Grothendieck and differential calculus[1]

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0. Introduction

Roughly speaking, differential calculus consists in this: given a good space X, a good function f on X, and points x, y in X, evaluate $f(y) - f(x)$ when x and y are *infinitesimally* close. What does one mean by "*infinitesimally* close"? If one just says "close", one would mean that the point (x, y) belongs to a neighborhood of the diagonal in $X \times X$. These neighborhoods lead to the notion of "uniform structure" in Bourbaki, Topologie Générale. But what would be an *infinitesimal neighborhood* of the diagonal? Grothendieck was the first one to give a rigorous definition of this notion. He did it in the context of analytic geometry, in two remarkable exposés at the Séminaire Cartan 1960-61 [26], which brought a totally new perspective on sheaves of differential forms and differential calculus in general.

Let me explain his construction in a simple case. Let X be a smooth complex analytic variety, of dimension n. Consider the diagonal map $i : X \to X \times X$. This is a closed immersion, defined by a coherent sheaf of ideals I fitting in the exact sequence

(0.1) $$0 \to I \to \mathcal{O}_{X \times X} \to \mathcal{O}_X \to 0.$$

In local coordinates (x_i) on X, (x_i, y_i) on $X \times X$, it is locally generated by $(y_i - x_i)$, $1 \leqslant i \leqslant n$. It coincides with $\mathcal{O}_{X \times X}$ outside the diagonal. Grothendieck considered the exact sequence deduced from (0.1) by dividing $\mathcal{O}_{X \times X}$ by I^2:

(0.2). $$0 \to I/I^2 \to \mathcal{O}_{X \times X}/I^2 \to \mathcal{O}_X \to 0.$$

He observed that:

(a) There is a canonical isomorphism

(0.3) $$I/I^2 \xrightarrow{\sim} \Omega^1_X,$$

where Ω^1_X is the sheaf of 1-forms on X, a vector bundle of rank n, with local basis (dx_i) $(1 \leqslant i \leqslant n)$.

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(b) If p_1^*, p_2^* are the two sections of $\mathcal{O}_{X\times X}/I^2 \to \mathcal{O}_X$ induced by the projections $X \times X \to X$, then the map

$$\mathcal{O}_X \to I/I^2,\ f \mapsto p_2^* f - p_1^* f$$

coincides via (1.3) with the derivation $d_X : \mathcal{O}_X \to \Omega^1_X$.

The verification is immediate: (a) follows from the exact sequence

$$0 \to I/I^2 \overset{d_{X\times X}}{\to} i^*\Omega^1_{X\times X} \to \Omega^1_X \to 0,$$

where the surjection is the sum map: $\Omega^1_X \oplus \Omega^1_X \to \Omega^1_X$, and (b) expresses that locally $y_i - x_i \bmod I^2$ corresponds to dx_i, I/I^2 is free with basis the classes of $y_i - x_i \bmod I^2$, and

$$f(y) - f(x) = \sum \partial f/\partial x_i(x)(y_i - x_i) \bmod I^2.$$

Grothendieck defined the *first infinitesimal neighborhood of the diagonal* as the closed analytic subspace $\Delta^1_X(1) \hookrightarrow X \times X$ defined by the ideal I^2. It has the same underlying space as X, and its ring of functions, $\mathcal{O}_{X\times X}/I^2$, also called sheaf of *principal parts* of order 1 and denoted $\mathcal{P}^1_X$, is by (0.2) an extension of $\mathcal{O}_X$ by the ideal of square zero $I/I^2 = \Omega^1_X$. By replacing I^2 by higher powers of I, one gets higher infinitesimal neighborhoods $\Delta^r_X(1)$ of the diagonal, leading to differential calculus of higher order.

At the same time, Grothendieck realized that one could reverse the steam, and *define* the sheaf of differentials Ω^1 by the I/I^2 formula, even if X is a singular analytic space, and even in a relative situation $X \to S$, and that in this way one obtained a theory with good functoriality properties. He also saw that one could develop a similar theory in algebraic geometry, for schemes, which he wrote up a couple of years later in EGA IV [29].

If $X \to S$ is a morphism of schemes, Grothendieck defines $\Omega^1_{X/S}$ as I/I^2, where I is the ideal of the diagonal immersion $X \to X \times_S X$, and the map $d_{X/S} : \mathcal{O}_X \to \Omega^1_{X/S}$ by $p_2^* - p_1^*$, where, as above, p_1^*, p_2^* are the two sections of $\mathcal{O}_{X\times_S X}/I^2 \to \mathcal{O}_X$ given by the two projections. The sheaf $\Omega^1_{X/S}$ is quasi-coherent (locally free of finite type when X/S is smooth), and $d_{X/S}$ is an $\mathcal{O}_S$-derivation. When X and S are affine, so that $X \to S$ corresponds to a homomorphism of rings $A \to B$, then $\Omega^1_{X/S}$ is the quasi-coherent sheaf defined by $\Omega^1_{B/A} := J/J^2$, where J is the kernel of the surjection $B \otimes_A B \to B$, $b_1 \otimes b_2 \mapsto b_1 b_2$, and $d_{X/S}$ corresponds to the A-derivation $d_{B/A} : B \to \Omega^1_{B/A}$, $d_{B/A}(b) = 1 \otimes b - b \otimes 1$. One checks that composition with $d_{B/A}$ defines a functorial isomorphism

$$\mathrm{Hom}_B(\Omega^1_{B/A}, M) \overset{\sim}{\to} \mathrm{Der}_A(B, M)$$

where $\mathrm{Der}_A(B, M)$ is the B-module of A-derivations of B in a B-module M, so that $\Omega^1_{B/A}$ thus defined is the module of *Kähler differentials* of B/A.[2]

