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SUR LE SECOND THÉORÈME PRINCIPAL

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SUR LE SECOND THÉORÈME PRINCIPAL

Résumé

La conjecture de Kobayashi stipule qu'une hypersurface générique $X_d \subset \mathbb{P}^{n+1}(\mathbb{C})$ de degré $d \geq 2n+1$ est hyperbolique complexe, un problème qui a attiré une grande attention récemment, avec l'espoir de mettre au point une théorie de Nevanlinna complète en dimension supérieure.

Dans la première partie de cette thèse, notre objectif est de construire des exemples d'hypersurfaces hyperboliques de l'espace projectif dont le degré soit aussi petit que possible. Tout d'abord, en tenant compte du niveau de troncation dans le Second Théorème Principal de Cartan, nous établissons l'hyperbolicité de complémentaires de certaines configurations d'hyperplans avec points de passages, ce qui étend un résultat classique de Bloch-Fujimoto-Green. Ceci nous permet d'amorcer un algorithme récent de Duval, basé sur la méthode de déformation de Zaidenberg, pour créer des sextiques hyperboliques dans $\mathbb{P}^3(\mathbb{C})$, et de construire ainsi des familles d'hypersurfaces hyperboliques $X_d \subset \mathbb{P}^{n+1}(\mathbb{C})$ de degré $d = 2n+2$ pour $2 \leq n \leq 5$. En adaptant cette technique aux dimensions supérieures, nous obtenons aussi des exemples d'hypersurfaces hyperboliques de degré $d \geq (\frac{n+3}{2})^2$ dans $\mathbb{P}^{n+1}(\mathbb{C})$.

Dans la deuxième partie, nous étudions le problème de diminuer le niveau de troncation dans le Second Théorème Principal de Cartan. Noguchi a conjecturé que dans ce théorème, pour une famille de 4 droites en position générale dans $\mathbb{P}^2(\mathbb{C})$, si une courbe holomorphe entière $f : \mathbb{C} \to \mathbb{P}^2(\mathbb{C})$ est supposée n'être pas algébriquement dégénérée, alors le niveau de troncation peut être abaissé à 1. En utilisation la théorie de recouvrement d'Ahlfors pour les surfaces, nous proposons une réponse positive dans le cas où la courbe f est proche d'une certaine courbe algébrique $\mathscr{C} \subset \mathbb{P}^2(\mathbb{C})$, au sens où l'ensemble d'accumulation de $f(\mathbb{C})$ à l'infini, le *cluster set* de f

$$\mathrm{Clu}(f) := \bigcap_{r>0} \big(\overline{f(\mathbb{C} \setminus \Delta_r)}\big)$$

est contenu dans $\mathscr{C}$.

Mots-clefs : Théorie de Nevanlinna, Hyperbolicité complexe, Théorie de Ahlfors, Conjecture de Kobayashi, Second Théorème Principal, courbe holomorphe, Courant de Nevanlinna, Lemme de Brody.

On the Second Main Theorem

Abstract

Kobayashi's conjecture asserts that a generic hypersurface $X_d \subset \mathbb{P}^{n+1}(\mathbb{C})$ having degree $d \geq 2n+1$ is complex hyperbolic, a problem that has attracted much attention recently, also with the hope of setting up a complete higher dimensional Nevanlinna theory.

In the first part of this thesis, our goal is to construct examples of hyperbolic hypersurfaces in projective spaces of degree as low as possible. First of all, taking into account the truncation level in Cartan's Second Main Theorem, we establish the hyperbolicity of complements of some configurations of hyperplanes with passage points, extending a classical result of Bloch-Fujimoto-Green. This allows us to launch a recent algorithm of Duval, based on the deformation method of Zaidenberg, on creating hyperbolic sextics in $\mathbb{P}^3(\mathbb{C})$, hence to construct families of hyperbolic hypersurfaces $X_d \subset \mathbb{P}^{n+1}(\mathbb{C})$ having degree $d = 2n+2$ for $2 \leq n \leq 5$. Adapting this technique to higher dimensional cases, we also obtain examples of hyperbolic hypersurfaces of degree $d \geq (\frac{n+3}{2})^2$ in $\mathbb{P}^{n+1}(\mathbb{C})$.

In the second part, we study the problem of decreasing the truncation level in Cartan's Second Main Theorem. It was conjectured by Noguchi that in this theorem, for a family of 4 lines in general position in $\mathbb{P}^2(\mathbb{C})$, if an entire holomorphic curve $f\colon \mathbb{C} \to \mathbb{P}^2(\mathbb{C})$ is assumed to be algebraically nondegenerate, then the truncation level can be decreased to 1. Using Ahlfors'theory of covering surfaces, we propose a positive answer in the case where the curve f is close to some algebraic curve $\mathscr{C} \subset \mathbb{P}^2(\mathbb{C})$, in the sense that the set of accumulation points of $f(\mathbb{C})$ at infinity, the *cluster set* of f

$$\mathrm{Clu}(f) := \bigcap_{r>0} \left(\overline{f(\mathbb{C} \setminus \Delta_r)}\right)$$

is contained in $\mathscr{C}$.

Keywords : Nevanlinna Theory, Complex hyperbolicity, Ahlfors' Theory, Kobayashi's conjecture, Second Main Theorem, holomorphic curve, Nevanlinna current, Brody's Lemma.

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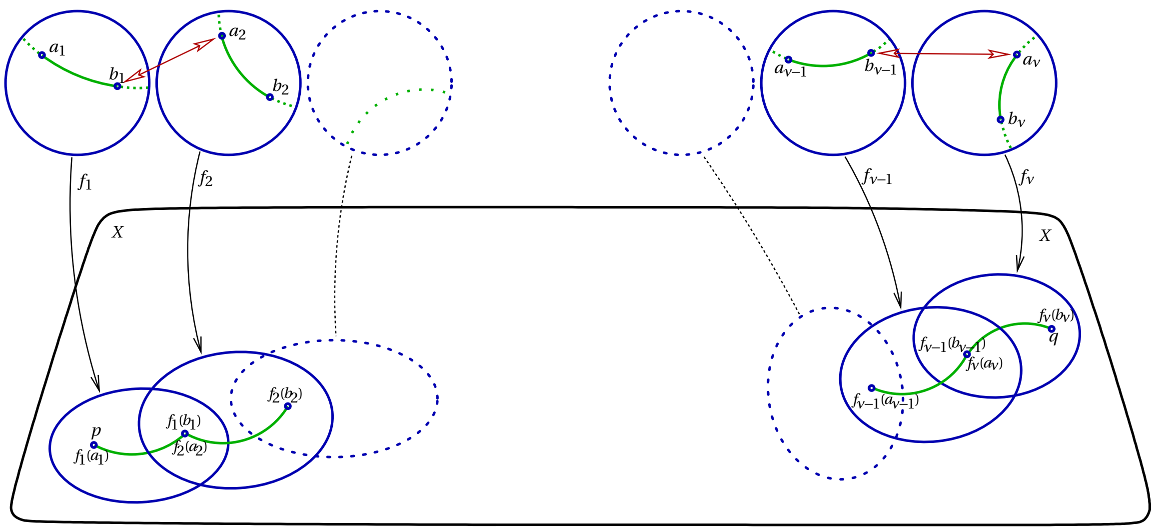
PRÉSENTATION DES RÉSULTATS

Soit X une variété complexe connexe. Pour deux points p, q de X, une chaîne holomorphe allant de p à q est une suite $\{(f_i, a_i, b_i)\}_{i=1}^{\nu}$, où f_i sont des courbes holomorphes du disque unité Δ à valeurs dans X, et a_i, b_i sont des points de Δ tels que

$$p = f_1(a_1),\ f_1(b_1) = f_2(a_2),\ \ldots\ldots,\ f_{\nu-1}(b_{\nu-1}) = f_\nu(a_\nu),\ f_\nu(b_\nu) = q.$$



La *pseudo-distance de Kobayashi* est alors définie par

$$d_X(p,q) := \inf\Big\{\sum_{i=1}^{\nu} \rho(a_i, b_i)\Big\},$$

où ρ est la distance de Poincaré sur Δ, et l'infimum est pris sur toutes les chaînes holomorphes de p à q. Cette pseudo-distance est la forme intégrée de la *pseudo-métrique infinitésimale de Kobayashi-Royden* sur X, définie par

$$F_X(v) := \inf\Big\{\lambda > 0 : \exists\, f \in \mathrm{Hol}(\Delta, X), f(0) = p, f'(0) = \frac{v}{\lambda}\Big\} \qquad (p \in X,\, v \in T_pX).$$

Par définition, la pseudo-distance de Kobayashi possède la propriété de distance décroissante, c'est-à-dire que pour toute application holomorphe $\varphi\colon X \to Y$ entre deux variétés complexes, pour tout couple (p,q) de points de X, on a

$$d_X(p,q) \geq d_Y(\varphi(p), \varphi(q)).$$

Grâce au Lemme de Schwarz-Pick, la pseudo-distance de Kobayashi sur le disque unité Δ coïncide avec la distance de Poincaré. En général, la pseudo-distance de Kobayashi peut-être dégénérée. Par exemple, si X est le plan complexe ou une courbe elliptique, alors $d_X \equiv 0$. Ceci nous conduit à la définition suivante :

Hyperbolicité au sens de Kobayashi. *La variété complexe X est hyperbolique au sens de Kobayashi si d_X est une vraie distance, i.e. si pour tout couple (p,q) de points distincts de X, $d_X(p,q) > 0$.*

En dimension 1, nous avons une théorie complète satisfaisante : une surface de Riemann compacte est hyperbolique au sens de Kobayashi si et seulement si son genre est ≥ 2. En utilisant le critère de Brody rappelé ci-dessous, cela revient au Théorème classique de Picard. Dès la dimension 2, il est très difficile de déterminer l'hyperbolicité au sens de Kobayashi en termes de la géométrie des espaces.

Par une *courbe entière*, nous entendons une courbe non constante $f : \mathbb{C} \to X$. Comme la pseudo-distance de Kobayashi satisfait la propriété de distance décroissante et comme $d_{\mathbb{C}} \equiv 0$, une variété complexe hyperbolique au sens de Kobayashi ne contient pas de courbe entière. La réciproque n'est pas vraie en général, mais elle est également vraie si X est *compacte* par un résultat de base de Brody [Bro78].

La production de courbes entières liées à la dégénérescence de la pseudo-métrique infinitésimale de Kobayashi-Royden sur les variétés complexes compactes que nous allons décrire maintenant est un outil efficace fondamental de la *méthode de déformation*, dont l'objectif est de créer des exemples d'hypersurfaces hyperboliques. Une *courbe de Brody* dans une *variéte complexe compacte X* munie d'une métrique hermitienne $\|\cdot\|$ est une courbe entière $f : \mathbb{C} \to X$ dont la dérivée $\|f'\|$ est bornée. Ces courbes résultent de limites de suites d'applications holomorphes, comme suit (voir [Bro78]).

Lemme de reparamétrisation de Brody. *Soit $f_k : \Delta \to X$ une suite d'applications holomorphes du disque Δ à valeurs dans une variété complexe compacte X. Si $\|f_k'(0)\| \to \infty$ lorsque $k \to \infty$, alors il existe un point $a \in \Delta$, une suite (a_k) convergeant vers a et une suite décroissante (r_k) de réels positifs convergeant vers 0 tels que la suite d'applications*

$$z \to f_k(a_k + r_k z)$$

converge uniformément sur tout compact de $\mathbb{C}$ vers une courbe de Brody, après extraction d'une sous-suite.

Par conséquent, on a une caractérisation bien connue de l'hyperbolicité au sens de Kobayashi.

Critère de Brody. *Une variéte complexe compacte X est hyperbolique au sens de Kobayashi si et seulement si elle ne contient pas de courbe entière.*

Le problème de la caractérisation de l'hyperbolicité au sens de Kobayashi d'une variété complexe X, dans le cas des variétés projectives, est motivé par la conjecture suivante [Kob70].

Conjecture de Kobayashi. *Une hypersurface générique $X_d \subset \mathbb{P}^{n+1}(\mathbb{C})$ de degré $d \geq 2n+1$ est hyperbolique.*

Bien qu'il ait été affirmé que toutes les hypersurfaces suffisamment générales dans l'espace projectif sont hyperboliques au sens de Kobayashi, peu d'exemples d'hypersurfaces hyperboliques ont été donnés. Nous limiterons notre ambition à trouver des exemples d'hypersurfaces hyperboliques de degré le plus bas possible. Même en petites dimensions, cette question est encore très ouverte, car jusqu'à présent, personne ne sait s'il existe une seule quintique hyperbolique dans $\mathbb{P}^3(\mathbb{C})$.

Un outil très efficace pour construire de tels exemples est la *méthode de déformation* introduite par Zaidenberg [Zai88], [SZ02a], [SZ05], [Zai09], qui se compose de deux ingrédients principaux. Le premier est le lemme de Brody qui doit être utilisé sous la forme suivante.

Suite de courbes entières. *Soit X une variéte complexe compacte et soit (f_k) une suite de courbes entières dans X. Alors il existe une suite de reparamétrisations $r_k : \mathbb{C} \to \mathbb{C}$ et une sous-suite de $(f_k \circ r_k)$ qui converge uniformément sur tout compact de $\mathbb{C}$ vers une courbe entière (où courbe de Brody).*

Le deuxième ingrédient principal dans la méthode de déformation est la persistance d'intersections qui s'énonce comme suit.

Stabilité des intersections. *Soit X une variéte complexe et soit $H \subset X$ une hypersurface analytique. Supposons qu'une suite (f_k) de courbes entières dans X converge vers une courbe entière f. Si $f(\mathbb{C})$ n'est pas contenue dans H, alors*

$$f(\mathbb{C}) \cap H \subset \lim_{k \to \infty} f_k(\mathbb{C}) \cap H.$$

L'idée générale de la méthode de déformation est de rechercher des hypersurfaces hyperboliques dans le pinceau linéaire $\{\Sigma_\epsilon\}_\epsilon$ d'hypersurfaces de degré d dans $\mathbb{P}^{n+1}(\mathbb{C})$ engendré par $S_0 = \{s_0 = 0\}$ et $S = \{s = 0\}$:

$$\Sigma_\epsilon = \big\{ s_0 + \epsilon s = 0 \big\},$$

où S_0 est une hypersurface *singulière* et où ϵ est assez petit. Ici, on considère le cas le plus singulier où S_0 est la réunion de d hyperplans génériques $H_1 = \{h_1 = 0\}, \dots, H_d = \{h_d = 0\}$. Supposons que les Σ_{ϵ_k} ne soient pas hyperboliques pour une suite (ϵ_k) convergeant vers 0. Alors le critère de Brody assure qu'il existe des courbes entières $f_{\epsilon_k} : \mathbb{C} \to \Sigma_{\epsilon_k}$.

Par le lemme de Brody, après reparamétrisation et extraction, on peut supposer que la suite (f_{ϵ_k}) converge vers une courbe entière $f : \mathbb{C} \to \cup_{i=1}^d H_i$. Par conséquent, on déduit du principe d'unicité qu'il existe un sous-ensemble $I \subset \{1, \dots, d\}$ de cardinalité maximale $|I| = n + 1 - n'$ tel que :

$$f(\mathbb{C}) \subset \bigcap_{i \in I} H_i \;\cong\; \mathbb{P}^{n'}(\mathbb{C}) \subset \mathbb{P}^{n+1}(\mathbb{C}) \qquad (1 \leq n' \leq n).$$

Maintenant, nous voudrions analyser la disposition de la courbe limite $f(\mathbb{C})$ par rapport à la famille d'hyperplans $H_j \cap \mathbb{P}^{n'}(\mathbb{C})$ dans $\mathbb{P}^{n'}(\mathbb{C})$ pour $j \not\in I$. Comme indiqué dans [Zai09], la *Stabilité des intersections* donne un autre principe de dégénérescence : la courbe limite $f(\mathbb{C})$ ne peut pas passer par des points arbitraires de $H_j \cap \mathbb{P}^{n'}(\mathbb{C})$. Plus précisément, il interdit $f(\mathbb{C})$ de rencontrer H_j ($j \not\in I$) en dehors des points de $S \cap H_j \cap \mathbb{P}^{n'}(\mathbb{C})$, car

$$f(\mathbb{C}) \cap H_j \subset \lim_{k\to\infty} f_{\epsilon_k}(\mathbb{C}) \cap H_j \subset \lim_{k\to\infty} \Sigma_{\epsilon_k} \cap H_j \subset S \cap H_j \qquad (\forall j \not\in I).$$

Par conséquent, la courbe limite $f(\mathbb{C})$ doit atterrir dans

$$\Big[\bigcap_{i\in I} H_i \Big\backslash \bigcup_{j\not\in I} H_j\Big] \bigcup \Big[\Big(\bigcap_{i\in I} H_i\Big) \cap \Big(\bigcup_{j\not\in I} H_j\Big) \cap S\Big] = \bigcap_{i\in I} H_i \Big\backslash \Big(\bigcup_{j\not\in I} H_j \backslash S\Big).$$

De cette analyse, l'hyperbolicité de suffisamment petites déformations de S_0 exige que tout complémentaire de la forme

$$\bigcap_{i\in I} H_i \Big\backslash \bigcup_{j\not\in I} H_j$$

devrait satisfaire la *propriété de non-percolation hyperbolique* [SZ02a] par une hypersurface appropriée S, c'est-à-dire ce complémentaire lui-même est hyperbolique et si l'on ajoute plus de points de passage comme $\cap_{i\in I} H_i \cap (\cup_{j\not\in I} H_j) \cap S$, l'hyperbolicité ne soit pas perdue.

Pour l'hyperbolicité du complémentaire de la réunion de q hyperplans génériques dans $\mathbb{P}^m(\mathbb{C})$, il faut et il suffit de prendre $q \geq 2m+1$ [Fuj72], [Gre72]. Cela donne à penser que l'hypersurface singulière S_0 devrait être la réunion de $d \geq 2n+2$ hyperplans génériques. Maintenant, le problème se réduit à trouver une hypersurface S satisfaisant la propriété de non-percolation hyperbolique. Il a été conjecturé [Zai03] que cette propriété devrait être valide pour des hypersurfaces génériques S, mais ici, une seule hypersurface appropriée est suffisante pour donner un exemple d'une hypersurface hyperbolique de degré d.

Duval [Duv14] introduit un algorithme pour créer une telle surface S dans $\mathbb{P}^3(\mathbb{C})$ par déformation. En commençant par l'hyperbolicité de tous les complémentaires de la forme $H_i \setminus \cup_{j\neq i} H_j$ et une surface générique, chaque étape de cet algorithme donne une surface qui autorise des points de passage sur une autre droite $H_j \cap H_i$, et après avoir épuisé tous les points de passage, on obtient une surface appropriée S.

En adaptant cette technique aux dimensions supérieures, la difficulté réside dans le *point de départ* de l'algorithme, qui exige non selement l'hyperbolicité de tous les complémentaires de la forme

$$\bigcap_{i\in I} H_i \Big\backslash \bigcup_{j\not\in I} H_j,$$

mais aussi l'hyperbolicité de certaines configurations d'hyperplans décrites comme suit.

Soit $\{H_i\}_{1\leq i\leq u}$, $u \geq 2v+1$ une famille d'hyperplans génériques dans $\mathbb{P}^v(\mathbb{C})$. Par un *étoile-sous-espace de dimension k*, nous entendons un complémentaires de la forme

$$P_{k,I_k} = \bigcap_{i\in I_k} H_i \Big\backslash \Big(\bigcup_{j\not\in I_k} H_j\Big),$$

où k est un nombre entier avec $0 \le k \le v-2$, et où $I_k = \{i_1, \ldots, i_{v-k}\}$ est un sous-ensemble de l'ensemble d'indices $\{1, \ldots, u\}$ ayant cardinalité $v-k$. Pour lancer l'algorithme, nous exigeons que tous les complémentaires de la forme

$$\bigcap_{i\in I} H_i \Big\backslash \Big(\bigcup_{j\in J} H_j \Big\backslash A_{m,n+1-|I|} \Big)$$

sont hyperboliques, où I et J sont deux sous-ensembles disjoints de l'ensemble d'indices $\{1, \ldots, d\}$ tels que $1 \le |I| \le n-1$, et $|J| = d+m+1-2|I|$ pour certains $0 \le m \le |I|-1$, et où $A_{m,n+1-|I|}$ est une collection d'au plus m étoile-sous-espaces provenant de la famille d'hyperplans $\{\cap_{i\in I} H_i \cap H_j\}_{j\in J}$ dans $\cap_{i\in I} H_i \cong \mathbb{P}^{n+1-|I|}(\mathbb{C})$.

La théorie de Nevanlinna dans l'espace projectif est un outil puissant pour étudier l'hyperbolicité de ces configurations. Soit $E = \sum \mu_\nu a_\nu$ un diviseur sur $\mathbb{C}$ avec $\mu_\nu \ge 0$ et soit $k \in \mathbb{N} \cup \{\infty\}$. En résumant les degrés k-tronqué du diviseur sur les disques par

$$n^{[k]}(t, E) := \sum_{|a_\nu|<t} \min\{k, \mu_\nu\} \qquad (t>0),$$

la *fonction de comptage tronqué au niveau* k de E est définie par

$$N^{[k]}(r, E) := \int_1^r \frac{n^{[k]}(t, E)}{t}\, \mathrm{d}t \qquad (r>1).$$

Quand $k = \infty$, on écrit $n(t, E)$, $N(r, E)$ au lieu de $n^{[\infty]}(t, E)$, $N^{[\infty]}(r, E)$. On note le diviseur des zéros d'une fonction méromorphe non-nulle φ par $(\varphi)_0$.

Soit $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ une courbe entière ayant une représentation réduite $f = [f_0 : \cdots : f_n]$ dans les coordonnées homogènes $[z_0 : \cdots : z_n]$ de $\mathbb{P}^n(\mathbb{C})$. Soit $D = \{Q = 0\}$ une hypersurface dans $\mathbb{P}^n(\mathbb{C})$ définie par un polynôme homogène $Q \in \mathbb{C}[z_0, \ldots, z_n]$ de degré $d \ge 1$. Si $f(\mathbb{C}) \not\subset D$, alors on définit la *fonction de comptage tronqué* de f par rapport à D par

$$N_f^{[k]}(r, D) := N^{[k]}\big(r, (Q \circ f)_0\big).$$

La *fonction de proximité* de f pour le diviseur D est définie par

$$m_f(r, D) := \int_0^{2\pi} \log \frac{\big\| f(re^{i\theta}) \big\|^d \, \|Q\|}{\big| Q(f)(re^{i\theta}) \big|} \frac{\mathrm{d}\theta}{2\pi},$$

où $\|Q\|$ est la valeur absolue maximale des coefficients de Q et

$$\big\| f(z) \big\| = \max\{|f_0(z)|, \ldots, |f_n(z)|\}.$$

Comme $\big|Q(f)\big| \le \|Q\| \cdot \|f\|^d$, on a $m_f(r, D) \ge 0$. Enfin, la *fonction ordre de Cartan* de f est définie par

$$\begin{aligned} T_f(r) :&= \frac{1}{2\pi} \int_0^{2\pi} \log \big\| f(re^{i\theta}) \big\| \, \mathrm{d}\theta \\ &= \int_1^r \frac{\mathrm{d}t}{t} \int_{\Delta_t} f^* \omega_n + O(1), \end{aligned}$$

où ω_n est la forme de Fubini–Study sur $\mathbb{P}^n(\mathbb{C})$.

Le noyau de la théorie de Nevanlinna se compose de deux théorèmes.

Premier Théorème Principal. *Soit* $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ *une courbe holomorphe et soit* D *une hypersurface de degré* d *dans* $\mathbb{P}^n(\mathbb{C})$ *telles que* $f(\mathbb{C}) \not\subset \mathrm{supp}(D)$*. Alors, pour tout nombre réel* $r > 1$*, on a*

$$m_f(r,D) + N_f(r,D) = d\,T_f(r) + O(1),$$

donc

$$N_f(r,D) \leq d\,T_f(r) + O(1).$$

Une courbe holomorphe $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ est *linéairement non-dégénérée* si son image n'est contenue dans aucun hyperplan. Pour deux fonctions $\varphi(r)$, $\psi(r)$ à valeurs dans $[0,\infty)$, on écrit

$$\varphi(r) \leq O(\psi(r)) \,\|$$

si l'inégalité est vraie en dehors d'un sous-ensemble borélien E de $[0,\infty)$ de mesure de Lebesgue finie.

Second Théorème Principal de Cartan [Car33]. *Soit* $\{H_i\}_{1\leq i\leq q}$ *une famille d'hyperplans en position générale dans* $\mathbb{P}^n(\mathbb{C})$*. Si* $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ *est une courbe holomorphe linéairement non-dégénérée, alors*

$$(q-n-1)\,T_f(r) \leq \sum_{i=1}^{q} N_f^{[n]}(r,H_i) + S_f(r),$$

où $S_f(r)$ *est un terme petit par rapport à* $T_f(r)$

$$S_f(r) = O(\log T_f(r) + \log r) \,\| .$$

Maintenant, soit $\{H_i\}_{1\leq i\leq 2n+1+m}$ une famille de $2n+1+m$ avec $m \geq 0$ hyperplans génériques dans l'espace projectif $\mathbb{P}^n(\mathbb{C})$. En utilisant la théorie de Nevanlinna-Cartan dans $\mathbb{P}^n(\mathbb{C})$, en tenant compte du niveau de troncation dans le Second Théorème Principal et en utilisant certaines estimations de base sur les nombres de contact de courbes entières le long de cycles de codimension ≥ 2, nous établissons que tous les complémentaires de la forme

$$\mathbb{P}^n(\mathbb{C}) \setminus (\cup_{i=1}^{2n+1+m} H_i \setminus A_{m,n}), \tag{0.0.1}$$

où $A_{m,n}$ est une collection d'au plus m étoile-sous-espaces provenant de $\{H_i\}_{1\leq i\leq 2n+1+m}$, sont hyperboliques dans les trois cas suivants

- $n = 2, m \leq 3$,
- $n = 3, m \leq 2$,
- $n = 4, m \leq 1$.

Ceci nous permet d'amorcer l'algorithme en dimensions $2 \leq n \leq 5$ pour créer une hypersurface $S \subset \mathbb{P}^{n+1}(\mathbb{C})$ de degré $d \geq 2n+2$ satisfaisant la propriéte de non-percolation hyperbolique par rapport à une famille de d hyperplans génériques, ce qui donne des exemples d'hypersurfaces hyperboliques de degré $d \geq 2n+2$ dans $\mathbb{P}^{n+1}(\mathbb{C})$.

Notre premier résultat, publié dans [Huy15], s'énonce comme suit.

Théorème I. *Soit n un nombre entier dans $\{2,3,4,5\}$. Soit $\{H_i\}_{1\leq i\leq 2n+2}$ une famille de $2n+2$ hyperplans génériques dans $\mathbb{P}^{n+1}(\mathbb{C})$ définies par $H_i=\{h_i=0\}$. Alors il existe une hypersurface $S=\{s=0\}$ de degré $2n+2$ telle que l'ypersurface*

$$\Sigma_\epsilon=\left\{\epsilon s+\Pi_{i=1}^{2n+2}h_i=0\right\}$$

est hyperbolique pour $\epsilon\neq 0$ assez petit.

Il est très naturel de se demander si cette affirmation est encore vraie en dimensions supérieures. Ce problème se réduit à prouver l'hyperbolicité de toutes les configurations d'hyperplans (0.0.1) sans restriction sur n et m, ce qui est encore très ouvert. Mais au moins l'algorithme peut être amorcé à partir d'un point de départ restreint qui exige que l'hypersurface singulière S_0 devrait être la réunion de $d\geq\left(\frac{n+3}{2}\right)^2$ hyperplans. Cela donne une légère amélioration du résultat de Shiffman et Zaidenberg [SZ02b], qui ont donné des exemples d'hypersurfaces hyperboliques de degré $\geq 4\,n^2$ dans $\mathbb{P}^{n+1}(\mathbb{C})$.

Notre second résultat, publié dans [Huy16], s'énonce comme suit.

Théorème II. *Soit $\{H_i\}_{1\leq i\leq q}$ une famille de $d\geq\left(\frac{n+3}{2}\right)^2$ hyperplans en position générale dans $\mathbb{P}^{n+1}(\mathbb{C})$ où $H_i=\{h_i=0\}$. Alors il existe une hypersurface $S=\{s=0\}$ de degré d telle que l'hypersurface*

$$\Sigma_\epsilon=\left\{\epsilon s+\Pi_{i=1}^{d}h_i=0\right\}$$

est hyperbolique pour $\epsilon\neq 0$ assez petit.

La limite de notre technique actuelle basée sur le Second théorème principal de Cartan nous conduit au problème de diminuer le niveau de troncation. Yamanoi [Yam04] a établi un Second théorème principal pour les courbes algébriquement non-dégénérées dans les variétés abéliennes avec la troncation 1, en autorisant le petit terme à être $\epsilon\,T_f(r)$. Noguchi, Winkelmann et Yamanoi [NWY08] ont obtenu un résultat similaire pour les courbes algébriquement non-dégénérées dans les variétés semi-abéliennes et comme application, ils ont obtenu une généralisation d'une conjecture de Green [Gre74] sur la dégénérescence de courbes holomorphes dans le complémentaire de deux droites et un cône dans le plan projectif.

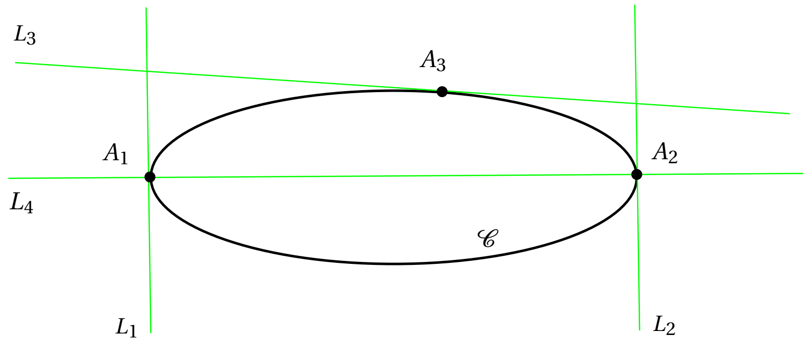
Diminuer le niveau de troncation dans le Second théorème principal de Cartan semble être un problème très difficile, même dans le cas le plus simple. Rappelons que pour une famille de 4 droites en position générale $\{L_i\}_{1\leq i\leq 4}$ dans le plan projectif, pour une courbe holomorphe linéairement non-dégénérée $f\colon\mathbb{C}\to\mathbb{P}^2(\mathbb{C})$, le Second théorème principal de Cartan stipule que

$$T_f(r)\leq\sum_{i=1}^{4}N_f^{[2]}(r,L_i)+o(T_f(r))\parallel. \tag{0.0.2}$$

Les fonctions de comptage tronquées dans la partie droite de cette inégalité sont optimales dans le sens où elles ne peuvent pas être tronquées au niveau 1. Autrement dit, il existe une certaine courbe holomorphe linéairement non-dégénérée $f\colon\mathbb{C}\to\mathbb{P}^2(\mathbb{C})$ telle que l'inégalité suivante n'est pas vraie

$$T_f(r)\leq\sum_{i=1}^{4}N_f^{[1]}(r,L_i)+o(T_f(r))\parallel. \tag{0.0.3}$$

Exemple : Soit $\mathscr{C}$ la conique unique qui est tangente à L_1, L_2 en $A_1 = L_1 \cap L_4$, $A_2 = L_2 \cap L_4$ respectivement, et qui est tangente à L_3 en un certain point A_3.



Comme $\mathscr{C} \setminus \{A_1, A_2\} \cong \mathbb{C}^*$ n'est pas hyperbolique, il contient une certaine courbe entière f. Par le choix de $\mathscr{C}$, les fonctions de comptage tronquées de f par rapport à L_i satisfont

$$
\begin{aligned}
N_f^{[2]}(r, L_i) &= 0 && (i \neq 3),\\
N_f^{[2]}(r, L_i) &= 2\, N_f^{[1]}(r, L_i) && (i = 3).
\end{aligned}
$$

En appliquant le Premier Théorème Principal et le Second Théorème Principal de Cartan, on obtient

$$
\begin{aligned}
N_f^{[2]}(r, L_3) \leq N_f(r, L_3) \leq\ & T_f(r) + O(1)\\
& \leq N_f^{[2]}(r, L_3) + \underbrace{\sum_{i \neq 3} N_f^{[2]}(r, L_i)}_{=0} + S_f(r).
\end{aligned}
$$

Donc toutes les inégalités sont les égalités modulo $S_f(r)$. En particulier, on a

$$
\begin{aligned}
T_f(r) &= N_f^{[2]}(r, L_3) + S_f(r)\\
&= 2\, N_f^{[1]}(r, L_3) + S_f(r)\\
&= 2 \sum_{i=1}^{4} N_f^{[1]}(r, L_i) + S_f(r),
\end{aligned}
$$

ce qui montre que l'inégalité 0.0.3 n'est pas vraie.

Dans l'exemple ci-dessus, la courbe f est linéairement non-dégénérée, mais son image est contenue dans une courbe algébrique $\mathscr{C} \subset \mathbb{P}^2(\mathbb{C})$). Les courbes entières satisfaisant cette propriété sont dites *algébriquement dégénérées*. Sinon, on dit que ces courbes sont algébriquement non-dégénérées. Jusqu'à ce jour, il n'y a pas de contre-exemple à la conjecture suivante de Noguchi.

Second Théorème Principal de type Cartan avec troncation 1. *Si $f : \mathbb{C} \to \mathbb{P}^2(\mathbb{C})$ est algébriquement non-dégénérée, alors* 0.0.3 *est vrai.*

Nous proposons une réponse positive dans un cas particulier où la courbe f est *proche* d'une certaine courbe algébrique $\mathscr{C}$ dans $\mathbb{P}^2(\mathbb{C})$, au sens où l'ensemble d'accumulation de $f(\mathbb{C})$ à l'infini (le *cluster set* de f)

$$\mathrm{Clu}(f) := \bigcap_{r>0} \big(\overline{f(\mathbb{C} \setminus \Delta_r)}\big)$$

est contenu dans $\mathscr{C}$.

Notre troisième résultat s'énonce comme suit.

Théorème III. *Soit* $\{L_i\}_{1\leq i\leq 4}$ *une famille de* 4 *droites en position générale dans le plan projectif* $\mathbb{P}^2(\mathbb{C})$ *et soit* $f : \mathbb{C} \to \mathbb{P}^2(\mathbb{C})$ *une courbe holomorphe algébriquement non-dégénérée. Si l'ensemble d'accumulation* $\mathrm{Clu}(f)$ *est contenue dans une courbe algébrique* $\mathscr{C} \subset \mathbb{P}^2(\mathbb{C})$, *alors*

$$T_f(r) \leq \sum_{i=1}^{4} N_f^{[1]}(r, L_i) + o(T_f(r)) \parallel .$$

Notre preuve se compose des trois ingrédients principaux suivants.

① *Courants positifs fermés associés à une courbe entière.*
Soit X une variété projective munie d'une forme de Kähler ω. On peut associer à une courbe entière $f : \mathbb{C} \to X$ dans X certains courants positifs fermés de bidimension $(1,1)$, qui sont appelés *courants de Nevanlinna*. Ces courants apparaissent notamment dans la preuve par M. McQuillan de la conjecture de Green-Griffiths pour les surfaces de type général dont les première et deuxième classes de Segre satisfont $c_1^2 - c_2 > 0$ [McQ98] (voir également [Bru99]). Ils sont construits en prenant la limite de certaines suites de courants positifs de masse bornée $\{\Phi_{r_k}\}_{k=1}^{\infty}$, où

$$\Phi_{r_k}(\eta) = \frac{\displaystyle\int_0^{r_k} \frac{\mathrm{d}t}{t} \int_{\Delta_t} f^*\eta}{\displaystyle\int_0^{r_k} \frac{\mathrm{d}t}{t} \int_{\Delta_t} f^*\omega},$$

pour toute $(1,1)$–forme lisse η sur X, et où $\{r_k\}_{k=1}^{\infty}$ est une suite de réels positifs convergeant vers ∞ telle que (il en existe)

$$\lim_{k\to\infty} \frac{\displaystyle\int_0^{r_k} \frac{\mathrm{d}t}{t} \,\mathrm{Length}\big(f(\partial\Delta_t)\big)}{\displaystyle\int_0^{r_k} \frac{\mathrm{d}t}{t} \int_{\Delta_t} f^*\omega} = 0.$$

② *L'intersection géométrique, l'intersection algébrique et leur relation.* L'intersection géométrique du courant limite Φ par rapport à un diviseur $Z \subset X$ est alors définie par

$$i_k(Z) = \int_0^{r_k} \mathrm{Card}\big(f(\Delta_t) \cap Z\big) \frac{\mathrm{d}t}{t},$$

$$i(Z) = \liminf_{k\to\infty} \frac{i_k(Z)}{T_{f,r_k}(\omega)},$$

où Card$\big(f(\Delta_t)\cap Z\big)$ est le nombre total de points d'intersection entre $f(\Delta_t)$ et Z. Avec les notations ci-dessus, le problème se réduit à démontrer que

$$\sum_{i=1}^{4} i(L_i) \geq 1.$$

Notons $[\Phi]$, $[Z]$ la classe d'homologie de Φ, Z dans $H^{1,1}(X,\mathbb{R})$. L'*intersection algébrique* de $[\Phi]$ et $[Z]$ est alors définie par

$$[\Phi]\cdot[Z] := \Phi(\omega_Z),$$

où ω_Z est une $(1,1)$–forme différentielle contenue dans la classe fondamentale de cohomologie $\{Z\}$. La relation suivante [Bru99] entre l'intersection géométrique $i(Z)$ et l'intersection algébrique $[\Phi]\cdot[Z]$ peut être considérée comme une conséquence du Premier Théorème Principal dans la théorie de Nevanlinna :

$$i(Z) \leq [\Phi]\cdot[Z].$$

③ *La théorie de recouvrement d'Ahlfors pour les surfaces.*

Présentons d'abord quelques notations et définitions. Une *surface à bord de type fini* est une région fermée sur une surface orientable compacte délimitée par un nombre fini de courbes fermées simples. Pour une surface à bord Σ, notons $\mathring{\Sigma}$ son intérieur et $\partial\Sigma$ son bord. Une application $f : \Sigma \to \Sigma_0$ entre deux surfaces à bord de type fini est dite *holomorphe* si f est holomorphe dans $\mathring{\Sigma}$ et se prolonge continûment jusqu'au bord. Supposons maintenant que Σ_0 est munie d'une métrique hermitienne ρ_0. Notons ρ la métrique tirée-en-arrière de ρ_0. On mesure toutes longueurs et aires sur Σ et Σ_0 par rapport à ρ et ρ_0. Une courbe est appelée *régulière* si elle est lisse par morceaux. Une région est appelée *régulière* si elle est délimitée par un nombre fini de courbes régulières. Supposons que Σ_0 est une région régulière. Le nombre de feuillets moyen de f est défini par

$$S := \frac{\mathrm{Aire}(\Sigma)}{\mathrm{Aire}(\Sigma_0)}.$$

De la même façon, on définit le nombre de feuillets au-dessus d'une région régulière D ou d'une courbe régulière γ sur Σ_0 par

$$S(D) := \frac{\mathrm{Aire}\big(f^{-1}(D)\big)}{\mathrm{Aire}(D)},$$

$$S(\gamma) := \frac{\mathrm{Longueur}\big(f^{-1}(\gamma)\big)}{\mathrm{Longueur}(\gamma)}.$$

Notons $L = \mathrm{Longueur}\big(\partial\Sigma \setminus f^{-1}(\partial\Sigma_0)\big)$ la longueur du bord relatif. La théorie d'Ahlfors se compose des deux analogues suivants du Premier Théorème Principal et du Second Théorème Principal dans la théorie de Nevanlinna classique.

Premier Théorème Principal d'Ahlfors. *Pour chaque région régulière D et chaque courbe régulière γ sur Σ_0, il existe une constante h qui est indépendante de Σ et f pour laquelle :*

$$|S - S(D)| \leq h\,L, \qquad |S - S(\gamma)| \leq h\,L.$$

Second Théorème Principal d'Ahlfors. *Il existe une constante h qui ne dépend que de* (Σ_0, ρ_0) *pour laquelle :*

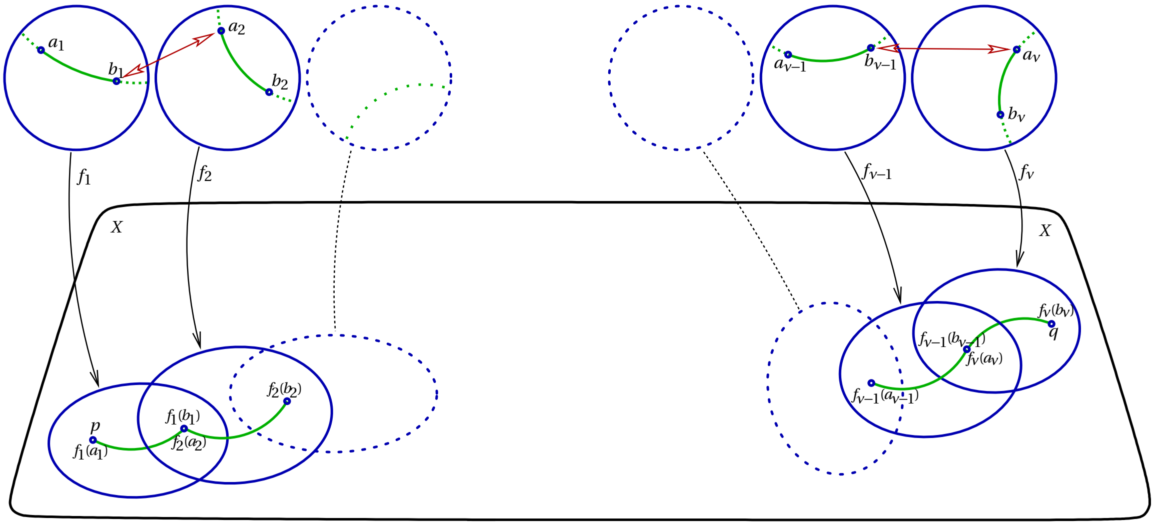
$$\min\{0, \chi(\Sigma)\} \leq S \cdot \chi(\Sigma_0) + hL.$$

Donnons maintenant une esquisse de la preuve du Théorème III. En nous basant sur ①, nous construisons simultanément les courants positifs fermés associés à la courbe holomorphe f et à ses relèvements dans le processus de résolution de la configuration de la courbe $\mathscr{C}$ et des quatre droites H_i. Ensuite, en appliquant la théorie d'Ahlfors décrite dans ③, qui est alors entièrement valable pour les applications quasiconformes $\pi \circ f$, où π est la projection définie dans un petit voisinage de certains diviseurs dans l'arbre de résolution, on obtient une relation entre les intersections géométriques et les masses du courant positif fermé le long de l'ensemble de certains diviseurs. Nous appliquons donc l'inégalité entre l'intersection algébrique et l'intersection géométrique ② pour les diviseurs exceptionnels à chaque étape du processus de résolution et et nous travaillons avec la combinatoire de l'arbre de résolution pour obtenir la conclusion.

