The posterity of *Residues and Duality*[1]

Luc Illusie

1. Derived categories

Derived and triangulated categories have changed the face of homological algebra. Since their introduction in [RD][2] and Verdier's notes [46], the theory has undergone considerable developments (among the most notable ones, let me just mention filtered derived categories, t-structures and perverse sheaves, derived ∞-categories). It is of common use today in almost all parts of algebraic geometry and homotopy theory. I will limit myself here to discussing a few points that are more or less directly related to [20] and [RD].

1.1. *Signs.* Sign problems have plagued homological algebra since the very beginning. Derived categories and derived functors and the ensuing formalism of *six operations* made it hard to elaborate a coherent system of conventions of signs. Deciding whether a given diagram involving canonical maps between various derived functors commutes is often a highly nontrivial matter. Grothendieck didn't pay too much attention to signs, considering that it was a matter of "routine verification". Some discrepancies in [RD] were observed by Deligne in ([16], Appendice, Signes). Here he proposes a system of conventions, which he thinks is coherent. They are based on those he developed in ([1], XVII, 1.1, 1.2) (see ([2], p. 312) for an erratum). A concise summary of them is given in ([10], 0.3). Sign conventions are also discussed in detail in section 1.3 of Conrad's book [14], the purpose of which is to provide proofs for delicate compatibilities left over in [RD] and straighten the signs. However, discrepancies were later found in it by Gabber (see Conrad's homepage for corrections and updates).

1.2. *Unbounded derived categories.* Leaving aside the questions of finiteness (such as coherence or constructibility of cohomology sheaves), it was long tacitly assumed that the definition of the usual derived functors such as Rf_*, Lf^*, $\otimes^L$, etc., required degree restrictions on their sources. For example, for a morphism of schemes $f : X \to Y$, denoting by $D(X)$ (resp.

[1] This text, written at the suggestion of R. Hartshorne, was supposed to be included in [25], of which I was to be a co-editor. The editors of the collection had given their agreement. However, in May 2023 they explained to me that Grothendieck's children would grant their permission to publish only if I withdrew from the project, which I did. Hence, despite the (pleasant) collaboration I had had with R. Hartshorne on this project from 2019 to 2023, I was not a co-editor, my text was removed, and all references to my name were eradicated, except for a one line of thanks at the end of the preface.

[2] [RD] stands for [23].

$D(Y)$) the derived category of $\mathcal{O}_X$-modules (resp. $\mathcal{O}_Y$-modules), if no assumption of finiteness or cohomological dimension was made, the functor Rf_* was defined only on the bounded below category $D^+(X)$. In the early 1960's Grothendieck asked whether, if one worked with *good* abelian categories (such as the category of $\mathcal{O}_X$-modules on a scheme, or a ringed topos, or more generally what is now called a *Grothendieck abelian category* ([29], 8.3.24)), one could get rid of these restrictions (see e.g. ([20], 2.2.3 (3))). It is only in 1988 that a satisfactory solution was proposed, by Spaltenstein [45], using the new notion of *homotopically injective* (resp. *projective*) *resolution*. A more systematic treatment is given in ([29], §§14, 18). However, this theory had no great bearing on duality, as duality theorems (whether global or local) require a combination of boundedness and finiteness assumptions. [3]

1.3. *Descent in derived categories.* As recalled in [24], morphisms or objects of derived categories in general do not glue on open covers. A notable exception was discovered long after [RD], namely *perverse sheaves* in the étale context ([9], 3.2.4). Grothendieck was of course well aware of the difficulty, and that led him to sketch his theory of *pseudo-complexes* in ([20], 5.2), which brought a partial solution to the gluing problem. I remember his excitement when in a talk at the IHES in the mid 1960's (see [27]) Deligne explained his theory of *cohomological descent* (later written up by Saint-Donat ([1], V bis)), which fully solved the problem. However, the conjectural approach via "pseudo-complexes" can be seen as a first attempt to defining a "higher" way of gluing in derived categories, which has nowadays been achieved by the enhancement of derived categories to derived ∞-categories. Classical derived functors can be enhanced to functors between derived ∞-categories, and objects of these enhanced categories can be glued on open covers, see ([34], [35], [36]). This formalism, which supersedes that of Deligne, is being increasingly used today. In [43], it is shown how it can be combined with inputs from Huber's adic geometry to yield another approach to the $f^!$ functor of [20] and [RD], and even to a full 6-operation formalism of coherent duality. See the end of section 3 for more references.

