Dimers and families of Cauchy-Riemann operators

Julien Dubédat

COLUMBIA UNIVERSITY
IN THE CITY OF NEW YORK

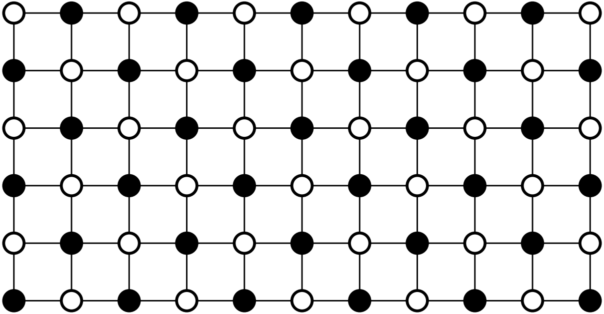
Bismutfest

Outline

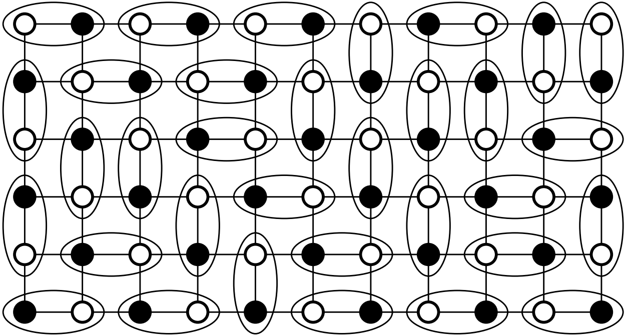
1. **Dimers**
   - Dimer model
   - 6-vertex model
   - Compactified free field
2. **Determinants**
   - ζ-regularization
   - Cauchy-Riemann operators
3. **Scaling limit**
   - Asymptotics
   - Insertions

Outline

Dimer configuration (perfect matching) of a graph $\Gamma = (V, E)$: $\mathfrak{m} \subset E$ s.t. each vertex is the endpoint of exactly one vertex in $\mathfrak{m}$.



Dimer configuration (perfect matching) of a graph $\Gamma = (V, E)$: $\mathfrak{m} \subset E$ s.t. each vertex is the endpoint of exactly one vertex in $\mathfrak{m}$.



Weighted graph $\Gamma = (V, E, (\omega_e)_{e \in E})$; $\mathfrak{m}$ a dimer configuration

$$\omega(\mathfrak{m}) = \prod_{e \in E} \omega_e$$

Partition function:

$$Z = Z_\omega = \sum_{\mathfrak{m}} \omega(\mathfrak{m})$$

Probability measure on dimer configurations ($\omega \geq 0$):

$$\mathbb{P}\{\mathfrak{m}\} = \frac{\omega(\mathfrak{m})}{Z}$$

Case: bipartite planar graphs: $V = V_B \sqcup V_W$ (black and white vertices), $|V_B| = |V_W|$.
Kasteleyn-Percus operator: $K_\omega : \mathbb{C}^{V_B} \to \mathbb{C}^{V_W}$,

$$K_\omega(b, w) = \begin{cases} \pm\omega(e) & \text{if } e = (bw) \in E \\ 0 & \text{otherwise} \end{cases}$$

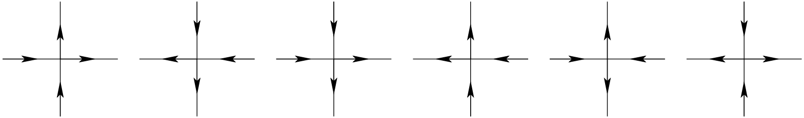
Kasteleyn enumeration

$$Z_\omega = \pm \det K_\omega$$

for well chosen signs. Then:

$$\mathbb{E}\left(\prod_{e\in\mathfrak{m}} \omega'(e)\right) = \frac{\det K_{\omega\omega'}}{\det K_\omega}$$

6-vertex model (square ice, F-model): Lieb, Baxter, ...
Configuration: orientation of a planar 4-regular graph with 2 incoming and 2 outgoing edges at each vertex.

