∞-categories in Algebraic Geometry
Université Paris–Saclay (Orsay)

Lecture 6: Koszul Duality for Algebras

Using the Barr–Beck–Lurie theorem, we have proven that if A is a small differential graded algebra over a field k (cf. Definition 5.18 in Lecture 5), then there is an equivalence

$$\mathrm{Ind}(\mathrm{Coh}_A) \simeq \mathrm{Mod}_{\mathfrak{D}^{(1)}(A)^{op}}.$$

Here $\mathfrak{D}^{(1)}(A) \simeq \mathbb{R}\mathrm{Hom}_A(k,k)$ is the Koszul dual of A, satisfying $\pi_*(\mathfrak{D}^{(1)}(A)) \cong \mathrm{Ext}^*_A(k,k)$.

Today, we will single out a certain property of algebras known as the *Koszul property*. It is often satisfied in practice, and makes the computation of $\mathfrak{D}^{(1)}(A)$ extremely simple.

6.1. **The Koszul property.** Let A be an augmented differential graded k-algebra with vanishing differentials, i.e. a homologically graded augmented k-algebra. Set $\overline{A} = \ker(A \to k)$.

Writing $TM = \bigoplus_{n\geq 0} M^{\otimes n}$, we consider the complex of graded A-modules

$$B(A) = \mathrm{Bar}(k,A,k) = (T(\overline{A}), d),$$

where $d([a_1|\dots|a_n]) = \sum_{i=2}^n (-1)^{\epsilon_i}[a_1|\dots|a_{i-1}a_i|\dots|a_n]$ with $\epsilon_i = (|a_1|+1)+\dots+(|a_{i-1}|+1)$.

An element $[a_1|\dots|a_n] \in (\overline{A}^{\otimes n})_i = \mathrm{Bar}_n(k,A,k)_i$ lies in "*internal degree*" $i = |a_1|+\dots+|a_n|$. Write $\mathrm{Tor}^A_*(k,k)_*$ for the bigraded A-module given by the homology of $B(A)$.

Remark 6.1. The chain complex $\mathrm{B}(A) = k \otimes^L_A k \in \mathrm{Mod}_A$ is obtained from the above chain complex of graded A-modules $B(A)$ by placing $[a_1|\dots|a_n]$ in homological degree $(|a_1|+1)+\dots+(|a_n|+1)$. Note the different fonts for B and B.

The key observation is that many algebras A as above admit an additional *Adams grading* indexed by the naturals. Write $A_i[w]$ for the component in homological degree i and Adams degree w, and assume that the augmentation induces an isomorphism $A_*[0] \cong k$.

The Bar construction then picks up a third grading satisfying

$$B(A)_n[w]_* = \bigoplus_{w_1+\dots+w_n=w} (\overline{A}[w_1]\otimes\dots\otimes\overline{A}[w_n])_*,$$

Hence, we obtain a chain complex of bigraded A-modules.

$$\begin{array}{ccccccccccc}
\dots & \to & 0 & \to & B(A)_3[3]_* & \to & B(A)_2[3]_* & \to & B(A)_1[3]_* & \to & 0\\
\dots & \to & 0 & \to & 0 & \to & B(A)_2[2]_* & \to & B(A)_1[2]_* & \to & 0\\
\dots & \to & 0 & \to & 0 & \to & 0 & \to & B(A)_1[1]_* & \to & 0\\
\dots & \to & 0 & \to & 0 & \to & 0 & \to & 0 & \to & k
\end{array}$$

Write $\mathrm{Tor}^A_n(k,k)[w]_i = \pi_n(B(A)[w]_i)$ for the component in homological degree n, internal degree i, and Adams degree w of the corresponding decomposition in homology.

Remark 6.2. More conceptually, $\mathrm{Tor}_n^A(k,k)_*[*]$ is the n^{th} left derived functor of $k \otimes_A (-)$ on the abelian category of bigraded A-modules. This allows us to use other resolutions of k.

