∞-Categories in Algebraic Geometry
Université Paris–Saclay (Orsay)

Lecture 5: Koszul Duality for Modules

In Lecture 1, we have used the classical Barr-Beck theorem to determine when two rings R and S have equivalent categories of left modules $\mathrm{Mod}_R^{\heartsuit} \cong \mathrm{Mod}_S^{\heartsuit}$. Namely, this happens precisely if there is a compact projective generator $Q \in \mathrm{Mod}_R^{\heartsuit}$ with $\mathrm{End}_R(Q)^{op} \cong S$.

Today, we will use Lurie's ∞-categorical monadicity theorem from last class to prove a similar statement for derived ∞-categories of chain complexes. In a second step, we will then use this generalisation to establish a Koszul duality for modules.

For these applications, we need to briefly discuss some further categorical constructions.

5.1. **Stable ∞-categories.** The axioms for stable ∞-categories capture the key properties of derived ∞-categories of chain complexes, just like abelian categories axiomatise the key properties of ordinary categories of modules. We define:

Definition 5.1 (Stable ∞-categories)**.** An ∞-category $\mathcal{C}$ is *stable* if

a) $\mathcal{C}$ is *pointed*, which means that $\mathcal{C}$ admits an object 0 which is both initial and final;
b) Every morphism $f : X \to Y$ in $\mathcal{C}$ admits a fibre $\mathrm{fib}(f)$ and a cofibre $\mathrm{cofib}(f)$, i.e. the following pullback and pushout squares exist in $\mathcal{C}$:

$$\begin{array}{ccc} \mathrm{fib}(f) & \longrightarrow & X \\ \downarrow & \lrcorner & \downarrow f \\ 0 & \longrightarrow & Y \end{array} \qquad\qquad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & \ulcorner & \downarrow \\ 0 & \longrightarrow & \mathrm{cofib}(f) \end{array}$$

c) A square in $\mathcal{C}$ of shape depicted below is a pullback if and only if it is a pushout.

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Z \end{array}$$

These axioms are equivalent to a priori stronger conditions (cf. [Lur, Proposition 1.1.3.4]).

Proposition 5.2. An ∞-category $\mathcal{C}$ is stable if and only if it has a zero object, admits finite limits and colimits, and a general square in $\mathcal{C}$ is a pullback if and only if it is a pushout.

Notation 5.3. Given an object X in a pointed ∞-category $\mathcal{C}$, we will write $\Sigma X = \mathrm{cofib}(X \to 0)$ for the *suspension* of X and $\Omega X = \mathrm{fib}(0 \to X)$ for the *loop object* of X.

Exercise 5.4. Prove that if $\mathcal{C}$ is stable, then Ω and Σ define inverse equivalences.

We then have the following key result (cf. [Lur, Proposition 1.1.4.1]):

Proposition 5.5. A functor $F : \mathcal{C} \to \mathcal{D}$ between stable ∞-categories preserves finite limits if and only if it preserves finite colimits.

5.2. **Spectra.** The primeval example of a stable ∞-category is the ∞-category of spectra, which is an analogue of the category of abelian groups in ordinary category theory.

We briefly outline its construction. Write $\mathcal{S}_* = \mathcal{S}_{*/}$ for the ∞-category of pointed spaces (cf. Lecture 2, Example 2.23.c). The one-point space $*$ is a zero object in $\mathcal{S}_*$, and we obtain a loops functor $\Omega : \mathcal{S}_* \to \mathcal{S}_*$.

Definition 5.6 (Spectra)**.** The ∞-category Sp if *spectra* is given by the homotopy limit of the following tower of ∞-categories:

$$\dots \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_*$$

Informally, spectra are sequences of pointed spaces $X_0, X_1, \dots$ with equivalences $\Omega X_{n+1} \simeq X_n$.

