Determinant line bundles in non-Kählerian geometry and instanton moduli spaces over class VII surfaces

Towards the classification of class VII surfaces

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1

- Let B, X be compact complex manifolds, $p : B \times X \to B$, $q : B \times X \to X$ the two projections and $\mathcal{E} \in \mathrm{Coh}(B \times X)$. The Fourier-Mukai transform of kernel $\mathcal{E}$ is the functor

$$\phi_{\mathcal{E}} : \mathrm{Coh}(X) \to \mathrm{Gr}(\mathrm{Coh}(B))$$

defined by

$$\phi_{\mathcal{E}}(\mathcal{F}) := R^{\bullet} p_*(\mathcal{E} \otimes q^*(\mathcal{F})) \ . \qquad (FM)$$

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- In projective algebraic geometry $\phi_{\mathcal{E}}$ can be lifted to a functor

$$\Phi_{\mathcal{E}} : \mathrm{D}^b(X) \to \mathrm{D}^b(B)$$

and has sense for a kernel $\mathcal{E} \in \mathrm{D}^b(B \times X)$. Such functors are extensively used in the literature in order to compare the derived categories associated with the two projective varieties.

2

- The non-Kählerian version of the GRR theorem [Bismut *"Hypoelliptic Laplacian and Bott-Chern cohomology*] computes the Chern character $\mathrm{ch}(\phi_{\mathcal{E}}(\mathcal{F}))$ in terms of the Chern classes of kernel $\mathcal{E}$, $\mathcal{F}$, in Bott-Chern cohomology of B assuming that certain technical conditions are satisfied ($\mathcal{E}$, $\mathcal{F}$ and the direct images are locally free).

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- We will use the simpler correspondence $\delta_{\mathcal{E}} : Coh(X) \to \mathrm{Pic}(B)$

$$\delta_{\mathcal{E}}(\mathcal{F}) := \det(R^{\bullet}p_*(\mathcal{E} \otimes q^*(\mathcal{F}))) = \lambda(\mathcal{E} \otimes q^*(\mathcal{F}))$$

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obtained by composing $\phi_{\mathcal{E}}$ with the determinant functor.
- This functor should be regarded as a method for constructing holomorphic line bundles on an unknown manifold B using a fixed kernel $\mathcal{E} \in \mathrm{Coh}(B \times X)$ and variable coherent sheaves $\mathcal{F}$ on the known manifold X.

3

- Specializing to line bundles $\mathcal{L} \in \mathrm{Pic}(X)$ one obtains a holomorphic map $\mathrm{Pic}(X) \to \mathrm{Pic}(B)$

$$\mathcal{L} \mapsto \det(R^\bullet p_*(\mathcal{E} \otimes q^*(\mathcal{L}))) = \lambda(\mathcal{E} \otimes q^*(\mathcal{L})) \ .$$

between Abelian complex groups.

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between Abelian complex groups.

- Recall that for a compact complex manifold X we have a canonical exact sequence

$$0 \to \mathrm{Pic}^0(X) \hookrightarrow \mathrm{Pic}(X) \xrightarrow{c_1} \mathrm{NS}(X) \to 0$$

where

$$\mathrm{NS}(X) := \ker(H^2(X,\mathbb{Z}) \to H^{0,2}(X,\mathbb{C}))$$

is the subgroup of classes whose image in $H^2_{\mathrm{DR}}(X,\mathbb{C})$ have a representative of type (1,1) and $\mathrm{Pic}^0(X)$ is connected.

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- **Example:** For *a class VII surface* X we have $\mathrm{NS}(X) = H^2(X, \mathbb{Z})$ and $\mathrm{Pic}^0(X) \simeq \mathbb{C}^*$ (non-compact!) .

4

- The connected component $\mathrm{Pic}^0(X)$ of $\mathcal{O}_X$ in $\mathrm{Pic}(X)$ fits in the diagram with exact rows and columns

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathrm{Pic}^0_{\mathrm{uf}}(X) & \longrightarrow & \mathrm{Pic}^0(X) & \xrightarrow{c_1^{\mathrm{BC}}} & H^{1,1}_{\mathrm{BC}}(X,\mathbb{R})^0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathrm{Pic}_{\mathrm{uf}}(X) & \longrightarrow & \mathrm{Pic}^{\mathrm{T}}(X) & \xrightarrow{c_1^{\mathrm{BC}}} & H^{1,1}_{\mathrm{BC}}(X,\mathbb{R})^0 & \longrightarrow & 0 \\
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\end{array}$$

- $H^{1,1}_{\mathrm{BC}}(X,\mathbb{R})^0 := \ker(H^{1,1}_{\mathrm{BC}}(X,\mathbb{R}) \to H^{1,1}_{\mathrm{DR}}(X,\mathbb{R}))$ and $\mathrm{Pic}_{\mathrm{uf}}(X)$ is the subgroup of holomorphic line bundles on X which admit a compatible flat unitary connection.

5

- The *maximal compact subgroup* of $\mathrm{Pic}^0(X)$ is the connected component of unit in $\mathrm{Pic}_{\mathrm{uf}}(X)$, and one has an identification

$$\mathrm{Pic}^0_{\mathrm{uf}}(X) = \{\rho \in \mathrm{Hom}(\pi_1(X, x_0), S^1) |\ c_1(\mathcal{L}_\rho) = 0\} \ ,$$

where, in general, for $\rho \in \mathrm{Hom}(\pi_1(X, x_0), \mathbb{C}^*)$, $\mathcal{L}_\rho$ denotes the flat holomorphic line bundle associated with ρ.