One checks that $d : \mathcal{O}_X \to \Omega^1_{X/S}$ uniquely extends to a complex

$$\Omega^\bullet_{X/S} = (\mathcal{O}_X \xrightarrow{d} \Omega^1_{X/S} \to \cdots \to \Omega^i_{X/S} \xrightarrow{d} \Omega^{i+1}_{X/S} \to \cdots),$$

called the *de Rham complex* of X/S, where $\Omega^i_{X/S} := \bigwedge^i_{\mathcal{O}_X} \Omega^1_{X/S}$, such that $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^i \alpha \wedge d\beta$ for α of degree i.

Grothendieck defined the *algebraic de Rham cohomology* of X/S by

$$H^*_{\mathrm{dR}}(X/S) = H^*(X, \Omega^\bullet_{X/S})$$

(hypercohomology of X for the Zariski topology with coefficients in $\Omega^\bullet_{X/S}$). This object enjoys nice functoriality properties. Its calculation led to rich developments.

Let k be a field. For X/k a standard projective space, a projective smooth curve, or an abelian variety, the de Rham cohomology groups $H^i_{\mathrm{dR}}(X/k)$ are finite dimensional k-vector spaces, with the same dimension (and same structure) as the transcendental Betti cohomology groups when $k = \mathbb{C}$. But in general, algebraic de Rham cohomology behaves in quite different ways in characteristic zero and in characteristic $p > 0$.

1. Algebraic de Rham cohomology in characteristic zero

Let X be a smooth scheme over $\mathbb{C}$. The set $X(\mathbb{C})$ of its complex points underlies a smooth complex analytic variety X^{an}. By the *holomorphic Poincaré lemma*[3] its de Rham complex $\Omega^\bullet_{X^{\mathrm{an}}}$ (where we omit $/\mathbb{C}$ for short) is a resolution of the constant sheaf $\mathbb{C}_X$: the sequence

$$0 \to \mathbb{C}_X \to \mathcal{O}_{X^{\mathrm{an}}} \to \Omega^1_{X^{\mathrm{an}}} \to \cdots$$

is exact. In other words, the augmentation

$$\mathbb{C}_X \to \Omega^\bullet_{X^{\mathrm{an}}}$$

is a quasi-isomorphism, which thus induces an isomorphism

$$H^*(X^{\mathrm{an}}, \mathbb{C}) \xrightarrow{\sim} H^*_{\mathrm{dR}}(X^{\mathrm{an}}), \tag{1.0}$$

[2] This module, defined by Kähler in 1953 [38], didn't attract much attention until Grothendieck used it extensively, already in [25].

[3] According to de Rham, ([21], p. 646) the C^∞ analogue of this lemma, namely the exactness of the augmented de Rham complex of global C^∞-forms on $\mathbb{R}^n$, attributed to Poincaré, was in fact first proved by Volterra ([53], pp. 407-422, 1889).

by which a class $\omega \in H^n_{\mathrm{dR}}(X^{\mathrm{an}})$ corresponds to the homomorphism

$$\mathrm{Hom}(H_n(X^{\mathrm{an}}, \mathbb{Z}), \mathbb{C}), \ \gamma \mapsto \int_\gamma \omega$$

(with a suitable sign convention, see ([17], I Th. 1.2, p. 14)).

On the algebraic side, the Poincaré lemma fails: $\Omega^\bullet_X := \Omega^\bullet_{X/\mathbf{C}}$ is far from being a resolution of $\mathbb{C}_X$ (as is already seen for X the affine line $\mathrm{Spec}(\mathbb{C}[t])$: the form $dt/(1+t)$ near zero has no algebraic primitive). However, we have a canonical map of ringed spaces

$$X^{\mathrm{an}} \to X, \tag{1.1}$$

inducing a map $\Omega^\bullet_X \to \Omega^\bullet_{X^{\mathrm{an}}}$ on de Rham complexes, hence on de Rham cohomology

$$H^*_{\mathrm{dR}}(X) \to H^*_{\mathrm{dR}}(X^{\mathrm{an}}). \tag{1.2}$$

Grothendieck proved the following surprising theorem.

Theorem 1.3 [28]. The map (1.2) is an isomorphism, and thus induces an isomorphism $H^*_{\mathrm{dR}}(X) \xrightarrow{\sim} H^*(X^{\mathrm{an}}, \mathbb{C})$.

In particular, $\dim_{\mathbf{C}} H^i_{\mathrm{dR}}(X) = b_i$, where $b_i = \dim_{\mathbf{C}} H^i(X^{\mathrm{an}}, \mathbb{C})$ is the transcental ith Betti number, and these algebraic de Rham cohomology spaces are homotopy invariants!