We will now state several important facts about the ∞-category Sp without proof; for a comprehensive treatment of spectra in the language of ∞-categories, we refer to [Lur], in particular Section 1.4.3.

a) The natural functor $\Omega^\infty : \mathrm{Sp} \to \mathcal{S}_*$ admits a left adjoint $\Sigma^\infty : \mathcal{S}_* \to \mathrm{Sp}$, which exhibits spectra as the *stabilisation* of spaces (the precise universal property of Sp is articulated in [Lur, Corollary 1.4.4.5]).
b) The functor Ω^∞ preserves filtered colimits, but it does *not* preserve geometric realisations.
c) Any $X \in \mathrm{Sp}$ is a canonical filtered colimit of pointed spaces $X \simeq \mathrm{colim}_n \Sigma^{\infty-n}\Omega^{\infty-n}X$, where $\Omega^{\infty-n} = \Omega^\infty \circ \Sigma^n$ and $\Sigma^{\infty-n} = \Omega^n \circ \Sigma^\infty$.
d) The ∞-categories $\mathcal{S}_*$ and Sp admit monoidal structures $\wedge$ and $\otimes$, both called *smash product*, and Σ^∞ is monoidal. In fact, both $\wedge$ and $\otimes$ define *symmetric monoidal structures.* We have not defined this notion yet, but this is a simple variation of Definition 3.7 in Lecture 3 (obtained by replacing Δ^{op} by the category of finite pointed sets Fin_*).
e) The ∞-category Sp admits a t-structure, which means that there are full subcategories $\mathrm{Sp}_{\geq 0}$ (*connective spectra*) and $\mathrm{Sp}_{\leq 0}$ (*coconnective spectra*), satisfying the following conditions:
   i) For $X \in \mathrm{Sp}_{\geq 0}$ and $Y \in \mathrm{Sp}_{\leq 0}$, we have $\mathrm{Map}_{\mathrm{Sp}}(X, \Sigma^{-1}Y) = 0$;
   ii) The functor Σ preserves $\mathrm{Sp}_{\geq 0}$ and the functor Ω preserves $\mathrm{Sp}_{\leq 0}$;
   iii) Any $X \in \mathrm{Sp}$ sits in a fibre sequence $\tau_{\geq 0}X \to X \to \tau_{\leq -1}X$ with $\tau_{\geq 0}X \in \mathrm{Sp}_{\geq 0}, \Sigma\tau_{\leq -1}X \in \mathrm{Sp}_{\leq 0}$.

   The heart $\mathrm{Sp}^\heartsuit = \mathrm{Sp}_{\geq 0} \cap \mathrm{Sp}_{\leq 0}$ of this t-structure is equivalent to $\mathrm{N(Ab)}$, the (nerve of the) ordinary category of abelian groups.
f) Using the monoidal structure $\otimes$ on Sp, we obtain an ∞-category Alg(Sp) of algebra objects (cf. Definition 3.14, Lecture 3) in Sp, which are usually called $\mathbb{E}_1$*-ring spectra.*
g) The full subcategory of Alg(Sp) spanned by all objects whose underlying spectrum lies in $\mathrm{Sp}^\heartsuit$ is equivalent to the (nerve of the) ordinary category of associative rings (cf. [Lur, Proposition 7.1.3.18]). Hence, we can identify rings with discrete $\mathbb{E}_1$-ring spectra.
h) Given an $\mathbb{E}_1$-ring $A \in \mathrm{Alg(Sp)}$, Definition 3.19 from Lecture 3 gives an ∞-category Mod_A of A-module objects, which we will refer to as *A-module spectra.* Here, we have used that the monoidal ∞-category Sp is naturally tensored over itself.
i) Given objects X, Y in a general stable ∞-category $\mathcal{C}$, the space $\mathrm{Map}_{\mathcal{C}}(X, Y)$ deloops to a spectrum $\underline{\mathrm{Map}}_{\mathcal{C}}(X, Y)$, whose n^{th} space is satisfies $\Omega^{\infty-n}\underline{\mathrm{Map}}_{\mathcal{C}}(X, Y) \simeq \mathrm{Map}_{\mathcal{C}}(X, \Sigma^n Y)$.

