∞-Categories in Algebraic Geometry
Université Paris–Saclay (Orsay)

Lecture 8: Koszul Duality for Commutative Algebras, Part II

Last lecture, we started setting up Koszul duality for augmented commutative k-algebras. Our plan was to generalise the contravariant Koszul duality functor

$$M \in \mathrm{Mod}_A \quad \mapsto \quad \mathbb{R}\mathrm{Hom}_A(M,k) \in \mathrm{Mod}_{\mathfrak{D}^{(1)}(A)}$$

for (left) modules over an associative k-algebra A. This functor can be built in four steps:

(1) Take the left derived tensor product to obtain a colimit-preserving functor

$$k \otimes_A^L (-) : \mathrm{Mod}_A \longrightarrow \mathrm{Mod}_k.$$

Its right adjoint is given by restriction of scalars along the augmentation $A \to k$;

(2) Postcompose with k-linear duality $(-)^\vee$ to obtain a limit-preserving functor

$$\begin{aligned} \mathrm{Mod}_A^{\mathrm{op}} &\longrightarrow \mathrm{Mod}_k \\ M \quad &\mapsto \quad (k \otimes_A^L M)^\vee \simeq \mathbb{R}\mathrm{Hom}_A(M,k); \end{aligned}$$

(3) Construct a differential graded k-algebra $\mathfrak{D}^{(1)}(A) = \mathbb{R}\mathrm{Hom}_A(k,k)$;

(4) Lift $\mathbb{R}\mathrm{Hom}_A(-,k) : \mathrm{Mod}_A^{\mathrm{op}} \to \mathrm{Mod}_k$ to a refined functor

$$\begin{tikzcd} & \mathrm{Mod}_{\mathfrak{D}^{(1)}(A)} \arrow[d] \\ \mathrm{Mod}_A^{\mathrm{op}} \arrow[ur, dashed] \arrow[r] & \mathrm{Mod}_k. \end{tikzcd}$$

To construct Koszul duality in the commutative setting, we make the following substitutes:

Associative algebra k	$\rightsquigarrow$	Identity monad $\mathbf{1} = \mathbb{L}\mathrm{Sym}_k^1$ on $\mathrm{Mod}_{k,\geq 0}$
Augmented algebra A	$\rightsquigarrow$	Augmented monad $\mathbb{L}\mathrm{Sym}_k^{\mathrm{nu}}$ on $\mathrm{Mod}_{k,\geq 0}$
∞-category Mod_A of chain complexes over A	$\rightsquigarrow$	∞-category $\mathrm{SCR}_k^{\mathrm{nu}}$ of nonunital simplicial commutative k-algebras
Restriction of scalars functor $\mathrm{Mod}_k \to \mathrm{Mod}_A$ defined via $A \to k$	$\rightsquigarrow$	Trivial algebra functor $\mathrm{sqz}^{\mathrm{nu}} : \mathrm{Mod}_{k,\geq 0} \to \mathrm{SCR}_k^{\mathrm{nu}}$ defined via $\mathbb{L}\mathrm{Sym}_k^{\mathrm{nu}} \to \mathbf{1}$
Extension of scalars functor $k \otimes_A^L (-) : \mathrm{Mod}_A \longrightarrow \mathrm{Mod}_k$	$\rightsquigarrow$	Cotangent fibre functor $\mathrm{cot}^{\mathrm{nu}} : \mathrm{SCR}_k^{\mathrm{nu}} \to \mathrm{Mod}_{k,\geq 0}$

Using the constructions introduced in Section 7.5 of last lecture, we can generalise step (1) and (2) above to the setting of nonunital simplicial commutative k-algebras as follows:

(1') Take the colimit-preserving cotangent fibre functor

$$\cot^{\mathrm{nu}} : \mathrm{SCR}_k^{\mathrm{nu}} \longrightarrow \mathrm{Mod}_{k,\geq 0}.$$

Its right adjoint is the trivial algebra functor $\mathrm{sqz}^{\mathrm{nu}} : \mathrm{Mod}_{k,\geq 0} \to \mathrm{SCR}_k^{\mathrm{nu}}$;

(2') Postcompose with k-linear duality $(-)^\vee$ to obtain a limit-preserving functor

$$(\mathrm{SCR}_k^{\mathrm{nu}})^{\mathrm{op}} \longrightarrow \mathrm{Mod}_k$$
$$R \mapsto \cot(R)^\vee$$

The goal of today's lecture is to also generalise steps (3) and (4) to the commutative setting.

