissible, critical and regular; is of type "*$3/2$*" if it is admissible, subcritical and regular; and finally is of type "*a*" with* $a \in (3/2; 2) \cup (2; 5/2)$ *if it is admissible, critical and non-generic with exponent* a*. For the border case* $a = 2$ *we will say that* $\mathbf{q}$ *is of type* 2 *if* $\mathbf{q}$ *is* equal *to the particular sequence given in (5.12) for the value* $a = 2$*. In all cases the constant* $p_{\mathbf{q}}$ *appears in the asymptotic of the negative tail of* ν *with respect to* $|k|^{-a}$*.*

5.1.4 Link with the $O(n)$ model

In order to motivate a bit more the introduction of the non-generic critical weights let us briefly recall some results for the $O(n)$ model on random quadrangulations proved in [20]. A loop-decorated quadrangulation $(\mathfrak{q}, \mathbf{l})$ is a planar map whose faces are all quadrangles on which non-crossing loops $\mathbf{l} = (l_i)_{i\geq 1}$ are drawn (see Fig. 2 in [20]). For simplicity we consider the so-called rigid model when loops can only cross quadrangles through opposite sides. We define a measure on such configurations by putting

$$W_{h,g,n}((\mathfrak{q}, \mathbf{l})) = g^{|\mathfrak{q}|} h^{|\mathbf{l}|} n^{\#\mathbf{l}},$$

for $g, h > 0$ and $n \in (0, 2)$ where $|q|$ is the number of faces of the quadrangulation, $|\mathbf{l}|$ is the total length of the loops and $\#\mathbf{l}$ is the number of loops. Provided that the measure $W_{h,g,n}$ has finite total mass one can use it to define random loop-decorated quadrangulations with a fixed number of vertices. Fix $n \in (0, 2)$. For most of the parameters (g, h) these random planar maps are "sub-critical" (believed to be tree like when large) or "generic critical" (believed to converge to the Brownian map). However, there is a fine tuning of g and h (actually a critical line) for which these planar maps may have different behaviors. More precisely, their gaskets, obtained by pruning off the interiors of the outer-most loops (see Fig. 4 in [20]) are precisely non-generic critical Boltzmann planar map in the sense of the last section where

$$a = 2 \pm \frac{1}{\pi} \arccos(n/2).$$

The case $a = 2 - \frac{1}{\pi} \arccos(n/2) \in (\frac{3}{2}; 2)$ is called the *dense phase* because the loops on the gasket are believed in the scaling limit to touch themselves and each other. The case $a = 2 + \frac{1}{\pi} \arccos(n/2) \in (2; \frac{5}{2})$ (which occurs at the extremity of the critical line) is called the *dilute phase* because the loops on the gasket are believed to be simple in the scaling limit and avoiding each other.

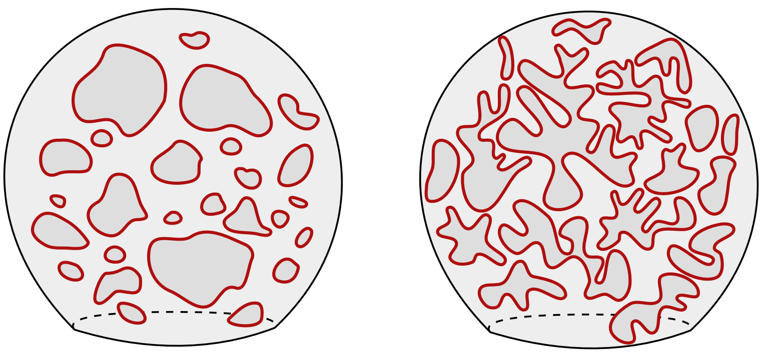


Figure 5.2: A schematic illustration of $\mathbf{q}$-Boltzmann RPM in the dilute (left) and dense phase (right).

5.2 Invariance principles for the perimeter processes

This section is devoted to describe the invariance principles for the process of the perimeters of the distinguished hole in a filled-in peeling exploration under $\mathbb{P}^{(\infty)}$, $\mathbb{P}_{\bullet}^{(\ell)}$ and $\mathbb{P}_{\infty}^{(\ell)}$. Recall from (3.4), (4.1) and (4.2) that $|\partial\bar{\mathfrak{e}}_n|$ the half-perimeter in a filled-in exploration under $\mathbb{P}_{\bullet}^{(\ell)}$, $\mathbb{P}_{\infty}^{(\ell)}$ or the algebraic variation thereof under $\mathbb{P}^{(\infty)}$ are respectively distributed as the Markov chains $S^{\downarrow}$, $S^{\uparrow}$ under $\mathbf{P}_{\ell}$ and the random walk S under $\mathbf{P}_0$. We will thus present the scaling limits in this section using the walk S and its relatives $S^{\downarrow}$ and $S^{\uparrow}$.

5.2.1 The case of the random walk S

Invariance principles for random walks with independent increments are standard tools in nowadays probability. We start by introducing the limiting processes involved which are stable Lévy processes (we refer to the book of Bertoin [12] for background). For $a \in [3/2; 5/2]$ let $(\Upsilon_a(t) : t \geq 0)$ be the $(a-1)$-stable Lévy process with positivity parameter $\varrho = \mathbb{P}(\Upsilon_t \geq 0)$ satisfying

$$(a-1)(1-\varrho) = \frac{1}{2}.$$

That is to say $(\Upsilon_a(t))_{t\geq 0}$ has no drift, no Brownian part and its Lévy measure has been normalized to

$$\Pi(\mathrm{d}x) = \cos(a\,\pi)\cdot\frac{\mathrm{d}x}{x^a}\mathbb{1}_{x>0} + \frac{\mathrm{d}x}{|x|^a}\mathbb{1}_{x<0}. \tag{5.14}$$

Notice that in the case $a = \frac{5}{2}$ or $a = \frac{3}{2}$, the process Υ_a has no positive jumps. Furthermore, in the case of $a = \frac{3}{2}$ it corresponds to the opposite of the $\frac{3}{2}$-stable subordinator (a pure jump process) and in the case $a = 2$ it is the symmetric Cauchy process without drift. Recall that under $\mathbf{P}_0$ the process $(S_n)_{n\geq 0}$ is a random walk with independent increments of law ν started from 0; the dependence in $\mathbf{q}$ hence in a in implicit in what follows.

Proposition 25 (Scaling limit for the random walk S)**.** *If $\mathbf{q}$ is of type $a \in [3/2; 5/2]$ then under $\mathbf{P}_0$ we have the following convergence in distribution for the Skorokhod topology*

$$\left(\frac{S_{[nt]}}{n^{1/(a-1)}}\right)_{t\geq 0} \xrightarrow[n\to\infty]{(d)} (\Upsilon_a(p_{\mathbf{q}}\cdot t))_{t\geq 0}.$$

Proof. By classical results (see [42]) the functional convergence towards the stable Lévy process is granted once we have the convergence of the finite-dimensional marginals or equivalently the convergence in distribution

$$\frac{S_n}{n^{1/(a-1)}} \xrightarrow[n\to\infty]{(d)} \Upsilon_a(p_{\mathbf{q}}) \overset{(d)}{=} (p_{\mathbf{q}})^{1/(a-1)}\Upsilon_a(1).$$

By the description of the tail asymptotics of ν given in the last section it follows that when $\mathbf{q}$ is of type a then ν is a probability distribution in the domain of attraction of an $(a-1)$-stable law. This means that $a_n^{-1}S_n - b_n$ converges to an $(a-1)$-stable law for some scaling sequence

(a_n) and centering sequence (b_n), see [17, Theorem 8.3.1]. By matching the Lévy measure Π with the tail asymptotics of ν it follows that one can take $a_n = n^{1/(a-1)}$ and it remains to show that the centering sequences (b_n) can be set to 0. This is always the case when $a \in (3/2; 2)$ since no centering is needed. In the case when $a \in (2; 5/2)$ the fact that the random walk $(S_i)_{i\geq 0}$ has mean 0 (Proposition 14) shows that the centering sequence can be set to 0 as well. If now $\mathbf{q}$ is of type 2, recall that it means that $\nu(k) = \frac{1}{4k^2-1}\mathbf{1}_{k\neq 0}$ by (5.13). We can then compute exactly in this particular case the form of (b_n) [do the calculation] and see that it can be taken equal to 0. (We believe that this scaling limit should also hold true for any critical and non-generic sequence with exponent $a = 2$ and not only the special case that we are dealing with). $\square$

For further use, let us note that under the assumptions of the above proposition, the local limit theorem [41, Theorem 4.2.1] implies that

$$\mathbf{P}_0(S_k = 0) \sim C_0 k^{-1/(a-1)} \tag{5.15}$$

as $k \to \infty$ for some $C_0 > 0$.

5.2.2 The cases of $S^\uparrow$ and $S^\downarrow$

Recall from Section 3.3 that since $h^\downarrow$ is harmonic for the walk S killed at first entrance of $\mathbb{Z}_{\leq 0}$ we can define the Markov chain $S^\downarrow$ as the $h^\downarrow$-transform of S and this chain corresponds to the evolution of the half-perimeter of the distinguished hole in peeling explorations under $\mathbb{P}_\bullet^{(\ell)}$. Also when $\mathbf{q}$ is critical, in particular if $\mathbf{q}$ is of type $a \in (3/2; 5/2]$ then we can also define $S^\uparrow$ via the $h^\uparrow$-transform which corresponds to the evolution of the half-perimeter of the distinguished hole in peeling explorations under $\mathbb{P}_\infty^{(\ell)}$. This chain is not defined when $a = \frac{3}{2}$. Recall also that under $\mathbf{P}_\ell$ the chains $S^\uparrow$ and $S^\downarrow$ start from $\ell \geq 1$.

We then introduce $\Upsilon_a^\uparrow$ and $\Upsilon_a^\downarrow$ the versions of the Lévy process Υ_a conditioned to stay positive and respectively to die continuously at 0 when it enters $\mathbb{R}_-$, these are obtained from continuous h-transforms of Υ_a using the functions $x \mapsto \sqrt{x}$ and $x \mapsto 1/\sqrt{x}$ respectively (see [28, 26] for the definition of these processes). Unless explicitly mentioned, the process $\Upsilon_a^\uparrow$ starts from 0 (this indeed has a meaning) and the process $\Upsilon_a^\downarrow$ is started from 1. The process $\Upsilon_{3/2}^\uparrow$ has no meaning since in the case $a = \frac{3}{2}$ the Lévy process $\Upsilon_{3/2}$ is just the opposite of a subordinator. Combining Proposition 25 with the recent invariance principles proved in [26] shows that:

Proposition 26 (Scaling limit for $S^\uparrow$ and $S^\downarrow$). *If $\mathbf{q}$ is of type $a \in [3/2; 5/2]$ we have the following convergences in distribution for the Skorokhod topology under $\mathbf{P}_\ell$ for $\ell \geq 1$, the first one being restricted to $a \neq 3/2$:*

$$\left(\frac{1}{n^{1/(a-1)}} \cdot S^\uparrow_{[nt]}\right)_{t\geq 0} \xrightarrow[n\to\infty]{(d)} (\Upsilon_a^\uparrow(\mathrm{p}_{\mathbf{q}} \cdot t))_{t\geq 0},$$
$$\left(\frac{1}{\ell} \cdot S^\downarrow_{[\ell^{(a-1)}t]}\right)_{t\geq 0} \xrightarrow[\ell\to\infty]{(d)} (\Upsilon_a^\downarrow(\mathrm{p}_{\mathbf{q}} \cdot t))_{t\geq 0}.$$

Here is a useful and geometric corollary of this scaling limit result:

Corollary 27 (The filled-in exploration discovers everything). *Suppose that* $\mathbf{q}$ *is of type* $a \in [3/2; 5/2]$. *Let* $(\bar{\mathfrak{e}}_n)_{n\geq 0}$ *be a filled-in peeling exploration under* $\mathbb{P}_\infty^{(\ell)}$ *for some algorithm* $\mathcal{A}$. *Then*

$$\bigcup_{n\geq 0} \bar{\mathfrak{e}}_n = \mathfrak{m}, \qquad \textit{almost surely.}$$

Proof. As in the proof of Lemma 17 one needs to prove that every "vertex" on the boundary of $\bar{\mathfrak{e}}_n$ is eventually swallowed by the peeling process and becomes an interior vertex of $\bar{\mathfrak{e}}_n$. We note however that conditionally on the past exploration, if ℓ is the length of the distinguished hole, then for any given vertex v on the boundary of the distinguished hole of $\bar{\mathfrak{e}}_n$ and any edge to peel on that hole there is probability of order

$$\sum_{k=[\ell/2]}^{\ell-1} b^{(\ell)}(*,k) \quad \text{or} \quad \sum_{k=[\ell/2]}^{\ell-1} b^{(\ell)}(k,*),$$

that the peeling of the edge $\mathcal{A}(\bar{\mathfrak{e}}_n)$ swallows in a finite region half of the boundary of the distinguished hole so that the vertex v becomes an inner vertex of $\bar{\mathfrak{e}}_{n+1}$. Using the exact values of $b_\infty^{(\ell)}$ described in Section 4.2.1 as well as the asymptotics for the tail of ν and the expression of $h^\uparrow$, it follows that the last display is of order ℓ^{1-a}. Since the half-perimeter of the only hole of $\bar{\mathfrak{e}}_n$ has the same law as $S^\uparrow$ the proof of the corollary boils down to proving that almost surely we have

$$\sum_{n=1}^{\infty} \left(\frac{1}{S_n^\uparrow}\right)^{a-1} = \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{n^{1/(a-1)}}{S_n^\uparrow}\right)^{a-1} = \infty.$$

We are exactly in the conditions to apply Jeulin's lemma [44, Proposition 4 c]. Since the proof is very short and elementary let us recall it here. We denote by $\mathcal{X}_n = \left(n^{1/(a-1)}/S_n^\uparrow\right)^{a-1}$ and suppose by contradiction that the random series $\sum \frac{1}{n}\mathcal{X}_n$ is bounded by $M \geq 0$ on an event A of probability at least $\varepsilon > 0$. Using the convergence of $\mathcal{X}_n$ towards the strictly positive random variable $\Upsilon_a^{-1}(\mathfrak{p}_\mathbf{q})$ implied by Proposition 26 we deduce that we can find $\delta > 0$ such that

$$\liminf_{n\to\infty} \mathbb{P}(\mathcal{X}_n \geq \delta) \geq 1 - \frac{\varepsilon}{2}.$$

Then taking expectation on the event A we deduce that

$$\mathbb{E}[\mathbf{1}_A \sum_{n\geq 1} \frac{1}{n}\mathcal{X}_n] = \sum_{n\geq 1} \frac{1}{n}\mathbb{E}[\mathbf{1}_A \mathcal{X}_n] \leq M.$$

But using the above display we deduce that $\mathbb{E}[\mathbf{1}_A \mathcal{X}_n] \geq \delta \max(0, \varepsilon - \mathbb{P}(\mathcal{X}_n < \delta))$ is asymptotically larger than $\delta\varepsilon/2$. Plugging back into the last display we find a contradiction because the series $\sum_{n\geq 1} \frac{1}{n}\varepsilon\delta/2$ is obviously not summable! □

Open question. *Is the last corollary true for any critical weight sequence?*

5.2.3 Case of $\mathbb{P}^{(\ell)}$: the locally largest cycle

In the case of peeling exploration under the finite Boltzmann measure $\mathbb{P}^{(\ell)}$, there is no canonical filled-in exploration. But recall that we defined in Section 3.1.4 a tailor made filled-in exploration $(\bar{\bar{\mathfrak{e}}}_n)_{n\geq 0}$ which follows the locally largest hole. The scaling limit for the half-perimeter process $|\partial\bar{\bar{\mathfrak{e}}}_n|$ is also known, and takes the following form: If $\mathbf{q}$ is of type $a \in [3/2; 5/2]$ under $\mathbb{P}^{(\ell)}$ we have the following convergence in distribution for the Skorokhod topology:

$$\left(\frac{1}{\ell} \cdot |\partial\bar{\bar{\mathfrak{e}}}_{[\ell^{(a-1)}t]}|\right)_{t\geq 0} \xrightarrow[\ell\to\infty]{(d)} (\aleph_a(p_{\mathbf{q}} \cdot t))_{t\geq 0}.$$

The process $\aleph_a$ is a little bit complicated to described: it is a self-similar Markov process with self-similarity index $a-1$ which starts from 1 and does not make negative jumps of more than half of its current size. By the classical Lamperti transformation [47], these processes can be written as time-change exponential of an underlying Lévy process ξ which in this case has a Lévy exponent Ψ_a with $\mathbb{E}[\exp(q\xi(t))] = \exp(t\Psi(q))$ for $q > 0$ where

$$\Psi_a(q) = \left(\frac{\Gamma(3-a)}{2\Gamma(4-2a)\cos(\pi(a-1))} + \frac{\Gamma(a)B_{\frac{1}{2}}(-a+1, 3-a)}{\pi}\right) q + \int_{\mathbb{R}} (\mathrm{e}^{qy} - 1 + q(1-\mathrm{e}^{y}))\, \Lambda_a(\mathrm{d}x) \tag{5.16}$$

where $B_{1/2}(u,v) = \int_0^{1/2} t^{u-1}(1-t)^{v-1}\mathrm{d}t$ is the incomplete Beta function and where Λ_a is the image of μ_a by the map $x \mapsto \ln x$ with

$$\mu_a(\mathrm{d}x) = \frac{\Gamma(a)}{\pi}\left(\frac{1}{(x(1-x))^a}\mathbb{1}_{1/2<x<1} + \cos(a\pi)\cdot\frac{1}{(x(x-1))^a}\mathbb{1}_{x>1}\right)\mathrm{d}x.$$

We do not give the details of the proof here and refer to Chapter **??**.

5.3 Scaling limit for the volume process

We will now study the volume of the explored map in filled-in peeling explorations. Recall that during the filled-in explorations under $\mathbb{P}^{(\ell)}_\bullet, \mathbb{P}^{(\ell)}, \mathbb{P}^{(\infty)}$ or $\mathbb{P}^{(\ell)}_\infty$ the non-distinguished finite holes created during the peeling are filled-in with independent Boltzmann maps of the proper perimeter. One should first start by studying the renormalized volume of Boltzmann maps with a large perimeter.

