Realization functors

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1 Weil cohomologies

1.1 Notations

The main reference for this talk is the book by Yves André :

References

[1] *Une introduction aux motifs (motifs purs, motifs mixtes, périodes)*, Panoramas et synthèses **17** (2004). Société Mathématique de France.

We fix a base field k. Let $\mathcal{V}$ be the category of smooth and projective varieties over k.

Let F be a field of coefficients. We shall assume that F is of characteristic zero. Let VecGr_F be the category of finite dimensional $\mathbf{Z}$-graded F-vector spaces (with Koszul rule).

1.2 Definitions

Weil cohomologies

Definition 1. A Weil cohomology is a *contravariant* functor $H\colon \mathcal{V} \to \mathrm{VecGr}_F^{\geq 0}$:

- $\dim H^2(\mathbf{P}^1) = 1$ (the Tate twist (1) is the tensor product with the *dual* of $H^2(\mathbf{P}^1)$);
- Künneth formula: $H(X) \otimes H(Y) \xrightarrow{\sim} H(X \times Y)$;
- Poincaré duality: there is a multiplicative trace map $H^{2d}(X)(d) \to F$ inducing perfect pairings $H^i(X) \otimes H^{2d-i}(X)(d) \to H^{2d}(X)(d) \to F$ for any $X \in \mathcal{V}$ that is connected and of dimension d;
- there is a cycle class map $\mathrm{cl}\colon CH^\star(X) \to H^{2\star}(X)(\star)$, contravariant in $X \in \mathcal{V}$, compatible with products and normalized with the trace map so that the trace of the cycle class of 0-cycles be given by the degree [1].

Remark 1. *If $H\colon \mathcal{V}^{\mathrm{opp}} \to \mathrm{VecGr}_F$ is a symmetric monoidal functor that leads to a Weil cohomology, then the cycle class is unique. It follows from the theory of Chern classes and the following diagram:*

$$\begin{array}{ccc} \bigoplus_{n\in\mathbb{N}} CH^n(X)_{\mathbf{Q}} & \xrightarrow{\mathrm{cl}} & \bigoplus_{n\in\mathbb{N}} H^{2n}(X)(n) \\ \sim\uparrow \mathrm{ch} & \nearrow & \\ K_0(X)_{\mathbf{Q}} & & \end{array}$$

where ch *is the Chern character (which is a morphism of rings).*

[1] We should also require that if $X = \mathbf{P}^1$, $\mathrm{cl}([\infty])$ is the canonical generator of $H^2(\mathbf{P}^1)(1)$.

Homological equivalence

Definition 2. A cycle $x \in CH^d(X) \otimes F$ is homologically equivalent to zero (with respect to the Weil cohomology H) if $\mathrm{cl}\, x = 0$ in $H^{2d}(X)(d)$. This is an adequate equivalence relation on cycles. We have functors

$$\mathrm{Mot}_{\mathrm{rat}} \to \mathrm{Mot}_{\mathrm{hom},F} \to \mathrm{Mot}_{\mathrm{num},F} \ .$$

Conjecture 1 (Standard conjecture D)**.** *The functor*

$$\mathrm{Mot}_{\mathrm{hom},F} \to \mathrm{Mot}_{\mathrm{num},F}$$

is an equivalence of categories, i.e. *a cycle is numerically equivalent to zero if and only if it is homologically equivalent to zero.*

Action of a Chow correspondence on H

Let X and Y be in $\mathcal{V}$. Let d_X be the dimension of X. Let $\alpha \in CH^{d_X}(X \times Y)$. The cycle class provides an element

$$\mathrm{cl}\, \alpha \in H^{2d_X}(X \times Y)(d_X) \ .$$

We may use the Künneth formula to think of it as a family of elements in

$$H^{2d_X - p}(X)(d_X) \otimes H^p(Y) \ ,$$

and then use the Poincaré duality to get elements in

$$H^p(X)^\vee \otimes H^p(Y) \simeq \mathbf{Hom}(H^p(X), H^p(Y)) \ .$$

We thus have defined the action $H(X) \to H(Y)$ of the Chow correspondence α.

Let $\mathrm{Mot}_{\mathrm{rat}}$ be the category of Chow motives. The Chow correspondence $\alpha \in CH^{d_X}(X \times Y)$ corresponds to a morphism

$$h(X) \to h(Y) \ .$$

We actually get a (covariant) symmetric monoidal functor

$$r_H \colon \mathrm{Mot}_{\mathrm{rat}} \to \mathrm{VecGr}_F$$

that extends the functor defined on $\mathcal{V}$ as there are canonical isomorphisms $r_H(h(X)) \simeq H(X)$ for all $X \in \mathcal{V}$.