2. The avatars of the $f^!$ functor

The central object in [20] and [RD] is the $f^!$ functor. As recalled in [24], Grothendieck in [20] discussed two ways of constructing it. In the appendix

[3](Added in November, 2023.) Bogdan Zavyalov (private communication) points out that nowadays it is possible to drop boundedness assumptions in Poincaré duality type results. For example, see ([37], Prop. 2.9.31) for the case of discrete adic spaces (and schemes), and ([50], Theorem 1.3.1/1.3.2) for étale cohomology of schemes/adic spaces. In fact, the categorical approach used to construct 6-functor formalisms in [37] or [50] *requires* functors to be defined on the unbounded categories $\mathcal{D}$.

to [RD] Deligne proposed a third construction.

2.1. *The three constructions.*

For a noetherian scheme Z, let $D^+_{\mathrm{coh}}(Z)$ denote the full subcategory of $D^+(Z, \mathcal{O}_Z)$ consisting of complexes with coherent cohomology sheaves. Let $f : X \to Y$ be a morphism of finite type between noetherian schemes. There are three ways of constructing

$$f^! : D^+_{\mathrm{coh}}(Y) \to D^+_{\mathrm{coh}}(X).$$

Each one requires additional hypotheses. Let me briefly recall the constructions.

(a) *Via embeddings into a smooth scheme.* Assume that we have a factorization $f = gi$, where $i : X \to Z$ is a closed immersion, and $g : Z \to Y$ is smooth, of pure relative dimension d. Using the *fundamental local isomorphism* (([20], 3.1.4), ([RD], III 7)), Grothendieck proves that, up to a transitive system of isomorphisms, the composite functor

$$i^!(g^*(-) \otimes \Omega^d_{Z/Y}[d]) : D^+_{\mathrm{coh}}(Y) \to D^+_{\mathrm{coh}}(X),$$

sending M to $i^!(g^*M \otimes \Omega^d_{Z/Y}[d])$, where $i^!L$ is $R\mathcal{H}om_{\mathcal{O}_Z}(i_*\mathcal{O}_X, L)$ restricted to X, is independent of the factorization. The corresponding limit of this system is denoted by $f^!$. He then shows (([20], 3.4), (RD III 11)) that, for f *projective*, $f^!$ is right adjoint to $Rf_* : D^+_{\mathrm{coh}}(X) \to D^+_{\mathrm{coh}}(Y)$. This uses the construction of a *trace map* $Rf_*f^! \to \mathrm{Id}$ (and reduction to the projection of a standard projective space onto its base scheme).

(b) *Using residual complexes and biduality.* Assume that Y admits a *dualizing complex* K_X, and that X and Y are of finite Krull dimension[4]. Then K_Y determines a codimension function on Y, hence a canonical filtration of the underlying set of Y by subsets stable under specialization, and a corresponding *Cousin complex* $E(K_Y)$, which is isomorphic to K_Y in $D^b_{\mathrm{coh}}(Y)$. Then $E(K_Y)$ is what Grothendieck calls a *residual complex* on Y, i.e., a dualizing complex, which, as a graded $\mathcal{O}_Y$-module is isomorphic to the direct sum over all points y of Y of $i_{y*}I(y)$, where $I(y) = H^{d_y}_{\{y\}}(\mathcal{O}_{Y,y})$ is an injective envelope of the residue field $k(y)$, with $d_y = \dim(\mathcal{O}_{Y,y})$, and $i_y : \{y\} = \mathrm{Spec}(k(y)) \to Y$. Using his (conjectural) theory of pseudo-complexes and rigidity properties of residual complexes, Grothendieck shows in [20] that the Cousin complexes of the various $E(i^!(g^*(E(K_Y)) \otimes \Omega^d_{Z/Y}[d]))$ for closed embeddings $i : U \hookrightarrow Z$ of open subschemes U of X in smooth schemes $g : Z \to Y$ can be glued to a

[4]This ensures that dualizing complexes on X and Y are bounded, and that pointwise dualizing complexes are dualizing.