5.3.1 Stable limit for the volume of Boltzmann maps

For $a \in [3/2; 5/2]$ we let $\xi_\bullet(a)$ be a positive $1/(a-\frac{1}{2})$-stable random variable with Laplace transform

$$\mathbb{E}[e^{-\lambda\xi_\bullet(a)}] = \exp\left(-(\Gamma(a+1/2)\lambda)^{\frac{1}{a-1/2}}\right). \tag{5.17}$$

In the case $a = 3/2$ this random variable is simply the constant 1. In general we have $\mathbb{E}[1/\xi_\bullet(a)] = \int_0^\infty \mathrm{d}x\, \exp(-x^{1/(a-1/2)})/\Gamma(a+\frac{1}{2}) = 1$ and we can define a random variable ξ_a by biasing $\xi_\bullet(a)$ by $x \to 1/x$, that is for any $f \geq 0$

$$\mathbb{E}[f(\xi_a)] = \mathbb{E}\left[f(\xi_\bullet(a))\frac{1}{\xi_\bullet(a)}\right].$$

Notice that ξ_a has mean $\mathbb{E}[\xi_a] = 1$. Recall that $|\mathfrak{m}|$ denotes the number of vertices of a map $\mathfrak{m}$.

Proposition 28. *Suppose that* $\mathbf{q}$ *is of type* $a \in [3/2; 5/2]$*. Then we have*

$$\mathbb{E}^{(\ell)}[|\mathfrak{m}|] \sim \mathsf{b}_\mathbf{q} \cdot \ell^{a-1/2} \quad \text{as } \ell \to \infty \qquad \text{where} \qquad \mathsf{b}_\mathbf{q} = \frac{2}{c_\mathbf{q}\, p_\mathbf{q} \sqrt{\pi}}, \tag{5.18}$$

and we have the convergence in distribution under $\mathbb{P}^{(\ell)}$

$$\ell^{-a+\frac{1}{2}}|\mathfrak{m}| \xrightarrow[\ell\to\infty]{(\mathrm{d})} \mathsf{b}_\mathbf{q} \cdot \xi_a. \tag{5.19}$$

It is good to notice that in the subcritical case $a = \frac{3}{2}$ the volume (i.e. the number of vertices) of a map $\mathfrak{m}$ under $\mathbb{P}^{(\ell)}$ is proportional to its perimeter (the random variable $\xi_{3/2}$ is just the constant 1). This indicates that the map is folded on its external face and is tree-like.

Proof. Recall from (2.8) that we have

$$\mathbb{E}^{(\ell)}[|\mathfrak{m}|] = \frac{W_\bullet^{(\ell)}}{W^{(\ell)}}.$$

It follows from Theorem 6 that we have the exact expression $W_\bullet^{(\ell)} = c_\mathbf{q}^\ell h^\downarrow(\ell)$. Combining this with the asymptotics for $W^{(\ell)}$ that we derived in Section 5.1 we get the first statement of the proposition. To prove the second statement it is sufficient (using the first point) to prove convergence in distribution of the rescaled volume of $\mathfrak{m}_\bullet$ under $\mathbb{P}_\bullet^{(\ell)}$ towards $\mathsf{b}_\mathbf{q} \cdot \xi_\bullet(a)$ (see [19, Proof of Proposition 4] for details). To show this, recall from (5.2) and the notation introduced there that we have

$$\mathbb{E}_\bullet^{(\ell)}[g^{|\mathfrak{m}|-1}] = \frac{1}{W_\bullet^{(\ell)}} \sum_{\mathfrak{m} \in \mathcal{M}^{(\ell)}} |\mathfrak{m}| w_\mathbf{q}(\mathfrak{m}) g^{|\mathfrak{m}|-1} = \frac{\left(g c_{\mathbf{q}_g}\right)^\ell h^\downarrow(\ell)}{c_\mathbf{q}^\ell h^\downarrow(\ell)} = \left(\frac{g c_{\mathbf{q}_g}}{c_\mathbf{q}}\right)^\ell,$$

where $c_{\mathbf{q}_g}/4$ is the smallest solution to $f_\mathbf{q}(gx) = 1 - \frac{1}{x}$. Recall from Section 5.1 the asymptotic expansion

$$\frac{g c_{\mathbf{q}_g}}{c_\mathbf{q}} = (1 - \kappa(1-g)^{1/(a-1/2)}) + o(1-g)^{1/(a-1/2)}.$$

Using this with $g = \exp(-\lambda \ell^{\frac{1}{2}-a})$ we find, using the relations between $c_\mathbf{q}, p_\mathbf{q}$ and $\mathsf{b}_\mathbf{q}$ that

$$\begin{aligned}\lim_{\ell\to\infty} \mathbb{E}_\bullet^{(\ell)}\left[\exp\left(-\lambda \ell^{\frac{1}{2}-a}|\mathfrak{m}_\bullet|\right)\right] = \lim_{\ell\to\infty}\left(\frac{g c_{\mathbf{q}_g}}{c_\mathbf{q}}\right)^\ell &= \exp(-\kappa \lambda^{1/(a-1/2)}) \\ &= \exp(-(\mathsf{b}_\mathbf{q} \cdot \lambda)^{1/(a-1/2)}) = \mathbb{E}[e^{-\lambda \mathsf{b}_\mathbf{q} \xi_\bullet(a)}],\end{aligned}$$

thereby proving the desired convergence in distribution. $\square$

Remark 11. In the above proposition we work with the number of vertices to measure the size of a map. However, all the results are true if one instead measures the volume in terms of the number of faces of the map provided that we change the constant $\mathsf{b}_\mathbf{q}$ into

$$\mathsf{b}_\mathbf{q}^F = \left(\frac{c_\mathbf{q}}{4} - 1\right) \mathsf{b}_\mathbf{q}.$$

5.3.2 Functional scaling limit for the volume and perimeter

We are now able to introduce the joint scaling limit for the perimeter and volume processes during a filled-in peeling exploration. For $a \in [3/2; 5/2]$, if $(\chi_t)_{t\geq}$ is a given càdlàg process we associate with it another process $\mathcal{V}_a(\chi)$ defined as follows. We let $\xi_a^{(1)}, \xi_a^{(2)}, \ldots$ be a sequence of independent real random variables distributed as the variable ξ_a of Proposition 28. We assume that this sequence is independent of the process χ and for every $t \geq 0$ we set

$$\mathcal{V}_a(\chi)(t) = \sum_{t_i \leq t} \xi_a^{(i)} \cdot |\Delta\chi(t_i)|^{a-\frac{1}{2}} \mathbf{1}_{\Delta\chi(t_i)<0}, \tag{5.20}$$

where $t_1, t_2, \ldots$ is a measurable enumeration of the jump times of χ. In general, $\mathcal{V}_a(\chi)$ may be infinite, but since $x \mapsto x^{a-\frac{1}{2}} \mathbf{1}_{x<0}$ integrates the Lévy measure of $(\Upsilon_a(t) : t \geq 0)$ in the neighborhood of 0 it is easy to check that $\mathcal{V}_a(\Upsilon_a)$ is a.s. finite for all $t \geq 0$. It can also be shown using absolute continuity relations with Υ_a that the processes $\mathcal{V}_a(\Upsilon_a^\uparrow)$ as well as $\mathcal{V}_a(\Upsilon_a^\downarrow)$ also make sense.

Theorem 9 (*Scaling limits of the filled-in peeling processes*)

Let $(\bar{\mathfrak{e}}_n)_{n\geq 0}$ be a filled-in exploration process and denote by $|\partial\bar{\mathfrak{e}}_n|$ and $|\bar{\mathfrak{e}}_n|$ the half-perimeter of the hole (or its algebraic variation in the case when the hole is of infinite perimeter) and the number of inner vertices of $\bar{\mathfrak{e}}_n$. Then if $\mathbf{q}$ is of type $a \in [3/2; 5/2]$ we have the following convergences in distribution in the sense of Skorokhod

$$
\begin{aligned}
&\text{under } \mathbb{P}^{(\infty)}, && \left(\frac{|\partial\bar{\mathfrak{e}}_{[nt]}|}{n^{\frac{1}{a-1}}}, \frac{|\bar{\mathfrak{e}}_{[nt]}|}{n^{\frac{a-1/2}{a-1}}}\right)_{t\geq 0} \xrightarrow[n\to\infty]{(d)} \left(\left(\Upsilon_a, \mathrm{b}_\mathbf{q}\cdot\mathcal{V}_a(\Upsilon_a)\right)(\mathrm{p}_\mathbf{q} t)\right)_{t\geq 0},\\
&\text{under } \mathbb{P}^{(\ell)}_\infty, && \left(\frac{|\partial\bar{\mathfrak{e}}_{[nt]}|}{n^{\frac{1}{a-1}}}, \frac{|\bar{\mathfrak{e}}_{[nt]}|}{n^{\frac{a-1/2}{a-1}}}\right)_{t\geq 0} \xrightarrow[n\to\infty]{(d)} \left(\left(\Upsilon_a^\uparrow, \mathrm{b}_\mathbf{q}\cdot\mathcal{V}_a(\Upsilon_a^\uparrow)\right)(\mathrm{p}_\mathbf{q} t)\right)_{t\geq 0},\\
&\text{under } \mathbb{P}^{(\ell)}_\bullet, && \left(\frac{|\partial\bar{\mathfrak{e}}_{[\ell^{(a-1)}t]}|}{\ell}, \frac{|\bar{\mathfrak{e}}_{[\ell^{(a-1)}t]}|}{\ell^{a-1/2}}\right)_{t\geq 0} \xrightarrow[\ell\to\infty]{(d)} \left(\left(\Upsilon_a^\downarrow, \mathrm{b}_\mathbf{q}\cdot\mathcal{V}_a(\Upsilon_a^\downarrow)\right)(\mathrm{p}_\mathbf{q} t)\right)_{t\geq 0}.
\end{aligned}
$$

Proof. The convergence of the first components in each of the above lines is given by Proposition 25 and Proposition 26. It thus remains to study the conditional distribution of the second component given the first one. We will do that in the first case since the arguments are the same in the two others cases. To simplify a bit notation we write $P_n = |\partial\bar{\mathfrak{e}}_n|$ and $V_n = |\bar{\mathfrak{e}}_n|$. Recall from the explicit transitions of the filled-in peeling process under $\mathbb{P}^{(\infty)}$ that the number of inner vertices in $\bar{\mathfrak{e}}_n$ can be written as

$$V_n = \sum_{i=0}^{n-1} \mathcal{X}^{(i)}_{P_i - P_{i+1}-1},$$

where $\mathcal{X}^{(i)}_j$ for $i \geq 0$ and $j \in \mathbb{Z}$ are independent random variables such that $\mathcal{X}^{(i)}_j$ has the same distribution as the number of vertices inside a $\mathbf{q}$-Boltzmann random map with perimeter $2j$ if $j \geq 0$ and is 0 otherwise. To simplify further we use the notation $\widetilde{\Delta}P_i = P_i - P_{i+1} - 1$ if $\Delta P_i \leq -1$ and 0 otherwise. Fix $\varepsilon > 0$ and set for $k \in \{1, 2, \dots, n\}$

$$V_k^{>\varepsilon} = \sum_{i=0}^{k-1} \mathcal{X}^{(i)}_{\widetilde{\Delta}P_i} \mathbf{1}_{\widetilde{\Delta}P_i > \varepsilon n^{1/(a-1)}}, \quad V_k^{\leq\varepsilon} = \sum_{i=0}^{k-1} \mathcal{X}^{(i)}_{\widetilde{\Delta}P_i} \mathbf{1}_{0\leq\widetilde{\Delta}P_i \leq \varepsilon n^{1/(a-1)}}. \tag{5.21}$$

We first observe that $n^{-(a-1/2)/(a-1)}\mathbb{E}[V_n^{\leq\varepsilon}]$ is small uniformly in n when ε is small. Indeed, using the bound $\mathbb{E}[\mathcal{X}_\ell] \leq C\ell^{a-1/2}$ provided in Proposition 28 we can write

$$
\begin{aligned}
\mathbb{E}[V_n^{\leq\varepsilon}] \leq C\sum_{i=0}^{n-1} \mathbb{E}\left[|\tilde{\Delta}P_i|^{a-1/2}\mathbf{1}_{\tilde{\Delta}P_i\leq\varepsilon n^{1/(a-1)}}\right] &= Cn\sum_{k=1}^{\varepsilon n^{1/(a-1)}} (k-1)^{a-1/2}\nu(-k)\\
&\leq C'\sqrt{\varepsilon}n^{(a-1/2)/(a-1)}.
\end{aligned}
\tag{5.22}$$

On the other hand, by Propositions 25 and 28 together with the fact that $(\Upsilon_a(t))_{t\geq 0}$ does not have jumps of size exactly $-\varepsilon$ almost surely, we deduce that jointly with the convergence of the rescaled perimeter towards Υ_a we have the following convergence in distribution for the Skorokhod topology

$$\left(n^{-\frac{a-1/2}{a-1}}\cdot V^{>\varepsilon}_{[nt]}\right)_{t\geq 0} \xrightarrow[n\to\infty]{(d)} \left(\mathrm{b}_\mathbf{q}\cdot\mathcal{V}_a^{>\varepsilon}(\Upsilon_a)(\mathrm{p}_\mathbf{q} t)\right)_{t\geq 0}, \tag{5.23}$$

where the process $\mathcal{V}_a^{>\varepsilon}(\chi)$ is defined as $\mathcal{V}_a(\chi)$ but only keeping the negative jumps of χ of absolute size larger than ε. Then, it is easy to verify that, for every $\delta > 0$ and any $t_0 > 0$ fixed we have

$$\mathbb{P}\Big(\sup_{0\le t\le t_0} |\mathcal{V}_a(\Upsilon_a)(t) - \mathcal{V}_a^{>\varepsilon}(\Upsilon_a)(t)| > \delta \Big) \xrightarrow[\varepsilon\to 0]{} 0.$$

Letting $\varepsilon \to 0$ we can use the last display, together with (5.23) and (5.22) to deduce the desired convergence in distribution. □

Bibliographical notes. The first estimates for the filled-in face-peeling process were performed by Angel [4] in the case of the UIPT. Scaling limits in distribution were computed in [35] in the case of the filled-in face-peeling process on critical triangulations and quadrangulations. These were later extended by Budd [23] to the present peeling process on **q**-Boltzmann maps of type $\frac{5}{2}$. The introduction of the non-generic critical weight sequences is due to [48], and the scaling limit for the filled-in peeling process when **q** is of type $(3/2; 5/2)$ is taken from [24].

Chapter VI : Percolation(s)

In this chapter, we first deal with random maps of the half-plane.

In this chapter we study percolations (face, dual-face and bond percolations) on random Boltzmann planar maps. We will focus on the case of the law $\mathbb{P}^{(\infty)}$ for which the calculations are the simplest. Our main tool of course will be the peeling process. However, contrary to the cases we considered so far, the peeling process will be driven by the percolation of the map which is itself random (so far we have considered only deterministic peeling process). We will first see that such explorations are indeed allowed and do not change the probability transitions. We also introduce in the next section the mean gulp and exposure which are very important geometric quantities.

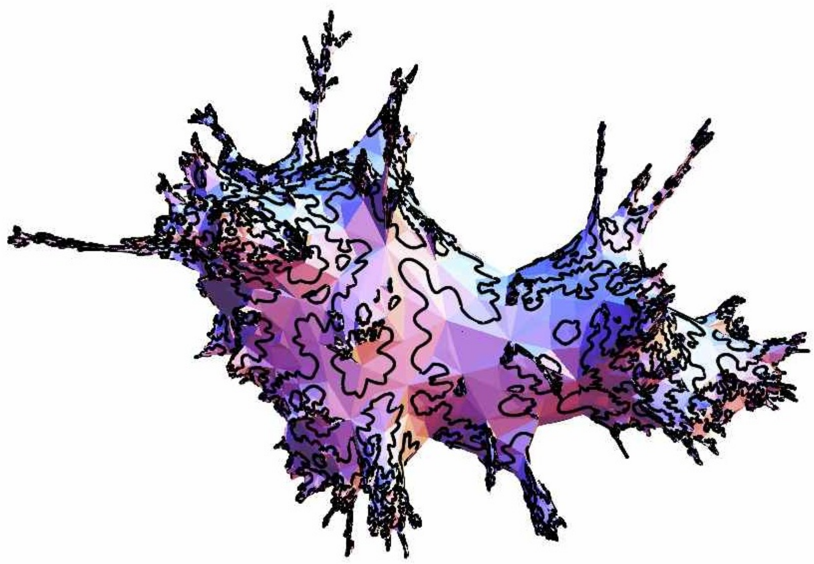


Figure 6.1: A random triangulation of the sphere together with the interfaces induced by a site percolation on it.

6.1 Prerequisites

6.1.1 Randomized peeling process

In the preceding chapters we have shown that under the Boltzmann distributions, the peeling process (with or without distinguished cycle, filled-in or not) is actually a Markov chain for which we computed the exact probability transitions. This was established when the peeling algorithm $\mathcal{A}$ is a deterministic function of the explored map $\mathfrak{e}, \bar{\mathfrak{e}}$ or $\mathfrak{e}^*$ when there is a distinguished cycle. Now, we can imagine that we choose the algorithm at random. More precisely, imagine that the peeling algorithm $\mathcal{A}$ depends on the explored map $\mathfrak{e}$ (possibly with a distinguished hole) together with another parameter ω that we may think as an additional randomness: $\mathcal{A}(\mathfrak{e}, \omega)$ is then the next edge to peel on the explored map $\mathfrak{e}$ given the additional parameter ω. Now, we suppose that the additional parameter ω which we see as a random variable[1] has law $\mathbf{P}$ and that it is independent of the underlying map $\mathfrak{m}$ sampled according to $\mathbb{P}^{(\ell)}, \mathbb{P}^{(\ell)}_\bullet, \mathbb{P}^{(\infty)}$ or $\mathbb{P}^{(\ell)}_\infty$. If we denote $\mathbb{P}$ generically for one of the last laws then

Proposition 29. *Under the law* $\mathbf{P}(\mathrm{d}\omega) \otimes \mathbb{P}(\mathrm{d}\mathfrak{m})$*, the law of the peeling process of* $\mathfrak{m}$ *with the "randomized" peeling algorithm* $\mathfrak{e} \mapsto \mathcal{A}(\mathfrak{e}, \omega)$ *is again a Markov chain with the same probability transitions as for a deterministic peeling algorithm.*

In other words, one can use the knowledge of the explored map together possibly with another source of randomness which is independent of the underlying map to choose which edge to peel for the next step. Usually, we use this source of randomness together with the map to define a stochastic process on the map itself: percolation, simple random walk, fpp percolation, SLE_6 process etc. Our explorations in these contexts be will "local" so that the peeling algorithm can always be written in the above form.

6.1.2 Mean gulp and exposure

Let $\mathbf{q}$ be an admissible weight sequence and recall from Definition 15 the probability measure ν which describes the asymptotic changes of the half-perimeter of large holes during a step of peeling. We shall encounter at many occasions the following quantities:

[1] We are a bit loose here on the state space of ω and $\mathbf{P}$, but we can think of $[0,1]^{\mathbb{N}}$ with the standard σ-field and the product Lebesgue measure. That will work for all our purposes.