From the Bar resolution above, it is clear that $\mathrm{Tor}_n^A(k,k)_*[w]$ vanishes whenever $n > w$. The following definition of Priddy asserts that vanishing also occurs for all $n < w$:

Definition 6.3. Let A be an augmented k-algebra with a homological grading and an Adams grading as above. A is said to be *Koszul* if for all $n \neq w$, we have

$$\mathrm{Tor}_n^A(k,k)_*[w] = \ker(B(A)_n[w]_* \to B(A)_{n-1}[w]_*)/\operatorname{im}(B(A)_{n+1}[w]_* \to B(A)_n[w]_*) = 0$$

Warning 6.4. In his original work [Pri70], Priddy calls these *homogeneous Koszul algebras*.

For simplicity, we will assume from now on that our ground field k satisfies $\mathrm{char}(k) \neq 2$.

Definition 6.5 (Polynomial and exterior algebras)**.** If $x_1, \dots, x_n$ are generators in Adams degree 1 and arbitrary homological degree, we define

$$k[x_1, \dots, x_n] := T(x_1, \dots, x_n)/(x_i \otimes x_j - x_j \otimes x_i);$$

$$E[x_1, \dots, x_n] := T(x_1, \dots, x_n)/(x_i \otimes x_j + x_j \otimes x_i).$$

As we have *not* imposed the Koszul sign rule, $k[x_1, \dots, x_n]$ need not be graded-commutative.

Before studying Koszul algebras in more detail, we give several simple examples.

Example 6.6. Consider $A = k[x]$ generated in Adams degree 1 and homological degree a. We use the following bigraded resolution of the A-module k:

$$\dots \to 0 \to \Sigma^a k[x][+1] \xrightarrow{1 \mapsto x} k[x] \to 0 \to \dots.$$

Here $[+1]$ denotes a shift by 1 in Adams grading and Σ^a is a shift by a in homological grading. Applying $k \otimes_{k[x]} (-)$, we obtain $\dots \to 0 \to \Sigma^a k[+1] \xrightarrow{0} k \to 0 \to 0 \to \dots$. Hence A is Koszul.

Example 6.7. Consider the exterior algebra $A = E[\epsilon] = k[\epsilon]/\epsilon^2$ on a generator in homological degree b and Adams degree 1. The bigraded A-module k admits a resolution

$$\dots \to \Sigma^{2b}(k[\epsilon]/\epsilon^2)[+2] \xrightarrow{1 \mapsto \epsilon} \Sigma^b(k[\epsilon]/\epsilon^2)[+1] \xrightarrow{1 \mapsto \epsilon} k[\epsilon]/\epsilon^2.$$

Applying $k \otimes_A (-)$ gives $\dots \to \Sigma^{2b} k[+2] \xrightarrow{0} \Sigma^b k[+1] \xrightarrow{0} k \to 0 \to \dots$, hence A is Koszul.

Exercise 6.8.

a) Show that if A, A' are Koszul algebras, then so is $A \otimes A'$.
b) Given generators $x_1, \dots, x_n$ in Adams degree 1 and arbitrary homological degree, show that both $k[x_1, \dots, x_n]$ and $E[x_1, \dots, x_n]$ are Koszul.

Exercise 6.9. Prove *directly* that the following algebras in homological degree 0 are Koszul:

(1) The polynomial algebra $k[x,y]$ with x, y in Adams degree 1.
(2) The exterior algebra $E(x,y)$ with x, y in Adams degree 1.
(3) The quantum algebra $A = T(x,y)/(x \otimes y - q y \otimes x)$ for any fixed nonzero scalar $q \in k^\times$.

6.2. **Quadratic generation.** We will prove that all Koszul algebras are of the following type:

Definition 6.10 (Quadratic algebras)**.** Given a graded k-vector space V and a homogeneous subspace $R \subset V \otimes V$, we define the following augmented k-algebra:

$$T(V;R) = T(V)/\langle R\rangle.$$

Here $\langle R\rangle$ denotes the two-sided ideal generated by the subspace R. The algebra $T(V;R)$ inherits a homological grading and an Adams grading, as the space of relations $R \subset V \otimes V$ is homogeneous for both the homological and the Adams grading on $T(V)$.

