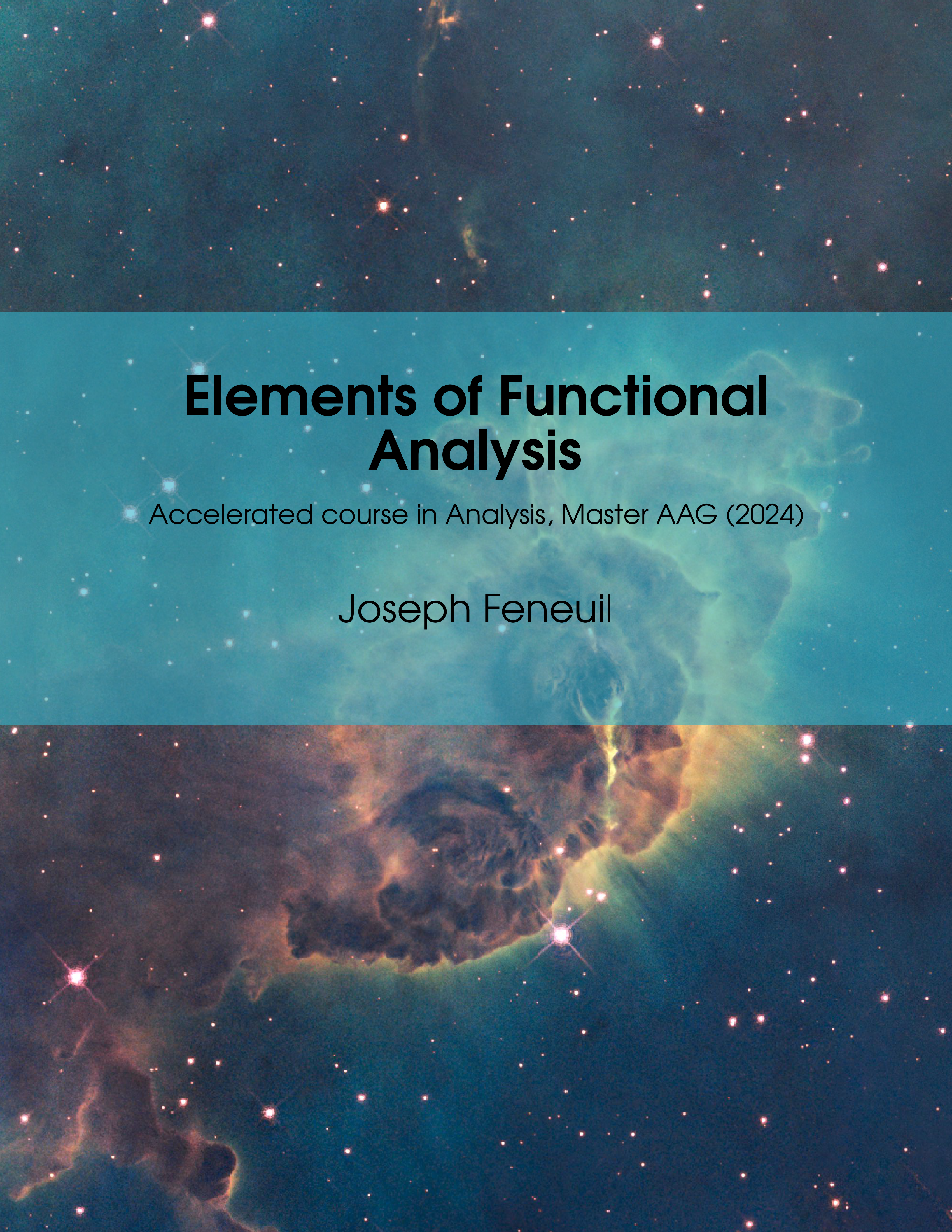

Elements of Functional Analysis

Accelerated course in Analysis, Master AAG (2024)

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LECTURES NOTES FOR THE ACCELERATED COURSE IN ANALYSIS "ELEMENTS OF FUNCTIONAL ANALYSIS", GIVEN FOR THE MASTER AAG OF THE UNIVERSITY PARIS-SACLAY.

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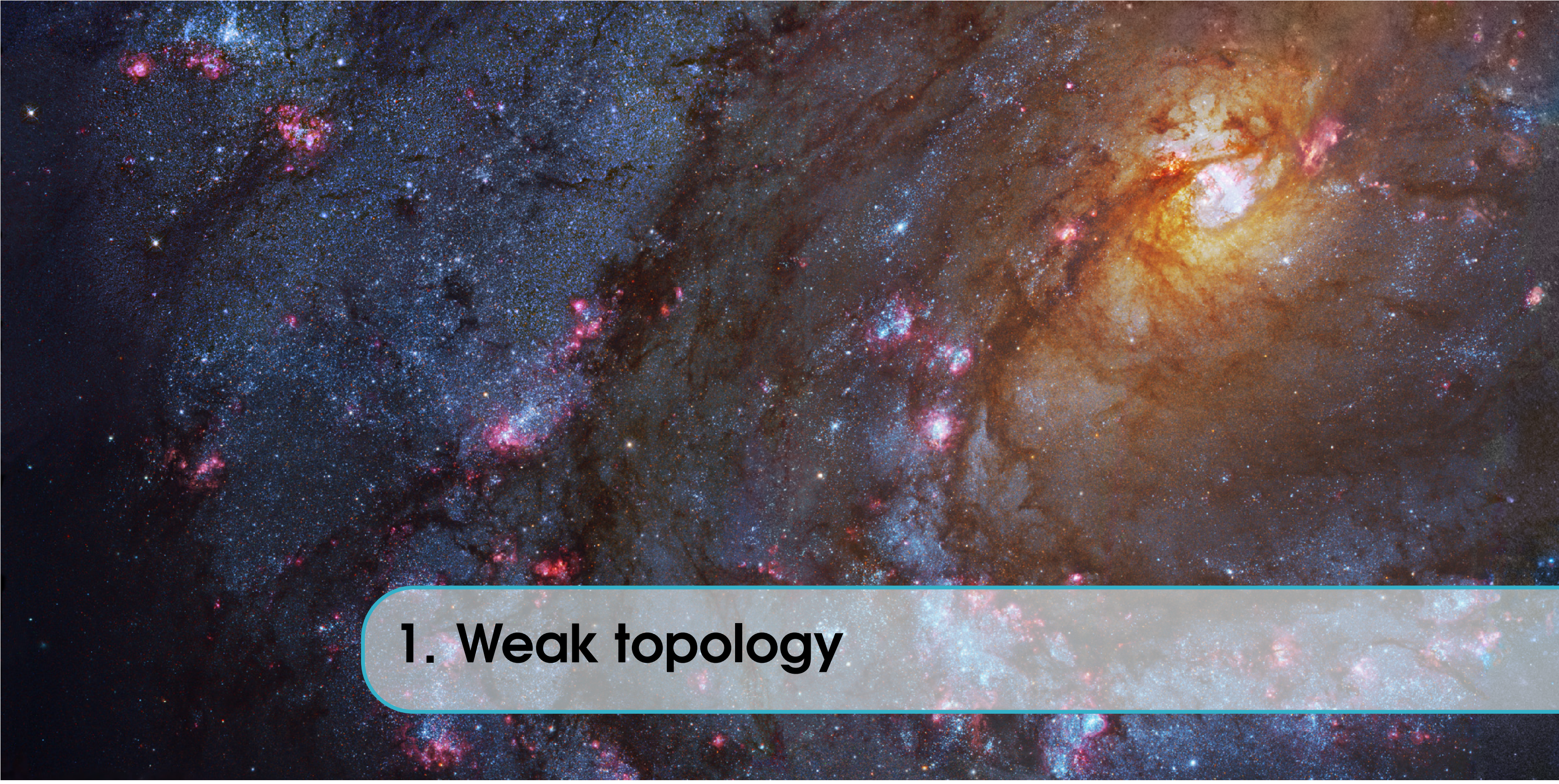
Contents



1. Weak topology

1.1 Duality, examples

Definition 1.1.1 Let X, Y be 2 normed vector spaces.

- $B(X,Y) := \{T : X \to Y \text{ linear and continuous}\}$.
- $B(X) := B(X,X)$.
- $X^* := B(X,\mathbb{R})$.
- $\forall x^* \in X^* \quad \forall x \in X \quad \langle x^*, x\rangle := x^*(x) =: \langle x, x^*\rangle$.

Lemma 1.1.1 Let X be a Banach space and $x^* \in X^*$. Then

$$\|x^*\|_{X^*} := \sup\{\langle x, x^*\rangle \; ; \; x \in X, \; \|x\| = 1\} < \infty.$$

Theorem 1.1.2 Let X be a normed vector space.
Then $(X^*, \|.\|_{X^*})$ is a Banach space.

Let us now give few duality spaces that should be known. First are the Hilbert spaces.

Theorem 1.1.3 Riesz representation theorem Let H be an Hilbert space, equipped with the scalar product $(.,.)$. Then $H^* = H$, in the sense that for any $x^* \in H^*$, there exists $h \in H$ such that

$$\langle x^*, x\rangle = (h, x) \qquad \text{for } x \in H.$$

Then we present the spaces of sequences.

Definition 1.1.2 We define for $1 \leq p < +\infty$

$$\ell^p = \ell^p(\mathbb{N}) := \{\mathbf{u} := (u_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}},\ \|\mathbf{u}\|_p := \Big(\sum_{n=0}^{\infty} |u_n|^p\Big)^{\frac{1}{p}} < +\infty\}.$$

When $p = +\infty$, we have

$$\ell^\infty = \ell^\infty(\mathbb{N}) := \{\mathbf{u} := (u_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}},\ \|\mathbf{u}\|_\infty := \sup_{n\in\mathbb{N}} |u_n| < +\infty\}$$

and

$$c_0 = c_0(\mathbb{N}) := \{\mathbf{u} := (u_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N}),\ \lim_{n\to\infty} u_n = 0\}.$$

Theorem 1.1.4 The space $(c_0)^*$ can be identified to ℓ^1 and, for $1 \leq p < \infty$, $(\ell^p)^*$ is isomorphic to $\ell^{p'}$, that is

- for any $x^* \in (c_0)^*$, we can find $\mathbf{u} = (u_n)_n \in \ell^1$ such that

$$\langle x^*, \mathbf{v}\rangle = \sum_{n=0}^{\infty} u_n v_n \qquad \text{for } \mathbf{v} = (v_n)_n \in c_0;$$

- for $p \in [1,\infty)$ and for any $x^* \in (\ell^p)^*$, we can find $\mathbf{u} = (u_n)_n \in \ell^{p'}$ - $\frac{1}{p} + \frac{1}{p'} = 1$ - such that

$$\langle x^*, \mathbf{v}\rangle = \sum_{n=0}^{\infty} u_n v_n \qquad \text{for } \mathbf{v} = (v_n)_n \in \ell^p.$$

We write $(c_0)^* = \ell^1$ and $(\ell^p)^* = \ell^{p'}$ for short.

When we deal with functions, we need to be a bit more careful. The L^p spaces, $p \in [1,\infty)$ are similar:

Theorem 1.1.5 Let (E,μ) be a measured space and $1 \leq p < \infty$. Then the dual of $L^p(E,\mu)$ is identified to $L^{p'}(E,\mu)$, in the sense that for any $x^* \in (L^p(E,\mu))^*$, there exists $f \in L^{p'}(E,\mu)$ such that

$$\langle x^*, g\rangle = \int_E f g\, d\mu \qquad \text{for } g \in L^p(E,\mu).$$

Remark 1.1 We recall that the L^p spaces the quotient of functions whose p-power is integrable by the set of functions that are zero except on a set of μ-measure zero. By notation abuse, we use the same notation for an element in $L^p(E,\mu)$ and one of its representative function. ■

Definition 1.1.3 — Radon measures. Let μ be a $\mathbb{R}$-Borel measure on $\mathbb{R}^n$. We define its variation $|\mu|$ as

$$|\mu|(F) := \sup_{\mathscr{A}_F} \sum_{A\in\mathscr{A}_F} |\mu(A)|,$$

where the supremum is taken over all finite collections of pairwise disjoint Borel measurable subsets of F.

We call $\mathscr{M}(\mathbb{R}^n)$ the space of $\mathbb{R}$-Borel measure on $\mathbb{R}^n$ such that $\|\mu\|_{\mathscr{M}} := |\mu|(\mathbb{R}^n) < +\infty$.

Proposition 1.1.6 The space $\mathscr{M}(\mathbb{R}^n)$ is complete. Moreover, any element $\mu \in \mathscr{M}(\mathbb{R}^n)$ is Radon[a], that is, for any Borel $B \subset \mathbb{R}^n$ and any $\varepsilon > 0$, there exists a compact $K \subset B$ and an open $U \supset B$ such that $|\mu|(U \setminus K) < \varepsilon$.

[a] result still holds if we replace $\mathbb{R}$ by a complete separable metric space

Theorem 1.1.7 — Riesz representation theorem. Let $C_0(\mathbb{R}^n)$ be the space of continuous function vanishing at ∞, that is $f \in C_0(\mathbb{R}^n)$ if for any $\varepsilon > 0$, there exists a compact $K = K_{f,\varepsilon} \Subset \mathbb{R}^n$ such that $|f| \leq \varepsilon$ on $\mathbb{R}^n \setminus K$.

Then the dual of the Banach space $C_0(\mathbb{R}^n)$ can be identified to $\mathscr{M}(\mathbb{R}^n)$[a], that is, for any $x^* \in (C_0(\mathbb{R}^n))^*$, there exists $\mu \in \mathscr{M}(\mathbb{R}^n)$ such that

$$\langle x^*, f \rangle = \int_{\mathbb{R}^n} f(x) d\mu(x) \qquad \text{for } f \in C_0(\mathbb{R}^n).$$

[a] still hold if $\mathbb{R}$ is replaced by a is a locally compact Hausdorff space

Exercise 1.1 Prove Theorem 1.1.2, that is that X^* is complete.

Exercise 1.2 Show that $(c_0)^* = \ell^1$.

1.2 Hahn-Banach theorem

1.2.1 Analytic Hahn-Banach theorem.

Let us start with a small easy result.

Proposition 1.2.1 Let X be a Banach space, and $\mathscr{S} \subset X$ be a **dense** subset. Let $T : \mathscr{S} \to X$ be linear and such that

$$\exists C > 0 \quad \forall x, y \in \mathscr{S} \quad \|T(x) - T(y)\| \leq C\|x - y\|.$$

Then there exists a unique $S \in B(X)$ such that $T(x) = S(x)$ for all $x \in \mathscr{S}$.