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- The Lie algebra of $\mathrm{Pic}^0_{\mathrm{uf}}(X)$ is

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- **Recall:** A Hermitian metric g on a complex n-manifold is called *Gauduchon* if $dd^c(\omega_g^{n-1}) = 0$. The degree

$$\deg_g : \mathrm{Pic}(X) \to \mathbb{R}$$

associated with such a metric is defined by

$$\deg_g(\mathcal{L}) := \int_X c_1(\mathcal{L},h) \wedge \omega_g^{n-1}\ (h \text{ Hermitian metric on } \mathcal{L})$$

6

- If X is a surface with $b_1(X)$ is odd then $\deg_g$ is surjective on $\mathrm{Pic}^0(X)$, and $\mathrm{Pic}^0_{\mathrm{uf}}(X) = \ker(\deg_g : \mathrm{Pic}^0(X) \to \mathbb{R})$

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Remark 1.1

Suppose X has a Hermitian metric g with $dd^c\omega_g = 0$. Then

$$\lambda(\mathcal{E} \otimes q^*(\mathrm{Pic}^0_{\mathrm{uf}}(X)) \subset \lambda(\mathcal{E}) \otimes \mathrm{Pic}^0_{\mathrm{uf}}(B) \ ,$$

hence perturbing the kernel $\mathcal{E}$ by a unitary flat line bundle on X will change the determinant by a unitary flat line bundle on M.

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- Indeed, the Chern forms of $\mathcal{E}$, $\mathcal{E} \otimes q^*(\mathcal{L})$ coincide when $\mathcal{L} \in \mathrm{Pic}^0_{\mathrm{uf}}(X)$, so the curvature of the corresponding determinant line bundles will be the same, by the recent theorem of Bismut which computes the curvature of the Quillen metric in the non-Kählerian framework [Bismut *"Hypoelliptic Laplacian and Bott-Chern cohomology*].

7

- **Problem:** Compute the linearization

$$l_{\mathcal{E}} : iH^1(X,\mathbb{R}) \to iH^1(B,\mathbb{R}) \text{ of the map}$$

$$\mathrm{Pic}^0_{\mathrm{uf}}(X) \ni \mathcal{L} \xmapsto{\delta_{\mathcal{E}}} \lambda(\mathcal{E} \otimes q^*(\mathcal{L})) \in \lambda(\mathcal{E}) \otimes \mathrm{Pic}^0_{\mathrm{uf}}(B)$$

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- How does $l_{\mathcal{E}} : iH^1(X,\mathbb{R}) \to iH^1(B,\mathbb{R})$ depend on the kernel $\mathcal{E}$? Is it a topological invariant? Compare $l_{\mathcal{E}}$ with $l_{\mathcal{E}\otimes q^*(\mathcal{T})}$ for a line bundle $\mathcal{T} \in \mathrm{Pic}(X)$.

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Theorem 1.2

Suppose X has a Hermitian metric g with $dd^c\omega_g = 0$. One has

$$l_{\mathcal{E}}(u) = p_*\left(q^*(u)\cup \mathrm{ch}(\mathcal{E})\cup \mathrm{td}(X)\right)^{(1)} .$$

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$$l_{\mathcal{E}}(u) = p_*\left(q^*(u)\cup \mathrm{ch}(\mathcal{E})\cup \mathrm{td}(X)\right)^{(1)} .$$

- Therefore $l_{\mathcal{E}}$ is determined by the Chern classes of $\mathcal{E}$, so has a topological character.

8

Proof of Theorem 1.2.

- Consider the projection

$$\tilde{p} : \mathrm{Pic}^0(X) \times B \times X \to \mathrm{Pic}^0(X) \times B =: \tilde{B} \ ,$$

and let $\mathcal{P}$ be a Poincaré line bundle on $\mathrm{Pic}^0(X) \times X$.

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- Apply Bismut curvature formula to the determinant line bundle (with respect to $\tilde{p}$) of the bundle

$$\tilde{\mathcal{E}} := p_{B\times X}^*(\mathcal{E}) \otimes p_{\mathrm{Pic}^0(X)\times X}^*(\mathcal{P}) \ .$$

We obtain an explicit formula for the curvature of a compatabible unitary connection $\tilde{A}$ of the holomorphic line bundle $\lambda(\tilde{\mathcal{E}})$ over the new base $\tilde{B} = \mathrm{Pic}^0(X) \times B$.

∎

9

Continuation of the proof of Theorem 1.2.

- For $u \in iH^1(X, \mathbb{R})$ consider the map

$$f_u : \mathbb{R} \times B \to \mathrm{Pic}^0(X) \times B \ , \ f_u(t, b) = (e^{tu}, b) \ ,$$

and the pull-back connection $f_u^*(\tilde{A})$ on the line bundle $f_u^*(\lambda(\tilde{\mathcal{E}}))$. Using a *temporal gauge* we obtain a 1-parameter family $(A_t)_{t\in\mathbb{R}}$ of unitary connections on a fixed Hermitian line bundle over B

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- The velocity of this path of connections is the coefficient of dt in the "mixed term" of the curvature $F_{f_u^*(\tilde{A})}$. The de Rham class of the velocity at 0 is precisely the class $l_{\mathcal{E}}(u)$ we need. We obtain
$l_{\mathcal{E}}(u) =$the coefficient of dt in $p_*\left(e^{dt\wedge q^*(u)} \cup \mathrm{ch}(\mathcal{E}) \cup \mathrm{td}(X)\right)^{(2)}$.

■

10

- Another approach (followed by Julien Grivaux, [see *"Variation of the holomorphic determinant bundle"*, `arXiv:1205.6170`]: Prove first a GRR formula in *Deligne cohomology*. The first Chern class in Deligne cohomology is just

$$[\mathrm{det}] : \mathrm{Coh}(-) \to \mathrm{Pic}(-)$$

so such a GRR formula will compute $\lambda(\mathcal{E})$ up to isomorphism, not just an invariant of it. It suffices to differentiate the obtained formula for $\lambda(\mathcal{E} \otimes q^*(e^{tu}))$ with respect to t.

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so such a GRR formula will compute $\lambda(\mathcal{E})$ up to isomorphism, not just an invariant of it. It suffices to differentiate the obtained formula for $\lambda(\mathcal{E} \otimes q^*(e^{tu}))$ with respect to t.
- **Problem:** Prove the GRR formula in the analytic framework using a cohomology theory which is finer than both de Rham cohomology $\oplus_k H^k_{\mathrm{DR}}(B, \mathbb{C})$ and Hodge cohomology $\oplus_k H^k(B, \Omega^k_B)$. Bismut solved this problem using Bott-Chern cohomology (assuming that the direct images are locally free).