Definition 20 (Mean exposure and gulp)**.** *The* mean exposure *(for Boltzmann maps with weight sequence* $\mathbf{q}$*) is the mean number of edges "exposed" during a peeling step of type* C_k *on the infinite hole under* $\mathbb{P}^{(\infty)}$ *that is:*

$$\mathbf{e_q} = \sum_{k\geq 1} b^{(\infty)}(k)(2k-1) = \sum_{k\geq 0} q_{k+1} c_{\mathbf{q}}^k (2k+1).$$

The mean gulp *is the mean number of edges swallowed on the boundary appart from the peeled edge, during a peeling step of type* $G_{\infty,k}$ *under* $\mathbb{P}^{(\infty)}$ *that is*

$$\mathbf{g_q} = \sum_{k\geq 0} b^{(\infty)}(\infty,k)(2k+1) = \sum_{k\geq 0} W^{(k)} c_{\mathbf{q}}^{-k-1}(2k+1).$$

The mean exposure and gulp may be infinite. In fact when $\mathbf{q}$ is of type a with $a \in (3/2; 2]$ then both $\mathbf{e_q}$ and $\mathbf{g_q}$ are infinite. When $\mathbf{q}$ is of type $a = \frac{3}{2}$ then only the mean gulp is infinite whereas the mean exposure is finite. Finally, when ν has a first moment, then these two quantities are finite. Furthermore, since in this case ν is centered (Proposition 14) we must have:

$$2\mathbf{g_q} + 1 = \mathbf{e_q}. \tag{6.1}$$

In the case of critical quadrangulations for example we have $\mathbf{e_q} = 3 \cdot b^{(\infty)}(2) = 3\frac{c_{\mathbf{q}}}{12} = 2$.

6.2 Face percolation

If $\mathfrak{m}$ is a planar map, a face-percolation of $\mathfrak{m}$ is a coloration of its faces in black and white. This can be modeled by a function $\eta : \mathsf{Faces}(\mathfrak{m}) \to \{0,1\}$ where 0 represents white faces and 1 represents black faces. Two faces are part of the same cluster in the percolation if they share an edge in common and if they have the same color (equivalently it corresponds to site percolation on the dual map $\mathfrak{m}^\dagger$). As usual in percolation theory we are interested in the probability that there exists an infinite cluster.

6.2.1 The primal exploration

In the case when $\mathfrak{m}$ is a map whose external face is of infinite degree we should change a little bit our definition and introduce a special boundary condition: Roughly speaking we image that the external face is colored in white, except for a little region on the right of the root edge. To be rigorous we can imagine that we split the initial root edge in order to create a face of degree 2 on its right (this face thus becomes the root face of the map). Then all the remaining inner faces

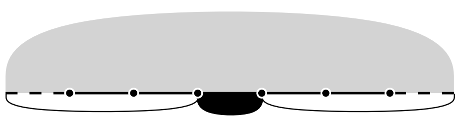


Figure 6.2: The initial boundary condition for face percolation.

of the map $\mathfrak{m}$ are independently of each other colored in black with probability $p \in (0,1)$ and in white otherwise. By denote by $\mathrm{P}_{\mathfrak{m}}^{p,\text{face}}$ the probability measure on percolations of $\mathfrak{m}$ obtained this way. We also denote by $\mathcal{C}$ the cluster of the little black region on the right of the root edge.

Theorem 10 (***Face percolation thresholds***)

Suppose that both e_q *and* g_q *are finite, then the percolation threshold for face percolation is* $p_{c,\text{face}} = (\mathsf{e}_\mathsf{q}+1)/(2\mathsf{e}_\mathsf{q})$*. More precisely when* $p \in (0,1)$ *we have*

$$\int \mathbb{P}^{(\infty)}(\mathrm{d}\mathfrak{m}) \int \mathrm{P}_{\mathfrak{m}}^{p,\text{face}}(\mathrm{d}\eta) \mathbf{1}_{\{|\mathcal{C}|=\infty\}} = \begin{cases} 0 & \text{if } p \le p_{c,\text{face}}, \\ > 0 & \text{if } p > p_{c,\text{face}}. \end{cases}$$

The above theorem actually gives an "annealed" percolation threshold since we average over the map $\mathfrak{m}$. However it is easy to show from its proof that $p_{c,\text{face}}$ actually corresponds to an almost sure (or "quenched") percolation threshold, see Proposition 31.

To prove the above theorem we use a particular (filled-in) peeling algorithm that explores the underlying map along the percolation. More precisely, imagine that after the first few steps of the exploration we have discovered a submap $\bar{\mathfrak{e}}_n \subset \mathfrak{m}$ which has the particularity that along the unique hole of infinite perimeter the inner faces of $\bar{\mathfrak{e}}_n$ which are part of the cluster of the root face form a connected black component and that all other faces incident to the boundary are white. See Fig. 6.3. The peeling algorithm $\mathcal{A}_{\text{face}}$ then selects the edge on the active boundary which is the left-most among those adjacent to an inner face belonging to the cluster of the root face. Clearly this algorithm can be written in the form $\mathcal{A}_{\text{face}}(\bar{\mathfrak{e}}_n, \omega)$ where ω is an external parameter (independent of the map $\mathfrak{m}$) from which the percolation configuration is retrieved (Exercise: prove it!).

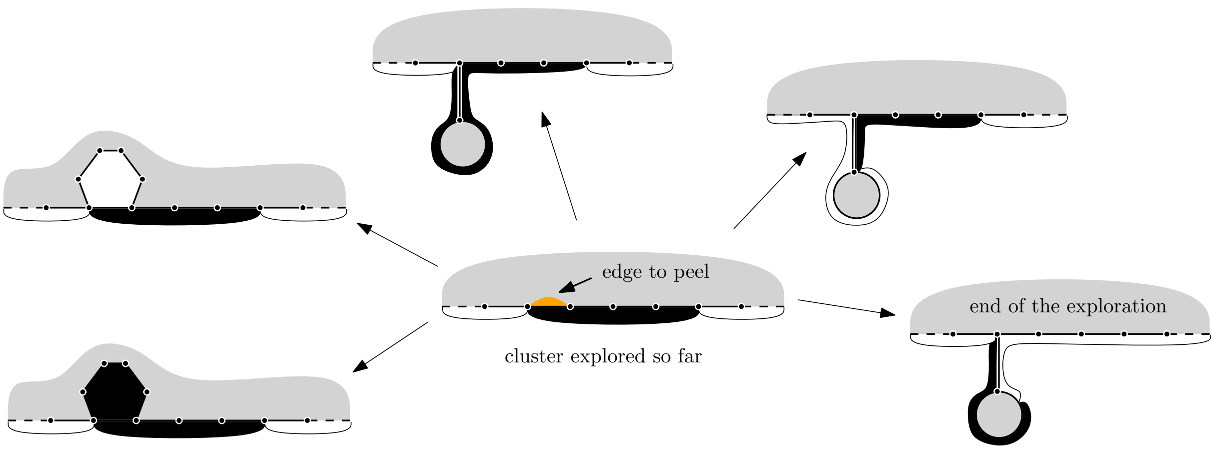


Figure 6.3: Illustration of the peeling algorithm used in the case of face-percolation. The part in gray have not been explored.

The key is that after one step of this peeling algorithm (and possibly filling-in the finite hole created) the explored map $\bar{\mathfrak{e}}_{n+1}$ we obtain is again of the above form and one can thus iterate the exploration. We write B_n for the number of inner faces of $\bar{\mathfrak{e}}_n$ adjacent to its unique infinite hole and which are part of the cluster of the root face. If $B_n = 0$ then the peeling algorithm stops, i.e. $\mathcal{A}_{\text{face}}(\bar{\mathfrak{e}}_n, \omega) = \dagger$. We have the following deterministic lemma:

Lemma 30. *Let $\mathfrak{m}$ be an infinite planar map of the half-plane. We suppose that $\mathfrak{m}$ carries a face percolation and has the above boundary condition. With the above notation we have*

$$|\mathcal{C}| = \infty \iff B_n > 0, \forall n \geq 0.$$

Proof. The proof should be clear from Fig. 6.3. In words, if the process survives for ever then we discover an infinite number of faces in the cluster of the root face. A contrario, if the process stops that means than some peeling step during an event of type $G_{\infty,k}$ has swallowed all the current "black" boundary of the hole thus caging the cluster of the origin into white faces, $\mathcal{C}$ is thus finite. Notice that the holes we fill-in during the exploration are all finite and thus cannot change the finite or infinite nature of $\mathcal{C}$. □

Proof of Theorem 10. With these ingredients, the proof is easy. It suffices to notice that under $\mathbb{P}^{(\infty)}(\mathrm{d}\mathfrak{m})\mathrm{P}_{\mathfrak{m}}^{p,\text{face}}(\mathrm{d}\eta)$ the process $(B_n)_{n\geq 0}$ roughly evolves as a random walk. More precisely the evolution $\Delta B_n = B_{n+1} - B_n$ is summarized in the next display:

$$\begin{array}{ll} \Delta B_n = -1 & \text{Event } \mathsf{G}_{k,\infty} \\ \Delta B_n = \sup(-B_n, -2k-2) & \text{Event } \mathsf{G}_{\infty,k} \\ \Delta B_n = -1 & \text{Event } \mathsf{C}_k \text{ face white} \\ \Delta B_n = -1 + (2k-1) & \text{Event } \mathsf{C}_k \text{ face black.} \end{array}$$

From this display we see that $(B_n)_{n\geq 0}$ has the same law as the random walk $(S_p(n))_{n\geq 0}$ started from 1, killed and set to 0 when it touches $\mathbb{Z}_{\leq 0}$, and with independent increments distributed as

$$\begin{array}{lll} \Delta S_p(n) = -2k & \text{with proba } \frac{1}{2}\nu(-k), & k \leq -1 \\ \Delta S_p(n) = 2k & \text{with proba } p \times \nu(k), & k \geq 0 \\ \Delta S_p(n) = -1 & \text{otherwise.} & \end{array}$$

The expectation of the above increment is $\delta = p\,\mathsf{e_q} - 1 - \mathsf{g_q}$. Clearly if $\delta > 0$ then S_p drifts towards infinity. In that case there is a positive probability that the walk S_p and *a fortiori* B stay positive forever. On the contrary if $\delta \leq 0$ then S_p almost surely visits the negative side and so B is eventually 0. Combining this with the relation $\mathsf{e_q} = 1 + 2\mathsf{g_q}$ gives our theorem. □

Proposition 31 (Quenched percolation threshold)**.** *Almost surely for $\mathbb{P}^{(\infty)}(\mathrm{d}\mathfrak{m})\mathrm{P}_{\mathfrak{m}}^{p,\text{face}}(\mathrm{d}\eta)$ there are no infinite black cluster inside the map $\mathfrak{m}$ (and not only starting from the root edge) if and only if $p \leq p_{c,\text{face}}$.*

Proof. Consider first the case when $p \leq p_{c,\text{face}}$. Then we know that $\mathbb{P}^{(\infty)}(\mathrm{d}\mathfrak{m})\mathrm{P}_{\mathfrak{m}}^{p,\text{face}}(\mathrm{d}\eta)$-almost surely the exploration of the cluster of the black root face terminates and leaves an unexplored region of law $\mathbb{P}^{(\infty)}(\mathrm{d}\mathfrak{m})\mathrm{P}_{\mathfrak{m}}^{p,\mathrm{face}}(\mathrm{d}\eta)$ but with a totally white boundary condition. Picking any edge on the boundary we may trigger a new peeling step until one discovers a black face which enables us to restart a percolation exploration which will again die out because $p \leq p_{c,\mathrm{face}}$. Iterating this, we can make sure that we explore the full map which thus contains only finite black clusters. Suppose now that $p > p_{c,\mathrm{face}}$. Since the cluster of the origin has a positive chance to be infinite, we deduce that for every $\varepsilon > 0$ there exists $M > 0$ such that if the percolation exploration starting at the root face is still alive after M steps of exploration then it will never die with probability at least $1-\varepsilon$. We denote by q the probability that the initial exploration runs for at least M steps. We thus start the percolation exploration at the root face and let it run until it either die or reaches M steps. In the second case we have an infinite cluster (to which the origin face belongs) with probability at least $1-\varepsilon$. In the first case we are left with a unexplored region of law $\mathbb{P}^{(\infty)}(\mathrm{d}\mathfrak{m})\mathrm{P}_{\mathfrak{m}}^{p,\mathrm{face}}(\mathrm{d}\eta)$ but with a totally white boundary condition which is furthermore *independent* of the exploration we have performed so far. We can then trigger a few peeling steps until we find a black face to restart the percolation exploration. Hence, the probability α to see an infinite cluster in $\mathfrak{m}$ is at least

$$\alpha \geq \underbrace{(1-\varepsilon)q}_{\text{second case}} + \underbrace{(1-q)\alpha}_{\text{first case}}.$$

Since ε was arbitrary the above display shows that $\alpha = 1$. □

Exercise 11. Let $\mathbf{q}$ be an arbitrary admissible weight sequence (in particular we do not suppose that $e_{\mathbf{q}}$ and $g_{\mathbf{q}}$ are finite). Show that $\mathbb{P}^{(\infty)}(\mathrm{d}\mathfrak{m})\mathrm{P}_{\mathfrak{m}}^{p,\mathrm{face}}(\mathrm{d}\eta)$-almost surely there is at most one infinite black cluster. (Hint: Use Proposition 14 to show that two black interfaces if they survive must collide).

6.2.2 Dual exploration

In this section we study the “dual” of the face percolation studied in the last section. By dual percolation we mean that if the origin cluster is blocked this is because there is a “dual” cluster in the dual percolation preventing it from going further. In particular we shall prove the unsurprising result that the corresponding thresholds equal 1 minus the initial one.

Face percolation is not self-dual. If two faces have only a common vertex but no common edge, they need not be part of the same white cluster, but two such faces do form a local barrier for connection of black faces. Hence the dual percolation of face percolation is face percolation but where two faces are part of the same cluster if they have the same color and if they share a vertex. Equivalently it corresponds to site percolation on the dual lattice where we add connections between sites whose dual faces share a vertex. This is known as the star-lattice in the case of $\mathbb{Z}^d$. We call it face′ percolation in the sequel.

Theorem 11 (*Dual face percolation thresholds*)

Suppose that both $e_{\mathbf{q}}$ and $g_{\mathbf{q}}$ are finite, then the percolation threshold for face' percolation *is* $p_{c,\text{face}'} = 1 - p_{c,\text{face}} = (\mathsf{e_q} - 1)/(2\mathsf{e_q})$. *Furthermore there is no percolation at criticality.*

The proof is the same as for Theorem 10 once we designed the proper peeling algorithm. The only change here is that we need to peel the edge which is immediately on the left of the left-most edge of the boundary of the infinite hole which is adjacent to an inner face of the explored map and part of the cluster of the root face. See Fig. 6.4. In this case the drift of the random walk

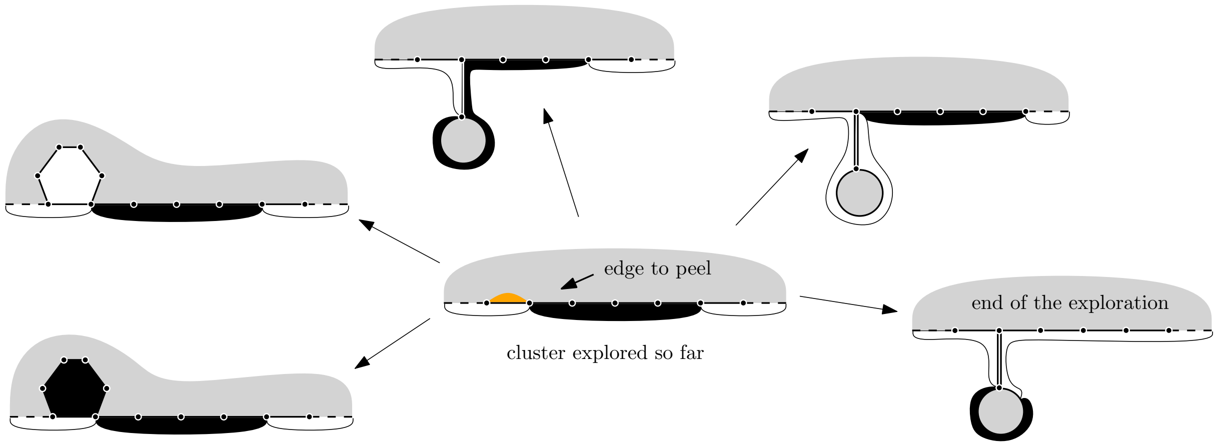


Figure 6.4: Illustration of the peeling algorithm use in the case of face' percolation.

associated to the length of the black component is $\delta = p\mathsf{e_q} - \mathsf{g_q}$ hence the critical threshold is $\mathsf{g_q}/\mathsf{e_q} = \frac{\mathsf{e_q}-1}{2\mathsf{e_q}}$.

6.2.3 The heavy tailed case

If $\mathbf{q}$ is of type $a \in [3/2; 2]$ then the mean gulp is infinite so we cannot apply directly the above considerations to study the face percolation. However, going through the proof of Theorem 10, it only suffices to study the random walk S_p whose increments are given by

$$\begin{array}{lll} \Delta S_p(n) = -2k & \text{with proba } \frac{1}{2}\nu(-k), & k \le -1 \\ \Delta S_p(n) = 2k & \text{with proba } p \times \nu(k), & k \ge 0 \\ \Delta S_p(n) = -1 & \text{otherwise.} & \end{array}$$

If $a \in [3/2, 2)$ we are in the regime where the value of $S_p(n)$ is of the same order of magnitude as its maximal jump so we deduce that the walk S_p oscillates no matter the value of $p \in (0, 1)$. In this regime we even have a more dramatic effect: the root face is separated from infinity by infinitely many cut faces, hence to percolate all these cut faces must have the same color which happens with probability 0 as long as $p \in (0, 1)$, see Proposition **??** for a similar phenomenon on the full plane.

When $\mathbf{q}$ is of type 2 a more interesting phenomenon appears. Recall that in this case we know explicitly the probability measure ν from (5.13) and that $\nu(k) = (4k^2 - 1)^{-1}\mathbf{1}_{k \neq 0}$. Looking at the above transitions for the walk S_p we see that the tails are unbalanced if $p \neq 1/2$ and balanced if and only if $p = 1/2$. Unfortunately, as pointed to us by Timothy Budd, even in the unbalanced case these walks are always oscillating and so there is never percolation as soon as $p \neq 1$.

