①

<u>Definition</u> $\Omega \subset \mathbb{R}^n$

- $W^{1,p}(\Omega) = \left\{ f \in L^p(\Omega) \,;\, \exists g_i \in L^p(\Omega) \text{ such that } \int_\Omega f \, \partial_i \varphi \, dx = -\int g_i \, \varphi \, dx \;\; \forall \varphi \in \mathcal{C}_0^\infty(\Omega) \right\}$

We say that g_i is the weak derivatives of f.

<u>Remark</u>:
- If $f \in W^{1,p}(\Omega) \cap \mathcal{C}^1(\Omega)$, then weak derivatives and the derivatives coincident (Integration by parts)
- The weak derivatives are unique (in L^p).
- The weak derivative of $|x|$ is $\mathbb{1}_{\{x>0\}} - \mathbb{1}_{\{x<0\}}$. The derivative at 0 is not defined, but it is OK since we see the weak derivatives as elements of L^p.

<u>Prop</u>: $W^{1,p}(\Omega)$ is complete with the norm $\|f\|_{1,p} = \|f\|_p + \sum_{i=1}^n \|g_i\|_p$.

<u>Proof</u>: Take $f_n \in W^{1,p}$ be a Cauchy sequence in $W^{1,p}$.

Then $f_n \to f$ in L^p.

$g_i^n \to g_i$ in L^p.

We just need to show that the derivatives of f are the g_i. Indeed, if $\varphi \in \mathcal{C}_0^\infty(\Omega)$

$$\int_\Omega f \underbrace{\partial_i \varphi}_{\in L^{p'}} dx = \lim_{n \to +\infty} \int_\Omega f_n \, \partial_i \varphi \, dx = \lim_{n \to +\infty} \int_\Omega g_n^i \, \varphi \, dx$$

$$= \int_\Omega g^i \varphi \, dx .$$

II Poincaré inequality

Th: Let $B \subset \mathbb{R}^n$ be a ball of radius r.
Then $\forall f \in W^{1,p}(B)$,

$$\fint\!\!\fint_{B\times B} |f(x)-f(y)|^p\,dx\,dy \leq r^p \fint_B |\nabla f|^p\,dz$$

Proof WLOG, $f \in C^1(B)$.
Observe that
$$|f(x)-f(y)| = \int_0^1 D f_{(y-x)} f(x+t(y-x))\,dt$$
$$\leq \int_0^1 |y-x|\,|\nabla f(x+t(y-x))|$$
$$\leq r\int_0^1 |\nabla f(x+t(y-x))|$$

So
$$\fint\!\!\fint_{B\times B} |f(x)-f(y)|^p\,dx\,dy \leq r^p \fint\!\!\fint_{B\times B}\int_0^1 |\nabla f(x+t(y-x))|^p\,dt\,dx\,dy$$

Let z be $x+t(y-x)$.
and we want to replace y by z.
so $dz = |t|^n dy$.
and $y \in B \Leftrightarrow \begin{cases} z \in B \\ y = \frac{z-(1-t)x}{t} \in B \end{cases}$.

So we further obtain

$$\leq \frac{r^p}{|B|}\fint_B \int_0^1 \int_B 1\!\!1_{\left\{\frac{z-(1-t)x}{t}\in B\right\}} |\nabla f(z)|^p \frac{dz}{t^n}\,dt\,dx$$

By Fubini,

$$= \frac{r^p}{|B|}\fint_B |\nabla f(z)|^p \int_0^1 \int_B 1\!\!1_{\left\{\frac{z-(1-t)x}{t}\in B\right\}} \frac{dx}{t^n}\,dt\,dz$$

So we want to compute / bound.

$$h(z) := \int_0^1 \int_B \mathbb{1}_{\{\frac{z-(1-t)x}{t} \in B\}}\, dx \frac{dt}{t^n}.$$

But $\int_B \mathbb{1}_{\{\underbrace{\frac{z-(1-t)x}{t}}\in B\}} dx \leq \min\{|B|, c(rt)^n\} = |B(0,rt)|$.

$\Rightarrow \left|x - \frac{z}{(1-t)}\right| < rt.$

So $h(z) \leq \int_0^1 |B(0,rt)| \frac{dt}{t^n} \leq c_n \int_0^1 r^n \cancel{t^n} \frac{dt}{\cancel{t^n}} = c_n r^n = |B|.$

We conclude that

$$⨏⨏_{B\times B} |f(x)-f(y)|^p\, dx\, dy \leq r^p ⨏_B |\nabla f(z)|^p\, dz \quad \text{as desired.}$$

Cor: (1) $⨏_B |f - f_B|^p dx \leq r^p ⨏_B |\nabla f|^p dx$ for any ball B of radius r for any $f \in W^{1,p}(B)$