The functor r_H factors through homological equivalence to give a faithful functor

$$\mathrm{Mot}_{\mathrm{hom},F} \to \mathrm{VecGr}_F \ .$$

Weil cohomologies and $\otimes$-functors

We can give a new (equivalent) definition of a Weil cohomology :

Definition 3. A Weil cohomology is a symmetric monoidal functor

$$r \colon \mathrm{Mot}_{\mathrm{rat}} \to \mathrm{VecGr}_F$$

such that the part of $r(\mathbf{L})$ of degree 2 is 1-dimensional [2].

$\mathbf{L}$ is the Lefschetz motive : $h(\mathbf{P}^1) = \mathbf{1} \oplus \mathbf{L}$, its $\otimes$-inverse is the Tate motive $\mathbf{T}$.

Remark 2. *We may replace* VecGr_F *by a more general* $\otimes$*-category so that* $\mathrm{Mot}_{\mathrm{rat}}$ *is the coefficient category of the universal Weil cohomology* $\mathcal{V}^{\mathrm{opp}} \to \mathrm{Mot}_{\mathrm{rat}}$.

[2] We should also require $r(h(X))$ be in nonnegative degrees.

1.3 Traces

Strong dualities (Dold, Puppe)

Let $\mathcal{T}$ be a $\otimes$-category.

Definition 4. Let M be an object of $\mathcal{T}$. We say that M admits a strong dual if there exists an object N of $\mathcal{T}$ and maps $\eta\colon \mathbf{1} \to M \otimes N$ and $\varepsilon\colon N \otimes M \to \mathbf{1}$ such that the following diagrams commute:

$$\begin{array}{ccc} M & \xrightarrow{\eta\otimes M} & M \otimes N \otimes M \\ & \searrow\!\!= & \downarrow{\scriptstyle M\otimes\varepsilon} \\ & & M \end{array} \qquad \begin{array}{ccc} N & \xrightarrow{N\otimes\eta} & N \otimes M \otimes N \\ & \searrow\!\!= & \downarrow{\scriptstyle \varepsilon\otimes N} \\ & & N \end{array}$$

In that case, the internal Hom. functor $\mathbf{Hom}(M, -)$ exists. We have $N \simeq M^\vee = \mathbf{Hom}(M, \mathbf{1})$ and there is a canonical isomorphism

$$M^\vee \otimes X \xrightarrow{\sim} \mathbf{Hom}(M, X)$$

for any $X \in \mathcal{T}$.

We say that $\mathcal{T}$ is rigid if its objects have strong duals.

Proposition 1. *The categories* VecGr_F *and* $\mathrm{Mot}_{\mathrm{rat}}$ *are rigid.*

In the case of $\mathrm{Mot}_{\mathrm{rat}}$, let $X \in \mathcal{V}$, $d = \dim X$. Let M be the motive of X and $N = M \otimes \mathbf{T}^d$. By definition (or by the projective bundle formula for Chow groups), there are isomorphisms

$$\mathrm{Hom}_{\mathrm{Mot}_{\mathrm{rat}}}(\mathbf{1}, M \otimes N) \simeq CH^d(X \times X) \simeq \mathrm{Hom}_{\mathrm{Mot}_{\mathrm{rat}}}(N \otimes M, \mathbf{1}) \,.$$

We define ε and η to be the morphisms corresponding to the cycle associated to the diagonal Δ_X in $X \times X$. We see that it makes $N = h(X) \otimes \mathbf{T}^d$ the strong dual of $M = h(X)$.

Definition of traces

Let $\mathcal{T}$ be a rigid $\otimes$-category.

Definition 5. Let $f\colon M \to M$ be an endomorphism in $\mathcal{T}$. We define the trace $\mathrm{tr}_{\mathcal{T}} f \in \mathrm{End}_{\mathcal{T}}(\mathbf{1})$ of f as the following composition:

$$\mathbf{1} \xrightarrow{\eta} M \otimes N \xrightarrow{f\otimes N} M \otimes N \simeq N \otimes M \xrightarrow{\varepsilon} \mathbf{1}$$

(N is the strong dual of M).

Proposition 2. *Let $F\colon \mathcal{T} \to \mathcal{T}'$ be a $\otimes$-functor between rigid $\otimes$-categories. Let $f\colon M \to M$ be an endomorphism in $\mathcal{T}$. Then there is an equality in* $\mathrm{End}_{\mathcal{T}'}(\mathbf{1})$ *:*

$$F(\mathrm{tr}_{\mathcal{T}} f) = \mathrm{tr}_{\mathcal{T}'} F(f) \,.$$

Lemma 6. *We have some formulas:*

$$\mathrm{tr}(f + g) = \mathrm{tr}\, f + \mathrm{tr}\, g \qquad \mathrm{tr}(g \circ f) = \mathrm{tr}(f \circ g)$$

$$\mathrm{tr}(\lambda \cdot f) = \lambda \cdot \mathrm{tr}\, f \qquad \mathrm{tr}({}^t f) = \mathrm{tr}\, f$$

