Asymptotic expansions of canonical metrics: methods and applications

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or ....

Asymptotics in geometric analysis,

A rhapsody in three parts

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Three short vignettes about the role of asymptotic analysis in problems involving metrics with special geometry.

1) Kähler-Einstein metrics with edges along a (SNC) divisor. Asymptotics at the divisor turn out to be crucial in proof of existence of these metrics.

2) The Weil-Peterson metric on $\mathcal{R}_\gamma$, the Riemann moduli space $\mathcal{R}_\gamma$ on a surface of genus γ. Goal: spectral geometry and index theory on $\mathcal{R}_\gamma$. Need asymptotics of g_{WP} at divisors of Deligne-Mumford compactification to be able to do things like define $\det \Delta_{WP}$, etc.

3) Poincaré-Einstein metrics (asymptotically hyperbolic, AdS/CFT). The renormalized volume is the basic action functional. Behaviour under Ricci flow.

Kähler-Einstein theory

Let Ω be a domain in $\mathbb{C}^n$. If $\phi \in \mathcal{C}^2(\Omega)$, then a typical complex Monge-Ampére (CMA) equation is a fully nonlinear partial differential equation of the form

$$\det\left(\mathrm{Id} + \sqrt{-1}\,\frac{\partial^2\phi}{\partial z_i \overline{\partial z_j}}\right) = F(z, \phi, \nabla\phi).$$

(Elliptic precisely when the matrix $\mathrm{Id} + \mathrm{Hess}_{\mathbb{C}}(\phi)$ is a positive definite Hermitian matrix.)

Intrinsic formulation when the domain Ω is replaced by a Kähler manifold (M, g).
If g is a Kähler metric and $\phi \in \mathcal{C}^2(M)$, define a new Hermitian $(1, 1)$ tensor g_ϕ by

$$(g_\phi)_{i\bar{\jmath}} = g_{i\bar{\jmath}} + \sqrt{-1}\frac{\partial^2 \phi}{\partial z_i \overline{\partial z_j}} = g_{i\bar{j}} + \sqrt{-1}\phi_{i\bar{\jmath}}.$$

This is a metric if the matrix on the right is Hermitian positive definite, and in this case, we write $\phi \in \mathcal{H}_g$.

Using the complex structure, convert Ric into a $(1,1)$ form

$$\rho_g = \sum_{i,\bar{j}} \mathrm{Ric}_{i\bar{j}} dz_i \wedge \overline{dz_j}.$$

g is KE if and only if $\rho_g = \mu\omega_g$ for some $\mu \in \mathbb{R}$.

Standard fact:

$$\frac{1}{2\pi i}\left[\rho_g\right] = c_1(M).$$

Assume that $c_1(M) = 0$ or else contains a positive or negative definite representative.

When $c_1(M) < 0$, respectively $c_1(M) = 0$, then there always exists g_ϕ which is KE with $\mu < 0$, respectively $\mu = 0$. (Aubin, Yau)

For $c_1(M) > 0$, existence if and only if K-stability. (Tian, Chen-Donaldson-Sun).

As a PDE, this amounts to solving the complex Monge-Ampère equation

$$\frac{\det\left(g_{i\bar{j}} + \sqrt{-1}\phi_{i\bar{j}}\right)}{\det\left(g_{i\bar{j}}\right)} = e^{F-\mu\phi},$$

where $F \in \mathcal{C}^{\infty}$ measures the discrepancy from g itself being Kähler-Einstein.

Kähler-Einstein edge metrics

Now suppose that (M, g) is Kähler as before, and that $D \subset M$ is a divisor with simple normal crossings: $D = D_1 \cup \ldots \cup D_N$ where each D_j is a smooth complex codimension one submanifold.