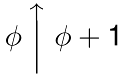


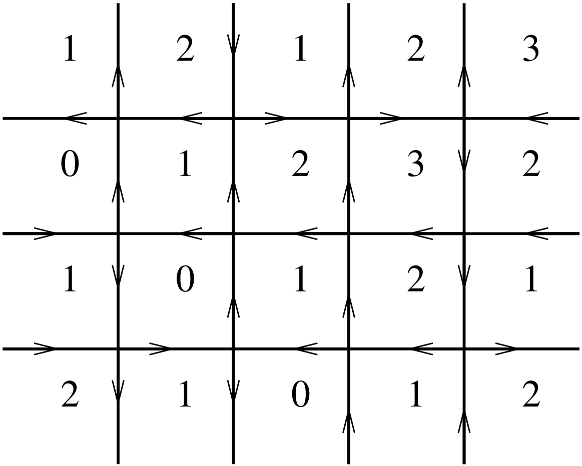
Configuration weight: product of vertex weights (a, a, b, b, c, c).

$$\mathbb{P}\{\text{configuration}\} \propto \text{configuration weight}$$

$$\Delta = \frac{a^2 + b^2 - c^2}{2ab}$$

Height (van Beijeren): defined on faces.



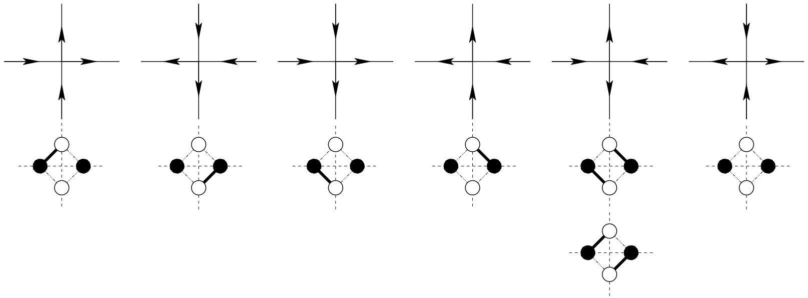


Periodic boundary conditions: toroidal graph

$$\mathbb{Z}^2/(N\mathbb{Z}+[N\tau]\mathbb{Z})$$

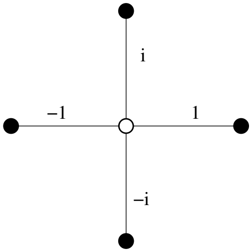
grid drawn on $\Sigma=\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$, $\Im\tau>0$.
Height: additively multivalued, with even periods.

At the "free fermion" point $\Delta = 0$, correspondence with dimer configuration on the medial graph:



Thurston: dimer height function

Square lattice, unit weights, mesh $\delta \ll 1$.



Up to gauge change: $(Kf)(w) \simeq 4\delta \frac{\partial}{\partial \bar{z}} f$.
Kasteleyn operator: finite difference approximation of $\bar{\partial}$.

Characteristic function

Kasteleyn matrix $K : \mathbb{C}^B \to \mathbb{C}^W$: discretization of $\bar\partial$ (finite difference operator); ϕ height field piecewise constant on faces.
For α a $(0,1)$-form, can define $K_\alpha(w,b) = K(w,b)e^{2i\Im\int_w^b \alpha}$.
Then K_α discretization of $\bar\partial + \alpha$ and

$$\mathbb{E}(\exp(2i\Im\int \phi\partial\alpha)) = \frac{\det(K_\alpha)}{\det(K)}\exp(\text{local compensation})$$

Tori

On a graph embedded in a torus Σ, the height function ϕ is additively multivalued; the current $d\phi = J + \bar{J}$, $J = \partial\phi$ is well defined.
To a spin structure $\chi : \pi_1(\Sigma) \to \{\pm 1\}$ corresponds a Kasteleyn matrix K_χ, discretization of $\bar{\partial}$ operating on the flat line bundle over Σ corresponding to χ. Then

$$\frac{\det(K_{\chi,\alpha})}{\det(K_\chi)} = \mathbb{E}(Q(\chi,\phi)\exp(-2i\Im\int J\wedge\alpha))\exp(\dots)$$

where $Q(\chi,\phi) = \pm 1$ depends on periods of ϕ mod 2 and can be eliminated by taking an appropriate linear combination over the four spin structures.

