Ricci flow on Fano manifolds

Gang Tian

Beijing University and Princeton University

(Based on my joint work with Zhenlei Zhang)

Let M be a Kähler manifold with a Kähler metric ω.

In local coordinates $z_1, \cdots, z_n$, the metric ω is given by a Hermitian positive matrix-valued function $(g_{i\bar{j}})$:

$$\omega = \sqrt{-1} \sum_{i,j=1}^{n} g_{i\bar{j}} dz_i \wedge d\bar{z}_j$$

ω being Kähler $\Leftrightarrow d\omega = 0$

We say ω is Kähler-Einstein if it is Kähler and Einstein, i.e.,

$$\mathrm{Ric}(\omega) \;=\; \lambda\,\omega,$$

where λ is a constant, say $-1, 0, 1$ after normalization, and $\mathrm{Ric}(\omega)$ denotes the Ricci curvature, in local coordinates,

$$\mathrm{Ric}(\omega) \;=\; -\sqrt{-1}\partial\bar{\partial}\log\det(g_{i\bar{j}}).$$

In 50s, E. Calabi started the study of Kähler-Einstein metrics on a compact Kähler manifold M.

A necessary condition is that M has definite first Chern class $c_1(M)$.

The existence of Kähler-Einstein metrics was established by

- Aubin, Yau independently in 1976 when $c_1(M) < 0$

- Yau in 1976 when $c_1(M) = 0$

The Ricci flow was introduced by Hamilton in early 80's:

$$\begin{cases} \dfrac{\partial g_{ij}}{\partial t} = -2R_{ij} \\ g(0) = \text{a given metric} \end{cases}$$

• For any initial metric, there is a unique solution $g(t)$ on $M \times [0, T)$ (Hamilton, DeTurck).

If $g(0)$ is Kähler, so is every $g(t)$. Hence, after scaling and reparametrization, we can consider the (normalized) Kähler-Ricci flow:

$$\begin{cases} \dfrac{\partial \omega(t)}{\partial t} = \lambda\, \omega(t) - \mathrm{Ric}(\omega(t)) \\ \omega(0) = \text{a given Kähler form } \omega_0, \end{cases}$$

where $\lambda = -1, 0, 1$. Here $\omega(t)$ is the Kähler form of $g(t)$.

From now on, we will denote a Kähler metric by its Kähler form.

In 1986, Cao proved that if $c_1(M) = \lambda[\omega_0]$, then the above Kähler-Ricci flow has a global solution $\omega(t)$ for $t \geq 0$, furthermore, he proved that if $\lambda \leq 0$, $\omega(t)$ converge to a Kähler-Einstein metric on M as t tends to ∞.

This gives an alternative proof of the Aubin-Yau Theorem and Calabi-Yau theorem.

In last ten years, there were many progresses on Kähler-Ricci flow and its singularity formation, referred as the analytic Minimal Model Program (aMMP). For instance, Tian-Zhang proved a sharp local existence theorem for the flow with any Kähler metric as initial value.

What about the limit of $\omega(t)$ when $\lambda = 1$?

In this case, $c_1(M) > 0$, that is, the underlying M is a Fano manifold.

Not every Fano manifold admits a Kähler-Einstein metric.

For now on, we always assume that M is Fano and $[\omega_0] = c_1(M)$.

There are obstructions:

1. The automorphism group of M being reductive due to Matsushima in 50s;

2. The Futaki invariant introduced by Futaki in 1983;

3. The K-stability introduced by myself in 1996 and reformulated by Donaldson in a more algebraic way in 2002.

Futaki invariant: It is a character of $\eta(M)$ of holomorphic vector fields on M defined by

$$f_M(X) = \int_M X(h_0)\,\omega_0^n,$$

where h_0 is chosen by

$$\mathrm{Ric}(\omega_0) - \omega_0 = \sqrt{-1}\partial\bar{\partial}h_0, \quad \int_M \left(e^{h_0} - 1\right)\omega_0^n = 0.$$

Futaki (1983): f_M is an invariant and vanishes if M admits a Kähler-Einstein metric.

K-stability:

By Kodaira, for $\ell >> 1$, any basis of $H^0(M, K_M^{-\ell})$ gives an embedding: $M \mapsto \mathbb{C}P^N$. So we may consider M as a subvariety in $\mathbb{C}P^N$. For any algebraic subgroup $G_0 = \{\sigma(t)\}_{t\in\mathbb{C}^*}$ of $SL(N+1, \mathbb{C})$, there is a unique limiting cycle

$$M_0 = \lim_{t\to 0} \sigma(t)(M) \subset \mathbb{C}P^N.$$

Ding-Tian (1992): If M_0 is normal, one can associate a generalized Futaki invariant $f_{M_0}(X)$.

M is called K-stable for ℓ if $f_{M_0}(X) \geq 0$ for any $G_0 \subset SL(N+1)$ with a normal M_0 and the equality holds if and only if M_0 is biholomorphic to M. M is K-stable if it is K-stable for all sufficiently large ℓ.