Locally, in coordinates each D_i can be described by an equation $\{z_i = 0\}$ for some choice of complex coordinates $(z_1, \ldots, z_n)$, and near intersections,

$$D_{i_1} \cap \ldots \cap D_{i_\ell} = \{z_{i_1} = \ldots = z_{i_\ell} = 0\}.$$

Problem (proposed by Tian in the early '90's, and more recently by Donaldson, mid 2000's)

Assume that $c_1(M) - \sum_{j=1}^{N}(1-\beta_j)c_1(L_{D_j}) = \mu[\omega]$ for some choice of constants $\beta_1, \ldots, \beta_N \in (0,1)$ and $\mu \in \mathbb{R}$. Can one then find a Kähler-Einstein metric with $\rho' = \mu\omega'$ in the same Kähler class as g and which is 'bent' with angle $2\pi\beta_j$ along D_j for every j?

This adds a small amount of flexibility to the problem.

Analogous situation: (M^3, g) hyperbolic 'conifold' (or cone-manifold). This is an incomplete stratified space which carries a hyperbolic metric on its open dense three-dimensional stratum, and which has singular set $\Sigma \subset M$ a geodesic one-dimensional network, along each smooth component of which M has an edge with fixed angle.

Back to the KE problem:

- Jeffres-M-Rubinstein, 2011; existence when D smooth, $\beta < 1$.
- M-Rubinstein, 2012. Existence in general case and resolution of Tian-Donaldson conjectures; general D, $\beta < 1$.

Following work by Jeffres, mid '90's (uniqueness for a given β); Berman, 2010, existence for general D, $\beta < 1/2$ (generalized solution, no asymptotic information);
Campagna-Guenancia-Paun, 2011, general D, $\mu \leq 0$, $\beta \leq 1/2$ (approximation techniques, again not much asymptotic information); Donaldson, 2011, D smooth, local deformation theory, $\beta \in (0, 1)$, all μ; Brendle, 2011, existence when D smooth, $\mu = 0$ and $\beta \leq 1/2$

Sidenote: Many things simplify in the "orbifold regime" $\beta \leq 1/2$.

Continuity method:

$$\frac{\det\left(g_{i\bar{\jmath}} + \sqrt{-1}\phi_{i\bar{\jmath}}\right)}{\det\left(g_{i\bar{\jmath}}\right)} = e^{tF-\mu\phi}, \qquad (\star)$$

and, as usual, the set $J = \{t \in [0,1] : \exists \text{ a solution to } (\star)\}$.

- J is nonempty ($0 \in J$ trivially).
- J is open
- J is closed.

One needs to start with an initial guess g_β; this is a Kähler metric with the correct edge geometry (and in particular angles $2\pi\beta_j$ along D_j.

In the flat space $\mathbb{C}^n$, for $D = \{z_1 = \ldots = z_k = 0\}$,

$$\omega_\beta = \frac{1}{2}\sqrt{-1}\sum_{j=1}^{k}|z_j|^{2\beta_j-2}|dz_j|^2 + \sum_{\ell=k+1}^{n}|dz_\ell|^2.$$

For the actual problem, choose a holomorphic section s_j on L_{D_j} and a Hermitian metric h_j on each of these line bundles, and set

$$\omega_\beta = \omega + \epsilon\sum_{j=1}^{k}\sqrt{-1}\partial\overline{\partial}|s_j|_{h_j}^{2\beta_j}$$

Here ω is a smooth Kähler form on M and $0 < \epsilon \ll 1$.

For this the cohomological condition
$c_1(M) - \sum(1-\beta_j)c_1(L_{D_j}) = \mu[\omega]$ is necessary!

There are difficulties with both the openness and closedness parts of the continuity argument:

For openness, must obtain nonstandard mapping properties of $L_{t_0} = \Delta_{g_{t_0}} + \mu$.

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For closedness, the most serious problem is that the method to derive the $\mathcal{C}^2$ a priori estimate by Aubin and Yau fails in this setting. It requires a lower bound on the bisectional curvature, but bisectional curvature of the initial model metric g_β is definitely NOT bounded below when any $\beta_j > 1/2$.