Lemma 7. *Let V be an object of* VecGr_F *and $f\colon V \to V$ be an endomorphism. Then,*

$$\mathrm{tr}_{\mathrm{VecGr}_F}(f\colon V \to V) = \sum_{n \in \mathbf{Z}} (-1)^n \,\mathrm{tr}_F(f\colon V^n \to V^n) \,.$$

Lefschetz's trace formula

Theorem 8. *Let $X \in \mathcal{V}$. Let $\alpha \in CH^{d_X}(X \times X)$ (which corresponds to an endomorphism $\alpha\colon h(X) \to h(X)$ in $\mathrm{Mot_{rat}}$). Let $[\Delta_X] \in CH^{d_X}(X \times X)$ be the class of the diagonal. Then there is an equality of integers:*

$$\deg(\alpha \cdot [\Delta_X]) = \sum_{n=0}^{2d_X} (-1)^n \operatorname{tr}(\alpha\colon H^n(X) \to H^n(X)) \ .$$

To prove this, we consider the $\otimes$-functor $r_H\colon \mathrm{Mot_{rat}} \to \mathrm{VecGr}_F$ and use the formula

$$\operatorname{tr}_{\mathrm{Mot_{rat}}}(\alpha) = \operatorname{tr}_{\mathrm{VecGr}_F}(H(\alpha)) \in F \ .$$

We have computed the right hand side in this equality. It remains to compute the left hand side.

Traces in $\mathrm{Mot_{rat}}$

Lemma 9. *For any map $\alpha\colon h(X) \to h(X)$ identified as an element $\alpha \in CH^d(X \times X)_{\mathbf{Q}}$, we have*

$$\operatorname{tr}_{\mathrm{Mot_{rat}}}(\alpha) = \deg(\alpha \cdot [\Delta_X]) \ .$$

Let $M = h(X)$ and $N = h(X) \otimes \mathbf{T}^d$, and ε and η like before. The composition

$$\mathbf{1} \xrightarrow{\eta} M \otimes N \xrightarrow{\alpha \otimes N} M \otimes N \simeq N \otimes M$$

is given by the transposition ${}^t\alpha$ of α in $CH^d(X \times X)_{\mathbf{Q}}$. Then, the composition

$$\mathbf{1} \xrightarrow{\eta} M \otimes N \xrightarrow{\alpha \otimes N} M \otimes N \simeq N \otimes M \xrightarrow{\varepsilon} \mathbf{1} \ .$$

is given by $\deg({}^t\alpha \cdot [\Delta_X]) = \deg(\alpha \cdot [\Delta_X])$.

2 Applications

2.1 Finite fields

Zeta functions over a finite field

Let $k = \mathbf{F}_q$ be a finite field.

Let X be a smooth and projective variety over k.

Definition 10. The Zeta function of $X/\mathbf{F}_q$ is :

$$Z(X,t) = \exp\left(\sum_{n=1}^{\infty} \#X(\mathbf{F}_{q^n}) \frac{t^n}{n}\right) \in \mathbf{Q}[[t]] \ .$$

We can consider the geometric Frobenius $F\colon X \to X$ (the identity on the underlying topological space and $x \mapsto x^q$ on the structural sheaf). It is a morphism of $\mathbf{F}_q$-schemes.

Lemma 11. *Let $F^n\colon X \to X$ be an iteration of the geometric Frobenius. Then,*

$$\operatorname{tr}_{\mathrm{Mot_{rat}}}(F^n\colon h(X) \to h(X)) = \#X(\mathbf{F}_{q^n}) \ .$$

The set $X(\mathbf{F}_{q^n})$ is in bijection with the set of fixed points of F^n acting on $X(\overline{\mathbf{F}_q})$. The differential of F^n is zero, so the intersection of the graph Gr_{F^n} of F^n and Δ_X in $X \times X$ is transversal. We thus have the equality

$$\deg([\mathrm{Gr}_{F^n}] \cdot [\Delta_X]) = \#X(\mathbf{F}_{q^n})$$

since all the intersection multiplicities are 1, which finishes the proof thanks to the computation of the traces in $\mathrm{Mot_{rat}}$.

Zeta functions in general $\otimes$-categories

Definition 12. Let $f\colon M \to M$ an endomorphism of an object in a rigid $\otimes$-category $\mathcal{T}$ (for instance $\mathrm{Mot}_{\mathrm{rat}}$ or VecGr_F). We define

$$Z(f,t) = \exp\left(\sum_{n=1}^{\infty} \mathrm{tr}_{\mathcal{T}}(f^n)\frac{t^n}{n}\right) \in F\,[[t]] \ ;$$

where $F = \mathrm{End}_{\mathcal{T}}(\mathbf{1}) \otimes \mathbf{Q}$ is the coefficient ring.