In the proper case, the proof is easy. By Serre's GAGA theorems, the map

$$H^j(X, \Omega^i_X) \to H^j(X^{\mathrm{an}}, \Omega^i_{X^{\mathrm{an}}})$$

is an isomorphism, which implies the result by the Hodge to de Rham spectral sequence $E_1^{ij} = H^j(X, \Omega^i_X) \Rightarrow H^{i+j}_{dR}(X)$. However, in the non-proper case, the proof is more difficult. Grothendieck uses Hironaka's resolution of singularities [33] to compactify X by a divisor with normal crossings at infinity, and concludes by a local calculation due to Atiyah–Hodge. This calculation, which involves differential forms with logarithmic poles, will be revisited by Deligne in his work on mixed Hodge theory ([15], [16]), and later will be one of the sources of the theory of logarithmic geometry.

Example. Let $X := \mathrm{Spec}(\mathbb{C}[t, t^{-1}])$ be the punctured affine line. Then $H^0_{\mathrm{dR}}(X) = \mathbb{C}$, $H^1_{\mathrm{dR}}(X) = \mathbb{C} \cdot (dt/t)$, and $H^n_{\mathrm{dR}}(X) = 0$ for $n > 1$. The class dt/t is a basis of $H^1(X^{\mathrm{an}}, \mathbb{C})$, sending the class of the circle in $H_1(X^{\mathrm{an}}, \mathbb{Z})$ to $2\pi i$.

Let X_0 be a smooth scheme over $\mathbb{Q}$, and let $X = X_0 \otimes \mathbb{C}$. Since $H^*_{\mathrm{dR}}(X_0/\mathbb{Q}) \otimes \mathbb{C} = H^*_{\mathrm{dR}}(X/\mathbb{C})$, the isomorphism of Theorem 1.3 gives a *period isomorphism*

$$H^*_{\mathrm{dR}}(X_0/\mathbb{Q}) \otimes \mathbb{C} \xrightarrow{\sim} H^*(X^{\mathrm{an}}, \mathbb{Q}) \otimes \mathbb{C}. \tag{1.3.1}$$

As $H^*(X^{\mathrm{an}}, \mathbb{Q}) = \mathrm{Hom}(H_*(X^{\mathrm{an}}, \mathbb{Z}), \mathbb{Q})$, the image by (1.3.1) of a class $\omega \in H^n_{\mathrm{dR}}(X_0/\mathbb{Q})$ corresponds to the homomorphism sending $\gamma \in H_n(X^{\mathrm{an}}, \mathbb{Z})$ to $\int_\gamma \omega \in \mathbb{C}$. These numbers are called *periods*. As the above example shows, they can be transcendental. Let $t(X)$ be the transcendence degree over $\mathbb{Q}$ of the subfield of $\mathbb{C}$ generated by those periods (for all n). Grothendieck asked for bounds for $t(X)$ and conjectured that $t(X)$ is the dimension of the image of the *motivic Galois group* of $\mathbb{Q}$ into $\mathrm{GL}(H^*(X^{\mathrm{an}}, \mathbb{Q}))$ ([2], 0.4). See [2] for the state of the art on this conjecture and related ones in 2016.

For $X/\mathbb{C}$ projective and smooth, X^{an} is Kähler, and the Hodge decomposition for $H^*(X^{\mathrm{an}}, \mathbb{C})$ gives an isomorphism

$$H^n_{\mathrm{dR}}(X) \xrightarrow{\sim} \bigoplus_{i+j=n} H^{ij}, \tag{1.4}$$

with $H^{ij} := H^j(X, \Omega^i_X)$ and $H^{ji} = \overline{H^{ij}}$. In particular, if $h^n := \dim H^n_{\mathrm{dR}}(X)$, $h^{ij} := \dim H^j(X, \Omega^i_X)$, we have

$$h^n = \sum_{i+j=n} h^{ij} \tag{1.5}$$

for all n, and the Hodge to de Rham spectral sequence degenerates at the first page. Moreover, we have the *Hodge symmetry* $h^{ij} = h^{ji}$. It was proved by Deligne ([14], Prop. 5.3) that (1.4) (hence (1.5) and Hodge symmetry) holds more generally for $X/\mathbb{C}$ only proper and smooth.

2. Algebraic de Rham cohomology in characteristic $p > 0$

Fix a prime number p.

If X is a scheme of characteristic p (i.e., a scheme over $\mathbb{F}_p$), X admits a natural endomorphism F_X, called the *Frobenius endomorphism*, which is the identity on the underlying space and is $a \mapsto a^p$ on $\mathcal{O}_X$. If X is a scheme over S, and S is of characteristic p, then, for a in $\mathcal{O}_X$, $d(a^p) = pa^{p-1}da = 0$ (where $d := d_{X/S}$), and for $\omega \in \Omega^i_{X/S}$, $d(a^p\omega) = a^p d\omega$, so that the de Rham complex $\Omega^\bullet_{X/S}$ is not just $\mathcal{O}_S$-linear, but also p-linear with respect to $\mathcal{O}_X$. The conjunction of $\mathcal{O}_S$-linearity and p-$\mathcal{O}_X$ linearity is best expressed by introducing the pull-back $X^{(1)}/S$ of X by $F_S : S \to S$, and the S-morphism $F : X \to X^{(1)}$, called the *relative Frobenius*, through which F_X canonically factors, and observing that $F_*\Omega^\bullet_{X/S}$, i.e., $\Omega^\bullet_{X/S}$, considered as a complex of $\mathcal{O}_{X^{(1)}}$-modules, has an $\mathcal{O}_{X^{(1)}}$-linear differential. In particular, the cohomology sheaves $\mathcal{H}^i(F_*\Omega^\bullet_{X/S})$ are $\mathcal{O}_{X^{(1)}}$-modules.