6.3 Bond percolation

We now move to bond percolation on random maps. Before giving the exploration process which may seem strange at first glance we start with a recreational section where we compute heuristically the bond percolation threshold using the face' percolation process. This might help the reader digest the next section.

6.3.1 A heuristic before the proof: adding faces of degree 2

Let $\mathbf{q}$ be an admissible weight sequence. Notice that from the admissibility criterion of Proposition 5 we must have $q_1 < 1$. We will now create a new planar map model by increasing the number of faces of degree 2. More precisely for $\alpha \in (0,1)$ we put the new weight sequence $\tilde{\mathbf{q}}$ given by

$$\begin{cases} \tilde{q}_k = \alpha^k q_k & \text{for } k \geq 2, \\ \tilde{q}_1 = \alpha \cdot q_1 + (1-\alpha). & \end{cases}$$

It is a straightforward calculation to see that this new weight sequence is again admissible (and critical if the former was) and the corresponding (smallest) solution to $f_{\tilde{\mathbf{q}}}(x) = 1 - \frac{1}{x}$ is $c_{\tilde{\mathbf{q}}} = c_{\mathbf{q}}/\alpha$. We also claim that the partition functions $W^{(\ell)}_{\mathbf{q}}$ (we stress the dependance in $\mathbf{q}$ here) are changed into into $W^{(\ell)}_{\tilde{\mathbf{q}}} = W^{(\ell)}_{\mathbf{q}} \alpha^{-\ell}$ for $\ell > 0$. One convenient way to see this is to express the partition function W by summing over skeletons: If $\mathfrak{m}$ is a (bipartite rooted) planar map with an external face of degree 2ℓ we write $< \mathfrak{m} >$ for the skeleton of the map which is obtained after contracting all the faces of degree 2 of the map $\mathfrak{m}$ (except the external face if $\ell = 1$), see Fig. 6.5. We denote

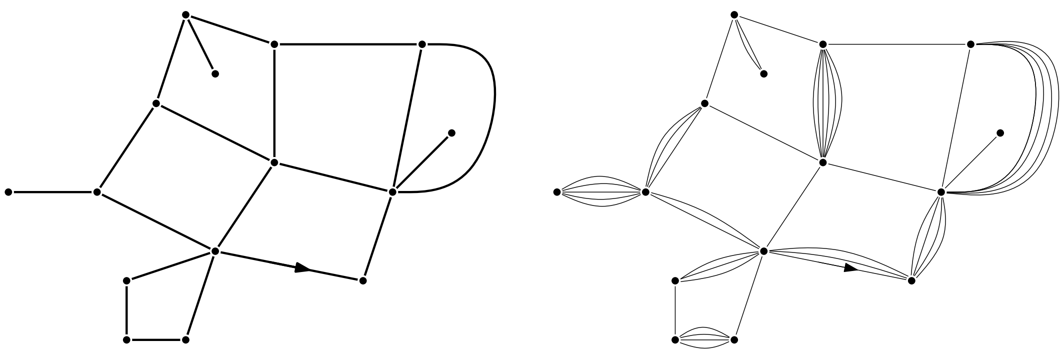


Figure 6.5: A planar bipartite map (on the right) and its skeleton (on the left).

by $\mathcal{S}^{(\ell)}$ the set of all (bipartite rooted) maps with external face of degree 2ℓ having no inner faces of degree 2. Using the above skeleton reduction we can write for any admissible weight sequence

q

$$W_{\mathbf{q}}^{(\ell)} = \sum_{\mathfrak{m}\in\mathcal{M}^{(\ell)}} \prod_{f\in \mathsf{Faces}(\mathfrak{m})\backslash\{f_r\}} q_{\deg(f)/2} = \sum_{\mathfrak{s}\in\mathcal{M}^{(\ell)}} \sum_{\mathfrak{m}\in\mathcal{M}^{(\ell)}:<\mathfrak{m}>=\mathfrak{s}} w_{\mathbf{q}}(\mathfrak{m})$$
$$= \sum_{\mathfrak{s}\in\mathcal{M}^{(\ell)}} w_{\mathbf{q}}(\mathfrak{s}) \sum_{\mathfrak{m}\in\mathcal{M}^{(\ell)}:<\mathfrak{m}>=\mathfrak{s}} q_1^{\#\text{Inner Faces of degree 2}}$$
$$= \sum_{\mathfrak{s}\in\mathcal{M}^{(\ell)}} w_{\mathbf{q}}(\mathfrak{s})(\frac{1}{1-q_1})^{\#\mathsf{Edges}(\mathfrak{s})}.$$

Writing the above display for the case of $\tilde{\mathbf{q}}$ we deduce that

$$W_{\tilde{\mathbf{q}}}^{(\ell)} = \sum_{\mathfrak{s}\in\mathcal{M}^{(\ell)}} w_{\tilde{\mathbf{q}}}(\mathfrak{s})(\frac{1}{1-\tilde{q}_1})^{\#\mathsf{Edges}(\mathfrak{s})} = \sum_{\mathfrak{s}\in\mathcal{M}^{(\ell)}} w_{\mathbf{q}}(\mathfrak{s})\alpha^{\#\mathsf{Edges}(\mathfrak{s})-\ell}(\frac{1}{\alpha}\frac{1}{1-q_1})^{\#\mathsf{Edges}(\mathfrak{s})} = W^{(\ell)}\alpha^{-\ell}.$$

From the above calculations we deduce that the step probability ν driving our favorite random walk is transformed into

$$\tilde{\nu}(k) = \alpha\nu(k) \text{ if } k \neq 0, \quad \tilde{\nu}(0) = \nu(0) + (1-\alpha) \text{ otherwise.}$$

Actually when $q_1 = 0$ the Boltzmann random maps with weight sequence $\tilde{\mathbf{q}}$ have a nice probabilistic interpretation in terms of the $\mathbf{q}$-Boltzmann ones:

Proposition 32 (Adding faces of degree 2)**.** *Let* $\mathfrak{m}$ *be distributed according to either* $\mathbb{P}_{\mathbf{q}}^{(\ell)}, \mathbb{P}_{\bullet,\mathbf{q}}^{(\ell)}, \mathbb{P}_{\infty,\mathbf{q}}^{(\ell)}$ *or* $\mathbb{P}_{\mathbf{q}}^{(\infty)}$ *where the admissible weight sequence* $\mathbf{q}$ *is such that* $q_1 = 0$*. Then split independently each edge of* $\mathfrak{m}$ *into a body of* $K \geq 0$ *parallel faces of degree* 2 *where* K *is distributed as a geometric variable of parameter* $(1-\alpha)$*. Then the resulting map in distributed according to the respective measure but with weight sequence* $\tilde{\mathbf{q}}$*.*

Proof. The proposition is easy to prove under the Boltzmann measure the other cases are obtained by passing to the limit. Indeed, from the above calculation, we see that under $\mathbb{P}_{\tilde{\mathbf{q}}}^{(\ell)}$ the skeleton $< \mathfrak{m} >$ is distributed according to $\mathbb{P}_{\mathbf{q}}^{(\ell)}$ and the number of faces of degree 2 attached to each edge of $< \mathfrak{m} >$ are independent geometric variable with probability distribution $\{\alpha(1-\alpha)^k : k \geq 0\}$. □

Now we will use this construction to (heuristically at least) compare face and bond percolation. Imagine that $\mathfrak{m}$ is a fixed planar map (without faces of degree 2). For $\alpha \in (0,1)$ we split all the edges of $\mathfrak{m}$ into bodies of independent number of parallel faces of degree 2 distributed as geometric variables of parameter $(1-\alpha)$. Denote by $\mathfrak{m}(\alpha)$ the resulting map. We then perform a face' percolation (the dual of face percolation) on $\mathfrak{m}(\alpha)$ with parameter $p \in (0,1)$. We now assume that the parameters α and p tend to 0 such that

$$p/\alpha = \beta > 0.$$

Then each face $f \in \mathsf{Faces}(\mathfrak{m})$ is more and more unlikely to be colored black in the resulting percolation and so eventually, the only way to build a cluster in the map is to use the additional faces of degree 2 (recall that in the face' percolation model, two faces of the same color which share a vertex are in the same cluster). With our fine tuning of the parameters α and p, since each edge of $\mathfrak{m}$ is first split into a geometric number of parallel faces and each of them is colored in black with probability p, the probability that none of the faces of degree 2 offspring of a given edge of $\mathfrak{m}$ is black is

$$\sum_{k=0}^{\infty} \alpha(1-\alpha)^k(1-p)^k = \frac{\alpha}{\alpha+p+\alpha p} \to \frac{1}{1+\beta}.$$

Hence we can state the following rough statement:

" Face' percolation with parameter p on $\mathfrak{m}(\alpha)$ as $\alpha, p \to 0$ with $p\alpha = \beta > 0$ asymptotically looks like bond percolation on $\mathfrak{m}$ with parameter $1 - \frac{1}{1+\beta}$."

Now if we perform this procedure for a random map $\mathfrak{m}$ under the law $\mathbb{P}_{\mathbf{q}}^{(\infty)}$ carrying a bond percolation with parameter $x > 0$. By the above trick and Proposition 32, we relate this model to a face' percolation on a map with law $\mathbb{P}_{\tilde{\mathbf{q}}}^{(\infty)}$ and a parameter p satisfying

$$p/\alpha = \beta, \quad \text{with} \quad \frac{\beta}{1+\beta} = x.$$

The mean exposure $\mathsf{e}_{\tilde{\mathbf{q}}}$ of this model is easily calculated and equals

$$\mathsf{e}_{\tilde{\mathbf{q}}} = (1-\alpha) + \alpha \mathsf{e}_{\mathbf{q}}.$$

Performing the easy algebra we can use Theorem 11 to see that there is no infinite cluster in this face' model if and only if we have $p \leq \frac{\mathsf{e}_{\tilde{\mathbf{q}}}-1}{2\mathsf{e}_{\tilde{\mathbf{q}}}}$ and as $\alpha \to 0$ this condition becomes

$$\frac{x}{1-x} \leq \frac{\mathsf{e}_{\mathbf{q}}-1}{2} \iff x \leq \frac{\mathsf{e}_{\mathbf{q}}-1}{\mathsf{e}_{\mathbf{q}}+1}.$$

We can thus guess that the critical percolation threshold for bond percolation is simply $p_{c,\text{bond}} = \frac{\mathsf{e}_{\mathbf{q}}-1}{\mathsf{e}_{\mathbf{q}}+1}$ and this is indeed the content of the main result of the next section.

Exercise 12. Use the above discrete relations between face' and bond percolation to actually prove that the critical percolation threshold $p_{c,\text{bond}}$ is equal to $\frac{\mathsf{e}_{\mathbf{q}}-1}{\mathsf{e}_{\mathbf{q}}+1}$ (the upper bound is easy, the lower bound a bit harder, but showing that there is no percolation at the critical point seems hard with this approach).

6.3.2 The true proof: give the edges a width!

In order to study bond percolation on random planar maps and to use the peeling process we shall need to see this process in a rather weird fashion at first sight. But the reader who has gone through the last section may find it more natural. Let $\mathfrak{m}$ be a planar map and consider its dual $\mathfrak{m}^\dagger$. We will imagine that each edge of $\mathfrak{m}^\dagger$ is actually a real segment of length given by an exponential variable of parameter 1 and that these lengths are independent for each edge of $\mathfrak{m}^\dagger$. Once this "random dilation" of the edges has been done, we throw a Poisson point process on the union of those dual edges with intensity $\lambda > 0$. That means that each dual edge of length $x > 0$ carries an independent number of "crosses" distributed according to a Poisson variable of mean λx. We say that the map $\mathfrak{m}$ is "decorated" and denote by $\tilde{\mathbb{P}}^{(\infty),\lambda}$ its law. This decoration induces a bond percolation on $\mathfrak{m}$: We will say that an edge of the primal lattice is black if its dual edge, carries a cross. By our construction, the probability that a given edge is white is thus

$$\int_0^\infty \mathrm{d}x \, e^{-x} e^{-\lambda x} = \frac{1}{1+\lambda},$$

and the colors of the edges of $\mathfrak{m}$ are independent. Since these color are independent we are in front of a bond percolation model on $\mathfrak{m}$ with parameter $p = \frac{\lambda}{1+\lambda}$ whose law we denote by $\mathrm{P}_{\mathfrak{m}}^{p,\mathrm{bond}}(\mathrm{d}\eta)$. This trick of declaring an edge black if the dual edge carries a cross, enables us to know for sure the color of an edge without discovering the faces to which it is adjacent. This will be key in the forthcoming peeling exploration. Actually, to be able to start the exploration we will force the root edge to be black which amount to put a cross at the origin of the dual root edge. As in the last section we write $\mathfrak{C}$ for the cluster of the origin where two edges are in the same cluster when they are of the same color and share a common vertex.

Theorem 12 (*Bond percolation thresholds*)

Let $\mathbf{q}$ *be a admissible critical weight sequence for which both* $\mathrm{e}_\mathbf{q}$ *and* $\mathrm{g}_\mathbf{q}$ *are finite. Then the percolation threshold for bond percolation on under* $\mathbb{P}^{(\infty)}$ *is* $p_{c,\mathrm{bond}} = \frac{\mathrm{e}_\mathbf{q}-1}{\mathrm{e}_\mathbf{q}+1}$, *that is we have*

$$\int \mathbb{P}^{(\infty)}(\mathrm{d}\mathfrak{m}) \int \mathrm{P}_{\mathfrak{m}}^{p,\mathrm{bond}}(\mathrm{d}\eta) \mathbf{1}_{\{|\mathfrak{C}|=\infty\}} = \begin{cases} 0 & \text{if } p \le p_{c,\mathrm{bond}}, \\ 1 & \text{if } p > p_{c,\mathrm{bond}}. \end{cases}$$

Proof. As expected, to prove the above theorem we will design a particular peeling algorithm that explores the underlying map to discover the cluster of the origin. This exploration will run on the decorated map $\mathfrak{m}$ of law $\tilde{\mathbb{P}}^{(\infty),\lambda}$ with $p = \frac{\lambda}{1+\lambda}$. More precisely, we will explore the stretched version of the dual $\mathfrak{m}^\dagger$. Imagine that after the first few steps of the exploration we have discovered a submap $\bar{\mathfrak{e}}_n \subset \mathfrak{m}$ which has the following form:

> (H): The status of all (dual) edges on the boundary is unknown, except for a connected segment of black edges. This means that a part of the corresponding dual edges has been explored and that we found a cross. We assume that the unexplored region is conditionally independent and distributed as $\tilde{\mathbb{P}}^{(\infty),\lambda}$.

More precisely, when doing the gluing operation with $\bar{\mathfrak{e}}_n$, one must glue also the parts of the dual edges which have been partially explored. The process then goes as follows: We explore the first dual edge whose status is unknown on the left of the black component. By doing so, we do not always trigger a peeling step, but just explore its dual edge until we find a cross. If we find a cross "inside" the dual edge, that means that the edge we where exploring is actually black and so we move on to the edge on its left. Notice that by the properties of exponential variables and Poisson process, the part of the dual edge which we have not yet explored is again of random length given by an exponential variable and decorated with a Poisson process of crosses. Hence our current exploration satisfies (H) again. If on the contrary we do not encounter a cross during this exploration, that means we have explored entirely the dual edge. In this case only, we trigger a peeling step which may lead to different possibilities as displayed in Fig. 6.6.

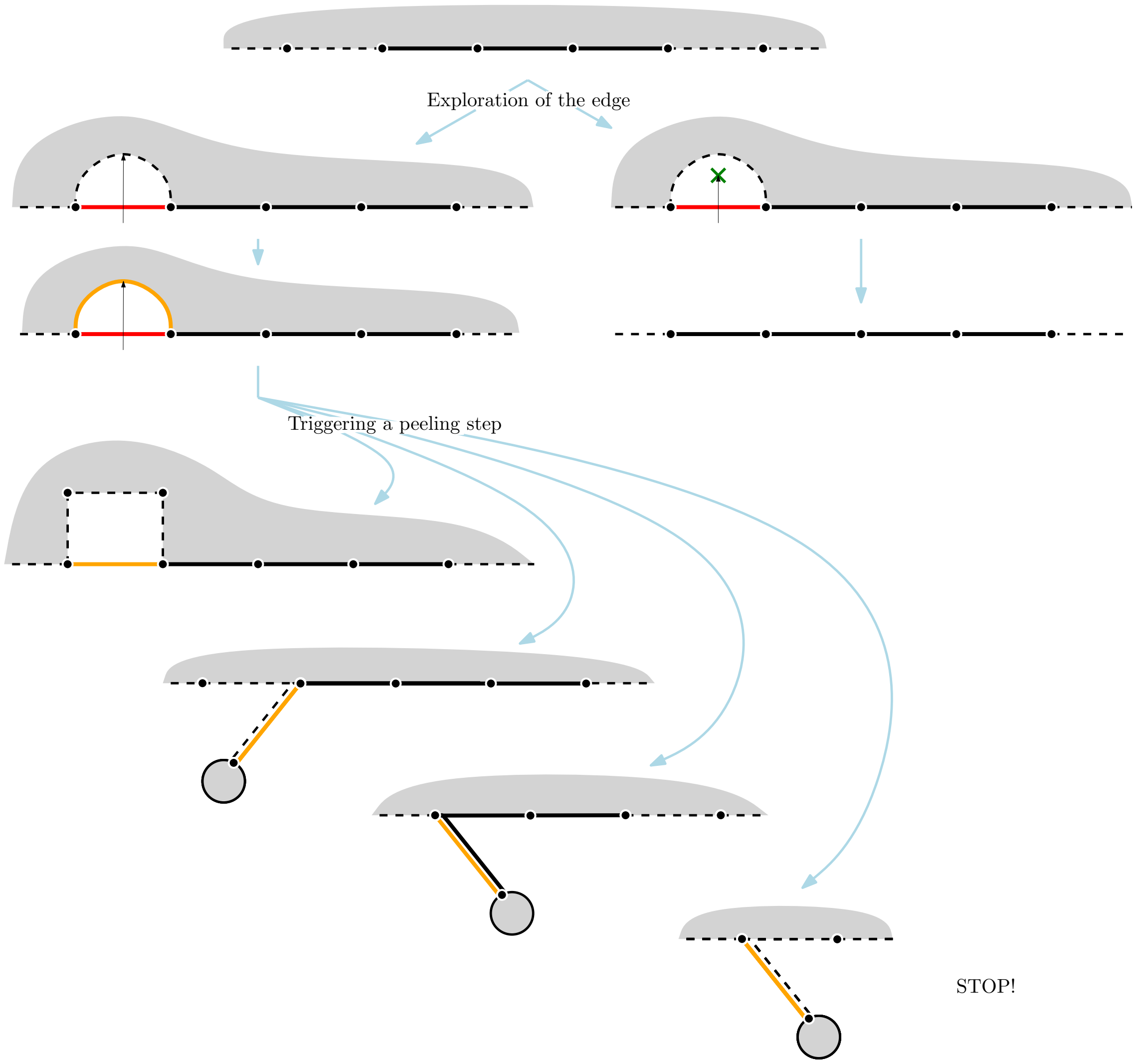


Figure 6.6: Illustration of the peeling process used in the case of bond percolation. Sometimes, we do not trigger a peeling step and just explore "the interior" of an edge to know more on its color.