Recall, if X and Y are orthonormal, then

$$\mathrm{Bisec}(X, Y) = \mathrm{Riem}(X, \overline{X}, Y, \overline{Y}).$$

New continuity path:

$$\frac{\det\left(g_{i\bar{\jmath}} + \sqrt{-1}\phi_{i\bar{\jmath}}\right)}{\det\left(g_{i\bar{\jmath}}\right)} = e^{F - s\phi}, \qquad (\star\star),$$

$-\infty < s \leq \mu$.

This was introduced by Rubinstein in his work on Kähler-Ricci iteration, which is a type of discretization of Kähler-Ricci flow.

Function spaces

Consider the simple edge case first (D smooth). Recall that

$$g \sim |z_1|^{2\beta}|dz_1|^2 + \ldots + |dz_n|^2.$$

Choose coordinates $z_1 = \rho e^{i\tilde{\theta}}$, $z' = (z_2, \ldots, z_n)$ and $y = (\mathrm{Re}\,(z'), \mathrm{Im}\,(z'))$.

Finally, set

$$r = \frac{\rho^{1+\beta}}{1+\beta}, \quad \theta = (1+\beta)\tilde{\theta}$$

and we use (r, θ, y).

In these coordinates

$$\Delta_g \sim \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\beta^2}{r^2}\frac{\partial^2}{\partial \theta^2} + \Delta_y.$$

Find function spaces on which Δ_g has good mapping and regularity properties.

There are (at least) two reasonable choices of function spaces:

$$\mathcal{C}_w^{k,\alpha}(M,D) \quad \text{based on differentiating by } \frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \theta}, \frac{\partial}{\partial y_j}$$

the wedge Hölder spaces (used by Donaldson, Brendle), and

$$\mathcal{C}_e^{k,\alpha}(M,D) \quad \text{based on differentiating by } r\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}, r\frac{\partial}{\partial y_j}$$

the edge Hölder spaces.

Both behave well with respect to dilations $(r,\theta,y) \longmapsto (\lambda r, \theta, \lambda y + y_0)$ (homogeneous of degrees -1 and 0, respectively).

Define the Hölder-Friedrichs domain:

$$\mathcal{D}^{k,\alpha}_{w/e} = \{u \in \mathcal{C}^{2,\alpha}_{w/e} : \Delta u \in \mathcal{C}^{k,\alpha}_{w/e}\}$$

Note: if $u \in \mathcal{C}^{2,\alpha}_{e}$, then we expect that $\Delta u = \mathcal{O}(r^{-2})$, so if $u \in \mathcal{D}^{0,\alpha}_{e}$, then it has at least some extra regularity properties near the edge which allow the cancellation to happen.

The usual mechanism:

$$u \sim a_{01}(y)\log r + a_{00}(y) + r^{\frac{1}{\beta}}(a_{11}(y)\cos\theta + a_{12}(y)\sin\theta) + r^2\tilde{u}(r,\theta,y).$$

The indicial roots of this problem are $\frac{k}{\beta}$, $k \in \mathbb{Z}$.

Friedrichs extension $\Longrightarrow$ the coefficent $a_{01}(y) \equiv 0$.

Big difference: if $\beta < 1/2$, then $1/\beta > 2$ so we only need to worry about the leading terms.

Regularity of the coefficients a_{ij} in y complicates matters – these lie in a homogeneous weighted Besov space $\dot{B}^s_{\infty,\infty}$, but to deal with this properly one would need to construct and study the Poisson operator.

Donaldson: consider the L^2 Friedrichs extension of the Laplacian and its Green function G. He proved 'by hand' that if we compose G with any two *wedge* derivatives, then the resulting operators

$$\partial \circ G,\ \partial\overline{\partial} \circ G,$$

are bounded on $\mathcal{C}_w^{0,\alpha}$.