In particular, if $V \subset X$ is a dense subspace and $\|T(x)\| \leq C\|x\|$ for all $x \in V$, then there exists a unique $S \in B(X)$ such that $T(x) = S(x)$ for all $x \in V$.

When $V \subset X$ is not a dense subspace, we prefer to use the Hahn-Banach theorem.

Theorem 1.2.2 — Helly, Hahn-Banach. Let E be a vector space over $\mathbb{R}$. Let $p : E \to \mathbb{R}$ be such that

$$p(\lambda x) = \lambda p(x) \quad \forall \lambda \in \mathbb{R}_+ \quad \forall x \in E,$$
$$p(x+y) \leq p(x) + p(y) \quad \forall x, y \in E.$$

Let $V \subset X$ be a subspace, and $\ell : V \to \mathbb{R}$ be a linear form such that

$$\forall x \in V \quad |\ell(x)| \leq p(x).$$

Then there exists a $\ell^* : E \to \mathbb{R}$ such that $\ell(x) = \ell^*(x)$ for all $x \in V$, and $|\ell^*(x)| \leq p(x) \quad \forall x \in E$.

Some important consequences deserve to be highlighted.

Corollary 1.2.3 — Hahn-Banach 1920's. Let X be a Banach space, and $V \subset X$ be a subspace. Let $\ell : V \to \mathbb{R}$ be linear and such that

$$\forall x \in V \quad |\ell(x)| \leq \|x\|.$$

Then there exists a $\ell^* \in X^*$ such that $\ell(x) = \ell^*(x)$ for all $x \in V$, and $\|\ell^*\|_{X^*} \leq 1$.

Corollary 1.2.4 Let X be a Banach space.

$$\forall x \in X \quad \exists x^* \in X^* \quad \begin{cases} \|x^*\|_{X^*} = 1, \\ \langle x^*, x \rangle = \|x\|. \end{cases}$$

Corollary 1.2.5 Let X, Y be Banach spaces, and $T \in B(X,Y)$. Define $T^* : Y^* \to X^*$ by

$$\langle T^*(y^*), x \rangle = \langle y^*, T(x) \rangle \quad \forall x \in X \quad \forall y^* \in Y^*.$$

Then $T^* \in B(Y^*, X^*)$ with $\|T^*\| = \|T\|$.

1.2.2 Geometric Hahn-Banach theorem

We need to notion of "*separation*".

Definition 1.2.1 Let X be a Banach space, $x^* \in X^*$, $\alpha \in \mathbb{R}$.

- $P_{\alpha,x^*} := \{x \in X \,;\, \langle x^*, x \rangle = \alpha\}$ is called a **hyperplane**.
- Two sets $A, B \subset X$ are called **separated** if there exists a hyperplane P_{α,x^*} such that

$$\begin{cases} \langle x^*, x \rangle \geq \alpha \quad \forall x \in A, \\ \langle x^*, x \rangle \leq \alpha \quad \forall x \in B. \end{cases}$$

- Two sets $A, B \subset X$ are called **strictly separated** if there exists a hyperplane P_{α,x^*} and an

$\varepsilon > 0$ such that:

$$\begin{cases} \langle x^*, x \rangle \leq \alpha - \varepsilon & \forall x \in A, \\ \langle x^*, x \rangle \geq \alpha + \varepsilon & \forall x \in B. \end{cases}$$

Theorem 1.2.6 — Hahn-Banach 1920's. Let $A, B \subset X$ be non-empty **convex** sets such that $A \cap B = \emptyset$. Assume that B is **open**. Then A and B are separated.

Corollary 1.2.7 Let $A, B \subset X$ be non-empty **closed convex** sets such that $A \cap B = \emptyset$. Assume that B is **compact**. Then A and B are strictly separated.

An important tool of the proof of the Hahn-Banach theorem is the Minkowski jauge.

Lemma 1.2.8 — Minkowski gauge. Let $A \subset X$ be a convex non-empty open set. Assume $0 \in A$. Define

$$\rho(x) := \inf\{\alpha > 0 \, ; \, \alpha^{-1} x \in A\} \quad \forall x \in X.$$

Then, for all $x, y \in X$ and $\beta > 0$,
- $\rho(\beta x) = \beta \rho(x)$.
- $\rho(x+y) \leq \rho(x) + \rho(y)$.
- There exists $M > 0$, such that for all $x \in X$, $\rho(x) \leq M\|x\|$.
- $A = \{x \in X \, ; \, \rho(x) < 1\}$.

Definition 1.2.2 Let $Y \subset X$ be a subspace.

$$Y^{\perp} := \{x^* \in X^* \, ; \, \langle x^*, y \rangle = 0 \quad \forall y \in Y\}.$$
$$Y^{\perp\perp} := \{x \in X \, ; \, \langle x, y^* \rangle = 0 \quad \forall y^* \in Y^{\perp}\}.$$

Theorem 1.2.9 Let $Y \subset X$ be a subspace. Then $\overline{Y} = Y^{\perp\perp}$. In particular, if $Y^{\perp} = \{0\}$ then Y is dense.

Exercise 1.3 Prove Corollary 1.2.5.

Exercise 1.4 Prove Theorem 1.2.9.

Exercise 1.5 Find an example of a two-dimensional Banach space X and a functional on one of its closed one-dimensional subspaces which has infinitely many extensions to a functional on X of the same norm.

Exercise 1.6 Show that $(\ell^\infty)^* \supsetneq \ell^1$. *(Hint: think about limits)*

1.3 Weak and weak-$*$ convergence

1.3.1 Weak convergence

Definition 1.3.1 Let X be a Banach space.
We write B_X for the **closed** ball $\{x \in X\ ;\ \|x\| \leq 1\}$ and S_X for the sphere $\{x \in X\ ;\ \|x| = 1\}$.

Lemma 1.3.1 Let $Y \subsetneq X$ be a closed subspace. Then

$$\forall \varepsilon > 0 \quad \exists x \in X \quad \|x\| = 1 \text{ and } \operatorname{dist}(x, Y) \geq 1 - \varepsilon.$$

■ **Remark 1.2** When X is an Hilbert, and $Y \subset X$ is a subspace, $\exists x \in \mathbb{S}_X$ such that $\operatorname{dist}(x, Y) = 1$. One can construct such x by considering unit vector in $Y^\perp$. ■

Theorem 1.3.2 Let X be a Banach space. If B_X is compact, then $\dim(X) < \infty$.

Proof of the lemma. Let $\varepsilon > 0$. Pick $x_0 \in Y^c$ and define $d := \operatorname{dist}(x_0, Y)$. Take now $y_0 \in Y$ such that $\|x_0 - y_0\| \leq \frac{d}{1-\varepsilon}$. Finally, define $x = \frac{x_0 - y_0}{\|x_0 - y_0\|}$, and observe that, for $y \in Y$

$$\|x - y\| = \frac{1}{\|x_0 - y_0\|}\left\|x_0 - \underbrace{y_0 + \|x_0 - y_0\| y}_{\in Y}\right\| \geq \frac{1}{\|x_0 - y_0\|} d \geq 1 - \varepsilon.$$

The lemma follows. □

Proof of the theorem. Assume $\dim X = \infty$. Then it means that there exists a countable family of linearly independent vectors $\{x_1, \ldots, x_n \ldots\}$. We constructthe family of subspaces $X_n := \operatorname{Span}\{x_1, \ldots, x_n\}$, which satisfies $X_n \subsetneq X_{n+1}$. Using the lemma, for any $n \in \mathbb{N}$, we can find $y_n \in \mathbb{S}_{X_{n+1}}$ - a unit vector in X_{n+1} - such that $\operatorname{dist}(y_{n+1}, X_n) > \frac{1}{2}$. It gives that for any $n, m \in \mathbb{N}$, $n \neq m$, $\|y_n - y_m\| \geq \frac{1}{2}$. Since the infinite collection of open sets $B(x_n, \frac{1}{2}) \cap B_X$ is non-overlapping and included in X, B_X cannot be compact. □

Sometimes, the norm topology is too strong and it is difficult to have convergence in norm. So we are looking at weaker topologies, where there are less open and close sets, where the convergence is easier, and hence where a set is more likely to be compact.

Definition 1.3.2 Given $\varepsilon > 0$, $x \in X$, $n \in \mathbb{N}$, $x_1^*, \ldots, x_n^* \in X^*$, we define

$$V_{\varepsilon, x, x_1^*, \ldots, x_n^*} := \{y \in X\ ;\ |\langle y - x, x_j^* \rangle| < \varepsilon \quad \forall j = 1, \ldots, n\}.$$

These sets form a base for the weakest topology associated with X^*. This topology is called the **weak topology** and is denoted by $\mathscr{T}_w$. If a sequence $(x_n)_{n \in \mathbb{N}}$ converges to x in $(X, \mathscr{T}_w)$, we write $x_n \xrightarrow[n \to \infty]{w} x$ or simply $x_n \rightharpoonup x$.

Exercise 1.7 A weakly open set is open (and a weakly closed set is closed).

Exercise 1.8 Show that for any $\alpha > 0$ and any $x^* \in X^*$, the set $U_{\alpha,x^*} := \{x \in X, < x,x^* >< \alpha\}$ is open.

Exercise 1.9 Show that the weak topology and the norm topology are the same for finite dimensional Banach spaces.

Lemma 1.3.3 Let $(x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ and $x \in X$.

$$x_n \xrightarrow[n\to\infty]{w} x \iff \forall x^* \in X^* \quad \langle x_n - x, x^*\rangle \underset{n\to\infty}{\to} 0.$$

Proposition 1.3.4 $\mathscr{T}_w$ is Hausdorff.

Proof of the lemma.
$(\implies)$ Let $x_n \rightharpoonup x$, and let $x^* \in X^*$. By definition of weak convergence, we have

$$\begin{aligned} &\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, x_n \in V_{\varepsilon,x,x^*} \\ \iff &\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, |\langle x_n - x, x^*\rangle| < \varepsilon \\ \iff &\langle x_n - x, x^*\rangle \to 0. \end{aligned}$$

$(\impliedby)$ Assume that $\langle x_n - x, x^*\rangle \to 0$ for all $x^* \in X^*$.

Let $\varepsilon > 0$, $N \in \mathbb{N}$, and $x_1^*, \ldots, x_N^* \in X^*$. By assumption, $\left\langle x_n - x, x_j^*\right\rangle \to 0$ for all $j = 1, \ldots, N$. In particular,

$$\begin{aligned} &\exists M \in \mathbb{N}, \forall n \geq M, \max_{j=1..N} |\langle x_n - x, x_j^*\rangle| < \varepsilon \\ &\exists M \in \mathbb{N}, \forall n \geq M, x_n \in V_{\varepsilon,x,x_1^*,\ldots,x_N^*} \end{aligned}$$

So we proved that $x_n \rightharpoonup x$. □

Proof of the proposition. Let $x, y \in X$, $x \neq y$. We want to find U_x, U_y open such that $x \in U_x$, $y \in U_y$, and $U_x \cap U_y = \emptyset$.

By Hahn-Banach, the two points are strictly separated, i.e. there exists $\alpha \in \mathbb{R}$, $\varepsilon > 0$, and $x^* \in X^*$ such that

$$\langle x^*, x\rangle \leq \alpha - \varepsilon < \alpha + \varepsilon \leq \langle x^*, y\rangle .$$

We take $U_x = \{z \in X, \langle x^*, z\rangle < \alpha\}$, and $U_y = \{z \in X, \langle x^*, z\rangle > \alpha\}$, which are not intersecting, and are open by Exercise 1.8. □

Theorem 1.3.5 — Mazur 1930's. Let X be a Banach space and $E \subset X$ be a convex set. Then the closure of E in $(X, \mathscr{T}_w)$ is equal to its closure in X.

Proof . Let E be closed and convex. We want to show that E^c is open. So take $x \in E^c$. By Hahn-Banach, E and x are strictly separated, so $\exists \alpha, x^*$ such that

$$\langle x, x^*\rangle < \alpha < \langle e, x^*\rangle \qquad \forall e \in E.$$

So $x \in U_{x^*,\alpha} := \{z \in X, \langle z, x^*\rangle < \alpha\} \subset E^c$, and $U_{x^*,\alpha}$ is open by Exercise 1.8. □

Corollary 1.3.6 Let $(x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ and $x \in X$ be such that $x_n \xrightarrow[n\to\infty]{w} x$.

- There exists $(y_n)_{n\in\mathbb{N}} \in (\text{convex}\{x_j \,;\, j \in \mathbb{N}\})^{\mathbb{N}}$ such that $y_n \xrightarrow[n\to\infty]{} x$.
- In particular, $\|x\| \le \liminf_{n\to\infty} \|x_n\|$.

Proof . **First point.** Let $E = \text{convex}\{x_j,\, j \in \mathbb{N}\}$. We have that $x \in \overline{E}^w = \overline{E}$ by Mazur's theorem. So there exists a sequence $y_n \in E$ such that $y_n \to x$.

Second point. Let $\ell := \liminf_{n\to\infty} \|x_n\|$. We can extract a subsequence $(x_{n_k})_k$ of $(x_n)_n$ such that $\|X_{n_k}\| \le \ell + \frac{1}{k}$. Of course, we still have $x_{n_k} \rightharpoonup x$, so the first point shows that

$$x \in \overline{\text{convex}\{x_{n_k},\, k \in \mathbb{N}\}} \subset (\ell + \frac{1}{k})B_X.$$

In particular $\|x\| \le \ell + \frac{1}{k}$. The result being true for all k, we have $\|x\| \le \ell$ as desired. □

1.3.2 Weak-$*$ convergence

The weak topology is nice, but we can sometimes have a even smaller topology: if the space under consideration is already a dual (X^*), then we can test the topology against elements of X instead of elements of X^{**}. Note that it will give a smaller topology because each element of X can be identified to an element of X^{**} via the following inclusive map.

$$\begin{aligned} i : X &\to X^{**} \\ x &\mapsto [x^* \mapsto \langle x^*, x\rangle]. \end{aligned}$$

Definition 1.3.3 Given $\varepsilon > 0$, $x^* \in X^*$, $n \in \mathbb{N}$, $x_1, \dots, x_n \in X$, we define

$$V_{\varepsilon, x^*, x_1, \dots, x_n} := \{y^* \in X^* \,;\, |\langle y^* - x^*, x_j\rangle| < \varepsilon \quad \forall j = 1, \dots, n\}.$$

These sets form a base for the weakest topology associated with $\{x^* \mapsto \langle x^*, x\rangle \,;\, x \in X\}$. This topology is called the **weak* topology** and is denoted by $\mathscr{T}_{w^*}$. If a sequence $(x_n^*)_{n\in\mathbb{N}}$ converges to x^* in $(X^*, \mathscr{T}_{w^*})$, we write $x_n^* \xrightarrow[n\to\infty]{w^*} x^*$ or simply $x_n^* \overset{*}{\rightharpoonup} x^*$.