Massless free field

(Σ, g) manifold with boundary: Δ_Σ (positive) Laplacian, $G_\Sigma = \Delta_\Sigma^{-1}$ Green kernel.
The free field on Σ is the Gaussian process with covariance operator G_Σ:

$$\mathbb{E}(e^{i\langle \phi, f\rangle}) = e^{-\frac{1}{2}\langle f, G_\Sigma f\rangle}$$

One can take $\mathcal{H}^{1-\frac{dim(\Sigma)}{2}-\varepsilon}(\Sigma) \subset C^\infty(\Sigma)'$ as abstract Wiener space (Gross).
For $\Sigma = [0, T]$, ϕ Brownian bridge.
If Σ compact, the field ϕ is defined modulo an additive constant.

Harmonic maps

$\phi : (\Sigma, g) \to (\Sigma', g')$. Classical solution: minimize

$$S(\phi) = \frac{g_0}{4\pi} \int_\Sigma |\nabla \phi|^2 dvol_\Sigma$$

subject to boundary conditions/topological constraints.
Quantized version: sample from "$e^{-S(\phi)}\mathcal{D}\phi$".
$\Sigma' = \mathbb{R}$: free field.

Compactified free field

$\phi : \Sigma \to \Sigma' = \mathbb{R}/2\pi r\mathbb{Z}$. Can write: $d\phi = \omega + d\psi$ where ω harmonic 1-form with periods in $2\pi r\mathbb{Z}$ and $\psi : \Sigma \to \mathbb{R}$. Then

$$S(\phi) = S(\psi) + \frac{g_0}{4\pi}\int_\sigma \omega \wedge *\omega$$

Take (ω, ψ) independent, ψ a free field,

$$\mathbb{P}(\omega = \omega_0) \propto e^{-\frac{g_0}{4\pi}\int_\Sigma \omega_0 \wedge *\omega_0}$$

with ω_0 in the lattice of harmonic forms with periods in $2\pi r\mathbb{Z}$.

Graph homomorphism model (Benjamini-Häggström-Mossel): sample uniformly from graph homomorphisms $f : \Gamma_1 \to \Gamma_2$ ($f(v_1) \sim f(v_2)$ if $v_1 \sim v_2$).

$\mathbb{Z}/M\mathbb{Z} \to \mathbb{Z}$ (mod constant). Limit ($M \to \infty$): Brownian bridge
$\mathbb{Z}/M\mathbb{Z} \to \mathbb{Z}/N\mathbb{Z}$, $N \sim r\sqrt{M} \gg 1$: compactified Brownian bridge ("instanton": winding).
$\mathbb{Z}/M\mathbb{Z} \to \mathbb{Z}^2$: 2D Brownian bridge.
$(\mathbb{Z}/M\mathbb{Z})^2 \to \mathbb{Z}$: uniform 6V model.

Graph homomorphism model (Benjamini-Häggström-Mossel): sample uniformly from graph homomorphisms $f : \Gamma_1 \to \Gamma_2$ ($f(v_1) \sim f(v_2)$ if $v_1 \sim v_2$).

$\mathbb{Z}/M\mathbb{Z} \to \mathbb{Z}$ (mod constant). Limit ($M \to \infty$): Brownian bridge
$\mathbb{Z}/M\mathbb{Z} \to \mathbb{Z}/N\mathbb{Z}$, $N \sim r\sqrt{M} \gg 1$: compactified Brownian bridge ("instanton": winding).
$\mathbb{Z}/M\mathbb{Z} \to \mathbb{Z}^2$: 2D Brownian bridge.
$(\mathbb{Z}/M\mathbb{Z})^2 \to \mathbb{Z}$: uniform 6V model.