In other words, although the 'real' derivatives may give problematic terms, the complex (z and $\bar{z}$) derivatives do not, and this is sufficient to understand issues related to the Laplacian of a Kähler metric, which is built out of these complex derivatives.

This turns out to be enough to handle the entire existence theory when all $\beta_j < 1/2$. (Brendle).

A closer look at G (using the theory of pseudodifferential edge operators) leads to:

Theorem (Jeffres-M-Rubinstein)

For all $\beta < 1$, the 'Riesz potential operators'

$$\frac{\partial}{\partial z_i} \circ G, \frac{\partial}{\partial \overline{z_j}} \circ G, \frac{\partial^2}{\partial z_i \overline{\partial z_j}} \circ G$$

are all bounded on $\mathcal{C}_e^{0,\alpha}$. If $\beta \leq 1/2$, then

$$\frac{\partial^2}{\partial r^2} \circ G, \frac{1}{r}\frac{\partial}{\partial r} \circ G, \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \circ G, \frac{1}{r}\frac{\partial^2}{\partial r \partial y_j} \circ G, \frac{\partial^2}{\partial y_i \partial y_j} \circ G$$

are all bounded on $\mathcal{C}_e^{0,\alpha}$.

$\mathcal{C}^0$ estimates can be handled almost as in the smooth closed case.

$\mathcal{C}^2$ estimate: based on an old inequality due to Chern and Lu, essentially a generalized Schwarz Lemma. This uses an *upper bound* on the bisectional curvature of the initial Kähler metric g and a lower bound on the Ricci curvature of the metric g_s along the continuity path. However, this lower bound is trivial precisely because g_s is a solution to a complex Monge-Ampère equation which states that its Ricci curvature is $s\omega_t - F \geq -C > -\infty$.

Remarkable fact: the initial Kähler edge metric g *does* have an upper bound on its bisectional curvature for all $\beta \leq 1$!

For D smooth this is based on some calculations by C. Li and worked out by Li and Rubinstein. A difficult calculation.

For general D, this is still true, and unfortunately an even more difficult calculation.

$\mathcal{C}_e^{2,\alpha}$ estimate (enter the role of asymptotics!):

It suffices to prove this estimate in Whitney cubes

$$B_{\epsilon,y_0} = \{(r,\theta,y) : \epsilon/2 \le r \le 2\epsilon,\ |y - y_0| \le 2\epsilon,\ \theta \in S^1\},$$

but these edge Hölder spaces are homogeneous with respect to dilation (in r and y), so it is actually enough to prove the estimate in cubes B_{1,y_0}, where it reduces to a now standard *local* version of the Evans-Krylov estimate.

We can now take a limit of solutions in $\mathcal{C}_e^{2,\alpha}$ (the convergence does not take place in this space but the limiting function does lie in this space).

Caveat: the openness argument *fails* for solutions/metrics in these edge spaces – need stronger regularity to make the linear theory work.

The way out: a regularity theorem. The limiting solution $u = u_{s_0}$ solves a complex Monge-Ampère equation. Must prove that it is polyhomogeneous along D, i.e.

$$u \sim \sum_{i,\ell=0}^{\infty} \left(r^{\frac{i}{\beta}+\ell} a_{i\ell 1} \cos(j\beta\theta) + a_{j\ell 2} \sin(j\beta\theta) \right)$$

with all $a_{i\ell j}(y) \in \mathcal{C}^\infty(D)$.

Hence this regularity theorem serves as a crucial intermediary, rather than a cosmetic afterthought, since it is what allows us to cycle back from the closedness to the openness argument.

The form of the expansion leads to some important and useful geometric facts: e.g. D is “totally geodesic” when $\beta < 2/3$.

When D has simple normal crossings:

First, blow up M along each of the D_j: $\widetilde{M} = [M; D_1; D_2; \ldots D_N]$. This is independent of order since the D_j are transverse to one another.