8.1. **The naive Koszul dual monad.** In step (3) above, we defined the Koszul dual of an augmented associative algebra A as $\mathfrak{D}^{(1)}(A) = \mathbb{R}\mathrm{Hom}_A(k,k)$. Our next goal is to construct a well-behaved Koszul dual monad of the augmented monad $\mathbb{L}\mathrm{Sym}_k^{\mathrm{nu}}$, which encodes nonunital simplicial commutative k-algebras.

To this end, we begin by observing that the functor

$$(\mathrm{SCR}_k^{\mathrm{nu}})^{\mathrm{op}} \longrightarrow \mathrm{Mod}_{k,\leq 0}$$
$$R \mapsto \cot^{\mathrm{nu}}(R)^\vee$$

preserves limits; its right adjoint is given by the assignment $V \mapsto \mathrm{sqz}^{\mathrm{nu}}(V^\vee)$.

By abstract nonsense, this adjunction gives rise to a canonical monad

$$T^{\mathrm{naive}}(-) = (\cot^{\mathrm{nu}}(\mathrm{sqz}^{\mathrm{nu}}((-)^\vee)))^\vee$$

on the full subcategory $\mathrm{Mod}_{k,\leq 0} \subset \mathrm{Mod}_k$ of coconnective chain complexes over k.

This monad T^{naive} suffers from two defects:

a) It is only defined on coconnective complexes, so will not recover the differential graded Lie algebra monad in characteristic 0;
b) It does not preserve sifted colimits, which, as we will see later, is a problem for applications in deformation theory.

To circumvent these obstacles, we will replace T^{naive} by a more well-behaved monad Lie_k^π, which is obtained by left Kan extending a certain restriction of T^{naive}. Making this idea precise will be the goal of the rest of this lecture, and will require several preliminaries.

8.2. **Tangent fibres via partition complexes.** We start by giving a concrete expression for cotangent fibres of trivial square-zero extensions in terms of the following simplicial sets:

Definition 8.1 (Doubly suspended partition complexes)**.** For each $n \geq 0$, we define a simplicial Σ_n-set $P(n)$ by specifying its set of k-simplices as

$$P(n)_k = \left\{ [\hat{0} = \sigma_0 \leq \sigma_1 \leq \ldots \leq \sigma_k = \hat{1}] \;\middle|\; \sigma_i \text{ are partitions of } \{1,\ldots,n\} \right\} \coprod \{*\},$$

where $\hat{0}$ is the discrete partition and $\hat{1}$ is the indiscrete partition of the set $\{1,\ldots,n\}$.

Degeneracy maps insert repeated partitions into chains and fix $*$. Face maps delete partitions from chains whenever this yields a "legal" chain starting in $\hat{0}$ and ending in $\hat{1}$; otherwise, they map to $*$.

Let us fix a simplicial k-vector space $V_\bullet$ with associated chain complex $|V_\bullet| \in \mathrm{Mod}_{k,\geq 0}$. As cot preserves geometric realisations, we obtain, by Corollary 7.20 of the last lecture, the following equivalence:

$$\cot^{\mathrm{nu}}(\mathrm{sqz}^{\mathrm{nu}}(|V_\bullet|)) \simeq |\operatorname{Bar}_\bullet(\mathrm{id}, \mathbb{L}\operatorname{Sym}^{\mathrm{nu}}, |V_\bullet|)|$$
$$\simeq \left\| \ \cdots \ \substack{\leftarrow\\[-2pt]\rightarrow\\[-2pt]\leftarrow\\[-2pt]\rightarrow\\[-2pt]\leftarrow\\[-2pt]\rightarrow} \ \bigoplus_{m\geq 1}\Big(\bigoplus_{n\geq 1}(V_\bullet)^{\otimes n}_{\Sigma_n}\Big)^{\otimes m}_{\Sigma_m} \ \substack{\rightarrow\\[-2pt]\leftarrow\\[-2pt]\rightarrow\\[-2pt]\leftarrow\\[-2pt]\rightarrow} \ \bigoplus_{n\geq 1}(V_\bullet)^{\otimes n}_{\Sigma_n} \ \substack{\leftarrow\\[-2pt]\rightarrow\\[-2pt]\rightarrow} \ V_\bullet \ \right\|.$$

For $(X, *)$ a pointed set, let $k[X]$ be the free k-module on X subject to the relation $0 \simeq *$.

