Higher analytic stacks, twisted complexes of holomorphic vector bundles, and the definition of the determinant

Ezra Getzler
Northwestern University

Happy Birthday, Jean-Michel!

"One could say that mathematics is the music of the mind"

The determinant

Following Grothendieck, Knudsen and Mumford showed in 1976 that the determinant line bundle of complexes of vector bundles on a projective variety X is natural with respect to quasi-isomorphisms. In particular, the determinant is well-defined in the derived category.

Goals of this talk

- Construct the n-stack of deformations of a complex of holomorphic vector bundles of length n on a compact complex manifold X.
- Extend Kuranishi's construction of the analytic stack of deformations of a vector bundle to complexes of vector bundles.
- Define the determinant on this n-stack.

This is a joint project with Kai Behrend.

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Banach algebras and Lie groups

First, we must explain what an n-stack is.

The open subset $G(A)$ of invertible elements of a Banach algebra A is a Lie group. A Lie group is an example of a 1-stack (more or less the same thing as a Lie groupoid).

When A^* is a differential graded Banach algebra, what replaces $G(A)$?

We will associate an analytic n-stack (Lie n-groupoid) to a differential graded Banach algebra A^* concentrated in degrees $(-n, \infty)$. In fact, we will represent this n-stack by its nerve $N_\bullet A$. This is a simplicial Banach analytic space.

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The nerve of a group

If n is a natural number, let $[n]$ be the category whose objects are the natural numbers $\{0, \ldots, n\}$, with a single morphism from i to j if $i \leq j$.

$$0 \longrightarrow 1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n$$

The **nerve** of a group G is the simplicial set whose n-simplices are the functors from $[n]$ to G (thought of as a category with a single object).

This set is denoted $N_n G$. In fact, $N_n G \cong G^n$. We have $N_0 G$ is the identity element, and $N_1 G$ is the set of elements of G.

This representation is more finite than one might fear: it is not hard to see that one may reconstruct the group from the 2-skeleton of its nerve.

We may also define the nerve of a groupoid (or of any small category): in this case, $N_0 G$ is the set of objects of the groupoid.

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Horns

The n-simplex Δ^n is the simplicial set whose m-simplices are functors from $[m]$ to $[n]$. It has a single non-degenerate n-simplex (corresponding to the identity map from $[n]$ to itself), and all of its non-degenerate simplices are faces of this one.

In particular, we have the ith face $\partial_i \Delta^n$, which is the $(n-1)$-simplex opposite the ith vertex: its geometric realization is the convex hull of the vertices $\{0, \ldots, \hat{\imath}, \ldots, n\}$.

The horn $\Lambda_i^n \subset \Delta^n$ is the union of all of the faces of the n-simplex that contain the ith vertex:

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$$\Lambda_i^n = \bigcup_{j \neq i} \partial_j \Delta^n.$$

Let $X_\bullet$ be a simplicial set. For each $0 \leq i \leq n$, there is a natural map

$$\lambda_{n,i}(X_\bullet) : X_n \to \mathrm{Hom}(\Lambda^n_i, X_\bullet)$$

from the n-simplices of the simplicial set to its horns.

For example, $\lambda_{1,0}$ and $\lambda_{1,1}$ take a 1-simplex to its **source** and **target**.

Theorem (Grothendieck)

A simplicial set $X_\bullet$ is the nerve of a **groupoid** *if and only if the maps*

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are bijections for $n > 1$. It is the nerve of a **group** *if, in addition, $X_0 \cong *$.*

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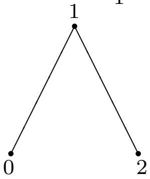
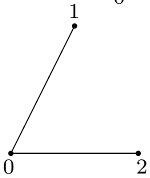
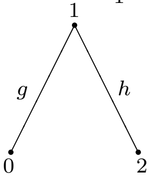