This is a manifold with corners, with coordinates $(r_1, \ldots, r_N, \theta_1, \ldots, \theta_N, y_1, \ldots, y_{2n-2N})$.

Blow up the intersections where any subcollection of the r's vanish, $\{r_{i_1} = \ldots = r_{i_\ell} = 0\}$, $\ell \leq N$. (only necessary on $M \times M$, not on M itself).

Solutions $u(r_1, \ldots, r_N, \theta_1, \ldots, \theta_N, y)$ are not polyhomogeneous in these coordinates, but only in some blown up picture.

The Weil-Peterson metric on moduli space

Let Σ be a compact surface of genus $\gamma \geq 2$.

Consider the Teichmüller space

$$\mathcal{T}_\gamma = \{\text{all conformal structures on } \Sigma\}/\,\mathrm{Diff}_0(\Sigma)$$

and the Riemann moduli space

$$\mathcal{R}_\gamma = \{\text{all conformal structures on } \Sigma\}/\,\mathrm{Diff}(\Sigma)$$

Here $\mathrm{Diff}_0(\Sigma)$ and $\mathrm{Diff}(\Sigma)$ are the groups of all diffeomorphism isotopic to the identity, respectively all diffeomorphisms, of Σ.

Note that

$$\mathcal{R}_\gamma = \mathcal{T}_\gamma / \operatorname{Map}(\Sigma),$$

where

$$\operatorname{Map}(\Sigma) = \operatorname{Diff}(\Sigma)/\operatorname{Diff}_0(\Sigma)$$

is the discrete mapping class group.

It is classical that $\mathcal{T}_\gamma \cong \mathbb{R}^{6\gamma-6}$. The picture is reminiscent of a hyperbolic quotient $\mathbb{H}^n/\Gamma$.

Moduli space $\mathcal{R}_\gamma$ has some interior orbifold singularities (we regard these as “trivial” and neglect them)

We can realize the elements of $\mathcal{T}_\gamma$ and $\mathcal{R}_\gamma$ by hyperbolic metrics g on Σ, modulo the relevant class of diffeomorphisms.

Natural metric on $\mathcal{T}_\gamma$:

For $\kappa_1, \kappa_2 \in T_g\mathcal{T}_\gamma$,

$$g_{WP}(\kappa_1, \kappa_2) = \int_\Sigma \langle \kappa_1, \kappa_2 \rangle_g \, dA_g$$

It was realized long ago that $(\mathcal{T}_\gamma, g_{WP})$ is not complete: there are paths leaving every compact set with finite length.

On the other hand, g_{WP} is invariant under Map(Σ), hence descends to $\mathcal{R}_\gamma$.

It turns out that $(\mathcal{R}_\gamma, g_{WP})$ has finite diameter. Consider its compactification

$$\overline{\mathcal{R}_\gamma}^{\mathrm{WP}}.$$

Remarkable fact: this compactification is identified with the Deligne-Mumford compactification $\overline{\mathcal{R}_\gamma}^{\mathrm{DM}}$, defined through algebraic geometry:

$$\overline{\mathcal{R}_\gamma}^{\mathrm{WP}} \cong \overline{\mathcal{R}_\gamma}^{\mathrm{DM}}$$

The singular set

$$\mathrm{Sing}(\overline{\mathcal{R}_\gamma}^{\mathrm{WP}}) = \overline{\mathcal{R}_\gamma}^{\mathrm{WP}} \setminus \mathcal{R}_\gamma$$

(again, forgetting the orbifold points) is a union of immersed divisors $D_1 \cup \ldots \cup D_N$.

Simple geometric description: choose a maximal collection of simple closed curves $c_1, \ldots, c_{N'}$ dividing Σ into pairs of pants; choose $c_1, \ldots, c_N$ amongst these which are not identified by mapping class group.