OVERVIEW OF THE MAIN RESULTS

Let X be a connected complex manifold. For two points p, q in X, a holomorphic chain connecting p and q is a sequence $\{(f_i, a_i, b_i)\}_{i=1}^{\nu}$, where f_i are holomorphic curves from the unit disk Δ to X, and a_i, b_i are points in Δ such that

$$p = f_1(a_1),\ f_1(b_1) = f_2(a_2),\ \ldots\ldots,\ f_{\nu-1}(b_{\nu-1}) = f_\nu(a_\nu),\ f_\nu(b_\nu) = q.$$



The *Kobayashi pseudodistance* is then defined by

$$d_X(p,q) := \inf\Big\{\sum_{i=1}^{\nu} \rho(a_i, b_i)\Big\},$$

where ρ is the Poincaré distance on Δ, and the infimum is taken over all holomorphic chains from p to q. This pseudodistance is the integrated form of the *Kobayashi-Royden infinitesimal pseudometric* on X, defined by

$$F_X(v) := \inf\Big\{\lambda > 0 : \exists\, f \in \mathrm{Hol}(\Delta, X), f(0) = p, f'(0) = \frac{v}{\lambda}\Big\} \qquad (p \in X, v \in T_pX).$$

It follows from the definition that the Kobayashi pseudodistance enjoys the distance decreasing property, namely for any holomorphic map $\varphi\colon X \to Y$ between complex manifolds, for all p, q in X, we have

$$d_X(p,q) \geq d_Y(\varphi(p), \varphi(q)).$$

Thanks to the Schwarz-Pick's Lemma, the Kobayashi pseudodistance on the unit disk Δ coincides with the Poincaré distance. In general, the Kobayashi pseudodistance may not be a distance. For example, if X is the complex plane $\mathbb{C}$ or an elliptic curve, then $d_X \equiv 0$. This leads to the following definition:

Kobayashi hyperbolicity. *The complex manifold X is said to be Kobayashi hyperbolic if d_X is a true distance, namely $d_X(p,q) > 0$ whenever $p \neq q$.*

In the one-dimensional case, we have a beautiful and complete picture: a compact Riemann surface is Kobayashi hyperbolic if and only if its genus is ≥ 2. By Brody's criterion recalled below, this boils down to the classical Picard's theorem. In higher-dimensional cases, it is very difficult to determine the Kobayashi hyperbolicity in terms of the geometry of the spaces.

By an *entire curve* in X, we mean a nonconstant holomorphic map $f : \mathbb{C} \to X$. Since the Kobayshi pseudodistance satisfies the distance decreasing property and since $d_{\mathbb{C}} \equiv 0$, a Kobayashi hyperbolic complex manifold must contain no entire curve. The converse is not true in general, but it is true at least when the manifold is *compact* by a basic result of Brody [Bro78].

The production of entire curves from the degeneracy of the Kobayashi–Royden infinitesimal pseudometric on compact complex manifolds that we are going to describe now is fundamental to the *deformation method*, an effective tool to create examples of hyperbolic hypersurfaces. A *Brody curve* in a *compact complex manifold* X equipped with a hermitian metric $\|\cdot\|$ is an entire curve $f : \mathbb{C} \to X$ whose derivative $\|f'\|$ is bounded. Such curves arise as limits of sequences of holomorphic maps as follows (see [Bro78]).

Brody's reparametrization lemma. *Let $f_k : \Delta \to X$ be a sequence of holomorphic maps from the unit disk to a compact complex manifold X. If $\|f_k'(0)\| \to \infty$ as $k \to \infty$, then there exist a point $a \in \Delta$, a sequence (a_k) converging to a and a decreasing sequence (r_k) of positive real numbers converging to 0 such that the sequence of maps*

$$z \to f_k(a_k + r_k z)$$

converges towards a Brody curve, after extracting a subsequence.

Consequently, we have a well known characterization of Kobayashi hyperbolicity.

Brody's criterion. *A compact complex manifold X is Kobayashi hyperbolic if and only if it contains no entire curve.*

The problem of characterizing the Kobayashi hyperbolicity of complex manifolds X, in the case of projective varieties, is motivated by the following conjecture due to Kobayashi [Kob70].

Kobayashi's conjecture. *A generic hypersurface $X_d \subset \mathbb{P}^{n+1}(\mathbb{C})$ having degree $d \geq 2n+1$ is hyperbolic.*

Although it was asserted that all sufficiently general hypersurfaces in projective space are Kobayashi hyperbolic, few examples of hyperbolic hypersurfaces were given. We restrict our ambition to find examples of hyperbolic hypersurfaces of degree as low as possible. This question is still wide open, even in low dimensional cases since until now, nobody knows whether there exists a single hyperbolic quintic in $\mathbb{P}^3(\mathbb{C})$.

A very effective tool to construct such examples is the *deformation method* introduced by Zaidenberg [Zai88], [SZ02a], [SZ05], [Zai09], which consists of two main ingredients. The first one is Brody's Lemma which shall be used under the following form.

Sequences of entire curves. *Let X be a compact complex manifold and let (f_k) be a sequence of entire curves in X. Then there exist a sequence of reparameterizations $r_k : \mathbb{C} \to \mathbb{C}$ and a subsequence of $(f_k \circ r_k)$ which converges towards an entire curve (or Brody curve).*

The second main ingredient in the deformation method is the persistence of intersections stated as follows.

Stability of intersections. *Let X be a complex manifold and let $H \subset X$ be an analytic hypersurface. Suppose that a sequence (f_k) of entire curves in X converges towards an entire curve f. If $f(\mathbb{C})$ is not contained in H, then*

$$f(\mathbb{C}) \cap H \subset \lim_{k\to\infty} f_k(\mathbb{C}) \cap H.$$

The general idea of the deformation method is to seek hyperbolic hypersurfaces in the linear pencil $\{\Sigma_\epsilon\}_\epsilon$ hypersurfaces of degree d in $\mathbb{P}^{n+1}(\mathbb{C})$ generated by $S_0 = \{s_0 = 0\}$ and $S = \{s = 0\}$:

$$\Sigma_\epsilon = \big\{s_0 + \epsilon s = 0\big\},$$

where S_0 is a *singular* hypersurface and where ϵ is chosen to be sufficiently small. Here, we consider the most singular case where S_0 is the union of d generic hyperplanes $H_1 = \{h_1 = 0\}, \dots, H_d = \{h_d = 0\}$. Suppose that Σ_{ϵ_k} are not hyperbolic for a sequence (ϵ_k) converging to 0. Then Brody's criterion insures that there exist entire curves $f_{\epsilon_k} : \mathbb{C} \to \Sigma_{\epsilon_k}$.

By Brody's lemma, after reparameterization and extraction, we may assume that the sequence (f_{ϵ_k}) converges to an entire curve $f : \mathbb{C} \to \cup_{i=1}^d H_i$. Hence we deduce from the uniqueness principle that there is a subset $I \subset \{1, \dots, d\}$ of maximal cardinality $|I| = n + 1 - n'$ such that:

$$f(\mathbb{C}) \subset \bigcap_{i\in I} H_i \;\cong\; \mathbb{P}^{n'}(\mathbb{C}) \subset \mathbb{P}^{n+1}(\mathbb{C}) \qquad (1 \leq n' \leq n).$$

Now we would like to analyse the disposition of the limit curve $f(\mathbb{C})$ with respect to the family of hyperplanes $H_j \cap \mathbb{P}^{n'}(\mathbb{C})$ in $\mathbb{P}^{n'}(\mathbb{C})$ for $j \not\in I$. As pointed out in [Zai09], the *Stability of intersections* yields a further degeneration principle: the limit curve $f(\mathbb{C})$ cannot pass through arbitrary points of $H_j \cap \mathbb{P}^{n'}(\mathbb{C})$. More precisely, it prohibits $f(\mathbb{C})$ from meeting H_j $(j \not\in I)$ outside the points of $S \cap H_j \cap \mathbb{P}^{n'}(\mathbb{C})$, since

$$f(\mathbb{C}) \cap H_j \;\subset\; \lim_{k\to\infty} f_{\epsilon_k}(\mathbb{C}) \cap H_j \;\subset\; \lim_{k\to\infty} \Sigma_{\epsilon_k} \cap H_j \;\subset\; S \cap H_j \qquad (\forall\, j \not\in I).$$

Consequently, the limit curve $f(\mathbb{C})$ must land inside

$$\Big[\bigcap_{i\in I} H_i \Big\backslash \bigcup_{j\notin I} H_j\Big] \bigcup \Big[\Big(\bigcap_{i\in I} H_i\Big) \cap \Big(\bigcup_{j\notin I} H_j\Big) \cap S\Big] = \bigcap_{i\in I} H_i \Big\backslash \Big(\bigcup_{j\notin I} H_j \backslash S\Big).$$

From this analysis, the hyperbolicity of sufficiently small deformations of S_0 requires that any complement of the form

$$\bigcap_{i\in I} H_i \Big\backslash \bigcup_{j\notin I} H_j$$

should enjoy the *hyperbolic non–percolation property* [SZ02a] through a suitable hypersurface S, namely this complement itself is hyperbolic and if one adds more passage points like $\cap_{i\in I} H_i \cap (\cup_{j\notin I} H_j) \cap S$, the hyperbolicity is not lost.

For the hyperbolicity of the complement of a union of q generic hyperplanes in $\mathbb{P}^m(\mathbb{C})$, it is necessary and sufficient to take $q \geq 2m+1$ [Fuj72], [Gre72]. This suggests that the singular hypersurface S_0 would be the union of $d \geq 2n+2$ generic hyperplanes. Now the problem reduces to finding a hypersurface S satisfying the hyperbolic non–percolation property. It was conjectured [Zai03] that this property should hold for generic hypersurfaces S, but here, a single suitable one is enough to give an example of a hyperbolic hypersurface of degree d.

Duval [Duv14] introduced an algorithm to create such a surface S in $\mathbb{P}^3(\mathbb{C})$ by deformation. Starting with the hyperbolicity of all complements of the form $H_i \setminus \cup_{j\neq i} H_j$ and a generic surface, each step of this algorithm gives a surface that allows passage points on one more line $H_j \cap H_i$, and after exhausting all passage points, a suitable surface S is obtained.

Adapting this technique in higher dimensional cases, the difficulty lies in the *starting point* of the algorithm, which requires not only the hyperbolicity of all complements of the form

$$\bigcap_{i\in I} H_i \Big\backslash \bigcup_{j\notin I} H_j,$$

but also the hyperbolicity of some configurations of hyperplanes described as follows.

Let $\{H_i\}_{1\leq i\leq u}$, $u \geq 2v+1$ be a family of generic hyperplanes in $\mathbb{P}^v(\mathbb{C})$. By a *star-subspace of dimension k*, we mean a complement of the form

$$P_{k,I_k} = \bigcap_{i\in I_k} H_i \Big\backslash \Big(\bigcup_{j\notin I_k} H_j\Big),$$

where k is an integer with $0 \leq k \leq v-2$, and where $I_k = \{i_1, \ldots, i_{v-k}\}$ is a subset of the index set $\{1, \ldots, u\}$ having cardinality $v-k$. To launch the algorithm, we require that all complements of the form

$$\bigcap_{i\in I} H_i \Big\backslash \Big(\bigcup_{j\in J} H_j \Big\backslash A_{m,n+1-|I|}\Big)$$

are hyperbolic, where I and J are two disjoint subsets of the index set $\{1, \ldots, d\}$ such that $1 \leq |I| \leq n-1$, and $|J| = d+m+1-2|I|$ with some $0 \leq m \leq |I|-1$, and where

$A_{m,n+1-|I|}$ is a collection of at most m star-subspaces coming from the family of hyperplanes $\{\cap_{i\in I} H_i \cap H_j\}_{j\in J}$ in $\cap_{i\in I} H_i \cong \mathbb{P}^{n+1-|I|}(\mathbb{C})$.

Nevanlinna theory in projective space is a strong tool to study the hyperbolicity of these configurations. Let $E = \sum \mu_\nu a_\nu$ be a divisor on $\mathbb{C}$ with $\mu_\nu \geq 0$ and let $k \in \mathbb{N} \cup \{\infty\}$. Summing the k-truncated degrees of the divisor on disks as

$$n^{[k]}(t, E) := \sum_{|a_\nu|<t} \min\{k, \mu_\nu\} \qquad (t>0),$$

the *truncated counting function at level* k of E is defined by

$$N^{[k]}(r, E) := \int_1^r \frac{n^{[k]}(t, E)}{t} \, \mathrm{d}t \qquad (r>1).$$

When $k = \infty$, we write $n(t, E)$, $N(r, E)$ instead of $n^{[\infty]}(t, E)$, $N^{[\infty]}(r, E)$. We denote the zero divisor of a nonzero meromorphic function φ by $(\varphi)_0$.

Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be an entire curve having a reduced representation $f = [f_0 : \cdots : f_n]$ in the homogeneous coordinates $[z_0 : \cdots : z_n]$ of $\mathbb{P}^n(\mathbb{C})$. Let $D = \{Q = 0\}$ be a hypersurface in $\mathbb{P}^n(\mathbb{C})$ defined by a homogeneous polynomial $Q \in \mathbb{C}[z_0, \ldots, z_n]$ of degree $d \geq 1$. If $f(\mathbb{C}) \not\subset D$, we define the *truncated counting function* of f with respect to D as

$$N_f^{[k]}(r, D) := N^{[k]}\big(r, (Q \circ f)_0\big).$$

The *proximity function* of f for the divisor D is defined as

$$m_f(r, D) := \int_0^{2\pi} \log \frac{\big\|f(re^{i\theta})\big\|^d \, \|Q\|}{\big|Q(f)(re^{i\theta})\big|} \frac{\mathrm{d}\theta}{2\pi},$$

where $\|Q\|$ is the maximum absolute value of the coefficients of Q and

$$\big\|f(z)\big\| = \max\{|f_0(z)|, \ldots, |f_n(z)|\}.$$

Since $\big|Q(f)\big| \leq \|Q\| \cdot \|f\|^d$, one has $m_f(r, D) \geq 0$. Finally, the *Cartan order function* of f is defined by

$$\begin{aligned} T_f(r) &:= \frac{1}{2\pi} \int_0^{2\pi} \log \big\|f(re^{i\theta})\big\| \, \mathrm{d}\theta \\ &= \int_1^r \frac{\mathrm{d}t}{t} \int_{\Delta_t} f^* \omega_n + O(1), \end{aligned}$$

where ω_n is the Fubini–Study form on $\mathbb{P}^n(\mathbb{C})$.

The core of Nevanlinna theory consists of two theorems.

First Main Theorem. *Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic curve and let D be a hypersurface of degree d in $\mathbb{P}^n(\mathbb{C})$ such that $f(\mathbb{C}) \not\subset \operatorname{supp}(D)$. Then for every real number $r > 1$, the following holds*

$$m_f(r, D) + N_f(r, D) = d\, T_f(r) + O(1),$$

hence

$$N_f(r, D) \leq d\, T_f(r) + O(1).$$

A holomorphic curve $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ is *linearly non-degenerate* if its image is not contained in any hyperplane. For functions $\varphi(r)$, $\psi(r)$ valued in $[0,\infty)$, we write

$$\varphi(r) \leq O(\psi(r)) \parallel$$

if the inequality holds outside a Borel subset E of $[0,\infty)$ of finite Lebesgue measure.

Cartan's Second Main Theorem [Car33]. *Let $\{H_i\}_{1\leq i\leq q}$ be a family of hyperplanes in general position in $\mathbb{P}^n(\mathbb{C})$. If $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ is a linearly nondegenerate holomorphic curve, then*

$$(q-n-1)\, T_f(r) \leq \sum_{i=1}^{q} N_f^{[n]}(r, H_i) + S_f(r),$$

where $S_f(r)$ is a small term compared with $T_f(r)$

$$S_f(r) = O(\log T_f(r) + \log r) \parallel .$$

Now, let $\{H_i\}_{1\leq i\leq 2n+1+m}$ be a family of $2n+1+m$ with $m \geq 0$ generic hyperplanes in the projective space $\mathbb{P}^n(\mathbb{C})$. Applying Nevanlinna-Cartan theory in $\mathbb{P}^n(\mathbb{C})$, taking into account the truncation level in Cartan's Second Main Theorem and using some basic estimates on the numbers of contacts of entire curves along cycles of codimension ≥ 2, we establish that all complements of the form

$$\mathbb{P}^n(\mathbb{C}) \setminus (\cup_{i=1}^{2n+1+m} H_i \setminus A_{m,n}), \tag{0.0.5}$$

in which $A_{m,n}$ is a collection of at most m star-subspaces coming from $\{H_i\}_{1\leq i\leq 2n+1+m}$, are hyperbolic in the following three circumstances

- $n = 2, m \leq 3$,
- $n = 3, m \leq 2$,
- $n = 4, m \leq 1$.

This allows us to launch the algorithm in dimension $2 \leq n \leq 5$ to create a hypersurface $S \subset \mathbb{P}^{n+1}(\mathbb{C})$ of degree $d \geq 2n+2$ satisfying the hyperbolic non–percolation property with respect to a family of d generic hyperplanes, which yields examples of hyperbolic hypersurfaces of degree $d \geq 2n+2$ in $\mathbb{P}^{n+1}(\mathbb{C})$.

Our first result, published as [Huy15], states as follows.

Theorem I. *Let n be an integer number in $\{2,3,4,5\}$. Let $\{H_i\}_{1\leq i\leq 2n+2}$ be a family of $2n+2$ generic hyperplanes in $\mathbb{P}^{n+1}(\mathbb{C})$ defined as $H_i = \{h_i = 0\}$. Then there exists a hypersurface $S = \{s = 0\}$ of degree $2n+2$ such that the hypersurface*

$$\Sigma_\epsilon = \left\{\epsilon s + \Pi_{i=1}^{2n+2} h_i = 0\right\}$$

is hyperbolic for sufficiently small complex $\epsilon \neq 0$.

It is very natural to ask whether this claim still holds in higher-dimensional cases. This problem reduces to proving the hyperbolicty of all configurations of hyperplanes

0.0.5 without restriction on n and m, which is still very open. But at least, the algorithm can be launched from a restricted starting point which requires that the singular hypersurface S_0 should be the union of $d \geq \left(\frac{n+3}{2}\right)^2$ hyperplanes. This yields a slight improvement of the result of Shiffman and Zaidenberg in [SZ02b], where examples of hyperbolic hypersurfaces of degree $\geq 4\,n^2$ in $\mathbb{P}^{n+1}(\mathbb{C})$ were given.

Our second result, published as [Huy16], states as follows.

Theorem II. *Let $\{H_i\}_{1\leq i\leq q}$ be a family of $d \geq \left(\frac{n+3}{2}\right)^2$ hyperplanes in general position in $\mathbb{P}^{n+1}(\mathbb{C})$ where $H_i = \{h_i = 0\}$. Then there exists a hypersurface $S = \{s = 0\}$ of degree d such that the hypersurface*

$$\Sigma_\epsilon = \left\{\epsilon s + \Pi_{i=1}^d h_i = 0\right\}$$

is hyperbolic for sufficiently small complex $\epsilon \neq 0$.

The limit of our current technique based on Cartan's Second Main Theorem leads us to the problem of decreasing the truncation level as low as possible. Yamanoi [Yam04] established a Second Main Theorem for algebraically nondegenerate curves in abelian varieties with the truncation level 1, allowing the remainder term to be $\epsilon\, T_f(r)$. Noguchi, Winkelmann and Yamanoi [NWY08] obtained a similar result for algebraically nondegenerate curves in semi-abelian varieties and as an application, they proved a generalization of a conjecture of Green [Gre74] about the degeneracy of holomorphic curves into the complement of two lines and one conic in the projective plane.

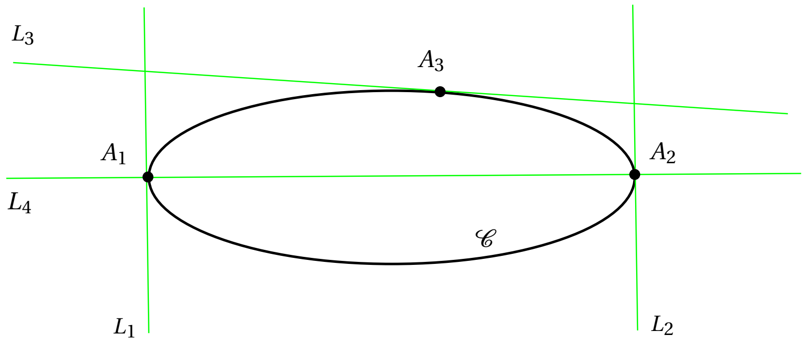
Decreasing the truncation level in Cartan's Second Main Theorem seems to be a very difficult problem, even in the simplest case. Recall that for a family of 4 lines in general position $\{L_i\}_{1\leq i\leq 4}$ in the projective plane, for a linearly nondegenerate holomorphic curve $f : \mathbb{C} \to \mathbb{P}^2(\mathbb{C})$, Cartan's Second Main Theorem states that

$$T_f(r) \leq \sum_{i=1}^{4} N_f^{[2]}(r, L_i) + o(T_f(r)) \parallel . \tag{0.0.6}$$

The truncated counting functions in the right hand side of this inequality are optimal in the sense that they can not be truncated to level one. In other words, there exists some linearly nondegenerate holomorphic curves $f : \mathbb{C} \to \mathbb{P}^2(\mathbb{C})$ such that the following inequality does not hold

$$T_f(r) \leq \sum_{i=1}^{4} N_f^{[1]}(r, L_i) + o(T_f(r)) \parallel . \tag{0.0.7}$$

Example: Let $\mathscr{C}$ be the unique conic which is tangent to L_1, L_2 at $A_1 = L_1 \cap L_4$, $A_2 = L_2 \cap L_4$ respectively, and which is tangent to L_3 at some point A_3.



Since $\mathscr{C} \setminus \{A_1, A_2\} \cong \mathbb{C}^*$ is not hyperbolic, it contains some entire curve f. By the choice of $\mathscr{C}$, the truncated counting functions of f with respect to L_i satisfy

$$N_f^{[2]}(r, L_i) = 0 \qquad (i \neq 3),$$
$$N_f^{[2]}(r, L_i) = 2\, N_f^{[1]}(r, L_i) \qquad (i = 3).$$

It follows from the First Main Theorem and Cartan's Second Main theorem that

$$\begin{aligned} N_f^{[2]}(r, L_3) \leq N_f(r, L_3) &\leq T_f(r) + O(1) \\ &\leq N_f^{[2]}(r, L_3) + \underbrace{\sum_{i\neq 3} N_f^{[2]}(r, L_i)}_{=0} + S_f(r). \end{aligned}$$

Hence, all inequalities are equalities modulo $S_f(r)$. In particular, we have

$$\begin{aligned} T_f(r) &= N_f^{[2]}(r, L_3) + S_f(r) \\ &= 2\, N_f^{[1]}(r, L_3) + S_f(r) \\ &= 2 \sum_{i=1}^{4} N_f^{[1]}(r, L_i) + S_f(r), \end{aligned}$$

which shows that the inequality 0.0.7 does not hold.

In the above example, the curve f is linearly nondegenerate but its image is contained in an algebraic curve $\mathscr{C} \subset \mathbb{P}^2(\mathbb{C})$). Such entire curves satisfying this property are said to be *algebraically degenerate*. Otherwise, they are said to be algebraically nondegenerate. Up to date, there is no counterexample to the following conjecture due to Noguchi.

Cartan's type Second Main Theorem with truncation 1. *If $f : \mathbb{C} \to \mathbb{P}^2(\mathbb{C})$ is algebraically nondegenerate, then* 0.0.7 *holds.*

We propose a positive answer to the above conjecture in a special case, namely when the curve f is *close* to some algebraic curve $\mathscr{C}$ in $\mathbb{P}^2(\mathbb{C})$, in the sense that the set of accumulation points of $f(\mathbb{C})$ at infinity (the cluster set of f) defined by

$$\mathrm{Clu}(f) := \bigcap_{r>0} \left(\overline{f(\mathbb{C} \setminus \Delta_r)} \right)$$

is contained in $\mathscr{C}$.

Our third result states as follows.

Theorem III. *Let* $\{L_i\}_{1\leq i\leq 4}$ *be a family of* 4 *lines in general position in the projective plane* $\mathbb{P}^2(\mathbb{C})$ *and let* $f:\mathbb{C}\to\mathbb{P}^2(\mathbb{C})$ *be an algebraically nondegenerate holomorphic curve. If its cluster set is contained in an algebraic curve* $\mathscr{C}\subset\mathbb{P}^2(\mathbb{C})$, *then*

$$T_f(r)\leq\sum_{i=1}^{4}N_f^{[1]}(r,L_i)+o(T_f(r))\parallel.$$

Our proof consists of the following three main ingredients.

① *Closed positive currents associated to holomorphic curves.*
For a complex projective variety X equipped with a Kähler form ω, one can associate to any nonconstant holomorphic curve $f:\mathbb{C}\to X$ closed positive currents of bidimension $(1,1)$, which are called *Nevanlinna currents*. Such currents have been considered for instance by M. McQuillan in his proof of the Green-Griffiths conjecture for surfaces of general type having second Segre class $c_1^2-c_2>0$ [McQ98] (see also [Bru99]). They are constructed by taking the limit of some sequences of positive currents of bounded mass $\{\Phi_{r_k}\}_{k=1}^{\infty}$, where

$$\Phi_{r_k}(\eta)=\frac{\displaystyle\int_0^{r_k}\frac{\mathrm{d}t}{t}\int_{\Delta_t}f^*\eta}{\displaystyle\int_0^{r_k}\frac{\mathrm{d}t}{t}\int_{\Delta_t}f^*\omega},$$

for all smooth $(1,1)$–forms η on X, and where $\{r_k\}_{k=1}^{\infty}$ is a sequence of positive real numbers converging to ∞ such that

$$\lim_{k\to\infty}\frac{\displaystyle\int_0^{r_k}\frac{\mathrm{d}t}{t}\,\mathrm{Length}\big(f(\partial\Delta_t)\big)}{\displaystyle\int_0^{r_k}\frac{\mathrm{d}t}{t}\int_{\Delta_t}f^*\omega}=0.$$

② *Geometric intersection, algebraic intersection and their relationship.* The geometric intersection of the limit current Φ with respect to a divisor $Z\subset X$ is then defined as

$$i_k(Z)=\int_0^{r_k}\mathrm{Card}\big(f(\Delta_t)\cap Z\big)\frac{\mathrm{d}t}{t},$$
$$i(Z)=\liminf_{k\to\infty}\frac{i_k(Z)}{T_{f,r_k}(\omega)},$$

where $\mathrm{Card}\big(f(\Delta_t)\cap Z\big)$ is the total number of intersection points between $f(\Delta_t)$ and Z.
With the above notation, the problem reduces to proving that

$$\sum_{i=1}^{4}i(L_i)\geq 1.$$

Denote by $[\Phi]$, $[Z]$ the cohomology classes of Φ, Z in $H^{1,1}(X,\mathbb{R})$. The algebraic intersection of $[\Phi]$ and $[Z]$ is then defined as

$$[\Phi]\cdot[Z]:=\Phi(\omega_Z),$$

where ω_Z is a differentiable $(1,1)$–form in the cohomology class $\{Z\}$. The following relationship [Bru99] between the geometric intersection $i(Z)$ and the algebraic intersection $[\Phi]\cdot[Z]$ can be regarded as a consequence of the First Main Theorem in Nevanlinna theory:

$$i(Z) \leq [\Phi]\cdot[Z].$$

③ *Ahlfors' theory of covering surfaces.*

We first introduce some notations and definitions. A *bordered surface of finite type* is a closed region on a compact orientable surface bounded by finite many simple closed curves. For a bordered surface Σ, we denote by $\mathring{\Sigma}$ its interior and by $\partial\Sigma$ its boundary. A map $f:\Sigma\to\Sigma_0$ between two bordered surfaces of finite type is said to be *holomorphic* if f is holomorphic in $\mathring{\Sigma}$ and extends continuously up to the boundary. Assume now Σ_0 is equipped with a hermitian metric ρ_0. We denote by ρ the pull–back of ρ_0. We measure all lengths and areas on Σ and Σ_0 with respect to ρ and ρ_0. A curve is called *regular* if it is piecewise smooth. A region is called *regular* if it is bounded by finitely regular curves. Assume now that Σ_0 is a regular region. The average number of sheets of f is defined by

$$S := \frac{\mathrm{Area}(\Sigma)}{\mathrm{Area}(\Sigma_0)}.$$

Similarly, we define the average number of sheets over a regular region D or a regular curve γ on Σ_0 as

$$S(D) := \frac{\mathrm{Area}(f^{-1}(D))}{\mathrm{Area}(D)},$$

$$S(\gamma) := \frac{\mathrm{Length}(f^{-1}(\gamma))}{\mathrm{Length}(\gamma)}.$$

Let $L = \mathrm{Length}\big(\partial\Sigma\setminus f^{-1}(\partial\Sigma_0)\big)$ be the length of the relative boundary. Ahlfors' theory consists of the two following analogues of the First Main Theorem and the Second Main Theorem in classical Nevanlinna theory.

Ahlfors' First Main Theorem. *For every regular region D and every regular curve γ on Σ_0, there exist a constant h which is independent of Σ and f such that*

$$|S - S(D)| \leq hL, \qquad |S - S(\gamma)| \leq hL.$$

Ahlfors' Second Main Theorem. *There exist a constant h depending only on (Σ_0,ρ_0) such that*

$$\min\{0,\chi(\Sigma)\} \leq S\cdot\chi(\Sigma_0) + hL.$$

We now give a sketch of the proof of the Theorem III. Based on ①, we construct simultaneously closed positive currents associated to the holomorphic curve f and its lifting curves in the resolution process of the configuration of the curve $\mathscr{C}$ and the four lines H_i. Next, Ahlfors' theory described in ③, which is also valid for any quasiconformal mapping $\pi\circ f$, where π is the projection defined in a small neighbourhood of some divisor in the resolution tree, yields a relationship between the geometric intersections and the masses of the closed positive current along the set of certain divisors. We then apply the inequality between algebraic intersection and geometric intersection in ② for exceptional divisors in each step of the resolution process and deal with the combinatorics on the resolution tree to reach the conclusion.

Chapter 1

EXAMPLES OF HYPERBOLIC HYPERSURFACES OF LOW DEGREE IN PROJECTIVE SPACES

Abstract

We construct families of hyperbolic hypersurfaces of degree $2n$ in the projective space $\mathbb{P}^n(\mathbb{C})$ for $3 \leq n \leq 6$.

1.1 Introduction and the main result

The Kobayashi conjecture states that a generic hypersurface $X_d \subset \mathbb{P}^n(\mathbb{C})$ of degree $d \geq 2n-1$ is hyperbolic. It is proved by Demailly and El Goul [DEG00] for $n=3$ and a very generic surface of degree at least 21. In [Pău08], Păun improved the degree to 18. In $\mathbb{P}^4(\mathbb{C})$, Rousseau [Rou07] was able to show that a generic three-fold of degree at least 593 contains no Zariski-dense entire curve, a result from which hyperbolicity follows, after removing divisorial components [DT10]. In $\mathbb{P}^n(\mathbb{C})$, for any n and for $d \geq 2^{(n-1)^5}$, Diverio, Merker and Rousseau [DMR10] established algebraic degeneracy of entire curves in X_d. An improvement of the effective degree bound in this result was given in [Dar16]. Recently, for any dimension n, a positive answer for generic hypersurfaces of degree $d \geq d(n) \gg 1$ very high was proposed by Siu [Siu15], and a strategy which is expected to give a confirmation of this conjecture for *very* generic hypersurfaces of degree $d \geq 2n$ was announced by Demailly [Dem15].

Concurrently, many authors tried to find examples of hyperbolic hypersurfaces of degree as low as possible. The first example of a *compact* Kobayashi hyperbolic manifold of dimension 2 is a hypersurface in $\mathbb{P}^3(\mathbb{C})$ constructed by Brody and Green [BG77]. Also, the first examples in all higher dimensions $n-1 \geq 3$ were discovered by Masuda and Noguchi [MN96], with degree large. So far, the best degree asymptotic is the square of dimension, given by Siu and Yeung [SY97] with $d = 16(n-1)^2$ and by Shiffman and Zaidenberg [SZ02b] with $d = 4(n-1)^2$. In $\mathbb{P}^3(\mathbb{C})$ many examples of low degree were given (see the reference of [Zai03]). The lowest degree found up to date is 6, given by Duval [Duv04]. Later, Ciliberto and Zaidenberg [CZ13] gave a new construction of hyperbolic surface of degree 6 and their method works for all degree $d \geq 6$ (hence, this is the first time when a hyperbolic surface of degree 7 was created). There are not so many examples of low degree hyperbolic hypersurfaces in $\mathbb{P}^4(\mathbb{C})$. We mention here an example of a hypersurface of degree 16 constructed by Fujimoto [Fuj01]. Various examples in $\mathbb{P}^5(\mathbb{C})$ and $\mathbb{P}^6(\mathbb{C})$ only appear in the cases of arbitrary dimension mentioned above.

Before going to introduce the main result, we need some notations and conventions. A family of hyperplanes $\{H_i\}_{1\leq i\leq q}$ with $q \geq n+1$ in $\mathbb{P}^n(\mathbb{C})$ is said to be in *general position* if any $n+1$ hyperplanes in this family have empty intersection. A hypersurface S in $\mathbb{P}^n(\mathbb{C})$ is said to be in *general position with respect to* $\{H_i\}_{1\leq i\leq q}$ if it avoids all intersection points of n hyperplanes in $\{H_i\}_{1\leq i\leq q}$, namely if:

$$S \cap \big(\cap_{i\in I} H_i\big) = \emptyset, \quad \forall I \subset \{1,\dots,q\}, |I| = n.$$

Now assume that $\{H_i\}_{1\leq i\leq q}$ is a family of hyperplanes of $\mathbb{P}^n(\mathbb{C})$ ($n \geq 2$) in general position. Let $\{H_i\}_{i\in I}$ be a subfamily of $n+2$ hyperplanes. Take a partition $I = J \cup K$ such that $|J|, |K| \geq 2$. Then there exists a unique hyperplane H_{JK} containing $\cap_{j\in J} H_j$ and $\cap_{k\in K} H_k$. We call H_{JK} a *diagonal hyperplane* of $\{H_i\}_{i\in I}$. The family $\{H_i\}_{1\leq i\leq q}$ is said to be *generic* if, for all disjoint subsets $I, J, J_1, \dots, J_k$ of $\{1,\dots,q\}$ such that $|I|, |J_i| \geq 2$ and $|I|+|J_i| = n+2$, $1 \leq i \leq k$, for every subset $\{i_1,\dots,i_l\}$ of I, the intersection between the $|J|$ hyperplanes H_j, $j \in J$, the k diagonal hyperplanes $H_{IJ_1}, \dots, H_{IJ_k}$, and the l hyperplanes $H_{i_1}, \dots, H_{i_l}$ is a linear subspace of codimension $\min\{k+l, |I|\} + |J|$, with the convention that when $\min\{k+l, |I|\} + |J| > n$, this intersection is empty. Such a generic condition naturally appears in our constructions, and it has the virtue of being preserved when passing to smaller-dimensional subspaces

Our aim is to prove that, for $3 \leq n \leq 6$, a small deformation of a union of generic $2n$ hyperplanes in $\mathbb{P}^n(\mathbb{C})$ is hyperbolic.

Main Theorem. *Let n be an integer number in $\{3,4,5,6\}$. Let $\{H_i\}_{1\leq i\leq 2n}$ be a family of $2n$ generic hyperplanes in $\mathbb{P}^n(\mathbb{C})$, where $H_i = \{h_i = 0\}$. Then there exists a hypersurface $S = \{s = 0\}$ of degree $2n$ in general position with respect to $\{H_i\}_{1\leq i\leq 2n}$ such that the hypersurface*

$$\Sigma_\epsilon = \left\{\epsilon s + \Pi_{i=1}^{2n} h_i = 0\right\}$$

is hyperbolic for sufficiently small complex $\epsilon \neq 0$.

Our proof is based on the technique of Duval [Duv14] in the case $n = 3$. By the deformation method of Zaidenberg and Shiffman [SZ02a], the problem reduces to finding a hypersurface S such that all complements of the form

$$\cap_{i\in I} H_i \setminus \left(\cup_{j\notin I} H_j \setminus S\right)$$

are hyperbolic. This situation is very close to the classical result of Fujimoto-Green [Fuj72], [Gre72]. To create such S, we proceed by deformation in order to allow points of intersection of S with more and more linear subspaces coming from the family $\{H_i\}_{1\leq i\leq 2n}$.

1.2 Notations and preparation

1.2.1 Nevanlinna theory and some applications

We recall some facts from Nevanlinna theory in the projective space $\mathbb{P}^n(\mathbb{C})$. Let $E = \sum \mu_\nu a_\nu$ be a divisor on $\mathbb{C}$ and let $k \in \mathbb{N} \cup \{\infty\}$. Summing the k-truncated degrees of the divisor on disks by

$$n^{[k]}(t,E) := \sum_{|a_\nu|<t} \min\{k, \mu_\nu\} \qquad (t>0),$$

the *truncated counting function at level* k of E is defined by

$$N^{[k]}(r,E) := \int_1^r \frac{n^{[k]}(t,E)}{t}\, \mathrm{d}t \qquad (r>1).$$

When $k = \infty$, we write $n(t,E)$, $N(r,E)$ instead of $n^{[\infty]}(t,E)$, $N^{[\infty]}(r,E)$. We denote the zero divisor of a nonzero meromorphic function φ by $(\varphi)_0$. Let $f\colon \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be an entire curve having a reduced representation $f = [f_0 : \cdots : f_n]$ in the homogeneous coordinates $[z_0 : \cdots : z_n]$ of $\mathbb{P}^n(\mathbb{C})$. Let $D = \{Q = 0\}$ be a hypersurface in $\mathbb{P}^n(\mathbb{C})$ defined by a homogeneous polynomial $Q \in \mathbb{C}[z_0, \ldots, z_n]$ of degree $d \geq 1$. If $f(\mathbb{C}) \not\subset D$, we define the *truncated counting function* of f with respect to D as

$$N_f^{[k]}(r, D) := N^{[k]}\big(r, (Q \circ f)_0\big).$$

The *proximity function* of f for the divisor D is defined as

$$m_f(r, D) := \int_0^{2\pi} \log \frac{\|f(re^{i\theta})\|^d \, \|Q\|}{\big|Q(f)(re^{i\theta})\big|} \frac{\mathrm{d}\theta}{2\pi},$$

where $\|Q\|$ is the maximum absolute value of the coefficients of Q and

$$\|f(z)\| = \max\{|f_0(z)|, \ldots, |f_n(z)|\}.$$

Since $\big|Q(f)\big| \leq \|Q\| \cdot \|f\|^d$, one has $m_f(r, D) \geq 0$. Finally, the *Cartan order function* of f is defined by

$$T_f(r) := \frac{1}{2\pi} \int_0^{2\pi} \log \|f(re^{i\theta})\| \, \mathrm{d}\theta.$$

It is known that [Ere10] if f is a Brody curve, then its order

$$\rho_f := \limsup_{r \to +\infty} \frac{T_f(r)}{\log r}$$

is bounded from above by 2. Furthermore, Eremenko [Ere10] showed the following.

Theorem 1.2.1. *If $f\colon \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ is a Brody curve omitting n hyperplanes in general position, then it is of order 1.*

Consequently, we have the following theorem.

Theorem 1.2.2. *If $f\colon \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ is a Brody curve avoiding the first n coordinate hyperplanes $\{z_i = 0\}_{i=0}^{n-1}$, then it has a reduced representation of the form*

$$[1 : e^{\lambda_1 z + \mu_1} : \cdots : e^{\lambda_{n-1} z + \mu_{n-1}} : g],$$

where g is an entire function and λ_i, μ_i are constants. If f also avoids the remaining coordinate hyperplane $\{z_n = 0\}$, then g is of the form $e^{\lambda_n z + \mu_n}$.

The core of Nevanlinna theory consists of two main theorems.

First Main Theorem. *Let $f\colon \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic curve and let D be a hypersurface of degree d in $\mathbb{P}^n(\mathbb{C})$ such that $f(\mathbb{C}) \not\subset D$. Then for every $r > 1$, the following holds*

$$m_f(r, D) + N_f(r, D) = d\, T_f(r) + O(1),$$

hence

$$N_f(r, D) \leq d\, T_f(r) + O(1). \tag{1.2.1}$$

A holomorphic curve $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ is said to be *linearly nondegenerate* if its image is not contained in any hyperplane. For non-negatively valued functions $\varphi(r)$, $\psi(r)$, we write

$$\varphi(r) \le \psi(r) \,\|$$

if this inequality holds outside a Borel subset E of $(0,+\infty)$ of finite Lebesgue measure. Next is the Second Main Theorem of Cartan [Car33].

Second Main Theorem. *Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ and let $\{H_i\}_{1\le i\le q}$ be a family of hyperplanes in general position in $\mathbb{P}^n(\mathbb{C})$. Then the following estimate holds:*

$$(q-n-1)\,T_f(r) \le \sum_{i=1}^{q} N_f^{[n]}(r,H_i) + S_f(r),$$

where $S_f(r)$ is a small term compared with $T_f(r)$

$$S_f(r) = o(T_f(r)) \,\| \,.$$

The next three theorems can be deduced from the Second Main Theorem.

Theorem 1.2.3. *Let $\{H_i\}_{1\le i\le n+2}$ be a family of hyperplanes in general position in $\mathbb{P}^n(\mathbb{C})$ with $n \ge 2$. If $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C}) \setminus \cup_{i=1}^{n+2} H_i$ is an entire curve, then its image lies in one of the diagonal hyperplanes of $\{H_i\}_{1\le i\le n+2}$.*

The following strengthened theorem is due to Dufresnoy [Duf44].

Theorem 1.2.4. *If a holomorphic map $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ has its image in the complement of $n+p$ hyperplanes $H_1,\ldots,H_{n+p}$ in general position, then this image is contained in a linear subspace of dimension $\left\lfloor \frac{n}{p} \right\rfloor$.*

As a consequence, we have the classical generalization of Picard's Theorem (case $n=1$), due to Fujimoto [Fuj72] (see also [Gre72]).

Theorem 1.2.5. *The complement of a collection of $2n+1$ hyperplanes in general position in $\mathbb{P}^n(\mathbb{C})$ is hyperbolic.*

For hyperplanes that are not in general position, we have the following result (see [Kob98], Theorem 3.10.15).

Theorem 1.2.6. *Let $\{H_i\}_{1\le i\le q}$ be a family of $q \ge 3$ hyperplanes that are not in general position in $\mathbb{P}^n(\mathbb{C})$. If $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C}) \setminus \cup_{i=1}^{q} H_i$ is an entire curve, then its image lies in some hyperplane.*

1.3 Starting lemmas

Let us introduce some notations before going to other applications. Let $\{H_i\}_{1\le i\le q}$ be a family of generic hyperplanes of $\mathbb{P}^n(\mathbb{C})$, where $H_i = \{h_i = 0\}$. For some integer $0 \le$

$k \leq n-1$ and some subset $I_k = \{i_1, \ldots, i_{n-k}\}$ of the index set $\{1, \ldots, q\}$ having cardinality $n-k$, the linear subspace $P_{k,I_k} = \cap_{i \in I_k} H_i \simeq \mathbb{P}^k(\mathbb{C})$ is called a *subspace of dimension k*. For a holomorphic mapping $f\colon \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$, we define

$$n_f(t, P_{k,I_k}) := \sum_{|z|<t, f(z) \in P_{k,I_k}} \min_{i \in I_k} \operatorname{ord}_z(h_i \circ f) \qquad (t>0),$$

where we take the sum only for z in the preimage of P_{k,I_k}, and

$$N_f(r, P_{k,I_k}) := \int_1^r \frac{n_f(t, P_{k,I_k})}{t} \, \mathrm{d}\, t \qquad (r>1). \tag{1.3.1}$$

We denote by P^*_{k,I_k} the complement $P_{k,I_k} \setminus \big(\cup_{i \notin I_k} H_i\big)$ which will be called a *star-subspace of dimension k*. We can also define $n_f(t, P^*_{k,I_k})$ and $N_f(r, P^*_{k,I_k})$. Assume now $q = 2n+1+m$ with $m \geq 0$. Consider complements of the form

$$\mathbb{P}^n(\mathbb{C}) \setminus (\cup_{i=1}^{2n+1+m} H_i \setminus A_{m,n}), \tag{1.3.2}$$

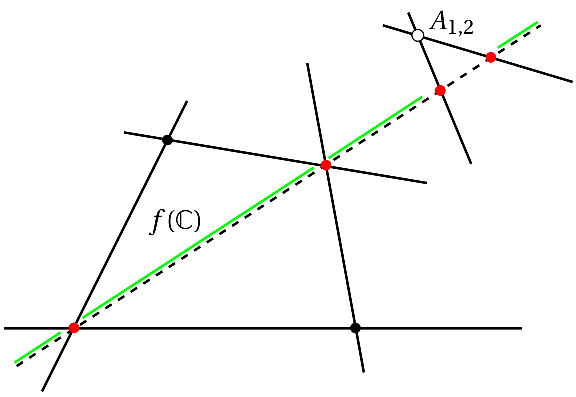
where $A_{m,n}$ is a set of at most m elements of the form P^*_{k,I_k} $(0 \leq k \leq n-2)$. We note that if $m = 0$, these complements are hyperbolic by Theorem 1.2.5.