In all the cases, the boundary conditions and the law of the unexplored map again satisfies (H) and so that we can continue the exploration. The process finishes if during such a peeling step we swallow entirely the black component.

It is easy to see that the above process goes on forever if and only if the cluster of the root edge in the bond percolation is infinite. As in Theorem 7 we thus look at the random process given by the length of the black boundary. It is again a random walk stopped when reaching $\mathbb{Z}_{\leq 0}$ whose increments have expectation $\delta = p - (1-p)\mathsf{g}_{\mathbf{q}}$. We then conclude as in the above proofs that the black boundary is eventually absorbed if and only if $\delta \leq 0 \iff p \leq (\mathsf{e}_{\mathbf{q}}-1)/(\mathsf{e}_{\mathbf{q}}+1)$. □

Exercise 13. Let $\mathbf{q}$ be an admissible sequence for which the mean gulp and exposure are not necessarily finite. Show that the critical point for bond percolation under $\mathbb{P}_{\mathbf{q}}^{(\infty)}$ is 1 (with percolation at criticality ☺) as soon as $\mathsf{g}_{\mathbf{q}} = \infty$.

Open question. *Can one similarly find a peeling algorithm to explore site percolation in the same spirit as [56]?*

6.4 The case of $\mathbb{P}_{\infty}^{(1)}$

We now use the result of the preceding section together with the scaling limit of the perimeter process to prove that the percolation thresholds in the case of the infinite Boltzmann map of the *plane* are the same as those of the half plane and that there is no percolation at criticality.

Theorem 13 (***Percolation thresholds in the plane***)

Let $\mathbf{q}$ be an admissible and critical weight sequence and suppose that $\mathsf{e}_{\mathbf{q}} < \infty$ and $\mathsf{g}_{\mathbf{q}} < \infty$. Then the percolation threshold for face and bond percolation respectively almost surely are $p_{c,\text{face}}$ and $p_{c,\text{bond}}$. Furthermore, there is no percolation at criticality.

[To be done.]

Bibliographical notes. Percolation on random maps was first studied in the pioneer work of Angel [4] where he proved that the critical threshold for site percolation on the UIPT is $\frac{1}{2}$. These results have then been extended to the case of bond and face percolations on triangulations and quadrangulations in [6]. Already in the work [6] a "universal" formula was found which is easily seen to match those of Theorem 10 and Theorem 12 although the under random map model and the peeling process considered in that work are different from the one we deal with in these pages. The critical value for site percolation on random quadrangulations remained unknown until the recent work of Richier [56].

Chapter VII : Metric explorations

In this chapter, we only deal with random maps of the plane.

In this chapter we study the geometry of the *duals* of the random maps sampled according to the measure $\mathbb{P}_\infty^{(1)}$ in the case when $\mathbf{q}$ is a weight sequence of type $a \in (3/2; 5/2]$ (in particular critical). We will see that the geometry of those maps are different depending on:

- when $a \in (3/2; 2)$, the so-called dilute case, we will see that the volume growth in $\mathfrak{m}^\dagger$ is polynomial in the radius with exponent $\frac{a-1/2}{a-2}$,
- when $a \in (2; 5/2)$, the so-called dense case, this volume growth changes dramatically and is exponential in the radius (or even blows up in the case of the FPP distances),

More precisely we will study both the usual graph distance $\mathrm{d}_{\mathrm{gr}}^\dagger$ on $\mathfrak{m}^\dagger$ and the distance induced by a first-passage percolation with exponential edge weights on $\mathfrak{m}^\dagger$. The two approaches follow roughly the same lines, but we start with the second one which is technically simpler.

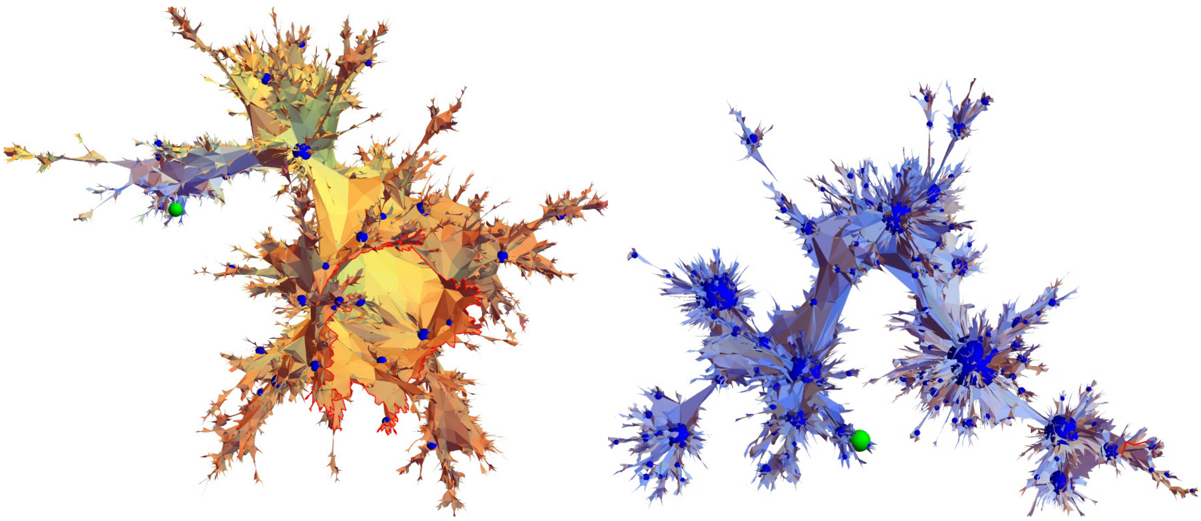


Figure 7.1: Two representations of the neighborhood of the root in infinite Boltzmann maps with large degree vertices in the dilute case (left) and dense case (right). The root is represented by a green ball, while the high degree vertices are represented by blue balls of size proportional to the degree. The boundary is colored red.

7.1 Eden model: exponential FPP distances on the dual

We first define the Eden model on a general map and link it to the uniform peeling. As in Section 6.3.2 the properties of the exponential law will be crucial in this section. Let $\mathfrak{m}$ be an (finite or infinite) planar map. On the dual map $\mathfrak{m}^\dagger$ of $\mathfrak{m}$ we sample independent weights x_e for each edge $e \in \mathsf{Edges}(\mathfrak{m}^\dagger)$ distributed according to the exponential law $\mathcal{E}(1) = e^{-x}\mathrm{d}x\mathbf{1}_{x>0}$. These weights can be used to modify the usual dual graph metric on $\mathfrak{m}^\dagger$ by considering the first-passage percolation distance: for $f_1, f_2 \in \mathsf{Faces}(\mathfrak{m})$

$$\mathrm{d}_{\mathrm{fpp}}(f_1, f_2) = \inf \sum_{e \in \gamma} x_e,$$

where the infimum is taken over all paths $\gamma : f_1 \to f_2$ in the dual map $\mathfrak{m}^\dagger$. We denote by $\mathrm{P}^{\mathrm{fpp}}_{\mathfrak{m}}$ the law of the $(x_e)_{e \in \mathsf{Edges}(\mathfrak{m})}$. This model (first-passage percolation with exponential edge weights on the dual graph) is often referred to as the Eden model on the primal map $\mathfrak{m}$. As in the last chapter, it is convenient in this section to view the edges of the map $\mathfrak{m}^\dagger$ as real segments of length x_e for $e \in \mathsf{Edges}(\mathfrak{m}^\dagger)$ glued together according to incidence relations of the map. This operation turns $\mathfrak{m}^\dagger$ into a continuous length space (but we keep the same notation) and the distance $\mathrm{d}_{\mathrm{fpp}}$ extends easily to all the points of this space.

Definition 21 (Fpp ball)**.** *For $t > 0$ we denote by*

$$\mathrm{Ball}^{\mathrm{fpp}}_t(\mathfrak{m})$$

the submap of $\mathfrak{m}$ whose associate connected subset of dual edges $\left(\mathrm{Ball}^{\mathrm{fpp}}_t(\mathfrak{m})\right)^\circ$ in $\mathfrak{m}^\dagger$ is the set of all dual edges which have been fully-explored by time $t > 0$, i.e. whose points are all at fpp*-distance less than t from the origin of $\mathfrak{m}^\dagger$ (the root-face of $\mathfrak{m}$). We put $\mathrm{Ball}^{\mathrm{fpp}}_0(\mathfrak{m}) = f_{\mathrm{r}}$.*

As usual, if $\mathfrak{m}$ is given with a distinguished point or is infinite and one ended, we can consider the hull $\overline{\mathrm{Ball}}^{\mathrm{fpp}}_t(\mathfrak{m})$ of $\mathrm{Ball}^{\mathrm{fpp}}_t(\mathfrak{m})$ obtained by filling-in the finite components of its complement in $\mathfrak{m}$ expect for the hole containing the distinguished point/infinity.

7.1.1 Uniform peeling

It is easy to see that there are jump times $0 = T_0 < T_1 < T_2 < \cdots$ for the process $\overline{\mathrm{Ball}}^{\mathrm{fpp}}_t(\mathfrak{m})$ and that almost surely (depending on the randomness of the x_e) the map $\overline{\mathrm{Ball}}^{\mathrm{fpp}}_{T_{i+1}}(\mathfrak{m})$ is obtained from $\overline{\mathrm{Ball}}^{\mathrm{fpp}}_{T_i}(\mathfrak{m})$ by the peeling of one edge and by filling-in the finite component possibly created. The following proposition only relies the randomness of the weights, the map $\mathfrak{m}$ is fixed.

Proposition 33 (Eden model and uniform peeling)**.** *If $\mathfrak{m}$ is an infinite map of the plane then under $\mathrm{P}^{\mathrm{fpp}}_{\mathfrak{m}}$ we have:*

- *the law of $(\overline{\mathrm{Ball}}^{\mathrm{fpp}}_{T_i}(\mathfrak{m}))_{i \geq 0}$ is that of a uniform filled-in peeling $\bar{\mathfrak{e}}_0 \subset \cdots \subset \bar{\mathfrak{e}}_n \subset \cdots \subset \mathfrak{m}$: conditionally on the past exploration, the next edge to peel is a uniform edge on the boundary of the explored part $\bar{\mathfrak{e}}_i$;*

- *conditionally on* $(\overline{\mathrm{Ball}}^{\mathrm{fpp}}_{T_i}(\mathfrak{m}))_{i\geq 0}$ *the variables* $\Delta T_i = T_{i+1} - T_i$ *are independent and distributed as exponential variables of parameter given by the perimeter (that is twice the half-perimeter) of unique hole of* $\bar{\mathfrak{e}}_i$.

Notice that the above uniform peeling, is of the type discussed in Section 6.1.1. Hence, when the underlying planar map $\mathfrak{m}$ is sampled according to $\mathbb{P}^{(\ell)}_\infty$ the transitions of the chain $\bar{\mathfrak{e}}_n$ are those of Section 4.2.1.

Proof. Fix $\mathfrak{m}$ and let us imagine the situation at time T_i for $i \geq 0$. We condition on the sigma-field $\mathcal{F}_i$ generated by all the exploration up to time T_i. Let us examine the edges in $\mathfrak{m}^\dagger$ which are dual to the boundary of $\bar{\mathfrak{e}}_i = \overline{\mathrm{Ball}}^{\mathrm{fpp}}_{T_i}(\mathfrak{m})$. These come in two types: *type-1* edges that are adjacent to a new face in the unexplored part (that is, if we peel one of those edges we have an event C), and *type-2* edges that link two faces adjacent to the boundary of the explored part (that is, if we peel one of these edges we have an event G_{k_1,k_2}). See Fig. 7.2.

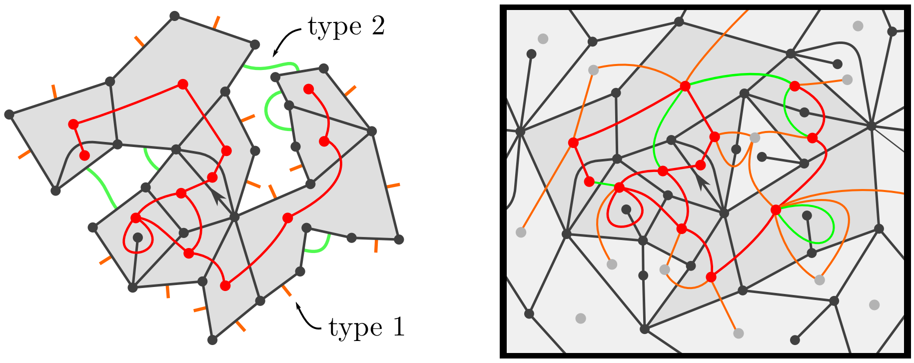


Figure 7.2: Illustration of the proof of Proposition 33. The edges of the first type are in orange and those of the second type are in green. Regardless of their number and locations, the next edge to peel can be taken uniformly on the boundary and the increase of time is given by an exponential variable of parameter given by the perimeter.

Let us consider an edge $e^{(1)}$ of the first type and denote by $e^{(1)}_-$ its extremity in the explored region. Since this edge has not been fully explored at time T_i, it follows that its weight $x_{e^{(1)}}$ satisfies $x_{e^{(1)}} > T_i - \mathrm{d}_{\mathrm{fpp}}(e^{(1)}_-, f_{\mathrm{r}})$ and furthermore by properties of exponential variables conditionally on $\mathcal{F}_i$

$$y_{e^{(1)}} := x_{e^{(1)}} - \left(T_i - \mathrm{d}_{\mathrm{fpp}}(e^{(1)}_-, f_{\mathrm{r}})\right)$$

has the law $\mathcal{E}(1)$ of an exponential variable of parameter 1. Let us now examine the situation for an edge $e^{(2)}$ of the second type. We denote by $e^{(2)}_-$ and $e^{(2)}_+$ its endpoints. Since $e^{(2)}$ is being explored from both sides but has not been fully explored by time T_i, we have that $x_{e^{(2)}} > \left(T_i - \mathrm{d}_{\mathrm{fpp}}(e^{(2)}_-, f_{\mathrm{r}})\right) + \left(T_i - \mathrm{d}_{\mathrm{fpp}}(e^{(2)}_+, f_{\mathrm{r}})\right)$ and by the same argument as above conditionally on $\mathcal{F}_i$

$$y_{e^{(2)}} := x_{e^{(2)}} - \left(T_i - \mathrm{d}_{\mathrm{fpp}}(e^{(2)}_-, f_{\mathrm{r}})\right) - \left(T_i - \mathrm{d}_{\mathrm{fpp}}(e^{(2)}_+, f_{\mathrm{r}})\right)$$

is again exponentially distributed. Of course, an edge of the second type is dual to two edges of the boundary of $\bar{\mathfrak{e}}_i$. Apart from this trivial identification, the variables y_e where e runs over the edges dual to the boundary of $\bar{\mathfrak{e}}_i$ are, conditionally on $\mathcal{F}_i$, independent of each other. Now, the time it takes until a new edge is fully explored is equal to

$$T_{i+1} - T_i = \inf\{y_e : e \text{ of the first type}\} \wedge \frac{1}{2} \inf\{y_e : e \text{ of the second type}\},$$

where the factor $1/2$ again comes from the fact that edges of the second type are explored from both sides. By the above independence property, $T_{i+1} - T_i$ is thus distributed as an exponential variable of parameter

$$T_{i+1} - T_i \overset{(d)}{=} \mathcal{E}(\#\{\text{edges of the first type}\} + 2\#\{\text{edges of the second type}\}) = \mathcal{E}(2\ell)$$

where 2ℓ is the perimeter of the hole of $\bar{\mathfrak{e}}_i$. That proves the second part of the proposition. To see that conditionally on $\mathcal{F}_i$ the next edge to peel is uniform on the boundary, we may replace for each edge $e^{(2)}$ of the second type the variable $\frac{1}{2}y_{e^{(2)}}$ of law $\mathcal{E}(2)$ by the minimum of two independent exponential variables $\tilde{y}_{e_1^{(2)}}$ and $\tilde{y}_{e_2^{(2)}}$ of law $\mathcal{E}(1)$ which we attach on the two edges dual to $e^{(2)}$ on the boundary of $\bar{\mathfrak{e}}_i$. Finally, everything boils down to assigning to each edge of the boundary of the explored map an independent exponential variable of parameter 1; the next edge to peel is the one carrying the minimal weight which is then uniform as desired. This completes the proof. □

By the above proposition the algorithm $\mathcal{A}$ which we use to discover $\mathfrak{m}$ along the first-passage percolation distance can clearly be written under the form $\mathcal{A}(\bar{\mathfrak{e}}, \omega)$ for an independent source of randomness ω which can be used to retrieve the information on the exponential weights. Hence under $\mathbb{P}_\infty^{(\ell)}(\mathrm{d}\mathfrak{m})\mathrm{P}_\mathfrak{m}^{\mathrm{fpp}}$ the exploration $\bar{\mathfrak{e}}_i = \overline{\mathrm{Ball}}_{T_i}^{\mathrm{fpp}}$ indeed has the law of a filled-in peeling exploration as described in Section 4.2.1.

7.1.2 The dilute case

We can now apply our results on the scaling limit of filled-in peeling processes (Proposition 9 to study the geometry of the balls of increasing radius for the fpp-distance on infinite Boltzmann maps of the plane.