If g is a hyperbolic metric, let $\ell_j(g)$ denote the length of the unique geodesic on (Σ, g) which is freely homotopic to c_j.

Fact: The sequence $[g_k]$ tends to infinity in $\mathcal{R}_\gamma$ if and only if at least some $\ell_j(g_k) \to 0$ as (some subsequence of) $k \to \infty$.

The double-points of D_j correspond to when two curves in the original collection which are identified under $\mathrm{Map}(\Sigma)$ have their lengths tending to 0.

The limits of these sequences correspond to noded surfaces, which carry complete finite area hyperbolic metrics, thus represent elements in unions of $\mathcal{R}_{\gamma',n'}$ (the n' corresponds to the number of punctures).

Near a given singular divisor D away from intersections, one has coordinates: length ℓ, twist (rotation) θ, and some coordinate system y on D itself (interior coordinates on the lower dimensional moduli space).

In terms of these coordinates:

$$g_{WP} \sim \frac{d\ell^2}{\ell} + \ell^3 d\theta^2 + dy^2 \quad \text{with appropriate constant factors}$$

This quasi-isometric form was first determined by Masur ('70's).

Later work: Wolpert ('90's), Yamada ('01). Improved from quasi-isometry to the beginnings of an expansion.

Also, Liu-Sun-Yau ('04), computed approximately 4 terms in expansion (actually, they obtained certain estimates on the curvature of the Ricci metric, so four derivatives of g_{WP} itself).

Near the crossing $D_K := D_1 \cap \ldots \cap D_k$, have length functions $\ell_1, \ldots, \ell_k$, as well as $\theta_1, \ldots, \theta_k$, and

$$g_{WP} \sim \sum_{j=1}^{k} \left(\frac{d\ell_j^2}{\ell_j} + \ell_j^3\right) + dy^2,$$

where y is a coordinate system on D_K.

Notable feature: the factors are asymptotically orthogonal at the $D_i \cap D_j$ (the curves c_i and c_j are a long distance apart!).

A slightly different form: use $r_j = \sqrt{\ell_j}$. Then

$$g_{WP} \sim \sum_{j=1}^{k} (dr_j^2 + r_j^6) + dy^2,$$

This has "crossing cusp-edges".

Unbounded sectional curvatures, etc.

Theorem (M-Swoboda)

The Weil-Peterson metric has a complete asymptotic expansion as all $r_j \searrow 0$, with all exponents in $\mathbb{N}$ (and possibly some log terms). Coefficients are "in principle" computable.

Our proof uses a global analysis formulation instead of the more traditional complex analytic formulation (holomorphic quadratic differentials, etc.). However, most of it is classical and could have been done long ago.

The difficulty: g_{WP} does not satisfy an equation (e.g. Monge-Ampere), so PDE enters only at a secondary level.

Before describing proof, here are some applications:

Consider Δ_{WP}, the Laplacian on functions on $\mathcal{R}_\gamma$ with respect to g_{WP}.

Theorem (Ji-M-Müller-Vasy)

Δ_{WP} is essentially self-adjoint, with discrete spectrum satisfying a Weyl law.

This allows one to study local and global spectral invariants.

1) Standard Weyl asymptotics: $N(\lambda) \sim c_n \mathrm{Vol}(\mathcal{R}_\gamma)\lambda^{3\gamma-3}$.

2) The heat trace $\mathrm{Tr}\, e^{-t\Delta_{WP}}$ has complete expansion as $t \searrow 0$ (Gell-Redman).
(Indeed, he constructs the entire heat kernel as a 'polyhomogeneous object' using methods of geometric microlocal analysis.)

3) $\det \Delta_{WP}$ makes sense.

Still to do: Obstructions to essential self-adjointness for Hodge Laplacian or Dirac operator?; signature theorem? (Gauss-Bonnet was accomplished by Liu-Sun-Yau); indices of other Dirac operators, etc.

