Kähler random walk & Lyapunov exponents.

Simplest setup:

C =  = $\overline{C}$ − S

smooth algebraic curve $/\mathbb{C}$ — projective curve — finite set

$(\mathcal{E}, \nabla)$ vector bundle $/C$ with flat connection $\Longleftrightarrow$ representation $\rho : \pi_1(C, c_0) \to GL(N, \mathbb{C})$

$N = \operatorname{rank} \mathcal{E}$

Assumption: $\forall\ s \in S$

Spec (monodromy ) $\subset \{ z \in \mathbb{C}^\times : |z| = 1 \}$

$\rightsquigarrow$ N real numbers $\lambda_1 \geq \cdots \geq \lambda_N$ — Lyapunov exponents of $C, (\mathcal{E}, \nabla)$.

<u>Definition</u>: 1) Pick $\forall$ Kähler metric g on C with area $= 1$

2) Pick norm $\|\cdot\|_x$ on $\mathcal{E}_x$ bounded at cusps

3) Pick $\forall$ initial point $c_0 \in C$

Consider random walk starting from c_0

For $t > 0$ get a holonomy matrix

$$H_t : \mathcal{E}_{c_0} \longrightarrow \mathcal{E}_{c_t}$$

singular values $s_1(t) \geq \dots \geq s_N(t) > 0$ $\quad \mathrm{Spec}\left(H_t^* H_t\right)^{1/2}$

$$\boxed{\lambda_i := \lim_{t \to +\infty} \frac{\log s_i(t)}{t}}$$

almost everywhere.

Existence of lim: multiplicative ergodic theorem (Oseledets, '68)

Assumption $\mathrm{Spec}((\circlearrowleft)_s) \subset U(1) \iff L^1$-condition

Independence on the choice of $(\|\cdot\|_x)_{x \in C}$: easy

Independence on the choice of Kähler metric:

$\Leftarrow$ Conformal invariance of (non)-parametrised random walk in 2d,

Martingales = harmonic functions. $\partial\bar\partial f = 0$.

Fact. $\underbrace{\lambda_1 + \dots + \lambda_N}_{\text{Lyapunov exponent for } \underset{\text{rank}=1}{\Lambda^N \mathcal{E}}, \nabla} = 0$

vanishing $\Leftarrow$ random walk is symmetric

$t \leftrightarrow -t$

Generalization:

Replace $\mathbb{C}$ $\rightsquigarrow$ (M, α) $\quad$ M: complex manifold $\dim_{\mathbb{C}} M = n$

$$\alpha \in \Gamma(M, \Omega^{n-1,n-1} \otimes_{C^\infty} C^{-\infty})$$

$$\alpha \geq 0 \quad \partial\alpha = \bar{\partial}\alpha = 0.$$

non-negative closed current

(e.g. $M = \mathbb{C}$ $\quad \alpha = 1$)

(M, α), $(\mathcal{E}, \nabla)$ under some L^1-conditions on α, $(\mathcal{E}, \nabla)$ at infinity

flat bundle

$\rightsquigarrow$ Lyapunov exponents.

Procedure: pick $\forall$ C^∞-density $\mu > 0$ $\quad \int_M \mu = 1$

generator of random walk

$$\Delta_{\alpha,\mu}(f) := \frac{i\partial\bar{\partial}f \wedge \alpha}{\mu}$$

apply multiplicative ergodic theorem.

Special case: $(\mathcal{E}, \nabla)$ variation of polarized Hodge structures of weight $\underline{1}$.

monodromy $\subset U(a,b) \subset GL(N,\mathbb{C})$
$N = a+b$.

holomorphic subbundle $\mathcal{E}^{1,0} \subset \mathcal{E}$ (not flat)

$\mathrm{rk}\, \mathcal{E}^{1,0} = a$ pseudohermitean form $|_{\mathcal{E}^{1,0}} > 0$.

$$\underbrace{\lambda_1 \geq \dots \geq \lambda_a}\geq 0 \geq \underbrace{\lambda_{a+1} \geq \dots \geq \lambda_{a+b}}$$

Theorem (essentially MK '96) G. Forni, ...

$$\sum_1^a \lambda_i = 2\int_C c_1\text{-form}(\mathcal{E}^{1,0})$$

Basic idea of the proof: special case $N = 2a$ monodromy $\subset Sp(2a, \mathbb{R})$