Proposition 34 (Distances in the uniform peeling)**.** *Let $(\bar{\mathfrak{e}}_n)_{n\geq 0}$ by the filled-in peeling exploration associated to the fpp-distance on $\mathfrak{m}^\dagger$ and recall the notation T_n. If $\mathbf{q}$ is of type $a \in (2; 5/2]$ then under $\mathbb{P}_\infty^{(1)}(\mathrm{d}\mathfrak{m})\mathrm{P}_\mathfrak{m}^{\mathrm{fpp}}$ we have the following convergence in distribution for the Skorokhod topology*

$$\left(\frac{|\partial\bar{\mathfrak{e}}_{[nt]}|}{n^{\frac{1}{a-1}}}, \frac{|\bar{\mathfrak{e}}_{[nt]}|}{n^{\frac{a-1/2}{a-1}}}, \frac{T_{[nt]}}{n^{\frac{a-2}{a-1}}}\right)_{t\geq 0} \xrightarrow[n\to\infty]{(d)} \left(\left(\Upsilon_a^\uparrow, \mathrm{b}_\mathbf{q}\mathcal{V}_a(\Upsilon^\uparrow), \frac{1}{2\mathrm{p}_\mathbf{q}}\int_0^\cdot \frac{\mathrm{d}u}{\Upsilon_a^\uparrow(u)}\right)(\mathrm{p}_\mathbf{q}t)\right)_{t\geq 0}.$$

The above result can easily be translated in geometric terms. Since for the above algorithm we have $\bar{\mathfrak{e}}_i = \mathrm{Ball}_{T_i}^{\mathrm{fpp}}(\mathfrak{m})$ from Section 7.1, if we denote by $|\overline{\mathrm{Ball}}_r^{\mathrm{fpp}}(\mathfrak{m})|$ and $|\partial\overline{\mathrm{Ball}}_r^{\mathrm{fpp}}(\mathfrak{m})|$ respectively the size (number of inner vertices) and the half-perimeter of the unique hole of $\overline{\mathrm{Ball}}_r^{\mathrm{fpp}}(\mathfrak{m})$. Then the above result shows that we have the following convergence in distribution in the sense of Skorokhod under $\mathbb{P}_\infty^{(\ell)}(\mathrm{d}\mathfrak{m})\mathrm{P}_{\mathfrak{m}}^{\mathrm{fpp}}$

$$\left(\frac{|\partial\overline{\mathrm{Ball}}_{[tn]}^{\mathrm{fpp}}(\mathfrak{m})|}{n^{\frac{1}{a-2}}}, \frac{|\overline{\mathrm{Ball}}_{[tn]}^{\mathrm{fpp}}(\mathfrak{m})|}{n^{\frac{a-1/2}{a-2}}}\right)_{t\geq 0} \xrightarrow[n\to\infty]{(d)} \left(\left(\Upsilon_a^{\uparrow}, \mathrm{b}_{\mathrm{q}}\cdot\mathcal{V}_a(\Upsilon_a^{\uparrow})\right)(\vartheta_{2\mathrm{p}_{\mathrm{q}}t})\right)_{t\geq 0}, \tag{7.1}$$

where for $t \geq 0$ we have put $\vartheta_t = \inf\{s \geq 0 : \int_0^s \frac{\mathrm{d}u}{\Upsilon_a^{\uparrow}(u)} \geq t\}$.

Remark 12. In the case when $a = \frac{5}{2}$ since $\Upsilon_{5/2}$ has no positive jumps, the first Lamperti transform $\Upsilon_a^{\uparrow}(\vartheta_t)$ has the same distribution as a the time-reversal of a branching process with branching mechanism $\phi(\lambda) = c\lambda^{3/2}$ started from $-\infty$ and conditioned to die at 0. See [35, Section 4.4]. This is not true for other values of a since the process Υ_a has positive and negative jumps.

Proof of Proposition 34. The joint convergence of the first two components is given by Theorem 9. We now prove the convergence of the third component jointly with the first two. To simplify notation we again write $P_n = |\partial\bar{\mathfrak{e}}_n|$ and $V_n = |\bar{\mathfrak{e}}_n|$. By Proposition 33 we known that conditionally on $(P_i, V_i)_{i\geq 0}$ we have

$$T_n = \sum_{i=0}^{n-1} \frac{\mathbf{e}_i}{2P_i},$$

where $\mathbf{e}_i$ are independent exponential variables of expectation 1. By the convergence of the first component in the proposition, we have

$$n^{-\frac{a-2}{a-1}} \sum_{i=[ns]+1}^{[nt]} \frac{1}{P_i} = \int_{n^{-1}([ns]+1)}^{n^{-1}([nt]+1)} \frac{\mathrm{d}u}{n^{\frac{-1}{a-1}}P_{[nu]}} \xrightarrow[n\to\infty]{(d)} \int_s^t \frac{\mathrm{d}u}{\Upsilon_a^{\uparrow}(\mathrm{p}_{\mathrm{q}}u)},$$

and, on the other hand,

$$\begin{aligned}&\mathbb{E}\left[\left(n^{-\frac{a-2}{a-1}} \sum_{i=[ns]+1}^{[nt]} \frac{\mathbf{e}_i}{P_i} - n^{-\frac{a-2}{a-1}} \sum_{i=[ns]+1}^{[nt]} \frac{1}{P_i}\right)^2 \middle| (P_k)_{k\geq 0}\right] \\ &= n^{-\frac{2a-4}{a-1}} \sum_{i=[ns]+1}^{[nt]} \frac{1}{(P_i)^2} = \frac{1}{n}\int_{n^{-1}([ns]+1)}^{n^{-1}([nt]+1)} \frac{\mathrm{d}u}{(n^{\frac{-1}{a-1}}P_{[nu]})^2}\end{aligned}$$

converges to 0 in probability as $n \to \infty$. It easily follows that

$$\left(n^{-\frac{a-2}{a-1}}\left(T_{[nt]} - T_{[n\varepsilon]}\right)\right)_{t\geq\varepsilon} \xrightarrow[n\to\infty]{(d)} \left(\frac{1}{2\mathrm{p}_{\mathrm{q}}}\int_{\mathrm{p}_{\mathrm{q}}\varepsilon}^{\mathrm{p}_{\mathrm{q}}t} \frac{\mathrm{d}u}{\Upsilon_a^{\uparrow}(u)}\right)_{t\geq\varepsilon}, \tag{7.2}$$

and this convergence holds jointly with the first two components considered in the proposition. Hence, to finish the proof of the proposition, it suffices to see that for any $\delta > 0$ we have

$$\lim_{\varepsilon\to 0}\sup_{n\geq 1} \mathbb{P}\left(n^{-\frac{a-2}{a-1}}\cdot T_{[n\varepsilon]} > \delta\right) = 0 \quad \text{and} \quad \lim_{\varepsilon\to 0}\mathbb{P}\left(\int_0^{\varepsilon}\frac{\mathrm{d}u}{\Upsilon_a^{\uparrow}(u)} > \delta\right) = 0,$$

so that we could harmlessly let $\varepsilon \to 0$ in (7.2). To prove the first claim we use the estimate (7.6) below to get

$$\mathbb{E}[T_{[n\varepsilon]}] = \mathbb{E}\left[\mathbb{E}\left[\sum_{i=0}^{[n\varepsilon]} \frac{\mathbf{e}_i}{2P_i} \middle| (P_i)_{i\geq 0}\right]\right] = \sum_{i=0}^{[n\varepsilon]} \mathbb{E}\left[\frac{1}{2P_i}\right] \underset{(7.6)}{\leq} C(\varepsilon n)^{\frac{a-2}{a-1}},$$

for some constant $C > 0$. The desired result follows from an application of Markov's inequality. The second statement just follows from the fact that $(\Upsilon_a^{\uparrow}(t))^{-1}$ is almost surely integrable around 0^+ since $a > 2$. One cheap way to see this is to take expectations in (7.2) and using Fatou's lemma together with the last calculation to get

$$\frac{1}{2\mathsf{p_q}}\mathbb{E}\left[\int_\varepsilon^1 \frac{\mathrm{d}u}{\Upsilon_a^{\uparrow}(u)}\right] \le C(1+\varepsilon^{\frac{a-2}{a-1}}).$$

Sending $\varepsilon \to 0$ we deduce that indeed $(\Upsilon_a^{\uparrow}(u))^{-1}$ is almost surely integrable around 0. □

As seen in the above proof it is key to estimate the expectation of $(S_n^{\uparrow})^{-1}$ where we recall that $S^{\uparrow}$ is the walk S conditioned to stay positive and has the same distribution as the half-perimeter process of filled-in exploration under $\mathbb{P}_\infty^{(\ell)}$. We write $\mathbb{P}_p$ and $\mathbb{E}_p$ for the probability and expectation under which S,$S^{\uparrow}$ and $S^{\downarrow}$ are started from $p \ge 1$.

Lemma 35. *Suppose that* $\mathbf{q}$ *is critical. For any* $p \ge 1$ *and* $n \ge 0$ *we have*

$$\mathbb{E}_p\left[\frac{1}{S_n^{\uparrow}}\right] = \frac{\mathbb{P}_p(S_n^{\downarrow} > 0)}{p}. \tag{7.3}$$

In particular,

$$\mathbb{E}_1\left[\frac{1}{S_n^{\uparrow}}\right] = 2\sum_{k=n+1}^{\infty}\frac{1}{k}\mathbb{P}_1(S_k = 0) \quad \text{and} \quad \sum_{n=0}^{\infty}\mathbb{E}_1\left[\frac{1}{S_n^{\uparrow}}\right] = 2\sum_{k=1}^{\infty}\mathbb{P}_1(S_k = 0). \tag{7.4}$$

Proof. The equality (7.3) follows directly from the definition of the $h^{\uparrow}$-transform and the exact forms of $h^{\uparrow}$ and $h^{\downarrow}$:

$$\begin{aligned}\mathbb{E}_p\left[\frac{1}{S_n^{\uparrow}}\right] &= \sum_{k=1}^{\infty}\frac{1}{k}\mathbb{P}_p(S_n^{\uparrow} = k) = \sum_{k=1}^{\infty}\frac{1}{k}\frac{h^{\uparrow}(k)}{h^{\uparrow}(p)}\mathbb{P}_p(S_i > 0 \text{ for } 1 \le i < n,\, S_n = k)\\ &= \frac{h^{\downarrow}(p)}{h^{\uparrow}(p)}\sum_{k=1}^{\infty}\frac{h^{\uparrow}(k)}{kh^{\downarrow}(k)}\mathbb{P}_p(S_n^{\downarrow} = k) = \frac{1}{p}\sum_{k=1}^{\infty}\mathbb{P}_p(S_n^{\downarrow} = k) = \frac{1}{p}\mathbb{P}_p(S_n^{\downarrow} > 0),\end{aligned}$$

which gives the first claim. For the remaining statements it suffices to consider $p = 1$. Since $\inf\{i : S_i^{\downarrow} = 0\}$ is a.s. finite, we may identify

$$\mathbb{E}\left[\frac{1}{S_n^{\uparrow}}\right] = \mathbb{P}_1(S_n^{\downarrow} > 0) = \frac{1}{h^{\downarrow}(1)}\sum_{k=n+1}^{\infty}\mathbb{P}_1(S_i > 0 \text{ for } 1 \le i < k,\, S_k = 0). \tag{7.5}$$

We now use the cycle lemma to re-interpret the probabilities in the sum (see [1, display before (1.7)]): For fixed $k > n \ge 0$ we can construct another sequence $(\tilde{S}_i)_{i\ge 0}$ by setting $\tilde{S}_i = 1 + S_n - S_{n-i}$ for $i \le n$, $\tilde{S}_i = S_n + S_k - S_{n+k-i}$ for $n < i < k$, and $\tilde{S}_i = S_i$ for $i \ge k$. Then clearly $(\tilde{S}_i)_{i\ge0}$ is equal in distribution to $(S_i)_{i\ge0}$ while the event $S_i > 0$, $1 \le i < k$, $S_k = 0$, is equivalent to $\tilde{S}_k = 0$ and the last maximum before time k occurring at time n. Since the probability of the former event does not involve n in its S-description, conditionally on $\tilde{S}_k = 0$ the probability of the latter is equal for each $n \in \{0, 1, \ldots, k-1\}$, and therefore

$$\mathbb{P}_1(S_i > 0 \text{ for } 1 \le i < k,\, S_k = 0) = \frac{1}{k}\mathbb{P}_1(S_k = 0).$$

Together with (7.5) and $h^{\downarrow}(1) = 1/2$ this implies the first equality in (7.4), while the second one follows from interchanging the sums over n and k. □

Now if $\mathbf{q}$ is of type $a \in (3/2; 5/2]$ the local limit estimates (5.15) together with the second point of the above lemma shows that for some constant $C > 0$ we have for any $n \geq 1$

$$\mathbb{E}_1[1/S_n^{\uparrow}] \leq Cn^{-1/(a-1)}. \tag{7.6}$$

7.1.3 The dense case

We now focus on the study of the dense phase corresponding to $a \in (3/2; 2)$. As we will see the geometry of the fpp-distances in this phase is much different since we show that distance from the origin to infinity in the map is finite! Recall that $\mathrm{d}_{\mathrm{fpp}}(\cdot,\cdot)$ is the first-passage percolation metric on $\mathfrak{m}^{\dagger}$ for which its edges are endowed with i.i.d. exponential weights. As usual f_{r} denotes the root face of $\mathfrak{m}$ which is the origin of $\mathfrak{m}^{\dagger}$.

Proposition 36. *When $\mathbf{q}$ is of type $a \in (3/2; 2)$ then under $\mathbb{P}_{\infty}^{(1)}$ we have*

$$\mathbb{E}\left[\mathrm{d}_{\mathrm{fpp}}(f_{\mathrm{r}}, \infty)\right] = \mathbb{E}[N_0] < \infty,$$

where $\mathrm{d}_{\mathrm{fpp}}(f_{\mathrm{r}}, \infty)$ is the infimum of the fpp*-length of all infinite paths in $\mathfrak{m}^{\dagger}$, and N_0 is the number of times the random walk $(S_i)_{i\geq 0}$ started at 1 visits 0.*

Proof. We do the filled-in peeling process on $\mathfrak{m}$ with the algorithm of Proposition 33 and recall the notation $(T_i)_{i\geq 0}$ of Section 7.1. The proposition boils down to computing the expectation of $T_{\infty} = \lim_{i\to\infty} T_i$. By Proposition 34, conditionally on the perimeter process $(P_i)_{i\geq 0}$ during the exploration, the increments $T_{i+1} - T_i$ are independent exponential variables of mean $1/(2P_i)$. Hence we have

$$\mathbb{E}[T_{\infty}] = \sum_{i=0}^{\infty} \mathbb{E}\left[\frac{1}{2P_i}\right] \underset{\text{Lem. 35}}{=} \sum_{k=1}^{\infty} \mathbb{P}_1(S_k = 0) = \mathbb{E}[N_0].$$

From the local limit theorem [41, Theorem 4.2.1] we have $\mathbb{P}_1(S_k = 0) \sim C_0\, k^{-1/(a-1)}$ as $k \to \infty$ for some constant $C_0 > 0$ and so when $a \in (3/2; 2)$ we have $\mathbb{E}[N_0] < \infty$ (in other words the walk $(S_i)_{i\geq 0}$ is transient whenever $a < 2$). □

7.1.4 The border case $a = 2$

[To be done.]

7.2 Graph distances on the dual

It does not seem easy to use the peeling process to systematically study the graph metric on the primal $\mathfrak{m}$ (this is because the degree of the faces are not bounded and so when discovering a new large face, one cannot *a priori* know what are the distances to the root of all of its incident vertices). However, as in [2, 35] for the face-peeling process it is still possible to use the peeling process in order to study the graph metric on the *dual* map $\mathfrak{m}^\dagger$. Specifically on the dual $\mathfrak{m}^\dagger$ we denote by $\mathrm{d}_{\mathrm{gr}}^\dagger$ the dual graph distance and if $f \in \mathsf{Faces}(\mathfrak{m})$ the dual distance to the root face $\mathrm{d}_{\mathrm{gr}}^\dagger(f, f_{\mathrm{r}})$ is called the *height* of f in $\mathfrak{m}$.

Definition 22 (Ball for $\mathrm{d}_{gr}^\dagger$)**.** *We define the dual ball of radius r by*

$$\mathrm{Ball}_r^\dagger(\mathfrak{m}),$$

the map made by keeping only the faces of $\mathfrak{m}$ that are at height less than or equal to r and cutting along all the edges which are adjacent on both sides to faces at height r. Equivalently, the corresponding connected subset

$$\left(\mathrm{Ball}_r^\dagger(\mathfrak{m})\right)^\circ$$

of dual edges in $\mathfrak{m}^\dagger$ is given by those edges of $\mathfrak{m}^\dagger$ which contain at least one endpoint at height strictly less than r. By convention we also put $\mathrm{Ball}_0^\dagger(\mathfrak{m})$ to be the root face of $\mathfrak{m}$.

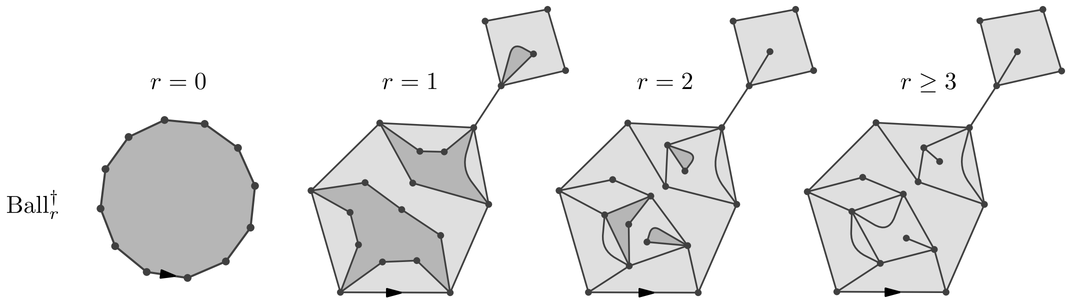


Figure 7.3: The ball of radius $0, 1, 2$ and 3 is some underlying planar map $\mathfrak{m}$.

Also, when $\mathfrak{m}$ is pointed or one-ended we write $\overline{\mathrm{Ball}}_r^\dagger(\mathfrak{m})$ for the hull of these balls, which are obtained by filling-in all the holes of $\mathrm{Ball}_r^\dagger(\mathfrak{m})$ inside $\mathfrak{m}$ which do not contain the distinguished point (or infinity).

7.2.1 Peeling along layers

We now define a peeling algorithm $\mathcal{L}^\dagger$ which explore the dual metric in $\mathfrak{m}$. Let $\mathfrak{e} \subset \mathfrak{m}$ be a submap of $\mathfrak{m}$ and suppose that:

> (H): There exists an integer $h \geq 0$ such all the inner faces adjacent to the holes of $\mathfrak{e}$ are at height h or $h+1$ in $\mathfrak{m}$. Suppose furthermore that the faces adjacent to a same hole in $\mathfrak{e}$ and which are at height h form a connected part on the boundary of that hole.