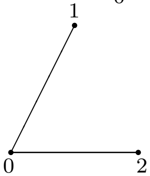
The horn Λ^2_0



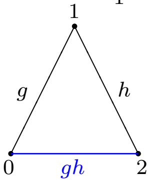
The horn Λ_1^2



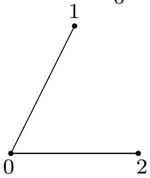
The horn Λ_0^2



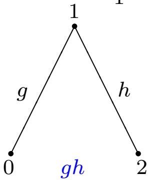
The horn Λ^2_1



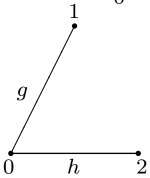
The horn Λ^2_0



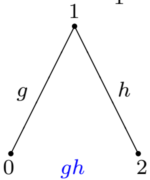
The horn Λ^2_1



The horn Λ^2_0



The horn Λ^2_1



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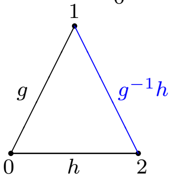

Several important notions from geometry and algebra are obtained by tweaking this theorem. For example, $X_\bullet$ is the nerve of a **category** if and only if the maps $\lambda_{n,i}(X_\bullet)$ are bijections for $0 < i < n$.

More relevant to this talk: a simplicial manifold $X_\bullet$ is the nerve of a **Lie groupoid** if and only if the maps $\lambda_{n,i}(X_\bullet)$ are diffeomorphisms for $n > 1$, and surjective submersions for $n = 1$.

The following definition is due to Duskin, in the discrete case, and Henriques, in the smooth case.

Definition

- A k-**groupoid** is a simplicial set such that

$$\lambda_{n,i}(X_\bullet) : X_n \to \mathrm{Hom}(\Lambda^n_i, X_\bullet)$$

  is a surjection for $n > 0$, and a bijection for $n > k$.
- A **Lie** k-**groupoid** is a simplicial Banach analytic space such that $\lambda_{n,i}(X_\bullet)$ is a surjective submersion for $n > 0$, and an isomorphism for $n > k$.

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The Maurer-Cartan set of a differential graded algebra

A Maurer-Cartan element of a differential graded algebra A^* is an element $\mu \in A^1$ of degree 1 satisfying the equation

$$\delta\mu + \mu^2 = 0.$$

Of course, this equation is familiar from the theory of connections on vector bundles: it is the equation for a connection to be flat.

Let $\delta_\mu : A^* \to A^{*+1}$ be the operator $\delta + [\mu, -]$. Then $\delta_\mu^2 = 0$. Maurer-Cartan elements of A^* correspond to deformations of the differential δ.

Definition

- If A^* is a differential graded algebra, $\mathrm{MC}(A)$ is the set of Maurer-Cartan elements.
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Examples of differential graded algebras

Let X be a topological space, with open cover $\mathcal{U} = \{U_i \subset X\}_{i \in I}$.

Let $\mathcal{A}^*$ be a sheaf of differential graded algebras over X. The normalized Čech complex of $\mathcal{A}$ is the graded vector space

$$\check{C}^k(\mathcal{U}, \mathcal{A}) = \bigoplus_{q=0}^{k} \bigoplus_{\substack{i_0,\ldots,i_q \in I \\ i_{j-1} \neq i_j \text{ for } 1 \leq j \leq q}} \Gamma(U_{i_0} \cap \cdots \cap U_{i_q}, \mathcal{A}^{k-q}).$$

The differential is

$$(da)_{i_0 \ldots i_q} = \delta a_{i_0 \ldots i_q} + \sum_{j=0}^{q} (-1)^{q-j}\, a_{i_0 \ldots \hat{\imath}_j \ldots i_q}|_{U_{i_0} \cap \cdots \cap U_{i_q}}$$

and the product is

$$(a \cup b)_{i_0 \ldots i_q} = \sum_{p=0}^{q} (-1)^{pq}\, a_{i_0 \ldots i_p}|_{U_{i_0} \cap \cdots \cap U_{i_p}} \cdot b_{i_p \ldots i_k}|_{U_{i_0} \cap \cdots \cap U_{i_k}}.$$

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The fat simplex

The special case where X is the geometric n-simplex

$$\Delta^n = \{(t_0, \dots, t_n) \in [0,1]^{n+1} \mid t_0 + \cdots + t_n = 1\}$$

covered by the open subsets $U_i = \{t_i > 0\}$ (the complements of the faces) plays a special role in this talk.

Let $\mathbb{\Delta}^n$ be the **fat simplex**

$$\mathbb{\Delta}^n = \operatorname{cosk}_0 \Delta^n.$$

It is the nerve of the groupoid whose objects are the vertices of Δ^n, with a single isomorphism between any two vertices.