Graph homomorphism model (Benjamini-Häggström-Mossel): sample uniformly from graph homomorphisms $f : \Gamma_1 \to \Gamma_2$ ($f(v_1) \sim f(v_2)$ if $v_1 \sim v_2$).

$\mathbb{Z}/M\mathbb{Z} \to \mathbb{Z}$ (mod constant). Limit ($M \to \infty$): Brownian bridge
$\mathbb{Z}/M\mathbb{Z} \to \mathbb{Z}/N\mathbb{Z}$, $N \sim r\sqrt{M} \gg 1$: compactified Brownian bridge ("instanton": winding).
$\mathbb{Z}/M\mathbb{Z} \to \mathbb{Z}^2$: 2D Brownian bridge.
$(\mathbb{Z}/M\mathbb{Z})^2 \to \mathbb{Z}$: uniform 6V model.

Graph homomorphism model (Benjamini-Häggström-Mossel): sample uniformly from graph homomorphisms $f : \Gamma_1 \to \Gamma_2$ ($f(v_1) \sim f(v_2)$ if $v_1 \sim v_2$).

$\mathbb{Z}/M\mathbb{Z} \to \mathbb{Z}$ (mod constant). Limit ($M \to \infty$): Brownian bridge
$\mathbb{Z}/M\mathbb{Z} \to \mathbb{Z}/N\mathbb{Z}$, $N \sim r\sqrt{M} \gg 1$: compactified Brownian bridge ("instanton": winding).
$\mathbb{Z}/M\mathbb{Z} \to \mathbb{Z}^2$: 2D Brownian bridge.
$(\mathbb{Z}/M\mathbb{Z})^2 \to \mathbb{Z}$: uniform 6V model.

Graph homomorphism model (Benjamini-Häggström-Mossel): sample uniformly from graph homomorphisms $f : \Gamma_1 \to \Gamma_2$ ($f(v_1) \sim f(v_2)$ if $v_1 \sim v_2$).

$\mathbb{Z}/M\mathbb{Z} \to \mathbb{Z}$ (mod constant). Limit ($M \to \infty$): Brownian bridge
$\mathbb{Z}/M\mathbb{Z} \to \mathbb{Z}/N\mathbb{Z}$, $N \sim r\sqrt{M} \gg 1$: compactified Brownian bridge ("instanton": winding).
$\mathbb{Z}/M\mathbb{Z} \to \mathbb{Z}^2$: 2D Brownian bridge.
$(\mathbb{Z}/M\mathbb{Z})^2 \to \mathbb{Z}$: uniform 6V model.

Outline

Ray-Singer ζ-regularization

For a trace class operator T, one can define a Fredholm determinant $\det_F(\mathrm{Id}+T)$.
For the Laplacian on Σ, spectrum $0<\lambda_1<\cdots$. Define

$$\zeta(s)=\sum_{i\geq 0}\lambda_i^{-s}=\frac{1}{\Gamma(s)}\int_0^\infty \mathrm{Tr}(e^{-t\Delta})t^{s-1}dt$$

$(\Re s>d/2)$ and by meromorphic continuation

$$\det_\zeta(\Delta)=\exp(-\zeta'(0))$$

For Σ Riemann surface, $L(\chi)$ line bundle corresponding to the unitary character χ, $\bar\partial:L(\chi)\to L(\chi)\bigotimes\Omega^{0,1}(\chi)$, the $\bar\partial$-analytic torsion is given by:

$$T(\Sigma,\chi)=\exp(-\frac{1}{2}\zeta'(0))$$

where ζ corresponds to $\bar\partial^*\bar\partial$.

Variational formula

For (M_t) a one-parameter family in $GL_n(\mathbb{C})$,

$$\frac{d}{dt}\log\det(M_t) = \operatorname{Tr}\left(\dot{M}_t M_t^{-1}\right)$$

For (M_t) a one-parameter family of generalized Laplacians on a compact manifold,

$$\frac{d}{dt}\log\det(M_t) = \mathrm{LIM}_{u\to 0}\operatorname{Tr}\left(\dot{M}_t M_t^{-1} e^{-uM_t}\right)$$

where

$$\mathrm{LIM}_{u\to 0}\sum_{k\leq 0} a_k u^{k/2} = a_0$$

.