When $X = Y$, then $\underline{\mathrm{End}}_{\mathcal{C}}(X) := \underline{\mathrm{Map}}_{\mathcal{C}}(X,X)$ can be equipped with the structure of an $\mathbb{E}_1$-ring spectrum, with multiplication given by composition (cf. [Lur, Remark 7.1.2.2]).

j) If A is an ordinary ring, then Mod_A can be identified with the unbounded derived ∞-category of A, whose objects are chain complexes of A-modules $\ldots \to M_2 \to M_1 \to \ldots$. Given ordinary R-modules M, N, we have $\mathrm{Ext}_R^{-*}(M,N) \cong \pi_*\left(\underline{\mathrm{Map}}_{\mathrm{Mod}_R}(M,N)\right)$.

We will discuss this point in more detail later.

5.3. **The Ind-construction.** Given an ∞-category $\mathcal{C}$, the presheaf ∞-category

$$\mathcal{P}(\mathcal{C}) := \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$$

freely adds small colimits (cf. [Lur09, Theorem 5.1.5.6]):

Proposition 5.7 (Universal property of the presheaf category)**.** Let $\mathcal{C}$ be a small ∞-category and $\mathcal{D}$ an ∞-category with small colimits. The Yoneda embedding $\mathcal{C} \to \mathcal{P}(\mathcal{C})$ induces an equivalence

$$\mathrm{Fun}^L(\mathcal{P}(\mathcal{C}), \mathcal{D}) \xrightarrow{\simeq} \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

between the ∞-category of small-colimit-preserving functors $\mathcal{P}(\mathcal{C}) \to \mathcal{D}$ and the ∞-category of all functors $\mathcal{C} \to \mathcal{D}$.

A variant of the $\mathcal{P}(-)$-construction only adds *filtered* colimits (cf. Definition 2.32, Lecture 2):

Definition 5.8 (Ind-construction)**.** Given a small ∞-category $\mathcal{C}$, let $\mathrm{Ind}(\mathcal{C}) \subset \mathcal{P}(\mathcal{C})$ be the full subcategory spanned by all functors $\mathcal{C}^{op} \to \mathcal{S}$ which preserve finite limits.

This construction satisfies the following universal property (cf. [Lur09, Proposition 5.3.5.10]

Proposition 5.9 (Universal property of the Ind-construction)**.** Let $\mathcal{C}$ be a small ∞-category and $\mathcal{D}$ be any ∞-category containing small filtered colimits. Restriction along the Yoneda embedding $\mathcal{C}$ induces an equivalence $\mathrm{Fun}_\omega(\mathrm{Ind}(\mathcal{C}), \mathcal{D}) \xrightarrow{\simeq} \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ between the ∞-category $\mathrm{Fun}_\omega(\mathrm{Ind}(\mathcal{C}), \mathcal{D})$ of filtered-colimit-preserving functors $\mathrm{Ind}(\mathcal{C}) \to \mathcal{D}$ and the ∞-category of all functors $\mathcal{C} \to \mathcal{D}$.

We now assume that $\mathcal{C}$ is an ∞-category with finite colimits, and state several key properties of the Ind-construction $\mathrm{Ind}(\mathcal{C})$:

a) The Yoneda embedding $j : \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$, $X \mapsto j(X) = \mathrm{Map}_{\mathcal{C}}(X, -)$ is fully faithful, preserves finite colimits and small limits, and $j(X) \in \mathrm{Ind}(\mathcal{C})$ is compact for all $X \in \mathcal{C}$;
b) The ∞-category $\mathrm{Ind}(\mathcal{C})$ admits small colimits;
c) If $\mathcal{C}$ is stable, then so is $\mathrm{Ind}(\mathcal{C})$.
d) Any $X \in \mathrm{Ind}(\mathcal{C})$ can be obtained as a filtered colimit $X = \mathrm{colim}_a\, j(X_a)$ of objects $X \in \mathcal{C}$. If $Y = \mathrm{colim}_b\, j(Y_b)$ is another such object, we can compute the mapping space as