The fat interval $\mathbb{\Delta}^1$ is sometimes written J. Its geometric realization is the sphere S^∞: it has two non-degenerate simplices in each dimension $(0, 1, 0, \dots)$ and $(1, 0, 1, \dots)$.

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The fat interval $\mathbb{\Delta}^1$ is sometimes written J. Its geometric realization is the sphere S^∞: it has two non-degenerate simplices in each dimension $(0, 1, 0, \dots)$ and $(1, 0, 1, \dots)$.

The fat simplex

The special case where X is the geometric n-simplex

$$\Delta^n = \{(t_0, \dots, t_n) \in [0,1]^{n+1} \mid t_0 + \cdots + t_n = 1\}$$

covered by the open subsets $U_i = \{t_i > 0\}$ (the complements of the faces) plays a special role in this talk.

Let $\mathbb{A}^n$ be the **fat simplex**

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The nerve of a differential graded algebra

The Čech complex $\check{C}^*(\{U_0, \dots, U_n\}, A)$ is isomorphic to $C^*(\mathbb{\Delta}^n, A)$. We have

$$C^k(\mathbb{\Delta}^n, A) = \bigoplus_{q=0}^{k} \bigoplus_{\substack{0\le i_0,\dots,i_q<n \\ i_{j-1}\neq i_j \text{ for } 1\le j\le q}} A^{k-q}.$$

Definition

The **nerve** $N_n A$ of a differential graded algebra A^* is the simplicial set

$$N_n A = \mathrm{MC}(C^*(\mathbb{\Delta}^n, A)).$$

More explicitly, an n-simplex in $N_n A$ is a collection

$$\mu = (a_{i_0\dots i_k} \in A^{1-k} \mid 0 \le i_0, \dots, i_k \le n \text{ and } i_j \neq i_{j+1}),$$

satisfying the equations

$$\delta a_{i_0\dots i_k} + \sum_{j=0}^{k} (-1)^{k-j}\, a_{i_0\dots\widehat{\imath}_j\dots i_k} + \sum_{j=0}^{k} (-1)^{jk}\, a_{i_0\dots i_j} a_{i_j\dots i_k} = 0.$$

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0-simplices and 1-simplices in the nerve

A 0-simplex in $N_\bullet A$ is a Maurer-Cartan element of A^*:

$$N_0 A \cong \mathrm{MC}(A).$$

Given a 1-simplex $(a_{i_0\dots i_k}) \in \mathrm{MC}(C^*(\Delta^1, A))$, the elements $\mu = a_0$ and $\nu = a_1$ are Maurer-Cartan elements.

A **morphism** $f : \mu \to \nu$ between Maurer-Cartan elements $\mu, \nu \in \mathrm{MC}(A)$ is an element $f \in A^0$ satisfying

$$\delta_\nu \cdot f = f \cdot \delta_\mu.$$

The elements $f = 1 + a_{01}$ and $g = 1 + a_{10}$ associated to a 1-simplex define morphisms $f : \mu \to \nu$ and $g : \nu \to \mu$.

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Quasi-isomorphisms

A **quasi-isomorphism** $f : \mu \to \nu$ is a morphism such that there exists a morphism $g : \nu \to \mu$ and homotopies $h, k \in A^{-1}$ satisfying the equations

$$\delta_\mu h = 1 - gf \qquad \delta_\nu k = 1 - fg.$$

The morphisms f and g associated to a 1-simplex are quasi-inverse to each other: take $h = a_{010}$ and $k = a_{101}$.

Theorem

A morphism $f : \mu \to \nu$ is a quasi-isomorphism if and only if there is a 1-simplex $(a_{i_0 \dots i_k}) \in \mathrm{MC}(C^*(\Delta^1, A))$ with $\mu = a_0$, $\nu = a_1$, and $f = 1 + a_{01}$.

When A^* is a differential graded Banach algebra, the set of quasi-isomorphisms is an **open** subset of $\mathrm{MC}(A) \times \mathrm{MC}(A) \times A^0$, generalizing the corresponding statement for invertible elements of a Banach algebra.