An augmented bigraded k-algebra A is *quadratic* if $A \cong T(V;R)$ for some $V, R \subset V \otimes V$.

Proposition 6.11. Every Koszul algebra is quadratic.

Proof. We will prove this result in two steps (following an argument presented in [Rez12]). Given $w > 1$, the assumption $\mathrm{Tor}_1^A(k,k)[w]_* = 0$ implies that the following map is surjective:

$$B(A)_2[w]_* = \bigoplus_{\substack{w_1+w_2=w\\ w_i>0}} (\overline{A}[w_1] \otimes \overline{A}[w_2])_* \longrightarrow \overline{A}[w]_* = B(A)_1[w]_*.$$

Hence A is generated in Adams degree 1. Applying $\mathrm{Bar}(-)$ to the surjection $T(\overline{A}[1]) \xrightarrow{f} A$ and taking the kernel gives an exact sequence of complexes of bigraded A-modules:

$$0 \to K \to \mathrm{Bar}(T(\overline{A}[1])) \to \mathrm{Bar}(A) \to 0. \tag{1}$$

In degree 1 of this chain complex, our sequence is given by $0 \to K_1 \to T(\overline{A}[1]) \to A \to 0$. To prove the result, it suffices to show that the following map is surjective for all $w > 2$:

$$\bigoplus_{\substack{w_1+w_2=w\\ w_i>0}} (K_1[w_1] \otimes \overline{A}[1]^{\otimes w_2} \oplus \overline{A}[1]^{\otimes w_1} \otimes K_1[w_2]) \longrightarrow K_1[w]. \tag{2}$$

Indeed, let us restrict attention to degree 1 and degree 2 of the chain complexes in (1):

$$\begin{array}{ccccccc}
0 \longrightarrow K_2[w] & \longrightarrow & \displaystyle\bigoplus_{\substack{w_1+w_2=w\\ w_i>0}} \overline{A}[1]^{\otimes w_1} \otimes \overline{A}[1]^{\otimes w_2} & \xrightarrow{f\otimes f} & \displaystyle\bigoplus_{\substack{w_1+w_2=w\\ w_i>0}} \overline{A}[w_1] \otimes \overline{A}[w_2] \\
\Big\downarrow \delta & & \Big\downarrow & & \Big\downarrow \\
0 \longrightarrow K_1[w] & \longrightarrow & \overline{A}[1]^{\otimes w} & \xrightarrow{f} & \overline{A}[w]
\end{array}$$

Consider the natural map

$$\bigoplus_{\substack{w_1+w_2=w\\ w_i>0}} \big((K_1[w_1] \otimes \overline{A}[1]^{\otimes w_2}) \oplus (\overline{A}[1]^{\otimes j_1} \otimes K_1[w_2])\big) \xrightarrow{\overline{\beta}} \bigoplus_{\substack{w_1+w_2=w\\ w_i>0}} \overline{A}[1]^{\otimes w_1} \otimes \overline{A}[1]^{\otimes w_2}.$$

and its lift $\displaystyle\bigoplus_{\substack{w_1+w_2=w\\ w_i>0}} \big((K_1[w_1] \otimes \overline{A}[1]^{\otimes w_2}) \oplus (\overline{A}[1]^{\otimes j_1} \otimes K_1[w_2])\big) \xrightarrow{\beta} K_2[w].$

The map β is surjective. Indeed, for any decompositions $w = w_1 + w_2$, we tensor the short exact sequences $K_1[w_1] \to \overline{A}[1]^{\otimes w_1} \to \overline{A}[w_1]$ and $K_1[w_2] \to \overline{A}[1]^{\otimes w_2} \to \overline{A}[w_2]$ to obtain a diagram