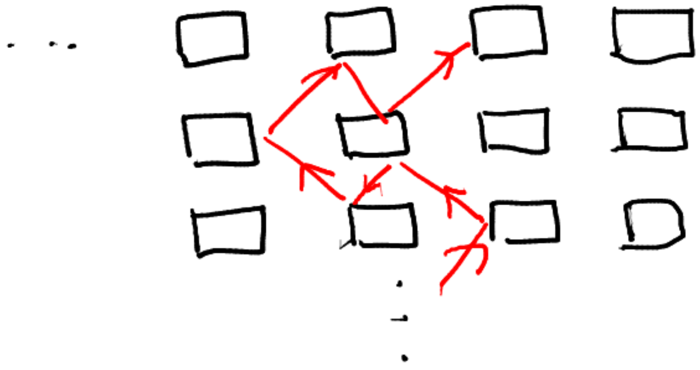
$\forall$ Real Lagrangian $L \subset \mathbb{C}^a \simeq \mathbb{R}^{2a}$

$$\log \|l_1 \wedge \dots \wedge l_a\|^2 = \underbrace{\log |\langle l_1 \wedge \dots \wedge l_a, \Omega^{a,0} \rangle|^2}_{\text{harmonic}} - \underbrace{\log |\Omega^{a,0} \wedge \bar{\Omega}^{a,0}|}_{\text{independent on } l_1 \dots l_a}$$

Applications

1° (Original motivation): Teichmüller dynamics

$\mathcal{M}$ foliated by $\mathbb{C}$ curves
transversal measure α



Time $T \to +\infty$

distance $\sim T^{1/3}$

(first observed by A. Zorich ~ 1994–1995)
in computer experiments

2° M. Möller 1207.5433.

$SU(2,1) >$ non-arithmetic subgroups of finite covolume (P. Deligne – G. Mostow)

as monodromy of hypergeom. equations.

Some of these groups ? commensurable or not?

Answer: no : $\quad$ Ball $\subset \mathbb{C}^2$ $\quad \boxed{M = \mathrm{Ball}/\Gamma}$ $\quad \rho : \Gamma \to SU(2,1) \cap SL(3, K)$

$[K:\mathbb{Q}] < \infty$

choose different $K \hookrightarrow \mathbb{C}$.

$\dashrightarrow \quad 5/17 \neq 7/22 \ldots$ $\quad$ Lyapunov exponents.

New observations

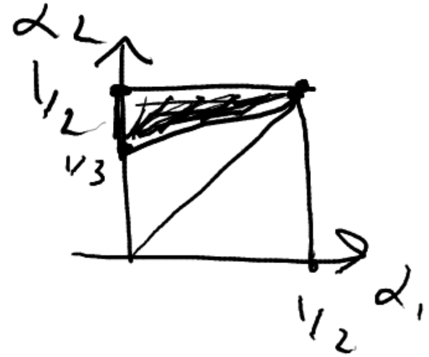
variations of Hodge structures of weight $= 3$ over $\mathbb{CP}^1 - 3$ pts.

().

$$T = \begin{pmatrix} 1 & & & 0 \\ 1 & 1 & & \\ 1/2 & 1 & 1 & \\ 1/6 & 1/2 & 1 & 1 \end{pmatrix} \qquad S = \begin{pmatrix} 1 & * & 0 & * \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ 0 & & & 1 \end{pmatrix}$$

$$\operatorname{Spec} TS = \exp\left(\pm 2\pi i\, \alpha_1, \pm 2\pi i\, \alpha_2\right)$$

$$0 \le \alpha_1 \le \alpha_2 \le \frac{1}{2}$$



Claim if $3\alpha_2 \ge \alpha_1 + 1$

then $\lambda_1 + \lambda_2 = 2\underbrace{(\alpha_1 + \alpha_2)}_{= \int c_1\text{-form of } \mathcal{E}^{3,0} \oplus \mathcal{E}^{2,1}}$

otherwise $\boxed{\lambda_1 + \lambda_2 > 2(\alpha_1 + \alpha_2)}$

Initially I've checked it in algebro-geometric cases 14 $= \underset{\text{good}}{7} + \underset{\text{bad}}{7}$

quinti case

(mirror dual to complete intersections $3CY \subset \mathbb{P}^{d_1 \ldots d_r}$)

Coincidence (?):

good cases $\Longleftrightarrow$ "thin" groups (∞ index in a lattice)

bad cases $\Longleftrightarrow$ arithmetic subgroup.