Exercise 1.10 Show that the weak-$*$ closure of a convex set is convex.

Lemma 1.3.7 Let X be a Banach space, and assume that $y^*, x_1^*, \dots, x_n^* \in X^*$ are such that $\bigcup_{j=1}^{n} \operatorname{Ker} x_j^* \subset \operatorname{Ker} y^*$. Then $y \in \operatorname{Span}\{x_1^*, \dots, x_n^*\}$.

Proof . Define the map

$$\begin{aligned} F : \; X &\to \mathbb{R}^{n+1} \\ x &\mapsto (\langle x, y^*\rangle, \langle x, x^*1\rangle, \dots, \langle x, x_n^*\rangle) \end{aligned}$$

By assumption, $(1,0\dots,0) \notin \mathrm{Im}(F)$, so by the Hahn-Banach theorem in $\mathbb{R}^{n+1}$, we can separate $\{(1,0,\dots,0)\}$ from $\Im(F)$, i.e. $\exists\alpha \in \mathbb{R}$, $\exists\lambda = (\lambda_0,\dots,\lambda_n) \in \mathbb{R}^{n+1}$ such that

$$\begin{aligned} &\langle \lambda,(1,0,\dots,0)\rangle < \alpha < \langle \lambda, F(x)\rangle \qquad \forall x \in X \\ \iff &\lambda_0 < \alpha < \lambda_0 \langle x,y^*\rangle + \sum_{j=1}^n \lambda_j \langle x,x_j^*\rangle \qquad \forall x \in X. \end{aligned}$$

We have that $\langle \lambda, F(0)\rangle = 0$, meaning that $\lambda_0 < \alpha < 0$. So

$$\langle y^*,x\rangle = -\sum_{j=1}^n \frac{\lambda_j}{\lambda_0} \langle x_j^*,x\rangle \qquad \forall x \in X,$$

i.e. $y^* \in \mathrm{Span}\{x_j^*,\ j = 1,\dots,n\}$. □

The lemma has two interesting consequences.

Proposition 1.3.8 Let X be a Banach space.
$(X^*,\mathscr{T}_{w^*}) = (X^*,\mathscr{T}_w) \iff X = X^{**}$.

Proof .
($\Longleftarrow$) If $X = X^{**}$ contains the same elements, i.e. the map i is bijective, it means that $\{\langle x^*,x\rangle\}_{x\in X} = \{\langle x^*,x^{**}\rangle\}$, and hence the neighborhood defining the weak and weak-∗ topologies are the same.

($\Longrightarrow$) The map $x^{**} \in (X^*)^*$ is weakly continuous. Indeed, the pre-image of open sets of $\mathbb{R}$ are weakly open. For instance, $(x^{**})^{-1}(-1,1) = V_{1,0,x^{**}}$. By assumption, x^{**} is then weakly-∗ continuous, meaning that $\exists\varepsilon > 0$, $\exists x_1,\dots,x_n \in X$ such that $V_{\varepsilon,0,x_1,\dots,x_n} \subset (x^{**})^{-1}(-1,1)$.

By Lemma 1.3.7, it suffices to show that $\bigcup_{j=1}^n \mathrm{Ker}\, x_j \subset \mathrm{Ker}\, x^{**}$, where we see x_j as elements of X^{**} via the injection i. Indeed, once the inclusion of kernel is proved, we will have that $x^{**} \in \mathrm{Span}\{i(x_1), dots, i(x_n)\}$, i.e. the map i is onto.

So let $y^* \in \bigcup_{j=1}^n \mathrm{Ker}\, x_j$, i.e. $\langle x_j,y^*\rangle = 0$ for all $j = 1,\dots,n$. As a consequence

$$\begin{aligned} &|\langle x_j,ty^*\rangle| = 0 \qquad \forall j = 1,\dots,n \forall t \in \mathbb{R} \\ \iff &ty^* \in V_{\varepsilon,0,x_1,\dots,x_j} \qquad \forall t \in \mathbb{R} \end{aligned}$$

By the choice of $V_{\varepsilon,0,x_1,\dots,x_j}$, we have then that $|tx^{**}(y^*)| < 1$ for all $t \in \mathbb{R}$, which forces $x^{**}(y^*) = 0$, i.e. $y^* \in \mathrm{Ker}\, x^{**}$. □

Lemma 1.3.9 — w^* Hahn-Banach. Let $B \subset X^*$ be w^*-closed and convex.
For $x_0^* \notin B$, we have that

$$\exists x_0 \in X \quad \exists\alpha \in \mathbb{R} \quad \forall y^* \in B \quad \langle x_0^*,x_0\rangle < \alpha \le \langle x_0,y^*\rangle.$$

Proof . By assumption, E^c is w*-open, which mean that for $x_0^* \in E^c$, there exists $\varepsilon > 0$, $x_1,\dots,x_n \in X$ such that $V_{\varepsilon,x_0^*,x_1,\dots,x_n} \subset E^c$.

E is convex by assumption, and it is easy to check that $V_{\varepsilon,x_0^*,x_1,\dots,x_n}$ is (open and) convex. So by Hahn-Banach, there exists $x_0^{**} \in X^{**}$ and $\alpha > 0$ such that

$$\langle x_0^{**},x^*\rangle \le \alpha \le \langle x_0^{**},y^*\rangle \qquad \forall x^* \in V, \forall y^* \in E.$$

We want to prove that x_0^{**} can be identified as an element of X, and for this, we will use Lemma 1.3.7 again. We want to prove that $\bigcup_{j=1}^{n} \mathrm{Ker}\, x_j \subset \mathrm{Ker}\, x^{**}$.

Let $x^* \in \bigcup_{j=1}^{n} \mathrm{Ker}\, x_j$.

$$\begin{aligned} & \langle x_0 + tx^*, x_j \rangle = \langle x_0^*, x_j \rangle \qquad \forall j = 0, \dots, n \, \forall t \in \mathbb{R} \\ \iff & x_0^* + tx^* \in V \qquad \forall t \in \mathbb{R} \end{aligned}$$

So by construction of x_0^{**},

$$\begin{aligned} & \alpha \geq \langle x^{**}, x_0^* + tx^* \rangle = \langle x_0^{**}, x_0^* \rangle + t \langle x_0^{**}, x^* \rangle \qquad \forall t \in \mathbb{R} \\ \iff & \langle x_0^{**}, x^* \rangle = 0 \\ \iff & x^* \in \mathrm{Ker}\, x_0^{**}. \end{aligned}$$

The lemma follows by noting that $\langle x_0^{**}, x_0^* \rangle < \alpha$, since we have $\langle x_0^{**}, x^* \rangle \leq \alpha$ for x^* in a neighborhood of x_0^*. □

The last lemma has an important consequence.

Theorem 1.3.10 — Goldstine. For any Banach space X, $i(B_X)$ is dense in $(B_{X^{**}}, \mathscr{T}_{w^*})$.

Proof. By contradiction, assume that $B_{X^{**}} \setminus \overline{i(B_X)}^{w*} \neq 0$. So there exists $x_0^{**} \in B_{X^{**}} \setminus \overline{i(B_X)}^{w*}$. By the w* Hahn-Banach, there exists $x_0^* \in X^*$ and $\alpha > 0$ such that

$$\langle x, x_0^* \rangle < \alpha \leq \langle x_0^*, x_0^{**} \rangle \qquad \forall x \in B_X.$$

So

$$\|x_0^*\|_{X^*} := \sup_{x \in B_X} \langle x, x_0^* \rangle < \langle x_0^*, x_0^{**} \rangle \leq \|x_0^*\|_{X^*}$$

since $x_0^{**} \in B_{X^{**}}$. The theorem follows. □

Let us move to the bigger results.

Theorem 1.3.11 $(B_{X^*}, \mathscr{T}_{w^*})$ is metrisable if and only if X is separable.

■ **Remark 1.3** If a topology is metrisable, then the compacity is equivalent to the sequential compacity. ■

Proof.

($\implies$) Let d be a distance that generate the weak-$*$ topology on B_{X^*}. For $n \in \mathbb{N}^*$, we define the open neiborhoods U_n

$$U_n := \{x^* \in B_{X^*}, \, d(x^*, 0) < \frac{1}{n}\}.$$

The set is an open neighborhood of 0, so by equivalence of the topologies, for each $n \in \mathbb{N}^*$, there exists $\varepsilon_n > 0$, $N_n \in \mathbb{N}$, and $x_{n,1}, \dots, x_{n,N_n} \in X$ such that

$$V_{\varepsilon,0,x_{n,1},\dots,x_{n,N_n}} \subset U_n.$$

Define $Y := \text{Span}\{x_{n,j},\, n \in \mathbb{N}^*,\, j = 1,\dots,N_n\}$, and we want to show that Y is dense, that is $Y^\perp = \{0\}$. So let $y^* \in Y^\perp$, which means that

$$\begin{aligned}
&\langle y^*, x\rangle = 0 \qquad \forall y \in Y\\
\iff &\left\langle y^*, x_{n,j}\right\rangle = 0 \qquad \forall n \in \mathbb{N}^*, j = 1,\dots,N_n\\
\implies &y^* \in V_{\varepsilon,0,x_{n,1},\dots,x_{n,N_n}} \subset U_n \qquad \forall n \in \mathbb{N}^*\\
\implies &d(y^*,0) < \frac{1}{n} \quad \forall n \in \mathbb{N}^* \implies d(y^*,0) = 0 \implies y^* = 0.
\end{aligned}$$

The first implication follows.

($\Longleftarrow$) Let X be separable, meaning that we can find a dense sequence $(x_j)_{j\in\mathbb{N}^*}$ in B_X. Define

$$|||x^*||| := \sum_{j=1}^{\infty} 2^{-j} |\left\langle x^*, x_j\right\rangle|.$$

$|||.|||$ is a norm on X^*, with $|||x^*||| \le \|x^*\|_{X^*}$ for all $x^* \in X^*$.

We want to prove the equality of the weak-∗ topology and the topology induced by $|||.|||$.

First, let $\varepsilon > 0$, $x_0^* \in B_{X^*}$, and $y_1,\dots,y_n \in X$. WLOG, we can take $y_k \in B_X$ instead. We need to find $\delta > 0$ such that $x^* \in V_{\varepsilon,x_0^*,y_1,\dots,y_n}$ whenever $|||x^* - x_0^*||| < \delta$. So let δ to be defined later, and take x^* such that $|||x^* - x_0^*||| < \delta$, then we have $|\left\langle x^* - x_0^*, x_j\right\rangle| < 2^j\delta$. Pick $j_1,\dots,j_n$ such that

$$\|x_{j_k} - y_k\| < \frac{\varepsilon}{4}$$

and then choose $\delta > 0$ such that $2^{j_k}\delta < \varepsilon/2$ for each $k = 1,\dots,n$. We have now

$$\begin{aligned}
|\langle x^* - x_0^*, y_k\rangle| &\le |\left\langle x^* - x_0^*, x_{j_k}\right\rangle| + |\left\langle x^* - x_0^*, y_k - x_{j_k}\right\rangle|\\
&< 2^{j_k}\delta + \|x^* - x_0^*\|\|x_{j_k} - y_k\|\\
&< \frac{\varepsilon}{2} + 2\frac{\varepsilon}{4} < \varepsilon,
\end{aligned}$$

i.e. $x^* \in V_{\varepsilon,x_0^*,y_1,\dots,y_n}$ as desired.

Second, we let $\delta > 0$ and $x_0^* \in, B_{X^*}$, and we want to find a weak-∗ neighborhood of x_0^* that is included in $B_{|||.|||}(x_0^*,\delta)$. For $x^* \in B_{X^*}$, we have

$$\begin{aligned}
|||x^* - x_0^*||| &= \sum_{j=1}^{\infty} 2^{-j}|\left\langle x^* - x_0^*, x_j\right\rangle| \le \sum_{j=1}^{N} 2^{-j}|\left\langle x^* - x_0^*, x_j\right\rangle| + \sum_{j=N+1}^{\infty} 2^{-j}.2 \qquad \text{since } x^*, x_0^* \in B_{X^*}\\
&\le \max_{j=1,\dots,N} |\left\langle x^* - x_0^*, x_j\right\rangle| + 2^{-N}.
\end{aligned}$$

Pick $N \in \mathbb{N}$ such that $2^{-N} < \delta/2$, and $\varepsilon = \delta/2$. So if $x^* \in V_{\varepsilon,x_0^*,x_1,\dots,x_n}$, we have

$$|||x^* - x_0^*||| < \frac{\delta}{2} + \frac{\delta}{2} = \delta,$$

i.e. $V_{\varepsilon,x_0^*,x_1,\dots,x_n} \subset B_{|||.|||}(x_0^*,\delta)$ as desired. □

1.3.3 Banach-Anaoglu Theorem

Theorem 1.3.12 (Banach-Alaoglu 1940) $(B_{X^*}, \mathscr{T}_{w^*})$ is compact.

We immediately have the following corollary

Corollary 1.3.13 — Banach-Anaoglu. Let X be a separable Banach space. Then (B_{X^*}, τ_{w^*}) is sequentially compact, meaning that any bounded sequence $x_n^* \in X^*$ has a weakly-$*$ converging subsequence.

■ **Remark 1.4** The proof of the Banach-Anaoglu theorem in the general (non-separable) case require the axiom of choice. Although we will not prove it, the corollary doesn't require the axiom of choice. ■

Proof. Let ϕ be the map

$$\begin{array}{rccc} \phi: & (B_{X^*}, \tau_{w^*}) & \to & \mathbb{R}^X \\ & x^* & \mapsto & (\langle x^*, x\rangle)_{x\in X}. \end{array}$$

Let $A = \Im(\phi)$, which is included in $\prod_{x\in X}[-\|x\|, \|x\|]$ since $\|x^*\| \leq 1$.

By Tychonoff's theorem, $\prod_{x\in X}[-\|x\|, \|x\|]$ is compact. So the theorem will be proved once we show that

1. A is closed,
2. ϕ is a homeomorphism between (B_{X^*}, τ_{w^*}) and A.

Indeed, it is a simple exercise to see that closed subsets of compacts sets and compacts, and compacts sets are preserved by homeomorphism.