11

Corollary 1.3

Let X be a complex surface and $\mathcal{E}$ is a holomorphic rank r bundle on $B \times X$ with $c_1(\mathcal{E}) \in p^*(H^2(B, \mathbb{Z})) + q^*(H^2(X, \mathbb{Z}))$ (so the mixed term in the Künneth decomposition of $c_1(\mathcal{E})$ vanishes). Then

$$l_{\mathcal{E}}(u) = \frac{1}{2r} p_*(q^*(u) \cup c_2(\mathrm{End}_0(\mathcal{E})) = \frac{1}{2r} c_2(\mathrm{End}_0(\mathcal{E}))/D_X(u)$$

In particular $l_{\mathcal{E}} = l_{\mathcal{E} \otimes q^*(\mathcal{T})}$ for any holomorphic line bundle $\mathcal{T}$ on X.

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In particular $l_{\mathcal{E}} = l_{\mathcal{E} \otimes q^*(\mathcal{T})}$ for any holomorphic line bundle $\mathcal{T}$ on X.

- The same formula gives the Donaldson μ-class associated with $D_X(u) \in H_3(X,\mathbb{R})$ on a moduli space of irreducible $\mathrm{PU}(r)$ connections. This gives an interesting geometric interpretation of this Donaldson class in Donaldson theory on complex surfaces.

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An interesting application

Theorem 1.4

In the conditions of the theorem suppose that

$$h^i(\mathcal{E}|_{X_b}) = 0 \ \ \forall b \in B \ \forall i \in \{0,1,2\} \ , \ \ X_b := \{b\} \times X \ .$$

Then the map

$$\mathrm{Pic}(X) \in \mathcal{L} \xmapsto{\delta_{\mathcal{E}}} \lambda(\mathcal{E} \otimes q^*(\mathcal{L})) \in \mathrm{Pic}(B)$$

is constant on every component $\mathrm{Pic}^c(X)$ *of* $\mathrm{Pic}(X)$.

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Proof.

- Using Grauert semicontinuity and the compactness of B we obtain: for every $\mathcal{L}$ in a neighborhood of $[\mathcal{O}_X]$ in $\mathrm{Pic}_0(X)$ one still has $h^i(\mathcal{E} \otimes \mathcal{L}|_{X_b}) = 0$. Therefore $\delta_{\mathcal{E}}$ is constant on $\mathrm{Pic}^0(X)$ so $l_{\mathcal{E}} = 0$.

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Proof.

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- Fix $\mathcal{T} \in \mathrm{Pic}^c(X)$. Using Corollary 1.3 we get $l_{\mathcal{E} \otimes q^*(\mathcal{T})} = 0$, hence $\delta_{\mathcal{E}}$ is constant on the compact real hypersurface

$$\mathcal{T} \otimes \mathrm{Pic}^0_{\mathrm{uf}} \subset \mathrm{Pic}^c(X) ,$$

hence is constant on $\mathrm{Pic}^c(X)$ because is holomorphic.

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Definition 2.1

The class VII of complex surfaces is defined by

$$VII := \{X \textit{complex surface} \mid b_1(X) = 1,\ \mathrm{kod}(X) = -\infty\}$$

The condition $\mathrm{kod}(X) = -\infty$ means $h^0(\mathcal{K}_X^{\otimes n}) = 0$ for every $n \in \mathbb{N}^*$. These surfaces are not classified yet. .

- *Topological invariants:* Let $X \in VII$. One has

1. $-c_1^2(X) = c_2(X) = b_2(X)$.
2. $b_+(X) = 0$, so the intersection form

$$q_X : H^2(X, \mathbb{Z})/\mathrm{Tors} \times H^2(X, \mathbb{Z})/\mathrm{Tors} \to \mathbb{Z}$$

is negative definite. *By the first Donaldson theorem* it follows that q_X is standard over $\mathbb{Z}$.

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- Therefore, putting $b := b_2(X)$, there exists a basis $(e_1, \ldots, e_b)$ of $H^2(X, \mathbb{Z})/\mathrm{Tors}$ such that

$$q_X(e_i, e_j) = -\delta_{ij} \ . \tag{1}$$

We can decompose $-c_1(X) = c_1(\mathcal{K}_X) = \sum_{i=1}^b x_i e_i$, where
  - $\sum x_i^2 = b$ because $c_1(\mathcal{K}_X)^2 = -b$,
  - x_i are all odd, because $c_1(\mathcal{K}_X)$ is a characteristic element, so $e_i \cdot c_1(\mathcal{K}_X) \equiv e_i^2 = -1 \bmod 2$.

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  - x_i are all odd, because $c_1(\mathcal{K}_X)$ is a characteristic element, so $e_i \cdot c_1(\mathcal{K}_X) \equiv e_i^2 = -1 \mod 2$.
- Therefore $x_i \in \{\pm 1\}$ and changing the signs of some e_i's if necessary, we may suppose that

$$c_1(\mathcal{K}_X) = \sum_{i=1}^{b} e_i \ . \qquad (2)$$

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$$q_X(e_i, e_j) = -\delta_{ij} \ . \tag{1}$$

We can decompose $-c_1(X) = c_1(\mathcal{K}_X) = \sum_{i=1}^b x_i e_i$, where

  - $\sum x_i^2 = b$ because $c_1(\mathcal{K}_X)^2 = -b$,
  - x_i are all odd, because $c_1(\mathcal{K}_X)$ is a characteristic element, so $e_i \cdot c_1(\mathcal{K}_X) \equiv e_i^2 = -1 \bmod 2$.

- Therefore $x_i \in \{\pm 1\}$ and changing the signs of some e_i's if necessary, we may suppose that

$$c_1(\mathcal{K}_X) = \sum_{i=1}^b e_i \ . \tag{2}$$

- A basis satisfying (1) and (2) (which is unique up to permutation) will be called the Donaldson basis of $H^2(X,\mathbb{Z})/\mathrm{Tors}$.