$$\begin{array}{ccccc}
K_1[w_1] \otimes K_1[w_2] & \longrightarrow & \overline{A}[1]^{\otimes w_1} \otimes K_1[w_2] & \longrightarrow & \overline{A}[w_1] \otimes K_1[w_2] \\
\downarrow & & \downarrow & & \downarrow \\
K_1[w_1] \otimes \overline{A}[1]^{\otimes w_2} & \longrightarrow & \overline{A}[1]^{\otimes w_1} \otimes \overline{A}[1]^{\otimes w_2} & \longrightarrow & \overline{A}[w_1] \otimes \overline{A}[1]^{\otimes w_2} \\
\downarrow & & \downarrow & & \downarrow \\
K_1[w_1] \otimes \overline{A}[w_2] & \longrightarrow & \overline{A}[1]^{\otimes w_1} \otimes \overline{A}[w_2] & \longrightarrow & \overline{A}[w_1] \otimes \overline{A}[w_2]
\end{array}$$

Since k-vector spaces are flat, all columns and rows are exact. A diagram chase shows that

$$(K_1[w_1] \otimes \overline{A}[1]^{\otimes w_2}) \oplus (\overline{A}[1]^{\otimes w_1} \otimes K_1[w_2]) \longrightarrow \ker(\overline{A}[1]^{\otimes w_1} \otimes \overline{A}[1]^{\otimes w_2} \to \overline{A}[w_1] \otimes \overline{A}[w_2])$$

is surjective, which implies that the map β is surjective as well.

As $H_*(\mathrm{Bar}(k, T(\overline{A}[1]), k)[w]_*) = \mathrm{Tor}_*^{T(\overline{A}[1])}(k,k)_*[w] = 0$ for all $w > 1$, the homology long exact sequence induced by (1) shows that $H_1(K_\bullet[w]) \cong \mathrm{Tor}_2^A(k,k)_*[w] = 0$ for all $w > 2$. As $K_0[w] = 0$, this implies that $K_2[w] \xrightarrow{\delta} K_1[w]$ is surjective for all $w > 2$, and hence $\bigoplus_{\substack{w_1+w_2=w \\ w_i>0}} \big((K_1[w_1] \otimes \overline{A}[1]^{\otimes w_2}) \oplus (\overline{A}[1]^{\otimes w_1} \otimes K_1[w_2])\big) \xrightarrow{\delta\circ\beta} K_1[w]$ is also surjective. □

6.3. **Dualising Koszul algebras.** Computing the dual of a Koszul algebra is not hard:

Theorem 6.12. Let A be a Koszul algebra with quadratic presentation $A = T(V;R)$. Assume that $A_i[n]$ and $T(\Sigma^{-1}V^\vee;\Sigma^{-2}R^\perp)_i$ are finite-dimensional for all i, n.

Then the Koszul dual is formal and given by $\mathfrak{D}^{(1)}(A) = \mathbb{R}\,\mathrm{Hom}_A(k,k) \simeq T(\Sigma^{-1}V^\vee;\Sigma^{-2}R^\perp)$, where $V^\vee = \mathrm{Map}_{\mathrm{Mod}_k}(V,k)$ and $R^\perp \subset V^\vee \otimes V^\vee$ is spanned by all $\phi\otimes\psi$ vanishing on $R \subset V \otimes V$.

Proof. Consider the differential graded coalgebra $\mathrm{B}(A) = (T(\Sigma\overline{A}), d)$, where

$$d([a_1|\ldots|a_m]) = \sum_{i=2}^{n} (-1)^{\epsilon_i}[a_1|\ldots|a_{i-1}a_i|\ldots|a_m]$$

with $\epsilon_i = (|a_1|+1)+\ldots+(|a_{i-1}|+1)$. An element $[a_1|\ldots|a_n] \in (\overline{A}^{\otimes n})_i$ lies in homological degree $i = |a_1|+\ldots+|a_n|+n$ in $\mathrm{B}(A)$. Comultiplication sends $[a_1|\ldots|a_m]$ to $\sum_k [a_1|\ldots|a_k] \otimes [a_{k+1}|\ldots|a_m]$.