$$\begin{aligned}\mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(X,Y) &\simeq \lim_b \mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(X, j(Y_b)) \\ &\simeq \lim_b \mathrm{colim}_a \mathrm{Map}_{\mathcal{C}}(j(X_a), j(Y_b)) \\ &\simeq \lim_b \mathrm{colim}_a \mathrm{Map}_{\mathcal{C}}(X_a, Y_b).\end{aligned}$$

The first equivalence is tautological, the second used that any object in the image of the Yoneda embedding is compact, and the third uses that j is fully faithful.

Definition 5.10 (Compact generation)**.** An ∞-category $\mathcal{D}$ is said to be *compactly generated* if there is a small ∞-category $\mathcal{C}$ with finite colimits and an equivalence $\mathcal{D} \simeq \mathrm{Ind}(\mathcal{C})$.

Remark 5.11. Many ∞-categories in nature are compactly generated. For example, the ∞-categories of spaces $\mathcal{S}$, spectra Sp, and module spectra Mod_R over a given $\mathbb{E}_1$-ring $R \in \mathrm{Alg}(\mathrm{Sp})$ satisfy this property. The easiest way to prove this is to exhibit all these ∞-categories as sifted-colimit-completions (cf. [Lur09, Proposition 5.5.8.10]).

Digression 5.12. There are also various ∞-categories of interest which are not compactly generated, such as the ∞-category of $K(n)$-local spectra in chromatic homotopy theory.

However, most of them fit into the more general framework of *presentable ∞-categories*, which we shall briefly outline. Given a regular cardinal κ, we say that a simplicial set is κ-small if its set of nondegenerate simplices has cardinality less than κ. For $\kappa = \omega$, this recovers the notion of a finite simplicial set. We can then define the notion of a κ-filtered ∞-category (cf. [Lur09, Definition 5.3.1.7]) generalising Definition 2.32 in Lecture 2, by allowing extensions over cones of all κ-small simplicial sets (rather than just finite ones). A generalisation of the Ind-construction, denoted by Ind_κ, then freely adds κ-filtered colimits.

An ∞-category $\mathcal{D}$ is said to be *presentable* if it can be written as $\mathcal{D} \simeq \mathrm{Ind}_\kappa(\mathcal{C})$ for *some* regular cardinal κ, where $\mathcal{C}$ is a small ∞-category containing all κ-small colimits. A list of equivalent conditions for presentability is given in [Lur09, Theorem 5.5.1.1].

Presentable ∞-categories $\mathcal{D} \simeq \mathrm{Ind}_\kappa(\mathcal{C})$ always admit small colimits; this is implied by the assumption that $\mathcal{C}$ admits κ-small colimits. If one removes this assumption, one obtains the notion of an *accessible* ∞-category.

5.4. **Colimit-preserving monads on Spectra.** The universal property of stabilisation (alluded to in Definition 5.6a)) implies that Sp is the free stable ∞-category generated by a single object, the *sphere spectrum* $\mathbb{S} = \Sigma^\infty(S^0)$ (cf. [Lur, Corollary 1.4.4.6.]):

Proposition 5.13. Given a compactly generated (or in fact presentable) stable ∞-category $\mathcal{D}$, evaluation at S induces an equivalence $\mathrm{Fun}^L(\mathrm{Sp},\mathcal{D}) \xrightarrow{\simeq} \mathcal{D}$. Here $\mathrm{Fun}^L(\mathrm{Sp},\mathcal{D}) \subset \mathrm{Fun}^L(\mathrm{Sp},\mathcal{D})$ is the full subcategory spanned by functors which preserve small colimits.