Note that the previous computations shows that

$$Z(X,t) = Z(F\colon h(X) \to h(X), t)$$

if X is a smooth and projective variety over $\mathbf{F}_q$.

Rationality of Zeta functions

Theorem 13. *Let $f\colon M \to M$ be a endomorphism of a motive in $\mathrm{Mot}_{\mathrm{rat}}$. If H is a Weil cohomology, then $Z(f,t)$ is a rational function. More precisely, if $P_n(t) = \det(\mathrm{id} - tf\colon H^n(X) \to H^n(X)) \in F\,[t]$ for any integer n, then*

$$Z(f,t) = \prod_{n\in\mathbf{Z}} P_n(t)^{(-1)^{n+1}} \ .$$

Using the realization functor $r_H\colon \mathrm{Mot}_{\mathrm{rat}} \to \mathrm{VecGr}_F$, we can replace $\mathrm{Mot}_{\mathrm{rat}}$ by VecGr_F. By "dévissage", one reduces to the case of the multiplication $F \to F$ by an element λ where $F \in \mathrm{VecGr}_F$ is in degree zero; it then reduces to the following identity :

$$Z(\lambda\colon F \to F, t) = \exp\left(\sum_{n=1}^{\infty} \frac{(\lambda t)^n}{n}\right) = \frac{1}{1-\lambda t} \ .$$

Remark 3.

$$\mathbf{Q}\,[[t]] \cap F(t) = \mathbf{Q}(t) \ .$$

Some formal formulas

The geometric Frobenius defines a $\otimes$-endomorphism F of the identity functor on $\mathrm{Mot}_{\mathrm{rat}}$. We can define the Zeta function of a motive M over $\mathbf{F}_q$ with respect to this endomorphism $F\colon M \to M$.

There are some formulas :

$$\begin{aligned} Z(M \otimes \mathbf{T}^d, q^d t) &= Z(M,t)\ ; \\ Z(M^\vee, \frac{1}{t}) &= (-t)^{\chi(M)} \prod_{n\in\mathbf{Z}} \det(H^n(f))^{(-1)^i} \cdot Z(M,t)\ . \end{aligned}$$

The integer $\chi(M)$ is the Euler characteristic of M (*i.e.* the trace of the identity on M).

Then, one may use the Poincaré duality isomorphism $h(X)^\vee \simeq h(X) \otimes \mathbf{T}^d$ to get the following functional equation:

Functional equation of the Zeta function

Theorem 14 (Functional equation)**.** *Let X be a smooth projective d-dimensional variety over $\mathbf{F}_q$.*

$$Z(X,t) = \varepsilon \cdot t^{-\chi(M)} q^{-\frac{d\chi(M)}{2}} Z(X, \frac{1}{q^d t}) \ ,$$

where $\varepsilon = (-1)^r$ where r is the multiplicity of $q^{\frac{d}{2}}$ as an eigenvalue of F acting on $H^{\frac{d}{2}}(X)$).

2.2 Numerical equivalence

Definition of numerical equivalence

Definition 15. Let $X \in \mathcal{V}$ and A be $\mathbf{Z}$ or a field F of characteristic zero, then a cycle x of codimension i on X (of dimension d) with coefficients in A is numerically equivalent to zero if for any cycle y of codimension $d-i$ on X, we have

$$\deg(x \cdot y) = 0 \in A \, ;$$

this is an adequate equivalence relation on cycle. We define $A^i_{\mathrm{num}}(X; A)$ to be the equivalence classes modulo cycles numerically equivalent to zero.

Exercise 1. *For any field F of characteristic zero, we have a canonical isomorphism*

$$A^i_{\mathrm{num}}(X) \otimes_{\mathbf{Z}} F \xrightarrow{\sim} A^i_{\mathrm{num}}(X; F) \, .$$

Finite generation

Theorem 16. *Assume that there exists a Weil cohomology over a field k with some coefficient field F (of characteristic zero). Then, for any $X \in \mathcal{V}$, the $\mathbf{Z}$-module $A^i_{\mathrm{num}}(X)$ is finitely generated and torsion free.*

There is a surjection of F-vector spaces

$$A^i_{\mathrm{hom}}(X; F) \to A^i_{\mathrm{num}}(X; F) \simeq A^i_{\mathrm{num}}(X) \otimes_{\mathbf{Z}} F \, .$$

We have an obvious injection $A^i_{\mathrm{hom}}(X; F) \to H^{2i}(X)(i)$ of F-vector spaces. So, $A^i_{\mathrm{num}}(X) \otimes \mathbf{Q}$ is finite dimensional. Use the embedding

$$A^i_{\mathrm{num}}(X) \to \mathrm{Hom}(A^{d-i}_{\mathrm{num}}(X), \mathbf{Z})$$

to prove that $A^i_{\mathrm{num}}(X)$ is a finitely generated group.