$f_B = ⨏_B f\, dx.$

~~(2) ... or $f \in W_0^{1,p}(B)$~~

(2) If Ω is bounded. Then we define $W_0^{1,p}(\Omega)$ as $\overline{C_0^\infty(\Omega)}^{\|\cdot\|_{1,p}}$ and we have that

$$\left(⨏_\Omega |f|^p dx\right)^{\frac{1}{p}} \leq C\,\mathrm{diam}(\Omega)\left(\int_\Omega |\nabla f|^p\right)^{\frac{1}{p}}$$

$\left(\int |\nabla f|^p\right)^{\frac{1}{p}}$ is a norm on $W_0^{1,p}(\Omega)$

Proof: (1) $⨏_B |f - f_B|^p dx = ⨏ \left| f - ⨏_B f\, dy\right|^p dx$

[Hölder, Jensen] $= ⨏ \left| ⨏_B (f(x) - f(y))\, dy\right|^p dx \leq ⨏⨏_{B\times B} |f(x)-f(y)|^p\, dx\, dy$

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2. Let B such that $\Omega \subset B$ and radius$(B) = \text{diam}(\Omega)$.

Then

$$\int_\Omega |f|^p dx = \int_{2B} |f|^p dx$$

$$= \frac{1}{|2B \setminus B|} \iint_{2B \times (2B \setminus B)} |f(x) - f(y)|^p dx$$

$$\leq \frac{1}{|2B \setminus B|} \iint_{2B \times 2B} |f(x) - f(y)|^p dx$$

$$\leq C \, \text{diam}(\Omega)^p \int_{2B} |Du|^p dx$$

$$= C \, \text{diam}(\Omega)^p \int_{\Omega} |Du|^p dx .$$

Th: The injection

$i : f \in W^{1,p}(\Omega) \longrightarrow f \in L^p(\Omega)$

is compact, when Ω is bounded.

Proof: We want to prove that the image of the unit ball is totally bounded.

Take $k \in \mathbb{N}$. Cover Ω by a finite number of balls of radius $1/2k$ (Possible by compactness), we call them $B_i = B(x_i; \frac{1}{2k})$. We construct φ_i such that $0 \leq \varphi_i \leq 1$ we can make them such that the B_i are overlapping at most 3 times.

$\sum \varphi_i = 1$ and $\text{supp}\, \varphi_i \subset 2B_i$ and we define the collection of functions

$$\mathcal{F} = \left\{ \sum_{i=1}^N \varphi_i a_i \; ; \text{ where } a_i \in [\![-k; k]\!] \right\}$$

The collection is finite ~~and we shall show that~~.

Take $f \in W^{1,p}(\Omega)$; $\|f\|_{1,p} \leq 1$.

Define $f_i = ⨍_{2B_i} f \, \varphi_i$ and then $\tilde{f} = \sum f_i \varphi_i$

and we have

$$\|f-\tilde{f}\|_p^p = \int_\Omega |f-\tilde{f}|^p \, dx$$

$$= \cancel{\int_\Omega \sum_{i=1}^N \int_{2B_i} |f-\tilde{f}|^p \varphi_i \, dx}$$

$$= \int_\Omega \left| \sum_{i=1}^N ⨍_{2B_i} [f(y)-f(z)] \, dy \; \varphi_i(z) \right|^p dz.$$

$\sum \varphi_i(z) = 1.$

$$\leq \sum_{i=1}^N \int_\Omega \left| ⨍_{2B_i} |f(y)-f(z)| \; \varphi_i(z) \right|^p dz.$$

supp $\varphi \subset 2B_i$
$\varphi_i \leq 1$

$$\leq \sum_{i=1}^N \int_{2B_i} ⨍_{2B_i} |f(y)-f(z)|^p \, dy \, dz.$$

$$\leq \sum_{i=1}^N \frac{1}{k^p} \int_{2B_i} |\nabla f|^p \, dz$$

$$\leq \frac{C}{k^k} \underbrace{\int_{2B} |\nabla f|^p dz}_{\leq 1.} \leq C \frac{1}{k^p}.$$

Take a_i such that $|f_i - a_i| < \frac{1}{k}$ and $\tilde{\tilde{f}} = \sum a_i \varphi_i$

Then $$\|\tilde{\tilde{f}} - \tilde{f}\|_p^p \leq \int_\Omega \left| \sum_i (a_i - f_i) \varphi_i \right|^p dx$$

$$\leq \frac{1}{k^p} |\Omega|.$$

Altogether $\tilde{\tilde{f}} \in \mathcal{F}$ and

$\|f - \tilde{\tilde{f}}\|_p \leq \frac{C}{k}$, which proves that is totally bounded $i(B_{1,p})$.

III Solution of the Laplacian.