These are trivial remarks. But in 1957 Cartier discovered a new operation on differential forms, which was to later control all differential calculus in positive characteristic (and, consequently, in mixed characteristic, too, for a large part). In a presentation due to Deligne and Katz ([39], 7.2), his result is the following.

Theorem 2.1. (Cartier, [13]). The maps

$$\mathcal{O}_X \to \mathcal{O}_X,\ a \mapsto a^p,$$

$$\Omega^1_{X/S} \to \Omega^1_{X/S},\ adf \mapsto a^p f^{p-1} df$$

induce $\mathcal{O}_{X^{(1)}}$-linear homomorphisms

$$\mathcal{O}_{X^{(1)}} \to \mathcal{H}^0(F_*\Omega^\bullet_{X/S}),$$

$$\Omega^1_{X^{(1)}/S} \to \mathcal{H}^1(F_*\Omega^\bullet_{X/S}),$$

which uniquely extend to a homomorphism of graded $\mathcal{O}_{X^{(1)}}$-algebras denoted[4]

$$C^{-1} : \bigoplus_i \Omega^i_{X^{(1)}/S} \to \bigoplus_i \mathcal{H}^i(F_*\Omega^\bullet_{X/S}).$$

When X/S is smooth, C^{-1} is an isomorphism.

The proof of the first part is a small calculation (that has recently reappeared in the theory of δ-structures and prisms [12]), and for the second one, an easy dévissage reduces to $X = \mathrm{Spec}(\mathbb{F}_p[t])$, where writing $\mathbb{F}_p[t]$ as the free module over $\mathbb{F}_p[t^p]$ with basis $1, t, \ldots, t^{p-1}$, one sees that H^0 (resp. H^1) has basis 1 (resp. $t^{p-1}dt$).

2.2. It follows from Theorem 2.1, that if $X = \mathrm{Spec}(A)$ is smooth, affine over a field k of characteristic p, then $H^*_{\mathrm{dR}}(X/k) \xrightarrow{\sim} \bigoplus \Omega^i_{A^{(p)}/k}$, hence is of infinite dimension over k if X is of positive dimension, in contrast with the case $k = \mathbb{C}$. On the other hand, if X/k is proper and smooth, then the Hodge spaces $H^j(X, \Omega^i_{X/k})$ have finite dimension h^{ij}, so that, by the Hodge to de Rham spectral sequence

$$E_1^{ij} = H^j(X, \Omega^i_{X/k}) \Rightarrow H^{i+j}_{\mathrm{dR}}(X/k)$$

the de Rham cohomology spaces $H^n_{\mathrm{dR}}(X/k)$ are also finite dimensional. However, in the late 1950's and early 1960's it was observed by Serre that

[4]For historical reasons: Cartier first defined $C : Z^i F_*\Omega^\bullet_{X/S} \to \Omega^i_{X^{(p)}/S}$

Hodge symmetry could fail [50], and by Mumford [46] that a strict inequality $h^n < \sum_{i+j=n} h^{ij}$ was possible. Concerning this second point, a progress was made in 1987.

Theorem 2.3 [18]. Assume k perfect and let $W(k)$ be the ring of Witt vectors on k.[5] Let X/k be proper and smooth. Assume that $\dim(X) \leqslant p$, and X is liftable to $W_2(k)$, i.e., there exists a proper, smooth scheme $\mathcal{X}/W_2(k)$ whose reduction $\mathcal{X} \otimes k$ on k is isomorphic to X. Then

$$h^n(X) = \sum_{i+j=n} h^{ij}(X) \tag{2.3.1}$$

for all n.

This is in fact a corollary of a stronger fact, namely, that assuming only X/k smooth of dimension $\leqslant p$ and liftable to $W_2(k)$, the de Rham complex $F_*\Omega^\bullet_{X/k}$, as an object of the derived category $D(X^{(1)}, \mathcal{O}_{X^{(1)}})$, is the sum of its shifted cohomology sheaves, i.e., thanks to the Cartier isomorphism 2.1, we have an isomorphism

$$\bigoplus_i \Omega^i_{X^{(1)}/k}[-i] \xrightarrow{\sim} F_*\Omega^\bullet_{X/k}$$

in $D(X^{(1)}, \mathcal{O}_{X^{(1)}})$.

A corollary of Theorem 2.3 was an algebraic proof of the Hodge degeneration (1.5). However, there remained the following question: does there exist X/k, proper, smooth of dimension $p+1$, liftable to $W_2(k)$, for which (2.3.1) fails for some n? A positive answer, in a stronger form, was recently given by A. Petrov.