2.3 Semi-simplicity

Jannsen's semi-simplicity theorem

Theorem 17. *For any characteristic zero coefficient field F, the category $\mathrm{Mot}_{\mathrm{num},F}$ of motives modulo numerical equivalence is a semi-simple abelian category.*

The major step is to prove that for any $X \in \mathcal{V}$, the algebra

$$\mathrm{End}_{\mathrm{Mot}_{\mathrm{num},F}}(h(X)) = A^{d_X}_{\mathrm{num}}(X \times X; F)$$

is finite dimensional and semi-simple. We may extend the coefficient field F so that there exists a Weil cohomology. Let $\mathcal{R} \subset \mathrm{End}_{\mathrm{Mot}_{\mathrm{hom},F}}(h(X))$ be the Jacobson radical. Let $f \in \mathcal{R}$. We want to prove that f is numerically equivalent to zero. Let g be any element in $\mathrm{End}_{\mathrm{Mot}_{\mathrm{hom},F}}(h(X))$.

$$\begin{aligned} \mathrm{tr}(g \circ f) &= 0 && \text{because } g \circ f \text{ is nilpotent,} \\ \mathrm{tr}(g \circ f) &= \deg(f \cdot {}^t g) && \text{(variant of the trace formula).} \end{aligned}$$

3 Examples

3.1 “Classical” Weil cohomologies

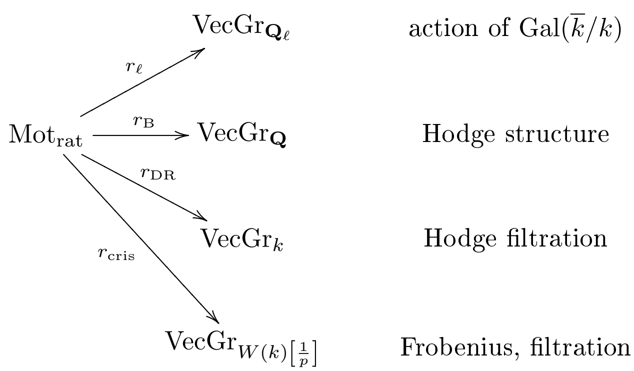
Classical Weil cohomologies

Let p be the characteristic of the base field k. We *define* the list of classical Weil cohomologies:

cohomology	groups	coeff.	restrictions
étale	$H^{\star}_{\ell}(X)$	$\mathbf{Q}_\ell$	$\ell \neq p, k \to k_{\mathrm{s}}$
Betti	$H^{\star}_{\mathrm{B}}(X)$	$\mathbf{Q}$	$\sigma \colon k \to \mathbf{C}$
algebraic De Rham	$H^{\star}_{\mathrm{DR}}(X)$	k	$p = 0$
crystalline	$H^{\star}_{\mathrm{cris}}(X)$	$W(k)\left[\frac{1}{p}\right]$	$p > 0$, k perfect

3.2 Realization functors

Realization functors on pure motives



3.3 Review of Hodge theory

Review of Hodge theory

Definition 18. A pure $\mathbf{Q}$-Hodge structure of weight $n \in \mathbf{Z}$ is a finite dimensional $\mathbf{Q}$-vector space V endowed with a decomposition of the $\mathbf{C}$-vector space

$$V \otimes_{\mathbf{Q}} \mathbf{C} = \bigoplus_{p+q=n} V^{p,q}$$

such that $\overline{V^{p,q}} = V^{q,p}$. The Hodge filtration on $V \otimes_{\mathbf{Q}} \mathbf{C}$ is defined by $\mathscr{F}^p(V \otimes_{\mathbf{Q}} \mathbf{C}) = \bigoplus_{p' \geq p} V^{p',q}$.

Theorem 19 (Classical Hodge theory)**.** *Let X be a compact $\mathbf{C}$-analytic variety. If there exists a Kähler metric on X, then $H^n(X, \mathbf{Q})$ is endowed with a pure $\mathbf{Q}$-Hodge structure of weight n.*

3.4 Comparison theorems

Comparison isomorphisms

There are several comparison isomorphisms if one extends scalars:

- $r_\ell \xrightarrow{\sim} r_{\mathrm{B}} \otimes_{\mathbf{Q}} \mathbf{Q}_\ell$, $k \subset \mathbf{C}$ (Artin);
- $r_B \otimes_{\mathbf{Q}} \mathbf{C} \xrightarrow{\sim} r_{\mathrm{DR}} \otimes_k \mathbf{C}$, $k \subset \mathbf{C}$ (Serre, Grothendieck);
- $r_p \otimes_{\mathbf{Q}_p} \mathrm{B}_{\mathrm{DR}} \simeq r_{\mathrm{DR}} \otimes_k \mathrm{B}_{\mathrm{DR}}$, $k/\mathbf{Q}_p$ algebraic (Fontaine, Tsuji, Faltings). B_{DR} is a p-adic period ring [3] which is a discrete valuation field with residue field $\mathbf{C}_p$;

[3] There are several such rings...