In $\mathbb{P}^2(\mathbb{C})$, a union of lines $\cup_{i=1}^q H_i$ is in general position if any three lines have empty intersection, and it is generic if in addition any three intersection points between three distinct pairs of lines are not collinear.

Lemma 1.3.1. *In $\mathbb{P}^2(\mathbb{C})$, if $m \leq 3$, all complements of the form* 1.3.2 *are hyperbolic.*

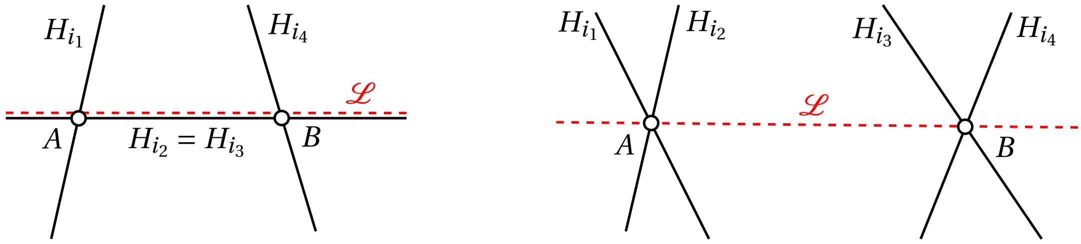
Proof. Without loss of generality, we can assume that $A_{m,2}$ is a set of m distinct points belonging to $\cup_{1 \leq i_1 < i_2 \leq 5+m} H_{i_1} \cap H_{i_2}$.

When $m = 1$, an entire curve $f\colon \mathbb{C} \to \mathbb{P}^2(\mathbb{C}) \setminus (\cup_{i=1}^6 H_i \setminus A_{1,2})$, if it exists, must avoid at least four lines.



By Theorem 1.2.3, its image lies in a diagonal line, which does not contain the intersection point of the two remaining lines by the generic condition. Hence, f must be contained in the complement of four points in a line. By Picard's theorem, f is constant, which is contradiction.

When $m = 2$, $A_{2,2}$ is a set consisting of two points A, B, where $A = H_{i_1} \cap H_{i_2}$, $B = H_{i_3} \cap H_{i_4}$. We denote by I the index set $\{i_1, i_2, i_3, i_4\}$, which has three elements if both A and B belong to a single line H_i and which has four elements otherwise.



Let $f\colon \mathbb{C} \to \mathbb{P}^2(\mathbb{C}) \setminus (\cup_{i=1}^7 H_i \setminus A_{2,2})$ be an entire curve. If $z \in f^{-1}(A)$, we have

$$\begin{aligned} \mathrm{ord}_z(h_{i_1} \circ f) &\geq 1, \\ \mathrm{ord}_z(h_{i_2} \circ f) &\geq 1. \end{aligned}$$

This implies

$$\min\{\mathrm{ord}_z(h_{i_1} \circ f), 2\} + \min\{\mathrm{ord}_z(h_{i_2} \circ f), 2\} \leq 3 \min_{1 \leq j \leq 2} \mathrm{ord}_z(h_{i_j} \circ f). \tag{1.3.3}$$

Hence, by summing this inequality

$$\sum_{|z|<t, f(z)=A} \min\{\mathrm{ord}_z(h_{i_1} \circ f), 2\} + \sum_{|z|<t, f(z)=A} \min\{\mathrm{ord}_z(h_{i_2} \circ f), 2\} \leq 3 \sum_{|z|<t, f(z)=A} \min_{1 \leq j \leq 2} \mathrm{ord}_z(h_{i_j} \circ f). \tag{1.3.4}$$

Similarly for h_{i_3}, h_{i_4} and $z \in f^{-1}(B)$, we have

$$\sum_{|z|<t, f(z)=B} \min\{\mathrm{ord}_z(h_{i_3} \circ f), 2\} + \sum_{|z|<t, f(z)=B} \min\{\mathrm{ord}_z(h_{i_4} \circ f), 2\} \leq 3 \sum_{|z|<t, f(z)=B} \min_{3 \leq j \leq 4} \mathrm{ord}_z(h_{i_j} \circ f). \tag{1.3.5}$$

By taking the sum of both sides of these inequalities and by integrating, we obtain

$$\sum_{i \in I} N_f^{[2]}(r, H_i) \leq 3\big(N_f(r, A) + N_f(r, B)\big). \tag{1.3.6}$$

Now, let $\mathscr{L} = \{\ell = 0\}$ be the line passing through A and B. Since $\ell = \alpha_1 h_{i_1} + \alpha_2 h_{i_2} = \alpha_3 h_{i_3} + \alpha_4 h_{i_4}$ for some $\alpha_1, \ldots, \alpha_4 \in \mathbb{C}$, the following inequalities hold

$$\begin{aligned} \min_{1 \leq j \leq 2} \mathrm{ord}_z(h_{i_j} \circ f) &\leq \mathrm{ord}_z(\ell \circ f) \qquad (z \in f^{-1}(A)), \\ \min_{3 \leq j \leq 4} \mathrm{ord}_z(h_{i_j} \circ f) &\leq \mathrm{ord}_z(\ell \circ f) \qquad (z \in f^{-1}(B)). \end{aligned} \tag{1.3.7}$$

Since $f^{-1}(A)$ and $f^{-1}(B)$ are two disjoint subsets of $f^{-1}(\mathscr{L})$, by taking the sum of both sides of these inequalities on discs and by integrating, we obtain

$$N_f(r, A) + N_f(r, B) \leq N_f(r, \mathscr{L}). \tag{1.3.8}$$

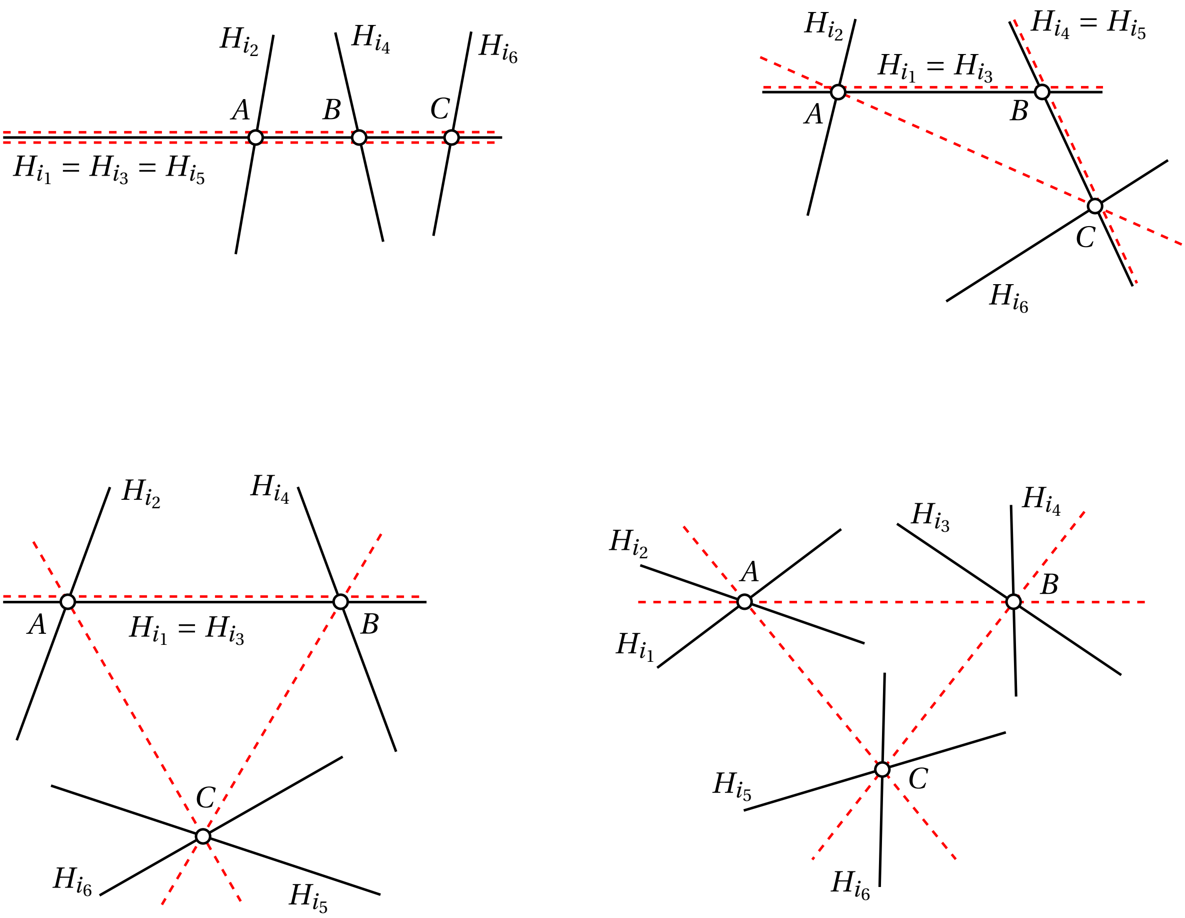
If f would be linearly nondegenerate, then starting from Cartan Second Main Theorem, and using 1.3.6, 1.3.8, we would get

$$4\, T_f(r) \leq \sum_{i=1}^{7} N_f^{[2]}(r, H_i) + S_f(r)$$

$$
\begin{aligned}
&= \sum_{i\in I} N_f^{[2]}(r, H_i) + S_f(r) \\
&\leq 3\big(N_f(r,A) + N_f(r,B)\big) + S_f(r) \\
&\leq 3\,N_f(r,\mathscr{L}) + S_f(r) \\
\text{[Use 1.2.1]} \quad &\leq 3\,T_f(r) + S_f(r), \qquad \text{1.3.9}
\end{aligned}
$$

which is absurd. Thus, any entire curve $f : \mathbb{C} \to \mathbb{P}^2(\mathbb{C}) \setminus (\cup_{i=1}^{7} H_i \setminus A_{2,2})$ must be contained in some line L. Furthermore, the number of points of intersection between L and $\cup_{i=1}^{7} H_i \setminus \{A, B\}$ is at least 3 by the generic condition. By Picard's Theorem, this contradicts the assumption that f is nonconstant.

When $m = 3$, $A_{3,2}$ is a set consisting of three points A, B, C, where $A = H_{i_1} \cap H_{i_2}$, $B = H_{i_3} \cap H_{i_4}$, $C = H_{i_5} \cap H_{i_6}$. In this case, the index set $J = \{i_1, i_2, i_3, i_4, i_5, i_6\}$ may contain 4, 5 or 6 elements.



Suppose to the contrary that there is an entire curve $f : \mathbb{C} \to \mathbb{P}^2(\mathbb{C}) \setminus (\cup_{i=1}^{8} H_i \setminus A_{3,2})$. Similarly as above, cf. 1.3.4, 1.3.5, 1.3.6, we can show in all the four illustrated cases

$$
\sum_{i\in J} N_f^{[2]}(r, H_i) \leq 3\big(N_f(r,A) + N_f(r,B) + N_f(r,C)\big).
$$

Next, let $\mathscr{C} = \{\mathsf{c} = 0\}$ be the degenerate cubic consisting of the three lines $AB = \{\ell_{AB} = 0\}$, $BC = \{\ell_{BC} = 0\}$, and $CA = \{\ell_{CA} = 0\}$. Similarly as in 1.3.7, we have

$$
\begin{aligned}
2 \min_{1\leq j\leq 2} \operatorname{ord}_z(h_{i_j} \circ f) &\leq \operatorname{ord}_z(\ell_{AB} \circ f) + \operatorname{ord}_z(\ell_{CA} \circ f)\\
&= \operatorname{ord}_z(\mathsf{c} \circ f) \qquad\qquad (z \in f^{-1}(A)).
\end{aligned}
$$

We also have two other inequalities for h_{i_3}, h_{i_4}, $z \in f^{-1}(B)$ and for h_{i_5}, h_{i_6}, $z \in f^{-1}(C)$. Summing these inequalities and integrating, we get

$$2\big(N_f(r,A) + N_f(r,B) + N_f(r,C)\big) \leq N_f(r,\mathscr{C}).$$

If the curve f is linearly nondegenerate, then by proceeding as we did in 1.3.9, we also get a contradiction.

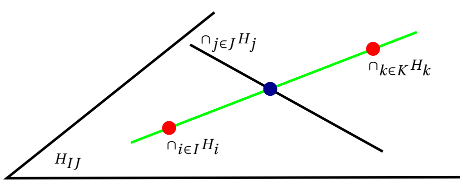
$$
\begin{aligned}
5\,T_f(r) &\leq \sum_{i=1}^{8} N_f^{[2]}(r,H_i) + S_f(r)\\
&\leq 3\big(N_f(r,A) + N_f(r,B) + N_f(r,C)\big) + S_f(r)\\
&\leq \frac{3}{2} N_f(r,\mathscr{C}) + S_f(r)\\
&\leq \frac{9}{2} T_f(r) + S_f(r).
\end{aligned}
$$

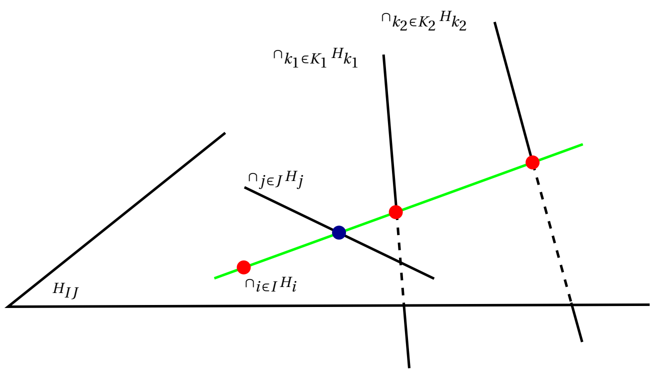
Thus the curve f must be contained in some line. By analyzing the position of this line with respect to $\{H_i\}_{1\leq i\leq 8} \setminus \{A,B,C\}$ and by using Picard's theorem, we conclude as above. □

In $\mathbb{P}^3(\mathbb{C})$, the generic condition for the family of planes $\{H_i\}_{1\leq i\leq q}$ excludes the following cases.

(1) There are three disjoint subsets I, J, K of $\{1,\dots,q\}$ with $|I| = 3$, $|J| = 2$, $|K| = 3$ such that the diagonal (hyper)plane H_{IJ} contains the point $\cap_{k\in K} H_k$.

(2) There are four disjoint subsets I, J, K_1, K_2 of $\{1,\dots,q\}$ with $|I| = 3$, $|J| = 2$, $|K_1| = |K_2| = 2$ such that the three points $(\cap_{k_1\in K_1} H_{k_1}) \cap H_{IJ}$, $(\cap_{k_2\in K_2} H_{k_2}) \cap H_{IJ}$ and $\cap_{i\in I} H_i$ are collinear.





Lemma 1.3.2. *In* $\mathbb{P}^3(\mathbb{C})$*, if* $m \leq 2$*, all complements of the form* 1.3.2 *are hyperbolic.*

Proof. Without loss of generality, we can assume that $A_{m,3}$ is a set of m elements belonging to:

$$\big(\cup_{1\leq i_1<i_2\leq 7+m}\,(H_{i_1}\cap H_{i_2})^*\big)\bigcup\big(\cup_{1\leq i_1<i_2<i_3\leq 7+m}\,H_{i_1}\cap H_{i_2}\cap H_{i_3}\big).$$

Suppose to the contrary that there exists a Brody curve $f\colon \mathbb{C}\to\mathbb{P}^3(\mathbb{C})\setminus(\cup_{i=1}^{7+m}H_i\setminus A_{m,3})$.

When $m=1$, the curve f must avoid at least five planes. By Theorem 1.2.4, its image is contained in some line L. By the generic condition, the number of intersection points between L and $\cup_{i=1}^{8}H_i\setminus A_{1,3}$ is at least 3. By Picard's theorem, f must be constant, which is a contradiction.

Next, we consider the case $m=2$. If $A_{2,3}=\{l_1^*,l_2^*\}$ where l_1,l_2 are lines, then the curve f avoids five planes, say $\{H_i\}_{1\leq i\leq 5}$. By Theorems 1.2.4 and 1.2.3, its image lands in some line $\mathscr{L}$, which is contained in a diagonal plane $\mathscr{P}$ of the family $\{H_i\}_{1\leq i\leq 5}$. We may assume that the plane $\mathscr{P}$ passes through the point $H_1\cap H_2\cap H_3$ and contains the line $H_4\cap H_5$. If the line $\mathscr{L}$ does not pass through the point $H_1\cap H_2\cap H_3$, then it intersects $\{H_i\}_{1\leq i\leq 3}$ in three distinct points, hence f is constant by Picard's theorem. Thus $\mathscr{L}$ must pass through the point $H_1\cap H_2\cap H_3$. In the plane $\mathscr{P}$, the curve f can pass through the points $l_1\cap\mathscr{P}$ and $l_2\cap\mathscr{P}$. But by the generic condition, cf. **(2)** above, the three points $H_1\cap H_2\cap H_3$, $l_1\cap\mathscr{P}$, $l_2\cap\mathscr{P}$ are not collinear. Hence, $f(\mathbb{C})$ is contained in a complement of at least three points in the line $\mathscr{L}$, which is impossible by Picard's theorem.

Two substantial cases remain:

(a) $A_{2,3}=\{A,l^*\}$, where A is a point and l is a line;

(b) the set $A_{2,3}$ consists of two points.

We treat case **(a)**. If both A and l are contained in some common plane H_i, then f avoids five planes. By Theorem 1.2.3, its image must be contained in some diagonal plane, which does not contain the point A by the generic condition. Hence f must avoid seven planes in general position, which is absurd by Theorem 1.2.5. Thus, we can assume that $A=H_1\cap H_2\cap H_3$ and $l^*=(H_4\cap H_5)\setminus\cup_{i\neq 4,5}H_i$. Hence f avoids four planes H_i $(6\leq i\leq 9)$.

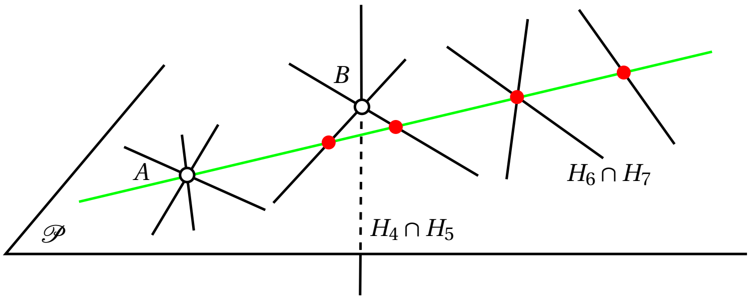
First, we show that f is linearly nondegenerate. Suppose to the contrary that $f(\mathbb{C})$ is contained in some plane $\mathscr{P}$. If $A\notin\mathscr{P}$, then f also avoids H_1, H_2, H_3, which is impossible by Theorem 1.2.5. Hence the plane $\mathscr{P}$ must pass through the point A. If $f(\mathbb{C})$ is contained in some line $\mathscr{L}\subset\mathscr{P}$, then $\mathscr{L}$ must also pass through A, for the same reason. Note that the number of intersection points between $\mathscr{L}$ and $\{H_i\}_{6\leq i\leq 9}$ is at least 2, and it equals 2 only if either $\mathscr{L}$ passes through some point $H_{i_1}\cap H_{i_2}\cap H_{i_3}$ $(6\leq i_1<i_2<i_3\leq 9)$ or $\mathscr{L}$ intersects two lines $H_{i_1}\cap H_{i_2}$, $H_{i_3}\cap H_{i_4}$ $(\{i_1,i_2,i_3,i_4\}=\{6,7,8,9\})$. If $\mathscr{L}$ has empty intersection with $H_4\cap H_5$, then f avoids at least four points in the line $\mathscr{L}$, hence it is constant. If $\mathscr{L}$ intersects $H_4\cap H_5$, then by considering the diagonal plane passing through A and containing $H_4\cap H_5$, the two cases where $|\mathscr{L}\cap\{H_i\}_{6\leq i\leq 9}|=2$ are excluded by the generic condition. Thus f always avoids three distinct points in $\mathscr{L}$, hence it is constant.

Consequently, we can assume that f does not land in any line in the plane $\mathscr{P}$. There are two possible positions of $\mathscr{P}$:

(a1) it is a diagonal plane containing A and some line in $\cup_{6\leq i_1<i_2\leq 9} H_{i_1}\cap H_{i_2}$;

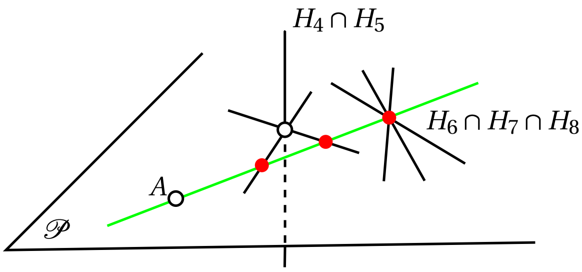
(a2) it does not contain any line in $\cup_{6\leq i_1<i_2\leq 9} H_{i_1}\cap H_{i_2}$.

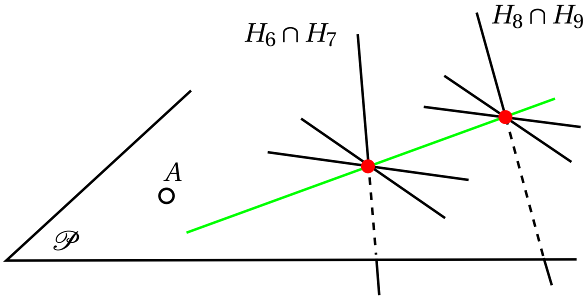
In case **(a1)**, assume that $\mathscr{P}$ contains the line $H_6\cap H_7$. Among $\{H_i\cap\mathscr{P}\}_{1\leq i\leq 9}$, two lines $H_6\cap\mathscr{P}$, $H_7\cap\mathscr{P}$ coincide, and dropping the line $H_1\cap\mathscr{P}$, by the generic condition, it remains seven lines $\{H_i\cap\mathscr{P}\}_{i\neq 1,7}$ in general position in $\mathscr{P}$.



Letting B be the intersection point of the line $l = H_4\cap H_5$ with the plane $\mathscr{P}$, the curve f lands in $\mathscr{P}\setminus(\cup_{1\leq i\leq 9}H_i\cap\mathscr{P})\setminus\{A,B\})$. As in 1.3.9, $f(\mathbb{C})$ is contained in some line, which is a contradiction.

Next, consider case **(a2)**.





If $\mathscr{P}$ contains some point in $\cup_{6\leq i_1<i_2<i_3\leq 9} H_{i_1}\cap H_{i_2}\cap H_{i_3}$, say $H_6\cap H_7\cap H_8$, then the curve f avoids three lines $H_i\cap\mathscr{P}$ $(6\leq i\leq 8)$, which are not in general position. By Theorem 1.2.6, $f(\mathbb{C})$ must be contained in some line, which is a contradiction. Therefore, $\mathscr{P}$ does not contain any point in $\cup_{6\leq i_1<i_2<i_3\leq 9} H_{i_1}\cap H_{i_2}\cap H_{i_3}$. But then the curve f avoids a collection of four lines $\{H_i\cap\mathscr{P}\}_{6\leq i\leq 9}$, which are in general position. By Theorem 1.2.3, its image must land in some diagonal line of this family, which is a contradiction.

Still in case **(a)**, we can therefore assume that f is linearly nondegenerate. Assume that the omitted planes H_6, H_7, H_8, H_9 are given in the homogeneous coordinates $[z_0 : z_1 : z_2 : z_3]$ by equations $\{z_i = 0\}$ $(0\leq i\leq 3)$. By Theorems 1.2.2, f has a reduced representation of the form

$$[1 : e^{\lambda_1 z+\mu_1} : e^{\lambda_2 z+\mu_2} : e^{\lambda_3 z+\mu_3}], \qquad 1.3.10$$

where λ_i, μ_i are constants with $\lambda_i \neq 0$ ($1 \leq i \leq 3$ and $\lambda_i \neq \lambda_j$ $(i \neq j)$. Let $\mathscr{D}$ be the diagonal plane passing through the point $A = H_1 \cap H_2 \cap H_3$ and containing the line $l = H_4 \cap H_5$. By similar arguments as in Lemma 1.3.1, cf. 1.3.7, 1.3.8, we can show that

$$N_f(r, A) + N_f(r, l^*) \leq N_f(r, \mathscr{D}). \tag{1.3.11}$$

From the elementary inequality

$$\min\{\mathrm{ord}_z(h_4 \circ f), 3\} + \min\{\mathrm{ord}_z(h_5 \circ f), 3\} \leq 4 \min_{4 \leq i \leq 5} \mathrm{ord}_z(h_i \circ f) \qquad (z \in f^{-1}(l^*)),$$

by taking the sum on disks and then by integrating, we get

$$N_f^{[3]}(r, H_4) + N_f^{[3]}(r, H_5) \leq 4\, N_f(r, l^*). \tag{1.3.12}$$

Next, we try to bound $N_f^{[3]}(r, H_i)$ $(1 \leq i \leq 3)$ from above in terms of $N_f(r, A)$. Since f is of the form 1.3.10, for any z_1, $z_2 \in f^{-1}(A)$, we have

$$f^{(k)}(z_1) = f^{(k)}(z_2) \qquad (k \in \mathbb{N}),$$

hence

$$\mathrm{ord}_{z_1}(h_i \circ f) = \mathrm{ord}_{z_2}(h_i \circ f) \qquad (1 \leq i \leq 3). \tag{1.3.13}$$

Thus, it suffices to consider the two cases:

(a3) $\mathrm{ord}_z(h_i \circ f) \leq 2$ for all $1 \leq i \leq 3$ and for all $z \in f^{-1}(A)$;

(a4) $\mathrm{ord}_z(h_i \circ f) \geq 3$ for some i with $1 \leq i \leq 3$ and for all $z \in f^{-1}(A)$.

In case **(a3)**, the elementary inequality

$$\sum_{i=1}^{3} \min\{\mathrm{ord}_z(h_i \circ f), 3\} \leq 5 \min_{1 \leq i \leq 3} \mathrm{ord}_z(h_i \circ f) \qquad (z \in f^{-1}(A)),$$

yields

$$N_f^{[3]}(r, H_1) + N_f^{[3]}(r, H_2) + N_f^{[3]}(r, H_3) \leq 5\, N_f(r, A). \tag{1.3.14}$$

Since f is linearly nondegenerate, we can proceed similarly as in 1.3.9

$$\begin{aligned}
5\, T_f(r) &\leq \sum_{i=1}^{9} N_f^{[3]}(r, H_i) + S_f(r) \\
&\leq 5\, N_f(r, A) + 4\, N_f(r, l^*) + S_f(r) \\
&= 5\big(N_f(r, A) + N_f(r, l^*)\big) - N_f(r, l^*) + S_f(r) \\
&\leq 5\, N_f(r, \mathscr{D}) - N_f(r, l^*) + S_f(r) \\
&\leq 5\, T_f(r) - N_f(r, l^*) + S_f(r).
\end{aligned} \tag{1.3.15}$$

This implies

$$N_f(r, l^*) = S_f(r)$$

and hence, by 1.3.12, we have

$$N_f^{[3]}(r, H_4) + N_f^{[3]}(r, H_5) = S_f(r).$$

Therefore, the first inequality of 1.3.15 can be rewritten as

$$5\,T_f(r) \le \sum_{i=1}^{3} N_f^{[3]}(r, H_i) + S_f(r).$$

By the First Main Theorem, the right-hand side of the above inequality is bounded from above by $3\,T_f(r) + S_f(r)$. Thus we get

$$5\,T_f(r) \le 3\,T_f(r) + S_f(r),$$

which is absurd.

Next, we consider case **(a4)**. Assume that $\operatorname{ord}_z(h_1 \circ f) \ge 3$ for all $z \in f^{-1}(A)$. Since f is of the form 1.3.10, we claim that

$$\operatorname{ord}_z(h_i \circ f) \le 2 \qquad {\scriptstyle (z\in f^{-1}(A), \quad 2\le i\le 3)}. \tag{1.3.16}$$

Indeed, if $\operatorname{ord}_z(h_i \circ f) \ge 3$ for some $z \in f^{-1}(A)$ and for some $2 \le i \le 3$, say $i = 2$, then $(e^{\lambda_1 z+\mu_1}, e^{\lambda_2 z+\mu_2}, e^{\lambda_3 z+\mu_3})$ is a solution of a system of six linear equations of the form

$$\begin{cases} 0 = a_{10} + a_{11}\,u + a_{12}\,v + a_{13}\,w, \\ 0 = \qquad a_{11}\,\lambda_1\,u + a_{12}\,\lambda_2\,v + a_{13}\,\lambda_3\,w, \\ 0 = \qquad a_{11}\,\lambda_1^2\,u + a_{12}\,\lambda_2^2\,v + a_{13}\,\lambda_3^2\,w, \\ 0 = a_{20} + a_{21}\,u + a_{22}\,v + a_{23}\,w, \\ 0 = \qquad a_{21}\,\lambda_1\,u + a_{22}\,\lambda_2\,v + a_{23}\,\lambda_3\,w, \\ 0 = \qquad a_{21}\,\lambda_1^2\,u + a_{22}\,\lambda_2^2\,v + a_{23}\,\lambda_3^2\,w, \end{cases}$$

where u, v, w are unknowns, and where a_{ij} $(0 \le i \le 3)$ are the coefficients of h_i $(1 \le i \le 2)$ in the homogeneous coordinate $[z_0 : z_1 : z_2 : z_3]$. Since λ_i are nonzero distinct constants, this forces the two linear forms h_1, h_2 to be linearly dependent, which is a contradiction.

It follows from 1.3.16 that

$$\min\{\operatorname{ord}_z(h_2 \circ f), 3\} + \min\{\operatorname{ord}_z(h_3 \circ f), 3\} \le 3 \min_{1\le i\le 3} \operatorname{ord}_z(h_i \circ f) \qquad {\scriptstyle (z\in f^{-1}(A))}.$$

By taking the sum on disks and by integrating, we get

$$N_f^{[3]}(r, H_2) + N_f^{[3]}(r, H_3) \le 3\,N_f(r, A). \tag{1.3.17}$$

We may therefore proceed similarly as in 1.3.15

$$\begin{aligned} 5\,T_f(r) &\le \sum_{i=1}^{9} N_f^{[3]}(r, H_i) + S_f(r) \\ &\le N_f^{[3]}(r, H_1) + 3\,N_f(r, A) + 4\,N_f(r, l^*) + S_f(r) \\ &\le N_f(r, H_1) + 4\big(N_f(r, A) + N_f(r, l^*)\big) - N_f(r, A) + S_f(r) \\ &\le T_f(r) + 4\,N_f(r, \mathscr{D}) - N_f(r, A) + S_f(r) \end{aligned}$$

$$\leq 5\,T_f(r) - N_f(r,A) + S_f(r). \qquad 1.3.18$$

This implies

$$N_f(r,A) = S_f(r).$$

By 1.3.17, we have

$$N_f^{[3]}(r,H_2) + N_f^{[3]}(r,H_3) = S_f(r).$$

Hence we can rewrite the first inequality of 1.3.18 and use First Main Theorem to get a contradiction

$$\begin{aligned} 5\,T_f(r) &\leq N_f^{[3]}(r,H_1) + N_f^{[3]}(r,H_4) + N_f^{[3]}(r,H_5) + S_f(r) \\ &\leq 3\,T_f(r) + S_f(r). \end{aligned}$$

Let us consider case **(b)**. Assume now $A_{2,3} = \{A,B\}$, where A, B are two points contained in $\cup_{1\leq i_1<i_2<i_3\leq 9} H_{i_1}\cap H_{i_2}\cap H_{i_3}$. There are three possibilities for the positions of A and B:

(b1) both A and B are contained in some line $H_i\cap H_j$;

(b2) both A and B are contained in some plane H_i but they are not contained in any line $H_i\cap H_j$;

(b3) there is no plane H_i containing both points A and B.

In case **(b1)**, the curve f avoids a family of five planes and, therefore, its image is contained in some diagonal plane of this family, which contains neither A nor B by the generic condition. Hence f avoids all planes H_i, which is absurd by Theorem 1.2.5.

Next, we consider case **(b2)**. Assume that $A = H_1\cap H_2\cap H_3$ and $B = H_1\cap H_4\cap H_5$, hence f avoids the 4 planes H_i $(6\leq i\leq 9)$. Similarly as in case **(a)**, the generic condition allows us to assume that f is linearly nondegenerate.

Since f avoids four planes, it is of the form 1.3.10 in some affine coordinates on $\mathbb{P}^3(\mathbb{C})$. Since f has no singular point, we have

$$\begin{aligned} \min_{i\in\{1,2,3\}} \mathrm{ord}_z(h_i\circ f) &= 1 \qquad (z\in f^{-1}(A)), \\ \min_{i\in\{1,4,5\}} \mathrm{ord}_z(h_i\circ f) &= 1 \qquad (z\in f^{-1}(B)). \end{aligned} \qquad 1.3.19$$

Hence by using these two equalities together with 1.3.16,

$$\sum_{i\in\{1,2,3\}} \min\{\mathrm{ord}_z(h_i\circ f),3\} \leq 6 = 6 \min_{i\in\{1,2,3\}} \mathrm{ord}_z(h_i\circ f), \qquad (z\in f^{-1}(A)),$$

$$\sum_{i\in\{1,4,5\}} \min\{\mathrm{ord}_z(h_i\circ f),3\} \leq 6 = 6 \min_{i\in\{1,4,5\}} \mathrm{ord}_z(h_i\circ f), \qquad (z\in f^{-1}(B)).$$

Thus, by taking the sum on disks of both sides of these inequalities and by integrating,

$$\sum_{i=1}^{5} N_f^{[3]}(r,H_i) \leq 6\big(N_f(r,A) + N_f(r,B)\big).$$

Next, using again that f is of the form 1.3.10, one can find two planes $\mathscr{P}_1 = \{\mathsf{p}_1 = 0\}$, $\mathscr{P}_2 = \{\mathsf{p}_2 = 0\}$ containing the line AB such that

$$\mathrm{ord}_z(\mathsf{p}_1 \circ f) \geq 2 \qquad (z \in f^{-1}(A)),$$

$$\mathrm{ord}_z(\mathsf{p}_2 \circ f) \geq 2 \qquad (z \in f^{-1}(B)).$$

Let $\mathscr{Q} = \{\mathsf{q} = \mathsf{p}_1\mathsf{p}_2 = 0\}$ be the degenerate quadric $\mathscr{P}_1 \cup \mathscr{P}_2$. We have

$$3 = 3 \min_{i\in\{1,2,3\}} \mathrm{ord}_z(h_i \circ f) \leq \mathrm{ord}_z(\mathsf{p}_1 \circ f) + \mathrm{ord}_z(\mathsf{p}_2 \circ f) = \mathrm{ord}_z(\mathsf{q} \circ f) \qquad (z \in f^{-1}(A)),$$

$$3 = 3 \min_{i\in\{1,4,5\}} \mathrm{ord}_z(h_i \circ f) \leq \mathrm{ord}_z(\mathsf{p}_1 \circ f) + \mathrm{ord}_z(\mathsf{p}_2 \circ f) = \mathrm{ord}_z(\mathsf{q} \circ f) \qquad (z \in f^{-1}(B)),$$

which implies, by integrating, that

$$3\big(N_f(r,A) + N_f(r,B)\big) \leq N_f(r,\mathscr{T}).$$

We proceed similarly as above to get a contradiction

$$\begin{aligned} 5\,T_f(r) &\leq \sum_{i=1}^{9} N_f^{[3]}(r,H_i) + S_f(r) \\ &\leq 6\big(N_f(r,A) + N_f(r,B)\big) + S_f(r) \\ &\leq 2\,N_f(r,\mathscr{T}) + S_f(r) \\ &\leq 4\,T_f(r) + S_f(r). \end{aligned}$$

Now, we consider case **(b3)**. Assume that $A = H_1 \cap H_2 \cap H_3$, $B = H_4 \cap H_5 \cap H_6$, when f avoids the three planes H_7, H_8, H_9. If $f(\mathbb{C})$ is contained in some plane $\mathscr{P}$, then it is not hard to see that $\mathscr{P}$ must pass through both A and B. Furthermore, by using Theorem 1.2.6, one can show that $\mathscr{P}$ does not pass through the point $C = H_7 \cap H_8 \cap H_9$. One can then always find 7 lines in general position in $\mathscr{P}$ among $\{H_i \cap \mathscr{P}\}_{1\leq i\leq 9}$. Hence one can use similar arguments as in Lemma 1.3.1, case $m = 2$, to get a contradiction. Thus, we can suppose that f is linearly nondegenerate.

Assume that the omitted planes H_7, H_8, H_9 are given in the homogeneous coordinates $[z_0 : z_1 : z_2 : z_3]$ by the equations $\{z_0 = 0\}$, $\{z_1 = 0\}$, $\{z_2 = 0\}$. Since $\{H_i\}_{1\leq i\leq 9}$ is a family of planes in general position, the planes H_i $(1 \leq i \leq 6)$ are given by

$$h_i = \sum_{j=0}^{3} a_{ij}\, z_j = 0,$$

with $a_{i3} \neq 0$ $(1 \leq i \leq 6)$. Set $l_{i_1,i_2} = H_{i_1} \cap H_{i_2}$ $(1 \leq i_1 < i_2 \leq 3)$, $l_{j_1,j_2} = H_{j_1} \cap H_{j_2}$ $(4 \leq j_1 < j_2 \leq 6)$. For $1 \leq i < j \leq 3$ or $4 \leq i < j \leq 6$, let $R_{i,j} = \{r_{i,j} = 0\}$ be the plane containing the lines AB, $l_{i,j}$ and let $T_{i,j} = \{t_{i,j} = a_{j3}\, h_i - a_{i3}\, h_j = 0\}$ be the plane passing through the point $C = [0:0:0:1]$ and containing the line $l_{i,j}$. We note that all $r_{i,j}$, $t_{i,j}$ are linear combinations of h_i and h_j with nonzero coefficients.

Since f avoids three planes, by Theorem 1.2.2 it has a reduced representation of the form

$$[1 : e^{\lambda_1 z + \mu_1} : e^{\lambda_2 z + \mu_2} : g], \tag{1.3.20}$$

where λ_1, λ_2, μ_1, μ_2 are constants with $\lambda_1 \neq \lambda_2$, $\lambda_1, \lambda_2 \neq 0$ and where g is an entire function. Since f has no singular point, we have

$$\min_{1\le i\le 3} \operatorname{ord}_z(h_i \circ f) = 1 \qquad (z\in f^{-1}(A)),$$
$$\min_{4\le j\le 6} \operatorname{ord}_z(h_j \circ f) = 1 \qquad (z\in f^{-1}(B)). \tag{1.3.21}$$

Since f is of the form 1.3.20, we claim that

$$\min\{\operatorname{ord}_z(h_{i_1}\circ f), \operatorname{ord}_z(h_{i_2}\circ f)\} \le 2 \qquad (z\in f^{-1}(A) \quad 1\le i_1<i_2\le 3),$$
$$\min\{\operatorname{ord}_z(h_{j_1}\circ f), \operatorname{ord}_z(h_{j_2}\circ f)\} \le 2 \qquad (z\in f^{-1}(B), \quad 4\le j_1<j_2\le 6). \tag{1.3.22}$$

Indeed, if one of these inequalities does not hold, say $\min\{\operatorname{ord}_z(h_1\circ f), \operatorname{ord}_z(h_2\circ f)\} \ge 3$ for some $z \in f^{-1}(A)$, then z is a solution of the following system of equations

$$\begin{cases} 0 = (t_{1,2}\circ f)(z), \\ 0 = (t_{1,2}\circ f)'(z), \\ 0 = (t_{1,2}\circ f)''(z). \end{cases}$$

Equivalently, $(e^{\lambda_1 z+\mu_1}, e^{\lambda_2 z+\mu_2})$ is a solution of a system of three linear equations of the form

$$\begin{cases} 0 = (a_{23}\,a_{10} - a_{13}\,a_{20}) + (a_{23}\,a_{11} - a_{13}\,a_{21})\,x + (a_{23}\,a_{12} - a_{13}\,a_{22})\,y, \\ 0 = (a_{23}\,a_{11} - a_{13}\,a_{21})\,\lambda_1\,x + (a_{23}\,a_{12} - a_{13}\,a_{22})\,\lambda_2\,y, \\ 0 = (a_{23}\,a_{11} - a_{13}\,a_{21})\,\lambda_1^2\,x + (a_{23}\,a_{12} - a_{13}\,a_{22})\,\lambda_2^2\,y, \end{cases}$$

where x, y are unknowns. Since $\lambda_1 \neq \lambda_2$, $\lambda_1, \lambda_2 \neq 0$, this implies that the two linear forms h_1, h_2 must be linearly dependent, which is a contradiction.

It follows from 1.3.21 and 1.3.22 that

$$\sum_{i=1}^{3} \min\{\operatorname{ord}_z(h_i\circ f), 3\} \le 6 \qquad (z\in f^{-1}(A)),$$
$$\sum_{j=4}^{6} \min\{\operatorname{ord}_z(h_j\circ f), 3\} \le 6 \qquad (z\in f^{-1}(B)). \tag{1.3.23}$$

Now we prove the following equality

Claim 1.3.1.

$$T_f(r) = N_f(r,A) + N_f(r,B) + S_f(r). \tag{1.3.24}$$

Proof. Since f is of the form 1.3.20 and since $t_{i,j}$ does not contain the term x_3, we have

$$\operatorname{ord}_{z_1}(t_{i_1,i_2}\circ f) = \operatorname{ord}_{z_2}(t_{i_1,i_2}\circ f) \qquad (z_1,z_2\in f^{-1}(A), 1\le i_1<i_2\le 3),$$
$$\operatorname{ord}_{z_1}(t_{j_1,j_2}\circ f) = \operatorname{ord}_{z_2}(t_{j_1,j_2}\circ f) \qquad (z_1,z_2\in f^{-1}(B), 4\le j_1<j_2\le 6). \tag{1.3.25}$$

Thus, it suffices to consider the four cases depending on f and $t_{i,j}$:

(b3.1) $\mathrm{ord}_z(t_{i_1,i_2}\circ f)=1$ for all $1\le i_1<i_2\le 3$, for all $z\in f^{-1}(A)$ and $\mathrm{ord}_z(t_{j_1,j_2}\circ f)=1$ for all $4\le j_1<j_2\le 6$, for all $z\in f^{-1}(B)$;

(b3.2) $\mathrm{ord}_z(t_{i_1,i_2}\circ f)\ge 2$ for some $1\le i_1<i_2\le 3$, for all $z\in f^{-1}(A)$ and $\mathrm{ord}_z(t_{j_1,j_2}\circ f)=1$ for all $4\le j_1<j_2\le 6$, for all $z\in f^{-1}(B)$;

(b3.3) $\mathrm{ord}_z(t_{i_1,i_2}\circ f)=1$ for all $1\le i_1<i_2\le 3$, for all $z\in f^{-1}(A)$ and $\mathrm{ord}_z(t_{j_1,j_2}\circ f)\ge 2$ for some $4\le j_1<j_2\le 6$, for all $z\in f^{-1}(B)$;

(b3.4) $\mathrm{ord}_z(t_{i_1,i_2}\circ f)\ge 2$ for some $1\le i_1<i_2\le 3$, for all $z\in f^{-1}(A)$ and $\mathrm{ord}_z(t_{j_1,j_2}\circ f)\ge 2$ for some $4\le j_1<j_2\le 6$, for all $z\in f^{-1}(B)$.