The graded differential graded coalgebra $\mathrm{B}(A) \simeq \bigoplus_w \mathrm{B}(A)[w]$ can be dualised in two ways:

(1) Applying $\mathrm{Map}_{\mathrm{Mod}_k}(-,k)$ gives the differential graded k-algebra $\mathfrak{D}^{(1)}(A) = \mathrm{B}(A)^\vee$;
(2) Taking the Adams-graded dual gives an Adams-graded differential graded k-algebra with

$$(\mathfrak{D}^{(1)})^{\mathrm{Gr}}(A)[n] := \mathrm{Map}_{\mathrm{Mod}_k}(\mathrm{B}(A)[n], k).$$

These are related by a multiplicative comparison map $\bigoplus_w (\mathfrak{D}^{(1)})^{\mathrm{Gr}}(A)[w] \to \mathfrak{D}^{(1)}(A)$ given by

$$\bigoplus_w \mathrm{B}(A)[w]^\vee \longrightarrow \prod_w \mathrm{B}(A)[w]^\vee \simeq \mathrm{B}(A)^\vee. \tag{3}$$

We begin by computing $\bigoplus_w \mathrm{B}(A)[w]^\vee$. Dualising the maps

$$\mathrm{Tor}_n^A(k,k)[n]_{*-n} \cong \ker\left(B(A)_n[n]_{*-n} \hookrightarrow B(A)_{n-1}[n]_{*-n}\right) \longrightarrow \mathrm{B}(A)[n]_*$$

gives maps $\mathrm{B}(A)[n]_*^\vee \to \mathrm{Ext}_A^n(k,k)[n]_{*+n}$. As A is Koszul, these assemble to an equivalence

$$\bigoplus_w \mathrm{B}(A)[n]_*^\vee \xrightarrow{\simeq} \bigoplus_n \mathrm{Ext}_A^n(k,k)[n]_{*+n}.$$

We can represent elements in

$$\mathrm{Ext}_A^n(k,k)[n]_{*+n} \cong \mathrm{coker}(\bigoplus_k (\overline{A}[1]^\vee)^{\otimes k} \otimes \overline{A}[2]^\vee \otimes (\overline{A}[1]^\vee)^{\otimes(n-k-1)} \longrightarrow (\overline{A}[1]^\vee)^{\otimes n})_{*+n}$$

by expressions $[\alpha_1|\dots|\alpha_n]$ with $\alpha_i \in \overline{A}[1]^\vee = V^\vee$. Here, we used $\dim_k(A_i[n]) < \infty$ for all i,n.

The product on $\bigoplus_w \mathrm{B}(A)[w]_*^\vee$ corresponds to the product on $\bigoplus_w \mathrm{Ext}_A^*(k,k)[w]_{*+n}$ sending elements represented by $[\alpha_1|\dots|\alpha_k]$ and $[\alpha_{k+1}|\dots|\alpha_m]$, respectively, to $[\alpha_1|\dots|\alpha_m]$. The image of $(V^{\otimes 2}/R)^\vee \cong \overline{A}[2]^\vee \longrightarrow (\overline{A}[1]^\vee)^{\otimes 2} \cong (V^\vee)^{\otimes 2}$ is spanned by all elements $\alpha \otimes \beta$ vanishing on R. Hence $\mathrm{Ext}_A^*(k,k)[2]_{*+2} \cong ((V^\vee)^{\otimes 2}/R^\perp)_{*+2} \cong ((\Sigma^{-1}V^\vee)^{\otimes 2}/\Sigma^{-2}R^\perp)_*$, and more generally $\mathrm{Ext}_A^*(k,k)[w]_* \cong (\Sigma^{-1}V^\vee)^{\otimes w}/\bigcup_k \Big((\Sigma^{-1}V^\vee)^{\otimes k} \otimes (\Sigma^{-2}R^\perp) \otimes (\Sigma^{-1}V^\vee)^{\otimes(w-k-1)}\Big)$..
These observations combine to give an equivalence $\bigoplus_w \mathrm{Ext}_A^*(k,k)[w]_* \simeq T(\Sigma^{-1}V^\vee;\Sigma^{-2}R^\perp)$. Since $T(\Sigma^{-1}V^\vee;\Sigma^{-2}R^\perp)_i$ is assumed to be finite-dimensional for all i, this also shows that the comparison map (3) is an equivalence. □

We illustrate Theorem 6.12 in several examples.