For simplicity below, we will say that the height of an edge of $\mathsf{Active}(\mathfrak{e})$ is the height of its incident inner face. If $\mathfrak{e}$ does not satisfy (H) then we put $\mathcal{L}^\dagger(\mathfrak{e}) = \dagger$, otherwise the next edge to peel $\mathcal{L}^\dagger(\mathfrak{e})$ is chosen as follows:

- If all edges on the boundaries of the holes of $\mathfrak{e}$ are at height h then $\mathcal{L}^\dagger(\mathfrak{e})$ is a deterministic edge on the boundary of one of its hole,
- Otherwise $\mathcal{L}^\dagger(\mathfrak{e})$ is a deterministic edge at height h such that the next edge in clockwise order around its hole is at height $h+1$.

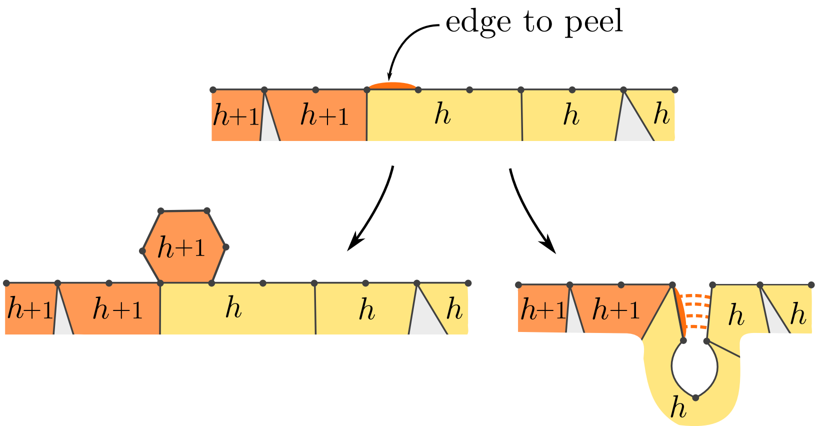


Figure 7.4: Illustration of the peeling using algorithm $\mathcal{L}^\dagger$.

It is easy to check by induction that if one uses the above algorithm starting at step $i=0$ to peel the edges of $\mathfrak{m}$, then for every $i \geq 0$ the explored map $\mathfrak{e}_i$ satisfies the hypothesis (H) and so the peeling process goes is never stopped before the map is entirely explored. Let us give a geometric interpretation of this peeling exploration. We denote by $\mathsf{H}(\mathfrak{e}_i)$ the minimal height in $\mathfrak{m}$ of an edge of $\mathsf{Active}(\mathfrak{e})$ and we put $\theta_r = \inf\{i \geq 0 : \mathsf{H}(\mathfrak{e}_i) = r\}$ for $r \geq 0$. With Definition 22 in mind, we easily prove by induction on $r \geq 0$ that:

$$\mathfrak{e}_{\theta_r} = \mathrm{Ball}_r^\dagger(\mathfrak{m}). \tag{7.7}$$

Also if $(\mathfrak{e}_i)_{i\geq 0}$ is the filled-in version of the peeling algorithm then we have

$$\bar{\mathfrak{e}}_{\theta_r} = \overline{\mathrm{Ball}}_r^\dagger(\mathfrak{m}). \tag{7.8}$$

7.2.2 Dilute phase

We turn the analogous of Proposition 34. In particular, recall the definition of $g_{\mathbf{q}}$ from Definition 20.

Theorem 14 (*Distances in the peeling by layers*)

Let $(\bar{\mathfrak{e}}_n)_{n\geq 0}$ by the filled-in peeling exploration with algorithm $\mathcal{L}^\dagger$ recall the notation $\mathsf{H}(\bar{\mathfrak{e}}_n)$. If $\mathbf{q}$ is of type $a \in (2;5/2]$ then under $\mathbb{P}_\infty^{(1)}(d\mathfrak{m})\mathrm{P}_{\mathfrak{m}}^{\mathrm{fpp}}$ we have the following convergence in *distribution for the Skorokhod topology*

$$\left(\frac{|\partial\bar{\mathfrak{e}}_{[nt]}|}{n^{\frac{1}{a-1}}}, \frac{|\bar{\mathfrak{e}}_{[nt]}|}{n^{\frac{a-1/2}{a-1}}}, \frac{\mathsf{H}(\bar{\mathfrak{e}}_{[nt]})}{n^{\frac{a-2}{a-1}}}\right)_{t\geq 0} \xrightarrow[n\to\infty]{(d)} \left(\left(\Upsilon_a^{\uparrow}, \mathsf{b_q}\mathcal{V}_a(\Upsilon^{\uparrow}), \frac{1+\mathsf{g_q}}{2\mathsf{p_q}}\int_0^{\cdot}\frac{\mathrm{d}u}{\Upsilon_a^{\uparrow}(u)}\right)(\mathsf{p_q}t)\right)_{t\geq 0}.$$

Let us again give a more geometric interpretation of the above result. Recall from (7.7) that the peeling process using algorithm $\mathcal{L}^{\dagger}$ discovers balls for the dual graph distance on B_∞ and we denote by $|\overline{\mathrm{Ball}}_r^{\dagger}(B_\infty)|$ and $|\partial\overline{\mathrm{Ball}}_r^{\dagger}(B_\infty)|$ respectively the size (number of inner vertices) and the half-perimeter of its unique hole of the hull of the ball of radius r for the dual distance. Then with the same notation as in (7.1) the above result implies the convergence in distribution in the sense of Skorokhod

$$\left(\frac{|\partial\overline{\mathrm{Ball}}_{[tn]}^{\dagger}(B_\infty)|}{n^{\frac{1}{a-2}}}, \frac{|\overline{\mathrm{Ball}}_{[tn]}^{\dagger}(B_\infty)|}{n^{\frac{a-1/2}{a-2}}}\right)_{t\geq 0} \xrightarrow[n\to\infty]{(d)} \left(\left(\Upsilon_a^{\uparrow}, \mathsf{b_q}\cdot\mathcal{V}_a(\Upsilon_a^{\uparrow})\right)(\vartheta_{2\mathsf{p_q}t/(1+\mathsf{g_q})})\right)_{t\geq 0}. \qquad (7.9)$$

[More soon...]

Chapter VIII : Simple and self-avoiding random walks

In this chapter we study the simple random walks on Boltzmann planar maps. We first recall the application of the circle packing theory to the Benjamini & Schramm theorem on recurrence of local limits of uniformly pointed maps. This theorem and its extension by Gurel-Gurevich & Nachmias theorem shows that under $\mathbb{P}_\infty^{(1)}$ the random map of the plane $\mathfrak{m}$ is almost surely recurrent. However the case of its dual is largely open and we present some partial results using the peeling process. The later is also a tool to measure the rate of escape the "subdiffusivity" exponent of the simple random walk.

8.1 Recurrence of the primal via circle packings

Let us start by giving a proper definition of simple random walk:

Definition 23. *If $\mathfrak{m}$ is a finite or infinite planar map, a simple random walk on $\mathfrak{m}$ is a Markov process taking values in the space of oriented edges of $\mathfrak{m}$, starting with the root edge, and such that if $\vec{e}$ is the current state then the next oriented edges is chosen uniformly among all the oriented edges outgoing from the target vertex of $\vec{e}$ independently of the past path. We denote by* $\mathrm{SRW}_{\mathfrak{m}}$ *its law.*

This definition is equivalent to a walk on the vertices which jumps to a uniform neighbor in the case of simple maps. In general, the presence of loops and multiple edges makes the transitions between vertices a little bit more complicated. The goal of this section is to explain the following result which is due to Bjornberg & Stefansson [18]:

Theorem 15 (*Recurrence of the primal map.*)

If $\mathbf{q}$ is a critical weight sequence then almost surely under $\mathbb{P}_\infty^{(1)}$ the map $\mathfrak{m}$ is recurrent for the simple random walk.

Actually, this theorem is a simple corollary of Theorem 8 (which was initially proved by Bjornberg and Stefansson) and a result of Gurel-Gurevich & Nachmias [39] which itself builds upon the Benjamini & Schramm theorem [11]. More precisely Benjamini & Schramm showed that any local limit of random finite planar maps such that the root edge is uniformly distributed on the map is recurrent provided that we have a uniform bound on the degrees of the vertices in the maps. Notice that the maps under $\mathbb{P}_n^{(1)}$ are indeed finite, the root edge is indeed distributed uniformly on the map[1], but the bounded degree condition is never true. This was the only

[1] after having glued the two edges of the root face together

obstruction to apply the Benjamini–Schramm theorem to the law $\mathbb{P}_\infty^{(1)}$. This problem has been solved by Gurel-Gurevich & Nachmias who showed that is suffices that the degree of the origin in the limit map has an exponential tail to apply the result. As an easy exercise we can check that this is indeed the case under $\mathbb{P}_\infty^{(1)}$ using the peeling process:

Proposition 37. *Let* $\mathbf{q}$ *be a critical weight sequence. Then there exists some* $c \in (0,1)$ *such that if* ρ *denotes the origin vertex of* $\mathfrak{m}$

$$\mathbb{P}_\infty^{(1)}(\deg(\rho) \geq k) \leq c^k.$$

Proof. We explore the neighborhood of the origin vertex using a peeling exploration that always peels the edge of the active boundary adjacent on the left of the origin vertex until it becomes an interior vertex. Each peeling step at most adds one unit to the degree of the origin and at each step there is a positive probability bounded from below by $c > 0$ that an event of type $\mathsf{G}_{k,0}$ occurs. On this event the origin vertex becomes an inner vertex of the explored map. The bound on the tail follows. □

Without giving the details, let us say a bit more on the proof by Benjamini & Schramm. At its core we find the theory of circle packing. We already mention this in Section 1.5 but this time one needs to deal with infinite circle packings. The Koebe–Andreev–Thurston theorem becomes an alternative in the case of infinite maps of the plane (recall Definition 4). In the following theorem due to He & Schramm [40], if $\mathcal{P}$ is an infinite circle packing in the plane the *carrier* of $\mathcal{P}$ is the subset of the plane made of the union of all the circles as well as the interstices between them.

Theorem 16 (*Infinite circle packing theorem*)

Let $\mathfrak{m}$ *be a infinite simple map of the plane. Then we have one of the mutually excluding alternatives:*

- *Parabolic case: either there is a circle packing whose carrier is* $\mathbb{R}^2$ *representing* $\mathfrak{t}$,
- *Hyperbolic case: or there is a circle packing whose carrier is* $\mathbb{D}$ *representing* $\mathfrak{t}$.

If furthermore $\mathfrak{m}$ *is a triangulation then in the first case the packing is unique up to rotation, translation and dilation, whereas in the second case it is unique up to Möbius transformation preserving the unit disk. Furthermore, if the vertex degrees of* $\mathfrak{m}$ *are bounded, the above dichotomy corresponds to the case when* $\mathfrak{m}$ *is recurrent (packing in* $\mathbb{R}^2$*) or transient (packing in* $\mathbb{D}$*).*

The above theorem can be seen as a discrete counter-part to the dichotomy for simple connected Riemann surfaces homeomorphic to the disk: either such a surface is conformally equivalent to the disk (and Brownian motion on the surface is transient) or it is conformally equivalent to the plane (and Brownian motion on the surface is recurrent). By the above theorem, the Benjamini & Schramm result boils down to showing that almost surely the circle packing of a limit of finite random map where the root edges is uniform is almost surely carried by $\mathbb{R}^2$.

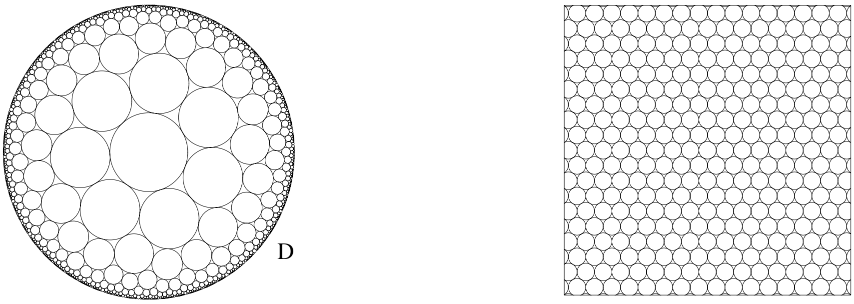


Figure 8.1: Illustration of Theorem 16: the 7-regular infinite triangulation is circle-packed in the disk (and is transient) whereas the 6-regular infinite triangulation is circle-packed in the plane (and is recurrent). Images of Kenneth Stephenson.

8.2 Simple random walk on the dual maps

The above discussion however leaves the question of the recurrent or transient of the *dual* map $\mathfrak{m}^\dagger$ under $\mathbb{P}_\infty^{(1)}$. Indeed, since we must perform the identification of the two faces of the root face to get a map where the root edge is uniformly distributed over all oriented edges, the degree of the origin in $\mathfrak{m}^\dagger$ sometimes does not have an exponential tail... In fact we show that these dual maps can actually be transient.

8.2.1 Transience of the dual maps in the dense case

We will see as a corollary of the study of the fpp-distance on $\mathfrak{m}^\dagger$ that these graphs are almost surely transient. The proof uses the method of random paths.

Corollary 38. *When* $\mathbf{q}$ *is of type* $a \in (3/2; 2)$ *the random lattice* $\mathfrak{m}^\dagger$ *distributed according to* $\mathbb{P}_\infty^{(1)}$ *is almost surely transient (for the simple random walk).*

Proof. We use the method of the random path [50, Section 2.5 page 41]. More precisely, the FPP model on $\mathfrak{m}^\dagger$ enables us to distinguish an infinite oriented path $\vec{\Gamma} : f_r \to \infty$ in $\mathfrak{m}^\dagger$ which is the shortest infinite path starting from the origin for the FPP-distance (almost sure uniqueness of this path is easy to prove under $\mathbb{P}_\infty^{(1)}(\mathrm{d}\mathfrak{m})\mathrm{Q}_{\mathfrak{m}}^{\mathrm{fpp}}(\mathrm{d}(x_e))$). From this path $\vec{\Gamma}$ one constructs a unit flow θ on the directed edges with source at f_r by putting for any oriented edge $\vec{e}$ of $\mathfrak{m}^\dagger$

$$\theta(\vec{e}) = \mathrm{Q}_{\mathfrak{m}}^{\mathrm{fpp}}(\vec{e} \in \vec{\Gamma}) - \mathrm{Q}_{\mathfrak{m}}^{\mathrm{fpp}}(\overleftarrow{e} \in \vec{\Gamma}).$$

To show that the energy of this flow is finite, we compare it to the expected FPP-length of $\vec{\Gamma}$ which is almost surely finite under $\mathbb{P}_\infty^{(1)}(\mathrm{d}\mathfrak{m})\mathrm{Q}_{\mathfrak{m}}^{\mathrm{fpp}}(\mathrm{d}(x_e))$ by Proposition 36. More precisely, if x_{e_0} denotes the exponential weight of a given edge e_0, and if $\mathrm{E}_{\mathfrak{m}}^{\mathrm{fpp}}$ denotes the expectation under $\mathrm{Q}_{\mathfrak{m}}^{\mathrm{fpp}}$, we just remark that there exists a constant $C > 0$ such that for any event A we have

$$\mathrm{E}_{\mathfrak{m}}^{\mathrm{fpp}}[x_{e_0} \mathbf{1}_A] \geq C\, \mathrm{Q}_{\mathfrak{m}}^{\mathrm{fpp}}(A)^2.$$

Indeed, if $\delta = \mathrm{Q}^{\mathrm{fpp}}_{\mathfrak{m}}(A)$ we have $\mathrm{E}^{\mathrm{fpp}}_{\mathfrak{m}}[x_e \mathbf{1}_A] \geq \mathrm{E}^{\mathrm{fpp}}_{\mathfrak{m}}[x_e \mathbf{1}_A \mathbf{1}_{x_e \geq \delta/2}] \geq \delta/2\, \mathrm{P}^{\mathrm{fpp}}_{\mathfrak{m}}(A \cap \{x_e \geq \frac{\delta}{2}\})$ and use the fact that $\mathrm{P}^{\mathrm{fpp}}_{\mathfrak{m}}(A \cap \{x_e \geq \frac{\delta}{2}\}) \geq \mathrm{P}^{\mathrm{fpp}}_{\mathfrak{m}}(A) + \mathrm{P}^{\mathrm{fpp}}_{\mathfrak{m}}(x_e \geq \frac{\delta}{2}) - 1 = \delta + e^{-\delta/2} - 1 \geq \delta/2$. Using this we can write

$$\begin{aligned} \sum_{\vec{e} \in \overrightarrow{\mathrm{Edges}}(\mathfrak{m}^\dagger)} \theta(\vec{e})^2 \leq 4 \sum_{e \in \mathrm{Edges}(\mathfrak{m}^\dagger)} \mathrm{P}^{\mathrm{fpp}}_{\mathfrak{m}}(e \in \Gamma)^2 &\leq \frac{4}{C} \sum_{e \in \mathrm{Edges}(\mathfrak{m}^\dagger)} \mathrm{E}^{\mathrm{fpp}}_{\mathfrak{m}}[\mathbf{1}_{e \in \Gamma} x_e] \\ &= \frac{4}{C} \mathrm{E}^{\mathrm{fpp}}_{\mathfrak{m}}[\mathrm{Length}_{\mathrm{fpp}}(\Gamma)] < \infty \quad a.s. \end{aligned}$$

This proves almost sure transience of the lattice as desired. □

Open question. *Let* $\mathbf{q}$ *be of type* $a \in [2; 5/2)$*, it is the case that* $\mathfrak{m}^\dagger$ *is transient under* $\mathbb{P}^{(1)}_\infty$*?*

8.2.2 Intersection and recurrence of SRW

Recall that a infinite graph g has the intersection property if for any two vertices x and y of the graph, the trajectories of two independent simple random walks started from x and y intersect (if so, then they intersect infinitely many often). In particular if a graph has the intersection property then it does not possess non trivial bounded harmonic function: it has the Liouville property.