Consider case **(b3.1)**. Since $t_{i,j}$ is a linear combination of h_i and h_j with nonzero coefficients, we have

$$\begin{aligned}\min\{\mathrm{ord}_z(h_{i_1}\circ f),\mathrm{ord}_z(h_{i_2}\circ f)\} &= 1 \qquad (z\in f^{-1}(A)\quad 1\le i_1<i_2\le 3),\\ \min\{\mathrm{ord}_z(h_{j_1}\circ f),\mathrm{ord}_z(h_{j_2}\circ f)\} &= 1 \qquad (z\in f^{-1}(B),\quad 4\le j_1<j_2\le 6).\end{aligned}$$

Using these equalities together with 1.3.21, we get

$$\begin{aligned}\sum_{i=1}^{3}\min\{\mathrm{ord}_z(h_i\circ f),3\} &\le 5 = 5\min_{1\le i\le 3}\mathrm{ord}_z(h_i\circ f) \qquad (z\in f^{-1}(A)),\\ \sum_{i=4}^{6}\min\{\mathrm{ord}_z(h_i\circ f),3\} &\le 5 = 5\min_{4\le i\le 6}\mathrm{ord}_z(h_i\circ f) \qquad (z\in f^{-1}(B)).\end{aligned} \tag{1.3.26}$$

By taking the sum on disks and by integrating these two inequalities, we obtain

$$N_f^{[3]}(r,H_1)+N_f^{[3]}(r,H_2)+N_f^{[3]}(r,H_3)\le 5\,N_f(r,A),$$

$$N_f^{[3]}(r,H_4)+N_f^{[3]}(r,H_5)+N_f^{[3]}(r,H_6)\le 5\,N_f(r,B).$$

Letting $\mathscr{B}$ be a plane passing through A and B, we proceed similarly as before

$$\begin{aligned}5\,T_f(r) &\le \sum_{i=1}^{9}N_f^{[3]}(r,H_i)+S_f(r)\\ &\le 5\,N_f(r,A)+5\,N_f(r,B)+S_f(r)\\ &\le 5\,N_f(r,\mathscr{B})+S_f(r)\\ &\le 5\,T_f(r)+S_f(r).\end{aligned} \tag{1.3.27}$$

Here, $S_f(r)=o(T_f(r))$ is negligible, hence all inequalities are equalities modulo $S_f(r)$. This gives 1.3.24, as wanted.

Next, we consider case **(b3.2)**. Let us set

$$\begin{aligned}E_{t,A} &= \{z\in\mathbb{C}\colon |z|<t, f(z)=A\},\\ E^{1}_{t,A,i} &= \{z\in\mathbb{C}\colon |z|<t, f(z)=A, \mathrm{ord}_z(h_i\circ f)=1\} \qquad (1\le i\le 3),\\ E^{\ge 2}_{t,A,i} &= \{z\in\mathbb{C}\colon |z|<t, f(z)=A, \mathrm{ord}_z(h_i\circ f)\ge 2\} \qquad (1\le i\le 3),\\ E_{t,B} &= \{z\in\mathbb{C}\colon |z|<t, f(z)=B\},\\ E^{1}_{t,B,i} &= \{z\in\mathbb{C}\colon |z|<t, f(z)=B, \mathrm{ord}_z(h_i\circ f)=1\} \qquad (4\le i\le 6),\end{aligned}$$

$$E^{\geq 2}_{t,B,i} = \{z \in \mathbb{C} \colon |z| < t, f(z) = B, \mathrm{ord}_z(h_i \circ f) \geq 2\} \qquad {\scriptstyle (4 \leq i \leq 6)}.$$

Assume that $\mathrm{ord}_z(t_{1,2} \circ f) \geq 2$ for all $z \in f^{-1}(A)$. Since $t_{1,2}$, $r_{1,2}$ are linear combinations of h_1 and h_2 with nonzero coefficients, we have

$$\begin{aligned} E^{\geq 2}_{t,A,1} &= E^{\geq 2}_{t,A,2}, \\ \mathrm{ord}_z(r_{1,2} \circ f) &\geq 2 \qquad {\scriptstyle (z \in E^{\geq 2}_{t,A,1})}. \end{aligned} \tag{1.3.28}$$

For the same reason

$$E^{1}_{t,A,1} = E^{1}_{t,A,2},$$

which yields

$$\sum_{i=1}^{3} \min\{\mathrm{ord}_z(h_i \circ f), 3\} \leq 5 \qquad {\scriptstyle (z \in E^{1}_{t,A,1})}. \tag{1.3.29}$$

Letting $\mathscr{R} = \{\mathsf{r} = r_{1,2}\, r_{4,5}\, r_{5,6}\, r_{4,6} = 0\}$ be the degenerate quartic $R_{1,2} \cup R_{4,5} \cup R_{5,6} \cup R_{4,6}$ whose four components pass through A and B, we have

$$\mathrm{ord}_z(\mathsf{r} \circ f) \geq 4 \qquad {\scriptstyle (z \in E_{t,A} \cup E_{t,B})}. \tag{1.3.30}$$

Furthermore, it follows from 1.3.28 that

$$\mathrm{ord}_z(\mathsf{r} \circ f) \geq 5 \qquad {\scriptstyle (z \in E^{\geq 2}_{t,A,1})}.$$

Using this inequality together with 1.3.23 and 1.3.21, we get

$$\sum_{i=1}^{3} \min\{\mathrm{ord}_z(h_i \circ f), 3\} \leq 6 = 6 \min_{1 \leq i \leq 3} \mathrm{ord}_z(h_i \circ f) \leq \frac{6}{5} \mathrm{ord}_z(\mathsf{r} \circ f) \qquad {\scriptstyle (z \in E^{\geq 2}_{t,A,1})}.$$

Combining 1.3.29, 1.3.21 and 1.3.30, we receive

$$\sum_{i=1}^{3} \min\{\mathrm{ord}_z(h_i \circ f), 3\} \leq 5 = 5 \min_{1 \leq i \leq 3} \mathrm{ord}_z(h_i \circ f) \leq \frac{5}{4} \mathrm{ord}_z(\mathsf{r} \circ f) \qquad {\scriptstyle (z \in E^{1}_{t,A,1})}.$$

Since $\mathrm{ord}_z(t_{j_1,j_2} \circ f) = 1$ for all $4 \leq j_1 < j_2 \leq 6$, for all $z \in f^{-1}(B)$, by similar arguments as in 1.3.26 and by using 1.3.30, we also have

$$\sum_{i=4}^{6} \min\{\mathrm{ord}_z(h_i \circ f), 3\} \leq 5 = 5 \min_{4 \leq i \leq 6} \mathrm{ord}_z(h_i \circ f) \leq \frac{5}{4} \mathrm{ord}_z(\mathsf{r} \circ f) \qquad {\scriptstyle (z \in E_{t,B})}.$$

By taking the sum on disks and by integrating these three inequalities, we obtain

$$\begin{aligned} \sum_{i=1}^{3} \int_1^r \frac{\sum_{z \in E^{\geq 2}_{t,A,1}} \min\{\mathrm{ord}_z(h_i \circ f), 3\}}{t}\, \mathrm{d}\,t &\leq 6 \int_1^r \frac{\sum_{z \in E^{\geq 2}_{t,A,1}} \min_{1 \leq i \leq 3} \mathrm{ord}_z(h_i \circ f)}{t}\, \mathrm{d}\,t \\ &\leq \frac{6}{5} \int_1^r \frac{\sum_{z \in E^{\geq 2}_{t,A,1}} \mathrm{ord}_z(\mathsf{r} \circ f)}{t}\, \mathrm{d}\,t, \\ \sum_{i=1}^{3} \int_1^r \frac{\sum_{z \in E^{1}_{t,A,1}} \min\{\mathrm{ord}_z(h_i \circ f), 3\}}{t}\, \mathrm{d}\,t &\leq 5 \int_1^r \frac{\sum_{z \in E^{1}_{t,A,1}} \min_{1 \leq i \leq 3} \mathrm{ord}_z(h_i \circ f)}{t}\, \mathrm{d}\,t \end{aligned} \tag{1.3.31}$$

$$\leq \frac{5}{4}\int_1^r \frac{\sum_{z\in E^1_{t,A,1}} \mathrm{ord}_z(\mathrm{r}\circ f)}{t}\,\mathrm{d}\,t, \qquad 1.3.32$$

$$\sum_{i=4}^{6}\int_1^r \frac{\sum_{z\in E_{t,B}} \min\{\mathrm{ord}_z(h_i\circ f),3\}}{t}\,\mathrm{d}\,t \leq 5\,N_f(r,B)$$
$$\leq \frac{5}{4}\int_1^r \frac{\sum_{z\in E_{t,B}} \mathrm{ord}_z(\mathrm{r}\circ f)}{t}\,\mathrm{d}\,t. \qquad 1.3.33$$

We then proceed similarly as before:

$$\begin{aligned}
5\,T_f(r) &\leq \sum_{i=1}^{9} N_f^{[3]}(r,H_i)+S_f(r)\\
&\leq \sum_{i=1}^{3} N_f^{[3]}(r,H_i)+\sum_{i=4}^{6} N_f^{[3]}(r,H_i)+S_f(r)\\
&= \sum_{i=1}^{3}\int_1^r \frac{\sum_{z\in E_{t,A}} \min\{\mathrm{ord}_z(h_i\circ f),3\}}{t}\,\mathrm{d}\,t+\sum_{i=4}^{6}\int_1^r \frac{\sum_{z\in E_{t,B}} \min\{\mathrm{ord}_z(h_i\circ f),3\}}{t}\,\mathrm{d}\,t+S_f(r)\\
&= \sum_{i=1}^{3}\Big(\int_1^r \frac{\sum_{z\in E^{\geq 2}_{t,A,1}} \min\{\mathrm{ord}_z(h_i\circ f),3\}}{t}\,\mathrm{d}\,t+\int_1^r \frac{\sum_{z\in E^{1}_{t,A,1}} \min\{\mathrm{ord}_z(h_i\circ f),3\}}{t}\,\mathrm{d}\,t\Big)\\
&\quad+\sum_{i=4}^{6}\int_1^r \frac{\sum_{z\in E_{t,B}} \min\{\mathrm{ord}_z(h_i\circ f),3\}}{t}\,\mathrm{d}\,t+S_f(r)\\
&\leq \frac{6}{5}\int_1^r \frac{\sum_{z\in E^{\geq 2}_{t,A,1}} \mathrm{ord}_z(\mathrm{r}\circ f)}{t}\,\mathrm{d}\,t+\frac{5}{4}\int_1^r \frac{\sum_{z\in E^{1}_{t,A,1}} \mathrm{ord}_z(\mathrm{r}\circ f)}{t}\,\mathrm{d}\,t\\
&\quad+\frac{5}{4}\int_1^r \frac{\sum_{z\in E_{t,B}} \mathrm{ord}_z(\mathrm{r}\circ f)}{t}\,\mathrm{d}\,t+S_f(r)\\
&= \frac{5}{4}\Big(\int_1^r \frac{\sum_{z\in E_{t,A}} \mathrm{ord}_z(\mathrm{r}\circ f)}{t}\,\mathrm{d}\,t+\int_1^r \frac{\sum_{z\in E_{t,B}} \mathrm{ord}_z(\mathrm{r}\circ f)}{t}\,\mathrm{d}\,t\Big)\\
&\quad+\Big(\frac{6}{5}-\frac{5}{4}\Big)\int_1^r \frac{\sum_{z\in E^{\geq 2}_{t,A,1}} \mathrm{ord}_z(\mathrm{r}\circ f)}{t}\,\mathrm{d}\,t+S_f(r)\\
&\leq \frac{5}{4}\,N_f(r,\mathscr{R})-\frac{1}{20}\int_1^r \frac{\sum_{z\in E^{\geq 2}_{t,A,1}} \mathrm{ord}_z(\mathrm{r}\circ f)}{t}\,\mathrm{d}\,t+S_f(r)\\
&\leq 5\,T_f(r)-\frac{1}{20}\int_1^r \frac{\sum_{z\in E^{\geq 2}_{t,A,1}} \mathrm{ord}_z(\mathrm{r}\circ f)}{t}\,\mathrm{d}\,t+S_f(r).
\end{aligned} \qquad 1.3.34$$

This implies

$$\int_1^r \frac{\sum_{z\in E^{\geq 2}_{t,A,1}} \mathrm{ord}_z(\mathrm{r}\circ f)}{t}\,\mathrm{d}\,t = S_f(r) \qquad 1.3.35$$

and whence all inequalities in 1.3.34 become equalities modulo $S_f(r)$, which gives

$$\sum_{i=1}^{6} N_f^{[3]}(r,H_i) = 5\,T_f(r)+S_f(r), \qquad 1.3.36$$

$$\sum_{i=1}^{3} N_f^{[3]}(r,H_i) = \frac{5}{4}\int_1^r \frac{\sum_{z\in E_{t,A}} \mathrm{ord}_z(\mathrm{r}\circ f)}{t}\,\mathrm{d}\,t+S_f(r), \qquad 1.3.37$$

$$\sum_{i=4}^{6} N_f^{[3]}(r,H_i) = \frac{5}{4}\int_1^r \frac{\sum_{z\in E_{t,B}} \mathrm{ord}_z(\mathrm{r}\circ f)}{t}\,\mathrm{d}\,t + S_f(r). \tag{1.3.38}$$

It follows from 1.3.33 and 1.3.38 that

$$\sum_{i=4}^{6} N_f^{[3]}(r,H_i) = 5\,N_f(r,B) + S_f(r). \tag{1.3.39}$$

Owing to 1.3.35, the two inequalities 1.3.31 become

$$\sum_{i=1}^{3}\int_1^r \frac{\sum_{z\in E_{t,A,1}^{\geq 2}} \min\{\mathrm{ord}_z(h_i\circ f),3\}}{t}\,\mathrm{d}\,t = S_f(r)$$

$$\int_1^r \frac{\sum_{z\in E_{t,A,1}^{\geq 2}} \min_{1\leq i\leq 3}\mathrm{ord}_z(h_i\circ f)}{t}\,\mathrm{d}\,t = S_f(r).$$

Hence

$$\sum_{i=1}^{3} N_f^{[3]}(r,H_i) = \sum_{i=1}^{3}\int_1^r \frac{\sum_{z\in E_{t,A,1}^{1}} \min\{\mathrm{ord}_z(h_i\circ f),3\}}{t}\,\mathrm{d}\,t + S_f(r), \tag{1.3.40}$$

$$N_f(r,A) = \int_1^r \frac{\sum_{z\in E_{t,A,1}^{1}} \min_{1\leq i\leq 3}\mathrm{ord}_z(h_i\circ f)}{t}\,\mathrm{d}\,t + S_f(r). \tag{1.3.41}$$

Combining 1.3.32, 1.3.40, 1.3.41, we get

$$\sum_{i=1}^{3} N_f^{[3]}(r,H_i) = 5\,N_f(r,A) + S_f(r). \tag{1.3.42}$$

The equality 1.3.24 follows from 1.3.36, 1.3.39, 1.3.42.

Case **(b3.3)** can be treated by similar arguments as for case **(b3.2)**.

Next, we consider case **(b3.4)**. Assume that

$$\mathrm{ord}_z(t_{1,2}\circ f) \geq 2 \qquad (z\in f^{-1}(A)),$$

$$\mathrm{ord}_z(t_{4,5}\circ f) \geq 2 \qquad (z\in f^{-1}(B)).$$

By similar argument as in 1.3.28, we have $E_{t,A,1}^{\geq 2} = E_{t,A,2}^{\geq 2}$, $E_{t,B,4}^{\geq 2} = E_{t,B,5}^{\geq 2}$, $E_{t,A,1}^{1} = E_{t,A,2}^{1}$, $E_{t,B,4}^{1} = E_{t,B,5}^{1}$, which implies

$$\mathrm{ord}_z(r_{1,2}\circ f) \geq 2 \qquad (z\in E_{t,A,1}^{\geq 2}), \tag{1.3.43}$$

$$\mathrm{ord}_z(r_{4,5}\circ f) \geq 2 \qquad (z\in E_{t,B,4}^{\geq 2}), \tag{1.3.44}$$

$$\sum_{i=1}^{3} \min\{\mathrm{ord}_z(h_i\circ f),3\} \leq 5 \qquad (z\in E_{t,A,1}^{1}),$$

$$\sum_{i=4}^{6} \min\{\mathrm{ord}_z(h_i\circ f),3\} \leq 5 \qquad (z\in E_{t,B,4}^{1}).$$

Letting $\mathscr{S} = \{\mathsf{s} = r_{12}\, r_{4,5} = 0\}$ be the degenerate quadric $R_{1,2} \cup R_{4,5}$, we see that

$$\operatorname{ord}_z(\mathsf{s}\circ f) = \operatorname{ord}_z(r_{1,2}\circ f) + \operatorname{ord}_z(r_{4,5}\circ f) \geq 2 \qquad (z\in E_{t,A}\cup E_{t,B}).$$

Furthermore, by using 1.3.43 and 1.3.44, we have

$$\operatorname{ord}_z(\mathsf{s}\circ f) = \operatorname{ord}_z(r_{1,2}\circ f) + \operatorname{ord}_z(r_{4,5}\circ f) \geq 3 \qquad (z\in E_{t,A,1}^{\geq 2}\cup E_{t,B,4}^{\geq 2}).$$

Similarly as in the previous case, by using these inequalities together with 1.3.21 and 1.3.23, we receive

$$\sum_{i=1}^{3} \min\{\operatorname{ord}_z(h_i\circ f), 3\} \leq 6 = 6 \min_{1\leq i\leq 3} \operatorname{ord}_z(h_i\circ f) \leq \frac{6}{3}\operatorname{ord}_z(\mathsf{s}\circ f) \qquad (z\in E_{t,A,1}^{\geq 2}),$$

$$\sum_{i=1}^{3} \min\{\operatorname{ord}_z(h_i\circ f), 3\} \leq 5 = 5 \min_{1\leq i\leq 3} \operatorname{ord}_z(h_i\circ f) \leq \frac{5}{2}\operatorname{ord}_z(\mathsf{s}\circ f) \qquad (z\in E_{t,A,1}^{1}),$$

$$\sum_{i=4}^{6} \min\{\operatorname{ord}_z(h_i\circ f), 3\} \leq 6 = 6 \min_{4\leq i\leq 6} \operatorname{ord}_z(h_i\circ f) \leq \frac{6}{3}\operatorname{ord}_z(\mathsf{s}\circ f) \qquad (z\in E_{t,B,4}^{\geq 2}),$$

$$\sum_{i=4}^{6} \min\{\operatorname{ord}_z(h_i\circ f), 3\} \leq 5 = 5 \min_{4\leq i\leq 6} \operatorname{ord}_z(h_i\circ f) \leq \frac{5}{2}\operatorname{ord}_z(\mathsf{s}\circ f) \qquad (z\in E_{t,B,4}^{1}).$$

By taking the sum on disks and by integrating these four inequalities, we obtain

$$\begin{aligned}\sum_{i=1}^{3}\int_1^r \frac{\sum_{z\in E_{t,A,1}^{\geq 2}} \min\{\operatorname{ord}_z(h_i\circ f),3\}}{t}\,\mathrm{d}t &\leq 6\int_1^r \frac{\sum_{z\in E_{t,A,1}^{\geq 2}} \min_{1\leq i\leq 3}\operatorname{ord}_z(h_i\circ f)}{t}\,\mathrm{d}t\\ &\leq 2\int_1^r \frac{\sum_{z\in E_{t,A,1}^{\geq 2}} \operatorname{ord}_z(\mathsf{s}\circ f)}{t}\,\mathrm{d}t,\end{aligned}$$

$$\begin{aligned}\sum_{i=1}^{3}\int_1^r \frac{\sum_{z\in E_{t,A,1}^{1}} \min\{\operatorname{ord}_z(h_i\circ f),3\}}{t}\,\mathrm{d}t &\leq 5\int_1^r \frac{\sum_{z\in E_{t,A,1}^{1}} \min_{1\leq i\leq 3}\operatorname{ord}_z(h_i\circ f)}{t}\,\mathrm{d}t\\ &\leq \frac{5}{2}\int_1^r \frac{\sum_{z\in E_{t,A,1}^{1}} \operatorname{ord}_z(\mathsf{s}\circ f)}{t}\,\mathrm{d}t,\end{aligned}$$

$$\begin{aligned}\sum_{i=4}^{6}\int_1^r \frac{\sum_{z\in E_{t,B,4}^{\geq 2}} \min\{\operatorname{ord}_z(h_i\circ f),3\}}{t}\,\mathrm{d}t &\leq 6\int_1^r \frac{\sum_{z\in E_{t,B,4}^{\geq 2}} \min_{4\leq i\leq 6}\operatorname{ord}_z(h_i\circ f)}{t}\,\mathrm{d}t\\ &\leq 2\int_1^r \frac{\sum_{z\in E_{t,B,4}^{\geq 2}} \operatorname{ord}_z(\mathsf{s}\circ f)}{t}\,\mathrm{d}t,\end{aligned}$$

$$\begin{aligned}\sum_{i=4}^{6}\int_1^r \frac{\sum_{z\in E_{t,B,4}^{1}} \min\{\operatorname{ord}_z(h_i\circ f),3\}}{t}\,\mathrm{d}t &\leq 5\int_1^r \frac{\sum_{z\in E_{t,B,4}^{1}} \min_{4\leq i\leq 6}\operatorname{ord}_z(h_i\circ f)}{t}\,\mathrm{d}t\\ &\leq \frac{5}{2}\int_1^r \frac{\sum_{z\in E_{t,B,4}^{1}} \operatorname{ord}_z(\mathsf{s}\circ f)}{t}\,\mathrm{d}t.\end{aligned}$$

Now, we proceed similarly as above

$$\begin{aligned}
5\,T_f(r) &\leq \sum_{i=1}^{9} N_f^{[3]}(r,H_i)+S_f(r)\\
&= \sum_{i=1}^{3}\int_1^r \frac{\sum_{z\in E_{t,A}}\min\{\mathrm{ord}_z(h_i\circ f),3\}}{t}\,\mathrm{d}\,t+\sum_{i=4}^{6}\int_1^r \frac{\sum_{z\in E_{t,B}}\min\{\mathrm{ord}_z(h_i\circ f),3\}}{t}\,\mathrm{d}\,t+S_f(r)\\
&= \sum_{i=1}^{3}\Big(\int_1^r \frac{\sum_{z\in E^1_{t,A,1}}\min\{\mathrm{ord}_z(h_i\circ f),3\}}{t}\,\mathrm{d}\,t+\int_1^r \frac{\sum_{z\in E^{\geq 2}_{t,A,1}}\min\{\mathrm{ord}_z(h_i\circ f),3\}}{t}\,\mathrm{d}\,t\Big)\\
&+ \sum_{i=4}^{6}\Big(\int_1^r \frac{\sum_{z\in E^1_{t,B,1}}\min\{\mathrm{ord}_z(h_i\circ f),3\}}{t}\,\mathrm{d}\,t+\int_1^r \frac{\sum_{z\in E^{\geq 2}_{t,B,4}}\min\{\mathrm{ord}_z(h_i\circ f),3\}}{t}\,\mathrm{d}\,t\Big)+S_f(r)\\
&\leq \frac{5}{2}\Big(\int_1^r \frac{\sum_{z\in E_{t,A}}\mathrm{ord}_z(\mathsf{s}\circ f)}{t}\,\mathrm{d}\,t+\int_1^r \frac{\sum_{z\in E_{t,B}}\mathrm{ord}_z(\mathsf{s}\circ f)}{t}\,\mathrm{d}\,t\Big)\\
&+ \Big(2-\frac{5}{2}\Big)\Big(\int_1^r \frac{\sum_{z\in E^{\geq 2}_{t,A,1}}\mathrm{ord}_z(\mathsf{s}\circ f)}{t}\,\mathrm{d}\,t+\int_1^r \frac{\sum_{z\in E^{\geq 2}_{t,B,4}}\mathrm{ord}_z(\mathsf{s}\circ f)}{t}\,\mathrm{d}\,t\Big)+S_f(r)\\
&\leq \frac{5}{2}N_f(r,\mathscr{S})-\frac{1}{2}\Big(\int_1^r \frac{\sum_{z\in E^{\geq 2}_{t,A,1}}\mathrm{ord}_z(\mathsf{s}\circ f)}{t}\,\mathrm{d}\,t+\int_1^r \frac{\sum_{z\in E^{\geq 2}_{t,B,4}}\mathrm{ord}_z(\mathsf{s}\circ f)}{t}\,\mathrm{d}\,t\Big)+S_f(r)\\
&\leq 5\,T_f(r)-\frac{1}{2}\Big(\int_1^r \frac{\sum_{z\in E^{\geq 2}_{t,A,1}}\mathrm{ord}_z(\mathsf{s}\circ f)}{t}\,\mathrm{d}\,t+\int_1^r \frac{\sum_{z\in E^{\geq 2}_{t,B,4}}\mathrm{ord}_z(\mathsf{s}\circ f)}{t}\,\mathrm{d}\,t\Big)+S_f(r).
\end{aligned}$$

This implies

$$\begin{aligned}
\int_1^r \frac{\sum_{z\in E^{\geq 2}_{t,A,1}}\mathrm{ord}_z(\mathsf{s}\circ f)}{t}\,\mathrm{d}\,t &= S_f(r),\\
\int_1^r \frac{\sum_{z\in E^{\geq 2}_{t,B,4}}\mathrm{ord}_z(\mathsf{s}\circ f)}{t}\,\mathrm{d}\,t &= S_f(r),\\
\sum_{i=1}^{6} N_f^{[3]}(r,H_i) &= 5\,T_f(r)+S_f(r),\\
\sum_{i=1}^{3} N_f^{[3]}(r,H_i) &= \frac{5}{2}\int_1^r \frac{\sum_{z\in E_{t,A}}\mathrm{ord}_z(\mathsf{r}\circ f)}{t}\,\mathrm{d}\,t+S_f(r),\\
\sum_{i=4}^{6} N_f^{[3]}(r,H_i) &= \frac{5}{2}\int_1^r \frac{\sum_{z\in E_{t,B}}\mathrm{ord}_z(\mathsf{r}\circ f)}{t}\,\mathrm{d}\,t+S_f(r).
\end{aligned}$$

By proceeding similarly as in 1.3.42, we receive

$$\begin{aligned}
\sum_{i=1}^{3} N_f^{[3]}(r,H_i) &= 5\,N_f(r,A)+S_f(r),\\
\sum_{i=4}^{6} N_f^{[3]}(r,H_i) &= 5\,N_f(r,B)+S_f(r).
\end{aligned}$$

Hence, the equality 1.3.24 also holds in this case. Claim 1.3.1 is thus proved. □

Next, since f is of the form 1.3.20, one can find a plane $\mathscr{K}=\{\mathsf{k}=0\}$ passing through A and C such that

$$\mathrm{ord}_z(\mathsf{k}\circ f)\geq 2 \qquad (z\in f^{-1}(A)).$$

Let $\mathscr{B}_i = \{\mathsf{b}_i = 0\}$ be the plane containing the two lines AB, $H_i \cap \mathscr{K}$ $(1 \leq i \leq 3)$. Since b_i is a linear combination of h_i and k with nonzero coefficients, we have

$$\operatorname{ord}_z(\mathsf{b}_i \circ f) \geq 2 \qquad {\scriptstyle (z \in E_{t,A,i}^{\geq 2})},$$

which yields

$$\sum_{i=1}^{3} \operatorname{ord}_z(\mathsf{b}_i \circ f) \geq 4 \qquad {\scriptstyle (z \in \cup_{i=1}^{3} E_{t,A,i}^{\geq 2})}. \tag{1.3.45}$$

Let $\mathscr{C} = \{\mathsf{c} = \mathsf{b}_1 \mathsf{b}_2 \mathsf{b}_3 = 0\}$ be the degenerate cubic $\cup_{1 \leq i \leq 3} \mathscr{B}_i$. It follows from 1.3.21 and 1.3.45 that

$$\min_{1\leq i\leq 3} \operatorname{ord}_z(h_i \circ f) = 1 \leq \frac{1}{4} \sum_{i=1}^{3} \operatorname{ord}_z(\mathsf{b}_i \circ f) = \frac{1}{4} \operatorname{ord}_z(\mathsf{c} \circ f) \qquad {\scriptstyle (z \in \cup_{i=1}^{3} E_{t,A,i}^{\geq 2})},$$

$$\min_{1\leq i\leq 3} \operatorname{ord}_z(h_i \circ f) = 1 \leq \frac{1}{3} \sum_{i=1}^{3} \operatorname{ord}_z(\mathsf{b}_i \circ f) = \frac{1}{3} \operatorname{ord}_z(\mathsf{c} \circ f) \qquad {\scriptstyle (z \in E_{t,A} \setminus \cup_{i=1}^{3} E_{t,A,i}^{\geq 2})},$$

$$\min_{4\leq i\leq 6} \operatorname{ord}_z(h_i \circ f) = 1 \leq \frac{1}{3} \sum_{i=1}^{3} \operatorname{ord}_z(\mathsf{b}_i \circ f) = \frac{1}{3} \operatorname{ord}_z(\mathsf{c} \circ f) \qquad {\scriptstyle (z \in E_{t,B})}.$$

By taking the sum on disks and by integrating these inequalities,

$$\int_1^r \frac{\sum_{z \in \cup_{i=1}^{3} E_{t,A,i}^{\geq 2}} \min_{1\leq i\leq 3} \operatorname{ord}_z(h_i \circ f)}{t} \,\mathrm{d}t \leq \frac{1}{4} \int_1^r \frac{\sum_{z \in \cup_{i=1}^{3} E_{t,A,i}^{\geq 2}} \operatorname{ord}_z(\mathsf{c} \circ f)}{t} \,\mathrm{d}t,$$

$$\int_1^r \frac{\sum_{z \in E_{t,A} \setminus \cup_{i=1}^{3} E_{t,A,i}^{\geq 2}} \min_{1\leq i\leq 3} \operatorname{ord}_z(h_i \circ f)}{t} \,\mathrm{d}t \leq \frac{1}{3} \int_1^r \frac{\sum_{z \in E_{t,A} \setminus \cup_{i=1}^{3} E_{t,A,i}^{\geq 2}} \operatorname{ord}_z(\mathsf{c} \circ f)}{t} \,\mathrm{d}t,$$

$$N_f(r,B) = \int_1^r \frac{\sum_{z \in E_{t,B}} \min_{4\leq i\leq 6} \operatorname{ord}_z(h_i \circ f)}{t} \,\mathrm{d}t \leq \frac{1}{3} \int_1^r \frac{\sum_{z \in E_{t,B}} \operatorname{ord}_z(\mathsf{c} \circ f)}{t} \,\mathrm{d}t.$$

By using these inequalities together with 1.3.24, we receive

$$\begin{aligned}
5\,T_f(r) &= 5\,N_f(r,A) + 5\,N_f(r,B) + S_f(r) \\
&= 5\Big(\int_1^r \frac{\sum_{z \in \cup_{i=1}^{3} E_{t,A,i}^{\geq 2}} \min_{1\leq i\leq 3} \operatorname{ord}_z(h_i \circ f)}{t} \,\mathrm{d}t + \int_1^r \frac{\sum_{z \in E_{t,A} \setminus \cup_{i=1}^{3} E_{t,A,i}^{\geq 2}} \min_{1\leq i\leq 3} \operatorname{ord}_z(h_i \circ f)}{t} \,\mathrm{d}t\Big) \\
&\quad + 5\int_1^r \frac{\sum_{z \in E_{t,B}} \min_{4\leq i\leq 6} \operatorname{ord}_z(h_i \circ f)}{t} \,\mathrm{d}t + S_f(r) \\
&\leq \frac{5}{3}\Big(\int_1^r \frac{\sum_{z \in E_{t,A}} \operatorname{ord}_z(\mathsf{c} \circ f)}{t} \,\mathrm{d}t + \int_1^r \frac{\sum_{z \in E_{t,B}} \operatorname{ord}_z(\mathsf{c} \circ f)}{t} \,\mathrm{d}t\Big) \\
&\quad + \Big(\frac{5}{4} - \frac{5}{3}\Big) \int_1^r \frac{\sum_{z \in \cup_{i=1}^{3} E_{t,A,i}^{\geq 2}} \operatorname{ord}_z(\mathsf{c} \circ f)}{t} \,\mathrm{d}t + S_f(r) \\
&\leq \frac{5}{3} N_f(r,\mathscr{C}) - \frac{5}{12} \int_1^r \frac{\sum_{z \in \cup_{i=1}^{3} E_{t,A,i}^{\geq 2}} \operatorname{ord}_z(\mathsf{c} \circ f)}{t} \,\mathrm{d}t + S_f(r) \\
&\leq 5\,T_f(r) - \frac{5}{12} \int_1^r \frac{\sum_{z \in \cup_{i=1}^{3} E_{t,A,i}^{\geq 2}} \operatorname{ord}_z(\mathsf{c} \circ f)}{t} \,\mathrm{d}t + S_f(r).
\end{aligned}$$

This implies

$$\int_1^r \frac{\sum_{z \in \cup_{i=1}^{3} E_{t,A,i}^{\geq 2}} \operatorname{ord}_z(\mathsf{c} \circ f)}{t} \,\mathrm{d}t = S_f(r). \tag{1.3.46}$$

By using 1.3.23 and 1.3.45, we get

$$\sum_{i=1}^{3} \min\{\mathrm{ord}_z(h_i \circ f), 3\} \le 6 \le \frac{3}{2}\,\mathrm{ord}_z(\mathtt{c} \circ f) \qquad (z \in \cup_{i=1}^{3} E_{t,A,i}^{\ge 2}),$$

which yields

$$\int_1^r \frac{\sum_{z \in \cup_{i=1}^{3} E_{t,A,i}^{\ge 2}} \min_{1\le i\le 3} \mathrm{ord}_z(h_i \circ f)}{t}\, \mathrm{d}\,t \le \frac{3}{2} \int_1^r \frac{\sum_{z \in \cup_{i=1}^{3} E_{t,A,i}^{\ge 2}} \mathrm{ord}_z(\mathtt{c} \circ f)}{t}\, \mathrm{d}\,t$$
$$\text{[Use 1.3.46]} \qquad = S_f(r). \qquad (1.3.47)$$

Moreover, we also have

$$\sum_{i=1}^{3} \min\{\mathrm{ord}_z(h_i \circ f), 3\} = 3 = 3 \min_{1\le i\le 3} \mathrm{ord}_z(h_i \circ f) \qquad (z \in E_{t,A} \setminus \cup_{i=1}^{3} E_{t,A,i}^{\ge 2}),$$

which implies, by integrating, that

$$\sum_{i=1}^{3} \int_1^r \frac{\sum_{z \in E_{t,A} \setminus \cup_{i=1}^{3} E_{t,A,i}^{\ge 2}} \min\{\mathrm{ord}_z(h_i \circ f), 3\}}{t}\, \mathrm{d}\,t \le 3 \int_1^r \frac{\sum_{z \in E_{t,A} \setminus \cup_{i=1}^{3} E_{t,A,i}^{\ge 2}} \min_{1\le i\le 3} \mathrm{ord}_z(h_i \circ f)}{t}\, \mathrm{d}\,t. \qquad (1.3.48)$$

By combining 1.3.47 and 1.3.48, we get

$$\begin{aligned}
\sum_{i=1}^{3} N_f^{[3]}(r, H_i) &= \sum_{i=1}^{3} \int_1^r \frac{\sum_{z \in E_{t,A}} \min\{\mathrm{ord}_z(h_i \circ f), 3\}}{t}\, \mathrm{d}\,t \\
&= \sum_{i=1}^{3} \Big(\int_1^r \frac{\sum_{z \in E_{t,A} \setminus \cup_{i=1}^{3} E_{t,A,i}^{\ge 2}} \min\{\mathrm{ord}_z(h_i \circ f), 3\}}{t}\, \mathrm{d}\,t \\
&\quad + \int_1^r \frac{\sum_{z \in \cup_{i=1}^{3} E_{t,A,i}^{\ge 2}} \min\{\mathrm{ord}_z(h_i \circ f), 3\}}{t}\, \mathrm{d}\,t \Big) + S_f(r) \\
&\le 3 \int_1^r \frac{\sum_{z \in E_{t,A} \setminus \cup_{i=1}^{3} E_{t,A,i}^{\ge 2}} \min_{1\le i\le 3} \mathrm{ord}_z(h_i \circ f)}{t}\, \mathrm{d}\,t + S_f(r) \\
&\le 3\, N_f(r, A) + S_f(r).
\end{aligned}$$

By symmetry, we also have

$$\sum_{i=4}^{6} N_f^{[3]}(r, H_i) \le 3\, N_f(r, B) + S_f(r).$$

Hence we can rewrite 1.3.27 to get a contradiction:

$$\begin{aligned}
5\, T_f(r) &\le \sum_{i=1}^{9} N_f^{[3]}(r, H_i) + S_f(r) \\
&\le 3\, N_f(r, A) + 3\, N_f(r, B) + S_f(r) \\
&\le 3\, N_f(r, \mathscr{B}) + S_f(r) \\
&\le 3\, T_f(r) + S_f(r).
\end{aligned}$$

□

In $\mathbb{P}^4(\mathbb{C})$, by the generic condition for the family of hyperplanes $\{H_i\}_{1\le i\le q}$, when $q \ge 10$, we see that, for all three disjoint subsets I, J, K of $\{1,\dots,q\}$ with $|I|\ge 2$, $|J|\ge 2$, $|I|+|J|=6$, $|K|=4$, the diagonal hyperplane H_{IJ} does not contain the point $\cap_{k\in K} H_k$.

Lemma 1.3.3. *In $\mathbb{P}^4(\mathbb{C})$, all complements of the form* 1.3.2 *are hyperbolic if $m=1$.*

Proof. We can assume that $A_{1,4}$ is a set consisting of one element in

$$\big(\cup_{1\le i_1<i_2\le 10}(H_{i_1}\cap H_{i_2})^*\big)\bigcup\big(\cup_{1\le i_1<i_2<i_3\le 10} H_{i_1}\cap H_{i_2}\cap H_{i_3}\big)^*\bigcup\big(\cup_{1\le i_1<i_2<i_3<i_4\le 10} H_{i_1}\cap H_{i_2}\cap H_{i_3}\cap H_{i_4}\big).$$

Suppose to the contrary that there is an entire curve $f\colon \mathbb{C}\to\mathbb{P}^4(\mathbb{C})\setminus\big(\cup_{i=1}^{10} H_i\setminus A_{1,4}\big)$. If $A_{1,4}$ is not a set of a point, then f avoids at least seven hyperplanes. By Theorem 1.2.4, its image is contained in a line L and we can continue to analyze the position of L with respect to $\cup_{i=1}^{10} H_i\setminus A_{1,4}$ to get a contradiction. Consider the remaining case where $A_{1,4}$ consists of a point, say $\cap_{i=1}^{4} H_i$. By Theorem 1.2.3, the curve f lands in some diagonal hyperplane of the family $\{H_i\}_{5\le i\le 10}$, which does not contain the point $\cap_{i=1}^{4} H_i$ by the generic condition. Hence, f must avoid all H_i $(1\le i\le 10)$, which is impossible by Theorem 1.2.5. □

1.3.1 Stability of intersections

We will also invoke the following known complex analysis fact.

Stability of intersections. *Let X be a complex manifold and let $H\subset X$ be an analytic hypersurface. Suppose that a sequence (f_n) of entire curves in X converges toward an entire curve f. If $f(\mathbb{C})$ is not contained in H, then we have*

$$f(\mathbb{C})\cap H\subset \lim f_n(\mathbb{C})\cap H.$$

1.4 Proof of the Main Theorem

We keep the notation of the previous section. Let S be a hypersurface of degree $2n$, which is in general position with respect to the family $\{H_i\}_{1\le i\le 2n}$. We would like to determine what conditions S should satisfy for Σ_ϵ to be hyperbolic. Suppose that Σ_{ϵ_k} is not hyperbolic for a sequence (ϵ_k) converging to 0. Then we can find entire curves $f_{\epsilon_k}\colon \mathbb{C}\to\Sigma_{\epsilon_k}$. By the Brody lemma, after reparameterization and extraction, we may assume that the sequence (f_{ϵ_k}) converges to an entire curve $f\colon\mathbb{C}\to\cup_{i=1}^{2n}H_i$. The curve $f(\mathbb{C})$ lands inside some hyperplane H_i. Moreover, it cannot land inside any subspace of dimension 1 (a line). Indeed, if $f(\mathbb{C})\subset\cap_{i\in I}H_i$ for some subset I of the index set $\mathbf{Q}=\{1,\dots,2n\}$ having cardinality $n-1$, then for all $j\in\mathbf{Q}\setminus I$, by stability of intersections, one has

$$f(\mathbb{C})\cap H_j\subset\lim f_{\epsilon_k}(\mathbb{C})\cap H_j\subset\lim\Sigma_{\epsilon_k}\cap H_j\subset S\cap H_j.$$

Thus $f(\mathbb{C})$ and H_j have empty intersection by the general position. Hence the curve $f(\mathbb{C})$ lands in

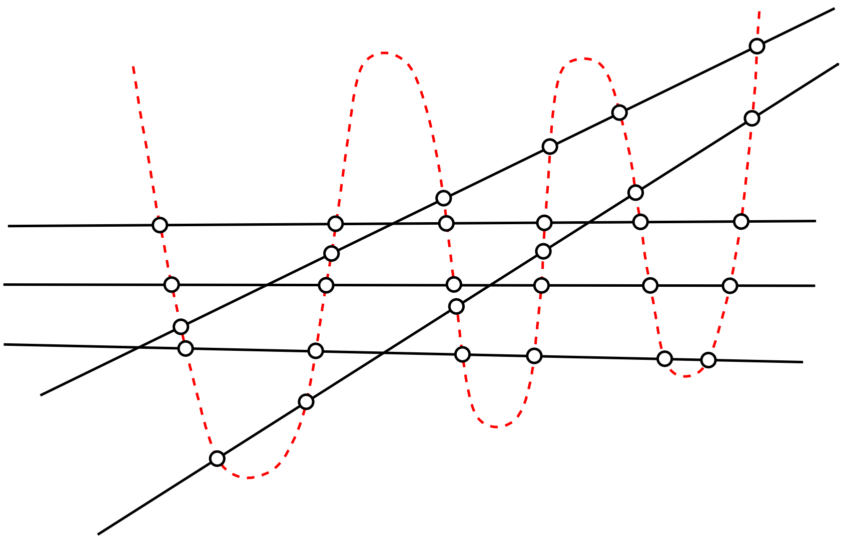
$$\cap_{i\in I}H_i\setminus\big(\cup_{j\in\mathbf{Q}\setminus I}H_j\big).$$

This is a contradiction, because the complement of $n+1$ ($n \geq 3$) points in a line is hyperbolic by Picard's theorem.

Now, let I be the largest subset of $\mathbf{Q}$ such that the curve $f(\mathbb{C})$ lands in $\cap_{i\in I} H_i$. We have $|I| \leq n-2$. By stability of intersections, $f(\mathbb{C}) \cap H_j$ is contained in S for all $l \in \mathbf{Q} \setminus I$. Therefore the curve $f(\mathbb{C})$ lands in

$$\cap_{i\in I} H_i \setminus \left(\cup_{j\in \mathbf{Q}\setminus I} H_j \setminus S\right). \tag{1.4.1}$$

So, the problem reduces to finding a hypersurface S of degree $2n$ such that all complements of the form 1.4.1 are hyperbolic, where I is a subset of $\mathbf{Q}$ of cardinality at most $n-2$. For example when $n=3$ ([Duv14]), we need to find a sextic curve S such that all complements of the form $H_i \setminus \left(\cup_{j\neq i} H_j \setminus S\right)$ are hyperbolic. In this case, we have the complement of five lines in the hyperplane H_i on which all points of intersection with S are deleted.