We prove (1). Let $\lambda = (\lambda_x)_{x\in X} \in \overline{A}$. We want to show that $\lambda \in A$, i.e.

(a) $x \to \lambda_x$ is linear,
(b) $|\lambda_x| \leq \|x\|$ for all $x \in X$.

For $\varepsilon > 0$ and $x_1, \ldots, x_n \in X$, define the basic neighborhoods of λ in the (cartesian) product topology

$$U_{\varepsilon, x_1, \ldots, x_n} := \{(\mu_x)_{x\in X} \in /R^X, |\mu_{x_j} - \lambda_{x_j}| < \varepsilon \text{ for } j = 1, \ldots, n\}.$$

Since $\lambda \in \overline{A}$, we have $U_{\varepsilon, x_1, \ldots, x_n} \cap A \neq \emptyset$.

Point (b) is now easy. Let $x \in X$, and take $\varepsilon > 0$. There exists $U_{\varepsilon, x} \cap A \neq \emptyset$, that is, there exists $x^* \in B^*$ such that $|\langle x^*, x\rangle - \lambda_x| < \varepsilon$. So

$$|\lambda_x| \leq |\langle x^*, x\rangle| + \varepsilon \leq \|x\| + \varepsilon$$

since $\|x^*\| \leq 1$. Since the inequality is true for all $\varepsilon > 0$, we deduce $|\lambda_x| \leq \|x\|$ as desired.

Point (a) is slightly more technical. Let $x_0, y_0 \in X$, $\alpha \in \mathbb{R}^*$, and $\varepsilon > 0$. We have that $U_{\min \varepsilon, \varepsilon/|\alpha|, x_0, y_0, x_0+\alpha y_0} \cap A \neq \emptyset$, so there is a $x^* \in B_{X^*}$ such that

$$\begin{cases} |\langle x^*, x_0\rangle - \lambda_{x_0}| < \varepsilon \\ |\langle x^*, y_0\rangle - \lambda_{y_0}| < \varepsilon/|\alpha| \\ |\langle x^*, x_0 + \alpha y_0\rangle - \lambda_{x_0+\alpha y_0}| < \varepsilon \end{cases}$$

As a consequence,

$$|\lambda_{x_0+\alpha y-0} - \lambda_{x_0} - \alpha\lambda_{y_0}| \leq |\lambda_{x_0+\alpha y_0} - \langle x^*, x_0 + \alpha y_0\rangle| + |\lambda_{x_0} - \langle x^*, x_0\rangle| + |\alpha||\lambda_{y_0} - \langle x^*, y_0\rangle| \leq 3\varepsilon$$

Since the inequality is true for all $\varepsilon > 0$, we have $\lambda_{x_0+\alpha y-0} = \lambda_{x_0} - +\alpha\lambda_{y_0}$ as desired.

We prove (2). First, note that ϕ is a bijection from B_{X^*} to A. Indeed,

$$\phi(x^*) = \phi(y^*) \iff \langle x^* - y^*, x\rangle = 0\ \forall x \in X \iff x^* = y^*.$$

So $B_{X^*} = \phi^{-1}(A)$. We want to show that ϕ^{-1} is continuous from A to (B_{X^*}, τ_{w^*}), i.e. for all $U \in B_{X^*}$ (basic) open set, $\phi(U)$ is open. So we wantto show that $\forall\varepsilon > 0$, $\forall x_0^* \in B_{X^*}$, $\forall n \in \mathbb{N}$, $\forall x_1, \dots x_n \in X$, $\phi(V_{\varepsilon,x_0^*,x_1,\dots,x_n} \cap B_{X^*})$ is open in A.

So let $\lambda \in \phi(V_{\varepsilon,x_0^*,x_1,\dots,x_n} \cap B_{X^*})$. It means that $\lambda = \phi(y_0^*)$ for some $y_0^* \in B_{X^*} \cap V_{\varepsilon,x_0^*,x_1,\dots,x_n}$, and we can also find $\delta > 0$ such that

$$|\langle y_0^* - x >_0^*, x_j\rangle| \leq (1-\delta)\varepsilon \qquad \forall j = 1,\dots,n. \tag{1.1}$$

We choose $U_{\varepsilon\delta/2,\lambda,x_1,\dots,x_n} := \{(\mu_x)_{x\in X}, |\mu_{x_j} - \lambda_{x_j}| < \varepsilon\delta/2$ for $j = 1\dots n\}$ as our neighborhood of λ, and our theorem will be proved if we can show that $U_{\varepsilon\delta/2,\lambda,x_1,\dots,x_n} \cap A \subset \phi(V_{\varepsilon,x_0^*,x_1,\dots,x_n} \cap B_{X^*})$.

Take then $\mu = (\mu_x)_{x\in X} \in U_{\varepsilon\delta/2,\lambda,x_1,\dots,x_n} \cap A$, which means in particular that there exists $y^* \in B_{X^*}$ such that $\mu = \phi(y^*)$ and we want to show that $y^* \in V_{\varepsilon,x_0^*,x_1,\dots,x_n}$. Indeed, for $j \in \{1,\dots,n\}$, one has

$$\begin{aligned}|\langle y^* - x_0^*, x_j\rangle| = |\mu_{x_j} - \lambda_{x_j} + \lambda_{x_j} - \langle x_0^*, x_j\rangle| \leq |\mu_{x_j} - \lambda_{x_j}| + |\langle y_0^* - x_0^*, x_j\rangle| \\ < \varepsilon\frac{\delta}{2} + (1-\delta)\varepsilon < \varepsilon.\end{aligned}$$

The theorem follows. □

Exercise 1.11 Let X be separable and $Y \subset X$ be a subspace. let $x^* \in X^*$. Prove that there exists $y^* \in Y^\perp$ such that

$$\operatorname{dist}(x^*, Y^\perp) := \inf\{x^* - z^*, z^* \in Y^\perp\} = \|x^* - y^*\|.$$

1.4 Reflexivity

Definition 1.4.1 A Banach space X is called **reflexive** if the canonical embedding

$$\begin{aligned} i : X &\to X^{**} \\ x &\mapsto [x^* \mapsto \langle x^*, x\rangle], \end{aligned}$$

is a bijection.

■ **Example 1.1** $\mathbb{R}^d$, Hilbert spaces, L^p for $1 < p < \infty$. ■

Proposition 1.4.1 Let X be reflexive, and $K \subset X$ be a convex closed and bounded subset. Then K is weakly compact.

Theorem 1.4.2 X is reflexive if and only if B_X is weakly compact.

Corollary 1.4.3 Let X be a Banach space. The following holds:

1. if X is isomorphic to a Banach space Y, then X reflexive if and only if Y is reflexive.
2. if X is reflexive, then every closed subspace of X is reflexive;
3. X is reflexive if and only if X^* is reflexive;

Proof of the Proposition. If X is reflexive, then $(B_{X^{**}}, \tau_{w^*}) = (B_X, \tau_w)$. So by Banach Anaoglu, (B_X, τ_w) is compact, and thus RB_X is also weakly compact. Since K is bounded, $K \subset RB_X$ for some $R > 0$. And since K is closed, Mazur's theorem gives that K is weakly closed. As a consequence, K is a weakly closed subset of a compact set, so it is weakly compact. □

Proof of the Theorem.
($\Longrightarrow$) is a direct consequence of the Banach Anaoglu theorem.

($\Longleftarrow$) Assume that (B_X, τ_w) is compact. Then $(i(B_X), \tau_{w^*})$ is also compact, and in partial it is weakly-$*$ closed. But since $i(B_X)$ is weakly-$*$ dense in $B_{X^{**}}$ (Goldstine theorem), we have $B_{X^{**}} = \overline{i(B_X)}^{w^*} = i(B_X)$. Moreover, we also have that $i(RB_X) = RB_{X^{**}}$, so i is indeed bijective. □

Proof of the Corollary.
Proof of (1): Let $T : X \to Y$ is an isomorphism, and X be reflexive. Take a neighborhood $V_{\varepsilon, T(x_0), y_1^*, \ldots, y_n^*} \subset Y$. We have that

$$\begin{aligned} T^{-1}(V_{\varepsilon, T(x_0), y_1^*, \ldots, y_n^*}) &= \{x \in X, |\langle T(x) - T(x_0), y_j^* \rangle| < \varepsilon \text{ for all } j = 1, \ldots, n\} \\ &= \{x \in X, |\langle x - x_0, T^*(y_j^*) \rangle| < \varepsilon \text{ for all } j = 1, \ldots, n\} \qquad \text{by def of } T^* \\ &= V_{\varepsilon, x_0, T^*(y_1^*), \ldots, T^*(Y_n^*)} \end{aligned}$$

so $T : (X, \tau_w) \to (Y, \tau_w)$ is continuous.

Now, since T is a isomorphism, we have $(B_Y, \tau_w) = \big(T(T^{-1}(B_Y)), \tau_w\big)$. Since T is an isomorphism, $T^{-1}(B_Y)$ is bounded and so it is a subset of a weakly compact of the form RB_X - the w-compacity of B_X coming from the fact that X is reflexive. Bythe weak continuity of T, $T^{-1}(B_Y)$ is weakly closed. $T^{-1}(B_Y)$ is weakly closed in the weakly compact set RB_X, so $T^{-1}(B_Y)$ is weakly compact. But since T is weakly continuous, its image $B_Y = T(T^{-1}(B_Y))$ is also weakly compact. We conclude with the theorem, that implies that Y is reflexive.

Proof of (2): If X is reflexive, then B_X is weakly compact. But B_{X_0} is closed and convex subset of B_X, so by Mazur's theorm, B_{X_0} is weakly closed in the weakly compact set B_X. Therefore, B_{X_0} is weakly compact, which means that X_0 is reflexive (by the theorem again).

Proof of (3): If X is reflexive, the weak and weak-$*$ topology of X^* coincide, and B_{X^*} is weakly compact by the Banach-Anaoglu theorem. The above theorem shows that X^* is reflexive. On the contrary, if X^* is reflexive, then X^{**} is too by what we just proved. But then $i(X)$ is a closed subspace of the reflexive space X^{**}, which means that $i(X) \approx X$ is reflexive by (1). □

Proposition 1.4.4 In a reflexive Banach space X, every bounded sequence $(x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ has a weakly converging subsequence $(x_{n_k})_{k\in\mathbb{N}}$.

■ **Remark 1.5** The above proposition is a simple consequence of the Banach-Anaoglu theorem when X is separable, but we also want the result in non-separable spaces. ■

Corollary 1.4.5 Let X be reflexive, and $K \subset X$ be convex, closed set. Then

$$\forall x \in X \quad \exists y \in K \quad \|x-y\| = \operatorname{dist}(x,K) =: \inf\{\|x-z\| \; ; \; z \in K\}.$$

Proof of the Proposition. Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence. Define $Y := \overline{\operatorname{Span}\{x_n, n \in \mathbb{N}\}} \subset X$, which is a closed and separable subspace of X. In particular, Y is reflexive of the previous corollary.

Assume that Y^* is separable, and conclude. If Y^* is separable, $(B_{Y^{**}}, \tau_{w^*})$ is metrizable, but $(B_{Y^{**}}, \tau_{w^*}) = (B_Y, \tau_w)$, so (B_Y, τ_w) is metrizable and compact (by Banach-Anaoglu and reflexivity), meaning that the sequence $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence.

Show that Y^* is separable. Pick $(y_n)_{n\in\mathbb{N}}$ be dense in $Y = Y^{**}$. Take then by the Hahn-Banach theorem a $(y_n^*)_{n\in\mathbb{N}} \in (\mathbb{S}_{Y^*})^{\mathbb{N}}$ such that $\langle y_n^*, y_n\rangle \geq \|y_n\|$.

It suffices to show that $\overline{\operatorname{Span}\{y_n^*, n \in \mathbb{N}\}} = Y^*$. We set $Z = \operatorname{Span}\{y_n^*, n \in \mathbb{N}\}$, and we want to prove that ${}^\perp Z = \{0\}$. So let $z \in Z^\perp$. We have $\langle z, y_n\rangle = 0$ for all $n \in \mathbb{N}$. Given $k \in \mathbb{N}$, by density of $\{y_n\}_n$ there exists $n_k \in \mathbb{N}$ such that $\|z - y_{n_k}\| \leq \frac{1}{k}$. So

$$\|y_{n_k}\| \leq \left\langle y_{n_k}^*, y_{n_k}\right\rangle = \left\langle y_{n_k}^*, y_{n_k} - z\right\rangle \leq \|y_n^*\|\|y_{n_k} - z\| \leq \frac{1}{k}$$

since $\|y_{n_k}^*\| = 1$ and $\|y_{n_k} - z\| \leq \frac{1}{k}$. We deduce that

$$\|z\| \leq \|y_{n_k}\| + \|y_{n_k} - z\| \leq \frac{2}{k}.$$

The inequality being true for all $k \in \mathbb{N}^*$, we conclude that $z = 0$. The proposition follows. $\square$

Proof of the Corollary. Let $x \in X$, and let $d := \operatorname{dist}(x,K)$. We can always find $y_n \in K^{\mathbb{N}}$ such that $\|x - y_n\| \leq d + \frac{1}{n}$. Obviously, the sequence y_n is bounded by $\|x\| + d + 1$, so by the proposition, there exists $y \in X$ such that $y_n \rightharpoonup y$. Let is check that y does the job.

Since K is convex and closed, so it is weakly closed by Mazur's theorem, and $y \in K$. We have $\|x - y\| \geq d$ since $y \in K$, and $\|x - y\| \leq \liminf \|x - y_n\| = d$ by the corollary of Mazur's theorem. Hence, $\|x - y\| = d$ as desired. $\square$

We have seen that in reflexive spaces, we have projections on closed convex sets. Finding a non-reflexive space where projections on convex sets don't exist is a hard question, so we shall forget about it. Another question is whether the projection is unique, like projections in Hilbert spaces. The answer is no in general, but it is true for some particular Banach spaces.

Definition 1.4.2 A Banach space X is called **uniformly convex** if

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x, y \in B_X \quad \|x - y\| \geq \varepsilon \implies \left\|\frac{x+y}{2}\right\| \leq 1 - \delta.$$

■ **Example 1.2** The L^p and ℓ^p spaces, when $p \in (1, \infty)$. ■

■ **Remark 1.6** Being a uniformly convex space is actually a property of the norm of the Banach space, and not of the topology. ■

Theorem 1.4.6 A uniformly convex Banach space is reflexive.