Taking $\mathcal{D} = \mathrm{Sp}$, we obtain an identification $\mathrm{Fun}^L(\mathrm{Sp},\mathrm{Sp}) \xrightarrow{\simeq} \mathrm{Sp}$. The left hand side carries a natural monoidal structure given by composition, and this can be taken as a definition of the smash product $\otimes$ on the right hand side Sp. However, more work is necessary to show that $\otimes$ *symmetric*; we refer to the beginning of [Lur, Section 4.8.2] for a discussion. The inverse of the above equivalence carries $X \in \mathrm{Sp}$ to $X \otimes (-)$. Passing to algebras, we deduce:

Proposition 5.14. Evaluation at $\mathbb{S}$ induces an equivalence $\mathrm{Alg}(\mathrm{Fun}^L(\mathrm{Sp},\mathrm{Sp})) \xrightarrow{\simeq} \mathrm{Alg}(\mathrm{Sp})$ between small-colimit-preserving monads on Sp and $\mathbb{E}_1$-ring spectra.

If an $\mathbb{E}_1$-ring $R = T_R(\mathbb{S})$ corresponds to a monad $T_R \in \mathrm{Alg}(\mathrm{Fun}^L(\mathrm{Sp},\mathrm{Sp}))$, then there is a canonical equivalence $\mathrm{Mod}_R \simeq \mathrm{Alg}_{T_R}(\mathrm{Sp})$.

5.5. **The Recognition Principle.** We will now develop a derived variant of Morita theory.

Let $\mathcal{C}$ be a compactly generated (or in fact presentable) ∞-category. Given any $Q \in \mathcal{C}$, the assignment $G_Q = \underline{\mathrm{Map}}_{\mathcal{C}}(Q,-) : \mathcal{C} \to \mathrm{Sp}$ preserves small limits. By a version of the adjoint functor theorem (cf. [Lur09, Corollary 5.5.2.9]), the functor G_Q admits a left adjoint $F_Q : \mathrm{Sp} \to \mathcal{C}$, which we will write as $F_Q(X) = X \otimes Q$. As notation suggests, the assignment $(X, Q) \mapsto F_Q(X) = X \otimes Q$ equips $\mathcal{C}$ with the structure of a Sp-tensored ∞-category. Note that by Proposition 5.13, F_Q is uniquely determined by the requirement that it preserves small colimits and sends the sphere spectrum $\mathbb{S}$ to $F_Q(\mathbb{S}) = Q$. We can now show:

Theorem 5.15 (Schwede–Shipley)**.** Let $\mathcal{C}$ be a compactly generated (or in fact presentable) stable ∞-category . Let $Q \in \mathcal{C}$ be an object satisfying the following properties:

a) Q is compact (cf. Lecture 2, Definition 2.35);
b) Q is a generator for $\mathcal{C}$, which means that $\underline{\mathrm{Map}}_{\mathcal{C}}(Q, D) \simeq 0$ implies $D \simeq 0$.

Then $G = \underline{\mathrm{Map}}_{\mathcal{C}}(Q, -) : \mathcal{C} \to \mathrm{Sp}$ is part of a monadic adjunction $F \dashv G$, the associated monad T preserves small colimits, and we obtain equivalences $\mathcal{C} \simeq \mathrm{Alg}_T(\mathrm{Sp}) \simeq \mathrm{Mod}_{\underline{\mathrm{End}}_{\mathcal{C}}(Q)^{op}}$.

Proof. We begin by checking that the right adjoint G (and hence T) preserves small colimits. Indeed, using Definition 5.6 c), we can write the functor G as

$$G(X) \simeq \mathrm{colim}_n\, \Sigma^{\infty-n}\Omega^{\infty-n}\underline{\mathrm{Map}}_{\mathcal{C}}(Q, X) \simeq \mathrm{colim}_n\, \Sigma^{\infty-n}\mathrm{Map}_{\mathcal{C}}(Q, \Sigma^n X).$$

Since Q is assumed to be compact, this composite of filtered-colimit-preserving functors must preserve filtered colimits. As G tautologically preserves finite (and in fact all) limits, it also preserves finite colimits Proposition 5.5. Any functor which preserves both finite and filtered colimits must preserve all small colimits.