We will construct such S by deformation, step by step. For $2 \leq l \leq n-1$, let Δ_l be a finite collection of subspaces of dimension $n-l$, in the sense of section 1.3. Let $D_l \not\in \Delta_l$ be another subspace of dimension $n-l$, defined as $D_l = \cap_{i\in I_{D_l}} H_i$. For a hypersurface $S = \{s=0\}$ in general position with respect to the family $\{H_i\}_{1\leq i\leq 2n}$ and $\epsilon \neq 0$, we set

$$S_\epsilon = \left\{\epsilon s + \Pi_{i\notin I_{D_l}} h_i^{n_i} = 0\right\},$$

where $n_i \geq 1$ are chosen (freely) so that $\sum_{i\notin I_{D_l}} n_i = 2n$. It is not hard to see that the hypersurface S_ϵ is also in general position with respect to the family $\{H_i\}_{1\leq i\leq 2n}$. We denote by $\overline{\Delta}_l$ the family of all subspaces of dimension $n-l$ ($2 \leq l \leq n$) with the convention $\overline{\Delta}_n = \emptyset$.

Lemma 1.4.1. *Assume that all complements of the form*

$$\cap_{i\in I} H_i \setminus \left(\cup_{j\in J} H_j \setminus (((\Delta_l \cup \overline{\Delta}_{l+1}) \cap S) \cup A_{m,n-|I|})\right) \tag{1.4.2}$$

are hyperbolic where I, J are two disjoint subsets of $\{1,\ldots,2n\}$ such that $|I| \leq n-2$, $|J| + 2|I| \geq 2n+1$ and $m \leq |J|+2|I|-(2n+1)$. Here, $A_{m,n-|I|}$ is a set of at most m star-subspaces *coming from the family of hyperplanes* $\{\cap_{i\in I}H_i \cap H_j\}_{j\in J}$ *in* $\cap_{i\in I}H_i \cong \mathbb{P}^{n-|I|}(\mathbb{C})$. *Then all complements of the form*

$$\cap_{i\in I} H_i \setminus \big(\cup_{j\in J} H_j \setminus (((\Delta_l \cup D_l \cup \overline{\Delta}_{l+1}) \cap S_\epsilon) \cup A_{m,n-|I|})\big) \qquad 1.4.3$$

are also hyperbolic for sufficiently small $\epsilon \neq 0$.

Proof. By the definition of S_ϵ, we see that $S_\epsilon \cap \big(\cap_{m\in M} H_m\big) = S \cap \big(\cap_{m\in M} H_m\big)$ when $M \cap (\mathbf{Q} \setminus I_{D_l}) \neq \emptyset$, hence

$$(\Delta_l \cup D_l \cup \overline{\Delta}_{l+1}) \cap S_\epsilon = ((\Delta_l \cup \overline{\Delta}_{l+1}) \cap S) \cup (D_l \cap S_\epsilon).$$

When $|I| \geq l$, using this, we observe that two complements 1.4.2, 1.4.3 coincide.

Assume therefore $|I| \leq l-1$. Suppose by contradiction that there exists a sequence of entire curves $(f_{\epsilon_k}(\mathbb{C}))_k$, $\epsilon_k \to 0$ contained in the complement

$$\cap_{i\in I} H_i \setminus \big(\cup_{j\in J} H_j \setminus (((\Delta_l \cup D_l \cup \overline{\Delta}_{l+1}) \cap S_{\epsilon_k}) \cup A_{m,n-|I|})\big).$$

By the Brody Lemma, we may assume that (f_{ϵ_k}) converges to an entire curve $f(\mathbb{C}) \subset \cap_{i\in I} H_i$. Our aim is to prove that the curve $f(\mathbb{C})$ lands in some complement of the form 1.4.2. Let $\cap_{k\in K} H_k$ be the smallest subspace containing $f(\mathbb{C})$. It is clear that $K \supset I$. Take an index j in $J \setminus K$. By stability of intersections, one has

$$\begin{aligned} f(\mathbb{C}) \cap H_j &\subset \lim f_{\epsilon_k}(\mathbb{C}) \cap H_j \\ &\subset ((\Delta_l \cup \overline{\Delta}_{l+1}) \cap S) \cup A_{m,n-|I|} \cup \lim(D_l \cap S_{\epsilon_k}). \end{aligned} \qquad 1.4.4$$

If the index j does not belong to I_{D_l}, then $H_j \cap D_l \cap S_{\epsilon_k} \subset \overline{\Delta}_{l+1} \cap S$. It follows from 1.4.4 that

$$f(\mathbb{C}) \cap H_j \subset ((\Delta_l \cup \overline{\Delta}_{l+1}) \cap S) \cup A_{m,n-|I|}. \qquad 1.4.5$$

If the index j belongs to I_{D_l}, noting that $\lim(D_l \cap S_{\epsilon_k})$ is contained in $D_l \cap (\cup_{i\notin I_{D_l}} H_i)$, again from 1.4.4, one has

$$f(\mathbb{C}) \cap H_j \subset ((\Delta_l \cup \overline{\Delta}_{l+1}) \cap S) \cup A_{m,n-|I|} \cup (D_l \cap (\cup_{i\notin I_{D_l}} H_i)). \qquad 1.4.6$$

Assume first that $K = I$. We claim that 1.4.5 also holds when the index $j \in J \setminus I$ belonging to I_{D_l}. Indeed, for the supplementary part in 1.4.6, we have

$$f(\mathbb{C}) \cap H_j \cap \big(D_l \cup_{i\notin I_{D_l}} H_i\big) \subset \cup_{i\notin I_{D_l}} (f(\mathbb{C}) \cap H_j \cap H_i),$$

so that 1.4.5 applies here to all $i \notin I_{D_l}$. Hence, the curve $f(\mathbb{C})$ lands inside

$$\cap_{i\in I} H_i \setminus \big(\cup_{j\in J} H_j \setminus (((\Delta_l \cup \overline{\Delta}_{l+1}) \cap S) \cup A_{m,n-|I|})\big),$$

contradicting the hypothesis.

Assume now that I is a proper subset of K. Let us set

$$A_{m,n-|I|,K} = \{X \cap (\cap_{k\in K} H_k) | X \in A_{m,n-|I|}\}.$$

This set consists of star-subspaces of $\cap_{k\in K} H_k \cong \mathbb{P}^{n-|K|}(\mathbb{C})$. Let $B_{m,K}$ be the subset of $A_{m,n-|I|,K}$ containing all star-subspaces of dimension $n-|K|-1$ (i.e., of codimension 1 in $\cap_{k\in K} H_k$), and let $C_{m,K}$ be the remaining part. A star-subspace in $B_{m,K}$ is of the form $(\cap_{k\in K} H_k \cap H_j)^*$ for some index $j \in J\setminus K$. Then let R denote the set of such indices j, so that

$$|R| = |B_{m,K}|.$$

We consider two cases separately, depending on the dimension of the subspace $Y = \cap_{k\in K} H_k \cap D_l$.

Case 1. Y is a subspace of dimension $n-|K|-1$. In this case, Y is of the form $(\cap_{k\in K} H_k)\cap H_y$ for some index y in I_{D_l}. It follows from 1.4.4, 1.4.5, 1.4.6 that the curve $f(\mathbb{C})$ lands inside the set

$$\cap_{k\in K} H_k \setminus \big(\cup_{j\in (J\setminus K)\setminus (R\cup\{y\})} H_j \setminus (((\Delta_l \cup \overline{\Delta}_{l+1})\cap S)\cup C_{m,K})\big).$$

Now we need to show that this set is of the form 1.4.2. First, we verify the corresponding required inequality between cardinalities

$$\begin{aligned}|(J\setminus K)\setminus (R\cup\{y\})| &\geq |J\setminus K| - |B_{m,K}| - 1\\ &\geq |J| - |J\cap K| - \big(|J| + 2|I| - 2n - 1 - |C_{m,K}|\big) - 1\\ &= 2(n-|K|) + |C_{m,K}| + 2|K\setminus I| - |J\cap K|\\ &\geq 2(n-|K|) + 1 + |C_{m,K}|,\end{aligned}$$

where the last inequality holds because I and J are two disjoint sets and I is a proper subset of K. Secondly, we verify that the set K is of cardinality at most $n-2$. Indeed, if $|K| = n-1$, then since S is in general position with respect to $\{H_i\}_{1\leq i\leq 2n}$, we see that

$$\cap_{k\in K} H_k \setminus\big(\cup_{j\in (J\setminus K)\setminus (R\cup\{y\})} H_j \setminus (((\Delta_l\cup\overline{\Delta}_{l+1})\cap S)\cup C_{m,K})\big) = \cap_{k\in K} H_k \setminus \big(\cup_{j\in (J\setminus K)\setminus (R\cup\{y\})} H_j \setminus C_{m,K}\big).$$

Owing to the inequality $|(J\setminus K)\setminus (R\cup\{y\})| \geq 3 + |C_{m,K}|$, the curve f lands in a complement of at least three points in a line. By Picard's theorem, f is constant, which is a contradiction.

Case 2. Y is a subspace of dimension at most $n-|K|-2$. In this case, the curve $f(\mathbb{C})$ lands inside

$$\cap_{k\in K} H_k \setminus \big(\cup_{j\in (J\setminus K)\setminus R} H_j \setminus (((\Delta_l\cup\overline{\Delta}_{l+1})\cap S)\cup C_{m,K}\cup Y^*)\big).$$

This set is of the form 1.4.2 since

$$|\{j \in (J\setminus K)\setminus R\}| \geq 2(n-|K|) + 1 + |C_{m,K}\cup Y^*|,$$

which also implies $|K| \leq n-2$ by similar argument as in **Case 1**.

The lemma is thus proved. □

End of proof of the Main Theorem. We now come back to the proof of the Main Theorem. Keep the notation as in Lemma 1.4.1. We claim that $\{\cap_{i\in I} H_i \cap H_j\}_{j\in J}$ is also a family of generic hyperplanes in the projective space $\cap_{i\in I} H_i \cong \mathbb{P}^{n-|I|}(\mathbb{C})$. Indeed, let $\mathscr{I}$, $\mathscr{J}$, $\mathscr{J}_1,\ldots,\mathscr{J}_k$ be disjoint subsets of J such that $|\mathscr{I}|, |\mathscr{J}_i| \geq 2$, $|\mathscr{I}| + |\mathscr{J}_i| = (n-|I|)+2$,

$1 \le i \le k$ and let $\{i_1, \ldots, i_l\}$ be a subset of $\mathscr{I}$. Let us set $\mathbf{I} = I \cup \mathscr{I}$; then the intersection between the $|\mathscr{J}|$ hyperplanes $H_j, j \in \mathscr{J}$, the k diagonal hyperplanes $H_{\mathbf{I}\mathscr{J}_1}, \ldots, H_{\mathbf{I}\mathscr{J}_k}$, and the $|I| + l$ hyperplanes H_i $(i \in I)$, $H_{i_1}, \ldots, H_{i_l}$ is a linear subspace of codimension $\min\{k + |I| + l, |\mathbf{I}|\} + |\mathscr{J}|$, with the convention that when $\min\{k + |I| + l, |\mathbf{I}|\} + |\mathscr{J}| > n$, this intersection is empty. Since

$$\min\{k + |I| + l, |\mathbf{I}|\} + |\mathscr{J}| = \min\{k + l, |\mathscr{I}|\} + |I| + |\mathscr{J}|$$

we deduce that in the projective space $\cap_{i \in I} H_i$, the intersection between the $|\mathscr{J}|$ hyperplanes $H_j, j \in \mathscr{J}$, the k diagonal hyperplanes $H_{\mathscr{I}\mathscr{J}_1}, \ldots, H_{\mathscr{I}\mathscr{J}_k}$, and the l hyperplanes $H_{i_1}, \ldots, H_{i_l}$ is a linear subspace of codimension $\min\{k + l, |\mathscr{I}|\} + |\mathscr{J}|$, with the convention that when $\min\{k + l, |\mathscr{I}|\} + |\mathscr{J}| > n - |I|$, this intersection is empty.

Starting point of the process by deformation: We start with the hyperbolicity of all complements of the forms

$$\cap_{i \in I} H_i \setminus \big(\cup_{j \in J} H_j \setminus A_{m,n-|I|} \big),$$

where I, J, $A_{m,n-|I|}$ are as in Lemma 1.4.1. More precisely,

- when $n = 3$, we start with the hyperbolicity of all complements $H_i \setminus \big(\cup_{j \neq i} H_j \big)$, which follows from Theorem 1.2.5 in $\mathbb{P}^2(\mathbb{C})$;
- when $n = 4$, we start with the hyperbolicity of all complements

$$H_i \setminus \big(\cup_{j \neq i} H_j \big),$$
$$\cap_{i \in I} H_i \setminus \big(\cup_{j \in J} H_j \setminus A_{1,2} \big) \qquad (|I| = 2, 5 + |A_{1,2}| \le |J| \le 6),$$

  which follows from Theorem 1.2.5 in $\mathbb{P}^3(\mathbb{C})$ and Lemma 1.3.1 for $m = 1$;
- when $n = 5$, we start with the hyperbolicity of all complements

$$H_i \setminus \big(\cup_{j \neq i} H_j \big),$$
$$\cap_{i \in I} H_i \setminus \big(\cup_{j \in J} H_j \setminus A_{1,3} \big) \qquad (|I| = 2, 7 + |A_{1,3}| \le |J| \le 8),$$
$$\cap_{i \in I} H_i \setminus \big(\cup_{j \in J} H_j \setminus A_{2,2} \big) \qquad (|I| = 3, 5 + |A_{2,2}| \le |J| \le 7),$$

  which follows from Theorem 1.2.5 in $\mathbb{P}^4(\mathbb{C})$, Lemma 1.3.2 for $m = 1$, and Lemma 1.3.1 for $m = 2$;
- when $n = 6$, we start with the hyperbolicity of all complements

$$H_i \setminus \big(\cup_{j \neq i} H_j \big),$$
$$\cap_{i \in I} H_i \setminus \big(\cup_{j \in J} H_j \setminus A_{1,4} \big) \qquad (|I| = 2, 9 + |A_{1,4}| \le |J| \le 10),$$
$$\cap_{i \in I} H_i \setminus \big(\cup_{j \in J} H_j \setminus A_{2,3} \big) \qquad (|I| = 3, 7 + |A_{2,3}| \le |J| \le 9),$$
$$\cap_{i \in I} H_i \setminus \big(\cup_{j \in J} H_j \setminus A_{3,2} \big) \qquad (|I| = 4, 5 + |A_{3,2}| \le |J| \le 8),$$

  which follows from Theorem 1.2.5 in $\mathbb{P}^5(\mathbb{C})$, Lemma 1.3.3 for $m = 1$, Lemma 1.3.2 for $m = 2$, and Lemma 1.3.1 for $m = 3$.

Details of the process by deformation: In the first step, we apply inductively Lemma 1.4.1 for $l = n - 1$ and get at the end a hypersurface S_1 such that all complements of the forms

$$\cap_{i \in I} H_i \setminus \big(\cup_{j \in J} H_j \setminus (S_1 \cup A_{m,n-|I|}) \big) \qquad (|I| = n-2),$$

$$\cap_{i\in I} H_i \setminus \big(\cup_{j\in J} H_j \setminus ((\overline{\Delta}_{n-1} \cap S_1) \cup A_{m,n-|I|})\big) \qquad {\scriptstyle (|I|\leq n-3)}$$

are hyperbolic. Considering this as the starting point of the second step, we apply inductively Lemma 1.4.1 for $l = n-2$. Continuing this process, we get at the end of the $(n-2)$th step a hypersurface $S = S_{n-2}$ satisfying the required properties. □

1.5 Some discussion

Actually, our method works for a family of at least $2n$ generic hyperplanes in $\mathbb{P}^n(\mathbb{C})$. We hope that the Main Theorem is true for all $n \geq 3$. As we saw above, the problem reduces to proving the following conjecture.

Conjecture. *All complements of the form* 1.3.2 *are hyperbolic.*

We already know it to be true for $n = 2$, since Lemma 1.3.1 holds generally, without restriction on m.

Lemma 1.5.1. *In* $\mathbb{P}^2(\mathbb{C})$, *all complements of the form* 1.3.2 *are hyperbolic*

Proof. Assume now $m \geq 4$ and $A_{m,2} = \{A_1, \ldots, A_m\}$, where $A_i = H_{i_1} \cap H_{i_2}$ $(1 \leq i \leq m)$. We denote by I the index set $\{i_j\colon 1 \leq i \leq m, 1 \leq j \leq 2\}$. Suppose to the contrary that there exists an entire curve $f\colon \mathbb{C} \to \mathbb{P}^2(\mathbb{C}) \setminus (\cup_{i=1}^{5+m} H_i \setminus A_{m,2})$. By the generic condition, we can assume that f is linearly nondegenerate. By similar arguments as in Lemma 1.3.1 (cf. 1.3.6), we have

$$\sum_{i\in I} N_f^{[2]}(r, H_i) \leq 3 \sum_{i=1}^{m} N_f(r, A_i).$$

Let $\mathscr{C}_m = \{\mathsf{c}_m = 0\}$ be an algebraic curve in $\mathbb{P}^2(\mathbb{C})$ of degree d passing through all points in $A_{m,2}$ with multiplicity at least k which does not contain the curve $f(\mathbb{C})$. Starting from the inequality

$$\min_{1\leq j\leq 2} \operatorname{ord}_z(h_{i_j} \circ f) \leq \frac{1}{k} \operatorname{ord}_z(\mathsf{c}_m \circ f) \qquad {\scriptstyle (z\in f^{-1}(A_i))}$$

and proceeding as in 1.3.8, we get

$$\sum_{i=1}^{m} N_f(r, A_i) \leq \frac{1}{k} N_f(r, \mathscr{C}_m).$$

We may then proceed similarly as in 1.3.9

$$\begin{aligned}(m+2)\, T_f(r) &\leq \sum_{i=1}^{5+m} N_f^{[2]}(r, H_i) + S_f(r)\\ &\leq 3 \sum_{i=1}^{m} N_f(r, A_i) + S_f(r)\\ &\leq \frac{3}{k} N_f(r, \mathscr{C}_m) + S_f(r)\\ &\leq \frac{3d}{k}\, T_f(r) + S_f(r).\end{aligned} \qquad 1.5.1$$

When $m \geq 5$, the following claim yields a concluding contradiction.

Claim 1.5.1. *If $m \geq 5$, we can find some curve $\mathscr{C}_m$ which does not contain $f(\mathbb{C})$ such that*

$$k > \frac{3d}{m+2}. \tag{1.5.2}$$

Indeed, the degree of freedom for the choice of a curve of degree d is

$$\frac{(d+1)(d+2)}{2} - 1.$$

We want $\mathscr{C}_m$ to pass through all points in $A_{m,2}$ with multiplicity at least k. The number of equations (with the coefficients of $\mathscr{C}_m$ as unknowns) for this is not greater than

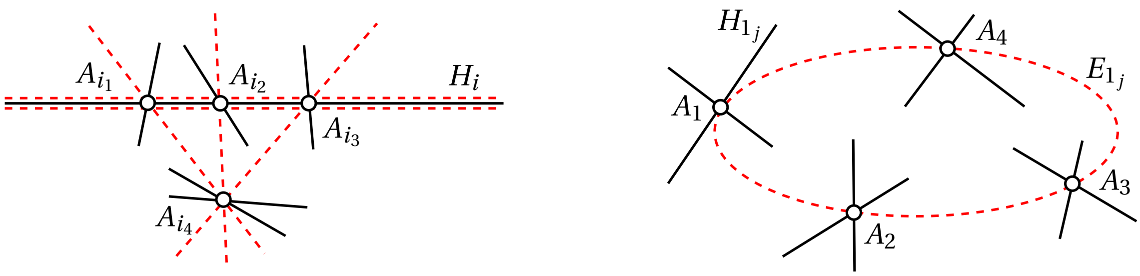
$$m\frac{k(k+1)}{2}.$$

Thus, for the existence of $\mathscr{C}_m$, it is necessary that

$$\frac{(d+1)(d+2)}{2} - 1 > m\frac{k(k+1)}{2}. \tag{1.5.3}$$

We try to find two natural numbers k, d satisfying 1.5.2 and 1.5.3. This can be done by choosing $d = (m+2)M$ and $k = 3M+1$ for large enough M. Using the remaining freedom in the choice of $\mathscr{C}_m$, we can choose it not containing $f(\mathbb{C})$, which proves the claim.

Next, we consider the remaining case where $m = 4$.



If there exists a collinear subset $\{A_{i_1}, A_{i_2}, A_{i_3}\}$ of $A_{4,2}$, then by the generic condition, it must be contained in some line H_i. Let A_{i_4} be the remaining point of the set $A_{4,2}$ and let $\mathscr{C}_4$ be the degenerate quintic consisting of the three lines $A_{i_j}A_{i_4}$ $(1 \leq j \leq 3)$ and of the line H_i with multiplicity 2. Since $\mathscr{C}_4$ passes through all points in $A_{4,2}$ with multiplicity at least 3, the inequality 1.5.2 is satisfied. By using 1.5.1, we get a contradiction.

Now we assume that any subset of $A_{4,2}$ containing three points is not collinear. Let $E_{i_j} = \{\mathsf{e}_{i_j} = 0\}$ $(1 \leq i \leq 4,\ 1 \leq j \leq 2)$ be the eight conics passing through all points of $A_{4,2}$, tangent to the line H_{i_j} at the point A_i $(1 \leq i \leq 4,\ 1 \leq j \leq 2)$. Let $\mathscr{E} = \{\mathsf{e} = 0\}$ be the degenerate curve of degree 16 consisting of all these E_{i_j}. We claim that f does not land in $\mathscr{E}$. Otherwise, it lands in some conic E_{i_j}. Since the number of intersection points between E_{i_j} and $\cup_{i=1}^{9} H_i \setminus A_{4,2}$ is > 3 and since any complement of three distinct points in an irreducible curve is hyperbolic, f must be constant, which is a contradiction.

Letting z be a point in $f^{-1}(A_i)$, we have

$$\mathrm{ord}_z(\mathrm{e}_{i_j} \circ f) \geq 1 \qquad (1 \leq i \leq 4, 1 \leq j \leq 2).$$

By the construction of E_{i_j}, if $\mathrm{ord}_z(h_{i_j} \circ f) \geq 2$ for some $1 \leq j \leq 2$, then we also have $\mathrm{ord}_z(\mathrm{e}_{i_j} \circ f) \geq 2$. Furthermore, if $\mathrm{ord}_z(h_{i_j} \circ f) \geq 2$ for all $1 \leq j \leq 2$, then $\mathrm{ord}_z(\mathrm{e}_{i_j} \circ f) \geq 2$ for all $1 \leq i \leq 4$, $1 \leq i \leq 2$. Thus, the following inequality holds:

$$\begin{aligned}\min\{\mathrm{ord}_z(h_{i_1}) \circ f, 2\} + \min\{\mathrm{ord}_z(h_{i_2}) \circ f, 2\} &\leq \frac{1}{3}\sum_{i=1}^{4}\sum_{j=1}^{2} \mathrm{ord}_z(\mathrm{e}_{i_j} \circ f) \\ &= \frac{1}{3}\mathrm{ord}_z(\mathrm{e} \circ f) \qquad (z \in f^{-1}(A_i)).\end{aligned}$$

This implies

$$\sum_{i \in I} N_f^{[2]}(r, H_i) \leq \frac{1}{3} N_f(r, \mathscr{E}).$$

We proceed similarly as before to derive a contradiction

$$\begin{aligned}6\,T_f(r) &\leq \sum_{i=1}^{9} N_f^{[2]}(r, H_i) + S_f(r) \\ &\leq \frac{1}{3} N_f(r, \mathscr{E}) + S_f(r) \\ &\leq \frac{16}{3} T_f(r) + S_f(r).\end{aligned}$$

Lemma 1.5.1 is thus proved. □

1.6 Examples of hyperbolic hypersurfaces in arbitrary dimension

The first examples of hyperbolic hypersurfaces in any dimension $n-1 \geq 3$ were discovered by Masuda and Noguchi [MN96], with high degree. Improving this result, such examples with lower degree asymptotic were given by Siu and Yeung [SY97] with $d(n) = 16(n-1)^2$, and by Shiffman and Zaidenberg [SZ02b] with $d(n) = 4(n-1)^2$.

Adapting the technique in the previous part, we improve the result of Shiffman and Zaidenberg [SZ02b] by proving that a small deformation of a union of $q \geq (\frac{n+2}{2})^2$ hyperplanes in general position in $\mathbb{P}^n(\mathbb{C})$ is hyperbolic.

Theorem II. *Let $\{H_i\}_{1 \leq i \leq q}$ be a family of $q \geq (\frac{n+2}{2})^2$ hyperplanes in general position in $\mathbb{P}^n(\mathbb{C})$, where $H_i = \{h_i = 0\}$. Then there exists a hypersurface $S = \{s = 0\}$ of degree q in general position with respect to $\{H_i\}_{1 \leq i \leq q}$ such that the hypersurface*

$$\Sigma_\epsilon = \big\{\epsilon s + \Pi_{i=1}^{q} h_i = 0\big\}$$

is hyperbolic for sufficiently small complex $\epsilon \neq 0$.

By similar arguments as in the first part, the problem reduces to finding a hypersurface S of degree q such that all complements of the form

$$\cap_{i\in I} H_i \setminus \big(\cup_{j\in \mathbf{Q}\setminus I} H_j \setminus S\big). \tag{1.6.1}$$

are hyperbolic, where I is an arbitrary subset of $\mathbf{Q}$ having cardinality at most $n-2$.

Starting point of the deformation process. Let $\{H_i\}_{1\leq i\leq q}$ be a family of hyperplanes in general position in $\mathbb{P}^n(\mathbb{C})$. For some integer $0\leq k\leq n-1$ and some subset $I_k=\{i_1,\dots,i_{n-k}\}$ of the index set $\{1,\dots,q\}$ having cardinality $n-k$, the linear subspace $P_{k,I_k}=\cap_{i\in I_k} H_i\simeq \mathbb{P}^k(\mathbb{C})$ will be called a *subspace of dimension k*. We will denote by P^*_{k,I_k} the complement $P_{k,I_k}\setminus\big(\cup_{i\not\in I_k} H_i\big)$, which we will call a *star-subspace of dimension k*. The process of constructing S by deformation will start with the following result, which is an application of Theorem 1.2.5.

Starting Lemma. *Let $\{H_i\}_{1\leq i\leq q}$ be a family of $q\geq (\frac{n+2}{2})^2$ hyperplanes in general position in $\mathbb{P}^n(\mathbb{C})$. Let I and J be two disjoint subsets of the index set $\{1,\dots,q\}$ such that $1\leq |I|\leq n-2$, and $|J|=q+m+1-2|I|$ with some $0\leq m\leq |I|-1$. Then all complements of the form*

$$\cap_{i\in I} H_i \setminus \big(\cup_{j\in J} H_j \setminus A_{m,n-|I|}\big) \tag{1.6.2}$$

are hyperbolic, where $A_{m,n-|I|}$ is a set of at most m star-subspaces coming from the family of hyperplanes $\{\cap_{i\in I} H_i\cap H_j\}_{j\in J}$ in the $(n-|I|)$-dimensional projective space $\cap_{i\in I}H_i\cong \mathbb{P}^{n-|I|}(\mathbb{C})$.

Proof. Suppose on the contrary that there exists an entire curve $f:\mathbb{C}\to\cap_{i\in I}H_i\setminus\big(\cup_{j\in J} H_j\setminus A_{m,n-|I|}\big)$. Since each star-subspace in $A_{m,n-|I|}$ is constructed from at most $n-|I|$ hyperplanes in the family $\{\cap_{i\in I}H_i\cap H_j\}_{j\in J}$, the curve f must avoid completely at least $|J|-m(n-|I|)$ hyperplanes in the projective space $\cap_{i\in I}H_i\cong\mathbb{P}^{n-|I|}(\mathbb{C})$. By the elementary estimate

$$\begin{aligned}
|J|-m(n-|I|) &= q+1-2|I|-m(n-|I|-1)\\
&\geq 2(n-|I|)+1+\Big[\Big(\frac{n+2}{2}\Big)^2-2n-(|I|-1)(n-|I|-1)\Big]\\
&\geq 2(n-|I|)+1,
\end{aligned}$$

and by using Theorem 1.2.5, we derive a contradiction. □

Deformation lemma. For $2\leq l\leq n-1$, let Δ_l be a finite collection of subspaces of dimension $n-l$ coming from the family $\{H_i\}_{1\leq i\leq q}$, possibly with $\Delta_l=\emptyset$, and let $D_l\not\in\Delta_l$ be another subspace of dimension $n-l$, defined as $D_l=\cap_{i\in I_{D_l}}H_i$. For an arbitrary hypersurface $S=\{s=0\}$ in general position with respect to the family $\{H_i\}_{1\leq i\leq q}$ and for $\epsilon\neq 0$, we set

$$S_\epsilon=\big\{\epsilon s+\Pi_{i\not\in I_{D_l}} h_i^{n_i}=0\big\},$$

where $n_i\geq 1$ are chosen (freely) so that $\sum_{i\not\in I_{D_l}} n_i=q$. Then the hypersurface S_ϵ is also in general position with respect to $\{H_i\}_{1\leq i\leq q}$. We denote by $\overline{\Delta}_l$ the family of all subspaces of dimension $n-l$ $(2\leq l\leq n)$, with the convention $\overline{\Delta}_n=\emptyset$. We shall apply inductively the following lemma.

Lemma 1.6.1. *Assume that all complements of the form*

$$\cap_{i\in I} H_i \setminus \big(\cup_{j\in J} H_j \setminus (((\Delta_l \cup \overline{\Delta}_{l+1}) \cap S) \cup A_{m,n-|I|})\big) \tag{1.6.3}$$

are hyperbolic where I and J are two disjoint subsets of the index set $\{1,\dots,q\}$ such that $1 \le |I| \le n-2$, and $|J| = q+m+1-2|I|$ with some $0 \le m \le |I|-1$, and where $A_{m,n-|I|}$ is a set of at most m star-subspaces coming from the family of hyperplanes $\{\cap_{i\in I} H_i \cap H_j\}_{j\in J}$ in $\cap_{i\in I} H_i \cong \mathbb{P}^{n-|I|}(\mathbb{C})$. Then all complements of the form

$$\cap_{i\in I} H_i \setminus \big(\cup_{j\in J} H_j \setminus (((\Delta_l \cup D_l \cup \overline{\Delta}_{l+1}) \cap S_\epsilon) \cup A_{m,n-|I|})\big) \tag{1.6.4}$$

are also hyperbolic for sufficiently small $\epsilon \neq 0$.

Proof. By the definition of S_ϵ, we see that $S_\epsilon \cap \big(\cap_{m\in M} H_m\big) = S \cap \big(\cap_{m\in M} H_m\big)$ when $M \cap (\mathbf{Q} \setminus I_{D_l}) \neq \emptyset$, hence

$$(\Delta_l \cup D_l \cup \overline{\Delta}_{l+1}) \cap S_\epsilon = ((\Delta_l \cup \overline{\Delta}_{l+1}) \cap S) \cup (D_l \cap S_\epsilon).$$

When $|I| \ge l$, using this, we observe that the two complements 1.6.3, 1.6.4 coincide.

Assume therefore $|I| \le l-1$. Suppose by contradiction that there exists a sequence of entire curves $f_{\epsilon_k}(\mathbb{C})$, $\epsilon_k \to 0$, contained in the complement 1.6.4 for $\epsilon = \epsilon_k$. By the Brody Lemma, we may assume that (f_{ϵ_k}) converges to an entire curve $f(\mathbb{C}) \subset \cap_{i\in I} H_i$. We are going to prove that the curve $f(\mathbb{C})$ lands in some complement of the form 1.6.3.

Let $\cap_{k\in K} H_k$ be the smallest subspace containing $f(\mathbb{C})$, so that I is a subset of K. Take an index j in $J \setminus K$. By stability of intersections, we have

$$\begin{aligned} f(\mathbb{C}) \cap H_j &\subset \lim f_{\epsilon_k}(\mathbb{C}) \cap H_j \\ &\subset ((\Delta_l \cup \overline{\Delta}_{l+1}) \cap S) \cup A_{m,n-|I|} \cup \lim(D_l \cap S_{\epsilon_k}). \end{aligned} \tag{1.6.5}$$

If the index j does not belong to I_{D_l}, then $H_j \cap D_l \cap S_{\epsilon_k} \subset \overline{\Delta}_{l+1} \cap S$. It follows from 1.6.5 that

$$f(\mathbb{C}) \cap H_j \subset ((\Delta_l \cup \overline{\Delta}_{l+1}) \cap S) \cup A_{m,n-|I|}. \tag{1.6.6}$$

If the index j belongs to I_{D_l}, noting that $\lim(D_l \cap S_{\epsilon_k})$ is contained in $D_l \cap (\cup_{i\notin I_{D_l}} H_i)$, hence from 1.6.5

$$f(\mathbb{C}) \cap H_j \subset ((\Delta_l \cup \overline{\Delta}_{l+1}) \cap S) \cup A_{m,n-|I|} \cup (D_l \cap (\cup_{i\notin I_{D_l}} H_i)). \tag{1.6.7}$$

Assume first that $K = I$. We claim that 1.6.6 also holds when the index $j \in J \setminus I$ belongs to I_{D_l}. Indeed, for the supplementary part in 1.6.7, we have

$$f(\mathbb{C}) \cap H_j \cap \big(D_l \cup_{i\notin I_{D_l}} H_i\big) \subset \cup_{i\notin I_{D_l}} (f(\mathbb{C}) \cap H_j \cap H_i),$$

so that 1.6.6 applies here to all $i \notin I_{D_l}$. Hence, the curve $f(\mathbb{C})$ lands inside

$$\cap_{i\in I} H_i \setminus \big(\cup_{j\in J} H_j \setminus (((\Delta_l \cup \overline{\Delta}_{l+1}) \cap S) \cup A_{m,n-|I|})\big),$$

contradicting the hypothesis.

Assume now that I is a proper subset of K. Let us set

$$A_{m,n-|I|,K} = \{X \cap (\cap_{k\in K} H_k) | X \in A_{m,n-|I|}\}.$$

This set consists of star-subspaces of $\cap_{k\in K} H_k \cong \mathbb{P}^{n-|K|}(\mathbb{C})$. Let $B_{m,K}$ be the subset of $A_{m,n-|I|,K}$ containing all star-subspaces of dimension $n-|K|-1$ (i.e. of codimension 1 in $\cap_{k\in K} H_k$), and let $C_{m,K}$ be the remaining part. A star-subspace in $B_{m,K}$ is of the form $(\cap_{k\in K} H_k \cap H_j)^*$ for some index $j \in J \setminus K$. Let then R denote the set of such indices j, so that

$$|R| = |B_{m,K}|.$$

We consider two cases separately, depending on the dimension of the subspace $Y = \cap_{k\in K} H_k \cap D_l$.

Case 1: Y is a subspace of dimension $n-|K|-1$. In this case, Y is of the form $(\cap_{k\in K} H_k) \cap H_y$ for some index y in I_{D_l}. It follows from 1.6.5, 1.6.6, 1.6.7 that the curve $f(\mathbb{C})$ lands inside

$$\cap_{k\in K} H_k \setminus \big(\cup_{j\in(J\setminus K)\setminus(R\cup\{y\})} H_j \setminus (((\Delta_l \cup \overline{\Delta}_{l+1}) \cap S) \cup C_{m,K})\big).$$

To conclude that this set is of the form 1.6.3, we need to show that

(1) $|(J \setminus K) \setminus (R \cup \{y\})| = q + m' + 1 - 2|K|$ with $|C_{m,K}| \leq m' \leq |K| - 1$;

(2) $|K| \leq n-2$.

Consider **(1)**. We need to verify the corresponding required inequality between cardinalities

$$|C_{m,K}| \leq |(J \setminus K) \setminus (R \cup \{y\})| - q + 2|K| - 1 \leq |K| - 1.$$

The right inequality is equivalent to

$$|(J \setminus K) \setminus (R \cup \{y\})| \leq |\{1, \ldots, q\} \setminus K|,$$

which is trivial. The left inequality follows from the elementary estimates

$$\begin{aligned} |(J \setminus K) \setminus (R \cup \{y\})| - q + 2|K| - 1 &\geq |J \setminus K| - |B_{m,K}| - q + 2|K| - 2 \\ &= |J| - |J \cap K| - |B_{m,K}| - q + 2|K| - 2 \\ &= (m - |B_{m,K}|) + (2|K| - 2|I| - |J \cap K| - 1) \\ &\geq |C_{m,K}|, \end{aligned}$$

where the last inequality holds because I and J are two disjoint sets and I is a proper subset of K.

Consider **(2)**. Suppose on the contrary that $|K| = n-1$. Since S is in general position with respect to $\{H_i\}_{1\leq i\leq 2n}$, we see that

$$\cap_{k\in K} H_k \setminus \big(\cup_{j\in(J\setminus K)\setminus(R\cup\{y\})} H_j \setminus (((\Delta_l \cup \overline{\Delta}_{l+1}) \cap S) \cup C_{m,K})\big) = \cap_{k\in K} H_k \setminus \big(\cup_{j\in(J\setminus K)\setminus(R\cup\{y\})} H_j \setminus C_{m,K}\big).$$

Since $|(J \setminus K) \setminus (R \cup \{y\})| \geq q + 3 - 2n + |C_{m,K}| \geq 3 + |C_{m,K}|$, the curve f lands in a complement of at least 3 points in a line. By Picard's Theorem, f is constant, which is a contradiction.

Case 2: Y is a subspace of dimension at most $n-|K|-2$. In this case, the curve $f(\mathbb{C})$ lands inside

$$\cap_{k\in K} H_k \setminus \big(\cup_{j\in(J\setminus K)\setminus R} H_j \setminus (((\Delta_l \cup \overline{\Delta}_{l+1}) \cap S) \cup C_{m,K} \cup Y^*)\big),$$

which is also of the form 1.6.3, since

$$|(J\setminus K)\setminus R| \geq q - 2|K| + 1 + |C_{m,K} \cup Y^*|,$$

and since $|K| \leq n-2$, by similar arguments as in **Case 1**.

The Lemma is thus proved. □

Inductive deformation process and end of the proof of Theorem II. We may begin by applying Lemma 1.6.1 for $l = n-1$ (with $\overline{\Delta}_n = \emptyset$), firstly with $\Delta_{n-1} = \emptyset$, and with some $D_{n-1} \in \overline{\Delta}_{n-1}$, since $(\Delta_{n-1} \cup \overline{\Delta}_n) \cap S = \emptyset$, hence the assumption of this lemma holds by the Starting Lemma. Next, we reapply Lemma 1.6.1 inductively until we exhaust all $D_{n-1} \in \overline{\Delta}_{n-1}$. We get at the end a hypersurface S_1 such that all complements of the forms

$$\cap_{i\in I} H_i \setminus \big(\cup_{j\in J} H_j \setminus (S_1 \cup A_{m,n-|I|})\big) \qquad (|I| = n-2)$$
$$\cap_{i\in I} H_i \setminus \big(\cup_{j\in J} H_j \setminus ((\overline{\Delta}_{n-1} \cap S_1) \cup A_{m,n-|I|})\big) \qquad (|I| \leq n-3)$$

are hyperbolic, since when $|I| = n-2$, two components $\cap_{i\in I} H_i \setminus \big(\cup_{j\in J} H_j \setminus ((\overline{\Delta}_{n-1} \cap S_1) \cup A_{m,n-|I|})\big)$ and $\cap_{i\in I} H_i \setminus \big(\cup_{j\in J} H_j \setminus (S_1 \cup A_{m,n-|I|})\big)$ are equal. Considering this as the starting point of the second step, we apply inductively Lemma 1.6.1 for $l = n-2$ and receive at the end a hypersurface S_2 such that all complements of the forms

$$\cap_{i\in I} H_i \setminus \big(\cup_{j\in J} H_j \setminus (S_2 \cup A_{m,n-|I|})\big) \qquad (n-3 \leq |I| \leq n-2)$$
$$\cap_{i\in I} H_i \setminus \big(\cup_{j\in J} H_j \setminus ((\overline{\Delta}_{n-2} \cap S_2) \cup A_{m,n-|I|})\big) \qquad (|I| \leq n-4)$$

are hyperbolic, for the same reason as in above. Continuing this process, we get at the end of the $(n-2)^{\text{th}}$ step a hypersurface $S = S_{n-2}$ such that all complements of the forms

$$\cap_{i\in I} H_i \setminus \big(\cup_{j\in J} H_j \setminus (S_{n-2} \cup A_{m,n-|I|})\big) \qquad (1 \leq |I| \leq n-2)$$

are hyperbolic. In particularly, by choosing $m = |I|-1$, whence $|J| = q - |I|$, and by choosing $A_{m,n-|I|} = \emptyset$, all complements of the form 1.6.1 are hyperbolic for $S = S_{n-2}$. □

Chapter 2

A GEOMETRIC SECOND MAIN THEOREM

Abstract

Using Ahlfors' theory of covering surfaces, we establish a Cartan's type Second Main Theorem in the complex projective plane $\mathbb{P}^2(\mathbb{C})$ with counting functions truncated to level 1 for entire holomorphic curves whose set of accumulation points at infinity is contained in a non-hyperbolic curve $\mathscr{C} \subset \mathbb{P}^2(\mathbb{C})$.

2.1 Introduction and Main Result

2.1.1 Nevanlinna theory in projective spaces

We recall some facts from Nevanlinna theory in the projective space $\mathbb{P}^n(\mathbb{C})$. Let $E = \sum \mu_\nu a_\nu$ be a divisor on $\mathbb{C}$ and let $k \in \mathbb{N} \cup \{\infty\}$. Summing the k-truncated degrees of the divisor on disks by

$$n^{[k]}(t, E) := \sum_{|a_\nu| < t} \min\{k, \mu_\nu\} \qquad (t > 0),$$

the *truncated counting function at level* k of E is defined by

$$N^{[k]}(r, E) := \int_1^r \frac{n^{[k]}(t, E)}{t} \, \mathrm{d}t \qquad (r > 1).$$

When $k = \infty$, we write $n(t, E)$, $N(r, E)$ instead of $n^{[\infty]}(t, E)$, $N^{[\infty]}(r, E)$. We denote the zero divisor of a nonzero meromorphic function φ by $(\varphi)_0$. Let $f\colon \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be an entire curve having a reduced representation $f = [f_0 : \cdots : f_n]$ in the homogeneous coordinates $[z_0 : \cdots : z_n]$ of $\mathbb{P}^n(\mathbb{C})$. Let $D = \{Q = 0\}$ be a hypersurface in $\mathbb{P}^n(\mathbb{C})$ defined by a homogeneous polynomial $Q \in \mathbb{C}[z_0, \ldots, z_n]$ of degree $d \geq 1$. If $f(\mathbb{C}) \not\subset D$, we define the *truncated counting function* of f with respect to D as

$$N_f^{[k]}(r, D) := N^{[k]}\big(r, (Q \circ f)_0\big).$$

The *proximity function* of f for the divisor D is defined as

$$m_f(r, D) := \int_0^{2\pi} \log \frac{\|f(re^{i\theta})\|^d \, \|Q\|}{\big|Q(f)(re^{i\theta})\big|} \frac{\mathrm{d}\theta}{2\pi},$$

where $\|Q\|$ is the maximum absolute value of the coefficients of Q and

$$\|f(z)\| = \max\{|f_0(z)|, \ldots, |f_n(z)|\}.$$

Finally, the *Cartan order function* of f is defined by

$$\begin{aligned} T_f(r) :&= \frac{1}{2\pi} \int_0^{2\pi} \log \|f(re^{i\theta})\| \, \mathrm{d}\theta \\ &= \int_1^r \frac{\mathrm{d}t}{t} \int_{\Delta_t} f^* \omega_n + O(1), \end{aligned}$$

where ω_n is the Fubini–Study form on $\mathbb{P}^n(\mathbb{C})$.

The core of Nevanlinna theory consists of two theorems.