Example 6.13. Let V be the graded k-vector space with basis $x_1,\dots,x_n$ in degree 0. Taking $R = \langle x_i \otimes x_j + x_j \otimes x_i\rangle$ gives the exteriour algebra $A = T(V;R) = E[x_1,\dots,x_n]$.

Consider the dual basis $x_1^*,\dots,x_n^*$ of $V^\vee$. An element $v^* = \sum_{i,j} \lambda_{ij}\ x_i^* \otimes x_j^* \in V^\vee \otimes V^\vee$ vanishes on R iff $\lambda_{ij} = -\lambda_{ji} = 0$ for all i,j, which happens iff $v^* \in R^\perp = \langle x_i^* \otimes x_j^* - x_j^* \otimes x_i^*\rangle$. Writing $y_i = \Sigma^{-1}(x_i^*)$, we deduce from Theorem 6.12 that

$$\mathfrak{D}^{(1)}(E[x_1,\dots,x_n]) \simeq k[y_1,\dots,y_n]$$

Exercise 6.14. Prove that there is an equivalence $\mathfrak{D}^{(1)}(k[y_1,\dots,y_n]) \simeq E[x_1,\dots,x_n]$.

This biduality is in fact a general phenomenon, which can be proven (under mild finiteness assumptions) using the Koszul complex. We refer to [BGS96, Secion 2.9] for a precise statement.

Remark 6.15. To see that the finiteness assumption in Theorem 6.12 is indeed necessary, consider the Koszul algebra $E[x]$ with x in Adams degree 1 and homological degree -1. Then $\mathfrak{D}^{(1)}(A) = k[[y]]$ is a power series ring generated by y in homological degree 0, while Theorem 6.12 would predict a polynomial ring.

6.4. **Poincaré Series.** Proposition 6.11 leads to the question whether every quadratic algebra is Koszul. To see that this is *not* the case, we use the following classical notion (in a form presented in [Ber14]):

Definition 6.16 (Poincaré Series)**.** Let $A = \bigoplus_{i,w} A_i[w]$ be an Adams-graded graded algebra with $\dim_k(A_i[w]) < \infty$ for all i, w. The *Poincaré series* of A is given by

$$A(t,z) = \sum_{i,w} \dim_k(A_i[w]) z^i t^w \in k[z,t].$$

We have the following criterion:

Proposition 6.17. If $A = \bigoplus_{i,w} A_i[w]$ is a Koszul algebra as in Theorem 6.12, then

$$P_{\mathfrak{D}^{(1)}(A)}(t,z) = P_A\left(-\frac{t}{z}, z\right)^{-1}.$$

Proposition 6.17 can be used to construct examples of quadratic algebras which are not Koszul. We work through an example of Lech:

Exercise 6.18. Consider the following quadratic algebra with its natural Adams grading, concentrated in homological degree $i = 0$:

$$A = k[x_1, x_2, x_3, x_4]/(x_1^2, x_2^2, x_3^2, x_4^2, x_1x_2 + x_3x_4)$$

(1) Show that $P_A(t,z) = 1 + 4w + 5w^2$.
(2) Compute $P_{\mathfrak{D}^{(1)}(A)}$, and use the answer to prove that A is not a Koszul algebra.

6.5. **PBW algebras.** Theorem 6.12 allows us to dualise an algebra $A = T(V;R)$ once we know that it is Koszul, but checking this property still requires some knowledge about the groups $\mathrm{Tor}^A_*(k,k)_*[*]$.

We will now introduce a simple condition on bases of V which implies that $T(V;R)$ is Koszul: Priddy's *PBW-property* (cf. [PP05, Chapter 4] for a more detailed treatment). Let V be a graded vector space with basis $x_1, \dots x_n$, and suppose that $R \subset V \otimes V$ is a homogeneous submodule of relations. Using the lexicographic order, we define

$$S = \{ (i,j) \mid x_i x_j \notin \mathrm{span}(x_r x_s)_{(r,s)<(i,j)} \subset (V \otimes V)/R \}.$$

Exercise 6.19. Show that the set $\{ x_i x_j \mid (i,j) \in S \}$ forms a basis of $(V \otimes V)/R$.