residual complex $E(K_X)$ on X associated with a dualizing complex K_X. A proof independent of the theory of pseudo-complexes is given in ([RD] VI). Issues in ([RD] IV 3.2, 3.4) concerning the equivalence between dualizing and residual complexes are fixed in ([14], 3.1.3), and a proof of the unchecked compatibilities ([RD], VI 3.1 VAR 5, VII 3.4 (a))) is given in ([14], 3.3.1). Then, putting $D_X := R\mathcal{H}om(-, K_X)$, $D_Y := R\mathcal{H}om(-, K_Y)$, the $f^!$ functor is defined as

$$f^! := D_X Lf^* D_Y : D^+_{\text{coh}}(Y) \to D^+_{\text{coh}}(X)$$

For f *proper*, $f^!$ thus defined is again shown to be a right adjoint to Rf_*, using a trace map defined by means of a *residue homomorphism*. The formulas stated in ([RD], III 9) for the corresponding *residue symbol* are proved in ([14], A2) . Various sign issues concerning it are discussed in [15].

(c) *Using formal adjunction.* Assume that Y is of finite Krull dimension and f is *compactifiable*, i.e., that there exists a factorization $f = gj$, where $j : X \to Z$ is an open immersion, and $g : Z \to Y$ is proper, with Z of finite Krull dimension. Using an analogue of Verdier's construction of the $f^!$ functor in the topological case [47], Deligne shows in ([RD], Appendix) that the functor $Rg_* : D^+_{\text{coh}}(Z) \to D^+_{\text{coh}}(Y)$ admits a right adjoint $g^!$. Then $f^!$ is defined as

$$j^* g^! : D^+_{\text{coh}}(Y) \to D^+_{\text{coh}}(X).$$

The fact that, up to a transitive system of isomorphisms, $j^* g^!$ is independent of the factorization is non-trivial.

In ([RD], Appendix) Deligne sketches one method to do it. For f not necessarily proper, he defines a functor $Rf_! : \text{pro}D^b_{\text{coh}}(X) \to \text{pro}D^b_{\text{coh}}(Y)$ between categories of pro-objects, by $Rf_! = Rg_* j_!$, where $j_!$ is a pro-coherent version of the extension by zero, obtained by sending a coherent sheaf $\mathcal{F}$ on X to the pro-coherent sheaf "$\varprojlim$" $I^n\overline{\mathcal{F}}$, where $\overline{\mathcal{F}}$ is a coherent extension of $\mathcal{F}$ on Z and I an ideal of $\mathcal{O}_Z$ defining a closed subscheme with support $Z - X$. The independence of $j^* g^!$ of the compactification is then reduced to the independence of $Rf_!$, which is easy by standard Artin-Rees-Krull arguments. By a different method, a full proof is given by Verdier in ([49], Cor. 1).

Verdier's construction of $f^!$ in the topological case as a right adjoint to $Rf_!$, and its subsequent use by Deligne, first in the coherent context in [RD, Appendix], and later in the étale one in ([1] XVIII) relies on an elementary lemma to the effect that any contravariant functor from a Grothendieck abelian category to abelian groups transforming arbitrary small inductive limits into projective ones is representable. A precursor of this lemma can be found in Gabriel's thesis. Variants and generalizations for triangulated categories (see ([29], 10.5.3, 14.2.3)), inspired by Brown's representability theorem, were later developed (and applied to coherent duality) by Neeman, Lipman et al. ([40], [33]). However, these apply to "classical derived categories". In the ∞-categorical setting, the adjoint functor theorem of ([34], Cor. 5.5.2.9) for presentable ∞-categories is in general needed.

2.2. *Comparison and base change.*

For each of the three constructions several natural compatibilities for composition and base change have to be checked. This is done only partially in the original texts. Also, while for f proper the fact that $f^!$ is right adjoint to Rf_* ensures the existence and uniqueness of isomorphisms between the three approaches in their common domain of definition, the extension of these to the non-proper case raises delicate questions.

Comparison between (a) and (b) is carried out in [RD] and with many more details in [14], where a main base change theorem for the trace map for proper, flat Cohen-Macaulay maps is proved ([14], th. 3.6.5).

Comparison between (a) (or (b)) and (c) is more difficult. Some cases (e. g., f compactifiable, flat, and locally of complete intersection) are sketched in ([RD], Appendix). The smooth case is (fully) treated by Verdier in ([49], th. 3). In addition, Verdier proves a base change theorem for $f^!$ for a flat base change ([49], th. 2), from which the calculation of $f^!G$, $G \in D^+_{\mathrm{coh}}(Y)$ for f of finite tor-dimension is reduced to the case where $G = \mathcal{O}_Y$. Another approach to this calculation is given in [28]. However, there remained delicate issues concerning explicit forms of the trace and residue maps and compatibility of $f^!$ with non-flat base change. These have been recently addressed by Nayak and Sastry ([38], [39]). One can also consult the stacks project [51].