Theorem 1.4.7 Let X be uniformly convex, and $K \subset X$ be convex and closed set. Then

$$\forall x \in X \quad \exists! y \in K \quad \|x - y\| = d(x, K).$$

We write $P_K(x) := y$.

Proof of the first theorem. Let X be a uniformly convex space. Assume by contradiction that X is not reflexive, that is

$$\exists x^{**} \in S_{X^{**}}, \exists \varepsilon > 0, \forall x \in B_X : \|i(x) - x^{**}\| > \varepsilon.$$

By uniform convexity, pick a $\delta > 0$ such that

$$\forall x, y \in X, \|x - y\| \geq \varepsilon \implies \left\| \frac{x+y}{2} \right\| \geq 1 - \delta.$$

Take now $x^* \in \mathbb{S}_{X^*}$ such that $\langle x^{**}, x^* \rangle > 1 - \delta/2$.

Define the weak-$*$ neighborhoof of x^{**}

$$V := V_{\delta/2, x^{**}, x^*} = \{y^{**} \in X^{**}, |\langle y^{**} - x^{**}, x^* \rangle| < \delta/2\}$$

By definition of the weak-$*$ topology, V is weak-$*$ open, so by Goldstine's lemma $i(B_X) \cap V \neq$ and there exists $x \in B_X$ such that $i(x) \in V$. Define the second set

$$W := (i(x) + \varepsilon B_{X^{**}})^c.$$

Since $B_{X^{**}}$ is weak-$*$ closed (by theBanach-Anaoglu theorem), the set W is weak-$*$ open, and a weak-$*$ neighborhood of x^{**}. Thus, by Goldstine's lemma again, $i(B_X) \cap V \cap W \neq \emptyset$ and there exists $y \in B_X$ such that $i(y) \in V \cap W$.

In one hand, we have the following estimates:

$$|\langle x^*, i(x) - x^{**} \rangle| < \delta/2 \qquad \text{since } i(x) \in V;$$

$$|\langle x^*, i(y) - x^{**} \rangle| < \delta/2 \qquad \text{since } i(y) \in V;$$

which altogether gives

$$|\langle x^*, i(x+y) - 2x^{**} \rangle| < \delta$$

or, by our choice of x^*

$$1 - \frac{\delta}{2} < |\langle x^*, x^{**} \rangle| < \frac{\delta}{2} + |\langle x^*, i([x+y]/2 \rangle| \leq \frac{\delta}{2} + \left\| \frac{x+y}{2} \right\|$$

since $x^* \in \mathbb{S}_{X^*}$. But in the other hand, by the construction of W, we have $\|x - y\| \geq \varepsilon$, so by uniform converxity, $\|(x+y)/2\| \leq 1 - \delta$. We conclude that

$$1 - \frac{\delta}{2} < \frac{\delta}{2} + 1 - \delta,$$

whence the contradiction and the fact that X is reflexive. □

Proof of the second theorem. By Corollary 1.4.5, we have existence. By contradiction, assume that we don't have uniqueness, that is

$$\exists x \in X, \exists y_1, y_2 \in K, \exists \varepsilon > 0 : \begin{cases} \|x - y_1\| = \|x - y_2\| = \mathrm{dist}(x, K) =: d \\ \|y_1 - y_2\| > \varepsilon. \end{cases}$$

Set $z_1 = \frac{x-y_1}{\|x-y_1\|}$ and $z_2 = \frac{x-y_2}{\|x-y_2\|}$, which are both unit vector in X. First, we have,

$$\|z_1 - z_2\| = \frac{1}{d}\|y_1 - y_2\| > \frac{\varepsilon}{d},$$

so by the uniform convexity, $\exists \delta > 0$ such that $\|z_1 + z_2\| \geq 2(1-\delta)$. We deduce that, since $(y_1 + y_2)/2 \in K$ by convexity,

$$d \leq \left\| x - \frac{y_1 + y_2}{2} \right\| = \frac{d}{2}\|z_1 - z_2\| \leq d(1-\delta) < \mathrm{dist}(x, K),$$

hence the contradiction. □

Exercise 1.12 Find a reflexive space X which is not uniformly convex.

Exercise 1.13 Let X be uniformly convex, and $(x_n)_n \in X^{\mathbb{N}}$ be a sequence that converges weakly to $x \in X$. Show that the following are equivalent:

- $\|x_n\| \to \|x\|$,
- $\|x_n - x\| \to 0$.

2. Bounded operators

2.1 Baire's theorem and its consequence

Theorem 2.1.1 (Baire 1899) Let X be a complete metric space, and $(O_n)_{n\in\mathbb{N}}$ be a family of open sets in X.

$$\overline{O_n} = X \quad \forall n \in \mathbb{N} \implies \overline{\bigcap_{n\in\mathbb{N}} O_n} = X.$$

or, equivalently, if $(F_n)_{n\in\mathbb{N}}$ are closed sets with empty interior, then $\bigcup_{n\in\mathbb{N}} F_n$ has empty interior.

Theorem 2.1.2 — Uniform Boundedness Principle, Banach-Steinhaus 1923. Let X,Y be Banach spaces, and $(T_n)_{n\in\mathbb{N}} \in B(X,Y)^{\mathbb{N}}$. Assume that

$$\forall x \in X \quad \exists C_x > 0 \quad \forall n \in \mathbb{N} \quad \|T_n x\| \leq C_x \|x\|.$$

Then

$$\exists C > 0 \quad \forall x \in X \quad \forall n \in \mathbb{N} \quad \|T_n x\| \leq C\|x\|.$$

Corollary 2.1.3 Let X be a Banach space and $B \subset X$. Then B is bounded if and only if $\{\langle x^*, x\rangle \; ; \; x \in B\}$ is bounded for all $x^* \in X^*$.

Theorem 2.1.4 (open mapping)
Let X,Y be Banach spaces and $T \in B(X,Y)$ be surjective.
There exists $\delta > 0$ such that $B(0,\delta) \subset T(B(0,1))$.

Corollary 2.1.5 Let X,Y be Banach spaces and $T \in B(X,Y)$ be bijective. Then $T^{-1} \in B(Y,X)$.

Theorem 2.1.6 (closed graph)
Let X,Y be Banach spaces, and $T : X \to Y$ be linear.
Then $G(T) := \{(x,T(x))\ ;\ x \in X\}$ is closed in $X \times Y$ if and only if $T \in B(X,Y)$.

Exercise 2.1 If a sequence converges weakly, then it is bounded.

Exercise 2.2 Let X be a Banach space and $Y \subset X$ be a closed subspace.

- Show that $(X/Y)^*$ is isomorphic to $Y^{\perp}$.
- Show that X/Y is reflexive whenever X is reflexive.

2.2 Complex Interpolation

We consider first a baby version of the complex interpolation.

Definition 2.2.1 If M is a $N \times N$ matrix, we define the norm of M seen as an operator from $(\mathbb{R}^N, \|.\|_p)$ to itself as

$$\|M\|_{p,p} := \sup_{a \in \mathbb{R}^N} \frac{\|Ma\|_p}{\|a\|_p}.$$

Exercise 2.3 Check that $\|M\|_{\infty,\infty} = \sup_i \sum_{j=1}^N |m_{ij}|$ and $\|M\|_{1,1} = \sup_j \sum_{i=1}^N |m_{ij}|$.

Lemma 2.2.1 — Schur. We have

$$\|M\|_{p,p} \leq \|M\|_{1,1}^{1/p'} \|M\|_{\infty,\infty}^{1/p}.$$

Proof. It is a simple consequence of the Hölder inequality. Indeed,

$$|(Ma)_i| = \left|\sum_{j=1}^N m_{ij}a_j\right| \leq \sum_{j=1}^N |m_{ij}||a_j| \leq \left(\sum_{j=1}^N |m_{ij}|\right)^{\frac{1}{p}} \left(\sum_{j=1}^N |m_{ji}||a_j|^p\right)^{1-\frac{1}{p}},$$

hence

$$\|Ma\|_p^p = \sum_{i=1}^N |(Ma)_i|^p \leq \|M\|_{\infty,\infty}^{p/p'} \sum_{i=1}^N \sum_{j=1}^N |m_{ij}||a_j|^p \leq \|M\|_{\infty,\infty}^{p/p'} \|M\|_{1,1} \|a\|_p^p.$$

The lemma follows. □

The Schur lemma is a special case of the more general statement

$$\|M\|_{p,p} \leq \|M\|_{q,q}^{1-\theta} \|M\|_{r,r}^{\theta},$$

where $1 \leq q < p < r \leq \infty$ and $\theta \in (0,1)$ are linked by the relation $\dfrac{1}{q} = \dfrac{1-\theta}{p} + \dfrac{\theta}{r}$.

Lemma 2.2.2 — Three lines lemma. Let S be the strip $\{z \in \mathbb{C}, 0 < \Re(z) < 1\}$. Suppose that $F : \overline{S} := \{\} \to \mathbb{C}$ is a bounded continuous function, holomorphic on S, with the bounds

$$\sup_{v\in\mathbb{R}} |F(iv)| \leq A_0 \qquad \text{and} \qquad \sup_{v\in\mathbb{R}} |F(1+iv)| \leq A_1.$$

Then for any $\theta \in (0,1)$, we have that

$$\sup_{v\in\mathbb{R}} |F(\theta+iv)| \leq A_0^{1-\theta} A_1^{\theta}.$$

■ **Remark 2.1** An inspection of the proof would show that the *a priori* boundedness of F can be relaxed. But it cannot be completely removed, as the function $F(z) = \exp(\exp(\pi(z-1)))$ is bounded on the lines $\Re(z) = 0$ and $\Re(z) = 1$ but unbounded on $\Re(z) = \frac{1}{2}$. ■

Theorem 2.2.3 — Riesz-Thorin interpolation theorem. Let $(\Omega, \mathscr{F}, \mu)$ and $(\Omega', \mathscr{F}', \mu')$ be two measured spaces and let $1 \leq p_0, p_1, q_0, q_1 \leq \infty$. Let T be a $\mathbb{C}$-linear operator on $L^{p_0}(\Omega) \cap L^{p_1}(\Omega)$ that satisfies

$$\|Tf\|_{L^{q_0}(\Omega')} \leq A_0 \|f\|_{L^{p_0}(\Omega)} \qquad \text{for } f \in L^{p_0}(\Omega) \cap L^{p_1}(\Omega)$$

and

$$\|Tf\|_{L^{q_1}(\Omega')} \leq A_1 \|f\|_{L^{p_1}(\Omega)} \qquad \text{for } f \in L^{p_0}(\Omega) \cap L^{p_1}(\Omega).$$

Let $\theta \in (0,1)$ and set p_θ, q_θ such that

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Then $T \in B(L^{p_\theta}, L^{q_\theta})$ in the sense that

$$\|Tf\|_{L^{q_\theta}(\Omega')} \leq A_0^{1-\theta} A_1^{\theta} \|f\|_{L^{p_\theta}(\Omega)} \qquad \text{for } f \in L^{p_0}(\Omega) \cap L^{p_1}(\Omega) \tag{2.1}$$

and T extends uniquely to a bounded operator on $L^{p_\theta}(\Omega)$.

Corollary 2.2.4 The previous theorem extends to $\mathbb{R}$-linear operators, as long as one replace the constant $A_0^{1-\theta} A_1^{\theta}$ in (2.1) by $2A_0^{1-\theta} A_1^{\theta}$. Moreover, the constant in (2.1) can be kept unchanged if $1 \leq p_i \leq q_i < \infty$ for $i \in \{0,1\}$.

Proof of the lemma. WLOG, we can assume that $A_0, A_1 > 0$. Let $\varepsilon > 0$ set set

$$G_\varepsilon(\zeta) = F(\zeta) A_0^{\zeta-1} A_1^{-\zeta} e^{\varepsilon\zeta^2} \quad \forall \zeta \in S.$$

For $\zeta = \theta + iv \in S$ $|e^{\varepsilon\zeta^2}| = |e^{\varepsilon(\theta^2 - v^2)}| \leq e^{\varepsilon}$, so G_ε is a bounded and continuous on $\overline{S}$ and holomorphic on S. We apply the maximum principle on the cube $Q_R = S \cap \{\zeta \in \mathbb{C}, |\Im\zeta| \leq R\}$, and we have that

$\sup_{Q_R}|G_\varepsilon| \leq \max_{\partial Q_R}|G_\varepsilon|$.

If $\zeta = i\nu$ with $|\nu| \leq R$, we have $|G_\varepsilon(\zeta)| \leq 1$. If $\zeta = 1+i\nu$ with $|\nu| \leq R$, we have $|G_\varepsilon(\zeta)| \leq e^{\varepsilon}$. If $\zeta = \theta \pm iR$, then $G_\varepsilon(\zeta) \leq C\sup_{0\leq x\leq 1} A_0^{\theta-1}A_1^{\theta}e^{1-R^2}$, where C is the bound of F on S. So for a fixed $\varepsilon > 0$, we can always find R_0 such that $R \geq R_0$, $Ce^{-\varepsilon R^2} \leq 1$, which means that

$$\sup_{\zeta\in S}|G_\varepsilon(x)| \leq e^{\varepsilon},$$

that is, for $\zeta \in S$,

$$|F(\zeta)| \leq |A_0^{1-\zeta}A_1^{\zeta}e^{\varepsilon(1-\zeta^2)}|,$$

that is, for $\nu \in \mathbb{R}$, $\theta \in (0,1)$, and $\varepsilon > 0$

$$|F(\theta+i\nu)| \leq A_0^{1-\theta}A_1^{\theta}e^{\varepsilon(1+\nu^2-\theta^2}.$$

We fix ν and we take $\varepsilon \to 0$, we obtain then that $|F(\theta+i\nu)| \leq A_0^{1-\theta}A_1^{\theta}$. The lemma follows. □

Proof of the theorem. WLOG, we can assume that $A_0, A_1 > 0$.

Case 1: $p_0 = p_1 = \infty$**.** Then, for all $f \in L^\infty(\Omega)$, one has $T(f) \in L^{q_0}(\Omega') \cap L^{q_1}(\Omega')$. Therefore,

$$\|T(f)\|_{q_\theta}^{q_\theta} = \int_{\Omega'} |Tf|^{\frac{(1-\theta)q_\theta}{q_0}q_0}|Tf|^{\frac{\theta q_\theta}{q_1}q_1}d\mu'.$$

But notice that the definition of q_θ implies that

$$\frac{(1-\theta)q_\theta}{q_0} + \frac{\theta q_\theta}{q_1} = 1,$$

meaning that by defining $r = \frac{q_0}{(1-\theta)q_\theta}$ and its Hölder conjugate $r' = \frac{q_1}{\theta q_\theta}$, and by using the Hölder inequality to the previous identity, we have

$$\|T(f)\|_{q_\theta}^{q_\theta} \leq \left(\int_{\Omega'}|Tf|^{q_0}d\mu'\right)^{\frac{1}{r}}\left(\int_{\Omega'}|Tf|^{q_1}d\mu'\right)^{\frac{1}{r'}} = \|T(f)\|_{q_0}^{(1-\theta)q_\theta}\|T(f)\|_{q_1}^{\theta q_\theta}$$

as desired.