- if $\mathscr{X}$ is a projective and smooth scheme over a complete valuation ring R (of unequal characteristic, with perfect residue field k), then there is a canonical isomorphism

$$H^\star_{\mathrm{DR}}(\mathscr{X}_\eta) \simeq H^\star_{\mathrm{cris}}(\mathscr{X}_s) \otimes_{W(k)[\frac{1}{p}]} K \ ,$$

where K is the quotient field of R (Berthelot-Ogus).

4 Absolute Hodge cycles, motivated cycles

4.1 Absolute Hodge cycles

Absolute Hodge cycles (Deligne)

We assume that the base field k is algebraically closed and of finite transcendance degree over $\mathbf{Q}$.

Definition 20. Let $X \in \mathcal{V}$. We define

$$H^n_{\mathbf{A}}(X) = H^n_{\mathrm{DR}}(X/k) \times \left(\prod_\ell H^n_{\text{ét}}(X;\mathbf{Z}_\ell) \right) \otimes \mathbf{Q} \ ;$$

it is a $k \times \mathbf{A}^{\mathrm{f}}$-module ($\mathbf{A}^{\mathrm{f}} = \hat{\mathbf{Z}} \otimes \mathbf{Q}$).

For any embedding $\sigma \colon k \to \mathbf{C}$, we have a comparison isomorphism:

$$H^n(X(\mathbf{C})_\sigma;\mathbf{Q}) \otimes (\mathbf{C} \times \mathbf{A}^{\mathrm{f}}) \xleftarrow{\sim} H^n_{\mathbf{A}}(X) \otimes_{k \times \mathbf{A}^{\mathrm{f}}} (\mathbf{C} \times \mathbf{A}^{\mathrm{f}}) \ .$$

Definition 21. An element $x \in H^{2n}_{\mathbf{A}}(X)(n)$ is a Hodge cycle with respect to some embedding $\sigma \colon k \to \mathbf{C}$ if

- the image of x in $H^{2n}_{\mathbf{A}}(X)(n) \otimes_{k \times \mathbf{A}^{\mathrm{f}}} (\mathbf{C} \times \mathbf{A}^{\mathrm{f}})$ lies in the rational subspace $H^{2n}(X(\mathbf{C})_\sigma;\mathbf{Q})$;
- the component of x in $H^{2n}(X(\mathbf{C})_\sigma;\mathbf{Q})(n)$ is in Hodge bidegree $(0,0)$.

The element x is an absolute Hodge cycle if it is a Hodge cycle for all embeddings $\sigma \colon k \to \mathbf{C}$.

Lemma 22. *For any $X \in \mathcal{V}$, and $x \in CH^d(X)$. The family of classes in cohomologies given by the various cycle classes of x provides an element in $H^{2d}_{\mathbf{A}}(X)(d)$ that is an absolute Hodge cycle.*

Definition 23. In the definition of $\mathrm{Mot}_\sim$, we may replace $A^\star_\sim(-)$ by absolute Hodge cycles in $H^{2\star}_{\mathbf{A}}(-)(\star)$ to define a Tannakian [4] category $\mathrm{Mot}_{\mathbf{AH}}$.

Remark 4. *We have an obvious faithful functor*

$$\mathrm{Mot}_{\mathrm{hom}} \to \mathrm{Mot}_{\mathbf{AH}} \ .$$

If the Tate conjecture or the Hodge conjecture is true, then it is an equivalence.

4.2 Motivated cycles

Improvement: Motivated cycles (André)

Let k be a field of characteristic zero and H be a classical Weil cohomology.

Conjecture 2 (Standard conjecture B)**.** *Let $X \in \mathcal{V}$, $d = \dim X$. Let D be an ample divisor on D. Then for any i, the upper injective map is surjective:*

$$\begin{array}{ccc} A^i_{\mathrm{hom},\mathbf{Q}}(X) & \xrightarrow{[D]^{d-2i}} & A^{d-i}_{\mathrm{hom},\mathbf{Q}}(X) \\ \downarrow & & \downarrow \\ H^{2i}(X)(i) & \xrightarrow[\textit{(hard Lefschetz)}]{\sim} & H^{2d-2i}(X)(d-i) \end{array}$$

[4] One has to change the commutativity constraint, see Sujatha's notes.

We want to enlarge morphisms in $\text{Mot}_{\text{hom},\mathbf{Q}}$ to force the standard conjecture B (of Lefschetz type) to be satisfied in that setting.