2.3. *Further developments.*

Let me discuss a few problems mentioned in [20] that were tackled later or are still of interest today.

(1) *The non-noetherian case*

Grothendieck insisted on eliminating noetherian hypotheses. To him it was the relative structure of a morphism that mattered, not its base. This is already apparent at various places of [20]. Over a non-noetherian scheme the notion of coherent sheaf is no longer adapted. In 3.2.4 Grothendieck introduces the notion of sheaf of modules of ∞-finite presentation, which he generalizes into that of *pseudo-coherent complex* in 4.1.1 (e). He also defines the notion of *pseudo-coherent morphism* in 5.4.5. These notions were to be systematically studied and developed in ([4], I – III). Complexes that are both pseudo-coherent and of finite tor-dimension, considered on p. 80, will be called *perfect* in [4]. They are interesting already in the noetherian case and they have since then played a crucial role in many parts of arithmetic geometry. In 5.8 (B) Grothendieck sketches the construction of the *determinant* of a perfect complex, a fundamental invariant, defined and studied in detail by Knudsen and Mumford in [30], and which is still an object of attention today in the framework of derived geometry. Partial generalizations of the duality theorem to the non-noetherian case (using the notion of pseudo-coherence) and to diagrams of schemes were given in [33] (see e.g., Cor. 4.4.2). However, a full 6-functor formalism in the non-noetherian case was constructed only recently [37], systematically using derived geometry and condensed mathematics.

(2) *Cotangent complex and self-intersection complex*

In ([20], 5.8) Grothendieck defines the (truncated of length one) *cotangent complex* of a homomorphism of rings $A \to B$. This is, historically, the first appearance of this object. As he told me, he got the idea from his work with Dieudonné on *imperfection modules* in ([6], 10.6): he conjectured that these modules should be (in good cases) interpreted as H_1 of a finer invariant, the cotangent complex of $A \to B$, whose H_0 would be the classical module of Kähler differentials. At about the same time, and independently, Lichtenbaum and Schlessinger defined a (truncated of length 2) cotangent complex of $A \to B$ [32]. However, the two problems mentioned by Grothendieck in ([20], 5.8), namely, (i) getting a theory giving a transitivity triangle for a composition $A \to B \to C$, (ii) globalizing to morphisms of schemes, were solved only a few years later. As for (ii), that was done by Grothendieck himself for the (truncated of length one) complex in [21]. Problem (i) was solved by Quillen [41] (a definition and study of the homology and cohomology objects of the cotangent complex was made independently by M. André [7]). The globalization of Quillen's construction was carried out in [26].

Already the cotangent complex played a key role in the formulation of the Riemann-Roch theorem for projective morphisms $f : X \to Y$ that are lci, i.e., *locally of complete intersection*[5], Y being quasi-compact and admitting an ample line bundle ([4], 3.6). It is also in [20] that the notion of lci morphism appears for the first time ([20], 4.8 C). The basic formula ([20], 5.8.2 (ii)), to the effect that for $X \to Y$ flat and lci, of relative dimension d, $f^!\mathcal{O}_Y = \det(L_{X/Y})[d]$ was not reproduced in [RD] (perhaps for lack of a suitable reference for the construction of the determinant of a perfect complex), and, curiously, not even in [14]. The flat lci morphisms, later called *syntomic morphisms*, were to play an important role in p-adic Hodge theory.

In ([20], p. 60) Grothendieck describes a construction of Cartier of the residue homomorphism for a morphism of rings $A \to B$, based on what

[5] In this case the truncated cotangent complex agrees with the full one.

he calls the *self-intersection complex* of B/A, namely $B \otimes^L_{B\otimes_A B} B$, which he denotes by $L_{\bullet B/A}$. The analogy of notation with the cotangent complex $L_{B/A}$ of B/A is not fortuitous. Deep relations between this self-intersection complex and $L_{B/A}$, for B flat over A, and in general, with $B\otimes^L_{B\otimes_A B}B$ replaced by $B\otimes^L_{B\otimes^L_A B}B$ were discovered by Quillen ([41], 8) in relation with *associative algebra cohomology* and *Hochschild cohomology*. This topic is still actively studied today in relation with p-adic Hodge theory.