We will now relate the simplicial sets $P(n)_\bullet$, defined using partitions, to the cotangent fibre of trivial square-zero extensions:

Exercise 8.2. Let $V_\bullet$ be a simplicial k-vector space with associated chain complex $V = |V_\bullet|$.

(1) Expand symmetric powers binomially to prove that $\cot^{\mathrm{nu}}(\mathrm{sqz}^{\mathrm{nu}}(V))$ is equivalent to the realisation of the following bisimplicial set:

$$\left\| \ \cdots \ \substack{\leftarrow\\[-2pt]\rightarrow\\[-2pt]\leftarrow\\[-2pt]\rightarrow\\[-2pt]\leftarrow\\[-2pt]\rightarrow} \ \bigoplus_{n\geq 1} k[P(n)_2] \underset{\Sigma_n}{\otimes} (V_\bullet)^{\otimes n} \ \substack{\rightarrow\\[-2pt]\leftarrow\\[-2pt]\rightarrow\\[-2pt]\leftarrow\\[-2pt]\rightarrow} \ \bigoplus_{n\geq 1} k[P(n)_1] \underset{\Sigma_n}{\otimes} (V_\bullet)^{\otimes n} \ \substack{\leftarrow\\[-2pt]\rightarrow\\[-2pt]\rightarrow} \ \bigoplus_{n\geq 1} k[P(n)_0] \underset{\Sigma_n}{\otimes} (V_\bullet)^{\otimes n} \ \right\|,$$

(2) Deduce the following equivalence:

$$\cot^{\mathrm{nu}}(\mathrm{sqz}^{\mathrm{nu}}(V)) \simeq \bigoplus_{n\geq 1} \widetilde{C}_\bullet(|P(n)|, k) \underset{\Sigma_n}{\otimes} (V_\bullet)^{\otimes n}.$$

Here $\widetilde{C}_\bullet(|P(n)|, k)$ are the k-valued chains on the geometric realisation $|P(n)|$ of $P(n)$.

(3) Write $\mathrm{Mod}^{\mathrm{ft}}_{k,\geq 0} \subset \mathrm{Mod}_k$ for the full subcategory spanned by all connective chain complexes with $\dim(\pi_i(V)) < \infty$.

Using the above formula, or otherwise, show that $V \in \mathrm{Mod}^{\mathrm{ft}}_{k,\geq 0}$ is coconnective and of finite type, then $\cot^{\mathrm{nu}}(\mathrm{sqz}^{\mathrm{nu}}(V)) \in \mathrm{Mod}^{\mathrm{ft}}_{k,\geq 0}$ shares the same property.

8.3. **The partition Lie algebra monad.** We return to our goal of replacing the naive Koszul dual monad T^{naive} on $\mathrm{Mod}_{k,\leq 0}$ by a more well-behaved monad defined on Mod_k – We briefly outline our construction:

Construction 1 (Partition Lie algebra monad)**.**

a) First, we observe that Exercise 8.2(3) implies that T^{naive} preserves the full subcategory $\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0} \subset \mathrm{Mod}_k$ of all chain complexes V which are coconnective and satisfy $\dim(\pi_i(V)) < \infty$ for all i. Hence $T^{\mathrm{naive}}|_{\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0}}$ acquires the structure of a monad.