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The nerve when $A^i = 0$, $i < 0$

If $A^i = 0$ for $i < 0$, the element $f \in A^0$ associated to a 1-simplex is a unit, with inverse g. This proves the following theorem.

Theorem

If $A^i = 0$, $i < 0$, then $N_\bullet A$ is the nerve of the **Deligne groupoid**, *associated to the action of the group $G(A) = \{f \in A^0 \mid f$ is invertible$\}$ on the Maurer-Cartan set $\mathrm{MC}(A)$:*

$$N_n A \cong \mathrm{MC}(A) \times G(A)^n.$$

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The stack of vector bundles

Consider the differential graded algebra of Čech cochains

$$A^* = \check{C}^*(\mathcal{U}, \mathrm{End}(\mathcal{O}^N)),$$

where $\mathcal{U}$ is a Stein cover of a complex manifold X.

The Maurer-Cartan elements of A^* are the 1-cocycles, i.e. vector bundles on X of rank N:

$$\mathrm{MC}(\check{C}^*(\mathcal{U}, \mathrm{End}(\mathcal{O}^N))) \cong \check{Z}^1(\mathcal{U}, \mathrm{GL}(\mathcal{O}, N)).$$

The group $G(\check{C}^*(\mathcal{U}, \mathrm{End}(\mathcal{O}^N)))$ of units is the group $\check{C}^0(\mathcal{U}, \mathrm{GL}(\mathcal{O}, N))$ of gauge transformations. We recover Kodaira and Spencer's moduli stack of vector bundles.

Working with the differential graded algebra $\check{C}^*(\mathcal{U}, \Omega^* \otimes \mathrm{End}(\mathcal{O}^N))$ instead, we obtain the stack of vector bundles of rank N on X with connection.

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The general case

Theorem

Let A^* be a differential graded algebra.

- *If $A^i = 0$ for $i \leq -k$, the nerve of A^* is a k-groupoid.*
- *If $A^i = 0$ for $i \leq -k$ and $i > 0$, the nerve of A^* is a k-group.*

Let A^* be a differential graded Banach algebra.

- *If $A^i = 0$ for $i \leq -k$, the nerve of A^* is a Lie k-groupoid.*

Even in the general case, $N_0 A \cong \mathrm{MC}(A)$. But the set of 1-simplices is now more complicated, and corresponds to elements of A^0 which are only quasi-invertible.

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Moduli of complexes of holomorphic vector bundles

Let X be a compact complex manifold, and let E^* be a complex of holomorphic vector bundles of length n.

Let $A^{0,q}(X, \mathrm{End}(E))$ be the $(0,q)$-forms with values in the graded vector bundle $\mathrm{End}(E)$, with coefficients in the Sobolev space H^{s-q}.

Theorem

If $s > \dim_{\mathbb{C}}(X)$, $A^{0,*}(X, \mathrm{End}(E))$ is a differential graded Banach algebra.

There is also a generalization where the complexes E^* are allowed to vary.

It follows that $N_\bullet A^{0,*}(X, \mathrm{End}(E))$ is a Lie n-groupoid. Of course, it is infinite-dimensional, so it is difficult to compare it to algebraic objects. For this, we should apply the technique of **Kuranishi**: let

$$\tilde{N}_\bullet A^{0,*}(X, \mathrm{End}(E)) \subset N_\bullet A^{0,*}(X, \mathrm{End}(E))$$

be the Lie n-subgroupoid obtained by imposing the gauge condition

$$\boxed{\bar{\partial}^* a_{i_0 \dots i_k} = 0}$$

on n-simplices.

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Equivalence of Lie n-groupoids

Let $\Delta^{m,n} = \Delta^m \times \Delta^n$ be the **prism**. Let $\Lambda_i^{m,n}$ be the **cup**

$$\Lambda_i^{m,n} = (\Lambda_i^m \times \Delta^n) \cup (\Delta^m \times \partial\Delta^n) \subset \Delta^{m,n}.$$

Definition

A simplicial morphism $f : X_\bullet \to Y_\bullet$ between Lie k-groupoids is an **equivalence** if, for each $n \geq 0$, the morphism

$$X_n \times_{Y_n} \operatorname{Hom}(\Delta^{1,n}, Y) \to \operatorname{Hom}(\partial\Delta^n, X) \times_{\operatorname{Hom}(\partial\Delta^n, Y)} \operatorname{Hom}(\Lambda_0^{1,n}, Y)$$

is a surjective submersion.