Families of Cauchy-Riemann operators

Start from L holomorphic bundle over Σ a Riemann surface, $\bar\partial : L \to L \bigotimes \Omega^{0,1}$. A CR operator is of the form:

$$\bar\partial + \alpha$$

where α a $(0, 1)$-form with values in L.
The family of CR operators has a natural complex structure.
Quillen defined a determinant line bundle over the space of CR operators and computed its curvature.
Bismut-Freed: families of first-order elliptic differential operators.

Variational formula

In a local coordinate, the inverse of $(\bar{\partial} + \alpha)$ near w has the expansion ($\alpha = a(z)d\bar{z}$):

$$(\bar{\partial} + \alpha)^{-1}(z, w) = e^{-2i\Im(a(w)\overline{z-w})}\left(\frac{1}{\pi(z-w)} + r_\alpha(w) + o(1)\right)$$

If $\alpha = \alpha_t$,

$$\frac{d}{dt}\log\det((\bar{\partial} + \alpha_t)^*(\bar{\partial} + \alpha_t)) = \int_\Sigma \dot{\alpha}_t \wedge r_{\alpha_t} dz$$

Outline

Scaling limit of the discrete height field (mesh $\delta \searrow 0$).
Dimer height function ($6V$ at $\Delta = 0$) in simply connected domains: converges to the free field (Kenyon, de Tilière, Kenyon-Okounkov-Sheffield,...) (moment method).
Questions: compactified limit (Boutillier-de Tilière)?
strong approximation ?
insertions ?
6V height field for $\Delta \in (-1, 1)$???

For a fixed mesh δ, discretizations K, K_α of $\bar\partial$, $\bar\partial + \alpha$; characteristic function of the height field expressed in terms of $\frac{\det(K_\alpha)}{\det(K)}$.
By the variational formulae for $\log\det(K_{u\alpha})$ and $\log\det(\bar\partial + u\alpha)$, need to show:

$$(K_{u\alpha})^{-1} = r_{u\alpha} + (\text{oscillating terms}) + L(\alpha)$$

Asymptotics

To estimate K_α^{-1} near w, set $\alpha = a(z)d\bar{z}$, $\lambda = a(w)$ and show that

$$K_\alpha^{-1}(.,w) - K_{\lambda d\bar{z}}^{-1}(.,w) \longrightarrow (\bar{\partial}+\alpha)^{-1}(.,w) - (\bar{\partial}+\lambda)^{-1}(.,w)$$

uniformly as $\delta \searrow 0$.

Identification of the limit

Kronecker's second limit formula:

$$T(\chi) = (2\Im\tau)^{1/2} e^{-2\pi(\Im z)^2/\Im\tau} |\theta_3(z)|^2$$

where $z = \varepsilon' + \varepsilon\tau$, $\chi(A) = e^{2i\pi\varepsilon'}$, $\chi(B) = e^{2i\pi\varepsilon}$.
Relates to the compactified free field via the ABMNV bosonization identity:

$$T(\chi) = \sum_{\omega} \exp\left(-\frac{\pi}{2}\int_{\Sigma} \omega \wedge *\omega + 2i\pi\varepsilon' \int_A \omega + 2i\pi\varepsilon \int_B \omega\right)$$

the sum bearing on harmonic 1-forms with integer periods.