15

- *Analytic invariants:* If $X \in VII$ then :
  1. $\mathrm{NS}(X) = H^2(X, \mathbb{Z})$,
  2. $h^1(X, \mathcal{O}_X) = 1$, hence, by Serre duality $h^1(X, \mathcal{K}_X) = 1$,
  3. There exists a *canonical* isomorphism $\mathrm{Pic}^0(X) \simeq \mathbb{C}^*$. We will denote by $\mathcal{L}_\zeta$ the line bundle which corresponds to $\zeta \in \mathbb{C}^*$.

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- The maximal compact subgroup of $\mathrm{Pic}^0(X)$ is $\mathrm{Pic}^0_{\mathrm{uf}}(X) \simeq S^1$ and $\deg_g$ defines a isomorphism

$$H^{1,1}_{\mathrm{BC}}(X, \mathbb{R})^0 \simeq {}^{\mathrm{Pic}^0(X)}\big/_{\mathrm{Pic}^0_{\mathrm{uf}}(X)} \xrightarrow{\simeq \deg_g} \mathbb{R}$$

16

- *Classification for $b_2 = 0$:* A *Hopf surface* is a complex surface H with $\tilde{H} \simeq \mathbb{C}^2 \setminus \{0\}$. The simplest Hopf surfaces are the primary Hopf surfaces: A primary Hopf surface is the quotient of $\mathbb{C}^2 \setminus \{0\}$ by a cyclic group, for instance by the group generated by a linear contraction. A primary Hopf surface is diffeomorphic to $S^1 \times S^3$ and has at least an elliptic curve.

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- An *Inoue surface* S is a class VII surface which is the quotient of $\mathbb{C} \times H \subset \mathbb{C}^2$ by a group of affine transformations. An Inoue surface has no curve at all.

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- An *Inoue surface* S is a class VII surface which is the quotient of $\mathbb{C} \times H \subset \mathbb{C}^2$ by a group of affine transformations. An Inoue surface has no curve at all.

Theorem 2.2

Let $X \in VII$ with $b_2(X) = 0$. Then X is biholomorphic to either a Hopf surface or an Inoue surface.

- It remains to describe the subclass $VII^{\min}_{b_2>0}$ of minimal class VII surfaces with $b_2 > 0$.

17

- *Kato surfaces:*
  A spherical shell (an SS) in a complex surface X is an open subset $U \subset X$ which is biholomorphic to a standard neighborhood of S^3 in $\mathbb{C}^2$. A spherical shell $U \subset X$ is called global (GSS) is $X \setminus U$ is connected.

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A Kato surface is a surface $X \in VII^{\text{min}}_{b_2>0}$ which contains a GSS.

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- *Kato surfaces are well understood:* they can be all obtained using a simple construction, and they can be classified, including the description of certain moduli spaces of Kato surfaces.
- Therefore Kato surfaces should be regarded as the *known* surfaces in $VII^{\min}_{b_2>0}$ and the following conjecture would solve the classification problem for class VII surfaces.

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Conjecture 1 (Nakamura, 1989)

Any surface $X \in VII^{\min}_{b_2>0}$ *is a Kato surface.*

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- Properties of Kato surfaces:
  1. Any Kato surface is a degeneration of a 1-parameter family of blown up primary Hopf surfaces, hence is diffeomorphic to

     $$(S^1 \times S^3)\# b\bar{\mathbb{P}}^2_{\mathbb{C}} \ ,$$

     hence if the conjecture is true, there will exist only one diffeomorphism type and only one deformation equivalence class of minimal class VII surfaces with fixed b_2.

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  2. Any Kato surface X has exactly $b_2(X)$ rational curves.
  3. Any Kato surface contains a cycle of rational curves.

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- Two important results show that the latter two properties play an important role in the classification of class VII surfaces:

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Any surface $X \in VII^{\min}_{b_2>0}$ with $b_2(X)$ rational curves is a Kato surface.

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Any surface $X \in VII^{\min}_{b_2>0}$ with $b_2(X)$ rational curves is a Kato surface.

Theorem 2.5 (Nakamura)

Any surface $X \in VII^{\min}_{b_2>0}$ with a cycle of rational curves is a degeneration of a 1-parameter family of blown up primary Hopf surfaces.

These results suggest the conjectures:

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Conjecture 2 ("Conjecture A")

Any surface $X \in VII^{\min}_{b_2>0}$ *has* $b_2(X)$ *rational curves.*

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Any surface $X \in VII^{\min}_{b_2>0}$ has a cycle of rational curves.

- By Theorem 2.4 Conjecture 2 will solve the classification problem up to biholomorphism.
- By Theorem 2.5 Conjecture 3 will solve the classification problem up to deformation equivalence (in particular up to diffeomorphism).

21

- One cannot easily obtain Conjecture 2 from Conjecture 3: If X has a cycle then it is the central fiber of a family $(X_z)_{z\in D}$, where X_z is a blown up Hopf surface for $z\neq 0$. Unfortunately the area of the exceptional curves $E^i_z\subset X_z$ diverges, so these curves do not converge to analytic cycles in $X=X_0$ as $z\to 0$.

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- This phenomenon (explosion of area) has been studied in "*Infinite bubbling in non-Kählerian geometry*" (Dloussky, –).
- If $X=X_0$ is a Kato surface then X has $b_2(X)$ rational curves, but these curves do not belong to the homology classes of the exceptional curves $E_z^i\subset X_z$. Therefore

Remark 2.6

The homology classes which are represented by holomorphic curves are not stable in holomorphic families of class VII surfaces.