Theorem 2.4 [47]. There exists a projective, smooth scheme $\mathcal{X}/W(k)$ of relative dimension $p+1$, such that its reduction $X := \mathcal{X} \otimes k$ has

$$h^p < \sum_{i+j=p} h^{ij}.$$

The proof is a tour de force. The identification and description of the obstruction to the degeneration at E_1 of the Hodge to de Rham spectral sequence, or rather, equivalently, of the degeneration at E_2 of the *conjugate spectral sequence* (coming from the canonical filtration of $F_*\Omega^\bullet_{X/k}$) uses the theory of *diffracted Hodge complexes* of Bhatt–Lurie–Drinfeld [10]. The core of the proof relies on an extremely ingenious description of extensions in

[5] $W(k) = \mathbb{Z}_p$ if $k = \mathbb{F}_p$ and $W_2(k) = W(k)/p^2W(k)$.

de Rham and Hodge cohomology complexes of abelian schemes over $W(k)$ endowed with group actions. See Section 4 for the definition of $\mathcal{X}$.

Let $\delta_p = \sum_{i+j=p} h_{ij} - h^p$. Petrov can show that, for his example, $\delta_p \geqslant 2$. He hopes that one could get $\delta_p = 1$ and then that δ_p could probably take any positive integer value.

3. Mixed characteristic: crystalline cohomology

The discovery by Grothendieck [30] (and, independently, by Katz–Oda [40]) of the algebraicity of the classical, analytic, *Gauss–Manin connection*, together with results of Monsky–Washnitzer on de Rham cohomology in the affine case [45], led him to conjecture (in [30]) the following.

Let k be a perfect field of characteristic $p > 0$. Let X/k be proper, smooth, and suppose it admits a proper, smooth lifting $\mathcal{X}/W(k)$. Then:

(a) $H^*_{\mathrm{dR}}(\mathcal{X}/W(k))$ does not depend on the lifting $\mathcal{X}$, in the sense that the H^*_{dR} of the various liftings are related by a transitive system of isomorphisms.

(b) For $n \geqslant 1$, one can define an inductive system of ringed sites

$$(X/W_n(k))_{\mathrm{crys}}, \tag{3.1}$$

called the *crystalline sites*, giving a compatible system of isomorphisms

$$H^*((X/W_n(k))_{\mathrm{crys}}, \mathcal{O}) \xrightarrow{\sim} H^*_{\mathrm{dR}}(X_n/W_n(k)), \tag{3.2}$$

for $X_n := \mathcal{X} \otimes W_n(k)$, yielding an isomorphism

$$H^*((X/W(k))_{\mathrm{crys}}, \mathcal{O}) := \varprojlim H^*((X/W_n(k))_{\mathrm{crys}}, \mathcal{O}) \xrightarrow{\sim} H^*_{\mathrm{dR}}(\mathcal{X}/W(k)). \tag{3.3}$$

Berthelot proved these conjectures in his thesis [6], where he developed the theory of crystalline sites, topoi, and crystalline cohomology in a much more general framework.

In the case of interest to us, the crystalline site $(X/W_n(k))_{\mathrm{crys}}$ is defined independently of the lifting $\mathcal{X}/W(k)$, which may not exist (see Serre's examples in ([37], 8.6)), and, in fact, for *any* scheme X/k. It is the category of $W_n(k)$-thickenings $U \hookrightarrow T$ of open subschemes of X, together with a DP (= divided power structure) $x \mapsto x^{[r]}$ on the ideal of T,[6] compatible with $p^{[r]} = p^r/r!$, with families $(U_i \hookrightarrow T_i)$ covering $U \hookrightarrow T$ if (T_i) Zariski covers T. In contrast with the étale site, this site is far from the geometric intuition. In particular, it has no final object. If X/k is smooth and $Y/W_n(k)$ is a smooth lifting, then $X \hookrightarrow Y$ covers the final object of the associated topos, and cohomology can be calculated by a certain Čech complex, called

[6] See ([51], Divided Powers. 07GK).

the *Čech-Alexander complex.* For $n = 1$ and $Y = X$, this complex is the cosimplicial ring

$$\mathcal{O}_{\widetilde{X^{\bullet}}} \tag{3.4}$$

formed by the rings of the divided power envelopes X, $\widetilde{X \times_k X}$, $X \times_k \widetilde{X \times_k X}$, ..., of the diagonal in the successive powers of X/k. A crucial point is the existence of an isomorphism:

$$\mathcal{O}_{\widetilde{X}^{\bullet}} \xrightarrow{\sim} \Omega^{\bullet}_{X/k} \tag{3.5}$$

in the derived category $D(X, k)$, compatible with the ring structures. This isomorphism is deep. It is a manifestation of Grothendieck's extraordinary insight into differential calculus. Berthelot's proof in ([6], V Th. 2.3.2), suggested by Grothendieck in ([30], 6.4, 6.5), using linearization of differential operators, is rather indirect. It is repeated in ([7], 6.14, 7.2). The construction of an explicit morphism of complexes from (3.4) to $\Omega^{\bullet}_{X/k}$ inducing an isomorphism in $D(X, k)$ after divided power completion of the left hand side is given in ([35], VIII, 1.4.4).

All proofs of (3.5) rely, in one way or another, on the following (trivial) observation.

Lemma 3.6. (DP Poincaré lemma) Let A be any commutative ring, and let $A\langle t\rangle$ be the divided power algebra $\Gamma_A(At)$ on the free module with basis t, and let $A\langle t\rangle \xrightarrow{d} A\langle t\rangle dt$ the differential graded algebra with $d(t^{[n]}) = t^{[n-1]}dt$. Then the sequence

$$0 \to A \to A\langle t\rangle \xrightarrow{d} A\langle t\rangle dt \to 0$$

of A-modules is exact.