Exercise 8.2(2) gives a quite explicit description of this restriction $T|_{\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0}}$. Indeed, if $V^\bullet$ is a cosimplicial k-module whose associated chain complex $\mathrm{Tot}(V^\bullet)$ is of finite type, then

$$T|_{\mathrm{Mod}^{\mathrm{ft}}_{k,\le 0}}(\mathrm{Tot}(V^\bullet)) \simeq \bigoplus_n \mathrm{Tot}\left(\widetilde{C}^\bullet(|P(n)_\bullet|, k) \otimes (V^\bullet)^{\otimes n}\right)^{\Sigma_n}.$$

Here $\widetilde{C}^\bullet(|P(n)_\bullet|, k)$ are the k-valued cosimplices of $|P(n)_\bullet|$, the functor $(-)^{\Sigma_n}$ takes strict fixed points, and the tensor product is computed in cosimplicial k-modules.

b) We then check that the functor $T|_{\mathrm{Mod}^{\mathrm{ft}}_{k,\le 0}}$
   - *is right complete*, which means that the canonical map

     $$\mathrm{colim}_n\, T(\tau_{\le -n} V) \xrightarrow{\simeq} T(V)$$

     is an equivalence for all $V \in \mathrm{Mod}^{\mathrm{ft}}_{k,\le 0}$;
   - *preserves finite coconnective geometric realisations*, which means that if $V_\bullet$ is a simplicial object in $\mathrm{Mod}^{\mathrm{ft}}_{k,\le 0}$ which is m-skeletal (for some m) and with $|V_\bullet| \in \mathrm{Mod}^{\mathrm{ft}}_{k,\le 0}$, then the canonical map $|T(V_\bullet)| \xrightarrow{\simeq} T(|V_\bullet|)$ is an equivalence.

c) Let us write $\mathrm{End}_\Sigma^{\mathrm{Mod}^{\mathrm{ft}}_{k,\le 0}} \subset \mathrm{End}(\mathrm{Mod}_k)$ for the full subcategory of sifted-colimit-preserving endofuctors of Mod_k which preserve $\mathrm{Mod}^{\mathrm{ft}}_{k,\le 0}$, and let

$$\mathrm{End}'_\sigma(\mathrm{Mod}^{\mathrm{ft}}_{k,\le 0}) \subset \mathrm{End}(\mathrm{Mod}^{\mathrm{ft}}_{k,\le 0})$$

be the full subcategory of right complete endofunctors of $\mathrm{Mod}^{\mathrm{ft}}_{k,\le 0}$ which preserve finite coconnective geometric realisations. We prove in [BM19, Corollary 3.17] that the following monoidal restriction functor is an equivalence:

$$\mathrm{End}_\Sigma^{\mathrm{Mod}^{\mathrm{ft}}_{k,\le 0}}(\mathrm{Mod}_k) \xrightarrow{\simeq} \mathrm{End}'_\sigma(\mathrm{Mod}^{\mathrm{ft}}_{k,\le 0}).$$

d) Using this equivalence, we extend the monad $T|_{\mathrm{Mod}^{\mathrm{ft}}_{k,\le 0}}$ from part (a) to obtain the monad Lie^π_k on Mod_k. This monad Lie^π_k preserves filtered colimits and geometric realisations, and if $V^\bullet$ is a cosimplicial K-module with associated chain complex $\mathrm{Tot}(V^\bullet)$, then

$$\mathrm{Lie}^\pi_k(\mathrm{Tot}(V^\bullet)) \simeq \bigoplus_n \mathrm{Tot}\left(\widetilde{C}^\bullet(\Sigma|\Pi_n|^\diamond, K) \otimes (V^\bullet)^{\otimes n}\right)^{\Sigma_n}.$$

Definition 8.3. The *∞-category of partition Lie algebras* is the ∞-category of algebras over the monad Lie^π_k. We will denote this ∞-category by $\mathrm{Alg}_{\mathrm{Lie}^\pi_k}$.

We have generalised step (3) from the very beginning of this lecture:

(3') Construct the monad Lie^π_k from the augmented monad $\mathbb{L}\mathrm{Sym}^*$.