Tian (1996): If M is a Fano manifold without holomorphic fields, M admits a Kähler-Einstein metric only if M is K-stable.

The converse, i.e., YTD conjecture in the case of Fano manifolds, has been proved last Fall. Thanks to Bismut, I lectured on my proof here last April.

Hence, one can not expect that $\omega(t)$ has a limit on M in general.

Conjecture (Hamilton, Tian, 90s): $(M, \omega(t))$ converges to a generalized shrinking Kähler-Ricci soliton $(M_\infty, \omega_\infty)$ in the Cheeger-Gromov topology.

Here $(M_\infty, \omega_\infty)$ is a compact metric space which is smooth outside a closed subset of Hausdorff codimension at least 4, moreover, ω_∞ is a smooth Kähler metric on the regular part of M_∞ satisfying:

$$\mathrm{Ric}(\omega_\infty) - \omega_\infty = \sqrt{-1}\partial\bar{\partial}u, \quad \nabla^{1,0}\partial u = 0.$$

It is known that this conjecture implies the YTD conjecture for Fano manifolds: If M is a K-stable, then M admits a Kähler-Einstein metric.

The K-stability condition on M assures that M_∞ in the HT conjecture coincides with M and g_∞ is Kähler-Einstein.

Reduction to scalar equation:

Write

$$\omega(t) = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi,$$

then the Kähler-Ricci flow becomes

$$\frac{\partial \varphi}{\partial t} = \log \left(\frac{(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi)^n}{\omega_0^n} \right) - \varphi - h_0, \quad \varphi(0) = 0,$$

where h_0 is determined by ω_0.

To solve the scalar equation, one needs to establish a prior estimates for $\varphi(t)$. In fact, by using Maximum principle, one can reduce to establish the C^0-estimate for $\varphi(t)$. But such an estimate does not hold for general Fano manifolds.

Using his W-functional, Perelman proved (2003):

• $\omega(t)$ has uniformly bounded scalar curvature and diameter;

• (Non-collapsing) There is a positive constant $c > 0$ such that

$$Vol(B_r(x, \omega(t))) \le c\, r^{2n}, \quad \forall x \in M,\ r \le \mathrm{diam}(M, \omega(t));$$

• $u(t)$, $|\nabla u(t)|$ and Δu are uniformly bounded, where $u(t) = \frac{\partial \varphi}{\partial t}$.

R. Ye, Q. Zhang (2007): There is a uniform bound on the Sobolev constants of $\omega(t)$, that is, there is a uniform A satisfying: for any f on M,

$$\left(\int_M |f|^{\frac{2n}{n-1}}\omega(t)^n\right)^{\frac{n-1}{n}} \leq A \int_M (|\nabla f|^2 + f^2)\omega(t)^n.$$

Q. Zhang (2011): There is a uniform constant C such that

$$Vol(B_r(x, \omega(t))) \leq Cr^{2n}.$$

It follows that any sequence $\omega(t_i)$ with $t_i \to \infty$ has a subsequence converging to a length space (M_∞, d_∞) in the Gromov-Hausdorff topology.

The conjecture of Hamilton-Tian can be then reduced to the regularity of (M_∞, d_∞).

Two supporting evidences:

• (Sesum-Tian, 2004): If the Ricci curvature of $\omega(t)$ is uniformly bounded, then the conjecture holds. The proof is based on a compactness theorem of Cheeger-Colding-Tian.

• (Tian-Zhu, 2007): If M has a Kähler-Einstein metric, then $\omega(t)$ converges to the Kähler-Einstein metric. This was first claimed by Perelman. Our proof used the K-energy introduced by Mabuchi.

Recently, Zhenlei Zhang and I solved the HT conjecture for dimension ≤ 3:

If $\dim M \leq 3$, then $(M, \omega(t))$ converges to a generalized Kähler-Ricci soliton $(M_\infty, \omega_\infty)$. Moreover, M_∞ is a normal variety.

The proof is based on the following two results.

1. (Tian-Zhang, 2013) There is a uniform bound on the L^4-norm of Ricci curvature along the normalized Kähler-Ricci flow.

This is proved by some delicate computations using Perelman's estimates and Bochner techniques.

Question: Is there a uniform bound on L^p-norm of Ricci curvature for some $p > n$?

2. (Tian-Zhang, 2013) If the L^p-norm of $\mathrm{Ric}(\omega(t))$ is uniformly bounded for some $p > n$, then for any $t_i \to \infty$, $\{\omega(t_i)\}$ contains a subsequence converging to a generalized Kähler-Ricci soliton $(M_\infty, \omega_\infty)$ such that M_∞ is a normal variety.

There are three ingredients in the proof of this.