By Proposition 5.14, the monad T is therefore given by $T(-) = R \otimes (-)$ for some $\mathbb{E}_1$-ring spectrum S. Unraveling the definition, we see that S is equivalent to $\mathrm{End}_{\mathcal{C}}(Q)^{op} = T(\mathbb{S})$, which implies the second asserted equivalence.

To prove the equivalence $\mathcal{C} \simeq \mathrm{Alg}_T(\mathrm{Sp})$, we apply Lurie's ∞-categorical Barr-Beck theorem. To verify that G is conservative, assume that G sends a morphism $f : X \to Y$ in $\mathcal{C}$ to an equivalence $G(f) : G(X) \to G(Y)$ in Sp. Since G preserves colimits, we have $G(\mathrm{cof}(f)) \simeq \mathrm{cof}(G(f)) \cong 0$, which implies that $\mathrm{cof}(f) \simeq 0$ since Q is a generator. Hence f is an equivalence. Since G preserves small colimits, it in particular preserves geometric realisations . The (crude) Barr–Beck–Lurie theorem (cf. Lecture 4, Theorem 4.6) shows that G induces an equivalence $\mathcal{C} \simeq \mathrm{Alg}_T(\mathrm{Sp})$. ☐

Remark 5.16. Any equivalence $\mathrm{Mod}_R \simeq \mathrm{Mod}_S$ between module ∞-categories of $\mathbb{E}_1$-ring spectra arises as in Theorem 5.15 (cf. [Lur, Section 4.8.4]).

Remark 5.17. If R and S are ordinary rings and $Q \in \mathrm{Mod}_R$ is a compact generator of $\mathcal{C} = \mathrm{Mod}_R$ for which $\mathrm{End}_Q(R)^{op}$ is the discrete ring spectrum S, then R and S have equivalent derived ∞-categories $\mathrm{Mod}_R \simeq \mathrm{Mod}_S$. This (of course) happens whenever R and S are Morita equivalent, but may also happen when R and S are *not* Morita equivalent. Example 3.25 in [Sch04] gives an example by considering two matrix rings over a field k:

$$R = \left\{ \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ 0 & x_{22} & x_{23} \\ 0 & 0 & x_{33} \end{pmatrix} \middle| \; x_{ij} \in k \right\} \qquad S = \left\{ \begin{pmatrix} y_{11} & y_{12} & y_{13} \\ 0 & y_{22} & 0 \\ 0 & 0 & y_{33} \end{pmatrix} \middle| \; y_{ij} \in k \right\}.$$

5.6. **Koszul Duality for Modules.** Let k be a field and $A \in \mathrm{Alg}^{\mathrm{aug}}(\mathrm{Mod}_k) = \mathrm{Alg}(\mathrm{Mod}_k)_{/k}$ an augmented algebra object in Mod_k, i.e. an augmented differential graded k-algebra.

The idea behind Koszul duality is to try to use the functor

$$G(-) := \underline{\mathrm{Map}}_{\mathrm{Mod}_k}(k, -) : \mathrm{Mod}_A \to \mathrm{Sp}$$

to produce an equivalence of ∞-categories, in spite of the fact that k is usually not compact (cf. Exercise 1.24. in Lecture 1).

We will be able to give an explicit Koszul equivalence if A is "small" in the following sense (cf. [Lur11, Definition 3.0.1]):

Definition 5.18 (Small algebras)**.** A differential graded k-algebra $A \in \mathrm{Alg}(\mathrm{Mod}_k)$ is *small* if

a) A is connective;
b) $\dim_k(\pi_*(A)) < \infty$;
c) Writing $\mathfrak{n}$ for the radical of A, the canonical map $k \xrightarrow{\simeq} \pi_0(A)/\mathfrak{n}$ is an equivalence.