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- Instantons and holomorphic bundles on complex surfaces
- A moduli space of instantons on class VII surfaces
- Existence of a cycle on class VII surfaces with small b_2

22

- Let (X, g) be a complex surface endowed with a Gauduchon metric. A holomorphic rank 2 bundle $\mathcal{E}$ on X is called
  - stable, if for every line bundle $\mathcal{L}$ and non-trivial morphism $\mathcal{L} \to \mathcal{E}$ one has $\deg(\mathcal{L}) < \frac{1}{2}\deg_g(\det(\mathcal{E}))$.
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- Let (E, h) be a differentiable Hermitian rank 2-bundle on X, $\mathcal{D}$ a fixed holomorphic structure on $D := \det(E)$. Denote by

$$\mathcal{M}^{\mathrm{st}}_{\mathcal{D}}(E)\ ,\ \mathcal{M}^{\mathrm{pst}}_{\mathcal{D}}(E)$$

the moduli sets of stable, respectively polystable holomorphic structures $\mathcal{E}$ on E inducing the fixed holomorphic structure $\mathcal{D}$ on $\det(E)$, modulo the complex gauge group $\mathcal{G}^{\mathbb{C}} := \Gamma(X, \mathrm{SL}(E))$.

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- $\mathcal{M}^{\rm st}_{\mathcal{D}}(E)$ has a natural complex subspace structure obtained using classical deformation theory. $\mathcal{M}^{\rm pst}_{\mathcal{D}}(E)$ can be endowed with a topology using the Kobayashi-Hitchin correspondence.

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- Let a be the Chern connection of the pair $(\mathcal{D}, \det(h))$.

$$\mathcal{G} := \Gamma(X, \mathrm{SU}(E)),$$

$$\mathcal{A}(E) := \text{ the space of unitary connections on } E$$

$$\mathcal{M}_a^{\mathrm{ASD}}(E) := \{A \in \mathcal{A}(E) |\ \det(A) = a,\ (F_A^0)^+ = 0\} / \mathcal{G}$$

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- We denote by $\mathcal{M}_a^{\mathrm{ASD}}(E)^*$ the open subspace defined by the condition "A is irreducible". This subspace has the structure of a real analytic space.
- The *Kobayashi-Hitchin correspondence* states that the map

$$A \mapsto \text{ the holomorphic structure defined by } \bar\partial_A$$

  induces a bijection $KH : \mathcal{M}_a^{\mathrm{ASD}}(E) \to \mathcal{M}_{\mathcal{D}}^{\mathrm{pst}}(E)$. More precisely

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- We have a commutative diagram

$$\begin{array}{ccc} \mathcal{M}_a^{\mathrm{ASD}}(E)^* & \hookrightarrow & \mathcal{M}_a^{\mathrm{ASD}}(E) \\ KH^* \downarrow \simeq & & \simeq \downarrow KH \\ \mathcal{M}_{\mathcal{D}}^{\mathrm{st}}(E) & \hookrightarrow & \mathcal{M}_{\mathcal{D}}^{\mathrm{pst}}(E) \end{array}$$

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- We endow $\mathcal{M}^{\mathrm{pst}}_{\mathcal{D}}(E)$ with the topology induced by KH.
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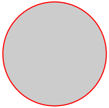
- The local structure of $\mathcal{M}_{\mathcal{D}}^{\mathrm{pst}}(E)$ around $\mathcal{R}$ can be studied using the Kobayashi-Hitchin correspondence and Donaldson theory.

25

- **Example1:** On a Kato surface with $b_2 = 1$ choosing in a suitable way the triple $(g, E, \mathcal{D})$ the resulting $\mathcal{M}^{\mathrm{pst}}_{\mathcal{D}}(E)$ is a disk whose boundary is the space of reductions $\mathcal{R}$ (a circle).



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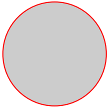

- **Example2:** On a Kato surface with $b_2 = 2$ choosing in a suitable way the triple $(g, E, \mathcal{D})$ the resulting $\mathcal{M}^{\mathrm{pst}}_{\mathcal{D}}(E)$ is a S^4 with two circles of reductions $\mathcal{R}_1$, $\mathcal{R}_2$. The complex structure of $\mathcal{M}^{\mathrm{st}}_{\mathcal{D}}(E)$ extends over $\mathcal{R}_2$ but note over $\mathcal{R}_1$.

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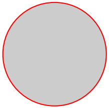

- **Example2:** On a Kato surface with $b_2 = 2$ choosing in a suitable way the triple $(g, E, \mathcal{D})$ the resulting $\mathcal{M}^{\mathrm{pst}}_{\mathcal{D}}(E)$ is a S^4 with two circles of reductions $\mathcal{R}_1$, $\mathcal{R}_2$. The complex structure of $\mathcal{M}^{\mathrm{st}}_{\mathcal{D}}(E)$ extends over $\mathcal{R}_2$ but note over $\mathcal{R}_1$.
- The Kobayashi-Hitchin correspondence has been first used by Donaldson as a tool to describe moduli spaces of instantons on algebraic surfaces. The unknown was $\mathcal{M}^{\mathrm{ASD}}_{a}(E)$ and the computable object was $\mathcal{M}^{\mathrm{pst}}_{\mathcal{D}}(E)$.

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- On non-algebraic surfaces: the appearance of *non-filtrable bundles* complicates the description of a moduli space $\mathcal{M}^{\mathrm{pst}}_{\mathcal{D}}(E)$. A rank 2 holomorphic bundle $\mathcal{E}$ on X is called *filtrable* if there exists a sheaf mono-morphism

$$0 \to \mathcal{L} \to \mathcal{E}$$

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- A filtrable bundle $\mathcal{E}$ fits in a short exact sequence

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for line bundles $\mathcal{M}$, $\mathcal{N}$ and a 0-dimensional l.c.i. $Z \subset X$.

- A non-filtrable bundle is stable with respect to *any* Gauduchon metric. There exists no classification method for non-filtrable bundles.

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- Let now X be a class VII surface and (E, h) a differentiable rank 2-bundle on X with

  $$c_2(E) = 0\ ,\ \det(E) = K_X \text{ (the underlying } \mathcal{C}^\infty \text{ bundle of } \mathcal{K}_X)\ ,$$

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  and its open subspace $\mathcal{M}^{\mathrm{st}} := \mathcal{M}^{\mathrm{st}}_{\mathcal{K}}(E)$ of stable bundles, which is a complex space of dimension $b := b_2(X)$.