Th: ~~The function~~ Ω bounded

Let $f \in L^2(\Omega)$

The functional $\mathcal{F}_f : W_0^{1,2} \to \mathbb{R}$.

$$\mathcal{F}_f(u) = \frac{1}{2}\int |\nabla u|^2 dx - \int f u \, dx$$

has a unique minimum, and the map $f \in L^2 \to u \in W_0^{1,2}$ is linear

Proof: (a) Since $\left|\int f u \, dx\right| \leq \|f\|_2 \|u\|_2 \leq C\|f\|_2 \|\nabla u\|_2$.

$$\leq \frac{1}{2}\left(C^2\|f\|_2^2 + \|\nabla u\|_2^2\right)$$

Then. $\mathcal{F}_f(u) \geq -\frac{C^2}{2}\|f\|_2^2$.

So $\mathcal{F}_f$ has a lower bound, so an infimum.

(b) Take u_n such that

$$\mathcal{F}_f(u_n) \leq \underbrace{\inf_{u \in W_0^{1,2}} \left(\mathcal{F}_f(u)\right)}_{= m} + \frac{1}{n}.$$

Then $(u_n)_n$ is bounded. Indeed

$$\frac{1}{2}\int |\nabla u_n|^2 dx \leq m + \frac{1}{n} + \underbrace{\int f u_n \, dx}_{\leq 2C^2\|f\|_2^2 + \frac{\|\nabla u\|_2}{8}}.$$

So $\frac{1}{4}\int |\nabla u_n|^2 dx \leq m + \frac{1}{n} + 2C^2\|f\|_2^2$.

so $\sup_{n \in \mathbb{N}} \|\nabla u_n\| < +\infty$.

By the Banach-Alaoglu theorem,

there exists $v \in W_0^{1,2}$ such that

$v_n \xrightarrow{w} v$.

We have $u \in W_0^{1,2}$ so $\mathcal{F}(u) \geq m$.

But $\mathcal{F}(u) \leq \liminf_{n\to\infty} \frac{1}{2}\int |\nabla u_n|^2 + \int v_n f$

$\leq m$.

So $\mathcal{F}(u) = m$.

(c). Since u is the minimum of $\mathcal{F}$, it means that

$D\mathcal{F}(u) = 0$ or

Characterization $\int \nabla u \cdot \nabla \varphi \, dx - \int f \varphi \, dx = 0 \quad \forall \varphi \in W_0^{1,2}$.

If u_1, u_2 are two minimum of $\mathcal{F}$, then.

$\int \nabla u_1 \cdot \nabla(u_2 - u_1) = \int f(u_2 - u_1)$

$\int \nabla u_2 \cdot \nabla(u_1 - u_2) = \int f(u_1 - u_2)$

By ~~s~~umming,

$\int |\nabla(u_1 - u_2)|^2 = 0$ so $\underline{u_1 = u_2}$.

cc: There are only 1 u that satisfies

(d). if we have u_f; u_g minima for $\mathcal{F}_f$, $\mathcal{F}_g$ respectively,

then $\int \nabla u_f \nabla \varphi \, dx - \int f \varphi = 0 \quad \forall \varphi \in W_0^{1,2}$

$\int \nabla u_g \nabla \varphi \, dx - \int g \varphi = 0 \quad \forall \varphi \in W_0^{1,2}$

So $\int \nabla(u_f + \lambda u_g) \cdot \nabla \varphi - \int (f + \lambda g)\varphi = 0 \quad \forall \varphi \in W_0^{1,2}$.

Meaning that $u_{f+\lambda g} = u_f + \lambda u_g$ as desired.

Decomposition of the Laplacian

We have constructed a map

$G: f \in L^2 \longrightarrow u_f \in W_0^{1,2}$ ~~L^2~~.

if $i : W^{1,2} \hookrightarrow L^2$, then $i \circ G$ is a compact operator on L^2,
so there exists $\{\mu_n\}$, $\{e_n\}$ onb.
eigenvalues

such that

$$i \circ G\left(\sum x_n e_n\right) = \sum \mu_n e_n x_n.$$

Note that e_n minimize $\int |\nabla u|^2 dx - \int \frac{e_n}{\mu_n} u \, dx$,
so
$$\int \nabla e_n \cdot \nabla \varphi \, dx = \frac{1}{\mu_n} \int e_n \varphi \, dx \quad \forall \varphi \in W_0^{1,2}.$$

so "
$$\int -\Delta e_n \, \varphi \, dx = \frac{1}{\mu_n} \int e_n \varphi \, dx \quad \forall \varphi \in W_0^{1,2}.$$

so $-\Delta e_n = \frac{1}{\mu_n} e_n$ "in the weak sense".

Morally, ~~$\frac{1}{\mu_n}$ are eigenvalues of~~ $-\Delta$ is an operator (although unbounded) that can be completely ~~known~~ described by its eigenvalues and eigenvectors.