For general $\alpha = \lambda d\bar{z} + \bar{\partial} f$, write $\bar{\partial} + \alpha = e^{-f}(\bar{\partial} + \lambda)e^{f}$, so that

$$r_\alpha = r_\lambda - \frac{1}{2\pi}\partial \Re f$$

and $\Re \int \alpha \wedge r_\alpha dz = (\ldots) - \frac{1}{2\pi}\int |\nabla \Re f|^2 dA$.
Finally,

$$\mathbb{E}(\exp((\ldots) + \int \phi \Delta f dA)) \longrightarrow (\ldots)\exp(\frac{1}{2\pi}\int |\nabla f|^2) dA$$

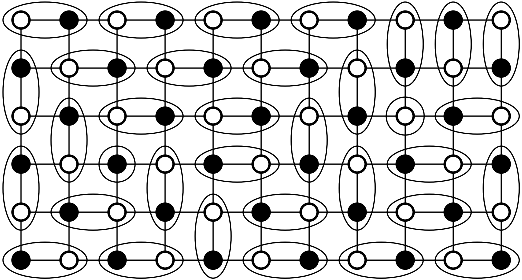
The dimer height field converges to a compactified GFF: $r = \frac{1}{2}$, $g_0 = 2$.
Conjecture (Nienhuis): $g_0 = \frac{8}{\pi}\arcsin(\frac{c}{2})$ for $(1, 1, 1, 1, c, c)$ 6V.

Insertions

Electric charges: $\mathbb{E}(\exp(i\pi s(\phi(f_2) - \phi(f_1))))$
$\phi(f_2) - \phi(f_1)$ has variance $\sim c \log |f_2 - f_1|$, takes values in $2\mathbb{Z}$
Coulomb Gas heuristic (Nienhuis): as if $\phi_{\mathrm{dimer}}(f) = \delta^{-2} \int_f \phi_{GFF}$
ie $\mathbb{E}(\dots) \sim c|f_2 - f_1|^{-2s^2}$
That heuristic breaks down at $s = \frac{1}{2}$.

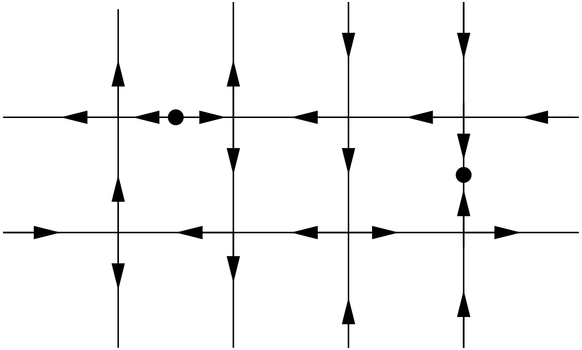
Insertions

Monomer correlation: $\frac{Z_{\Gamma \setminus \{b,w\}}}{Z_\Gamma}$



Fisher-Stephenson conjecture: $\propto |b - w|^{-1/2}$ [Hartwig, Ciucu]

6V formulation: vortices.



Height gains additive monodromy at the magnetic charges.

Families of CR operators

Scalar fluctuations: $(\bar{\partial} + \bar{\partial} g)_g$.

Instanton: $(\bar{\partial} + \lambda d\bar{z})_\lambda$

Electric charges: punctures at insertions, unitary line bundle over the punctured surface; isomonodromic deformation.

Magnetic charges: $\simeq$ divisors at insertions+ (-1) monodromy around insertions.

Families of CR operators

Scalar fluctuations: $(\bar{\partial} + \bar{\partial} g)_g$.
Instanton: $(\bar{\partial} + \lambda d\bar{z})_\lambda$
Electric charges: punctures at insertions, unitary line bundle over the punctured surface; isomonodromic deformation.
Magnetic charges: $\simeq$ divisors at insertions+ (-1) monodromy around insertions.

Families of CR operators

Scalar fluctuations: $(\bar{\partial} + \bar{\partial} g)_g$.
Instanton: $(\bar{\partial} + \lambda d\bar{z})_\lambda$
Electric charges: punctures at insertions, unitary line bundle over the punctured surface; isomonodromic deformation.
Magnetic charges: $\simeq$ divisors at insertions+ (-1) monodromy around insertions.

Families of CR operators

Scalar fluctuations: $(\bar{\partial}+\bar{\partial}g)_g$.
Instanton: $(\bar{\partial}+\lambda d\bar{z})_\lambda$
Electric charges: punctures at insertions, unitary line bundle over the punctured surface; isomonodromic deformation.
Magnetic charges: $\simeq$ divisors at insertions+ (-1) monodromy around insertions.