Case 2: $\min\{p_0,p_1\} < \infty$**.** In this case, $p_\theta < \infty$. Let $a = \sum_{k=1}^{N}\alpha_k\mathbb{1}_{A_k}$ and $b = \sum_{\ell=1}^{M}\beta_\ell\mathbb{1}_{B_\ell}$ be two simple integrable functions on Ω and Ω' respectively. We want to prove that

$$\left|\int_{\Omega'}T(a).b\,d\mu'\right| \leq A_0^{1-\theta}A_1^{\theta}\|a\|_{p_\theta}\|b\|_{q'_\theta}.$$

Indeed, by duality and the density of the simple functions in $L^{p_\theta}(\Omega)$ and $L^{q'_\theta}(\Omega)$, we will obtain that T extends uniquely to a bounded operator in$B(L^{p_\theta}, L^{q_\theta})$, with norm smaller than $A_0^{1-\theta}A_1^{\theta}$.

We shall modify a and b a bit by defining p_z and q_z as

$$\frac{1}{p_z} = \frac{1-z}{p_0} + \frac{z}{p_1} \text{ and } \frac{1}{q'_z} = \frac{1-z}{q'_0} + \frac{z}{q'_1}$$

and then

$$f_z(\omega) = \mathbb{1}_{a\neq 0}|a(\omega)|^{p_\theta/p_z}\frac{a(\omega)}{|a(\omega)|} = \sum_{k=1}^{N}|\alpha_k|^{p_\theta/p_z}\frac{\alpha_k}{|\alpha_k|}\mathbb{1}_{A_k}(\omega),$$

$$g'_z(\omega') = \mathbb{1}_{b\neq 0}|b(\omega')|^{q'_\theta/q'_z}\frac{b(\omega)}{|b(\omega)|} = \sum_{\ell=1}^{M}|\beta_\ell|^{q_\theta/q_z}\frac{\beta_\ell}{|\beta_\ell|}\mathbb{1}_{B_\ell}.$$

Obviously, since a, b are simple functions, we have $g'_z \in L^{q'_0}\cap L^{q'_1}$, and $f_z \in L^{p_0}\cap L^{p_1}$, the latter implying in turn that $T(f_z)\in L^{q_0}\cap L^{q_1}$. We define then

$$F(z) = \int_{\Omega'} T(f_z)(\omega').q_z(\omega')\,d\mu'(\omega') = \sum_{k=1}^{N}\sum_{\ell=1}^{M}|\alpha_k|^{p_\theta/p_z}\frac{\alpha_k}{|\alpha_k|}|\beta_\ell|^{q_\theta/q_z}\frac{\beta_\ell}{|\beta_\ell|}\int_{B_\ell} T(\mathbb{1}_{A_k})\,d\mu',$$

which is holomorphic in $\mathbb{C}$ (sum of exponential functions) so obviously holomorphic on $S=\{\Re(z)\in(0,1)\}$ and continuous on $\overline{S}$. Moreover, it is easy to check that F is bounded on the strip $\overline{S}$, and for $\nu\in\mathbb{R}$, that

$$|F(i\nu)| \le A_0\|f_{i\nu}\|_{p_0}\|g_{i\nu}\|_{q'_0} \le A_0\|a\|_{p_\theta}^{p_\theta/p_0}\|b\|_{q'_\theta}^{q'_\theta/q'_0}$$

and

$$|F(i\nu)| \le A_1\|f_{i\nu}\|_{p_1}\|g_{i\nu}\|_{q'_1} \le A_1\|a\|_{p_\theta}^{p_\theta/p_1}\|b\|_{q'_\theta}^{q'_\theta/q'_1}.$$

By the 3 lines lemma, we have

$$|F(\theta)| = \left|\int_{\Omega'} T(a)(\omega').b(\omega')\,d\omega'\right| \le A_0^{1-\theta}A_1^{\theta}\|a\|_{p_\theta}\|b\|_{q'_\theta}.$$

The theorem follows. □

Proof of the corollary. If T is a real linear map, then we define the complex linear map as

$$T_{\mathbb{C}}(u+iv) := T(u)+iT(v).$$

By the triangle inequality, we immediately have $\|T\|_{p\to q}\le\|T_{\mathbb{C}}\|_{p\to q}\le 2\|T\|_{p\to q}$, which gives the first part of the corollary.

We want to prove that if $1\le p_i\le q_i<\infty$ the constant stays the same. It suffices to prove that $\|T\|_{p\to q} = \|T_{\mathbb{C}}\|_{p\to q}$ when T when $p\le q$. First, observe that if $z=a+iv$, we have

$$|z| = \frac{1}{\|\gamma\|_q}(\mathbb{E}|a\gamma_1+b\gamma_2|^q)^{\frac{1}{q}},$$

where γ,γ_1,γ_2 are some independent real valued standard Gaussian random variable Indeed, if $|z|=|a+ib|=1$, then $a\gamma_1+b\gamma_2$ is also a real valued standard Gaussian random variable on a space $(\tilde{\Omega},\tilde{\mu})$, and so $(\mathbb{E}|a\gamma_1+b\gamma_2|^q)^{\frac{1}{q}} = \|\gamma\|_q$. The general case follows by scaling.

Using this identity, we have now

$$\|\gamma\|^q_{L^q(\tilde{\Omega})}\|T_{\mathbb{C}}(u+iv)\|^q_{L^q(\Omega')} = \|\gamma\|^q_q \int_{\Omega'} |T(u)+iT(v)|^q\,d\mu' = \int_{\Omega'} \mathbb{E}|\gamma_1 T(u)+\gamma_2 T(v)|^q\,d\mu'$$
$$= \int_{\tilde{\Omega}} \left(\int_{\Omega'} |T(\gamma_1 u+\gamma_2 v)|^q\,d\mu'\right)^{\frac{q}{q}} d\tilde{\mu} \le \|T\|^q_{p\to q} \int_{\tilde{\Omega}} \left(\int_{\Omega} |\gamma_1 u+\gamma_2 v|^p\,d\mu\right)^{\frac{q}{p}} d\tilde{\mu}$$
$$\le \|T\|^q_{p\to q} \left(\int_{\Omega} \left[|\gamma_1 u+\gamma_2 v|^q \mu\right]^{\frac{p}{q}} d\mu\right)^{\frac{q}{p}} = \|T\|^q_{p\to q}\|\gamma\|^q_q\|u+iv\|^q_{L^p(\Omega)}$$

where the last but one inequality is by the generalized Minkowski inequality. The corollary follows. □

Let us give an example of application.

Definition 2.2.2 On $\mathbb{R}^d$, we defined the normalized Lebesgue measure $dm = (2\pi)^{-d/2}dx$, and then the Fourier transform

$$\hat{f}(\xi) = \mathscr{F}(f)(\xi) := \int_{\mathbb{R}^d} f(x)\exp(-ix\cdot\xi)\,dm(x),\ \xi \in \mathbb{R}^d.$$

Corollary 2.2.5 For $p \in [1,2]$, we have

$$\|\hat{f}\|_{L^{p'}(\mathbb{R}^d)} \le \|f\|_{L^p(\mathbb{R}^d)}.$$

Proof. The Plancherel theorem says that the Fourier transform $\mathscr{F}$ is an isometry on $L^2(\mathbb{R}^d, dm)$. It is easy to check that $\|\hat{f}\|_\infty \le \|f\|_{L^1(\mathbb{R}^d,dm)}$. So the corollary of the Riesz-Thorin interpolation theorem gives the result. □

3. Spectral theory

3.1 Spectral theorem for compact symmetric operators

Definition 3.1.1 Let H be a Hilbert space.

- $B(H) := \{T : H \to H \text{ linear and continuous}\}$.
- $K(H) := \{T \in B(H)\ ;\ \overline{T(B_H)} \text{ is compact}\}$.

Let $T \in B(H)$.

- $N(T) = \{h \in H\ ;\ T(h) = 0\}$.
- $\rho(T) = \{\lambda \in \mathbb{R}\ ;\ \lambda I - T \text{ is a bijection}\}$.
- $\sigma(T) = \rho(T)^c$ is called the **spectrum** of T.
- $EV(T) = \{\lambda \in \sigma(T)\ ;\ N(\lambda I - T) \neq \{0\}\}$.
- T is called **self-adjoint** if

$$\langle T(h), g\rangle = \langle h, T(g)\rangle \quad \forall g, h \in H.$$

Theorem 3.1.1 Let $T \in B(H)$ be self-adjoint. Define

$$m := \inf\{\langle T(h), h\rangle\ ;\ h \in H,\ \|h\| = 1\}$$
$$M := \sup\{\langle T(h), h\rangle\ ;\ h \in H,\ \|h\| = 1\}.$$

Then m, M are finite, and $\sigma(T) \subset [m, M]$.
Moreover $\|T\| = \max\{|m|, |M|\}$, and

$$T \in K(H) \implies m, M \in EV(T).$$

Corollary 3.1.2 Let $T \in K(H)$ be self-adjoint. Then

$$\sigma(T) \subset \{0\} \implies T = 0.$$

Lemma 3.1.3 Let $T \in K(H)$. Then $\sigma(T)\backslash\{0\} = EV(T)\backslash\{0\}$.

Theorem 3.1.4 — Hilbert 1906. Let H be a separable Hilbert space, and $T \in K(H)$ be self-adjoint. Then there exists an orthonormal basis $(e_n)_{n\in\mathbb{N}}$, and a sequence $(\lambda_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ such that

$$T(e_n) = \lambda_n e_n \quad \forall n \in \mathbb{N}.$$

3.2 Compact operators on Banach spaces.

Definition 3.2.1 A bounded linear operator $T \in B(X)$ is called compact if $\overline{T(B_X)}$ is compact. We write $T \in K(X)$.

Exercise 3.1 If (M,d) is a complete metric space, then M is precompact if and only if M is totally bounded, that is

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \exists (x_1, \dots, x_N) \in M : M \subset \bigcup_{i=1}^{N} B(x_j, \varepsilon).$$

Proposition 3.2.1 Let $(T_n)_{n\in\mathbb{N}} \in B(X)^{\mathbb{N}}$ and $T \in B(X)$ be such that $R(T_n)$ is finite dimensional, and $\|T_n - T\|_{B(X)} \underset{n\to\infty}{\to} 0$. Then $T \in K(X)$.

Proof. First, any finite finite rank operator is compact. Indeed, if $\dim T_n(X) < \infty$, then $\overline{T(B_X)}$ is a closed and bounded subset of the finite dimensional set $T_n(X)$, so $\overline{T(B_X)}$ is compact.

We want to prove that $T(B_X)$ is totally bounded. So let $\varepsilon > 0$ and pick n such that $\|T - T_n\| < \varepsilon/3$. But since T_n is totally bounded, there exists $x_1, \dots, x_n$ such that $T_n(B_X) \subset \bigcup_{j=1}^{N} B(T_n(x_j), \varepsilon/3)$. So for all $x \in B_X$, we pick j such that $\|T_n(x) - T_n(x_j)\| < \varepsilon/2$ and we have

$$\|T(x) - T(x_j)\| \le \|T(x) - T_n(x)\| + \|T_n(x) - T_n(x_j)\| + \|T_n(x_j) - T(x_j)\| \le \varepsilon,$$

meaning that $T(B_X) \subset \bigcup_{i=1}^{N} B(T(x_j), \varepsilon)$, $T(B_X)$ is totally bounded, and then that T is compact. □

■ **Remark 3.1** With the same proof, we can show that a limit of compact operators is still a compact operator. ■

■ **Remark 3.2** Every limit of finite rank operator is compact, but is every compact operator a limit of finite rank operator. The answer is no in general (the counterexample is complicated), but it is true if X is Hilbert or more generally have a (Schauder) basis. ■

■ **Example 3.1** Given $f \in L^2$, let $T(f) \in L^2(0,1)$ be the weak solution of $-T(f)'' + T(f) = f$, that exists with the Lax-Milgram theorem. Using the compactness in the Sobolev embedding, we can show that $T \in K(L^2)$. ■

Proposition 3.2.2 Let X be reflexive and separable, and $T \in B(X)$.

Then $T \in K(X)$ if and only if

$$\forall (x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}} \quad \forall x \in X \quad x_n \xrightarrow[n\to\infty]{w} x \implies \|T(x_n - x)\| \underset{n\to\infty}{\to} 0.$$

Proof. Let X be reflexive and separable.

($\implies$) Let $T \in K(X)$, and then take $(x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$, $x \in X$ such that $x_n \rightharpoonup x$.

We want to prove that $T(x_n) \to T(x)$. Assume by contradiction that there exists $\varepsilon > 0$ and a subsequence $\exists (x_{n_k})_{k\in\mathbb{N}}$ such that $\|T(x_{n_k} - T(x)\| > \varepsilon$ for all $k \in \mathbb{N}$.

The sequence $T(x_{n_k})$ is bounded, since $x_n \rightharpoonup x$. Therefore $T(x_{n_k})$ is precompact (i.e. relatively compact) and there exists a subsequence $T(x_{n_{k_l}})$ and $y \in X$ such that $\|T(x_{n_{k_l}} - y\| \to 0$ as $l \to \infty$. In particular, $T(x_{n_{k_l}} \rightharpoonup y$, that is

$$\forall x^* \in X^*, \left\langle T(x_{n_{k_l}}), x^* \right\rangle \to \langle y, x^* \rangle \text{ as } l \to \infty$$

which is equivalet to

$$\forall x^* \in X^*, \left\langle x_{n_{k_l}}, T^*(x^*) \right\rangle \to \langle y, x^* \rangle \text{ as } l \to \infty.$$

On the other hand, $\left\langle x_{n_{k_l}}, T^*(x^*) \right\rangle \to \langle x, T^*(x^*) \rangle$, since $x_n \rightharpoonup x$. We deduce that $y = T(x)$ and thus

$$\|T(x_{n_{k_l}} - T(x)\| \to 0 \text{ as } l \to \infty,$$

which contradicts our earlier statement.