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- The fundamental objects used in our program to prove existence of curves on class VII surfaces: the moduli space

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  and its open subspace $\mathcal{M}^{\mathrm{st}} := \mathcal{M}^{\mathrm{st}}_{\mathcal{K}}(E)$ of stable bundles, which is a complex space of dimension $b := b_2(X)$.
- Rough idea of the strategy: prove that the same filtrable bundle can be written as en extension in two different ways. This yields a non-trivial (and non-isomorphic) morphism of line bundles, whose vanishing locus will be a curve.

- Compactness: $\mathcal{M}$ is compact (– and Buchdahl).
  This is proved using a the KH correspondence and a combination of gauge theoretical and complex geometric arguments.

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- Every $\mathcal{R}_{\{I,\bar{I}\}}$ has a compact neighborhood homeomorphic to $\mathcal{R}_{\{I,\bar{I}\}} \times$ [cone over $\mathbb{P}^{b-1}_{\mathbb{C}}$].

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- Symmetry and twisted reductions: $\mathcal{M}$ comes with a involution $\otimes\mathcal{L}_0$, where $\mathcal{L}_0 \in \mathrm{Pic}^0(X)$ is the non-trivial square root of $[\mathcal{O}_X]$. This involution has 2^{b-1} fixed points, called *twisted reductions*.

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- **Example 1:** For $b_2 = 1$: $\mathcal{M}$ is a (possibly non-connected) compact Riemann surface with boundary and $\partial\mathcal{M}$ is a circle of reductions. It contains a single twisted reduction.
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- Put again $b := b_2(X)$ and let $(e_1, \ldots, e_b)$ the Donaldson basis of $H^2(X, \mathbb{Z})$. Let $\mathcal{E}$ be rank 2 bundle on X with

$$\det(\mathcal{E}) = \mathcal{K}_X \ , \ c_2(\mathcal{E}) = 0 \ ,$$

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- Then $\mathcal{E}/\mathcal{L}$ is locally free and $c_1(\mathcal{L}) = e_I := \sum_{i \in I} e_i$ for a subset $I \subset \{1, \dots, b\}$. Therefore
- Any filtrable bundle in our moduli space is an extension of the form

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{K}_X \otimes \mathcal{L}^\vee \to 0 \ , \tag{3}$$

  where $c_1(\mathcal{L}) = e_I$ for an index set $I \subset \{1, \dots, b\}$.

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- A crucial role in the proof is played by $\mathcal{M}^{\rm st}_{I_m}$ associated with the maximal index set $I_m := \{1, \ldots, b\}$. One has

$$\mathcal{M}^{\rm st}_{I_m} = \{\mathcal{A}, \mathcal{A}'\}$$

where $\mathcal{A}$ is the *canonical extension* of X, defined as the essentially unique non-trivial extension of the form

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(note $h^1(\mathcal{K}_X) = 1$ by Serre duality) and $\mathcal{A}' := \mathcal{A} \otimes \mathcal{L}_0$.

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(note $h^1(\mathcal{K}_X) = 1$ by Serre duality) and $\mathcal{A}' := \mathcal{A} \otimes \mathcal{L}_0$.
- For $I \neq I_m$: If X has no curves in certain homology classes (which we assume for simplicity!) $\mathcal{M}^{\rm st}_I$ is a $\mathbb{P}^{b-|I|-1}_{\mathbb{C}}$-fibration over a punctured disk, these fibrations are pairwise disjoint, and the closure $\bar{\mathcal{M}}^{\rm st}_I$ in $\mathcal{M}$ contains the circle $\mathcal{R}_{I,\bar{I}}$.

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- $\mathcal{M}^{\mathrm{st}}$ is a smooth b-dimensional manifold and the local structure around a circle of reductions is known (topologically).
- The locus of filtrable stable bundles decomposes as

$$\bigcup_{I\subset I_m} \mathcal{M}_I^{\mathrm{st}} \ ,$$

where

$$\mathcal{M}_{I_m}^{\mathrm{st}} = \{\mathcal{A},\mathcal{A}'\} \ , \mathcal{A}' := \mathcal{A}\otimes\mathcal{L}_0$$

and for $I \neq I_m$ the space $\mathcal{M}_I^{\mathrm{st}}$ is a $\mathbb{P}_{\mathbb{C}}^{b-|I|-1}$-fibration over a punctured disk. The closure of this fibration contains $\mathcal{R}_{\{I,\bar{I}\}}$.

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Theorem 3.1

Any minimal class VII surface X with $b_2(X) \in \{1, 2, 3\}$ has a cycle.

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- I will explain the proof for $b_2 = 1$ and *a new proof* for $b_2 = 2$ which generalizes to higher b_2 and makes use of Bismut results about the determinant line bundle in non-Kählerian geometry.

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- I will explain the proof for $b_2 = 1$ and *a new proof* for $b_2 = 2$ which generalizes to higher b_2 and makes use of Bismut results about the determinant line bundle in non-Kählerian geometry.
- Strategy of the proof (in general): Use the following

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Proposition 3.2

If the canonical extension $\mathcal{A}$ can be written as an extension in a different way, then X has a cycle. In particular, if $\mathcal{A}$ belongs to $\mathcal{M}_I^{\mathrm{st}}$ for $I \neq I_m$ or coincides with a twisted reduction, then X has a cycle.

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Proof.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K}_X & \stackrel{i}{\longrightarrow} & \mathcal{A} & \stackrel{p}{\longrightarrow} & \mathcal{O}_X & \longrightarrow & 0 \\ & & & & j\uparrow & \nearrow p\circ j & & & \\ & & & & \mathcal{L} & & & & \end{array}$$

$p \circ j$ is non-zero (because $\mathcal{L}$ is a different kernel) and non-isomorphism, because the canonical extension is non-split. Therefore $\mathrm{im}(p \circ j) = \mathcal{O}_X(-D)$ where $D > 0$ is the vanishing divisor of $p \circ j$. Restrict the diagram to D taking into account that j is a bundle embedding. We get $\omega_D := \mathcal{K}_X(D)_D$ is trivial on D, so D is a cycle.