The algebra $k\langle t\rangle$ has a natural comultiplication, which makes it the Hopf algebra of a commutative unipotent group scheme[7] $\mathbf{G}_a^{\sharp}$ (over k). The morphism of group schemes $\mathbf{G}_a^{\sharp} \to \mathbf{G}_a$ given by $k[t] \to k\langle t\rangle$ defines a ring stack

$$\mathbf{G}_a^{dR} := [\mathbf{G}_a^{\sharp} \to \mathbf{G}_a],$$

which in turn, by Bhatt–Lurie's transmutation ([9], 2.3.8), yields a stack

$$X^{dR},\ R \mapsto X(\mathbf{G}_a^{dR}(R)),$$

a gerbe on $X^{(1)}$ banded by the DP-envelope $T^{\sharp}_{X^{(1)}}$ of the tangent sheaf of $X^{(1)}/k$. We have

$$H^*(X^{dR}, \mathcal{O}) \xrightarrow{\sim} H^*_{\mathrm{dR}}(X/k).$$

[7] In the sense of ([19], II 9) ($\mathbf{G}_a^{\sharp}$ is not of finite type).

The cosimplicial ring (3.4) can be re-interpreted as the ring of the Čech simplicial complex associated with the cover $X \to X^{dR}$ (transmuted from the cover $\mathbf{G}_a \to \mathbf{G}_a^{dR}$). This viewpoint leads to a new proof of (3.5) ([9], 2.5.6). A finer result (taking into account the conjugate filtration of $F_*\Omega^\bullet_{X/k}$) is given in ([9], 2.7.2 (3)).

In his famous letter to Tate dated May, 1966, Grothendieck said that the characteristic properties of a crystal were rigidity and ability to grow in an appropriate neighborhood. Crystals in $\mathcal{O}$-modules on the crystalline site $(X/W_n(k))_{\mathrm{crys}}$ are defined as sheaves of $\mathcal{O}$-modules for which the transition maps from one thickening to another are isomorphisms ([6], IV 1.1). A typical example is *de Rham cohomology of liftings* X_n of X/k to $W_n(k)$, incarnating the crystal $R\Gamma((X/W_n(k))_{\mathrm{crys}}, \mathcal{O})$ on $(\mathrm{Spec}(k)/\mathrm{Spec}(W_n(k))_{\mathrm{crys}}$: vertical rigidity, horizontal growth:

$$\begin{tikzcd} R\Gamma_{\mathrm{crys}}(X/k) \arrow[d, "="] & R\Gamma_{\mathrm{crys}}(X/W_n(k)) \arrow[l] \arrow[d, "="] \\ R\Gamma_{\mathrm{dR}}(X/k) & R\Gamma_{\mathrm{dR}}(X_n/W_n(k)) \arrow[l] \end{tikzcd}$$

A stronger crystalline property of this cohomology was recently proven by S. Mondal ([43], Theorem 5.0.1): $R\Gamma_{\mathrm{crys}}(X/W_n(k))$ is the unique (functorial in X) multiplicative lift of $R\Gamma_{\mathrm{dR}}(X/k)$ to $W_n(k)$.

Crystalline cohomology has been intensely studied during the past 50 years and has been the subject of numerous reviews. I will just briefly recall a few main facts and sketch a couple of recent developments.

Let $K := \mathrm{Frac}(W(k))$. For X/k proper and smooth, $H^*_{\mathrm{crys}}(X/W(k))$ is a finite anti-commutative $W(k)$-algebra ($R\Gamma_{\mathrm{crys}}(X/W(k))$ in $D(W(k))$ is a perfect complex, and a derived commutative algebra), and

$$X \mapsto H^*_{\mathrm{crys}}(X/W(k)) \otimes K$$

is a Weil cohomology (satisfying Künneth, Poincaré duality, and having a cycle class theory), with *good Betti numbers*: by Katz–Messing [41]

$$(b_i)_{\mathrm{crys}}(X) := \dim_K H^i_{\mathrm{crys}}(X/W(k)) \otimes K = \dim_{\mathbb{Q}_\ell} H^i(X_{\overline{k}}, \mathbb{Q}_\ell)$$

($\ell \neq p$), and if X lifts to characteristic zero, then $(b_i)_{\mathrm{crys}}(X)$ is the ith Betti number of the resulting complex variety over $\mathbb{C}$.

A distinctive feature of crystalline cohomology is its *Frobenius lift.* The endomorphism F_X of X induces an endomorphism φ of $H^*_{\mathrm{crys}}(X/W(k))$ which is semi-linear with respect to the automorphism σ of $W(k)$ deduced from the Frobenius automorphism of k. This endomorphism φ is an *isogeny*, i.e., $\varphi\otimes\mathbb{Q}_p$ is an isomorphism: more precisely, if X/k is of dimension d, there exists a σ^{-1}-linear endomorphism v of $H^*_{\mathrm{crys}}(X/W(k))$ such that $\varphi v = v\varphi = p^d$. By [41] again, for X/k proper and smooth, and $k = \mathbb{F}_q$, with $q = p^a$, then $\det(1-\varphi^a t, H^i_{\mathrm{crys}}(X/W(k)))$ belongs to $\mathbb{Z}[t]$ and coincides with the ℓ-adic analogue $\det(1-F_q t, H^i(X_{\overline{k}}, \mathbb{Q}_\ell))$ for $\ell \neq p$.