First Main Theorem. *Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic curve and let D be a hypersurface of degree d in $\mathbb{P}^N(\mathbb{C})$ such that $f(\mathbb{C}) \not\subset \mathrm{supp}(D)$. Then for every real number $r > 1$, the following holds*

$$m_f(r, D) + N_f(r, D) = d\, T_f(r) + O(1).$$

A holomorphic curve $f:\mathbb{C}\to\mathbb{P}^n(\mathbb{C})$ is *linearly non-degenerate* if its image is not contained in any hyperplane. For functions $\varphi(r)$, $\psi(r)$ valued in $[0,\infty)$, we write

$$\varphi(r)\leq O(\psi(r))\ \|$$

if the inequality holds outside a Borel subset E of $[0,\infty)$ of finite Lebesgue measure.

Cartan's Second Main Theorem [Car33]. *Let $\{H_i\}_{1\leq i\leq q}$ be a family of hyperplanes in general position in $\mathbb{P}^n(\mathbb{C})$. If $f:\mathbb{C}\to\mathbb{P}^n(\mathbb{C})$ is a linearly nondegenerate holomorphic curve, then*

$$(q-n-1)\,T_f(r)\leq\sum_{i=1}^{q}N_f^{[n]}(r,H_i)+S_f(r),$$

where $S_f(r)$ is a small term compared with $T_f(r)$

$$S_f(r)=O(\log T_f(r)+\log r)\ \|\,.$$

2.1.2 The main result

In dimension $n=2$, with the smallest possible number $q=4$ of lines $\{L_i\}_{1\leq i\leq 4}$, Cartan's Second Main Theorem reads as

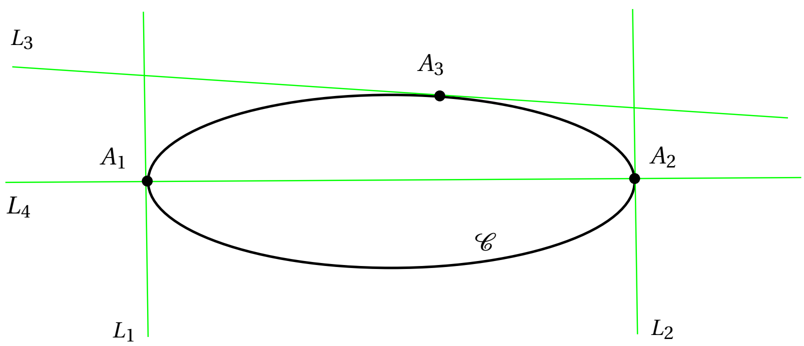
$$T_f(r)\leq\sum_{i=1}^{4}N_f^{[2]}(r,L_i)+o(T_f(r))\ \|, \tag{2.1.1}$$

where $f:\mathbb{C}\to\mathbb{P}^2(\mathbb{C})$ is a linearly nondegenerate curve. The 2–truncated counting functions in the right hand side of this inequality is optimal in the sense that they can not be truncated to level 1.

Lemma. *In $\mathbb{P}^2(\mathbb{C})$, for any family of 4 lines $\{L_i\}_{1\leq i\leq 4}$ in general position, there exits a linearly nondegenerate $f:\mathbb{C}\to\mathbb{P}^2(\mathbb{C})$ such that the following inequality does not hold*

$$T_f(r)\leq\sum_{i=1}^{4}N_f^{[1]}(r,L_i)+o(T_f(r))\ \|\,. \tag{2.1.2}$$

Proof. Indeed, let $\mathscr{C}$ be the unique conic which is tangent to L_1, L_2 at $A_1=L_1\cap L_4$, $A_2=L_2\cap L_4$, respectively and which is tangent to L_3 at some point A_3.



Since $\mathscr{C} \setminus \{A_1, A_2\} \cong \mathbb{C}^*$ is not hyperbolic, it contains some nonconstant holomorphic curve $f(\mathbb{C})$. By the choice of $\mathscr{C}$, it is not hard to verify the following equalities

$$N_f^{[2]}(r, L_i) = 0, \qquad (i \neq 3)$$
$$N_f^{[2]}(r, L_i) = 2\, N_f^{[1]}(r, L_i). \qquad (i = 3)$$

Using these equalities together with the First Main Theorem, we obtain

$$\begin{aligned} T_f(r) &= 2\, N_f^{[1]}(r, L_3) + S_f(r) \\ &= 2 \sum_{i=1}^{4} N_f^{[1]}(r, L_i) + o(T_f(r)), \end{aligned}$$

which shows that the inequality 2.1.2 does not hold. □

In the above example, the curve f is linearly nondegenerate, but its image is contained in an algebraic curve $\mathscr{C} \subset \mathbb{P}^2(\mathbb{C})$. Such entire curves satisfying this property are said to be *algebraically degenerate*. Up to date, there is no counterexample to the following

Conjecture. *If $f : \mathbb{C} \to \mathbb{P}^2(\mathbb{C})$ is algebraically non-degenerate, then* 2.1.2 *holds.*

We propose a positive answer for the above conjecture with an additional assumption on the *cluster set* of $f : \mathbb{C} \to \mathbb{P}^2(\mathbb{C})$, defined to be

$$\mathrm{Clu}(f) := \bigcap_{r>0} \left(\overline{f(\mathbb{C} \setminus \Delta_r)} \right).$$

Main Theorem. *Let $\{L_i\}_{1 \leq i \leq 4}$ be a family of lines in general position in the projective plane. Let $f : \mathbb{C} \to \mathbb{P}^2(\mathbb{C})$ be an algebraically nondegenerate holomorphic curve. If its cluster set is contained in an algebraic curve $\mathscr{C} \subset \mathbb{P}^2(\mathbb{C})$, then*

$$T_f(r) \leq \sum_{i=1}^{4} N_f^{[1]}(r, L_i) + o(T_f(r)). \qquad 2.1.4$$

A curve f satisfying the condition in this theorem must approach $\mathscr{C}$ without being contained in $\mathscr{C}$. To confirm that this assumption is meaningful, we shall prove at the end of this chapter that every nonhyperbolic curve $\mathscr{C} \subset \mathbb{P}^2(\mathbb{C})$ is the cluster set of some algebraically nondegenerate holomorphic curve $f : \mathbb{C} \to \mathbb{P}^2(\mathbb{C})$.

2.1.3 Strategy of the proof of the Main Theorem

First, we take a good resolution of singularities for the configuration $\mathscr{C} \cup \mathscr{L}$ of the curve $\mathscr{C}$ and the family of four lines $\mathscr{L} = \{L_i\}_{1 \leq i \leq 4}$

$$\mathbb{P}^2_n(\mathbb{C}) \xrightarrow{\sigma_n} \mathbb{P}^2_{n-1}(\mathbb{C}) \to \cdots \to \mathbb{P}^2_j(\mathbb{C}) \to \cdots \to \mathbb{P}^2_1(\mathbb{C}) \xrightarrow{\sigma_1} \mathbb{P}^2_0(\mathbb{C}) = \mathbb{P}^2(\mathbb{C}).$$

We then construct simultaneously closed positive currents T_j on $\mathbb{P}^2_j(\mathbb{C})$ associated to the initial curve $f: \mathbb{C} \to \mathbb{P}^2(\mathbb{C})$ and the lifted curves $\tilde{f}_j: \mathbb{C} \to \mathbb{P}^2_j(\mathbb{C})$ at each step of the resolution process. The total number of intersections between $f(\mathbb{C})$ and $\mathscr{L}$ is bounded from below by the total number of intersections between the lifted curve $\tilde{f}_n(\mathbb{C})$ and the family of divisors in $(\sigma_1 \circ \cdots \circ \sigma_n)^{-1}(\mathscr{C} \cup \mathscr{L})$ consisting of the strict transforms of L_i and a collection of exceptional divisors in the resolution tree.

Next, we estimate the masses of the current T_n along divisors in $\mathbb{P}^2_n(\mathbb{C})$ by applying Ahlfors' theory and Riemann-Hurwitz's formula. In the last step, we deal with the combinatorics on the resolution tree and apply the inequality between geometric intersections and algebraic intersections.

2.2 A brief summary on Ahlfors' theory

Topology of surfaces

A *bordered surface of finite type* is a closed region on a compact orientable surface bounded by finitely many smooth simple closed curves bounding disks. For a bordered surface Σ, we denote by $\mathring{\Sigma}$ its interior and by $\partial\Sigma$ its boundary. The Euler characteristic of Σ is

$$\chi(\Sigma) = 2 - 2g - k,$$

where g is the number of handles and k is the number of boundary components.

A map $f : \Sigma \to \Sigma_0$ between two bordered surfaces of finite type is said to be holomorphic if f is holomorphic on $\mathring{\Sigma}$ and extends continuously up to the boundary. From now on in this section, we assume that f is a non constant holomorphic map. A point p on $\mathring{\Sigma}$ is is a ramification point of f if the differential of f at p vanishes, namely if $\frac{\partial f}{\partial z}(p) = 0$ with respect to some local coordinate z near p. With

$$\mathrm{ord}_f(p) = \min\{n \geq 1 : f^{(n)}(p) \neq 0\},$$

the *ramification* of f at p is

$$r_f(p) = \mathrm{ord}_f(p) - 1.$$

When $f(\partial\Sigma) \subset \partial\Sigma_0$, f is a ramified covering. In this case, the function $d(q) = \sum_{f(p)=q} \mathrm{ord}_f(p)$ on Σ_0 is independent of q and called the *degree* of f, denoted by $\deg f$. The relationship between the Euler characteristics of Σ and Σ_0 is then given by the Riemann–Hurwitz formula

$$\chi(\Sigma) = \deg f \cdot \chi(\Sigma_0) - \sum_{p \in \Sigma} r_f(p), \tag{2.2.1}$$

from which follows the inequality

$$\chi(\Sigma) \leq \deg f \cdot \chi(\Sigma_0). \tag{2.2.2}$$

Ahlfors' theory [Ahl35] can be considered as a generalization of the above facts to the case when f is not proper. Equivalently, we allow the *relative boundary* of f, defined as $\partial\Sigma \setminus f^{-1}(\partial\Sigma_0)$, to be nonempty.

Ahlfors' theory

We keep the notation Σ, Σ_0, f and we suppose that Σ_0 is equipped with a smooth conformal metric ρ_0. We denote by ρ the pull–back of ρ_0 by f. We measure all lengths and areas on Σ and Σ_0 with respect to ρ and ρ_0. The *average number of sheets of* f is defined by

$$S := \frac{\mathrm{Area}(\Sigma)}{\mathrm{Area}(\Sigma_0)}.$$

When f is a ramified covering, $S = \deg f$.

A region is called *regular* if it is bounded by finitely many piecewise smooth curves. Generally, we define the *average number of sheets over a regular region* $D_0 \subset \Sigma_0$ as

$$S(D_0) := \frac{\mathrm{Area}(f^{-1}(D_0))}{\mathrm{Area}(D_0)},$$

with $S = S(\Sigma_0)$.

Let $L = \mathrm{Length}\big(\partial\Sigma \setminus f^{-1}(\partial\Sigma_0)\big)$ be the length of the relative boundary. Ahlfors' theory contains two theorems corresponding to the First Main Theorem and the Second Main Theorem in Nevanlinna theory. We would like to mention that Ahlfors' theory can be extended to quasi-conformal mappings (see [Sal14]) for further discussion).

Theorem 2.2.1. *Given* (Σ_0, ρ_0)*, there exists a constant* $h = h(\Sigma_0, \rho_0) > 0$ *such that for every* Σ*, for every* $f\colon \Sigma \to \Sigma_0$*, for every region* $D_0 \subset \Sigma_0$*,*

$$|S(\Sigma_0) - S(D_0)| \leq \frac{h}{\mathrm{Area}(D_0)}\, L. \tag{2.2.3}$$

Theorem 2.2.2. *Given* (Σ_0, ρ_0)*, there exists a constant* $h = h(\Sigma_0, \rho_0) > 0$ *such that for every* $f\colon \Sigma \to \Sigma_0$*,*

$$\chi^-(\Sigma) \leq S(\Sigma_0) \cdot \chi(\Sigma_0) + h\,L, \tag{2.2.4}$$

where $\chi^-(\Sigma)$ *denotes* $\min\{0, \chi(\Sigma)\}$.

The above theorem is interesting only when the surface Σ_0 has negative Euler characteristic.

Let $f : \mathbb{C} \to \mathbb{P}^1(\mathbb{C})$ be a nonconstant meromorphic function and let $a_1, \ldots, a_q$ be q distinct points in the Riemann sphere $\mathbb{P}^1(\mathbb{C})$. Nevanlinna's Second Main Theorem states that

$$(q-2)\, T_f(r) \leq \sum_{i=1}^{q} N_f^{[1]}(r, a_i) + o(T_f(r)) \parallel. \tag{2.2.5}$$

Let us give a brief explanation how this classical result can be recovered by using Ahlfors' theory [Tsu59]. Let w be a complex coordinate on the affine part $\mathbb{C} \subset \mathbb{P}^1(\mathbb{C})$. All lengths and areas on $\mathbb{P}^1(\mathbb{C})$ are measured by using the conformal metric associated to the Fubini–Study form

$$\omega_1 = dd^c \log(1 + |w|^2),$$

so that

$$\int_{\mathbb{P}^1(\mathbb{C})} \omega_1 = 1.$$

Now, let $\{D_j\}_{j=1}^{q}$ be a collection of $q \geq 3$ open disks D_j with disjoint closure such that $a_j \in D_j$ and let Σ_0 be the bordered surface in $\mathbb{P}^1(\mathbb{C})$ obtained by removing from $\mathbb{P}^1(\mathbb{C})$ all D_j, so that its Euler characteristic $\chi(\Sigma_0) = 2 - q$ is negative.

For a positive number t, a connected component Ω of $f^{-1}(D_j) \cap \overline{\Delta}_t$ is called an *island* over D_j if it is relatively compact in Δ_t, otherwise it is called a *peninsula*.

Let $\{F^{is}_{j,\mu}\}_{1 \leq \mu \leq \mu_j}$, $\{F^{pe}_{j,\mu}\}_{1 \leq \nu \leq \nu_j}$ denote the families of islands and peninsulas over D_j, respectively. It is clear that

$$\overline{\Delta}_t = \Big[\bigcup_{1 \leq j \leq q} \bigcup_{1 \leq \nu \leq \nu_j} F^{pe}_{j,\nu} \Big] \bigcup \Big[\bigcup_{1 \leq j \leq q} \bigcup_{1 \leq \mu \leq \mu_j} F^{is}_{j,\mu} \Big] \bigcup \Sigma_t,$$

where the remaining part Σ_t consists of a finite number of connected surfaces:

$$\Sigma_t = \bigcup_{\eta} \Sigma_{t,\eta}$$

such that each $\Sigma_{t,\eta}$ is a covering surface of Σ_0. If $F^{is}_{j,\mu}$ is a *simple* connected island then $1 = \chi(F^{is}_{j,\mu}) = \chi^-(F^{is}_{j,\mu}) + 1$, otherwise $0 \geq \chi(F^{is}_{j,\mu}) = \chi^-(F^{is}_{j,\mu})$. Similarly, $\chi(\Sigma_{t,\eta}) = \chi^-(\Sigma_{t,\eta}) + 1$ if and only if $\Sigma_{t,\eta}$ is *simple* connected.

Now, let k be the number of simple connected components of Σ_t and let $\bar{n}(t, D_j)$ be the number of simple connected islands over D_j. For generic t, after removing from $\overline{\Delta}_t$ all peninsulas, we receive the surface F_t consisting of $\mathsf{h} \geq \mathsf{k}$ disjoint simply connected surfaces, so that its Euler characteristic is given by

$$\begin{aligned} \mathsf{h} = \chi(F_t) &= \sum_{j=1}^{q} \sum_{\mu=1}^{\mu_j} \chi\big(F^{is}_{j,\mu}\big) + \sum_{\eta} \chi(\Sigma_{t,\eta}) \\ &= \sum_{j=1}^{q} \Big(\sum_{\mu=1}^{\mu_j} \chi^-\big(F^{is}_{j,\mu}\big) + \bar{n}(t, D_j) \Big) + \sum_{\eta} \chi^-(\Sigma_{t,\eta}) + \mathsf{k}. \end{aligned}$$

Consequently, one has

$$\begin{aligned} \sum_{j=1}^{q} \bar{n}(t, D_j) &= - \sum_{j=1}^{q} \sum_{\mu=1}^{\mu_j} \chi^-\big(F^{is}_{j,\mu}\big) - \sum_{\eta} \chi^-(\Sigma_{t,\eta}) + \mathsf{h} - \mathsf{k} \\ &\geq - \sum_{\eta} \chi^-(\Sigma_{t,\eta}). \end{aligned} \qquad 2.2.6$$

Since each $\Sigma_{t,\eta}$ is a covering surface of Σ_0, by applying the Ahlfors' Second Main Theorem 2.2.2 for $f|_{\Sigma_{t,n}}$, we receive

$$-\chi^-(\Sigma_{t,\eta}) \geq (q-2)\, S(\Sigma_{t,\eta}) - h\, L(\Sigma_{t,\eta}),$$

where $S(\Sigma_{t,\eta})$ and $L(\Sigma_{t,\eta})$ denote the average number of sheets and the length of the relative boundary of $\Sigma_{t,\eta}$, respectively. Taking the sum of both sides of these inequalities, we get

$$- \sum_{\eta} \chi^-(\Sigma_{t,\eta}) \geq (q-2)\, S(\Sigma_t) - h \sum_{\eta} L(\Sigma_{t,\eta}),$$

where $S(\Sigma_t)$ is the average number of sheets of Σ_t. Since $|S(\Sigma_t) - S(\Delta_t)| \leq h\,L(\Delta_t)$ by Ahlfors' First Main Theorem 2.2.1 and since the total length of the relative boundaries of all $\Sigma_{t,\eta}$ is bounded from above by $L(\Delta_t)$, it follows from the above inequality and 2.2.6 that

$$\sum_{j=1}^{q} \bar{n}(t, D_j) \geq (q-2)\, S(\Delta_t) - h\, L(\Delta_t)$$
$$= (q-2) \int_{\Delta_t} f^*\omega_1 - h\, L(\Delta_t).$$

Since each a_j lies in the interiors of D_j ($1 \leq j \leq q$) and since the restriction of f on an island is a ramified covering over some D_j, it is evident that $n_f^{[1]}(r, a_j) \geq \bar{n}(r, D_j)$. Hence

$$\sum_{j=1}^{q} n_f^{[1]}(t, a_j) \geq \sum_{j=1}^{q} \bar{n}(t, D_j) \geq (q-2) \int_{\Delta_t} f^*\omega_1 - h\, L(\Delta_t). \qquad 2.2.7$$

Using Length–Area principle (see [Hay64]), one sees that the complex plane $\mathbb{C}$ is *regularly exhaustible in Ahlfors' sense*, namely it satisfies

$$\lim_{r\to\infty} \frac{\mathrm{Length}\big(f(\partial\Delta_r)\big)}{\mathrm{Area}\big(f(\Delta_r)\big)} = 0 \, \| \, .$$

Hence the length of the relative boundary $L(\Delta_t)$ is negligible compared with $S(\Delta_t)$. Therefore the inequality 2.2.7 can be regarded as an analog of the classical Second Main Theorem of Nevanlinna without log integration.

Dividing 2.2.7 by t and taking integration from 1 to r, we receive

$$\sum_{j=1}^{q} N_f^{[1]}(t, a_j) \geq \sum_{j=1}^{q} \bar{N}(r, D_j) \geq (q-2)\, T_f(r) - h \int_1^r \frac{\mathrm{d}t}{t} L(\Delta_t).$$

Hence, it suffices to control the error term

$$S_f(r) := h \int_1^r \frac{\mathrm{d}t}{t} L(\Delta_t).$$

Here we could obtain only a weaker result compared with the classical Second Main Theorem, see [Din39], [Wil57], [Mil69] for concerned discussions.

2.3 Nevanlinna currents

2.3.1 Construction of Nevanlinna currents

Let (X, ω) be a complex projective variety of dimension n equipped with a Kähler form ω and let $f : \mathbb{C} \to X$ be a nonconstant holomorphic curve. Our aim in this part is to associate to f a closed positive current of bidimension $(1,1)$. For any smooth $(1,1)$–form η on X, we define

$$T_{f,r}(\eta) = \int_1^r \frac{\mathrm{d}t}{t} \int_{\Delta_t} f^*\eta. \qquad 2.3.1$$

Let $A^{(1,1)}(X)$ denote the set of smooth $(1,1)$–forms on X. Consider the family of positive currents of bounded mass $\{\Phi_r\}_{r>0}$ defined as

$$\Phi_r(\eta) = \frac{T_{f,r}(\eta)}{T_{f,r}(\omega)} \qquad (r>0, \eta \in A^{(1,1)}(X)).$$

Theorem – Definition. *There exist infinitely many sequences* $\{r_k\}$ *converging to* ∞ *such that the sequence of currents* $\{\Phi_{r_k}\}$ *converges in weak topology to a closed positive current* $\Phi \in A^{(1,1)}(X)'$. *Such limit currents are called* Nevanlinna currents *for* f.

Proof. From any sequence $\{r_k\}$ converging to ∞, by Banach–Alaoglu's theorem, we can always find a subsequence $\{r_{k_l}\}$ such that the sequence of currents $\{\Phi_{r_{k_l}}\}$ converges to a positive current. We would like to determine which conditions $\{r_k\}$ must satisfy so that the limit current $\Phi = \lim_{l\to\infty} \Phi_{r_{k_l}}$ is closed.

For any smooth 1–form β on X, using Stokes' formula and the compactness of X, we receive

$$\begin{aligned}\left|T_{f,r}(\mathrm{d}\beta)\right| &\leq \int_1^r \frac{\mathrm{d}t}{t} \int_{\partial\Delta_t} |f^*\beta| \\ &\leq C(\beta) \int_1^r \mathrm{Length}_\omega\left(f(\partial\Delta_t)\right) \frac{\mathrm{d}t}{t},\end{aligned}$$

where $C(\beta)$ is a positive constant which is independent of f, r. Set

$$L_{f,r}(\omega) = \int_1^r \mathrm{Length}_\omega\left(f(\partial\Delta_t)\right) \frac{\mathrm{d}t}{t}.$$

Then for the closedness of the limit current Φ, it suffices to choose the sequence $\{r_k\}$ such that

$$\lim_{k\to\infty} \frac{L_{f,r_k}(\omega)}{T_{f,r_k}(\omega)} = 0.$$

For this, we use the following result presented in [Bru99]. □

Ahlfors' lemma. *Under the above assumptions and notations, for any positive number* $\epsilon > 0$, *the set*

$$\left\{ r > 1 : \frac{L_{f,r}(\omega)}{T_{f,r}(\omega)} \geq \epsilon \right\}$$

has finite Lebesgue measure.

Proof. Suppose that in the polar coordinate (ρ, θ) of $\mathbb{C}$, the pull-back of ω by f is of the form

$$f^*\omega = F^2(\rho,\theta)\rho\,\mathrm{d}\rho \wedge \mathrm{d}\theta,$$

where $F^2(\rho,\theta)$ is smooth and $F(\rho,\theta) \geq 0$ is at least continuous. Then for each $t > 0$, the length of $f(\partial\Delta_t)$ and the area of $f(\Delta_t)$ are given by

$$\mathrm{Length}_\omega\left(f(\partial\Delta_t)\right) = t \int_0^{2\pi} F(t,\theta)\,\mathrm{d}\theta,$$

$$\operatorname{Area}_\omega\big(f(\Delta_t)\big)=\int_0^t \rho\,\mathrm{d}\rho\int_0^{2\pi}F(\rho,\theta)^2\,\mathrm{d}\theta,$$

respectively. Using Cauchy-Schwarz's inequality, we get

$$\begin{aligned}
L_{f,r}(\omega) &= \int_1^r\int_0^{2\pi} tF(t,\theta)\,\mathrm{d}\theta\frac{\mathrm{d}t}{t}\\
&\leq \Big(\int_1^r\int_0^{2\pi}\mathrm{d}\theta\frac{\mathrm{d}t}{t}\Big)^{\frac{1}{2}}\cdot\Big(\int_1^r\int_0^{2\pi}t^2F(t,\theta)^2\,\mathrm{d}\theta\frac{\mathrm{d}t}{t}\Big)^{\frac{1}{2}}\\
&\leq \big(2\pi\log r\big)^{\frac{1}{2}}\cdot\big(\operatorname{Area}_\omega\big(f(\Delta_r)\big)\big)^{\frac{1}{2}}\\
&= \big(2\pi r\log r\big)^{\frac{1}{2}}\cdot\Big(\frac{\mathrm{d}}{\mathrm{d}r}T_{f,r}(\omega)\Big)^{\frac{1}{2}}.
\end{aligned}$$

Dividing the above inequality by $T_{f,r}$, we obtain

$$\frac{L_{f,r}(\omega)}{T_{f,r}(\omega)}\leq\Big(\frac{2\pi r\log r}{T_{f,r}(\omega)^2}\frac{\mathrm{d}\,T_{f,r}(\omega)}{\mathrm{d}r}\Big)^{\frac{1}{2}}. \qquad 2.3.2$$

Since f is nonconstant, the order function $T_{f,r}(\omega)$ is strictly increasing and has at least a logarithmic growth rate. As a consequence, we have

$$\int_1^\infty\frac{2\pi r\log r}{T_{f,r}(\omega)^2}\frac{\mathrm{d}\,T_{f,r}(\omega)}{\mathrm{d}r}\frac{\mathrm{d}r}{2\pi r\log r}=\lim_{R\to\infty}\Big(\frac{1}{T_{f,1}(\omega)}-\frac{1}{T_{f,R}(\omega)}\Big)<\infty. \qquad 2.3.3$$

Ahlfors' Lemma follows from 2.3.2, 2.3.3 and the fact that

$$\int_1^\infty\frac{\mathrm{d}r}{2\pi r\log r}=\infty.$$

□

It is known that any Nevanlinna current is of mass 1. We would like to mention that there is a basic version of these results without log integration $\int\frac{\mathrm{d}t}{t}(\cdot)$ described as follows. For any non-constant holomorphic curve $f\colon\mathbb{C}\to X$, for any positive number $\epsilon>0$, the set

$$\Big\{r>0:\frac{\operatorname{Length}_\omega\big(f(\partial\Delta_r)\big)}{\operatorname{Area}_\omega\big(f(\Delta_r)\big)}\Big\}$$

has finite Lebesgue measure. As a consequence, there exist infinitely many sequences of positive real numbers $\{r_k\}$ converging to ∞ such that

$$\lim_{r_k\to\infty}\frac{\operatorname{Length}_\omega\big(f(\partial\Delta_{r_k})\big)}{\operatorname{Area}_\omega\big(f(\Delta_{r_k})\big)}=0.$$

For any such sequence $\{r_k\}$, the following sequence of positive currents

$$\Big\{\frac{\int_{\Delta_{r_k}}f^*\eta}{\int_{\Delta_{r_k}}f^*\omega}\Big\}_k \qquad (\eta\in A^{(1,1)}(X))$$

converges, after exacting a subsequence, to a closed positive current. This limit current is called an *Ahlfors current* for f.

2.3.2 Geometric intersections

Let Z be a divisor on X with $f(\mathbb{C}) \not\subset Z$. Summing without multiplicity the total number of intersection points on disks and integrating, introduce

$$i_{f,r_k}(Z) = \int_1^{r_k} \mathrm{Card}\big(\big(f(\Delta_t) \cap Z\big)\frac{\mathrm{d}t}{t},$$

where $\{r_k\}_{k=1}^{\infty}$ is a sequence giving birth to a Nevanlinna current T for f. Then define the *geometric intersection* of T with Z as

$$i_T(Z) = \liminf_{k\to\infty} \frac{i_{f,r_k}(Z)}{T_{f,r_k}(\omega)}.$$

It is clear that $i_T(Z) \geq 0$. When no confusion can arise, we often abbreviate it as $i(Z)$.

2.3.3 Singularities of Nevanlinna currents

Throughout this subsection, let (X,ω) be a smooth complex projective surface equipped with a Kähler form ω and let $f : \mathbb{C} \to X$ be a nonconstant holomorphic curve. The singularities of Ahlfors currents for f were studied in [Duv06], which extends to Nevanlinna currents.

Theorem 2.3.1. *If an irreducible algebraic curve $\mathscr{C} \subset X$ is charged by some Nevanlinna current (or Ahlfors current), then its genus is equal to* 0 *or* 1.

Now, let $D = \cup_{i=1}^{q} D_i$ be a divisor of *simple normal crossing type* on X, namely $\{D_i\}_{1\leq i\leq q}$ is a collection of pairwise transverse smooth curves, any three curves having empty intersection. Denote by D_i^* the complement $D_i \setminus \cup_{j\neq i} D_j$. The following application of Riemann–Hurwitz's formula and Ahlfors' theory shall play an important role in the proof of the Main Theorem.

Proposition 2.3.1. *Let T be a Nevanlinna current for f. If T is supported in D, namely*

$$T = \sum_{i=1}^{q} \lambda_i \, [D_i],$$

where $\lambda_i \geq 0$ ($1 \leq i \leq q$), then

$$-\sum_{i=1}^{q} \lambda_i \, \chi(D_i^*) \leq i(D). \tag{2.3.4}$$

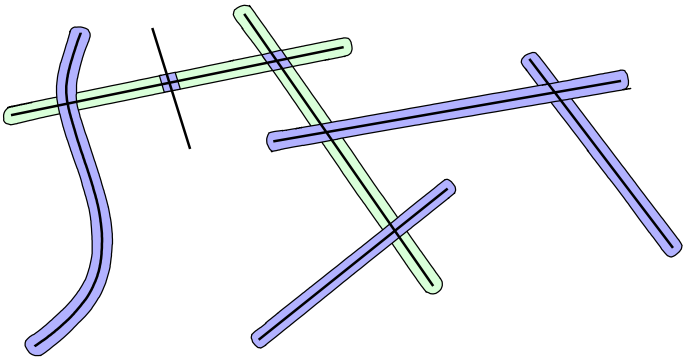
Proof. An intersection point between two curves in $\{D_i\}_{1\leq i\leq q}$ is called a *double point*. Let $\{A_\iota\}_{\iota\in I}$ be the set consisting of all double points. For each index i, denote by κ_i the number of double points in the divisor D_i. The Euler characteristic of D_i^* is then given by

$$\chi(D_i^*) = 2 - 2g_i - \kappa_i,$$

where g_i is the genus of the curve D_i. If $\chi(D_i^*) < 0$, then D_i is said to be *stable*. If $\chi(D_i^*) = 1$, then D_i is said to be *free*. It is not hard to check that D_i is free if and only if $g_i = 0$ and $\kappa_i = 1$. Denote by $\mathscr{E}_{free}$ the set consisting of all free divisors, by $\mathscr{E}_{stable}$ the set consisting of all stable divisors, and by $\mathscr{E}_{non-stable}$ the set consisting of all non-stable divisors.

Let $\{U_{i,\epsilon}\}_\epsilon$ $(0 < \epsilon < \delta)$ be a family of neighbourhoods of D_i such that there exists the projection $\pi_i\colon U_{i,\epsilon} \to D_i$ whose composition with f is a quasi–conformal mapping. Let $\{V_{\iota,\epsilon}\}_\epsilon$ be a family of neighbourhoods of the double A_ι. Set

$$W_\epsilon = \big(\cup_{i\in\mathscr{E}_{stable}} U_{i,\epsilon}\big) \setminus \Big(\big(\cup_{E\in\mathscr{E}_{non-stable}} U_{i,\epsilon}\big) \cup \big(\cup_{\iota\in I} V_{\iota,\epsilon}\big)\Big),$$
$$W_\epsilon^c = \big(\cup_{E\in\rangle_{non-stable}} U_{i,\epsilon}\big) \cup \big(\cup_{\iota\in I} V_{\iota,\epsilon}\big).$$



Now, for each $t > 0$, set $\Sigma_{i,\epsilon,t} = f^{-1}(U_{i,\epsilon}) \cap \Delta_t$ and denote by $\varphi_{i,\epsilon,t} : \Sigma_{i,\epsilon,t} \to D_i$ the restriction of $\pi_i \circ f$ on $\Sigma_{i,\epsilon,t}$. Let $\mathbf{C}_{W_\epsilon,t}$ be the set of all components of $f(\Delta_t)$ in W_ϵ and denote by $\mathsf{b}_{\epsilon,t}$ the total number of all boundary components of all $\mathfrak{C} \in \mathbf{C}_{W_\epsilon,t}$. We separate $\mathbf{C}_{W_\epsilon,t}$ into two disjoint subsets, depending on the properness of the projections π_i.

Let $\mathbf{C}_{p,W_\epsilon,t}$ be the subset of $\mathbf{C}_{W_\epsilon,t}$ consisting of all components $\mathfrak{C}$ such that the restriction on $\mathfrak{C}$ of some projection π_i is proper. Since the Euler characteristic of a component $\mathfrak{C} \in \mathbf{C}_{p,W_\epsilon,t}$ such that $\pi_i|_{\mathfrak{C}}$ is proper is given by

$$\chi(\mathfrak{C}) = 2 - \mathsf{b}_{\mathfrak{C}},$$

where $\mathsf{b}_{\mathfrak{C}}$ is the number of all boundary components of $\mathfrak{C}$, it follows from Riemann–Hurwitz's formula that

$$\mathsf{b}_{\mathfrak{C}} \geq 2 + (\kappa_i + 2\,g_i - 2)\deg(\pi_i|_{\mathfrak{C}}) \qquad (\mathfrak{C} \in \mathbf{C}_{p,W_\epsilon,t}).$$

Denote by $\mathsf{b}_{p,\epsilon,t}$ the total number of all boundary components of all $\mathfrak{C} \in \mathbf{C}_{p,W_\epsilon,t}$. Taking the sum of both sides of the above equalities, we obtain

$$\underbrace{\sum_{\mathfrak{C}\in\mathbf{C}_{p,W_\epsilon,t}} \mathsf{b}_{\mathfrak{C}}}_{\mathsf{b}_{p,\epsilon,t}} \geq 2\,|\mathbf{C}_{p,W_\epsilon,t}| + \sum_{\mathfrak{C}\in\mathbf{C}_{p,W_\epsilon,t}} (\kappa_i + 2\,g_i - 2)\deg(\pi_i|_{\mathfrak{C}}). \tag{2.3.5}$$

Now, let $\mathbf{C}_{np,W_\epsilon,t}$ be the complement of $\mathbf{C}_{p,W_\epsilon,t}$ in $\mathbf{C}_{W_\epsilon,t}$. Denote by $\mathsf{b}_{np,\epsilon,t}$ the total number of all boundary components of all $\mathfrak{C} \in \mathbf{C}_{np,W_\epsilon,t}$. We are going to apply Ahlfors' theory for $\varphi_{i,\epsilon,t}$ to estimate the number of all non-proper components $\mathsf{b}_{np,\epsilon,t}$. First, we show that there exists a positive number ϵ satisfying the following two properties.

(a) Firstly, the length of the image under the map f of $\partial\Sigma_{i,\epsilon,t}$ is negligible (compared with the order function):

$$\int_1^{r_k} \text{Length}\big(f(\partial\Sigma_{i,\epsilon,t})\big)\,\frac{\mathrm{d}t}{t} = o\big(T_{f,r_k}(\omega)\big). \tag{2.3.6}$$

(b) Secondly, the Euler characteristic of $\Sigma_{i,\epsilon,t}$ is also a negligible quantity:

$$\int_1^{r_k} -\chi(\Sigma_{i,\epsilon,t})\,\frac{\mathrm{d}t}{t} = o\big(T_{f,r_k}(\omega)\big). \tag{2.3.7}$$

Consider the first required property. Fix a positive real number $\delta > 0$ so that in any neighbourhood $U_{i,3\delta}$, we always have the projection π_i. Starting with the co-area formula and using the fact that the current T is supported in D, we obtain

$$\begin{aligned}\int_1^{r_k}\Big(\int_\delta^{2\delta} \text{Length}\big(f(\Delta_t)\cap\partial U_{i,s}\big)ds\Big)\frac{\mathrm{d}t}{t} &= \int_1^{r_k} \text{Area}\big(f(\Delta_t)\cap(U_{i,2\delta}\setminus U_{i,\delta})\big)\,\frac{\mathrm{d}t}{t}\\ &= o\big(T_{f,r_k}(\omega)\big),\end{aligned}$$

which implies

$$\int_\delta^{2\delta}\Big(\int_1^{r_k} \text{Length}\big(f(\partial\Sigma_{i,s,t})\big)\frac{\mathrm{d}t}{t}\Big)\,ds = o\big(T_{f,r_k}(\omega)\big).$$

Hence, by choosing $\epsilon\in[\delta,2\delta]$ such that

$$\int_1^{r_k} \text{Length}\big(f(\partial\Sigma_{i,\epsilon,t})\big)\,\frac{\mathrm{d}t}{t} = \inf\Big\{\int_1^{r_k} \text{Length}\big(f(\partial\Sigma_{i,s,t})\big)\,\frac{\mathrm{d}t}{t}, \delta < s < 2\delta\Big\},$$

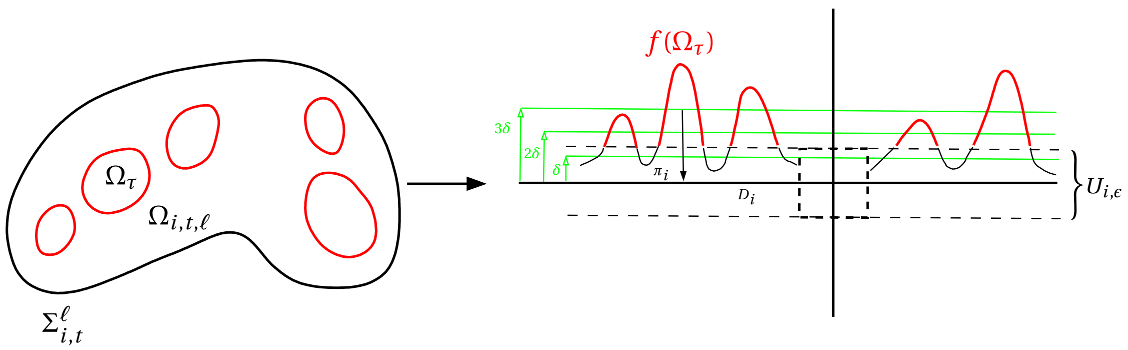
2.3.6 is satisfied.

Now, we consider the second required property. Each connected component $\Sigma_{i,t}^\ell$ of $\Sigma_{i,\epsilon,t}$ is the complement of a union of finite small disks in a large disk, namely it is of the form:

$$\Sigma_{i,t}^\ell = \Omega_{i,t,\ell}\Big\backslash\Big(\bigcup_{1\le\tau\le a_{i,t,\ell}} \overline{\Omega}_\tau\Big),$$

where $\Omega_{i,t,\ell}$ is a disk, Ω_τ $(1\le\tau\le a_{i,t,\ell})$ are open disks such that $\overline{\Omega}_\tau\subset\Omega_{i,t,\ell}$ and $f(\partial\Omega_\tau)\subset\overline{U}_{i,\epsilon}$. The Euler characteristic of $\Sigma_{i,t}^\ell$ is then computed by counting the number of small disks:

$$\chi(\Sigma_{i,t}^\ell) = 1 - a_{i,t,\ell}.$$



The total number of such small disks is approximately equal to the number of intersection points between f and $\partial U_{i,\epsilon}$. Since each small disk Ω_τ with $f(\Omega_\tau) \subset U_{i,3\delta}$ will be projected onto D_i, we don't take into account these disks when compute $\chi(\Sigma_{i,\epsilon,t})$. Hence, $\chi(\Sigma_{i,\epsilon,t})$ is approximately equal to the number of small disks Ω_τ such that $f(\Omega_\tau)$ intersects with $\partial U_{i,\epsilon}$ and $\partial U_{i,3\delta}$. A small disk that satisfies this property is called a *counting* disk. To estimate the number of these disks, we first note that each counting disk Ω_τ contains a domain Ω'_τ such that

$$f(\Omega'_\tau) \subset U_{i,3\delta} \setminus U_{i,\epsilon},$$
$$f(\partial\Omega'_\tau) \subset \partial(U_{i,3\delta} \setminus U_{i,\epsilon}).$$

Since the current T is supported in D, the total sum of areas of images of all such Ω'_τ under f is a negligible quantity. By Lelong's Theorem, for each above Ω'_τ, there exists a positive constant c depending only on (X,ω) such that

$$\text{Area}\big(f(\Omega'_\tau)\big) \geq c\,(3\delta - \epsilon)^2.$$

Hence the total number of such Ω'_τ is negligible, so is the total number of counting disks. Thus 2.3.7 is also satisfied.

We now comeback to the proof of the proposition. Denote by $\mathbf{C}_{np,i,\epsilon,t}$ the set of non proper components over divisor E and by $\mathsf{b}_{np,i,\epsilon,t}$ the total number of boundary components of all $\mathfrak{C} \in \mathbf{C}_{np,i,\epsilon,t}$. Applying Ahlfors' theory for the map $\varphi_{i,\epsilon,t}$ and for $\mathbf{C}_{np,i,\epsilon,t}$, there exists a constant $h_i > 0$ such that

$$\mathsf{b}_{np,i,\epsilon,t} \geq \sum_{\mathfrak{C}\in\mathbf{C}_{np,i,\epsilon,t}} (\kappa_i + 2g_i - 2)\int_{\mathfrak{C}} \pi_i^*\,\omega_{D_i} + \min\{0, \chi(\Sigma_{i,\epsilon,t})\} - h_i\,\text{Length}\big(f(\partial\Sigma_{i,\epsilon,t})\big),$$

where ω_{D_i} is the Kähler form on D_i. Taking the sum of both sides of these inequalities, we receive

$$\sum_{i\in\mathscr{E}_{stable}} \mathsf{b}_{np,i,\epsilon,t} \geq \sum_{i\in\mathscr{E}_{stable}} \sum_{\mathfrak{C}\in\mathbf{C}_{np,i,\epsilon,t}} (\kappa_i + 2g_i - 2)\int_{\mathfrak{C}} \pi_i^*\,\omega_{D_i} + \sum_{i\in\mathscr{E}_{stable}} \big[\min\{0, \chi(\Sigma_{i,\epsilon,t})\} - h_i\,\text{Length}\big(f(\partial\Sigma_{i,\epsilon,t})\big)\big],$$

or equivalently

$$\mathsf{b}_{np,\epsilon,t} \geq \sum_{\mathfrak{C}\in\mathbf{C}_{np,W_\epsilon,t}} (\kappa_i + 2g_i - 2)\int_{\mathfrak{C}} \pi_i^*\,\omega_{D_i} + \sum_{i\in\mathscr{E}_{stable}} \big[\min\{0, \chi(\Sigma_{i,\epsilon,t})\} - h_i\,\text{Length}\big(f(\partial\Sigma_{i,\epsilon,t})\big)\big]. \quad 2.3.8$$

Summing both sides of 2.3.5 and 2.3.8, we obtain

$$\mathsf{b}_{\epsilon,t} \geq \sum_{\mathfrak{C}\in\mathbf{C}_{W_\epsilon,t}} (\kappa_i + 2g_i - 2)\int_{\mathfrak{C}} \pi_i^*\,\omega_{D_i} + 2\,|\mathbf{C}_{p,W_\epsilon,t}| + \mathfrak{h}(t), \quad 2.3.9$$

where

$$\mathfrak{h}(t) = \sum_{i\in\mathscr{E}_{stable}} \big[\min\{0, \chi(\Sigma_{i,\epsilon,t})\} - h_i\,\text{Length}\big(f(\partial\Sigma_{i,\epsilon,t})\big)\big]$$

satisfies

$$\int_1^{r_k} \mathfrak{h}(t)\,\frac{\mathrm{d}t}{t} = o\big(T_{f,r_k}(\omega)\big).$$

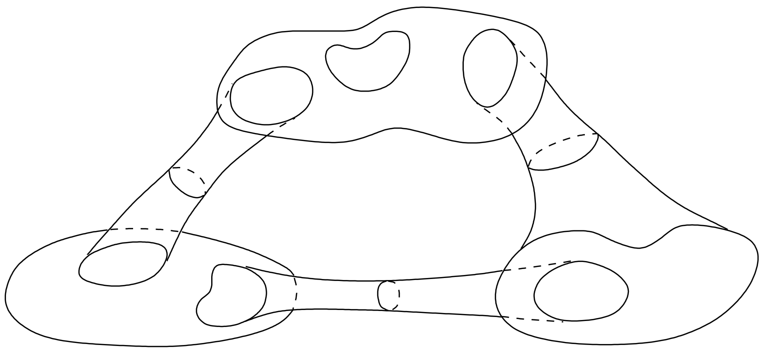
Next, let $\mathbf{C}_{W_\epsilon^c,t}$ be the set consisting of all components of $f(\Delta_t)$ in W_ϵ^c having some boundary components counted in $\mathsf{b}_{\epsilon,t}$. A component $\mathfrak{C} \in \mathbf{C}_{W_\epsilon^c,t}$ is said to be *simple* if it has only one boundary component, otherwise it is said to be *multiple*. Denote by $\mathsf{s}_{\epsilon,t}$, $\mathsf{m}_{\epsilon,t}$ the number of all simple, multiple components of $\mathbf{C}_{W_\epsilon^c,t}$, respectively. It's clear from the definition that

$$\mathsf{b}_{\epsilon,t} \geq \mathsf{s}_{\epsilon,t} + 2\,\mathsf{m}_{\epsilon,t}. \tag{2.3.10}$$

Now, let $G_{\epsilon,t}$ be the undirected graph defined by

- A vertex of $G_{\epsilon,t}$ is an element of $\mathbf{C}_{p,W_\epsilon,t} \cup \mathbf{C}_{W_\epsilon^c,t}$,
- There is an edge between two vertices $\mathfrak{C}_1$, $\mathfrak{C}_2$ if they have a common boundary component.