Equivalence of Lie 0-groupoids is isomorphism of Banach analytic spaces.

The equivalences form a **saturated** subcategory of the category of Lie k-groupoids: if $f : X \to Y$ and $g : Y \to Z$ are morphisms such that gf is an equivalence and either f or g is an equivalence, then so is the other.

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Kuranishi gauge as an equivalence

Warning: The statement of the following theorem is only approximate.

Theorem

- $\tilde{N}_\bullet A^{0,*}(X, \mathrm{End}(E))$ *is a finite-dimensional Lie n-groupoid.*
- *The inclusion $\tilde{N}_\bullet A^{0,*}(X, \mathrm{End}(E)) \subset N_\bullet A^{0,*}(X, \mathrm{End}(E))$ is an equivalence of Lie n-groupoids.*

This is an analytic version of a theorem of Hirschowitz and Simpson (there is also a "derived version" of this theorem by Toën, Vacquié and Vezzosi).

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The determinant line bundle

Maurer-Cartan elements of $A^{0,*}(X, \mathrm{End}(E))$ are twisted deformations:

$$\mu = \mu_{[0]} + \mu_{[1]} + \dots,$$

where $\mu_{[q]} \in A^{0,q}(X, \mathrm{Hom}(E^*, E^{*+1-q}))$.

The section $\mu_{[0]}$ deforms the differential of E^*, $\mu_{[1]}$ deforms the $\bar{\partial}$-operator, $\mu_{[2]}$ is a homotopy expressing the error in the Kodaira-Spencer equation for $\mu_{[1]}$ *et cetera*. Such twisted deformations are familiar from the work of Bismut, Gillet and Soulé.

We want to define the determinant of a twisted complex, in such a way that it is invariant under quasi-isomorphism. This was done by Knudsen and Mumford in 1976, following Grothendieck. Their formulas used choices of local frames. Knudsen gave a direct construction in 2002 which relied instead on auxilliary choices associated to the quasi-isomorphism.

This auxilliary data is just a lift of the quasi-isomorphism to a 1-simplex in our nerve.

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We want to define the determinant of a twisted complex, in such a way that it is invariant under quasi-isomorphism. This was done by Knudsen and Mumford in 1976, following Grothendieck. Their formulas used choices of local frames. Knudsen gave a direct construction in 2002 which relied instead on auxilliary choices associated to the quasi-isomorphism.

This auxilliary data is just a lift of the quasi-isomorphism to a 1-simplex in our nerve.

The determinant line bundle

Maurer-Cartan elements of $A^{0,*}(X, \mathrm{End}(E))$ are twisted deformations:

$$\mu = \mu_{[0]} + \mu_{[1]} + \dots,$$

where $\mu_{[q]} \in A^{0,q}(X, \mathrm{Hom}(E^*, E^{*+1-q}))$.

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The determinant of the twisted complex E_μ is the line bundle

$$\det(E) = \bigotimes_{i \text{ even}} \Lambda^{\mathrm{rk}(E_i)} E^i \otimes \bigotimes_{i \text{ odd}} \left(\Lambda^{\mathrm{rk}(E_i)} E^i\right)^{-1}$$

with holomorphic structure defined by the Maurer-Cartan form

$$\mathrm{Str}(\mu_{[1]}) \in A^{0,1}(X).$$

Theorem

Let $f : \mu \to \nu$ be the quasi-isomorphism of twisted deformations associated to a 1-simplex $\mathbb{\mu}$. There is a canonical trivialization of the determinant line bundle $\det(E_\mu)^{-1} \otimes \det(E_\nu)$ associated to the contracting homotopy

$$\begin{pmatrix} h & a_{0101} \\ g & -k \end{pmatrix} = \begin{pmatrix} a_{010} & a_{0101} \\ 1 + a_{10} & -a_{101} \end{pmatrix}$$

for the differential

$$\delta_E + \bar\partial + \mathrm{ad}\begin{pmatrix} \mu & f \\ 0 & -\nu \end{pmatrix}.$$

Maybe this is evidence that the nerve we have explained here is the “right” realization of the moduli n-stack of (twisted) deformations of a complex of holomorphic vector bundles.