Example. Let A/k be an abelian variety of dimension g. Then

$$H^*_{\mathrm{crys}}(A/W(k)) = \bigwedge^* H^1_{\mathrm{crys}}(A/W(k)).$$

The $W(k)$-module $H^1_{\mathrm{crys}}(A/W(k))$ is free of rank $2g$. Equipped with $F := \varphi$ and $V := p^{-1}F$, it is canonically isomorphic to the Dieudonné module $M(A[p^\infty])$ of the p-divisible group associated with A. This result, due to Grothendieck (see ([42], p. VI)), was the starting point of the so-called *crystalline Dieudonné theory*, developed by Grothendieck, Messing, Mazur–Messing, Berthelot–Breen–Messing in the 1970's and early 1980's, and recently revived in the prismatic context by several people: [49], [3], [24], [44].

Note that there exists an abelian scheme $\mathcal{A}/W(k)$ lifting A, and we have a canonical isomorphism

$$H^*_{\mathrm{crys}}(A/W(k)) \xrightarrow{\sim} H^*_{\mathrm{dR}}(\mathcal{A}/W(k)).$$

Though the Frobenius endomorphism of A rarely lifts to $\mathcal{A}$,[8] it does produce an endomorphism φ of the de Rham cohomology of $\mathcal{A}/W(k)$.

Together with its isogeny φ, the $W(k)$-module $H^i_{\mathrm{crys}}(X/W(k))$ becomes an *F-crystal*. As such, it has a *Dieudonné–Manin slope decomposition*

$$H^i_{\mathrm{crys}}(X/W(k)) \otimes \mathbb{Q}_p = \bigoplus_{\lambda\in\mathbb{Q}} (H^i_{\mathrm{crys}}(X/W(k)) \otimes \mathbb{Q}_p)_\lambda,$$

where for $\lambda = s/r$ in canonical form, $(H^i_{\mathrm{crys}}(X/W(k)) \otimes \mathbb{Q}_p)_\lambda$ is the part of *slope* λ, which over $W(\overline{k}) \otimes \mathbb{Q}_p$ becomes isomorphic to a sum of copies of $(W(\overline{k})_\sigma[F]/(F^r - p^s))\otimes\mathbb{Q}_p$ (see ([19], IV 3)). The relation between the *Newton polygon* associated with this decomposition and the Hodge numbers of X/k, and their variation in families (starting with Grothendieck's observation that the Newton polygon rises under specialization), have been the subject of a great number of studies in the 1970's and 1980's, by Katz, Mazur, Ogus, Oort, and many others. The theory of the *de Rham-Witt complex* [36] gave some

[8] It does so only when A is ordinary and $\mathcal{A}$ is its so-called *canonical lifting*.

geometric insight into these questions, which are still actively pursued today with the new technology provided by the Drinfeld–Bhatt–Lurie prismatic stacks ([10], [20]) and the theory of prismatic F-gauges [9].

But undoubtedly the most important development of crystalline cohomology was the study of its relation with p-adic étale cohomology, which became known as *p-adic Hodge theory.*

The starting point is this. Let $\mathcal{X}/W(k)$ be proper and smooth, and let $X := \mathcal{X} \otimes k$, $\mathcal{X}_K := \mathcal{X} \otimes K$. Two different kinds of cohomology groups are associated with $\mathcal{X}_K$:

(a) $H^i_{\mathrm{dR}}(\mathcal{X}_K/K)$, canonically isomorphic to $H^i_{\mathrm{crys}}(X/W(k)) \otimes K$, which is a finite dimensional K-vector space, of dimension b_i, equipped with the automorphism φ, and the *Hodge filtration* $\mathrm{Fil}^\bullet$ coming from the (degenerate) Hodge to de Rham spectral sequence;

(b) $H^i(\mathcal{X}_{\overline{K}}, \mathbb{Q}_p)$, a finite dimensional $\mathbb{Q}_p$-vector space, of the same dimension b_i, equipped with the continuous action of the Galois group $G_K := \mathrm{Gal}(\overline{K}/K)$.

Grothendieck observed that, for $\mathcal{X}$ an abelian scheme and $i = 1$, both (a) and (b) characterize $\mathcal{X}$ up to isogeny,[9] and that led him to (boldly) conjecture that in general (a) and (b) should be related by *mysterious functors.* A solution was proposed about ten years later by Fontaine, in the form of a conjectural *period isomorphism* [22], ([23], III 6.1.4)[10]

$$B_{\mathrm{crys}} \otimes_K H^*_{\mathrm{dR}}(\mathcal{X}_K/K) \xrightarrow{\sim} B_{\mathrm{crys}} \otimes_{\mathbb{Q}_p} H^*(\mathcal{X}_{\overline{K}}, \mathbb{Q}_p), \tag{3.7}$$

where B_{crys} is a big K-algebra, equipped with the three types of structure (φ, $\mathrm{Fil}^\bullet$, G_K-action). This isomorphism (3.7) should be compatible with the three structures, and $H^*_{\mathrm{dR}}(\mathcal{X}_K/K)$ (resp. $H^*(\mathcal{X}_{\overline{K}}, \mathbb{Q}_p)$) should be recovered from the left (resp. right) hand side by a simple algebraic operation. After several special cases were treated by Fontaine, Fontaine–Messing, Kato, (3.7) was first proved in general by Tsuji [52] (together with a variant involving semistable reduction). Different proofs were provided later by Faltings, Niziol, Beilinson.