In particular, for any $(i,j) \notin S$, we can write

$$x_i x_j = \sum_{(r,s)<(i,j)} c_{ij}^{rs} x_r x_s \in T(V;R)$$

for uniquely determined scalars $c_{ij}^{rs} \in k$.

Definition 6.20. We say that $x_1, \dots, x_n$ is a *PBW basis* for the quadratic algebra $T(V;R)$ if the following polynomials form a basis for $T(V;R)$:

$$\{x_{j_1} x_{j_2} \dots x_{j_n} \in T(V;R) \mid (j_1,j_2),(j_2,j_3),\dots,(j_{n-1},j_n) \in S\}.$$

For $n = 0$, the above product is 1 by convention.

Priddy then established the following useful criterion:

Theorem 6.21. If $A = T(V;R)$ admits a PBW-basis, then A is a Koszul algebra.

Example 6.22. For $V = \langle x_1, x_2 \rangle$ and $R = \langle x_1^2, x_1x_2+x_2x_1, x_2^2 \rangle$, we have $E(x_1,x_2) = T(V;R)$. We observe that $S = \{(1,2)\}$. Since $\{1, x_1, x_2, x_1x_2\}$ is a basis for $E(x_1,x_2)$, Theorem 6.21 gives an alternative proof that $E(x_1,x_2)$ is Koszul.

Theorem 6.12 then has the following consequence:

Corollary 6.23. Let $x_1,\dots,x_n$ be a PBW-basis for $A = T(V;R)$. Assume that all $A_i[n]$ and $T(\Sigma^{-1}V^\vee; \Sigma^{-2}R^\perp)_i$ are finite-dimensional. Write $x_ix_j = \sum_{(r,s)<(i,j)} c_{ij}^{rs} x_r x_s \in A$ for $(i,j) \notin S$. Then $\mathfrak{D}^{(1)}(A)$ is generated by $y_1,\dots,y_n$ with y_i in degree $-|x_i|-1$ subject to the relations:

$$(-1)^{\nu_{i,j}} y_i y_j + \sum_{(k,l)\notin S} (-1)^{\nu_{k,l}} c_{rs}^{ij} y_r y_s = 0 \quad \text{if} \quad (i,j) \in S.$$

Here $\nu_{ab} = |y_a| + (|y_a|-1)(|y_b|-1)$.

We conclude this lecture by stating two classical applications of Koszul algebras in topology.

6.6. **Application 1: The Homology of Loop Spaces.** Koszul duality can be used to compute the homology of loop spaces (cf. [Ber14] for a detailed treatment):

Proposition 6.24. Let X be a simply connected space whose algebra of rational cochains $C^*(X;\mathbb{Q}) \simeq H^*(X,\mathbb{Q})$ is both formal and Koszul.

Then the homology of the loop space of X is given by the Koszul dual of $H^*(X;\mathbb{Q})$:

$$H_*(\Omega X;\mathbb{Q}) \cong \mathfrak{D}^{(1)}(H^*(X;\mathbb{Q})).$$

Example 6.25. For $X = S^2$, the cochain algebra $C^*(X,\mathbb{Q})$ is given by $E[x]$ with x in homological degree -2. As this algebra is Koszul, we deduce $H_*(\Omega S^2;\mathbb{Q}) \simeq \mathfrak{D}^{(1)}(E[x]) \simeq \mathbb{Q}[y]$ with y in degree 1. An alternative proof uses the James splitting $\Sigma\Omega\Sigma S^1 \simeq \Sigma \bigvee_{m>0} S^m$.

6.7. **Application 2: The Adams Spectral Sequence.** We begin by recalling the Steenrod algebra, which is of key importance in topology:

Definition 6.26 (The Steenrod algebra)**.** The *Steenrod algebra* $\mathcal{A}$ (at $p=2$) is the associative algebra generated by elements $\mathrm{Sq}^0, \mathrm{Sq}^1, \mathrm{Sq}^2, \dots$ subject to the following relations:

(1) $\mathrm{Sq}^0 = 1$;
(2) If $i < 2j$, then $\mathrm{Sq}^i\,\mathrm{Sq}^j = \sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} \mathrm{Sq}^{i+j-k}\,\mathrm{Sq}^k$.