We claim that $G_{\epsilon,t}$ is a tree. Indeed, suppose to the contrary that there is a simple cycle defined by the sequence of vertices $\mathfrak{C}_1,\ldots,\mathfrak{C}_e$ in $G_{\epsilon,t}$. By gluing all two consecutive vertices, we receive a bordered surface Σ with a handle on it. This is a contradiction since Σ is contained in a disk.



Since $G_{\epsilon,t}$ is a tree, the number of its edges is bounded from above by the number of its vertices:

$$\mathsf{b}_{\epsilon,t} \leq |\mathbf{C}_{p,W_\epsilon,t}| + \mathsf{s}_{\epsilon,t} + \mathsf{m}_{\epsilon,t}. \tag{2.3.11}$$

It follows from 2.3.10, 2.3.11 that $\mathsf{m}_{\epsilon,t} \leq |\mathbf{C}_{p,W_\epsilon,t}|$ and therefore, by using again 2.3.11, we get

$$2\,|\mathbf{C}_{p,W_\epsilon,t}| + \mathsf{s}_{\epsilon,t} \geq \mathsf{b}_{\epsilon,t}. \tag{2.3.12}$$

Now we would like to estimate the total number of simple components $\mathsf{s}_{\epsilon,t}$. Let $i_{\epsilon,t}(D)$ be the number of intersection points in W_ϵ^c between $f(\Delta_t)$ and D. Let $\mathbf{C}_{W_\epsilon^c,t,free}$ denote the set consisting of all complements in $\mathbf{C}_{W_\epsilon^c,t}$ over some free divisors. Since each simple component either intersects with D or contributes to the mass of the current T along some free divisors, one has

$$\mathsf{s}_{\epsilon,t} \leq i_{\epsilon,t}(D) + \sum_{\mathfrak{C}\in\mathbf{C}_{W_\epsilon^c,t,free}} \int_{\mathfrak{C}} \pi_i^*\,\omega_{D_i}.$$

Using this inequality together with 2.3.9, 2.3.12 and noting that $\kappa_i + 2g_i - 2 = -1$ for each free divisor D_i, we obtain

$$\sum_{\mathfrak{C}\in \mathbf{C}_{W_\epsilon,t}\cup \mathbf{C}_{W_\epsilon^c,t,free}} (\kappa_i + 2g_i - 2)\int_{\mathfrak{C}} \pi_i^* \omega_{D_i} + \mathfrak{h}(t) \le i_{\epsilon,t}(D).$$

Since $(\pi_i)_*(\mathbb{1}_{U_{i,\epsilon}} T) = \lambda_i [D_i]$, after integrating, dividing by $T_{f,r_k}(\omega)$ and taking the limit both sides of the above inequality, we receive 2.3.4. □

2.4 Resolution of singularities

Let $\mathscr{S}$ is a smooth complex surface and let P be a point in $\mathscr{S}$. A *blowing up* of $\mathscr{S}$ at P is a smooth surface $\mathscr{T}$ together with a holomorphic map $\phi\colon \mathscr{T} \to \mathscr{S}$ such that $E = \phi^{-1}(P) \cong \mathbb{P}^1(\mathbb{C})$ is a smooth rational curve and $\mathscr{T}\setminus E$ is isomorphic to $\mathscr{S}\setminus\{P\}$. The surface $\mathscr{T}$ is unique up to isomorphism. The curve E is called the *exceptional divisor* of the blowing up.

Now, let $\mathscr{C}$ be a reduced curve in $\mathscr{S}$. The *strict transform* $\tilde{\mathscr{C}}$ of $\mathscr{C}$ with respect to the blowing up $(\mathscr{T},\phi)$ of $\mathscr{S}$ at P is the closure of $\phi^{-1}(\mathscr{C}\setminus\{P\})$ in $\mathscr{T}$

$$\tilde{\mathscr{C}} := \overline{\phi^{-1}(\mathscr{C}\setminus\{P\})}.$$

When $P \notin \mathscr{C}$, the strict transform $\tilde{\mathscr{C}} = \phi^{-1}(\mathscr{C})$ coincides with the inverse image. But when $P \in \mathscr{C}$, the exceptional divisor must be added

$$\phi^{-1}(\mathscr{C}) = \tilde{\mathscr{C}} \cup E.$$

Now, assume that P is a point of multiplicity m in $\mathscr{C}$. The *total transform* $\phi^*(\mathscr{C})$ with respect to the blowing up $(\mathscr{T},\phi)$ of $\mathscr{S}$ at P is the divisor in $\mathscr{T}$ defined as

$$\phi^*(\mathscr{C}) := \tilde{\mathscr{C}} + mE,$$

whose support is contained in $\phi^{-1}(\mathscr{C})$.

For a divisor $E \subset \mathscr{T}$, we denote by $[E]$ its homology class in $H_{1,1}(\mathscr{T},\mathbb{R})$. The following well-known result shall be repeatedly used to compute algebraic intersections.

Theorem 2.4.1. *Let $\mathscr{C}$ be a curve in a surface $\mathscr{S}$ and let $P \in \mathscr{C}$ be a point of multiplicity m. If $\phi\colon \tilde{\mathscr{S}} \to \mathscr{S}$ is the blowing up at P with the exceptional divisor E, then*

$$[E]\cdot[E] = -1.$$

Furthermore, for any two curves $\mathscr{F}, \mathscr{G}$ in $\mathscr{S}$

$$[\phi^*\mathscr{F}]\cdot[E] = 0,$$
$$[\phi^*\mathscr{F}]\cdot[\phi^*\mathscr{G}] = [\mathscr{F}]\cdot[\mathscr{G}].$$

Consequently, with $\mathscr{F} = \mathscr{G} = \mathscr{C}$ so that $\tilde{\mathscr{C}} = \phi^*(\mathscr{C}) - mE$

$$[\tilde{\mathscr{C}}]\cdot[E] = m,$$
$$[\tilde{\mathscr{C}}]\cdot[\tilde{\mathscr{C}}] - [\mathscr{C}]\cdot[\mathscr{C}] = -m^2.$$

A *resolution of singularities* for a curve $\mathscr{C}$ in a smooth surface $\mathscr{S}$ is a smooth surface $\mathscr{T}$ together with a holomorphic map $\phi\colon \mathscr{T} \to \mathscr{S}$ isomorphic outside the singularities

$$\phi|_{\mathscr{T}\setminus\phi^{-1}(\mathrm{Sing}(\mathscr{C}))} \cong \mathscr{S} \setminus \mathrm{Sing}(\mathscr{C}),$$

such that the total inverse image $\phi^{-1}(\mathscr{C})$ is of *normal crossings type*. It is known (see [Wal04]) that resolutions of singularities exist for every plane curve by means of a finite sequence of blowing ups.

2.4.1 The genus formula for plane curves

Let $\mathscr{C} \subset \mathbb{P}^2(\mathbb{C})$ be a plane curve and let P_0 be a singular point of $\mathscr{C}$. Let $(\widetilde{\mathbb{P}}^2(\mathbb{C}), \phi)$ be a resolution of singularity for $\mathscr{C}$ by repeatedly blowing up. Assume that $\{\sigma_i\}_{1\le i\le k}$ be the sequence of blowing ups starting at the point P_0, where σ_{i+1} $(0 \le i \le k-1)$ is the blowing up at the point P_i in the strict transform $\mathscr{C}^i$ of $\mathscr{C}^{i-1}$ with the convention that $\mathscr{C}^0 = \mathscr{C}$. Denote by E_i the exceptional divisor of the blowing up σ_i. For simplicity of notation, we write E_i to denote its strict transformations in the next blowing ups σ_j $(j \ge i+1)$. A point of the exceptional divisor E_i is said to be an *infinitely near point of the i–order* to P_0.

The following formulas shall be used in the sequel. The reader is referred to [Wal04, Chapter 6, 7] for proofs.

Theorem 2.4.2. *The Milnor number of a curve $\mathscr{C} \subset \mathbb{P}^2(\mathbb{C})$ at a singular point $P \in \mathscr{C}$ is given by*

$$\mathrm{Mil}_P = \sum_Q m_Q(m_Q - 1) - r_P + 1,$$

where r_P is the number of local branches of $\mathscr{C}$ passing through P, where the sum is extended over infinitely near points Q in some strict transform of $\mathscr{C}$ at which the multiplicity m_Q is at least 2.

Theorem 2.4.3. *Let $\mathscr{C} \subset \mathbb{P}^2(\mathbb{C})$ be a plane curve of degree d. For a given point P in $\mathscr{C}$, set*

$$\delta_P = \mathrm{Mil}_P + r_P - 1.$$

Then the normalisation $\tilde{\mathscr{C}}$ of $\mathscr{C}$ has Euler characteristic

$$\chi = 3d - d^2 + \sum_{P\in \mathrm{Sing}\mathscr{C}} \delta_P.$$

In particular, when the curve $\mathscr{C}$ is irreducible, the Riemann surface $\tilde{\mathscr{C}}$ has genus

$$g = \frac{1}{2}(d-1)(d-2) - \sum_{P\in \mathrm{Sing}\mathscr{C}} \frac{1}{2}\delta_P.$$

2.5 Proof of the Main Theorem

2.5.1 Resolution of singularities and constructions of close positive currents

Suppose that the cluster set of f is contained in a reduced algebraic curve $\mathscr{C} \subset \mathbb{P}^2(\mathbb{C})$, then any Nevanlinna current (or Ahlfors current) for f is supported on $\mathscr{C}$. We can further assume that these currents charge all irreducible components of $\mathscr{C}$. By Theorem 2.3.1, these components must be rational or elliptic. Let $\mathscr{C}_j$ $(1 \leq j \leq m)$ be the irreducible components of the curve $\mathscr{C}$, let d_j be the degrees of $\mathscr{C}_j$ and let g_j be the genus of $\mathscr{C}_j$. Let

$$\widetilde{\mathbb{P}}^2_{\mathrm{N}}(\mathbb{C}) \xrightarrow{\sigma_{\mathrm{N}}} \widetilde{\mathbb{P}}^2_{\mathrm{N}-1}(\mathbb{C}) \to \cdots \to \widetilde{\mathbb{P}}^2_{\mathrm{K}}(\mathbb{C}) \xrightarrow{\sigma_{\mathrm{K}}} \widetilde{\mathbb{P}}^2_{\mathrm{K}-1}(\mathbb{C}) \to \cdots \to \widetilde{\mathbb{P}}^2_{1}(\mathbb{C}) \xrightarrow{\sigma_{1}} \widetilde{\mathbb{P}}^2_{0}(\mathbb{C}) = \mathbb{P}^2(\mathbb{C})$$

be a resolution of singularities for the configuration of the curve $\mathscr{C}$ and the family $\mathscr{L} = \{L_i\}_{1\leq i\leq 4}$, where $\sigma_{\mathrm{K}}\colon \widetilde{\mathbb{P}}^2_{\mathrm{K}}(\mathbb{C}) \to \widetilde{\mathbb{P}}^2_{\mathrm{K}-1}(\mathbb{C})$ are the blowing ups of $\widetilde{\mathbb{P}}^2_{\mathrm{K}-1}(\mathbb{C})$ at some points $P_{\mathrm{K}-1}$. Denote by $\mathscr{C}^{\mathrm{K}} \subset \widetilde{\mathbb{P}}^2_{\mathrm{K}}(\mathbb{C})$ the strict transform of $\mathscr{C}^{\mathrm{K}-1}$, by $\mathscr{C}_j^{\mathrm{K}}$ the strict transform of $\mathscr{C}_j^{\mathrm{K}-1}$, by $\mathscr{L}^{\mathrm{K}} \subset \widetilde{\mathbb{P}}^2_{\mathrm{K}}(\mathbb{C})$ the strict transform of the family $\mathscr{L}^{\mathrm{K}-1}$, and by $\widetilde{f}_{\mathrm{K}}\colon \mathbb{C} \to \widetilde{\mathbb{P}}^2_{\mathrm{K}}(\mathbb{C})$ the lifted curves, with the convention that $\mathscr{C}^0 = \mathscr{C}$, $\mathscr{L}^0 = \mathscr{L}$ and $\widetilde{f}_0 = f$. Let $E_{\mathrm{K}} = \sigma_{\mathrm{K}}^{-1}(P_{\mathrm{K}-1})$ be the exceptional divisor of the blowing up σ_{K}. For simplicity of notation, we continue to write E_{K} to denote its strict transforms in the next blowing ups σ_ℓ $(\ell \geq \mathrm{K}+1)$.

Now we would like to construct *simultaneously* closed positive currents T_{K} associated to $\widetilde{f}_{\mathrm{K}}$ by similar arguments as in the previous section. We firs recall the following standard result.

Lemma 2.5.1. *Let (X,ω) be a Kähler surface. If $\phi\colon \widetilde{X} \to X$ is a blowing up of X at some point P with the exceptional divisor $E = \phi^{-1}(P)$, then there exists a Kähler form $\widetilde{\omega}$ on $\widetilde{X}$ such that*

$$\widetilde{\omega} = \phi^*\omega - \epsilon\,\Theta_{h_E}, \tag{2.5.1}$$

where Θ_{h_E} is the curvature of some hermitian metric h_E on the line bundle $\mathscr{O}(E)$ and $\epsilon > 0$ is a positive constant. □

Now, for a non-constant holomorphic curve $g : \mathbb{C} \to X$ whose image is not contained in the critical value of ϕ, we always have the lifted curve $\widetilde{g} : \mathbb{C} \to \widetilde{X}$. Based on the First Main Theorem, we obtain:

Observation 2.5.1.

$$T_{\widetilde{g},r}(\widetilde{\omega}) \leq T_{g,r}(\omega) + O(1).$$

Proof. We follow the proof of [Bru99, Lemma 1]. Taking pull-back by $\widetilde{g}$, dividing by t and integrating from 1 to r the equality 2.5.1, we receive

$$\begin{aligned} T_{\widetilde{g},r}(\widetilde{\omega}) &= T_{\widetilde{g},r}(\phi^*\omega) - \epsilon\, T_{\widetilde{g},r}(\Theta_{h_E}) \\ &= T_{g,r}(\omega) - \epsilon\, T_{\widetilde{g},r}(\Theta_{h_E}). \end{aligned} \tag{2.5.2}$$

Let s of be a section of $\mathcal{O}(E)$ with $E = \{s = 0\}$ and $\|s\| \leq 1$. Using Poincaré-Lelong's formula, we get

$$\Theta_{h_E} = \delta_E - \mathrm{d}\mathrm{d}^c \log \|s\|^2,$$

where δ_E is the current of integration over E and the equation is in the sense of currents. Now, taking pull-backs by $\widetilde{g}$, dividing by t and integrating from 1 to r, we obtain

$$T_{\widetilde{g},r}(\Theta_{h_E}) = \int_1^r \frac{\mathrm{d}t}{t} \int_{\Delta_t} \widetilde{g}^* E + 2\int_1^r \frac{\mathrm{d}t}{t} \int_{\Delta_t} \mathrm{d}\mathrm{d}^c \Big[\log \frac{1}{\|s \circ \widetilde{g}\|}\Big].$$

Using Jensen's formula, we rewrite the above equality as

$$T_{\widetilde{g},r}(\Theta_{h_E}) = \int_1^r \frac{\mathrm{d}t}{t} \int_{\Delta_t} \widetilde{g}^* E + \frac{1}{2\pi} \int_0^{2\pi} \log \frac{1}{\|s \circ \widetilde{g}(r\, e^{i\theta})\|}\, \mathrm{d}\theta - \frac{1}{2\pi} \int_0^{2\pi} \log \frac{1}{\|s \circ \widetilde{g}(e^{i\theta})\|}\, \mathrm{d}\theta. \tag{2.5.3}$$

Since

$$\int_1^r \frac{\mathrm{d}t}{t} \int_{\Delta_t} \widetilde{g}^* E \geq 0$$

and since $\|s\| \leq 1$, it follows from 2.5.2 and 2.5.3 that

$$\begin{aligned} T_{\widetilde{g},r}(\widetilde{\omega}) &\leq T_{g,r}(\omega) + \frac{1}{2\pi} \int_0^{2\pi} \log \frac{1}{\|s \circ \widetilde{g}(e^{i\theta})\|}\, \mathrm{d}\theta \\ &= T_{g,r}(\omega) + O(1). \end{aligned}$$

□

Lemma 2.5.1 allows us to construct inductively Kähler forms ω_2^{K} on $\widetilde{\mathbb{P}}^2_{\mathrm{K}}(\mathbb{C})$ satisfying

(1) $\omega_2^0 = \omega_2$,

(2) for $1 \leq \mathrm{K} \leq \mathrm{N}$, the pull-back of $\omega_2^{\mathrm{K}-1}$ expresses as

$$(\sigma_{\mathrm{K}})^* \omega_2^{\mathrm{K}-1} = \omega_2^{\mathrm{K}} + \epsilon_{\mathrm{K}} \Theta_{h_{E_{\mathrm{K}}}},$$

where $\Theta_{h_{E_{\mathrm{K}}}}$ is the curvature form of some hermitian metric $h_{E_{\mathrm{K}}}$ on the line bundle $\mathcal{O}(E_{\mathrm{K}})$ and $\epsilon_{\mathrm{K}} > 0$ is a positive constant.

For $1 \leq \mathrm{K} \leq \mathrm{N}$, we infer from Observation 2.5.1 that

$$\begin{aligned} T_{\widetilde{f}^*_{\mathrm{K}},r}(\omega_2^{\mathrm{K}}) &\leq T_{\widetilde{f}^*_{\mathrm{K}-1},r}(\omega_2^{\mathrm{K}-1}) + O(1) \\ &\leq \cdots \\ &\leq T_{f,r}(\omega_2) + O(1). \end{aligned}$$

Hence, by using similar arguments as in the proof of Ahlfors' Lemma, we deduce that for each $\epsilon > 0$, the set

$$\Big\{ r > 0 : \frac{L_{\widetilde{f}_{\mathrm{K}},r}(\omega_2^{\mathrm{K}})}{T_{f,r}(\omega_2)} \geq \epsilon,\ 0 \leq \mathrm{K} \leq \mathrm{N} \Big\}$$

is of finite Lebesgue measure. This implies

$$\lim_{r\to\infty} \frac{L_{\widetilde{f}_{\mathrm{K}},r}(\omega_2^{\mathrm{K}})}{T_{f,r}(\omega_2)} = 0 \;\| \qquad (0 \leq \mathrm{K} \leq \mathrm{N}).$$

Consequently, there exist infinitely many sequences of positive numbers $\{r_k\}_{k=1}^{\infty}$ tending to ∞ and giving simultaneously closed positive currents associated to $\widetilde{f}_{\mathrm{K}}$:

$$T_{\mathrm{K}}(\eta) = \lim_{k\to\infty} \frac{T_{\widetilde{f}_{\mathrm{K}},r_k}(\eta)}{T_{f,r_k}(\omega_2)} \qquad \Big(\eta \in A^{(1,1)}\big(\widetilde{\mathbb{P}}^2_{\mathrm{K}}(\mathbb{C})\big),\ 0 \le \mathrm{K} \le \mathrm{N}\Big).$$

When $n = 0$, T_0 is a Nevanlinna current for f, hence it is of mass 1. Since T_0 is supported on $\mathscr{C}$ and charges all irreducible component $\mathscr{C}_j$, it is of the form

$$T_0 = \sum_{j=1}^{m} \mu_j [\mathscr{C}_j],$$

where $\mu_j > 0$ are positive real numbers such that $\sum_{j=1}^{m} \mu_j d_j = 1$. By construction, the currents T_{K} $(0 \le \mathrm{K} \le \mathrm{N})$ are of the form

$$T_{\mathrm{K}} = \sum_{j=1}^{m} \mu_j \big[\mathscr{C}_j^{\mathrm{K}}\big] + \sum_{1\le \ell \le \mathrm{K}} \lambda_\ell [E_\ell],$$

where $\lambda_\ell \ge 0$.

2.5.2 Geometric intersections and algebraic intersections

Let Z be a divisor on $\widetilde{\mathbb{P}}^2_{\mathrm{K}}(\mathbb{C})$ with $\widetilde{f}_{\mathrm{K}}(\mathbb{C}) \not\subset Z$. Similarly as in the previous section, we can define the geometric intersection of the closed positive current T_{K} with Z as follows. First, taking the sum without multiplicity the total number of intersection points on disks and integrating, introduce

$$i_{\widetilde{f}_{\mathrm{K}},r_k}(Z) = \int_1^{r_k} \mathrm{Card}\Big(\big(\widetilde{f}_{\mathrm{K}}(\Delta_t) \cap Z\Big)\frac{\mathrm{d}t}{t},$$

where $\{r_k\}_{k=1}^{\infty}$ is a sequence giving birth to closed positive currents $T_0, \dots, T_{\mathrm{N}}$. Then define the *geometric intersection* of T_{K} with Z as

$$i_{T_{\mathrm{K}}}(Z) = \liminf_{k\to\infty} \frac{i_{\widetilde{f}_{\mathrm{K}},r_k}(Z)}{T_{f,r_k}(\omega_2)}.$$

When it is understood that $Z \subset \widetilde{\mathbb{P}}^2_{\mathrm{K}}(\mathbb{C})$, often we will abbreviate it as

$$i(Z) \ge 0.$$

Proposition 2.5.1. *In order to prove the Main Theorem, it suffices to show that for any sequence of positive numbers $\{r_k\}_{k=1}^{\infty}$ satisfying*

$$\lim_{r_k\to\infty} \frac{L_{\widetilde{f}_{\mathrm{K}},r_k}(\omega_2^{\mathrm{K}})}{T_{f,r_k}(\omega_2)} = 0 \qquad (0 \le \mathrm{K} \le \mathrm{N}),$$

that defines closed positive currents T_{K} as in the previous subsection, we have

$$\sum_{i=1}^{4} i(L_i) \ge 1. \tag{2.5.4}$$

Proof. Indeed, for such above sequence $\{r_k\}_{k=1}^{\infty}$, it follows form the definition of geometric intersection that

$$\liminf_{k\to\infty} \frac{\sum_{i=1}^4 N^{[1]}(r_k, L_i)}{T_f(r_k)} \geq 1.$$

Hence

$$\sum_{i=1}^{4} N^{[1]}(r_k, L_i) \geq T_f(r_k) - o\big(T_f(r_k)\big),$$

for all sequence $r_k \to \infty$ outside a set E of finite Lebesgue measure, or equivalently

$$\sum_{i=1}^{4} N^{[1]}(r_, L_i) \geq T_f(r) - o\big(T_f(r)\big) \parallel .$$

□

Now, denote by $[T_{\mathrm{K}}]$, $[Z]$ the cohomology classes of T_{K}, Z in $H^{1,1}\big(\widetilde{\mathbb{P}}^2_{\mathrm{K}}(\mathbb{C}),\mathbb{R}\big)$, respectively. The algebraic intersection of $[T_{\mathrm{K}}]$ and $[Z]$ is then defined as

$$[T_{\mathrm{K}}]\cdot[Z] := T_{\mathrm{K}}(\omega_Z),$$

where ω_Z is a differentiable $(1,1)$-form in the cohomology class $\{Z\}$. The following relationship between $i(Z)$ and $[T_{\mathrm{K}}]\cdot[Z]$ can be regarded as a consequence of the First Main Theorem in Nevanlinna theory.

Proposition 2.5.2. *The geometric intersection is bounded from above by the algebraic intersection:*

$$0 \leq i(Z) \leq [T_{\mathrm{K}}]\cdot[Z]. \qquad 2.5.5$$

Proof. Our arguments are similarly as in the proof of Observation 2.5.1. Fix a hermitian metric h on the line bundle $\mathscr{O}(Z)$ and denote by Θ_h its curvature form. Taking a section s of $\mathscr{O}(Z)$ with $Z = \{s = 0\}$ and $\|s\| \leq 1$. Using Poincaré-Lelong's formula, we get

$$\Theta_h = \delta_Z - \mathrm{d}\mathrm{d}^c \log \|s\|^2,$$

where δ_Z is the current of integration over Z and the equation is in the sense of currents. Taking pull-backs by $\widetilde{f}_{\mathrm{K}}$, dividing by t and integrating from 1 to r_k, we obtain

$$T_{\widetilde{f}_{\mathrm{K}},r_k}(\Theta_h) = \int_1^{r_k} \frac{\mathrm{d}t}{t} \int_{\Delta_t} \widetilde{f}_{\mathrm{K}}^{\,*} Z + 2\int_1^{r_k} \frac{\mathrm{d}t}{t} \int_{\Delta_t} \mathrm{d}\mathrm{d}^c \Big[\log \frac{1}{\|s\circ \widetilde{f}_{\mathrm{K}}\|}\Big].$$

Using Jensen's formula, we rewrite the above equality as

$$T_{\widetilde{f}_{\mathrm{K}},r_k}(\Theta_h) = \int_1^{r_k} \frac{\mathrm{d}t}{t} \int_{\Delta_t} \widetilde{f}_{\mathrm{K}}^{\,*} Z + \frac{1}{2\pi}\int_0^{2\pi} \log \frac{1}{\|s\circ \widetilde{f}_{\mathrm{K}}(r_k e^{i\theta})\|}\,\mathrm{d}\theta - \frac{1}{2\pi}\int_0^{2\pi} \log \frac{1}{\|s\circ \widetilde{f}_{\mathrm{K}}(e^{i\theta})\|}\,\mathrm{d}\theta.$$

This yields

$$T_{\widetilde{f}_{\mathrm{K}},r_k}(\Theta_h) \geq i_{\widetilde{f}_{\mathrm{K}},r_k}(Z) + \frac{1}{2\pi}\int_0^{2\pi} \log \frac{1}{\|s\circ \widetilde{f}_{\mathrm{K}}(r_k e^{i\theta})\|}\,\mathrm{d}\theta - \frac{1}{2\pi}\int_0^{2\pi} \log \frac{1}{\|s\circ \widetilde{f}_{\mathrm{K}}(e^{i\theta})\|}\,\mathrm{d}\theta,$$

by definition of the geometric intersection. Using the assumption $\|s\| \leq 1$, after dividing by $T_{f,r_k}(\omega_2)$ and taking the limit of both sides of the above inequality, we obtain 2.5.5. □

2.5.3 Applications of Riemann-Hurwitz formula and Ahlfors' theory

In the resolution tree

$$\Big[\bigcup_{1\le j\le m}\mathscr{C}_j^{\mathrm{N}}\Big]\bigcup\Big[\bigcup_{1\le i\le 4}\mathscr{L}_i^{\mathrm{N}}\Big]\bigcup\Big[\bigcup_{1\le \mathrm{K}\le \mathrm{N}}E_{\mathrm{K}}\Big]\subset\widetilde{\mathbb{P}}^2_{\mathrm{N}}(\mathbb{C}),$$

an intersection point between two divisors is called a *double point*. Let κ_D denote the number of double points in the divisor D. Proposition 2.3.1 still holds when we replace the Nevanlinna current T by the closed positive current T_{N}. Using this fact together with Proposition 2.5.2, we obtain:

Proposition 2.5.3. *In the resolution tree, we have*

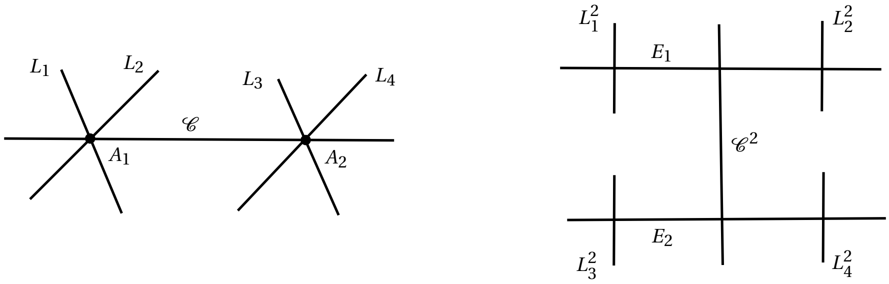
$$\sum_{j=1}^{m}\mu_j(\kappa_{\mathscr{C}_j^{\mathrm{N}}}+2g_j-2)+\sum_{\mathrm{K}=1}^{\mathrm{N}}(\kappa_{E_{\mathrm{K}}}-2)\lambda_{\mathrm{K}}\le i(\mathscr{L}^{\mathrm{N}})+i(\mathscr{C}^{\mathrm{N}})+\sum_{\mathrm{K}=1}^{\mathrm{N}}i(E_{\mathrm{K}}) \qquad 2.5.6$$

$$\le i(\mathscr{L}^{\mathrm{N}})+[T_{\mathrm{N}}]\cdot[\mathscr{C}^{\mathrm{N}}]+\sum_{\mathrm{K}=1}^{\mathrm{N}}i(E_{\mathrm{K}}). \qquad 2.5.7$$

2.5.4 Some examples

Before giving proof of the Main Theorem, we would like to treat some special cases. Recall that from the very definition of the geometric intersection, it suffices to prove that $i(\mathscr{L})\ge 1$.

1. Consider the case where $\mathscr{C}$ is the diagonal line passing through $A_1=L_1\cap L_2$, $A_2=L_3\cap L_4$. After blowing up the two points A_1, A_2 with exceptional divisors E_1, E_2, we receive the resolution tree containing the strict transforms $\mathscr{C}^2$, $\mathscr{L}^2=\{L_i^2\}_{1\le i\le 4}$ of $\mathscr{C}$, $\mathscr{L}$, respectively and the exceptional divisors E_1, E_2. The closed positive current associated to the lifted curve $\tilde{f}_2$ constructed in the first part is of the form
$$T_2=\big[\mathscr{C}^2\big]+\lambda_1[E_1]+\lambda_2[E_2],$$
for some non-negative constants $\lambda_1,\lambda_2\ge 0$.



We first use Theorem 2.4.1 to compute the intersection numbers $\big[\mathscr{C}^2\big]^2=\big(\big[\mathscr{C}^2\big]^2-[\mathscr{C}]^2\big)+[\mathscr{C}]^2=-2+1=-1$, $\big[\mathscr{C}^2\big]\cdot[E_i]=1$ $(1\le i\le 2)$. Next, by applying Proposition 2.5.3, we obtain

$$i(\mathscr{L}^2)+i(E_1)+i(E_2)+[T_2]\cdot[\mathscr{C}^2]\ge\lambda_1+\lambda_2.$$

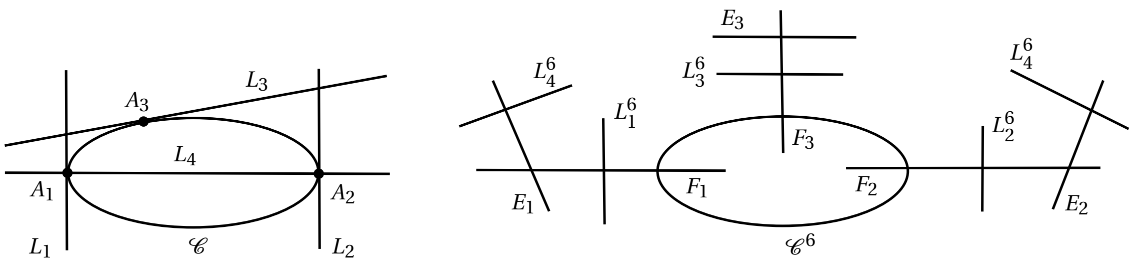
Now, note that for any $z \in \mathbb{C}$, if $\widetilde{f}_2(z) \in E_i$ then $f(z) = A_i$. One can deduce from this fact that

$$\sum_{i=1}^{4} i(L_i) \geq i(\mathscr{L}^2) + i(E_1) + i(E_2).$$

Combining the two above inequalities, we get

$$\begin{aligned}\sum_{i=1}^{4} i(L_i) &\geq \lambda_1 + \lambda_2 - [T_2]\cdot[\mathscr{C}^2] \\ &= \lambda_1 + \lambda_2 - \left(\left[\mathscr{C}^2\right]^2 + \lambda_1\left[\mathscr{C}^2\right]\cdot[E_1] + \lambda_2\left[\mathscr{C}^2\right]\cdot[E_2]\right) \\ &= \lambda_1 + \lambda_2 - (-1 + \lambda_1 + \lambda_2) \\ &= 1.\end{aligned}$$

2. We comeback to the counter example in the first part of this note, where $\mathscr{C}$ is the conic which is tangent to L_1, L_2 at $A_1 = L_1 \cap L_4$, $A_2 = L_2 \cap L_4$, respectively and which is tangent to L_3 at some point A_3. We have a good resolution for this configuration after six blowing ups described as follows. First, we blow up at the three points A_i with exceptional divisors E_i. We then continue to blow up at the three intersection points between E_i and the strict transform of the conic $\mathscr{C}$ with exceptional divisors F_i. After this process, we receive a resolution tree containing the strict transforms $\mathscr{C}^6$, $\mathscr{L}^6 = \{L_i^6\}_{1\leq i\leq 4}$, respectively and six exceptional divisors E_i, F_i $(1 \leq i \leq 3)$.



The closed positive current associated to the lifted curve $\widetilde{f}_6$ is of the form

$$T_6 = \frac{1}{2}\left[\mathscr{C}^6\right] + \lambda_1[E_1] + \lambda_2[E_2] + \lambda_3[E_3] + \nu_1[F_1] + \nu_2[F_2] + \nu_3[F_3],$$

for some non-negative constants λ_i, ν_i. Note that E_3 is a free divisor and all F_i are stable divisors. Applying Proposition 2.5.3, we get

$$\underbrace{i(\mathscr{L}^6) + i(E_1) + i(E_2) + i(E_3) + i(F_1) + i(F_2) + i(F_3)}_{\leq \sum_{i=1}^4 i(L_i)} + [T_6]\cdot[\mathscr{C}^6] \geq \frac{1}{2} + \nu_1 + \nu_2 + \nu_3 - \lambda_3,$$

which yields

$$\sum_{i=1}^{4} i(L_i) + [T_6]\cdot[\mathscr{C}^6] \geq \frac{1}{2} + \nu_1 + \nu_2 + \nu_3 - \lambda_3,$$

by similar arguments as in the first example. Next, we compute the intersection number $[T_6]\cdot[\mathscr{C}^6]$ and use Proposition 2.5.2 to estimate λ_3:

$$
\begin{aligned}
[T_6]\cdot[\mathscr{C}^6] &= \frac{1}{2}\big([\mathscr{C}^6]^2-[\mathscr{C}]^2\big)+\nu_1[\mathscr{C}^6]\cdot[F_1]+\nu_2[\mathscr{C}^6]\cdot[F_2]+\nu_3[\mathscr{C}^6]\cdot[F_3]+\frac{1}{2}[\mathscr{C}]^2\\
\text{[Use Theorem 2.4.1]}\qquad &= \frac{1}{2}\cdot(-6)+\nu_1+\nu_2+\nu_3+\frac{1}{2}\cdot 4\\
&= \nu_1+\nu_2+\nu_3-1,
\end{aligned}
$$

$$
\begin{aligned}
0 &\le [T_3]\cdot[E_3]\\
&= \Big(\frac{1}{2}[\mathscr{C}^3]+\lambda_1[E_1]+\lambda_2[E_2]+\lambda_3[E_3]\Big)\cdot[E_3]\\
&= \frac{1}{2}-\lambda_3.
\end{aligned}
$$

Combining the three above estimates, we receive $\sum_{i=1}^4 i(L_i)\ge 1$, as wanted.

2.5.5 Estimations of intersections numbers - end of the proof

In general case, we can not estimate directly $\sum_{i=1}^4 i(L_i)$ from Proposition 2.5.3 because of the complexity of the resolution tree. The key technique to overcome this difficulty is to view this quantity *locally*, site by site in the resolution tree near singular points of the configuration $\mathscr{C}\cup\mathscr{L}$. First, we rewrite the inequality 2.5.7 as follows

$$
i(\mathscr{L}^{\mathrm{N}})+\sum_{\mathrm{K}=1}^{\mathrm{N}} i(E_{\mathrm{K}})+\sum_{j=1}^{m}\mu_j(2-2g_j)+[T_0]\cdot[\mathscr{C}]\ge\sum_{j=1}^{m}\mu_j\kappa_{\mathscr{C}_j^{\mathrm{N}}}+\sum_{\mathrm{K}=1}^{\mathrm{N}}(\kappa_{E_{\mathrm{K}}}-2)\lambda_{\mathrm{K}}-\big([T_{\mathrm{N}}]\cdot[\mathscr{C}^{\mathrm{N}}]-[T_0]\cdot[\mathscr{C}]\big), \tag{2.5.8}
$$

where the quantity $[T_{\mathrm{N}}]\cdot[\mathscr{C}^{\mathrm{N}}]-[T_0]\cdot[\mathscr{C}]$ can be viewed locally. Let $\mathrm{Sing}_{(\mathscr{C}\cup\mathscr{L})}$ denote the set of singular points of the configuration $\mathscr{C}\cup\mathscr{L}$. Let

$$
\mathrm{Sing}_{(\mathscr{C}\cup\mathscr{L})}=\mathrm{Sing}_0\cup\mathrm{Sing}_1\cup\mathrm{Sing}_2
$$

be the partition of $\mathrm{Sing}_{(\mathscr{C}\cup\mathscr{L})}$ determined by

$$
\mathrm{Sing}_v=\{Q\in\mathrm{Sing}_{(\mathscr{C}\cup\mathscr{L})}:\text{there are } v \text{ lines of } \mathscr{L} \text{ passing through } Q\}\qquad (0\le v\le 2).
$$

Note that locally in any site near some singular point, we always have $\sum_{i=1}^4 i(L_i)\ge i(\mathscr{L}^{\mathrm{N}})$. For a point $Q\in\mathrm{Sing}_2$, for a divisor E in the site near Q, for any $z\in\mathbb{C}$, if $\widetilde{f_{\mathrm{N}}}(z)\in E$ then $f(z)=Q$. Hence for any site near some point in Sing_2, the local contribution to $\sum_{i=1}^4 i(L_i)$ is $\ge$ the sum of the local contribution to $i(\mathscr{L}^{\mathrm{N}})$ and all $i(E)$, where E are exceptional divisors in this site. Note also that for any exceptional divisor E_{K}, one has $i(E_{\mathrm{K}})\le[T_{\mathrm{N}}]\cdot[E_{\mathrm{K}}]$ by Proposition 2.5.2. Thanks to these remarks, we deduce from 2.5.8 that

$$
\begin{aligned}
\sum_{i=1}^4 i(L_i)+\sum_{j=1}^{m}\mu_j(2-2g_j)+[T_0]\cdot[\mathscr{C}] &\ge \sum_{Q\in\mathrm{Sing}_{(\mathscr{C}\cup\mathscr{L})}}\Big(\sum_{j=1}^{m}\mu_j\kappa_{\mathscr{C}_j^{\mathrm{N}}}+\sum_{\mathrm{K}=1}^{\mathrm{N}}(\kappa_{E_{\mathrm{K}}}-2)\lambda_{\mathrm{K}}-\big([T_{\mathrm{N}}]\cdot[\mathscr{C}^{\mathrm{N}}]-[T_0]\cdot[\mathscr{C}]\big)\Big)_Q\\
&\quad-\sum_{Q\in\mathrm{Sing}_0\cup\mathrm{Sing}_1}\Big(\sum_{\mathrm{K}=1}^{\mathrm{N}}[T_{\mathrm{N}}]\cdot[E_{\mathrm{K}}]\Big)_Q,
\end{aligned} \tag{2.5.9}
$$

where $\mathscr{A}_Q$ denotes the local contribution of $\mathscr{A}$ near Q.

Lemma 2.5.2. *For a point $Q \in \mathrm{Sing}_{(\mathscr{C}\cup\mathscr{L})}$, denote by Mil_Q^j the Milnor number of $\mathscr{C}_j$ at Q and by r_Q^j the number of branches of $\mathscr{C}_j$ at Q. Set*

$$\delta_Q^j = \mathrm{Mil}_Q^j + r_Q^j - 1,$$

then the local contribution of the right hand side of 2.5.9 *at Q is bounded from below by*

$$\sum_{j=1}^m \mu_j \delta_Q^j + \sum_{j=1}^m \mu_j \sum_{k\neq j} (\mathscr{C}_j \cdot \mathscr{C}_k)_Q + \sum_{j=1}^m \mu_j (\mathscr{C}_j \cdot \mathscr{L})_Q.$$

Proof. Without lost of generality, we may assume that $\{E_\kappa, 1 \leq \kappa \leq \mathrm{N}_Q\}$ is the set consisting of exceptional divisors starting from the point $Q = P_0$ in the resolution tree. We first introduce some notations and basic properties. For two indexes κ_1, κ_2 with $\kappa_1 > \kappa_2$, the exceptional divisor E_{κ_1} is said to be *proximate* to E_{κ_2} if E_{κ_2} intersects E_{κ_1} when E_{κ_1} appears in the resolution tree. It is clear that E_κ is always proximate to $E_{\kappa-1}$. Furthermore, if E_{κ_1} is proximate to E_{κ_2} for some $\kappa_2 < \kappa_1 - 1$, then E_{κ_1-1} is also proximate to E_{κ_2}. Set

$$\mathrm{Prox}_{E_\kappa} := \{\ell \leq \kappa - 1 : E_\kappa \text{ is proximate to } E_\ell\},$$

$$\mathrm{Child}_{E_\kappa} := \{\ell \geq \kappa + 1 : E_\ell \text{ is proximate to } E_\kappa\}.$$

Denote by p_κ the cardinality of Prox_{E_κ}. If E_κ appears in the resolution tree by blowing up an intersection point of two exceptional divisors, then $p_\kappa = 2$. Otherwise $p_\kappa = 1$. Let a_κ denote the self-intersection $[E_\kappa] \cdot [E_\kappa]$. Note that at its first appearance, E_κ has self-intersection number -1 and this is diminished by 1 each time a point on E_κ is blown up. Hence,

$$|\mathrm{Child}_{E_\kappa}| = -1 - a_\kappa. \qquad 2.5.10$$

Now, in the resolution tree, let n_κ be the number of intersection points between E_κ and the other exceptional divisors E_ℓ ($\ell \neq \kappa$). Its relationship to p_κ is described as follows.