■

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- In order to complete the proof it "suffices" to prove **The remarkable incidence:** The bundle $\mathcal{A}$ belongs to

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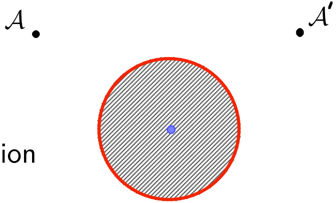

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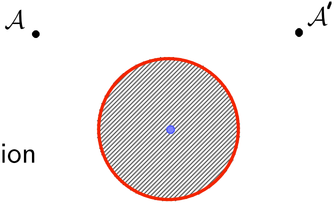

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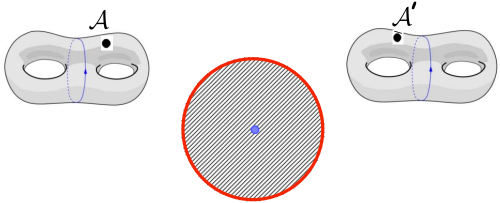

- Therefore either the remarkable incidence holds (*and the conjecture is proved*), or the connected component of $\mathcal{A}$ in $\mathcal{M}$ is a closed Riemann surface $Y \subset \mathcal{M}^{\mathrm{st}}$ which has at most two filtrable points.
- The latter possibility is ruled out by the following

Proposition 3.3

Suppose that X is a complex surface with $a(X) = 0$, E a differentiable rank 2 bundle over X, Y a closed Riemann surface and

$$f : Y \to \mathcal{M}^{\mathrm{simple}}(E) \ , \ y \mapsto [\mathcal{E}_y]$$

a holomorphic map. There exists a locally free sheaf $\mathcal{T}$ of rank 1 or 2 on X, a non-empty Zariski open set $U \subset X$ and for every $y \in Y$ a sheaf monomorphism $\mathcal{T} \to \mathcal{E}_y$ which is a bundle embedding on U. In particular the bundles $\mathcal{E}_y$ are either all filtrable or all non-filtrable.

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Proof.

Prove that f admits a universal (classifying) bundle $\mathcal{E}$ on $Y \times X$, interpret $\mathcal{E}$ as a family of bundles on Y parameterized by X. Use the fact that Y is algebraic and $a(X) = 0$. ∎

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- A priori there are three possibilities: Y contains 0, 1 or 2 circles. We will prove that the first two cases lead to a contradiction, and the third possibility implies again the remarkable incidence.
- Case 0: Suppose Y does not contain any circle of reductions. Then Y is a smooth compact surface contained in $\mathcal{M}^{\mathrm{st}}$ and it contains at most two filtrable bundles ($\mathcal{A}$ and $\mathcal{A}'$).
- The embedding $Y \hookrightarrow \mathcal{M}^{\mathrm{st}}$ has a universal bundle $\mathcal{E}$ on $Y \times X$. This is proved "*Instantons and holomorphic curves on class VII surfaces*" in full generality.

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- For $\mathcal{T} \in \mathrm{Pic}^e(X)$ one obtains

$$h^0(\mathcal{E}_y \otimes p_X^*(\mathcal{T})) = h^2(\mathcal{E}_y \otimes p_X^*(\mathcal{T})) = 0 \ \forall y \in Y,$$

so $h^1(\mathcal{E}_y \otimes p_X^*(\mathcal{T})) \equiv 1$ by Riemann-Roch theorem.

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$$\mathcal{L}_{\mathcal{T}} := R^1(p_Y)_*(\mathcal{E} \otimes p_X^*(\mathcal{T}))$$

on Y for every $\mathcal{T} \in \mathrm{Pic}^e(X)$.

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  on Y for every $\mathcal{T} \in \mathrm{Pic}^e(X)$.
- One has obviously $\mathcal{L}_{\mathcal{T}} = \lambda(\mathcal{E} \otimes p_X^*(\mathcal{T}))^\vee$. The main result of the first section Theorem 1.4 applies and shows that the isomorphism type of $\mathcal{L}_{\mathcal{T}}$ is independent of $\mathcal{T}$.

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- Consider the rank 2-bundle on $Y \times X$.

$$\mathcal{F}_{\mathcal{T}} := \mathcal{E} \otimes p_X^*(\mathcal{T}) \otimes p_Y^*(\mathcal{L}_{\mathcal{T}})^{\vee} .$$

The contradiction will be obtained computing $h^1(\mathcal{F}_{\mathcal{T}})$ using the Leray spectral sequences associated with the two projections p_Y, p_X. We will obtain different results.

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- We have

$$R^0(p_Y)_*(\mathcal{F}_{\mathcal{T}}) = R^0(p_Y)_*(\mathcal{E} \otimes p_X^*(\mathcal{T})) \otimes \mathcal{L}_{\mathcal{T}}^{\vee} = 0$$

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- The Leray spectral sequence associated with p_Y has

$$E_2^{p,0} = 0 , \ E_2^{0,1} = H^0(R^1(p_Y)_*(\mathcal{F}_{\mathcal{T}})) \simeq \mathbb{C} ,$$

hence

$$E_\infty^{1,0} = 0 , \ E_\infty^{0,1} \simeq \mathbb{C} \Rightarrow h^1(\mathcal{F}_{\mathcal{T}}) = 1 .$$

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- Projecting onto X we get

$$R^0(p_X)_*(\mathcal{F}_\mathcal{T}) = \mathcal{T} \otimes R^0(p_X)_*(\mathcal{E} \otimes p_Y^*(\mathcal{L}_\mathcal{T}^\vee)) \ .$$