($\impliedby$) Let $(y_n)_{n\in\mathbb{N}} \in \overline{T(B_X}$. For $n \in \mathbb{N}$, pick $x_n \in B_X$ such that $\|T(x_n) - y_n\| < \frac{1}{n}$. By Banach Anaoglu's theorem (X is reflexive and separable), we can find a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $x_{n_k} \rightharpoonup x$, and by assumption, we have then $\|T(x_{n_k}) - T(x)\| \to 0$ as $k \to \infty$. We have now

$$\|y_{n_k} - T(x)\| \leq \|y_{n_k} - T(x_{n_k})\| + \|T(x_{n_k}) - T(x)\| \to 0.$$

We deduce that $\overline{T(B_X)}$ is sequentially compact, so compact (X separable). □

The above proposition (the converse) is false if we do not assume reflexivity, as shown by the following result.

Lemma 3.2.3 If $u^{(k)} \rightharpoonup u \in \ell^1(\mathbb{N})$, then $u^{(k)} \to u \in \ell^1(\mathbb{N})$, that is weakly convergent sequences in $\ell^1(\mathbb{N})$ are strongly convergent.

Proof. Let $u^{(k)}$ be a sequence in $\ell^1(\mathbb{N})$ which converges weakly to 0. Assume by contradiction that $u^{(k)}$ does not converges to 0, meaning that (up to a subsequence) $\|u^{(k)}\| > \varepsilon$ for all $k \in \mathbb{N}$.

Take $k_1 = 1$. There exists N_1 such that $\sum_{n>N_1} |u_n^{(k_1)}| < \varepsilon/5$, so there exists $\delta_1, \ldots, \delta_n \in \{-1, 1\}$ such that $\sum_{n=1}^{N_1} \delta_n u_n^{(k_1)} > 4\varepsilon/5 > 3\varepsilon/5$. Moreover, for any choice of δ_n, $n \geq N_1$, we have

$$\sum_{n\in\mathbb{N}} \delta_n u_n^{(k_1)} > \varepsilon/5.$$

Now, since $u^{(k)} \rightharpoonup 0$, there exists $k_2 > k_1$ such that $\sum_{k\leq N_1} |u_n^{(k_2)}| < \varepsilon/5$. Take similarly $N_2 > N_1$ such that $\sum_{n>N_2} |u_n^{(k_2)}| < \varepsilon/5$ and there exists $\delta_{N_1+1}, \ldots \delta_{N_2} \in \{-1, 1\}$ such that $\sum_{n=N_1+1}^{N_2} \delta_n u_n^{(k_1)} > 3\varepsilon/5$, and we also have that $\sum_{n\in\mathbb{N}} \delta_n u_n^{(k_2)} > \varepsilon/5$ for any choice of δ_n, $n > N_2$.

By repeating the process, we find a subsequence $u^{(k_l)}$ and an element $\delta = (\delta_n) \in \{-1,1\}^{\mathbb{N}} \subset \ell^\infty(\mathbb{N})$ such that

$$\left\langle u^{k_l}, \delta \right\rangle_{\ell^1,\ell^\infty} = \sum_{n\in\mathbb{N}} \delta_n u_n^{k_l} > \varepsilon/5 \qquad \text{for all } l \in \mathbb{N},$$

meaning that u^{k_l} doesn't converge weakly to 0, whence the contradiction. □

Proposition 3.2.4 Let X be separable, and $T \in K(X)$.
Then $T^* \in K(X^*)$.

Proof. X is separable, meaning that we want to prove that $T^*(B_{X^*})$ is sequentially (pre-)compact.
So let $(x_n^*)_{n\in\mathbb{N}} \in B_{X^*}^{\mathbb{N}}$. By the Banach-Anaoglu theorem, there exists a subsequence $x_{n_k}^*$ and a limit $x^* \in X^*$ such that $x_{n_k}^* \xrightarrow{*} x^*$. We want to show that $T^*(x_{n_k}^*) \to T(x^*)$. Indeed, we have

$$\|T^*(x_{n_k}^*) - T^*(x^*)\|_{X^*} = \sup_{x\in B_X} \left\langle x_{n_k}^* - x^*, T(x) \right\rangle = \sup_{y\in \overline{T(B_X)}} \left\langle x_{n_k}^* - x^*, y \right\rangle$$

If the right-hand above tends to 0, then we obtain $T^*(x*_{n_k}) \to T^*(x^*)$, which proves that $T^* \in K(X^*)$.

So we just need to prove the following claim:

If K is totally bounded, if $z_k^* \in B_{X^*}$ is such that $\langle z_k^*, y\rangle \to 0$ for all $y \in K$, then $\sup_{y\in K} \langle z_k^*, y\rangle \to 0$.

The argument is standard. Let $\varepsilon > 0$. By the total boundedness, we can find $y_1, \dots, y_N$ such that $K \subset \bigcup_{j=1}^N B(y_j, \varepsilon/2\|\|)$, and then we can find $k_0 \in \mathbb{N}$ such that $k \geq k_0$ implies $|\left\langle z_k^*, y_j \right\rangle| \leq \varepsilon/2$ for all $j = 1, \dots, N$. For all $y \in K$ and all $k \geq k_0$, we can find y_j such that $\|y_j - y\| < \varepsilon/2$ and then

$$|\langle z_k^*, y\rangle| \leq |\left\langle z_k^*, y - y_j \right\rangle| + |\left\langle z_k^*, y_j \right\rangle| \leq \varepsilon.$$

The claim and then the proposition follow. □

3.3 Fredholm theory

Definition 3.3.1 Let $T \in B(X)$.

- $N(T) = \{x \in X \,;\, T(x) = 0\}$.
- $\rho(T) = \{\lambda \in \mathbb{R} \,;\, \lambda I - T \text{ is a bijection}\}$.
- $\sigma(T) = \rho(T)^c$ is called the **spectrum** of T.
- $EV(T) = \{\lambda \in \sigma(T) \,;\, N(\lambda I - T) \neq \{0\}\}$.
- $\sigma_{ap}(T) := \{\lambda \in \mathbb{R} \,;\, \exists (x_n)_{n\in\mathbb{N}} \in S_X^{\mathbb{N}} \quad \|\lambda x_n - T(x_n)\| \underset{n\to\infty}{\to} 0\}$.

Proposition 3.3.1 Let $T \in B(X)$ and $\lambda \in \mathbb{R}$.
The following assertions are equivalent.

1. $\lambda \notin \sigma_{ap}(T)$.
2. $\exists C > 0 \quad \forall x \in X \quad \|\lambda x - T(x)\| \geq C\|x\|$.
3. $N(\lambda - T) = \{0\}$ and $R(\lambda - T)$ is closed.

Proof. (1) $\implies$ (2). By contraposition, assume that (2) is false. Then,

$$\forall n \in \mathbb{N}^*, \exists x_n \in X, \|\lambda x_n - T(x_n)\| < \frac{1}{n}\|x_n\|.$$

The x_n cannot be 0, so by setting $y_n = x_n/\|x_n\| \in S_X$, we have

$$\forall n \in \mathbb{N}^*, \exists y_n \in S_X, \|\lambda y_n - T(y_n)\| < \frac{1}{n},$$

i.e. $\lambda \in \sigma_{ap}(T)$.

(2) $\implies$ (3). Let $x \in N(\lambda - T)$, so $0 = \|\Lambda x - Tx\| \geq C\|x\|$. Thus $x = 0$ and $N(\lambda - T) = \{0\}$. Let $(x_n)_n \in X^{\mathbb{N}}$ such that $y_n := \lambda x_n - T(x_n)$ converges to some $y \in X$, in particular y_n is a Cauchy sequence So

$$\|x_n - x_m\| \leq \frac{1}{C}\|\lambda(x_n - x_m) - T(x_n - x_m)\| \to 0.$$

So x_n is a Cauchy sequence, meaning $\|x_n - x\| \to 0$, so by continuity of $\lambda - T$, $\|\lambda x_n - T(x_n) - (\lambda x - T(x))\| \to 0$. This proves that $y = \lambda x - T(x) \in R(\lambda - T)$ and $R(\lambda - T)$ is closed.

(3) $\implies$ (1). $R(\lambda - T)$ is a closed subspace of a Banach space, so it is a Banach space. So, as $N(\lambda - T) = \{0\}$, the map $\lambda - T$ is a bijection from X to $R(\lambda - T)$. By the open mapping theorem, $(\lambda - T)^{-1} \in B(R(\lambda - T), X)$.

Assume by contradiction that $\lambda \in \sigma_{ap}(T)$, so $\exists (x_n)_n \in S_X^{\mathbb{N}}$ such that $\|\Lambda x_n - Tx_n\| \to 0$. We deduce that

$$1 = \|x_n\| = \|(\lambda - T)^{-1}(\lambda - T)(x_n)\| \leq \|\lambda - T\|\|(\lambda - T)(x_n)\| \to 0,$$

hence the contradiction. □

Theorem 3.3.2 Let X be a separable Banach space. Let $T \in K(X)$ and $\lambda \in \mathbb{R}^*$. Then $N(\lambda - T)$ has finite dimension, and $R(\lambda - T)$ has finite codimension. We say that $\lambda - T$ is a **Fredholm operator**.

■ **Remark 3.3** This starts the Fredholm theory. One of the main result in this theory is the Atiyah–Singer index theorem, that states that, for an elliptic differential operator on a compact manifold, the analytical index (related to the dimension of the space of solutions) is equal to the topological index (defined in terms of some topological data). ■

Proof. Without loss of generality, $\lambda = 1$.

Step 1: $\dim N(I - T) < +\infty$.

$N(I - T)$ is a Banach space. $N(I - T)$ is closed, so it is a Banach space. Moreover,

$$\forall x \in B_{N(I-T)}, x = T(x) \in \overline{T(B_X)}$$

So $B_{N(I-T)}$ is a closed subset in the compact set $\overline{T(B_X)}$, so $B_{N(I-T)}$ is compact and $N(I - T)$ is finite dimensional.

Step 2: $R(I - T)$ is closed.

Let $(u_n)_n \in X^{\mathbb{N}}$, $f \in X$ such that $u_n - T(u_n) \to f$.

Define for $n \in \mathbb{N}$, $d_n = \text{dist}(u_n, N(I-T))$, and then pick $v_n \in N(I-T)$ such that $\|u_n - v_n\| \leq d_n + \frac{1}{n}$.

Step 2(a): Assume for now that $\|u_n - v_n\|$ is bounded. Since T is compact, we can find a subsequence u_{n_j} and $g \in X$ such that $T(u_{n_j} - v_{n_j}) \to g$ as $j \to \infty$.

Then, since $v_{n_j} \in N(I-T)$, we have

$$u_{n_j} - v_{n_j} = u_{n_j} - T(v_{n_j}) = u_{n_j} - T(u_{n_j}) + T(u_{n_j} - v_{n_j}) \to f + g \text{ as } j \to \infty.$$

We deduce that, using $v_{n_j} \in N(I-T)$ again

$$(I-T)(f+g) = \lim_{j\to\infty} (I-T)(u_{n_j} - v_{n_j}) = \lim_{j\to\infty} (I-T)u_{n_j} = f.$$

We prove that $f \in R(I-T)$ and thus that $R(I-T)$ is closed, assuming that $\|u_n - v_n\|$ is bounded.

Step 2(b): We want to prove that $\|u_n - v_n\|$ is bounded.

Assume by contradiction that $\|u_n - v_n\|$ is unbounded, meaning that up to taking a subsequence, we can assume that $\|u_n - v_n\| \to \infty$. Moreover, up to another subsequence, and since T is compact, we can assume that

$$T\Big(\frac{u_n - v_n}{\|u_n - v_n\|}\Big) \to y$$

Define $w_n := \frac{u_n - v_n}{\|u_n - v_n\|}$. In one hand, we have

$$\|(I-T)w_n\| = \frac{\|(I-T)u_n\|}{\|u_n - v_n\|} \to \frac{\|f\|}{\infty} = 0.$$

On the other hand, since $v_n \in N(I-T)$,

$$\text{dist}(w_n, N(I-T)) = \text{dist}\Big(\frac{u_n}{\|u_n - v_n\|}, N(I-T)\Big) = \frac{d_n}{\|u_n - v_n\|} \geq \frac{d_n}{d_n + \frac{1}{n}} \to 1.$$

This two facts are contradictory, indeed, by the first fact

$$(I-T)y = \lim_{n\to\infty} T(I-T)w_n = 0,$$

meaning that $y \in N(I-T)$. So we have

$$0 = \text{dist}(y, N(I-T) = \lim_{n\to\infty} \text{dist}(T(w_n), N(I-T)) = \lim_{n\to\infty} \text{dist}(w_n - w_n + T(w_n), N(I-T)) = \lim_{n\to\infty} \text{dist}(w_n, N(I-T) \geq 1,$$

since $(I-T)w_n \to 0$ and $\text{dist}(w_n, N(I-T)) \geq 1$. The contradiction follows.

Step 3: Main idea of the rest of the proof Assume that we have

$$\exists n_0 \in \mathbb{N} : X = N((I-T)^{n_0}) \oplus R((I-T)^{n-0}). \tag{S}$$

Then $(I-T)^{n-0}$ can be written as $I - S_{n_0}$, where S_{n_0} is a compact operator. So by step 1, $\dim N((I-T)^{n_0}) < +\infty$. By (S), we have then that $\text{codim} R((I-T)^{n_0}) < +\infty$, and thus

$$R((I-T)) \supset R((I-T)^{n_0})$$

has finite codimension.

So we now want to prove (S). To that objective, we write

$$K_n := N((I-T)^n) \text{ and } M_n := R((I-T)^n).$$

Of course, for $n \in \mathbb{N}$, we have $M_{n+1} \subset M_n$, and $K_n \subset K_{n+1}$. We want to prove

$$\exists n_0 \in \mathbb{N} : \forall n \geq n_0, K_n = K_{n_0} \tag{S1}$$

and

$$\exists n_0 \in \mathbb{N} : \forall n \geq n_0, M_n = M_{n_0}. \tag{S2}$$

Note that if $K_{n_0} = K_{n_0+1}$, we have $K_n = K_{n_0}$ for $n \geq n_0$. Indeed, assume that $K_{n_0+1} = K_{n_0}$, and take $y \in K_{n_0+2}$, then $T(y) \in K_{n_0+1} = K_{n_0}$, so $T^{n_0+1}(y) = 0$, i.e. $y \in K_{n_0+1} = K_{n_0}$. Similarly, if $M_{n_0} = M_{n_0+1}$, then $M_n = M_{n_0}$ for all $n \geq n_0$.