Some ideas in proof of essential self-adjointness:

Define $\mathrm{Dom}_{\min}(\Delta_{WP})$, the graph closure of Δ_{WP} in L^2 over the core domain $\mathcal{C}_0^\infty((\mathcal{R}_\gamma)_{\mathrm{reg}})$, and $\mathrm{Dom}_{\max}(\Delta_{WP})$, the space of all $u \in L^2$ such that $\Delta_{WP} u \in L^2$.

Must show that these domains are equal. It suffices to check that if $u \in \mathrm{Dom}_{\max}(\Delta_{WP})$, then u can be approximated in graph norm by $u_j \in \mathcal{C}_0^\infty$, and for this it is enough to check that u has some a priori decay, $u \in (r_1 \dots r_N)^\eta L^2$, $\eta > 0$.

This is done by deriving a priori bounds for $||w_\delta(r)u||_{L^2}$ where

$$w_\delta(r) = \prod (\delta + r_j)^{-\eta}$$

using a chain of inequalities (including Hardy inequality with precise constant).

Amusing feature: this proof is borderline; works more easily for metrics of the form $dr^2 + r^{2k}d\theta^2 + g_D$ for $k > 3$; still works for $k = 3$ (Weil-Peterson case), but approximation is weaker.

Analogous to the fact that essential self-adjointness for scalar Laplacian on $\mathcal{C}_0^\infty(\mathbb{R}^n)$ holds for $n \geq 4$ but borderline for $n = 4$. (Same radial part).

Ideas in proof of asymptotics of g_{WP}:

First, identify $T_g\mathcal{R}_\gamma$ with

$$S_{\mathrm{TT}}(g) = \{\kappa \in S^2(T^*\Sigma) : \mathrm{tr}^g\kappa = 0,\ \delta^g\kappa = 0\}.$$

the transverse-traceless tensors. This is orthogonal to the Diff(Σ) orbit in Met^{-1}, the space of *all* hyperbolic metrics. (The gauge condition).

Consider a simple loop c with $\ell_g(c) \to 0$. Model: hyperbolic collar,

$$g_{\mathrm{model}} = ds^2 + \ell^2 \cosh^2 s \, d\phi^2$$

(on Σ), $|s| \leq |\log \ell|$.

$\dot{g} = 2\ell \cosh^2 s \, d\phi^2$. Need to put in gauge.

$$\kappa = \dot{g} - \delta^* \omega - \frac{1}{2}\mathrm{tr}(\dot{g} - \delta^* \omega) g$$

where

$$(\nabla^* \nabla - \mathrm{Ric})\omega = (\Delta + 2)\omega = \delta^g \dot{g}.$$

Eventually, one needs to find a uniform inverse for $P_\ell = \Delta_\ell + 2$ which is polyhomogeneous as $\ell \searrow 0$.

One could do this directly (separation of variables, etc.) for model cylinder, and then glue together with parametrix on the rest of Σ (still need to do work to make sure that true inverse is polyhomogeneous).

One technical tool: harmonic map parametrization by Mike Wolf. Let (Σ_0, g_0) be the noded surface (with $\ell = 0$). This is complete, noncompact, with two cusp ends. Wolf produces a *real analytic* parametrization of a neighbourhood of g_0 in $\overline{\mathcal{R}_\gamma}$ using infinite energy harmonic maps

$$U_g : (\Sigma_0 . g_0) \longrightarrow (\Sigma, g).$$

We use this to pull all computations back to the fixed surface Σ_0.

The other difficulty is that $L^2 - S_{\mathrm{TT}}(g_0) = 0$, so the projector onto $S_{\mathrm{TT}}(g)$ disappears in limit. (It has a perfectly nice limit on the infinite model cylinder $ds^2 + \cosh^2 s\, d\phi^2$, $s \in \mathbb{R}$, which is the front face of the natural blowup.)