Definition of motivated cycles

We can define a category Cohom like $\text{Mot}_\sim$ but so as to have

$$\text{Hom}_{\text{Cohom}}(h(X), h(Y)) = H^{2d_X}(X \times Y)(d_X) \simeq \mathbf{Hom}(H(X), H(Y)) .$$

Definition 24. There exists a smallest $\mathbf{Q}$-linear pseudoabelian sub-$\otimes$-category Mot_{mot} of Cohom containing $\text{Mot}_{\text{hom},\mathbf{Q}}$ and such that for any $X \in \mathcal{V}$ and D an ample divisor on X, the upper injective map is bijective :

$$\begin{array}{ccc} A^i_{\text{mot}}(X) & \xrightarrow{[D]^{d-2i}} & A^{d-i}_{\text{mot}}(X) \\ \downarrow & & \downarrow \\ H^{2i}(X)(i) & \xrightarrow[\text{(hard Lefschetz)}]{\sim} & H^{2d-2i}(X)(d-i) \end{array}$$

where $A^n_{\text{mot}}(X) = \text{Hom}_{\text{Mot}_{\text{mot}}}(\mathbf{L}^n, h(X))$ are "motivated cycles".

Remark 5. *The faithful functor* $\text{Mot}_{\text{hom},\mathbf{Q}} \to \text{Mot}_{\text{mot}}$ *is an equivalence of categories if and only if the standard conjecture B (Lefschetz) is true.*

Proposition 3. *The category* Mot_{mot} *does not depend on the classical Weil cohomology and there is an obvious faithful functor* $\text{Mot}_{\text{mot}} \to \text{Mot}_{\mathbf{AH}}$.

Proposition 4 ("$B \Rightarrow C$")**.** *For any* $X \in \mathcal{V}$, *the Künneth projectors in* $\text{End}_{\text{Cohom}}(h(X))$ *are defined in* Mot_{mot}.

Proposition 5. Mot_{mot} *is a neutral Tannakian category. (⇒ unconditional definition of the motivic Galois group).*

4.3 Hodge cycles on abelian varieties

Hodge cycles on abelian varieties

Theorem 25 (Deligne)**.** *Let A be an abelian variety over an algebraically closed field k embedded in* $\mathbf{C}$. *Any Hodge cycle is an absolute Hodge cycle.*

Theorem 26 (André)**.** *Let A be an abelian variety over an algebraically closed field k embedded in* $\mathbf{C}$. *Any Hodge cycle is a motivated cycle.*

5 Mixed realizations

5.1 Abelian category of mixed realizations

Absolute Hodge style's mixed realizations (Jannsen, Deligne)

Let k be a field embeddable in $\mathbf{C}$ and $\overline{k}$ be an algebraic closure of k.

Definition 27 (sketch)**.** The abelian category MR_k of mixed realizations is the category whose objects are families of objects:

- H_{DR} is a k-vector space with a Hodge filtration and a weight filtration;
- H_σ (for any embedding $\sigma\colon k \to \mathbf{C}$) is a mixed $\mathbf{Q}$-Hodge structure;
- H_ℓ (for any prime number ℓ) is a $\mathbf{Q}_\ell$-vector space with an action of $\text{Gal}(\overline{k}/k)$;

with comparison isomorphisms.

Proposition 6. MR_k *is a* $\mathbf{Q}$*-neutral Tannakian category.*

Problem 28. *Define objects in such a way that they would have a "geometric origin".*

Definition 29. Mixed motives are defined by Jannsen to be the sub-Tannakian category of MR_k generated by $H(U)$ for any smooth variety U over k.

Problem 30. *There is no unconditional good notion of an abelian category of mixed motives.*

5.2 Triangulated categories of mixed motives

Triangulated categories of mixed motives

$$\begin{array}{lll} \mathbf{Sm}_k & \xrightarrow{\text{covariant}} \mathrm{DM}_{\mathrm{gm}}(k) & \quad \text{(Voevodsky)} \\ & \searrow^{\text{contravariant}} & \\ & \qquad \mathscr{DM}(k) & \quad \text{(Levine)} \end{array}$$

Theorem 31 (Levine, Ivorra)**.**

- $\mathrm{DM}_{\mathrm{gm}}(k)^{\mathrm{opp}} \simeq \mathscr{DM}(k)$ *(k of characteristic zero);*
- $\mathrm{DM}_{\mathrm{gm}}(k;\mathbf{Q})^{\mathrm{opp}} \simeq \mathscr{DM}(k;\mathbf{Q})$ *(k perfect).*

Theorem 32 (Voevodsky)**.** *There is a canonical functor*

$$\mathrm{Mot}_{\mathrm{rat}}(k)^{\mathrm{opp}} \to \mathrm{DM}_{\mathrm{gm}}(k)$$

that is fully faithful.