Given any space X, the $\mathbb{F}_2$-valued cohomology $H^*(X,\mathbb{F}_2)$ is equipped with a natural action by the Steenrod algebra satisfying the following well-known conditions:

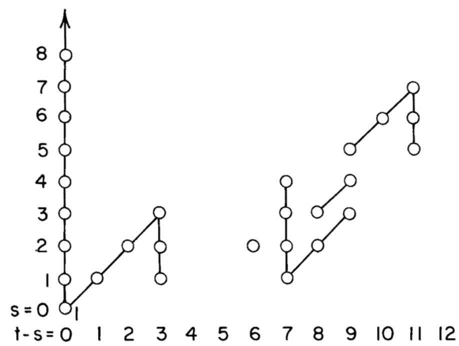
a) $\mathrm{Sq}^n : H^*(X,\mathbb{F}_2) \to H^{*+n}(X,\mathbb{F}_2)$ shifts degree by n;
b) $\mathrm{Sq}^n(x) = 0$ if $x \in H^m(X,\mathbb{F}_2)$ wirth $m < n$;
c) $\mathrm{Sq}^n(x) = x \cup x$ for $x \in H^n(X,\mathbb{F}_2)$;
d) $\mathrm{Sq}^n(x \cup y) = \sum_{a+b=n} \mathrm{Sq}^a(x) \cup \mathrm{Sq}^b(y)$.

Descent along the morphism of $\mathbb{E}_\infty$-rings $S \to \mathbb{F}_2$ can be used to prove the following result of Adams – we refer to [Lur10, Lecture 8] for a more detailed discussion.

Theorem 6.27 (Adams spectral sequence)**.** There is a spectral sequence of signature

$$E_s^{s,t} = \mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \quad \Rightarrow \quad \pi_*^s(S)_2^\wedge$$

We depict the E_2-term of this spectral sequence (in Adams convention), cf. [Rav78]:



It is therefore an important computational problem to compute the algebra $Ext_{\mathcal{A}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$, and our previous discussion leads to the hope that Koszul algebras might be helpful.

Since the defining relations of $\mathcal{A}$ are not homogeneous, the natural grading on the tensor algebra $T(\mathrm{Sq}^0, \mathrm{Sq}^1, \ldots)$ does *not* descend to an Adams-grading on $\mathcal{A}$. However, it induces an ascending filtration whose n^{th} stage $F^n(\mathcal{A})$ is spanned by all products of at most n generators $\mathrm{Sq}^0, \mathrm{Sq}^1, \ldots$.

The associated graded $\mathrm{Gr}(\mathcal{A})$ of this filtration admits an Adams grading, and Theorem 6.21 can be used to prove that $\mathrm{Gr}(\mathcal{A})$ is a Koszul algebra. Priddy then refines the analysis carried out in Theorem 6.12 to prove the following result:

Theorem 6.28. The Koszul dual $\mathfrak{D}^{(1)}(\mathcal{A})$ of the Steenrod algebra is given by the Λ-algebra, which is the differential graded $\mathbb{F}_2$-algebra generated by $\lambda_1, \lambda_2, \lambda_3, \ldots$ subject to relations

$$\lambda_a \lambda_b = \sum_{j=2b}^{\lfloor \frac{2(a+b)}{3} \rfloor} \binom{a-j-1}{j-2b} \lambda_j \lambda_{a+b-j} \quad \text{if } a \geq 2b > 0$$

with differential

$$\delta(\lambda_a) = \sum_{j=1}^{\lfloor \frac{2a}{3} \rfloor} \binom{a-j-1}{j} \lambda_j \lambda_{a-j}$$

The Λ-algebra provides a valuable tool in the computation of stable and unstable homotopy groups of spheres; we refer to [Rav03, Chapter 3] for a detailed discussion.

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