To generalise step (4), we want to construct a partition Lie algebra for any augmented simplicial commutative k-algebra. To this end, we will use the category $\mathrm{Poly}^{\mathrm{aug}}_k$ of augmented commutative k-algebras of the form

$$k[x_1, \ldots, x_n] \to k$$

from Variant 7.13 of last lecture. By [Lur04, Proposition 3.2.14], we know that $\mathrm{cot}^{\mathrm{aug}}(A)^\vee$ belongs to $\mathrm{Mod}^{\mathrm{ft}}_{k,\le 0}$ for any $A \in \mathrm{Poly}^{\mathrm{aug}}_k$. Hence the tautological T^{naive}-algebra structure on $\cot^{\mathrm{aug}}(A)^\vee \simeq \cot^{\mathrm{nu}}(\mathrm{I}A)^\vee$ equips this chain complex with a Lie_k^π-algebra structure. We therefore obtain a functor $\mathrm{Poly}_k^{\mathrm{aug}} \to \mathrm{Alg}_{\mathrm{Lie}_k^\pi}^{\mathrm{op}}$, which allows us to define:

(4') The Koszul duality functor

$$\mathfrak{D} : \mathrm{SCR}_k^{\mathrm{aug}} \simeq \mathcal{P}_\Sigma(\mathrm{Poly}_k^{\mathrm{aug}}) \longrightarrow \mathrm{Alg}_{\mathrm{Lie}_k^\pi}^{\mathrm{op}}$$

is the unique sifted-colimit-preserving extension of the above functor $\mathrm{Poly}_k^{\mathrm{aug}} \to \mathrm{Alg}_{\mathrm{Lie}_k^\pi}^{\mathrm{op}}$.

8.4. **Partition Lie algebras in characteristic zero.** In this section, we will show that partition Lie algebras reduce to the following classical notion in characteristic zero:

Definition 8.4 (Differential graded Lie algebras)**.** Let k be a field of characteristic zero. A *differential graded Lie algebra* ('DGLA') over k is a complex

$$\ldots \to \mathfrak{g}_2 \to \mathfrak{g}_1 \to \mathfrak{g}_0 \to \mathfrak{g}_{-1} \to \mathfrak{g}_{-2} \to \ldots$$

with a bilinear map $[-,-] : \mathfrak{g}_i \times \mathfrak{g}_j \to \mathfrak{g}_{i+j}$ satisfying the following rules:

(Antisymmetry) $\quad [x,y] = (-1)^{|x||y|+1}[y,x]$

(Jacobi identity) $\quad (-1)^{|x||z|}[[x,y],z] + (-1)^{|z||y|}[[z,x],y] + (-1)^{|y||x|}[[y,z],x] = 0$

(Leibniz rule) $\quad d([x,y]) = [dx,y] + (-1)^{|x|}[x,dy]$.

The category $\mathbf{dgla}_k$ of differential graded Lie algebras admits the structure of a left proper combinatorial model category (c.f. e.g. [Lur11, Proposition 2.1.10]) whose weak equivalences are the quasi-isomorphisms and whose fibrations are the levelwise surjections.

We write dgla_k for the underlying ∞-category of $\mathbf{dgla}_k$.

Recall from Exercise 7.12 of last lecture that if k is a field of characteristic 0, the category $\mathbf{cdga}_k$ of commutative differential graded k algebras carries a model structure whose weak equivalences are the quasi-isomorphisms and whose fibrations are the levelwise surjections. Write cdga_k for the underlying ∞-category of $\mathbf{cdga}_k$.

We will also need the model category $\mathbf{cdga}_k^{\mathrm{aug}} = (\mathbf{cdga}_k)_{/k}$ of augmented commutative differential graded k-algebras and its underlying ∞-category $\mathrm{cdga}_k^{\mathrm{aug}}$.