Poincaré-Einstein metrics and renormalized volume

Final vignette: Let (M, g) be a Poincaré-Einstein manifold. This means:

i) M^n is the interior of a smooth compact manifold with boundary.

ii) $g = \rho^{-2}\overline{g}$ where $\overline{g}$ is a smooth metric on $\overline{M}$ and ρ is a defining function for $\partial\overline{M}$ ($\rho = 0$ precisely on this boundary and $d\rho \neq 0$ there).

iii) $\mathrm{Ric}(g) = -(n-1)g$.

Basic (string) action

$$\mathrm{Ren} - \mathrm{Vol}(M, g) = \underset{\epsilon=0}{\mathrm{F.P.}} \mathrm{Vol}\,(M \cap \{x \geq \epsilon\})$$

To make sense of this:

Associated to g is its conformal infinity,
$c(g) = [\rho^2 g|_{T\partial M}] = [\overline{g}|_{T\partial M}]$, which is a conformal class on ∂M.

(Graham-Lee). To any choice of representative $h_0 \in c(g)$, there is a unique defining function x such that

$$g = \frac{dx^2 + h(x)}{x^2}, \quad h(x) \sim h_0 + xh_1 + \dots$$

In fact (Fefferman-Graham), when n is even, the expansion of h has no odd terms $x^j h_j$ for $j \leq n-2$, and $h_{n-1} \in S_{\mathrm{TT}}(h_0)$. (For n odd, there are no odd terms for $j \leq n-1$, but then $x^n \log x \tilde{h}_n$, etc.). (Vindicated by Chrusciel-Delay-Lee-Skinner).

To define the renormalized volume we use one of these *special* boundary defining functions.

Theorem (Henningson-Skenderis, Graham-Witten)

This definition is well-defined when n even. (When n odd, there is a conformal anomaly.)

Can define renormalized volume for all conformally compact ($g = \rho^{-2}\overline{g}$) metrics provided the same evenness condition holds (for all special boundary defining functions).

Basic question: Let n be even and (M^n, g_0), n conformally compact and even up to order $n-2$ with special properties of h_{n-1}. Let $g(t)$ be solution of Ricci flow with initial condition g_0. How does renormalized volume evolve under flow?

To do this, we must first establish that the class of polyhomogeneous conformally compact metrics is invariant under Ricci flow (accomplished by Bahuaud '11).

Simple case: suppose (M, g_0) is Poincaré-Einstein through order n, i.e. $E(g_0) = \mathrm{Ric}(g_0) + (n-1)g_0$ is $\mathcal{O}(x^{n+1})$. (Qing-Shi-Wu). Method: compute evolution of E and use maximum principle.

Theorem (Bahuaud-M-Woolgar)

If (M^n, g_0) is polyhomogeneous and even to order n, then the solution $g(t)$ to Ricci flow has the same property for $t < T$ (time of existence of flow). The renormalized volume $\mathrm{Ren} - \mathrm{Vol}(g(t))$ is monotone in t.

Thus

$$g(t) = \frac{1}{x_t^2}\Big(dx_t^2 + h_0 + x_t^2 h_2(t) + \ldots + x_t^{n-2} h_{n-2}(t) + x_t^{n-1} h_{n-1}(t) + \ldots\Big)$$

The difficult part is to show that if the initial metric is even to order m then $g(t)$ is even to order m. We first do this for the gauged Ricci-de Turck flow and then show that the diffeomorphism which takes the solution of the gauged flow to that for the ungauged flow preserves this property.

This relies on linear theory, specifically the existence of a (filtered) even subcalculus of the heat calculus; one must show that the heat kernel for the Lichnerowicz Laplacian of an even (to order m) metric lies in this even subcalculus, etc.

A further trick: "test" against the solution of the Ricci flow $\tilde{g}(t)$ which has the same h_0 and which is asymptotically P-E.

Happy Birthday Jean-Michel!