Theorem 17 (*Intersection property*)

Let $\mathbf{q}$ *be a weight sequence of type* $a \in [3/2; 5/2]$ *then* $\mathfrak{m}^\dagger$ *almost surely has the intersection property under* $\mathbb{P}^{(\ell)}_\infty$*. In particular it is almost surely Liouville.*

To prove this theorem we will use a peeling exploration along the simple random walk on $\mathfrak{m}$. Under $\mathrm{SRW}_{\mathfrak{m}}$ we write by $(\vec{E}_n)_{n \geq 0}$ the oriented edges visited by the walk then this induces a peeling exploration we setting $\mathfrak{e}^\circ_n = \{\vec{E}_k : 0 \leq k \leq n\}$. This is not exactly a peeling exploration since it might be that $\mathfrak{e}_n = \mathfrak{e}_{n+1}$ if the edge $\vec{E}_n$ has already been previously visited by the walk. However, erasing the repetitions this indeed defines a peeling exploration whose filled-in version (under $\mathbb{P}^{(1)}_\infty$) is as usual denoted by

$$\bar{\mathfrak{e}}_0 \subset \bar{\mathfrak{e}}_1 \subset \cdots \subset \mathfrak{m}.$$

The algorithm we used to define this exploration is randomized, but it is easy to see that we can encode all the randomness part of the simple random walk into a variable ω independent of $\mathfrak{m}$ such that the above exploration is indeed a filled-in peeling process for the randomized algorithm that we denote by $\mathcal{A}_{\mathrm{walk}}(\mathfrak{e}, \omega)$. Geometrically, if θ_n is the number of steps of the walk inside the explored map $\overline{\mathfrak{e}_n}$ then the former correspond to the hull of the trace of the walk up to time θ_n. Yet, otherwise said, we let the simple random walk move freely and when it is necessary to trigger a new peeling step to discover the edge it wants to go through we do so and immediately fill-in the finite hole we may create on the way.

Proof of Theorem 17. Under $\mathbb{P}^{(1)}_\infty(\mathrm{d}\mathfrak{m})\mathrm{SRW}_{\mathfrak{m}}(\mathrm{d}(\vec{e}_n)_{n \geq 0})$ we explore the map $\mathfrak{m}$ using a filled-in peeling exploration under algorithm $\mathcal{A}_{\mathrm{walk}}$. Since $\mathbf{q}$ is of type $a \in (3/2; 5/2]$, by Corollary 27 we know that

$$\bigcup_{n\geq 0} \bar{\mathfrak{e}}_n = \mathfrak{m} \qquad \text{almost surely.}$$

With the geometric interpretation of $\mathfrak{e}_n$ in term of the random walk $(\vec{e}_n)_{n\geq 0}$ this means that the hull of the trace $\{\vec{e}_n : n \geq 0\}$ almost surely equal to the full map. In particular the complement of the vertices visited by the walk is made of finite components only and so any other random walk must intersect the vertex-trajector of $(\vec{e}_n)_{n\geq 0}$ almost surely. □

As corollary here is a special case of Benjamini–Schramm theorem:

Corollary 39. *If* $\mathbf{q}$ *is critical and finitely supported then* $\mathfrak{m}^\dagger$ *is a.s. recurrent under* $\mathbb{P}_\infty^{(1)}$.

Proof. It is known by an older result of Benjamini & Schramm [10] that a planar graph with bounded vertex degrees is recurrent if and only if it has no bounded harmonic functions. Clearly if $\mathbf{q}$ is finitely supported then $\mathfrak{m}$ has bounded vertex degrees under $\mathbb{P}_\infty^{(1)}$ and is Liouville by the above result (a finitely supported critical weight sequence is necessarily of type $5/2$). Hence it must be almost surely recurrent. □

[Much more soon...]

Chapter I : Coding of bipartite maps with labeled trees

In this section we quickly present the coding of (bipartite) planar maps via labeled trees, based on a variant of the construction of Schaeffer and Bouttier–Di Francesco–Guitter. This coding enables a quick proof of Proposition 5 and Theorem 6 which were the key enumerative inputs in these lecture notes.

A.1 Bouttier–Di Francesco – Guitter coding of bipartite maps

A.1.1 From maps to trees

Let $\mathfrak{m}_\bullet = (\mathfrak{m}, \rho)$ be a pointed bipartite planar map with a external face of degree 2ℓ and perform the following operations:

1. Draw a vertex in each face of $\mathfrak{m}$ (including the external face). The new vertices are considered black ($\bullet$) and the old ones white ($\circ$). Label each white vertex by its distance to the distinguished vertex ρ. Since the map is bipartite, the labels of any two adjacent vertices differ exactly by one.

2. For a face f of $\mathfrak{m}$ and a white vertex adjacent to f, link the white vertex to the black vertex inside f if the next white vertex in the clockwise order around f has a smaller label.

3. Remove the edges of $\mathfrak{m}$ and the vertex ρ. It can be shown that the resulting graph is a tree [21].

4. Let v_0 be the black vertex corresponding to the external face of $\mathfrak{m}$. By removing v_0 and its adjacent edges, we obtain a forest of cyclically ordered trees, rooted at the neighbors of v_0. Finally, we choose uniformly at random one of the trees to be the first one, subtract the labels by a constant so that the label of the root vertex of this first tree becomes zero.

With a moment of thought on the Step 2 of the above construction, one observes that

(i) Each internal face of degree $2k$ in $\mathfrak{m}$ gives rise to a black vertex of degree k in the forest, and the forest is composed of p trees.

(ii) Given a black vertex of degree k, the possible labels on its (white) neighbors are exactly those which, when read in the clockwise order around the black vertex, can decrease at most by 1.

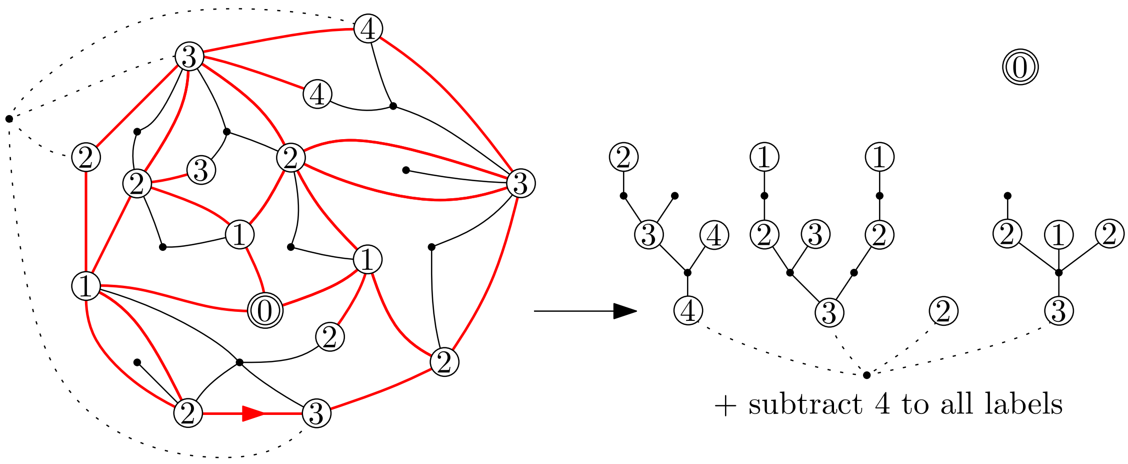


Figure A.1: Illustration of the construction of a forest of ℓ mobiles from a pointed bipartite planar map with a boundary of perimeter 2ℓ. The first mobile in the forest is not specified by the map and is chosen uniformly at random among all the mobiles.

Remark 13. Let v be a black vertex of degree ℓ and fix the label of one of its neighbor (to 0 say). Then the number of well-labelings of the white vertices around v is given by the number of walks in $\mathbb{Z}$ which starts at 0, come back at 0 after ℓ steps, and whose steps are in $\{-1, 0, 1, 2, 3, 4, ...\}$. Up to adding 2 to each increment, this number is given by the number of partitions of 2ℓ into ℓ positive integers which is classical to compute

$$\mathsf{N}(\ell) := \binom{2\ell - 1}{\ell - 1}.$$

A *mobile* is a rooted plane tree whose vertices at even (resp. odd) generations are white (resp. black). We say that a forest of mobiles $(\mathfrak{t}_1, \cdots, \mathfrak{t}_p)$ is *well-labeled* if (a) the root vertex of $\mathfrak{t}_1$ has label 0, (b) the labels satisfy the constraint in the observation (ii) above, and (c) the labels of the roots of $\mathfrak{t}_1, \cdots, \mathfrak{t}_p$ satisfy the similar constraint.

The above construction thus associate to $\mathfrak{m}_\bullet$ a forest $\mathfrak{f}$ of ℓ well-labeled mobiles (modulo the addition of randomness needed to chose the first tree in the forest) and we shall denote

$$\mathfrak{f} = \mathsf{Mob}(\mathfrak{m}_\bullet).$$

A.1.2 From trees to maps

Let us now present the inverse construction. Start with the forest $\mathfrak{f}$ of ℓ mobiles $\mathfrak{t}_1, \cdots, \mathfrak{t}_\ell$ which we imagine drawn on the plane once grafted in clockwise order on a cycle of length ℓ. We then perform the usual "Schaeffer construction" by doing the contour of the mobiles and linking any corner associated to a vertex of label i to the next corner in the contour associated with a vertex of label $i - 1$. If i is the minimal label then we link this corner to an additional vertex ρ put in the infinite face of the embedding. The edges can be drawn in a non-crossing fashion and after erasing the embedding of the cycle and the mobiles, we are left with a bipartite map with a distinguished vertex ρ. The external face of the map is the face that "encloses" the cycle on which the mobiles have been grafted. The root edge of the map is not prescribed by the forest and is taken uniformly at random on an edge of the external face of degree 2ℓ (so that the external face is on its right). We denote by $\mathsf{BDG}(\mathfrak{f})$ the resulting pointed random map. As usual in Schaeffer-type constructions, the labeling of the above forest has a geometric interpretation in terms of the map but since we do not use it we do not bother to enter the details. It should be clear on a drawing that modulo the rooting of the map we have

$$\mathfrak{m}_\bullet = \mathsf{BDG}(\mathsf{Mob}(\mathfrak{m}_\bullet)). \tag{A.1}$$

A.2 Distribution of the forest of mobiles

Let us now describe the effect of this coding on the Boltzmann measure. Recall from (2.6) the definition of the measure $w_{\mathbf{q}}$ on bipartite planar maps. We now compute the image measure of $w_{\mathbf{q}}$ by the mapping Mob (which has an additional randomness in it for the choice of the root mobile). But before doing so we first transform a mobile into a simple plane tree by yet another mapping due to Janson & Stefansson [43].

A.2.1 Janson & Stefansson's trick

In [43, Section 3], Janson & Stefánsson discovered a mapping which transforms a mobile into a rooted plane tree by keeping the same set of vertices, but changing the set of edges so that every white vertex is mapped to a leaf, and every black vertex of degree k is mapped to an internal vertex with k children. We refer to [34, Section 3.2] for details of this transformation and the curious reader may have a look at the figure below and try to guess how the bijection works. We denote by JS this transformation.

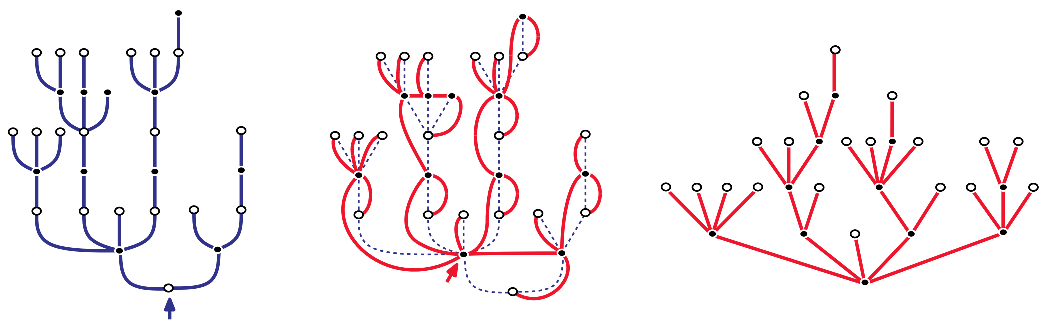


Figure A.2: Illustration of the Janson & Stefánsson transformation.

Proposition 40. *Let $\ell \geq 1$, then the image measure of $w_{\mathbf{q}}$ on $\mathcal{M}_{\bullet}^{(\ell)}$ by the chain of mappings*

$$\mathsf{JS} \circ \mathsf{ForgetLabels} \circ \mathsf{Mob}$$

is the measure on forests of ℓ trees defined by

$$\tilde{w}_{\mathbf{q}}(\mathfrak{f}) = \binom{2\ell-1}{\ell-1} \prod_{u \in \mathfrak{f}} \tilde{q}_{k_u},$$

where $\tilde{q}_\ell = q_\ell \binom{2\ell-1}{\ell-1}$ and $\tilde{q}_0 = 1$ and where k_u is the number of children of u in the forest $\mathfrak{f}$.

Proof. Let $\mathsf{f} = (\mathsf{t}_1, \mathsf{t}_2, \dots, \mathsf{t}_\ell)$ be ℓ plane trees. Let us first see which forest of ℓ mobiles give rise to such a forest of trees after applying Janson–Stefansson mapping. The unlabeled forest of mobiles $\mathfrak{f} = (\mathfrak{t}_1, \dots, \mathfrak{t}_\ell)$ can be retrieved by applying the inverse Janson–Stefansson mapping and in particular the degrees of the black vertices are given by the number of children in the forest, hence by

$$\{k_u : u \in \mathsf{f}\},$$

where k_u is the number of children of a vertex u. Now, to such a forest $\mathfrak{f}$ may correspond a lot of well-labeled forests. Specifically, using Remark 13 around each black vertex (and around the origins of the mobiles) since the label of the (white) origin of the first mobile is fixed to 0, there are exactly

$$\mathsf{N}(\ell) \prod_{v \in \mathrm{BlackVertices}(\mathfrak{f})} \mathsf{N}(\deg(v)) = \mathsf{N}(\ell) \prod_{u \in \mathsf{f}} \mathsf{N}(k_u),$$

possible well-labelings of the forest where we put $\mathsf{N}(0) = 1$. Now, fix a labeling of the forest $\mathfrak{f}$ and let us see which pointed map $\mathfrak{m}_\bullet$ can give rise to this forest by the Mob construction. By (A.1), up to the location of the root edge, the pointed map $\mathfrak{m}_\bullet$ can be recovered by applying the Bouttier–Di Francesco–Guitter construction to the forest of well-labeled mobiles. In particular since the degree of the inner faces of the map are twice those of the black vertices of the forest we deduce that the $w_{\mathbf{q}}$ weight of such a map (if rooted) is given by

$$\prod_{v \in \mathrm{BlackVertices}(\mathfrak{f})} q_{\deg(v)} = \prod_{u \in \mathsf{f}} q_{k_u},$$

where we put $q_0 = 1$ by convention. Let us assume first that the unrooted pointed map $\mathsf{BDG}(\mathfrak{f})$ has no symmetry, i.e. that rooting the map on each of the 2ℓ edges of its boundary yield 2ℓ different rooted pointed maps. Then for each of these 2ℓ maps the Mob construction returns $\mathfrak{f}$ with probability $1/\ell$ (the probability to choose the right mobile as first one). In the case of symmetry, the fewer number of maps obtained by rooting on the boundary is exactly compensated by the larger probability to get the forest (since the latter also inherits the symmetries of the map), we leave the details here to the reader. In total, gathering the above equation we deduce that the $w_{\mathbf{q}}$ weight of all the maps $\mathfrak{m}_\bullet$ such that $\mathsf{JS} \circ \mathsf{ForgetLabels} \circ \mathsf{Mob}(\mathfrak{m}_\bullet) = \mathfrak{f}$ is

$$2\ell \cdot \frac{1}{\ell} \cdot \prod_{u \in \mathsf{f}} q_{k_u} \cdot \mathsf{N}(\ell) \prod_{u \in \mathsf{f}} \mathsf{N}(k_u).$$

□

A.2.2 Proof of the enumeration results

Proof of Proposition 5 and Theorem 6. By the above proposition $w_{\mathbf{q}}(\mathcal{M}_{\bullet}^{(\ell)})$ is finite (for one $\ell \geq 1$ or equivalently for all $\ell \geq 1$) if and only if we have

$$\sum_{\mathfrak{t} \in \mathsf{FinitePlaneTrees}} \sum_{u \in \mathfrak{t}} \tilde{q}_{k_u} < \infty. \tag{A.2}$$

Checking whether the above sum is finite is standard, call x the potential value of the series and notice that by a recursive decomposition at the root vertex (a tree is either equal to a single vertex or a vertex of degree k with k trees attached to them) we should have

$$x = 1 + \sum_{k \geq 1} \tilde{q}_k x^k.$$

This equation is exactly the one considered in Proposition 5, and it is easy to see that (A.2) is finite if and only if the above equation admits a solution in $x \geq 0$. In this case, the weight sequence $\mathbf{q}$ is admissible and the series (A.2) is equal to the smallest of such solutions which we call $Z_{\mathbf{q}}$ (which is also equal to $4c_{\mathbf{q}}$) and we deduce thanks to the above proposition that

$$W_{\bullet}^{(\ell)} = 2\binom{2\ell - 1}{\ell - 1}(Z_{\mathbf{q}})^{\ell} = c_{\mathbf{q}}^{\ell} h^{\downarrow}(\ell),$$

which is exactly the statement of Theorem 6. □

A.2.3 Back to criticality

With the notation of the above section, if $\mathbf{q}$ is admissible, then it is easy to see that upon normalizing $w_{\mathbf{q}}$ on $\mathcal{M}_{\bullet}^{(\ell)}$ to make it a probability measure, the distribution of the ℓ plane trees obtained by pushing the last distribution via $\mathsf{JS} \circ \mathsf{ForgetLabels} \circ \mathsf{Mob}$ is just given by ℓ independent Galton–Watson trees whose offspring distribution is given by

$$\mu_{\mathrm{JS}}(k) = \tilde{q}_k Z_{\mathbf{g}}^{k-1}, \quad k \geq 0,$$

where we recall that $\tilde{q}_0 = 1$ by convention. The above display indeed defines an offspring distribution when $\mathbf{q}$ is admissible and it is an easy calculus exercise to see that the notion of criticality for the weight sequence $\mathbf{q}$ (Definition 9) is equivalent to criticality for the offspring distribution μ_{JS} i.e. having mean equal to 1. Similarly, we have the following dictionary for the notions appearing in Section 5.1:

$\mathbf{q}$ subcritical	μ_{JS} of mean < 1.
$\mathbf{q}$ critical	μ_{JS} of mean 1.
$\mathbf{q}$ regular critical	μ_{JS} of mean 1 and some exponential moment.
$\mathbf{q}$ is of type $a \in (3/2; 5/2)$	μ_{JS} of mean 1 and has a polynomial tail of order a.

Bibliographical notes. The coding of planar maps via label trees first appeared in the seminal work of Cori & Vauquelin [31] and later made clear and popularized by Schaeffer [57]. The coding we used here is an extension of Schaeffer's initial construction due to Bouttier–Di Francesco–Guitter [21] and more precisely the variant presented in [16]. The study of the induced distribution of trees is found first in [51]. The presentation here is partially taken from our work [29]. Notice that the use of bijections with trees is one of the most useful tool to study geometric properties of random planar maps, but in these lecture notes we prefer not to use it and focused on the peeling process itself.

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