To compare differential graded Lie algebras with partition Lie algebras, we will rely on the following well-known construction (cf. e.g. [Lur11, Construction 2.2.13]):

Construction 2 (Chevalley–Eilenberg complex)**.** Given a differential graded Lie algebra $\mathfrak{g} \in \mathbf{dgla}_k$, consider its *homological Chevalley–Eilenberg complex* $\mathrm{CE}_*(\mathfrak{g}) = (\mathrm{Sym}^*(\mathfrak{g}[1]), D)$. Here $\mathrm{Sym}^*(\mathfrak{g}[1])$ is the sum of all symmetric powers of the underlying graded vector space of $\mathfrak{g}[1]$. The differential D sends the product of homogeneous elements x_i in degree p_i to

$$\begin{aligned} D(x_1 \ldots x_n) = &\sum_{1\le i\le n} (-1)^{p_1+\ldots+p_{i-1}} x_1 \ldots x_{i-1} dx_i x_{i+1} \ldots x_n \\ &+ \sum_{1\le i<j\le n} (-1)^{p_i(p_{i+1}+\ldots+p_{j-1})} x_1 \ldots x_{i-1} x_{i+1} \ldots x_{j-1} [x_i, x_j] x_{j+1} \ldots x_n. \end{aligned}$$

Write $\mathrm{CE}^*(\mathfrak{g})$ for the linear dual of $\mathrm{CE}_*(\mathfrak{g})$, and define a graded-commutative multiplication on $\mathrm{CE}^*(\mathfrak{g})$ by declaring the product of $f \in \mathrm{CE}^p(\mathfrak{g})$ and $g \in \mathrm{CE}^q(\mathfrak{g})$ to be the element $fg \in \mathrm{CE}^{n+m}(\mathfrak{g})$ satisfying

$$(fg)(x_1 \ldots x_n) = \sum_{S,T} \epsilon(S,T) f(x_{i_1} \ldots x_{i_m}) g(x_{j_1} \ldots x_{j_{n-m}}).$$

Here $x_i \in \mathfrak{g}_{r_i}$ are homogeneous elements, the sum is indexed by disjoint sets $S = \{i_1, \ldots, i_m\}$, $T = \{j_1, \ldots, j_{n-m}\}$ with $S \cup T = \{1, \ldots, n\}$ and $r_{i_1} + \ldots + r_{r_m} = p$, and the sign $\epsilon(S,T)$ is given by $\epsilon(S,T) = \prod_{i \in S, j \in T, i<j} (-1)^{r_i r_j}$.

Remark 8.5. Let $\mathrm{U}\mathfrak{g}$ be the universal enveloping algebra of $\mathfrak{g}$. There are weak equivalences $\mathrm{CE}_*(\mathfrak{g}) \simeq k \otimes^L_{\mathrm{U}\mathfrak{g}} k$ and $\mathrm{CE}^*(\mathfrak{g}) \simeq \mathbb{R}\operatorname{Hom}_{\mathrm{U}\mathfrak{g}}(k,k)$ (cf. [Lur11, Remarks 2.2.11 and 2.2.14]).

Noting that $\mathrm{CE}^*(\mathfrak{g})$ is naturally augmented, the above construction defines a functor

$$\mathrm{CE}^* : \mathbf{dlga}_k^{\mathrm{op}} \to \mathbf{cdga}_k^{\mathrm{aug}}.$$

Inverting weak equivalences, we obtain an induced functor

$$\mathrm{dgla}_k^{\mathrm{op}} \to \mathrm{cdga}_k^{\mathrm{aug}}$$

from the ∞-category of differential graded Lie algebras to the ∞-category of augmented commutative differential graded k-algebras. This functor preserves limits and therefore admits a left adjoint

$$\mathfrak{D}^{\mathrm{dg}} : \mathrm{cdga}_k^{\mathrm{aug}} \to \mathrm{dgla}_k^{\mathrm{op}}$$

These ingredients allow us to prove:

Proposition 8.6. Let k be a field of characteristic zero. The composite

$$\mathrm{Alg}_{\mathrm{Lie}_k^\pi} \to \mathrm{Mod}_k \xrightarrow{\Sigma^{-1}} \mathrm{Mod}_k$$

of the forgetful functor and the shift functor lifts to a canonical equivalence

$$\mathrm{Alg}_{\mathrm{Lie}_k^\pi} \xrightarrow{\simeq} \mathrm{dgla}_k$$

along the forgetful functor $\mathrm{dgla}_k \to \mathrm{Mod}_k$.