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- The case $\mathrm{rk}(p_X)_*(\mathcal{E} \otimes p_Y^*(\mathcal{N}))) \geq 3$ implies $a(Y) > 0$, so Y is covered by divsiors. We would get a Riemann surface containing both filtrable and non-flitrable bundles (contradicts Proposition 3.3)

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- Use Bochner vansihing theorem for H^0 [see S. Kobayashi: *"Differential Geometry of Complex vector bundles"*]

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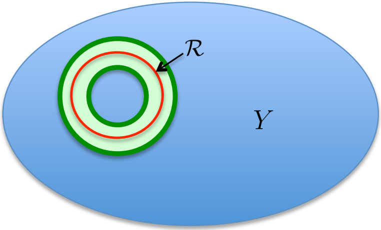

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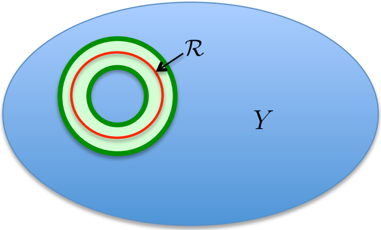

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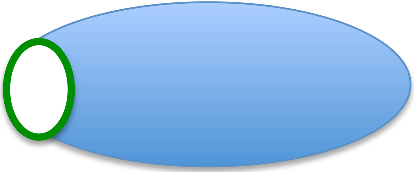

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- But the restriction of the Donaldson class $\mu(\eta)$, where η is a generator of $H_1(X, \mathbb{Z})$, is the fundamental class of ∂N. Contradiction.

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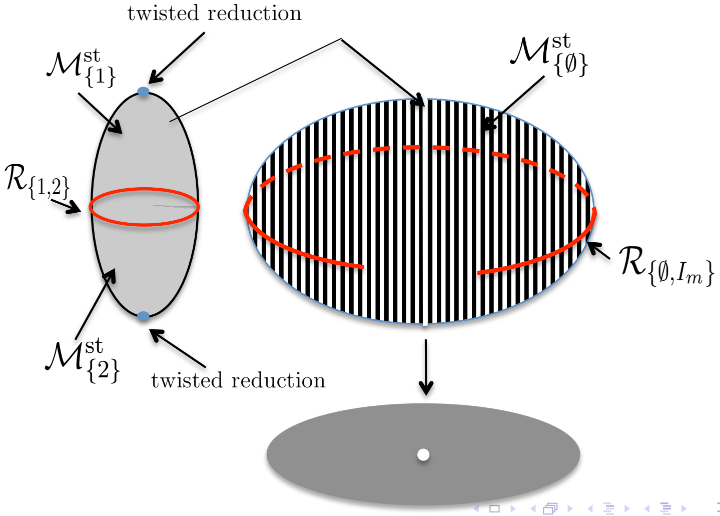

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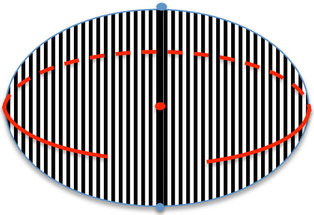

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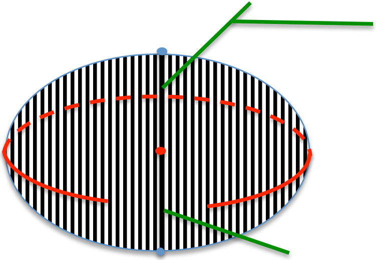

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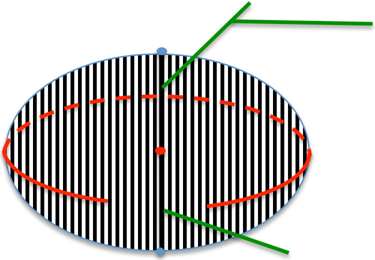

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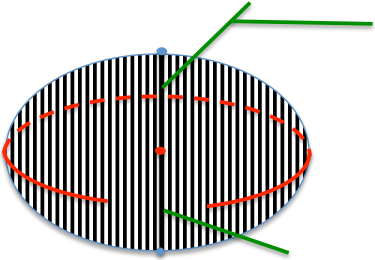
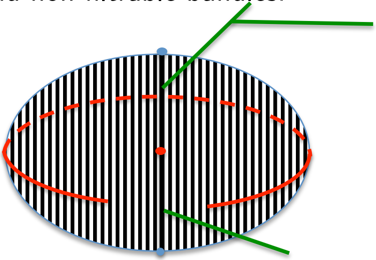


- The fiber over $0 \in D$ is a tree of rational curves: the known curve $\mathcal{M}^{\mathrm{st}}_{\{1\}} \cup \mathcal{M}^{\mathrm{st}}_{\{2\}} \cup \mathcal{R}_{\{\{1\},\{2\}\}} \cup \{$two twisted reductions$\}$ and unknown ("green") curves, whose generic points must be non-filtrable.

43

- The point $\mathcal{A}$ cannot belong to an unknown (green) curve, because if it did, we would get again a Riemann surface containing both filtrable and non-filtrable bundles.



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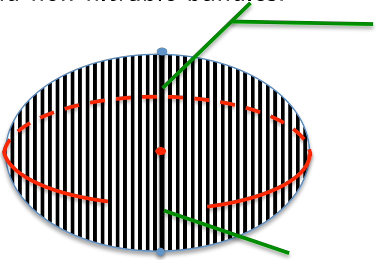

- Therefore $\mathcal{A}$ belongs to the "grey & blue locus", i.e. to the union:

$$\mathcal{M}^{\rm st}_{\emptyset} \cup \mathcal{M}^{\rm st}_{\{1\}} \cup \mathcal{M}^{\rm st}_{\{2\}} \cup \mathcal{R}_{\{\{1\},\{2\}\}} \cup \{\text{two twisted reductions}\} \ .$$

The remarkable incidence holds again, hence X has a cycle.

44

- The difficulty for general b_2:
  Main difficulty: rule out the situation when $\mathcal{A}$ belongs to an unknown stratum Y (consisting generically of non-filtrable bundles) but contains a circle of reductions. Such a stratum is not a complex space.

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- THANK YOU!

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