This was not the end of the story. Indeed, (3.7) is an isomorphism over $\mathbb{Q}_p$, and Grothendieck was dreaming of a comparison between $H^*_{\mathrm{dR}}(\mathcal{X}/W(k))$ and $H^*(\mathcal{X}_{\overline{K}}, \mathbb{Z}_p)$ that would not neglect torsion. He even asked whether one

[9] In the sense that the contravariant functor from the category of abelian schemes $\mathcal{X}/W(k)$ lifting X/k, up to isogeny, to the category of data (a) $(H^1_{\mathrm{dR}}(\mathcal{X}_K/K), \varphi, \mathrm{Fil}^1)$ (resp. (b) $(H^1(\mathcal{X}_{\overline{K}}, \mathbb{Q}_p)$, G_K-action)) is fully faithful, as follows from results of Serre-Tate, Grothendieck-Messing, Fontaine, Tate.

[10] See Note III 4, p. 402 of the second edition for remarks on the history of the proof.

could compare the lengths of the corresponding torsion submodules. This problem was recently solved by Bhatt–Morrow–Scholze and Bhatt–Scholze, by the theory of *integral p-adic Hodge theory* [11] and *prismatic cohomology* [12]. These new theories somehow interpolate between the two sides of (3.7). Concerning torsion, the result is simple: one has

$$\lg_{W(k)} H^*_{\mathrm{dR}}(\mathcal{X}/W(k))_{\mathrm{tors}} \geqslant \lg_{\mathbb{Z}_p} H^*(\mathcal{X}_{\overline{K}}, \mathbb{Z}_p)_{\mathrm{tors}}. \tag{3.8}$$

The inequality can be strict, and in case of equality, the structures of elementary divisors can be different ([11], 2.10, 2.11).

For non-smooth schemes, algebraic de Rham cohomology does not behave well. It was already noted by Grothendieck in EGA IV ([27], 0_{IV} 20.6.26) that for an A-algebra B, written as a quotient of a polynomial algebra P by an ideal J, the complex (in degrees -1 and 0)

$$J/J^2 \overset{d_{P/A}}{\to} B \otimes_P \Omega^1_{P/A}$$

does not depend, up to homotopy, on the choice of the surjective homomorphism $P \to B$, and was a finer invariant than its $H^0 = \Omega^1_{B/A}$ and H^{-1} (called *imperfection module* in *loc. cit.*). This is the first appearance of the *cotangent complex* $L_{B/A}$, in the form of its canonical truncation in degree $\geqslant -1$. Grothendieck returned to this in ([32], 5.8), and suggested the use of this (truncated) complex in the formulation and proof of the Riemann-Roch theorem for locally complete intersection morphisms in SGA 6 ([5], Exp. VIII). He globalized it in [31]. A non-truncated version, in the affine case, was constructed independently by M. André [1] and D. Quillen [48]. Globalization on ringed toposes was made in [34] and [35] (which contain applications to deformation theory suggested by Grothendieck). Since then, the theory of the cotangent complex has undergone many developments. Let me just mention that a natural extension of it was the construction of *derived de Rham cohomology*. Introduced in ([35], VIII), it has been extensively studied since 2011, starting with seminal papers by B. Bhatt [8] and A. Beilinson [4].

4. Petrov's example

Choose an elliptic curve E over $W(k)$ with supersingular reduction $E_k := E \otimes k$ (i.e. $E_k[p](\overline{k}) = 0$,[11] or equivalently, the Frobenius endomorphism is zero on $H^1(E_k, \mathcal{O})$). Then $E[p]$ is a finite, flat, commutative $\mathbb{F}_p$-module scheme over $W(k)$. Let $q = p^2$ and consider $E[p] \otimes_{\mathbb{F}_p} \mathbb{F}_q^{\oplus p}$, which is a sum of $2p$ copies of $E[p]$. Let $\mathrm{SL}_p(\mathbb{F}_q)$ act on $E[p] \otimes_{\mathbb{F}_p} \mathbb{F}_q^{\oplus p}$ by the trivial action on $E[p]$ and its natural action on $\mathbb{F}_q^{\oplus p}$, and define

$$G := \mathrm{SL}_p(\mathbb{F}_q) \ltimes (E[p] \otimes_{\mathbb{F}_p} \mathbb{F}_q^{\oplus p}).$$

[11] As usual, $-[p]$ denotes the kernel of multiplication by p.

This is a (non-commutative) finite flat group scheme over $W(k)$. By the Godeaux–Serre–Raynaud approximation technique, one can choose a projective, smooth scheme $\mathcal{X}$ over $W(k)$, equipped with a G-torsor P, such that the classifying map $\mathcal{X} \to BG$ defined by P induces, on the reductions $X = \mathcal{X}_k$ and BG_k on k, an injection $H^{p+1}(BG_k, \mathcal{O}) \hookrightarrow H^{p+1}(X, \mathcal{O})$.

Then, for X, we have $h^p < \sum_{i+j=p} h^{ij}$.

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