Proof. We consider the following pair of adjunctions:

$$\begin{tikzcd} \mathrm{cdga}_k^{\mathrm{aug}} \arrow[r, shift left, "\mathfrak{D}^{\mathrm{dg}}"] \arrow[r, phantom, "\perp"] & \mathrm{dgla}_k^{\mathrm{op}} \arrow[l, shift left, "\mathrm{CE}^*"] \arrow[r, shift left, "\mathrm{forget}_{\mathrm{dgla}}"] \arrow[r, phantom, "\perp"] & \mathrm{Mod}_k^{\mathrm{op}} \arrow[l, shift left, "\mathrm{free}_{\mathrm{dgla}}"] \end{tikzcd}$$

By a straightforward computation (cf. [Lur11, Proposition 2.2.15]), we have

$$\mathrm{CE}^{\mathrm{dg}}(\mathrm{free}_{\mathrm{dgla}}(V)) \simeq k \oplus \Sigma^{-1} V^\vee = \mathrm{sqz}^{\mathrm{aug}}(\Sigma^{-1} V^\vee).$$

Taking adjoints, we obtain an equivalence

$$\mathrm{forget}_{\mathrm{dgla}}(\mathfrak{D}^{\mathrm{dg}}(A)) \simeq \Sigma^{-1} \cot^{\mathrm{aug}}(A)^\vee.$$

Hence the composite of the above adjunctions is

$$\begin{tikzcd} \mathrm{cdga}_k^{\mathrm{aug}} \arrow[r, shift left, "\Sigma^{-1}\cot(-)^\vee"] \arrow[r, phantom, "\perp"] & \mathrm{Mod}_k^{\mathrm{op}} \arrow[l, shift left, "\mathrm{sqz}^{\mathrm{aug}}((\Sigma-)^\vee)"] \end{tikzcd}$$

Inserting the unit $\mathrm{id} \to \mathfrak{D}^{\mathrm{dg}} \circ \mathrm{CE}^*$, we obtain a map of monads

$$\mathrm{Lie}_k^{\mathrm{dg}}(-) = \mathrm{forget}_{\mathrm{dgla}} \circ \mathrm{free}_{\mathrm{dgla}}(-) \longrightarrow \Sigma^{-1}(\cot^{\mathrm{aug}}(\mathrm{sqz}^{\mathrm{aug}}(\Sigma-)^\vee)^\vee) \simeq \Sigma^{-1} T^{\mathrm{naive}} \Sigma(-)$$

Observe that $\mathrm{Lie}_k^{\mathrm{dg}}$ preserves the full subcategory $\mathrm{Mod}_{k,\leq -1}^{\mathrm{ft}} \subset \mathrm{Mod}_k$ of (-1)-coconnective chain complexes V with $\dim(\pi_i(V)) < \infty$ for all i, as Lie brackets decrease degree.

By (a variant of) [Lur07, Lemma 2.3.5], the unit map $\mathfrak{g} \to \mathfrak{D}^{\mathrm{dg}}(\mathrm{CE}^*(\mathfrak{g}))$ is an equivalence for all differential graded Lie algebras $\mathfrak{g}$ with underlying chain complex in $\mathrm{Mod}_{k,\leq -1}^{\mathrm{ft}}$. Hence the above transformation of monads $\mathrm{Lie}_k^{\mathrm{dg}}(-) \to \Sigma^{-1} T^{\mathrm{naive}} \Sigma(-)$ restricts to an equivalence on $\mathrm{Mod}_{k,\leq -1}^{\mathrm{ft}}$. We obtain an equivalence of monads

$$(\Sigma \,\mathrm{Lie}_k^{\mathrm{dg}}\, \Sigma^{-1})|_{\mathrm{Mod}_{k,\leq 0}^{\mathrm{ft}}} \simeq \mathrm{Lie}_k^{\pi}|_{\mathrm{Mod}_{k,\leq 0}^{\mathrm{ft}}}$$

As $\Sigma \,\mathrm{Lie}_k^{\mathrm{dg}}\, \Sigma^{-1}$ preserves sifted colimits, Construction 1(3) gives an equivalence of monads $\Sigma \,\mathrm{Lie}_k^{\mathrm{dg}}\, \Sigma^{-1} \simeq \mathrm{Lie}_k^{\pi}$, which implies the claim. □

References

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