Note that small algebras are canonically augmented. The main point of interest is that we can explicitly describe a certain subcategory of Mod_A if particular interest (cf. [Lur11, Remark 3.4.2]):

Proposition 5.19. The thick subcategory (cf. Definition 1.26, Lecture 1) $\mathrm{Thick}_A(k)$ of Mod_A generated by k is equivalent to the full subcategory $\mathrm{Coh}_A \subset \mathrm{Mod}_A$ spanned by all $M \in \mathrm{Mod}_A$ for which $\dim_k(\pi_*(M)) < \infty$.

Remark 5.20. Module spectra lying in $\mathrm{Coh}_A \subset \mathrm{Mod}_A$ are said to be *coherent*.

Proof of Proposition 5.19. Since Coh_A is evidently thick and contains k, the inclusion $\mathrm{Thick}_A(k) \subset \mathrm{Coh}_A$ is evident. For the converse, pick $M \in \mathrm{Coh}_A$ nonzero. We can then chose n minimal with $\pi_n(M) \neq 0$ and an element $x \in \pi_n(M)$ which is annihilated by the (nilpotent) augmentation ideal $\ker(\pi_0(A) \to k)$. The cofibre sequence $\Sigma^n k \xrightarrow{x} M \to M'$ induces a long exact sequence on homotopy groups, which shoes that $\dim_k(\pi_*(M')) < \dim_k(\pi_*(M))$. By induction, we can therefore assume that $M' \in \mathrm{Thick}_A(k)$, which also implies the claim for M as $\Sigma^n k \in \mathrm{Thick}_A(k)$. ☐

We deduce the following well-known result (cf. [Lur11, Remark 3.4.5]), which goes back to the work of Beilinson-Ginzburg-Soergel ([BGS96]):

Theorem 5.21. For A a small $\mathbb{E}_1$-ring as above, there is an equivalence of stable ∞-categories $\mathrm{Ind}(\mathrm{Coh}_A) \simeq \mathrm{Mod}_{\mathfrak{D}^{(1)}(A)^{op}}$ for $\mathfrak{D}^{(1)}(A) := \mathrm{End}_{\mathrm{Mod}_A}(k)$.

Proof. We start with the ∞-category $\mathrm{Ind}(\mathrm{Coh}_A) \simeq \mathrm{Ind}(\mathrm{Thick}_A(k))$, which is stable as $\mathrm{Thick}_A(k)$ has this property. Like any element in the image of the Yoneda embedding, k is a compact object. To see that k is also a generator, assume that $\underline{\mathrm{Map}}_{\mathcal{C}}(k, M) = 0$ and consider the full subcategory $\{N \in \mathrm{Thick}_A(k) \mid \underline{\mathrm{Map}}_{\mathcal{C}}(N, M) = 0\} \subset \mathrm{Thick}_A(k)$. Since it is thick, we deduce that any $N \in \mathrm{Thick}_A(k)$ satisfies $\underline{\mathrm{Map}}_{\mathcal{C}}(N, M) = 0$. In particular, this applies to M itself, and we have $\underline{\mathrm{Map}}_{\mathcal{C}}(M, M) = 0$. This implies that $M = 0$. ☐

References

[BGS96] Alexander Beilinson, Victor Ginzburg, and Wolfgang Soergel, *Koszul duality patterns in representation theory*, Journal of the American Mathematical Society **9** (1996), no. 2, 473–527.

[Lur] Jacob Lurie, *Higher algebra*, Preprint from the author's web page.

[Lur09] ______, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009. MR 2522659

[Lur11] ______, *Derived algebraic geometry X: Formal moduli problems*, Preprint from the author's web page (2011).

[Sch04] Stefan Schwede, *Morita theory in abelian, derived and stable model categories*, Structured ring spectra **315** (2004), 33–86.