Observation 2.5.2. *One has*

$$2 - n_\kappa = 2 - p_\kappa - \sum_{\ell \in \mathrm{Child}_{E_\kappa}} (2 - p_\ell).$$

Proof. The quantity $2 - n_\kappa$ can be computed inductively by tracing E_κ, form the first time it appears until the end of the resolution process. When E_κ appears, it intersects with the other exceptional divisors at p_κ points. Hence we set $2 - n_{\kappa,0} = 2 - p_\kappa$. Next, when a point A_1 in E_κ is blown up with exceptional divisor E_{ℓ_1}, we determine $2 - n_{\kappa,1}$ as follows. If A_1 does not belong to any other exceptional divisor $E_\mu, \mu < \kappa$, then after the blowing up, E_κ still intersects with all old exceptional divisors. Since E_κ also intersects with E_{ℓ_1}, we set $2 - n_{\kappa,1} = 2 - n_{\kappa,0} - 1$. Note that in this case $p_{\ell_1} = 1$, hence $2 - n_{\kappa,1} = 2 - n_{\kappa,0} - 1 = 2 - p_\kappa - (2 - p_{\ell_1})$. If $A_1 = E_\kappa \cap E_\mu$ for some $\mu < \kappa$, then the number of intersection points between E_κ and the other exceptional divisors does not change, since it intersects with the new exceptional divisor E_{ℓ_1}, but has empty intersection with E_μ. Hence we set $2 - n_{\kappa,1} = 2 - n_{\kappa,0}$. Note that in this case $p_{\ell_1} = 2$, thus $2 - n_{\kappa,1} = 2 - n_{\kappa,0} = 2 - p_\kappa - (2 - p_{\ell_1})$. Continue this process until the end, we obtain $2 - n_\kappa = 2 - n_{\kappa,|\mathrm{Child}_{E_\kappa}|} = 2 - p_\kappa - \sum_{\ell \in \mathrm{Child}_{E_\kappa}} (2 - p_\ell)$, as wanted. □

Next, let $\{B_{j,\nu}\}_{1\le\nu\le \mathrm{R}_j}$ be the collection of branches of the component $\mathscr{C}_j$ at Q. Similarly as in the first part of this section, we use the notation $B^{\mathrm{K}}_{j,\nu}$ to denote the strict transform of $B^{\mathrm{K}-1}_{j,\nu}$, with the convention that $B^0_{j,\nu} = B_{j,\nu}$. Let $E_{a_{j,\nu}}$ be the exceptional divisor in the resolution tree that intersects $B^{\mathrm{N}_Q}_{j,\nu}$. The number of double points $\kappa_{E_{a_{j,\nu}}}$ is then given by

$$\kappa_{E_{a_{j,\nu}}} = n_{a_{j,\nu}} + 1 + |E_{a_{j,\nu}} \cap \mathscr{L}^{\mathrm{N}_Q}|. \tag{2.5.11}$$

Let m^{K}_j, $m^{\mathrm{K}}_{j,\nu}$ $(1 \le \mathrm{K} \le \mathrm{N}_Q)$ be the multiplicity of $\mathscr{C}^{\mathrm{K}-1}_j$, $B^{\mathrm{K}-1}_{j,\nu}$ at $P_{\mathrm{K}-1}$, respectively. Note that the sequence $\{m^{\mathrm{K}}_{j,\nu}\}^{\mathrm{N}_Q}_{\mathrm{K}=1}$ is of the form $m^1_{j,\nu}, \cdots, m^{a_{j,\nu}-1}_{j,\nu}, 1, 0, \cdots, 0$. Note also that

$$m^{\mathrm{K}}_j = \sum_{\nu=1}^{\mathrm{R}_j} m^{\mathrm{K}}_{j,\nu}. \tag{2.5.12}$$

The following equality is a basic property of the proximity relation (see [Wal04, Proposition 3.5.1]):

$$m^{\mathrm{K}}_{j,\nu} = \sum_{\ell\ge \mathrm{K}+1,\, \ell\in \mathrm{Prox}_{E_{\mathrm{K}}}} m^{\ell}_{j,\nu} \qquad (1 \le \mathrm{K} \le a_{j,\nu}-1).$$

As a consequence, one has

Observation 2.5.3.

$$m^1_{j,\nu} = \sum_{\mathrm{K}\ge 2,\, p_{\mathrm{K}}=2} m^{\mathrm{K}}_{j,\nu} + 1 \qquad (1 \le j \le m; 1 \le \nu \le \mathrm{R}_j).$$

Proof. Taking the sum of both sides of the $a_{j,\nu}$ above equalities, we receive

$$\sum_{\mathrm{K}=1}^{a_{j,\nu}-1} m^{\mathrm{K}}_{j,\nu} = \sum_{\mathrm{K}=1}^{a_{j,\nu}-1} \sum_{\ell\ge \mathrm{K}+1,\, \ell\in \mathrm{Prox}_{E_{\mathrm{K}}}} m^{\ell}_{j,\nu},$$

which yields

$$\begin{aligned} m^1_{j,\nu} &= \sum_{\mathrm{K}=1}^{a_{j,\nu}-1} \Big(\sum_{\ell\ge \mathrm{K}+1,\, \ell\in \mathrm{Prox}_{E_{\mathrm{K}}}} m^{\ell}_{j,\nu} - m^{\mathrm{K}+1}_{j,\nu} \Big) + 1 \\ &= \sum_{\mathrm{K}\ge 2,\, p_{\mathrm{K}}=2} m^{\mathrm{K}}_{j,\nu} + 1. \end{aligned}$$

□

Now we are ready to prove Lemma 2.5.2. Using Theorem 2.4.1 and Theorem 2.4.2, the local contribution of $\big([T_{\mathrm{N}}]\cdot[\mathscr{C}^{\mathrm{N}}] - [T_0]\cdot[\mathscr{C}]\big)$ at Q can be computed as follows

$$\begin{aligned} &\big([T_{\mathrm{N}}]\cdot[\mathscr{C}^{\mathrm{N}}] - [T_0]\cdot[\mathscr{C}]\big)_Q \\ &= \Big(\sum_{j=1}^{m} \mu_j \sum_{\nu=1}^{\mathrm{R}_j} [B^{\mathrm{N}_Q}_{j,\nu}] + \sum_{\mathrm{K}=1}^{\mathrm{N}_Q} \lambda_{\mathrm{K}}[E_{\mathrm{K}}] \Big)\cdot \Big(\sum_{j=1}^{m}\sum_{\nu=1}^{\mathrm{R}_j} [B^{\mathrm{N}_Q}_{j,\nu}] \Big) - \Big(\sum_{j=1}^{m} \mu_j \sum_{\nu=1}^{\mathrm{R}_j} [B_{j,\nu}] \Big)\cdot \Big(\sum_{j=1}^{m}\sum_{\nu=1}^{\mathrm{R}_j} [B_{j,\nu}] \Big) \\ &= \sum_{j=1}^{m} \mu_j \Big([\sum_{\nu=1}^{\mathrm{R}_j} B^{\mathrm{N}_Q}_{j,\nu}]^2 - [\sum_{\nu=1}^{\mathrm{R}_j} B_{j,\nu}]^2 \Big) + \sum_{j=1}^{m} \mu_j \Big(\sum_{\nu_1=1}^{\mathrm{R}_j} \sum_{1\le k\le m,\, k\ne j} \sum_{\nu_2=1}^{\mathrm{R}_k} \big([B^{\mathrm{N}_Q}_{j,\nu_1}]\cdot[B^{\mathrm{N}_Q}_{k,\nu_2}] - [B_{j,\nu_1}]\cdot[B_{k,\nu_2}] \big) \Big) \end{aligned}$$

$$
\begin{aligned}
&+\Big(\sum_{\mathrm{K}=1}^{\mathrm{N}_Q}\lambda_{\mathrm{K}}[E_{\mathrm{K}}]\Big)\cdot\Big(\sum_{j=1}^{m}\sum_{\nu=1}^{\mathrm{R}_j}[B_{j,\nu}^{\mathrm{N}_Q}]\Big)\\
&=\sum_{j=1}^{m}\mu_j\Big(\sum_{\nu=1}^{\mathrm{R}_j}\big([B_{j,\nu}^{\mathrm{N}_Q}]^2-[B_{j,\nu}]^2\big)+2\sum_{1\le\nu<\eta\le\mathrm{R}_j}\big([B_{j,\nu}^{\mathrm{N}_Q}]\cdot[B_{j,\eta}^{\mathrm{N}_Q}]-[B_{j,\nu}]\cdot[B_{j,\eta}]\big)\Big)\\
&+\sum_{j=1}^{m}\mu_j\Big(\sum_{1\le k\le m,k\neq j}\sum_{\nu_1=1}^{\mathrm{R}_j}\sum_{\nu_2=1}^{\mathrm{R}_k}\big([B_{j,\nu_1}^{\mathrm{N}_Q}]\cdot[B_{k,\nu_2}^{\mathrm{N}_Q}]-[B_{j,\nu_1}]\cdot[B_{k,\nu_2}]\big)\Big)\\
&+\sum_{j=1}^{m}\sum_{\nu=1}^{\mathrm{R}_j}\lambda_{a_{j,\nu}}[B_{j,\nu}^{\mathrm{N}_Q}]\cdot[E_{a_{j,\nu}}]\\
&=\sum_{j=1}^{m}\mu_j\Big(-\sum_{\nu=1}^{\mathrm{R}_j}\sum_{\mathrm{K}=1}^{\mathrm{N}_Q}(m_{j,\nu}^{\mathrm{K}})^2-2\sum_{1\le\nu<\eta\le\mathrm{R}_j}\sum_{\mathrm{K}=1}^{\mathrm{N}_Q}m_{j,\nu}^{\mathrm{K}}m_{j,\eta}^{\mathrm{K}}\Big)\\
&+\sum_{j=1}^{m}\mu_j\Big(\sum_{1\le k\le m,\,k\neq j}\sum_{\nu_1=1}^{\mathrm{R}_j}\sum_{\nu_2=1}^{\mathrm{R}_k}\Big(-\sum_{\mathrm{K}=1}^{\mathrm{N}_Q}m_{j,\nu_1}^{\mathrm{K}}m_{k,\nu_2}^{\mathrm{K}}\Big)\Big)+\sum_{j=1}^{m}\sum_{\nu=1}^{\mathrm{R}_j}\lambda_{a_{j,\nu}}\\
&=-\sum_{j=1}^{m}\mu_j\sum_{\mathrm{K}=1}^{\mathrm{N}_Q}\Big(\sum_{\nu=1}^{\mathrm{R}_j}m_{j,\nu}^{\mathrm{K}}\Big)^2-\sum_{j=1}^{m}\mu_j\sum_{1\le k\le m,\,k\neq j}\sum_{\mathrm{K}=1}^{\mathrm{N}_Q}\Big(\sum_{\nu_1=1}^{\mathrm{R}_j}m_{j,\nu_1}^{\mathrm{K}}\Big)\Big(\sum_{\nu_2=1}^{\mathrm{R}_k}m_{k,\nu_2}^{\mathrm{K}}\Big)+\sum_{j=1}^{m}\sum_{\nu=1}^{\mathrm{R}_j}\lambda_{a_{j,\nu}}\\
&=-\sum_{j=1}^{m}\mu_j\sum_{\mathrm{K}=1}^{\mathrm{N}_Q}(m_j^{\mathrm{K}})^2-\sum_{j=1}^{m}\mu_j\sum_{1\le k\le m,\,k\neq j}\sum_{\mathrm{K}=1}^{\mathrm{N}_Q}(m_j^{\mathrm{K}})(m_k^{\mathrm{K}})+\sum_{j=1}^{m}\sum_{\nu=1}^{\mathrm{R}_j}\lambda_{a_{j,\nu}}\\
&=-\sum_{j=1}^{m}\mu_j\sum_{\mathrm{K}=1}^{\mathrm{N}_Q}\big[(m_j^{\mathrm{K}})^2-m_j^{\mathrm{K}}\big]-\sum_{j=1}^{m}\mu_j\sum_{1\le k\le m,\,k\neq j}\sum_{\mathrm{K}=1}^{\mathrm{N}_Q}(m_j^{\mathrm{K}})(m_k^{\mathrm{K}})+\sum_{j=1}^{m}\sum_{\nu=1}^{\mathrm{R}_j}\lambda_{a_{j,\nu}}-\sum_{j=1}^{m}\mu_j\sum_{\mathrm{K}=1}^{\mathrm{N}_Q}m_j^{\mathrm{K}}\\
&=-\sum_{j=1}^{m}\mu_j(\mu_Q^j+r_Q^j-1)-\sum_{j=1}^{m}\mu_j\sum_{k\neq j}(\mathscr{C}_j\cdot\mathscr{C}_k)_Q+\sum_{j=1}^{m}\sum_{\nu=1}^{\mathrm{R}_j}\lambda_{a_{j,\nu}}-\sum_{j=1}^{m}\mu_j\sum_{\mathrm{K}=1}^{\mathrm{N}_Q}m_j^{\mathrm{K}}. \qquad 2.5.13
\end{aligned}
$$

Next, by very definition of a_{K}, n_{K}, the local contribution of $\sum_{\mathrm{K}=1}^{\mathrm{N}}[T_{\mathrm{N}}]\cdot[E_{\mathrm{K}}]$ at Q is given by

$$
\begin{aligned}
\Big(\sum_{\mathrm{K}=1}^{\mathrm{N}}[T_{\mathrm{N}}]\cdot[E_{\mathrm{K}}]\Big)_Q&=\Big(\sum_{j=1}^{m}\mu_j\sum_{\nu=1}^{\mathrm{R}_j}[B_{j,\nu}^{\mathrm{N}_Q}]+\sum_{\mathrm{K}=1}^{\mathrm{N}_Q}\lambda_{\mathrm{K}}[E_{\mathrm{K}}]\Big)\cdot\Big(\sum_{\mathrm{K}=1}^{\mathrm{N}_Q}[E_{\mathrm{K}}]\Big)\\
&=\Big(\sum_{j=1}^{m}\mu_j\sum_{\nu=1}^{\mathrm{R}_j}[B_{j,\nu}^{\mathrm{N}_Q}]\Big)\cdot\Big(\sum_{\mathrm{K}=1}^{\mathrm{N}_Q}[E_{\mathrm{K}}]\Big)+\Big(\sum_{\mathrm{K}=1}^{\mathrm{N}_Q}\lambda_{\mathrm{K}}[E_{\mathrm{K}}]\Big)\cdot\Big(\sum_{\mathrm{K}=1}^{\mathrm{N}_Q}[E_{\mathrm{K}}]\Big)\\
&=\sum_{j=1}^{m}\mu_j\sum_{\nu=1}^{\mathrm{R}_j}[B_{j,\nu}^{\mathrm{N}_Q}]\cdot[E_{a_{j,\nu}}]+\sum_{\mathrm{K}=1}^{\mathrm{N}_Q}\lambda_{\mathrm{K}}[E_{\mathrm{K}}]^2+\sum_{\mathrm{K}=1}^{\mathrm{N}_Q}\lambda_{\mathrm{K}}[E_{\mathrm{K}}]\cdot\Big(\sum_{\ell=1,\ell\neq\mathrm{K}}^{\mathrm{N}_Q}[E_\ell]\Big)\\
&=\sum_{j=1}^{m}\mu_j\mathrm{R}_j+\sum_{\mathrm{K}=1}^{\mathrm{N}_Q}\lambda_{\mathrm{K}}a_{\mathrm{K}}+\sum_{\mathrm{K}=1}^{\mathrm{N}_Q}\lambda_{\mathrm{K}}n_{\mathrm{K}}. \qquad 2.5.14
\end{aligned}
$$

We consider three cases depending on the position of Q with respect to the family $\mathscr{L}$.

Case 1: the point Q is contained in Sing_0. Since any exceptional divisor in the site near Q does not intersect $\mathscr{L}^{\mathrm{N}}$, we infer from 2.5.11 that the local contribution of the right hand side of 2.5.9 at Q is given by

$$\sum_{j=1}^{m}\sum_{\nu=1}^{\mathrm{R}_j}\lambda_{a_{j,\nu}}+\sum_{j=1}^{m}\mu_j\mathrm{R}_j-\sum_{\kappa=1}^{\mathrm{N}_Q}(2-n_\kappa)\lambda_\kappa-\Big(\big([T_\mathrm{N}]\cdot[\mathscr{C}^\mathrm{N}]-[T_0]\cdot[\mathscr{C}]\big)+\sum_{\kappa=1}^{\mathrm{N}}[T_\mathrm{N}]\cdot[E_\kappa]\Big)_Q,$$

which is equal to

$$\sum_{j=1}^{m}\mu_j\delta_Q^j+\sum_{j=1}^{m}\mu_j\sum_{k\neq j}(\mathscr{C}_j\cdot\mathscr{C}_k)_Q+\sum_{j=1}^{m}\mu_j\sum_{\kappa=1}^{\mathrm{N}_Q}m_j^\kappa-\sum_{\kappa=1}^{\mathrm{N}_Q}(2+a_\kappa)\lambda_\kappa,$$

by using 2.5.13, 2.5.14. Since $\sum_{j=1}^{m}\mu_j(\mathscr{C}_j\cdot\mathscr{L})_Q=0$, it suffices to verify the following inequality

$$\sum_{\kappa=1}^{\mathrm{N}_Q}(2+a_\kappa)\lambda_\kappa\le\sum_{j=1}^{m}\mu_j\sum_{\kappa=1}^{\mathrm{N}_Q}m_j^\kappa. \tag{2.5.15}$$

For each $1\le\kappa\le\mathrm{N}_Q$, by applying Proposition 2.5.2, we get

$$\begin{aligned}
0&\le[T_\kappa]\cdot[E_\kappa]\\
&=\Big(\sum_{j=1}^{m}\mu_j\sum_{\nu=1}^{\mathrm{R}_j}\big[B_{j,\nu}^\kappa\big]+\sum_{\ell=1}^{\kappa}[E_\ell]\Big)\cdot[E_\kappa]\\
\text{[Use Theorem 2.4.1]}\quad&=\sum_{j=1}^{m}\mu_j\sum_{\nu=1}^{\mathrm{R}_j}m_{j,\nu}^\kappa+\sum_{\ell\in\mathrm{Prox}_{E_\kappa}}\lambda_\ell-\lambda_\kappa\\
\text{[Use 2.5.12]}\quad&=\sum_{j=1}^{m}\mu_jm_j^\kappa+\sum_{\ell\in\mathrm{Prox}_{E_\kappa}}\lambda_\ell-\lambda_\kappa,
\end{aligned}$$

or equivalently

$$\lambda_\kappa\le\sum_{j=1}^{m}\mu_jm_j^\kappa+\sum_{\ell\in\mathrm{Prox}_{E_\kappa}}\lambda_\ell.$$

Summing both sides of these inequalities, we receive

$$\begin{aligned}
\sum_{\kappa=1}^{\mathrm{N}_Q}\lambda_\kappa&\le\sum_{j=1}^{m}\mu_j\sum_{\kappa=1}^{\mathrm{N}_Q}m_j^\kappa+\sum_{\kappa=1}^{\mathrm{N}_Q}\sum_{\ell\in\mathrm{Prox}_{E_\kappa}}\lambda_\ell,\\
&=\sum_{j=1}^{m}\mu_j\sum_{\kappa=1}^{\mathrm{N}_Q}m_j^\kappa+\sum_{\kappa=1}^{\mathrm{N}_Q}|\mathrm{Child}_{E_\kappa}|\lambda_\kappa\\
\text{[Use 2.5.10]}\quad&=\sum_{j=1}^{m}\mu_j\sum_{\kappa=1}^{\mathrm{N}_Q}m_j^\kappa+\sum_{\kappa=1}^{\mathrm{N}_Q}(-1-a_\kappa)\lambda_\kappa,
\end{aligned}$$

which yields 2.5.15.

Case 2: the point Q is contained in Sing_1, i.e. there is only one line L of the family $\mathscr{L}$ containing the point Q. Assume that in the resolution tree, L^{N_Q} intersects the exceptional divisor E_{a_L}. We first observe that

$$\sum_{j=1}^{m}\mu_j(\mathscr{C}_j\cdot\mathscr{L})_Q=\sum_{j=1}^{m}\mu_j\sum_{\nu=1}^{\mathrm{R}_j}\big(B_{j,\nu}\cdot L\big)$$

$$= \sum_{j=1}^{m} \mu_j \sum_{\nu=1}^{\mathrm{R}_j} \sum_{\kappa=1}^{a_L} m_{j,\nu}^{\kappa}$$
$$= \sum_{j=1}^{m} \mu_j \sum_{\kappa=1}^{a_L} m_j^{\kappa}. \qquad 2.5.16$$

Using 2.5.11 and noting that $\kappa_{E_{a_L}} = n_{a_L} + 1 + |E_{a_L} \cap \mathscr{C}^{\mathrm{N}}|$, the local contribution of the right hand side of 2.5.9 at Q is given by

$$\sum_{j=1}^{m} \sum_{\nu=1}^{\mathrm{R}_j} \lambda_{a_{j,\nu}} + \sum_{j=1}^{m} \mu_j \mathrm{R}_j - \sum_{\kappa=1}^{\mathrm{N}_Q} (2 - n_\kappa)\lambda_\kappa - \Big(\big([T_{\mathrm{N}}]\cdot[\mathscr{C}^{\mathrm{N}}] - [T_0]\cdot[\mathscr{C}]\big) + \sum_{\kappa=1}^{\mathrm{N}} [T_{\mathrm{N}}]\cdot[E_\kappa]\Big)_Q + \lambda_{a_L},$$

which is equal to

$$\sum_{j=1}^{m} \mu_j \delta_Q^j + \sum_{j=1}^{m} \mu_j \sum_{k \neq j} (\mathscr{C}_j \cdot \mathscr{C}_k)_Q + \sum_{j=1}^{m} \mu_j \sum_{\kappa=1}^{\mathrm{N}_Q} m_j^\kappa - \sum_{\kappa=1}^{\mathrm{N}_Q} (2 + a_\kappa)\lambda_\kappa + \lambda_{a_L},$$

by 2.5.13, 2.5.14. Hence, using this fact together with 2.5.16, it's enough to prove that

$$\sum_{\kappa=1}^{\mathrm{N}_Q} (2 + a_\kappa)\lambda_\kappa - \lambda_{a_L} \leq \sum_{j=1}^{m} \mu_j \sum_{\kappa=1}^{\mathrm{N}_Q} m_j^\kappa - \sum_{j=1}^{m} \mu_j \sum_{\kappa=1}^{a_L} m_j^\kappa. \qquad 2.5.17$$

Since $\kappa - 1 \in \mathrm{Prox}_{E_\kappa}$, $\lambda_\kappa \geq 0$ $(1 \leq \kappa \leq \mathrm{N}_Q)$, we have

$$\lambda_\kappa + \Big(\sum_{j=1}^{m} \mu_j m_j^\kappa - \lambda_\kappa + \lambda_{\kappa-1}\Big) \leq \sum_{j=1}^{m} \mu_j m_j^\kappa + \sum_{\ell \in \mathrm{Prox}_{E_\kappa}} \lambda_\ell \qquad (1 \leq \kappa \leq a_L),$$

with the convention that $\lambda_0 = 0$. For each index $\kappa > a_L$, applying Proposition 2.5.2 for T_κ and E_κ, we receive

$$\lambda_\kappa \leq \sum_{j=1}^{m} \mu_j m_j^\kappa + \sum_{\ell \in \mathrm{Prox}_{E_\kappa}} \lambda_\ell \qquad (a_L + 1 \leq \kappa \leq n_Q),$$

as in **Case 1**. Summing both sides of these inequalities, we obtain

$$\sum_{\kappa=1}^{\mathrm{N}_Q} \lambda_\kappa + \sum_{j=1}^{m} \mu_j \sum_{\kappa=1}^{a_L} m_j^\kappa - \lambda_{a_L} \leq \sum_{j=1}^{m} \mu_j \sum_{\kappa=1}^{\mathrm{N}_Q} m_j^\kappa - \sum_{\kappa=1}^{\mathrm{N}_Q} (-1 - a_\kappa)\lambda_\kappa,$$

which yields the inequality 2.5.17.

Case 3: Q is contained in Sing_2, i.e. Q is the intersection point of two lines in $\mathscr{L}$, says L_1 and L_2. Assume that in the resolution tree, L_u^{N} $(1 \leq u \leq 2)$ intersects the exceptional divisors $E_{a_{L_u}}$. Since each branch $B_{j,\nu}$ intersects transversally with L_1 or L_2, we observe that

$$B_{j,\nu} \cdot \mathscr{L} = m_{j,\nu}^1 + \sum_{\kappa=1}^{a_{L_{j,\nu}}} m_{j,\nu}^\kappa,$$

where $L_{j,\nu}$ is the remaining line that $B_{j,\nu}$ may not intersect transversally. Consequently, one has

$$\sum_{j=1}^{m} \mu_j (\mathscr{C}_j \cdot \mathscr{L})_Q = \sum_{j=1}^{m} \mu_j \sum_{\nu=1}^{\mathrm{R}_i} (B_{j,\nu} \cdot \mathscr{L})$$

$$= \sum_{j=1}^{m} \mu_j \sum_{\nu=1}^{\mathrm{R}_j} \Big(m_{j,\nu}^1 + \sum_{\kappa=1}^{a_{L_{j,\nu}}} m_{j,\nu}^{\kappa} \Big). \qquad 2.5.18$$

Similarly as in the two above cases, the local contribution of the right hand side of 2.5.9 at Q is given by

$$\sum_{j=1}^{m} \sum_{\nu=1}^{\mathrm{R}_j} \lambda_{a_{j,\nu}} + \sum_{j=1}^{m} \mu_j \mathrm{R}_j - \sum_{\kappa=1}^{\mathrm{N}_Q} (2-n_\kappa)\lambda_\kappa - \big([T_{\mathrm{N}}]\cdot[\mathscr{C}^{\mathrm{N}}] - [T_0]\cdot[\mathscr{C}] \big)_Q + \lambda_{a_{L_1}} + \lambda_{a_{L_2}}.$$

If follows from 2.5.13 that this quantity is equal to

$$\sum_{j=1}^{m} \mu_j \delta_Q^j + \sum_{j=1}^{m} \mu_j \sum_{k\neq j} (\mathscr{C}_j \cdot \mathscr{C}_k)_Q + \sum_{j=1}^{m} \mu_j \sum_{\nu=1}^{\mathrm{R}_j} \sum_{\kappa=1}^{\mathrm{N}_Q} m_{j,\nu}^{\kappa} + \sum_{j=1}^{m} \mu_j \mathrm{R}_j - \sum_{\kappa=1}^{\mathrm{N}_Q} (2-n_\kappa)\lambda_\kappa + \lambda_{a_{L_1}} + \lambda_{a_{L_2}}. \qquad 2.5.19$$

Using this fact together with 2.5.18, the problem reduces to proving that

$$\sum_{\kappa=1}^{\mathrm{N}_Q} (2-n_\kappa)\lambda_\kappa - \lambda_{a_{L_1}} - \lambda_{a_{L_2}} + \sum_{j=1}^{m} \mu_j \sum_{\nu=1}^{\mathrm{R}_j} m_{j,\nu}^1 - \sum_{j=1}^{m} \mu_j \mathrm{R}_j \leq \sum_{j=1}^{m} \mu_j \sum_{\nu=1}^{\mathrm{R}_j} \sum_{\kappa=a_{L_{j,\nu}}+1}^{\mathrm{N}_Q} m_{j,\nu}^{\kappa}. \qquad 2.5.20$$

First, we infer from Observation 2.5.2 that

$$\begin{aligned}
\sum_{\kappa=1}^{\mathrm{N}_Q} (2-n_\kappa)\lambda_\kappa - \lambda_{a_{L_1}} - \lambda_{a_{L_2}} &= (2-n_1)\lambda_1 + \sum_{\kappa=2}^{\mathrm{N}_Q} (2-n_\kappa)\lambda_\kappa - \lambda_{a_{L_1}} - \lambda_{a_{L_2}} \\
&= (2-n_1)\lambda_1 + \sum_{\kappa=2}^{\mathrm{N}_Q} \Big((2-p_\kappa) - \sum_{\ell \in \mathrm{Child}_{E_\kappa}} (2-p_\ell) \Big) \lambda_\kappa - \lambda_{a_{L_1}} - \lambda_{a_{L_2}} \\
&= \Big((2-n_1) + \sum_{\ell \in \mathrm{Child}_{E_1}} (2-p_\ell) \Big) \lambda_1 - \lambda_{a_{L_1}} - \lambda_{a_{L_2}} \\
&\quad + \sum_{\kappa=2}^{\mathrm{N}_Q} (2-p_\kappa) \Big(\lambda_\kappa - \sum_{\ell \in \mathrm{Prox}_{E_\kappa}} \lambda_\ell \Big) \\
&= 2\lambda_1 - \lambda_{a_{L_1}} - \lambda_{a_{L_2}} + \sum_{\kappa \geq 2, p_\kappa = 1} (\lambda_\kappa - \lambda_{\kappa-1}) \\
&= -\sum_{\kappa=2}^{a_{L_1}} (\lambda_\kappa - \lambda_{\kappa-1}) - \sum_{\kappa=2}^{a_{L_2}} (\lambda_\kappa - \lambda_{\kappa-1}) + \sum_{\kappa \geq 2, p_\kappa = 1} (\lambda_\kappa - \lambda_{\kappa-1}) \\
&\leq \sum_{\kappa \geq a_{L_{j,\nu}}+1,\, p_\kappa = 1} (\lambda_\kappa - \lambda_{\kappa-1}) \\
\text{[Use Proposition 2.5.2]} \qquad &\leq \sum_{j=1}^{m} \mu_j \sum_{\nu=1}^{\mathrm{R}_j} \sum_{\kappa \geq a_{L_{j,\nu}}+1,\, p_\kappa = 1} m_{j,\nu}^{\kappa}. \qquad 2.5.21
\end{aligned}$$

Next, using Observation 2.5.3, we rewrite the remaining term in the left hand side of 2.5.20 as:

$$\sum_{j=1}^{m} \mu_j \sum_{\nu=1}^{\mathrm{R}_j} m_{j,\nu}^1 - \sum_{j=1}^{m} \mu_j \mathrm{R}_j = \sum_{j=1}^{m} \mu_j \sum_{\nu=1}^{\mathrm{R}_j} \big(m_{j,\nu}^1 - 1 \big)$$

$$= \sum_{j=1}^{m} \mu_j \sum_{\nu=1}^{\mathrm{R}_j} \sum_{\kappa\geq 2,\, p_\kappa=2} m_{j,\nu}^{\kappa}$$

$$= \sum_{j=1}^{m} \mu_j \sum_{\nu=1}^{\mathrm{R}_j} \sum_{\kappa\geq a_{L_{j,\nu}}+1,\, p_\kappa=2} m_{j,\nu}^{\kappa}, \qquad 2.5.22$$

where the last equality holds since $p_\kappa = 1$ for all $1 \leq \kappa \leq a_{L_{j,\nu}}$. The inequality 2.5.20 follows directly from 2.5.21 and 2.5.22.

□

We now come back to the proof of the Main Theorem. It follows from Proposition 2.5.3 and Lemma 2.5.2 that

$$\sum_{i=1}^{4} i(L_i) + \sum_{j=1}^{m} \mu_j (2-2g_j) + [T_0]\cdot[\mathscr{C}] \geq \sum_{j=1}^{m} \mu_j \sum_{Q} \delta_Q^j + \sum_{j=1}^{m} \mu_j \sum_{k\neq j} \sum_{Q} (\mathscr{C}_j \cdot \mathscr{C}_k)_Q + \sum_{j=1}^{m} \mu_j \Big(\sum_{Q} (\mathscr{C}_j \cdot \mathscr{L})_Q \Big). \qquad 2.5.23$$

The algebraic intersection of T_0 and $\mathscr{C}$ is given by

$$[T_0]\cdot[\mathscr{C}] = \Big(\sum_{j_1=1}^{m} \mu_{j_1} [\mathscr{C}_{j_1}] \Big) \cdot \Big(\sum_{j_2=1}^{m} [\mathscr{C}_{j_2}] \Big) = \sum_{j_2=1}^{m} \Big(\sum_{j_1=1}^{m} \mu_{j_1} d_{j_1} \Big) d_{j_2} = \sum_{j=1}^{m} d_j.$$

Hence, using Theorem 2.4.3

$$\sum_{j=1}^{m} \mu_j \sum_{Q} \delta_Q^j = \sum_{j=1}^{m} \mu_j \big((d_j-1)(d_j-2) - 2\, g_j \big) = \sum_{j=1}^{m} \mu_j (d_j^2 - 3\, d_j) + \sum_{j=1}^{m} \mu_j (2 - 2\, g_j),$$

and noting that

$$\sum_{j=1}^{m} \mu_j \sum_{k\neq j} \sum_{Q} (\mathscr{C}_j \cdot \mathscr{C}_k)_Q = \sum_{j=1}^{m} \mu_j \sum_{k\neq j} d_j d_k,$$

$$\sum_{j=1}^{m} \mu_j \sum_{Q} (\mathscr{C}_j \cdot \mathscr{L})_Q = 4 \sum_{j=1}^{m} \mu_j d_j,$$

the inequality 2.5.23 can be rewritten as

$$\sum_{i=1}^{4} i(L_i) \geq \sum_{j=1}^{m} \mu_j (d_j^2 - 3\, d_j) + \sum_{j=1}^{m} \mu_j \sum_{k\neq j} d_j d_k + 4 \sum_{j=1}^{m} \mu_j d_j - \sum_{j=1}^{m} d_j$$

$$
\begin{aligned}
&= \sum_{j=1}^{m} \mu_j d_j^2 + \sum_{j=1}^{m} \mu_j \sum_{k\neq j} d_j d_k - \sum_{j=1}^{m} d_j + (4-3) \sum_{j=1}^{m} \mu_j d_j \\
&= \Big(\sum_{j=1}^{m} \mu_j d_j\Big)\Big(\sum_{j=1}^{m} d_j\Big) - \sum_{j=1}^{m} d_j + 1 \\
&= 1 \cdot \Big(\sum_{j=1}^{m} d_j\Big) - \sum_{j=1}^{m} d_j + 1 \\
&= 1
\end{aligned}
$$

The Main Theorem is thus proved.

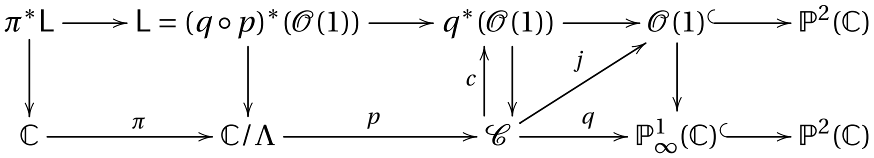
2.6 Construction of algebraically nondegenerate curve having cluster set contained in a given nonhyperbolic curve

In this section, we will give an example of holomorphic curve satisfying the condition on the main theorem. When $\mathscr{C}$ is a line, such an example was given by Da Costa in [dC13], and its method can be generalized to arbitrary nonhyperbolic curves.

Theorem 2.6.1. *Let $\mathscr{C} \subset \mathbb{P}^2(\mathbb{C})$ be an irreducible nonhyperbolic curve. Then there exists an algebraically nondegenerate holomorphic curve $f : \mathbb{C} \to \mathbb{P}^2(\mathbb{C})$ having cluster set contained in $\mathscr{C}$.*

Proof. Let $[z_0 : z_1 : z_2]$ be the homogeneous coordinates of $\mathbb{P}^2(\mathbb{C})$ and let $\mathbb{P}^1_\infty(\mathbb{C}) := \{z_0 = 0\}$ be the line at infinity. Fix a point $A \notin \mathbb{P}^1_\infty(\mathbb{C})$ and let $q : \mathscr{C} \to \mathbb{P}^1_\infty(\mathbb{C})$ be the restriction to $\mathscr{C}$ of the projection from A to the line $\mathbb{P}^1_\infty(\mathbb{C})$. Since $\mathscr{C}$ is not hyperbolic, its universal cover is either $\mathbb{P}^1(\mathbb{C})$ or an elliptic curve $\mathbb{C}/\Lambda$, where $\Lambda \cong \mathbb{Z}\times\mathbb{Z}$ is a lattice in $\mathbb{C}$. Hence, using any doubly periodic elliptic function $\mathbb{C}/\Lambda \to \mathbb{P}^1$, we can always find a nonconstant holomorphic mapping $p : \mathbb{C}/\Lambda \to \mathscr{C}$.

Let L be the pull-back by $q \circ p$ of the line bundle $\mathscr{O}(1)$ on $\mathbb{P}^1_\infty(\mathbb{C})$ and let $\pi : \mathbb{C} \to \mathbb{C}/\Lambda$ be the canonical projection.



Since the line bundle L has a metric of strict positive curvature, its Chern number is strict positive. Hence it admits a hermitian metric h' such that its curvature cohomologous to $\alpha\, \mathrm{d}\mathrm{d}^c\big(|z|^2\big)$ for some $\alpha > 0$, namely their different is of the form $\mathrm{d}\mathrm{d}^c\varphi$ for some smooth real function φ on $\mathbb{C}/\Lambda$. Thus, after replacing the initial metric h' by $h'e^{-\varphi}$, we receive the hermitian metric h having the associated curvature of the form

$$\Theta_h = \alpha\, \mathrm{d}\mathrm{d}^c\big(|z|^2\big).$$

For a section k trivializing the line bundle $\pi^*\mathsf{L}$, using Lelong-Poincaré equation, one has
$$\mathrm{d}\mathrm{d}^c\left(\log\|k\|_h^2\right) = -\alpha\,\mathrm{d}\mathrm{d}^c\left(|z|^2\right),$$
which implies that $\log\|k\|_h^2 + \alpha\,|z|^2$ is a harmonic function. Therefore, we can write

$$\log\|k\|_h^2 + \alpha\,|z|^2 = \mathrm{Re}\,g,$$

for some holomorphic function g. Hence by taking the exponential both sides of the above equation, we obtain
$$\|k\|_h^2 . e^{\alpha\,|z|^2} = |e^g|,$$
or equivalently
$$\|e^{-g/2}\,k\|_h^2 = e^{-\alpha\,|z|^2}.$$
Thus, one can find a trivial section s_1 of $\pi^*\mathsf{L}$ such that $\|s_1\|_h^2 = e^{-\alpha\,|z|^2}$. Now, since the line bundle $\mathscr{O}(1) \to \mathbb{P}^1_\infty(\mathbb{C})$ can be regarded as the pencil of lines though the point A, the embedding $j\colon \mathscr{C} \to \mathscr{O}(1) \subset \mathbb{P}^2(\mathbb{C})$ induces a section $c\colon \mathscr{C} \to q^*(\mathscr{O}(1))$. Let s_2 be the section of $\pi^*\mathsf{L}$ induced by c in the diagram. The section $s = s_1 + s_2$ induces an algebraically nondegenerate holomorphic curve $f\colon \mathbb{C} \to \mathbb{P}^2(\mathbb{C})$ having the cluster set contained in $\mathscr{C}$, since for $r \to \infty$, the image $f(\mathbb{C}\setminus\Delta_r)$ is contained in the $e^{-\alpha\,r^2}$–neighbourhood of $\mathscr{C}$. □

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Titre : SUR LE SECOND THÉORÈME FONDAMENTAL

Mots clés : Théorie de Nevanlinna, Hyperbolicité complexe, Théorie de Ahlfors, Conjecture de Kobayashi, Second Théorème Principal, courbe holomorphe, Courant de Nevanlinna, Lemme de Brody

Résume : La conjecture de Kobayashi stipule qu'une hypersurface générique $X_d \subset \mathbb{P}^{n+1}(\mathbb{C})$ de degré $d \geq 2n+1$ est hyperbolique complexe, un problème qui a attiré une grande attention récemment, avec l'espoir de mettre au point une théorie de Nevanlinna complète en dimension supérieure.
Dans la première partie de cette thèse, notre objectif est de construire des exemples d'hypersurfaces hyperboliques de l'espace projectif dont le degré soit aussi petit que possible. Tout d'abord, en tenant compte du niveau de troncation dans le Second Théorème Principal de Cartan, nous établissons l'hyperbolicité de complémentaires de certaines configurations d'hyperplans avec points de passages, ce qui étend un résultat classique de Bloch-Fujimoto-Green. Ceci nous permet d'amorcer un algorithme récent de Duval, basé sur la méthode de déformation de Zaidenberg, pour créer des sextiques hyperboliques dans $\mathbb{P}^3(\mathbb{C})$, et de construire ainsi des familles d'hypersurfaces hyperboliques $X_d \subset \mathbb{P}^{n+1}(\mathbb{C})$ de degré $d = 2n+2$ pour $2 \leq n \leq 5$.
En adaptant cette technique aux dimensions supérieures, nous obtenons aussi des exemples d'hypersurfaces hyperboliques de degré $d \geq (\frac{n+3}{2})^2$ dans $\mathbb{P}^{n+1}(\mathbb{C})$.
Dans la deuxième partie, nous étudions le problème de diminuer le niveau de troncation dans le Second Théorème Principal de Cartan. Noguchi a conjecturé que dans ce théorème, pour une famille de 4 droites en position générale dans $\mathbb{P}^2(\mathbb{C})$, si une courbe holomorphe entière $f\colon \mathbb{C} \to \mathbb{P}^2(\mathbb{C})$ est supposée n'être pas algébriquement dégénérée, alors le niveau de troncation peut être abaissé à 1. En utilisation la théorie de recouvrement d'Ahlfors pour les surfaces, nous proposons une réponse positive dans le cas où la courbe f est proche d'une certaine courbe algébrique $\mathscr{C} \subset \mathbb{P}^2(\mathbb{C})$, au sens où l'ensemble d'accumulation de $f(\mathbb{C})$ à l'infini, le *cluster set* de f

$$\mathrm{Clu}(f) := \bigcap_{r>0} \left(\overline{f(\mathbb{C}\setminus\Delta_r)}\right)$$

est contenu dans $\mathscr{C}$.

Title : ON THE SECOND MAIN THEOREM

Keywords : Nevanlinna Theory, Complex hyperbolicity, Ahlfors' Theory, Kobayashi's conjecture, Second Main Theorem, holomorphic curve, Nevanlinna current, Brody's Lemma

Abstract : Kobayashi's conjecture asserts that a generic hypersurface $X_d \subset \mathbb{P}^{n+1}(\mathbb{C})$ having degree $d \geq 2n+1$ is complex hyperbolic, a problem that has attracted much attention recently, also with the hope of setting up a complete higher dimensional Nevanlinna theory.
In the first part of this thesis, our goal is to construct examples of hyperbolic hypersurfaces in projective spaces of degree as low as possible. First of all, taking into account the truncation level in Cartan's Second Main Theorem, we establish the hyperbolicity of complements of some configurations of hyperplanes with passage points, extending a classical result of Bloch-Fujimoto-Green. This allows us to launch a recent algorithm of Duval, based on the deformation method of Zaidenberg, on creating hyperbolic sextics in $\mathbb{P}^3(\mathbb{C})$, hence to construct families of hyperbolic hypersurfaces $X_d \subset \mathbb{P}^{n+1}(\mathbb{C})$ having degree $d = 2n+2$ for $2 \leq n \leq 5$.
Adapting this technique to higher dimensional cases, we also obtain examples of hyperbolic hypersurfaces of degree $d \geq (\frac{n+3}{2})^2$ in $\mathbb{P}^{n+1}(\mathbb{C})$.
In the second part, we study the problem of decreasing the truncation level in Cartan's Second Main Theorem. It was conjectured by Noguchi that in this theorem, for a family of 4 lines in general position in $\mathbb{P}^2(\mathbb{C})$, if an entire holomorphic curve $f\colon \mathbb{C} \to \mathbb{P}^2(\mathbb{C})$ is assumed to be algebraically nondegenerate, then the truncation level can be decreased to 1. Using Ahlfors'theory of covering surfaces, we propose a positive answer in the case where the curve f is close to some algebraic curve $\mathscr{C} \subset \mathbb{P}^2(\mathbb{C})$, in the sense that the set of accumulation points of $f(\mathbb{C})$ at infinity, the *cluster set* of f

$$\mathrm{Clu}(f) := \bigcap_{r>0} \left(\overline{f(\mathbb{C}\setminus\Delta_r)}\right)$$

is contained in $\mathscr{C}$.

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