5.3 Contravariant realization functors

Contravariant triangulated realization functors

$$\begin{array}{ccccc} \mathrm{DM}_{\mathrm{gm}}(k)^{\mathrm{opp}} & \xrightarrow{\text{(Huber)}} & \mathrm{D}_{\mathrm{MR}_k} & \xrightarrow{\sigma:\, k \to \mathbf{C}} & \mathrm{D^b}(\mathrm{MHS}_{\mathbf{Q}}) \\ \downarrow{\scriptstyle\text{(Ivorra)}} & & \downarrow & \searrow & \\ \mathrm{D^b_c}(k_{\text{ét}};\mathbf{Z}_\ell) & \xrightarrow{\otimes \mathbf{Q}_\ell} & \mathrm{D^b}(k_{\text{ét}},\mathbf{Q}_\ell) & & \mathrm{D^b}(\mathrm{Vec}_k) \end{array}$$

The hard part in these constructions is to get functoriality of complexes computing cohomologies with respect to *finite correspondences.*

Remark 6. *These functors obviously lead to "regulators". If* $X \in \mathbf{Sm}_k$*, by definition,*

$$H^p(X,\mathbf{Z}(q)) = \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}(k)}(M(X),\mathbf{Z}(q)\,[p]) \ .$$

For instance, the étale realization functor gives a map

$$H^p(X,\mathbf{Z}(q)) \to H^p_{\acute{e}t,\mathrm{cont}}(X,\mathbf{Z}_\ell(q)) \ .$$

Using his definition of a motivic category $\mathscr{DM}(k)$, Levine constructed a mixed realization functor

$$\mathscr{DM}(k) \to \mathrm{D^b_{MR_k}}$$

that provides Betti, étale, Hodge, etc. realizations.

However, it is not clear whether or not these functors coincide with the ones defined on Voevodsky's category.

5.4 Covariant realization functors

Covariant triangulated realization functors

Theorem 33 (Suslin, Voevodsky). *There is a "trivial" covariant étale realization functor*

$$\mathrm{DM}(k) \to \mathrm{DM}_{\acute{e}t}(k; \mathbf{Z}/\ell^\nu) \simeq \mathrm{D}(k_{\acute{e}t}, \mathbf{Z}/\ell^\nu) ,$$

at least if k is virtually of finite ℓ-cohomological dimension.

Howerer, it is not clear whether this functor is dual to Ivorra's.

Let $E\colon \mathbf{Sch}_k^{\mathrm{opp}} \to C(\mathrm{Vec}_F^\infty)$ with additional data and properties:

- F is of characteristic 0;
- multiplicative structure and Künneth formula;
- Mayer-Vietoris property (Nisnevich descent);
- homotopy invariance and cohomology of $\mathbf{P}^1$;
- proper descent.

Theorem 34 (Cisinski, Déglise). *Then, there is a representable covariant $\otimes$-realization functor*

$$\mathrm{DM}(k; F) \to \mathrm{D}(\mathrm{Vec}_F^\infty) \simeq \mathrm{VecGr}_F^\infty$$

that maps the motive of a smooth variety X to the dual of $E(X)$.

Vec_F^∞ is the category of F-vector spaces (not necessarily finite dimentional).

They get

- De Rham realization: $\mathrm{DM}(k; k) \to \mathrm{D}(\mathrm{Vec}_k)$ (in characteristic zero);
- rigid realization: if R is a complete discrete valuation ring of unequal characteristic with quotient field K and perfect residual field k, then they constructs a $\otimes$-functor

$$\mathrm{DM}(k) \to \mathrm{D}(\mathrm{Vec}_K) .$$

However, their convention on twists prevents them from keeping the Galois action on the étale realization.

5.5 Generalizations over a base scheme

Motivic coefficients and realizations

Let S be a noetherian separated scheme.

- Levine actually defined $\mathscr{DM}(S)$, and a "mixed Hodge modules" realization functor if S is a smooth variety over $\mathbf{C}$;
- Cisinski and Déglise defined $\mathrm{DM}(S)$;
- Ivorra defined $\mathrm{DM}_{\mathrm{gm}}(S)$ (it is a full subcategory of $\mathrm{DM}(S)$) and a functor

$$\mathrm{DM}_{\mathrm{gm}}(S)^{\mathrm{opp}} \to \mathrm{D}^+(S; \mathbf{Z}_\ell) ,$$

  and a "moderate" version, for instance, if K is a number field

$$\mathrm{DM}_{\mathrm{gm}}(K)^{\mathrm{opp}} \to \mathrm{colim}_S\, \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Spec}\, \mathscr{O}_S; \mathbf{Z}_\ell)$$

  where S go through finite sets of finite places of K.

Theorem 35 (Cisinski, Déglise, Ayoub). *There exists a six operations formalism for the categories* $\mathrm{DM}(S)$*. For any $f\colon T \to S$, there are functors $(f^\star, f_\star)$, and for $f\colon T \to S$ "quasi-projective", functors $(f_!, f^!)$, a map $f_! \to f_\star$ which is an isomorphism if f is projective.*

Remark 7 (Bloch). *These categories do not see "nilpotents":* $\mathrm{DM}(S) \simeq \mathrm{DM}(S_{r\acute{e}d})$.