One can try to repeat the argument

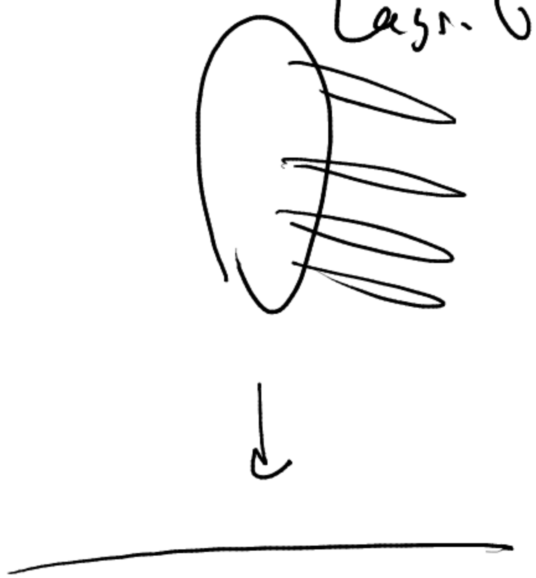
Breaks down if real Lagrangian

$$L \subset \mathbb{R}^4$$

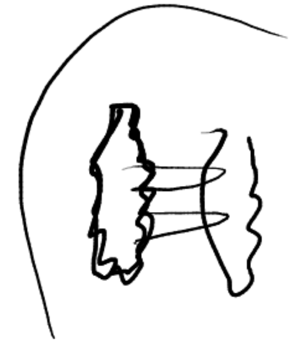
is <u>not</u> transversal to $\underbrace{\mathcal{E}^{3,0} \oplus \mathcal{E}^{2,1}}_{\text{middle term of Hodge filtration}}$.

<u>Heuristics</u>:

Lagr. Grassmannian $(\mathbb{R}^4)$



2-dim. $\supset$ foliation



$\mathbb{C}P^1$ $-$ 3 pts

fractal S^1s covariantly constant = support of random walk on the total space $\tilde{\mathcal{M}}$

$\mathrm{Supp}(\tilde{\mu})$ everywhere transversal to $E^{3,0} \oplus E^{2,1}$

Quintic case:

$$\psi_0(z) = \sum_{n\geq 0} \frac{(5n)!}{(n!)^5} z^n$$

$$\psi_1(z) = \psi_0(z)\log z + \sum_{n\geq 1} \frac{(5n)!}{(n!)^5}\left(\sum_{k=n+1}^{5n} \frac{1}{k}\right) z^k$$

$$\vdots$$

$$F(z) = \det\begin{pmatrix} \psi_0 & \psi_0' \\ \psi_1 & \psi_1' \end{pmatrix} \in \mathbb{Z}[[z]]$$

<u>Claim</u>: $F \neq 0$ on $\overbrace{\mathbb{CP}^1 - \{0, \frac{1}{5^5}, \infty\}}^{\text{universal cover}}$

$$q \rightarrow \log\left(\frac{z}{5^5} F\left(\frac{z}{5^5}\right)\right)\Big|_{z=\lambda(q)}$$

has <u>bounded</u> Taylor coefficients.

$$\lambda(q) = q\left(\frac{\sum q^{n^2+n}}{\sum q^{n^2}}\right)^4$$

$\lambda: 0 < |q| < 1$

$\rightarrow \mathbb{CP}^1 - \{0,1,\infty\}$

More general:

$$\begin{array}{c} (\mathcal{F}, \nabla) \\ \downarrow \\ C \end{array}$$

variation of Hodge structures, irreducible

$$\mathcal{F}^\vee \supset \underline{(\mathcal{F}^\vee)^{n,0} \simeq \mathbb{C}^1}$$

"Calabi-Yau".

Assume $\exists$ flat multivalued section σ of $\mathcal{F}$

$$\langle \sigma, (\mathcal{F}^\vee)^{n,0} \rangle \neq 0 \quad \text{everywhere}$$

$\Longrightarrow$ The leading Lyapunov exponent

$$\lambda_1((\mathcal{F}, \nabla)) = 2 \int_C c_1\text{-form of } (\mathcal{F}^\vee)^{n,0},$$

In all examples $\mathcal{F} = \Lambda^a(\underset{\mathbb{C}^{a+1}}{\overset{\shortparallel}{\mathcal{E}}})$

Lie-theoretic reasons for $\neq 0$:

G^{comp} compact Lie group

$G_{\mathbb{C}}$ complex reductive group.

Hodge data: $\widetilde{U(1)} \longrightarrow G^{comp}$

2:1 cover

$\sqrt{1} \longmapsto$ central element

$$\cdots \quad \mathbb{C} \otimes \mathfrak{g} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}^n$$

$$\mathfrak{g} = \mathfrak{g}^{even} \oplus \mathfrak{g}^{odd}$$

$$\dashrightarrow \quad \tilde{\mathfrak{g}}_{\mathbb{R}} = \mathfrak{g}^{even} \oplus \sqrt{-1}\, \mathfrak{g}^{odd}$$

Lie algebra of a real form $G_{\mathbb{R}}$.

<u>Question</u>: When $\exists$ $\mathbb{C}$-repre. V of $G_{\mathbb{C}}$ + highest vector v^* in V^* + vector $v \in V - 0$

$(v^*, G_{\mathbb{R}} \cdot v) \neq 0$ everywhere?

Original example $\quad G_{\mathbb{R}} = Sp(2n, \mathbb{R})$

$$V = \Lambda^n_{prim}(\mathbb{C}^{2n})$$

There are other examples related to Shimura varieties of exceptional type. $E_8, \ldots$