Equivariant Analytic Torsion for $K3$ Surfaces with Involution

"Control, index, traces and determinants" in honor of Professor Jean-Michel Bismut

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May 28, 2013

Contents

Determinant of Laplacian for elliptic curves

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The Laplacian of (E_τ, g_τ) is the differential operator defined as

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Definition (Spectral Zeta Function)

The spectral zeta function of (E_τ, g_τ) is defined as

$$\zeta_\tau(s) := \sum_{\lambda \in \sigma(\square_\tau)\setminus\{0\}} \lambda^{-s} = \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \left(\frac{\pi^2 |m\tau + n|^2}{\Im\tau}\right)^{-s}.$$

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$$\tau(E_\tau) = 4\|\eta(\tau)\|^{-4} = 4(\Im\tau)^{-1}\left|e^{2\pi i\tau}\prod_{n>0}\left(1-e^{2\pi i n\tau}\right)^{24}\right|^{-1/6}$$

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Extension of the Kronecker-Ray-Singer theorem to $K3$ surfaces with involution.

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Fact (Kodaira)

Every $K3$ surface is diffeomorphic to a Kummer surface

$$\widetilde{T^4/\pm 1} = \frac{T^4 - \{\text{points of order } 2\}}{\pm 1} \amalg E_1 \amalg \ldots \amalg E_{16}, \qquad E_i \cong \mathbb{P}^1$$

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$$H_+^2(X,\mathbb{Z}) := \{v \in H^2(X,\mathbb{Z});\ \iota^* v = v\}.$$

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$$\{\textit{isometry classes of primitive, 2-elementary, Lorentzian } M \subset \mathbb{L}_{K3}\}$$

$$\Updownarrow$$

$$\{(r(M), \ell(M), \delta(M));\ M \subset \mathbb{L}_{K3} \textit{ primitive, 2-elementary, Lorentzian}\}$$

where $\delta(M) \in \{0, 1\}$ *is the "parity" of* M.

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For a lattice Λ with $\mathrm{sign}(\Lambda) = (2, r(\Lambda) - 2)$, define

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The period of (X, ι) is defined by

$$\begin{aligned} \varpi(X, \iota) := \left[\alpha\left(H^0(X, \Omega_X^2)\right)\right] &\in \Omega_{M^\perp} / O(M^\perp) \\ &\subset \mathbb{P}(M^\perp \otimes \mathbb{C}) / O(M^\perp) \end{aligned}$$

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where $\mathcal{D}_{M^\perp}$ *is the discriminant divisor of* $\Omega_{M^\perp}$

$$\mathcal{D}_{M^\perp} := \sum_{d \in M^\perp,\, d^2=-2} H_d$$

$$H_d := \{[\eta] \in \Omega_{M^\perp};\ \langle \eta, d\rangle = 0\}$$

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$\Longrightarrow X^{\iota} = \emptyset$ or every C_i is a compact Riemann surface (Nikulin).

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$$\gamma : \text{Ricci-flat} \quad \Longleftrightarrow \quad \frac{\eta \wedge \bar{\eta}}{\gamma^2/2!} = \frac{\mathrm{Vol}(X, \gamma)}{\|\eta\|_{L^2}^2}$$

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- *There exist an "automorphic form"* Φ_M *on* $\Omega^+_{M^\perp}$ *for* $O^+(M^\perp)$ *and an integer* $\nu \in \mathbb{Z}_{>0}$ *such that*

$$\tau_M = \|\Phi_M\|^{-1/2\nu}, \qquad \mathrm{div}(\Phi_M) = \nu\,\mathcal{D}_{M^\perp}.$$

- On the open part of the moduli space $\Omega^+_{M^\perp} \setminus \mathcal{D}_{M^\perp}$, the equation

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$\Longrightarrow \exists$ automorphic form Φ_M of weight $((r(M)-6)\nu, 4\nu)$, $\nu \gg 1$ s.t.

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Up to a constant, $\Phi_{[M\perp d]}$ is the quasi-pullback of Φ_M:

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Corollary

Every Φ_M *is obtained from* $\Phi_{\langle 2 \rangle}$, $\Phi_{\mathbb{U}(2)}$, $\Phi_{\mathbb{U}(2) \oplus \mathbb{E}_8(2)}$ *by applying quasi-pullbacks successively*

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where $\mathcal{W} \subset L \otimes \mathbb{R}$ is a "Weyl chamber", $\varrho \in L \otimes \mathbb{Q}$ is the "Weyl vector", $\mathbf{1}_L \in L^\vee/L$ is the unique element s.t. $\langle \mathbf{1}_L, \alpha\rangle \equiv \alpha^2 \mod \mathbb{Z}$ $(\forall\, \alpha \in L^\vee/L)$.

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$r(\Lambda) < 20 \implies \Psi_\Lambda(z, \eta_{1^{-8}2^8 4^{-8}}\vartheta^{12-r(\Lambda)})^8$ *is a (possibly meromorphic) automorphic form on* $L \otimes \mathbb{R} + \sqrt{-1}\mathcal{C}_L^+ \cong \Omega_\Lambda^+$ *for* $O^+(\Lambda)$*, whose weight, zeros and poles are computed explicitly from the Fourier coefficients of* $\eta_{1^{-8}2^8 4^{-8}}\vartheta^{12-r(\Lambda)}$ *and its modular transformation.*

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Except for the possible 12 *triples* $[M] = (r(M), \ell(M), \delta(M))$ *given by* $(r, r, 1)\,(1 \leq r \leq 5)$, $(2, 2, 0)$, $(2, 0, 0)$, $(10, \ell, 0)\,(0 \leq \ell \leq 8,\ \ell \equiv 0\,(2))$, *the following equality holds, up to a constant depending only on* M

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This 2-elementary K3 surface is denoted by $(X_{(S,C)}, \iota_{(S,C)})$ and is called a double Del Pezzo surface associated to (S, C).

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Discriminant of $|-2K_S|$ as a section of the Hodge bundle

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$$\omega_\xi := \bigwedge_{i\in I} \mathrm{Res}_{C_\xi}\left(\frac{s_i}{\sum_{\alpha\in A}\xi_\alpha\sigma_\alpha}\right) \in \det H^0(C_\xi, \Omega^1_{C_\xi}) \setminus \{0\}$$

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- $\Delta_S(\xi)\in\mathbb{Z}[\xi_\alpha]_{\alpha\in A}$: defining Eq of the projective dual of $(S,-2K_S)$ $\Longleftrightarrow$ defining Eq of the discriminant locus of $|-2K_S|$

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which is equipped with the canonical involution

$$\iota\colon X \ni (y, \xi) \to (y, -\xi) \in X$$

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By assigning Y *the period of* $(\widetilde{X}, \theta)$*, the moduli space of log-Enriques surfaces with* k*-singularities* $(k > 0)$ *is isomorphic to the modular variety*

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where $\mathcal{D}_{\Lambda_k} = \sum_{d \in \Lambda_k,\, d^2=-1} d^{\perp}$ *is the Heegner divisor of norm* -1*-vectors.*

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is the Ricci-flat volume form on Y induced by Ξ.

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N.B. $\Psi_{\Lambda_k(2)}$ is an automorphic form on the Kähler moduli of a Del Pezzo surface of degree k vanishing exactly on the divisor of norm -1-vectors