(3) *Complexes of differential operators*

In ([20], 3.5) Grothendieck proposes an extension of the duality theorem to complexes of differential operators. As far as I know, the problem has not yet been addressed in general. Over complex manifolds, duality theorems involving $\mathcal{D}$-modules and complexes of differential operators are established in [42].

(4) *Homology, cycle classes, Lefschetz-Verdier formula*

Let $f : X \to S$ be a morphism of finite type between noetherian schemes of finite dimension, where S admits a dualizing complex K_S and $K_X := f^! K_S$ is the corresponding dualizing complex on X. In ([20], 8.4), elaborating on ideas he had sketched in his talk at the 1958 ICM, Grothendieck defines, for $L \in D^-(S, \mathcal{O}_S)$, the *homology groups* of X with coefficients in L as

$$H^{-i}R\mathrm{Hom}(L, K_X) = H^{-i}(X, R\mathcal{H}om(L, K_X)) = \mathrm{Ext}^{-i}(L, K_X).$$

This definition was later transposed in the settings of topological spaces (by Verdier ([17], VI)), and étale cohomology (by Grothendieck in the oral seminar SGA 5 [6]). Since then it has often been abusively called *Borel-Moore homology*, by reference to [11], where no derived categories, *a fortiori*, no dualizing complexes appear. Grothendieck intended to write in section 9 properties of variance for these groups, and use them to construct a theory of *cycle classes* of Hodge style, satisfying compatibilities with cup-products and Gysin maps. As explained in [24], this part was not written. In [18] El Zein discusses variants of these cycle classes in de Rham and Hodge homology groups.

In his letter to Serre of Aug. 8, 1964 ([13], p. 165), Grothendieck thus comments on the Woods-Hole Lefschetz fixed point formula that Serre had described in the first section of his letter of Aug. 2–3, 1964: "Le n. 1 ne m'excite guère, malgré les jolies applications ; le théorème des points fixes lui-même ne me semble pas plus qu'un exercice sur un air connu !". In [20], at the end of *Commentaires*, Grothendieck hinted that such fixed point formulas

[6] Written up later by Laumon ([17], VIII) and Deligne ([2], Cycle).

would follow from his duality formalism. It was at that time that Verdier was elaborating his general Lefschetz fixed point formula for cohomological correspondences in étale cohomology, later called the *Lefschetz-Verdier formula* [48], ([3], III), relying on the étale analogue of the duality formalism of [RD]. A variant of this formula in the coherent setting was later established in ([3], III 6. Appendice), generalizing the Woods-Hole formula. Hodge cohomology classes and residues of ([RD], III 9) play a crucial role in it.

3. Glimpses of duality in other contexts

The duality formalism of [RD] has inspired similar theories in many other contexts. A full description of them, or even of the state of the art today, would require a whole volume. I will list some of them below, with no references, as the literature is too large.

• *Topological spaces*: Verdier (1965).

• *Complex analytic spaces*: Ramis-Ruget, Verdier (1971).

• *Étale cohomology*: Artin-Grothendieck-Verdier, Deligne, Gabber, and many others (1964 –).

• *$\mathcal{D}$-modules and mixed Hodge theory*: Bernstein, Deligne, Kashiwara-Schapira, Mebkhout, M. Saito, and many others (1970 –).

• *Crystalline cohomology, rigid cohomology*: Berthelot, Ekedahl, Kedlaya, and many others (1970 –).

It is by far the étale theory that has undergone the most extensive developments (up to today). In many respects, it has served as a model for the others.

Quite recently there has been a renewal of interest in the general formalism of the six operations[7] in various settings, in connection with the new tools provided by the language of derived geometry and the theory of condensed mathematics, see [43], [8], [12], [44], [50], [37].

For a historical study of Grothendieck's work, ideas and reflections on the theme of duality in the light of [22], see L. Lafforgue's essay [31].

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[7] "formalisme des six opérations" was Grothendieck's terminology in [22]. Nowadays, "6-functor formalism" is a preferred terminology.

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[51] The Stacks project. Tag 08XG *Dualizing complexes*. Tag 0DWE *Duality for schemes*.

[RD] = [23]