Step 4: (S1) + (S2) $\implies$ (S)**.** Let n_0 such that $K_n = K_{n_0}$ and $M_n = M_{n_0}$ for all $n \geq n_0$. We want to prove that $K_{n_0} \cap M_{n_0} = \{0\}$ and $K_{n_0} + M_{n_0} = X$.

Let $x \in K_{n_0} \cap M_{n_0}$. There exists $y \in X$ such that $x = (I-T)^{n_0} y$ and $(I-T)^{n_0}(x) = 0$, meaning that $(I-T)^{2n_0} y = 0$. Thus $y \in K_{2n_0} = K_{n_0}$ and $x = (I-T)^{n_0} = 0$. We deduce that $K_{n_0} \cap M_{n_0} = \{0\}$.

Let $x \in X$. Since $M_{2n_0} = M_{n_0}$, there exists $y \in X$ such that $(I-T)^{n_0} x = (I-T)^{2n-0} y$. So $(I-T)^{n_0}(x - (I-T)^{n-0} y) = 0$, meaning that $x - (I-T)^{n-0} y \in K_{n_0}$. So

$$x = (I-T)^{n_0} y + [x - (I-T)^{n_0} y] \in M_{n_0} + K_{n_0}.$$

Step 4 follows.

Step 5: Proof of (S1)**.** By contradiction, assume that (S1) is false, and so for $n \in \mathbb{N}$, $K_n \subsetneq K_{n+1}$.

We have $T(K_n) \subset T(K_{n+1})$, and we want to prove that $T(K_n) \subsetneq T(K_{n+1})$. Assume again by contradiction that $T(K_n) = T(K_{n+1})$. Then for any $x_n \in K_{n+1}$, there is a $y_{n-1} \in K_n$ such that $T(x_n) = T(y_{n-1})$, so

$$x_n = \underbrace{x_n - T(x_n)}_{\in (I-T)K_{n+1} \subset K_n} + \underbrace{T(y_{n-1})}_{\in K_n} \in K_n,$$

hence $K_{n+1} = K_n$, which contradicts $K_n \subsetneq K_{n+1}$.

We have now $T(K_n) \subsetneq T(K_{n+1})$ for $n \in \mathbb{N}$. By Riesz lemma, there exists $(x_n)_n \in S_X^{\mathbb{N}}$ such that $x_n \in K_{n+1}$ and $\operatorname{dist}(T(x_n), T(K_n)) > \frac{1}{2}$. Let $n, m \in \mathbb{N}$, $n > m$, $T(x_m) \in T(K_n)$, so $\|T(x_n) - T(x_m)\| \geq \frac{1}{2}$. Thus $T(x_n)$ does not have a converging subsequence, which contradicts the fact that T is compact.

Step 6: Proof of (S2)**.** Assume by contradiction that (S2) is false, that is $M_{n+1} \subsetneq M_n$ for all $n \in \mathbb{N}$. By Riesz's lemma, there exists $(x_n)_n \in S_X^{\mathbb{N}}$ such that $x_n \in M_n$ and $\operatorname{dist}(x_n, M_{n+1}) > \frac{1}{2}$. So for $n, m \in \mathbb{N}$, $n > m$,

$$\begin{aligned}\|T(x_n) - T(x_m)\| = \|x_m + \underbrace{T(x_m) - x_m}_{\in M_{m+1}} - \underbrace{x_n}_{} \in M_n \subset M_{m+1} + \underbrace{(I-T)(x_n)}_{\in M_{n+1} \subset M_{m+1}} \\ \geq \operatorname{dist}(x_m, M_{n+1}) \geq \frac{1}{2}.\end{aligned}$$

This contradicts $T \in K(X)$. The Theorem follows. □

Corollary 3.3.3 Let X be a separable Banach space, $T \in K(X)$, and $\lambda \in \mathbb{R}^*$. The following assertions are equivalent.
1. $\lambda \in \rho(T)$.
2. $N(\lambda - T) = \{0\}$.
3. $R(\lambda - T) = X$.

Proof. WLOG, $\lambda = 1$. We want to show that
1. $N(I-T) = \{0\} \implies R(I-T) = X$,
2. $R(I-T) = X \implies N(I-T) = \{0\}$.

(1). By the proof of theorem, we have (S), that is there exists $n_0 \in \mathbb{N}$ such that

$$X = N((I-T)^{n_0}) \oplus R((I-T)^{n-0}).$$

Let $x \in N((I-T)^{n_0})$. Then $(I-T)(I-T)^{n_0-1}x = 0$. But since $N(I-T) = \{0\}$, we have $(I-T)^{n_0-1}x$, and inductively, $x = 0$. It means that $N((I-T)^{n_0}) = \{0\}$ and so $X = R((I-T)^{n-0}) \subset R(I-T)$.

Let us show (2) from (1) by duality. Assume that $R(I-T) = X$. Let $x^* \in N(I-T^*)$ and $x \in X$, so $\exists y \in X$ such that $x = (I-T)y$.

$$\langle x, x^* \rangle = \langle (I-T)y, x^* \rangle = \langle y, (I-T)^* x^* \rangle = 0.$$

Thus, $N(I-T^*) = \{0\}$ and by (1) - since T^* is compact - $R(I-T^*) = X^*$.

Let $x \in N(I-T)$, $x^* \in X^*$. There exists $y^* \in X^*$ such that $x^* = (I-T^*)y^*$.

$$\langle x, x^* \rangle = \langle x, (I-T^*)y^* \rangle = \langle (I-T)x, y \rangle = 0,$$

so $x = 0$ and $N(I-T) = \{0\}$. □

Corollary 3.3.4 Let X be a separable Banach space, and $T \in K(X)$. Then $\sigma(T)\backslash\{0\} = EV(T)\backslash\{0\}$, and $EV(T)\backslash\{0\}$ is either finite or a sequence that tends to 0.

Proof. By the previous corollary, if $\lambda \in \sigma(T) \setminus \{0\}$, then $N(I-T) \supsetneq \{0\}$, i.e. $\lambda \in EV(T)$. Assume that $\sigma(T) \setminus \{0\}$ has an infinite number of values. Let a sequence $\lambda_n \in \sigma(T) \setminus \{0\}$, $\lambda_n \neq \lambda_m$ when $n \neq m$, and let $x_n \in S_X$ be eigenvectors, i.e. such that $T(x_n) = \lambda_n x_n$. We claim that $(x_n)_n$ is a linearly independent family.

Indeed, assume by induction that $\{x_1, \ldots, x_n\}$ is a independent family, and $x_{n+1} = \sum_{j=1}^n \alpha_j x_j$, and then by contradiction,

$$\sum_{j=1}^n \lambda_{n+1} \alpha_j x_j = \lambda_{n+1} x_{n+1} = T(x_{n+1}) = \sum_{j=1}^n \alpha_j T(x_j) = \sum_{j=1}^n \alpha_j \lambda_j x_j,$$

i.e.

$$\sum_{j=1}^n (\lambda_{n+1} - \lambda_j) \alpha_j x_j = 0.$$

By linear independence, $(\lambda_{n+1} - \lambda_j)\alpha_j = 0$, and then $\alpha_j = 0$ since $\lambda_{n+1} \neq \lambda_j$, and then $0 = x_{n+1} \in S_X$, hence the contradiction.

We repeat the overused argument from the proof of the theorem. Let $Y_n = \text{Span}\{x_1, \dots, x_n\}$, and we have $Y_n \subsetneq Y_{n+1}$. By Riesz's lemma, $\exists y_n \in S_X$ such that $y_n \in Y_n$ and $\text{dist}(y_n, Y_{n-1}) \geq \frac{1}{2}$. For any $n \in \mathbb{N}$, $\exists \alpha_j^{(n)}$ such that $y_n = \sum_{j=1}^n \alpha_j^{(n)} x_j$. So for all $n > m$,

$$\begin{aligned}\|T(y_n) - T(y_m)\| &= \Big\| \sum_{j=1}^{n} \alpha_j^{(n)} \lambda_j x_j - | \sum_{j=1}^{m} \alpha_j^{(m)} \lambda_j x_j \Big\| \\ &= \Big\| \lambda_n y_n + \underbrace{\sum_{j=1}^{n-1} \alpha_j^{(n)} \lambda_j x_j - | \sum_{j=1}^{m} \alpha_j^{(m)} \lambda_j x_j}_{\in Y_{n-1}} \Big\| \\ &= \frac{1}{2} |\lambda_n|.\end{aligned}$$

If λ_n does not converges to 0, then T would not be compact. □

A. Complements

A.1 Tychonoff's theorem

Proposition A.1.1 Let X be a non-empty topological space. The following are equivalent:
- X is compact,
- If $\mathscr{C}$ is a collection of closed sets such that any finite subcollection has non-empty intersection - we say that $\mathscr{C}$ has the *finite intersection property* - then $\mathscr{C}$ has non-empty intersection.

Exercise A.1 Prove the above proposition.

Lemma A.1.2 — Zorn. Suppose that a partially ordered set $\mathbb{D}$ has the property that every chain in $\mathbb{D}$ - i.e. any totally ordered subset of $\mathbb{D}$ - has an upper bound (in $\mathbb{D}$). Then $\mathbb{D}$ has a maximal element.

Proof. It is actually equivalent to the axiom of choice! □

Definition A.1.1 — Product topology. Let I be a nonempty set, and let $\{(X_i, \tau_i)\}_{i\in I}$ is a collection of topological spaces.
- The **cartesian product** of the family $\{X_i\}_{i\in I}$ is the set $X := \prod_{i\in I} X_i$, whose elements are the functions x that maps $i \in I$ to an element $x(i) \in X_i$.
- The **coordinate mappings** are the elements $p_i : X \to X_i$ such that $p_i(x) = x(i)$.
- The **product topology** of $X = \prod_{i\in I} X_i$ is the smallest topology generated by the open sets $p_i^{-1}(U_i)$, where $i \in I$ and U_i are open sets of X_i.

■ **Remark A.1** If $I = \{1, \dots, n\}$ has a finite number of elements, then the product topology is the one generated by the sets $U_1 \times \cdots \times U_n$, where U_j is open in X_j.

But if I is infinite, the topology is generated by the sets $\prod_{i\in I} U_i$, where $U_i \subset X_i$ is open, and $U_i = X_i$ except for a finite number of values of I. In this sense, it is very relatable to the weak and weak-$*$ topology. ■

Theorem A.1.3 — Tychonoff. The product of any family of compact spaces is a compact space.

Proof. Let (X_i, τ_i) be a compact topological space, and define $X = \prod_{i \in I} X_i$. Without loss of generality, we can assume that $X_i \neq \emptyset$ for each $i \in I$, (otherwise $X = \emptyset$ and there is nothing to prove).

We want to prove that X is compact via the characterization given in Proposition A.1.1.

Step 1: We consider first the set $\mathbb{D}$ of all collections of subsets of X (closed or not) that contains $\mathscr{C}$ and that has the finite intersection property. $\mathbb{D} \ni \mathscr{C}$, so it is not empty, and we place a partial order on $\mathbb{D}$ (the inclusion). We want to show that it has a maximal element with Zorn's lemma.

Therefore, we take a totally ordered $\mathbb{T}$, and we want to show that the upper bound $\mathscr{B} := \bigcup_{\mathscr{T} \in \mathbb{T}} \mathscr{T}$ is in $\mathbb{D}$, i.e. that $\mathscr{B} \in \mathbb{D}$ has the finite intersection property. Take then $T_1, \ldots, T_k \in \mathscr{B}$, each T_j belongs to some $\mathscr{T}_j \in \mathbb{T}$. Since $\mathbb{T}$ is totally ordered, the collection $\{\mathscr{T}_j\}$ has a maximal element $\mathscr{T}_{max} \in \mathbb{T}$ for which $T_1, \ldots, T_k \in \mathscr{T}_{max}$. But $\mathscr{T}_{max}$ has the finite intersection property, so $\bigcap_{j=1..k} T_k \neq \emptyset$. We conclude that $\mathscr{B}$ has the finite intersection property.

By Zorn's lemma, $\mathbb{D}$ has at least one maximal element $\mathscr{M}$.

Step 2: We define for each $i \in I$ the collection $\mathscr{X}_i = \{\overline{p_i(M)}, M \in \mathscr{M}\}$ of sets of X_i. Since $\mathscr{M}$ has the finite intersection property, $\mathscr{X}_i$ has it too, and since X_i is compact, there exists $y_i \in Y_i := \bigcup_{M \in \mathscr{M}} \overline{p_i(M)} \subset X_i$. We want to show that the element $y = \{y_i\}_{i \in I} \in \overline{M}$ for all $M \in \mathscr{M}$.

First, note that by maximality of $\mathscr{M}$,

$$\text{any finite intersection of elements of } \mathscr{M} \text{ belongs to } M. \tag{A.1}$$

Then, if U_i is an open set in X_i that contains y_i, the fact that $y_i \in \overline{p_i(M)}$ for all $M \in \mathscr{M}$ implies that $p_i(M) \cap U_i \neq \emptyset$, and hence $M \cap p_i^{-1}(U_i) \neq \emptyset$ for all $M \in \mathscr{M}$. Thanks to (A.1), it means that $\mathscr{M} \cup \{p_i^{-1}(U_i), i \in I\}$ has the finite intersection property, hence by maximality,

$$p_i^{-1}(U_i) \in \mathscr{M} \text{ for all } i \in I. \tag{A.2}$$

Finally, let U be a basic neighborhood of y in X. By definition of the product topology, U has the form $U = \prod_{i=1}^k p_{i_j}^{-1}(U_{i_j})$, where $i_j \in I$ and $U_{i_j} \in \tau_{i_k}$. Since $\mathscr{M}$ has the finite intersection property, and by (A.2), we have $M \cap U = M \cap p_{i_1}^{-1}(U_{i_1}) \cap \cdots \cap p_{i_k}^{-1}(U_{i_k}) \neq \emptyset$ for all $M \in \mathscr{M}$. Since this is true for all basic neighborhood of y, we have $y \in \overline{M}$ as desired.

Conclusion. From step 2, we have

$$x \in \bigcap_{M \in \mathscr{M}} \overline{M} \subset \bigcap_{C \in \mathscr{C}} \overline{C} = \bigcap_{C \in \mathscr{C}} \overline{C}$$

since $\mathscr{C} \subset \mathscr{M}$ and elements of $\mathscr{C}$ are closed. As a consequence, for all collection of closed sets with the finite intersection property, we have $\bigcap_{C \in \mathscr{C}} C \neq \emptyset$. The theorem follows. □