1. Extend Cheeger-Colding's and Cheeger-Colding-Tian's theories to metrics with L^p-bounded Ricci curvature:

Assume (M_i, g_i) converge to (M_∞, d_∞) in the GH topology with

$$\int_{M+i} |\mathrm{Ric}(g_i)|^p dv(g_i) \leq \Lambda, \quad vol(B_r(x, g_i)) \geq \kappa r^m,$$

where $2p > m = \dim M_i$, $x \in M_i$ and $r \leq 1$.

Then $M_\infty = \mathcal{R} \cup \mathcal{S}$ satisfying: $\mathcal{S}$ is a closed subset of codimension ≥ 2, $\mathcal{R}$ is $C^{1,\alpha}$-smooth and d_∞ is induced by a C^α-metric g_∞ on $\mathcal{R}$, where $\alpha < 2 - \frac{m}{p}$.

Moreover, if (M_i, g_i) are Kähler, $\mathcal{S}$ is of codimension ≥ 4.

2. Regularity of Kähler-Ricci flow:

If the above $(M_i, g_i) = (M, \omega(t_i))$ arise from a solution $\omega(t)$ of the normalized Kähler-Ricci flow, then by using an extension of Perelman's pseudo-locality and the regularity theory of Ricci flow, $\mathcal{R}$ is smooth and g_∞ is smooth, furthermore, g_i converge to g_∞ in the smooth topology outside $\mathcal{S}$, i.e., in the Cheeger-Gromov topology.

To see g_∞ is a Kähler-Ricci soliton on $\mathcal{R}$, we use the W-functional of Perelman.

For any Kähler metric ω on M with $[\omega] = c_1(M)$,

$$\mathcal{W}(\omega, f) \,=\, \int_M (s + |\nabla f|^2 + f - n)\, e^f\, dv.$$

Let f solve the following along the Kähler-Ricci flow:

$$\frac{\partial f}{\partial t} = -\Delta f + |\nabla f|^2 + \Delta u,$$

then

$$\frac{d}{dt}\mathcal{W}(\omega(t), f(t)) = \int_M (|\nabla^{1,0}\bar{\partial}(u - f)|^2 + |\nabla^{1,0}\partial f|^2) dv.$$

This implies that $|\nabla^{1,0}\partial u|(t_i)$ goes to 0 in the L^2-norm, so $\omega(t_i)$ converges to a Kähler-Ricci soliton on $\mathcal{R}$.

3. Partial C^0-estimate for Kähler-Ricci flow:

As I pointed out before, the normality of M_∞ follows from a version of the partial C^0-estimate for Kähler-Ricci flow.

By Kodaira, for $\ell >> 0$, any basis of $H^0(M, K_M^{-\ell})$ embeds M into a certain projective space $\mathbb{C}P^N$. For any $t > 0$, choose a Hermitian metric $H(t)$ on $K_M^{-\ell}$ with curvature $\omega(t)$, then we have an induced inner product on $H^0(M, K_M^{-\ell})$, let $\{S_a\}$ be any orthonormal basis with respect to this inner product, define

$$\rho_{t,\ell}(x) = \sum H(t)(S_a, S_a)(x).$$

Partial C^0-estimate: There are $c_k = c(k, n, \beta_0) > 0$ for $k \geq 1$ such that for a sufficiently large ℓ

$$\rho_{t,\ell}(x) \geq c_\ell.$$

This can be proved by similar arguments in the proof of the partial C^0-estimate for Kähler-Einstein metrics (Tian, Donaldson-Sun) once we establish corresponding analytic tools: Gradient estimate and the L^2-estimate for $\bar{\partial}$-operators.

1. Gradient estimate: There is $C = C(n, \omega_0)$ such that for any $\ell > 0$ and $\sigma \in H^0(M, K_M^{-\ell})$, we have

$$\sup_M \left(|\sigma|_t^2 + \ell^{-1} |\nabla \sigma|_t^2 \right) \le C \, \ell^n \int_M |\sigma|^2 \omega(t)^n.$$

Here $|\cdot|_t$ denotes the norm induced by $\omega(t)$.

2. L^2**-estimate**: There is $\ell_0 = \ell_0(\omega_0, n)$ such that for $\ell \geq \ell_0$ and $t \geq 0$ and $\tau \in \Lambda^{0,1}(M, K_M^{-\ell})$ with $\bar{\partial}\tau = 0$, we can find ϑ satisfying:

$$\bar{\partial}\vartheta = \tau, \quad \int_M |\vartheta|_t^2 \omega(t)^n \leq \frac{1}{\ell} \int_M |\tau|_t^2 \omega(t)^n.$$

This is a modification of the standard L^2-estimate.

With the above two estimates, for any $x \in M$, we can transplant constant functions on tangent cones at x to a section $\sigma \in H^0(M, K_M^{-\ell})$ whose norm is bounded uniformly from below near x.

This leads to the partial C^0-estimate. Then the known arguments, e.g., as in Chi Li's thesis, show that M_∞ is a normal variety.