THE HYPOELLIPTIC LAPLACIAN AND CHERN-GAUSS-BONNET

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This paper is dedicated to the memory of Professor S.S. Chern

Abstract. We construct a new Hodge theory on the cotangent bundle of a Riemannian manifold X. The corresponding Laplacian is a second order hypoelliptic operator, which is self-adjoint with respect to a Hermitian form whose signature is (∞, ∞). This Hodge theory interpolates between the classical Hodge theory on X and the geodesic flow on T^*X. We also give results obtained with G. Lebeau on the analysis of the hypoelliptic Laplacian and on the hypoelliptic analytic torsion. Finally we explain the connections of this construction with Chern's proof of Chern-Gauss-Bonnet.

Introduction

The purpose of this paper is to describe a deformation of the classical Hodge theory of a compact Riemannian manifold X, whose corresponding Laplacian is a hypoelliptic operator on the cotangent bundle T^*X.

This construction came from the author's attempt to develop the Hodge theory of the loop space LX of X, and to construct the Witten deformation [Wit82] of the Hodge Laplacian of LX which would be associated to the energy functional E. Such a Witten deformation, if it existed, would interpolate between the Hodge Laplacian $\square^{LX}$ on LX and the Morse theory for E, whose critical points are the closed geodesics in X. There is indeed no Hodge theory on LX, one difficulty being the construction of a L^2 scalar product on the de Rham complex of LX. Still one can think of our construction as being the semiclassical limit of the non existing Hodge theory of LX.

Needless to say, the construction of the hypoelliptic Laplacian can be done without any explicit reference to the loop space LX. Still many of the remarkable properties of this operator can be anticipated if one accepts the fact it is the 'shadow' of a Hodge theory to be on LX.

Another impetus came from the realization of the fact that many properties of the Witten deformation are related to an infinite dimensional version

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of the proof by Chern [Che44] of Chern-Gauss-Bonnet. Indeed our strategy was to try finding what exotic Hodge theory corresponded to a formally defined supersymmetric path integral associated to the energy functional E on LX.

This paper is organized as follows. In section 1, we construct the adjoint of the de Rham operator d^{T^*X} with respect to an exotic bilinear form on the de Rham complex of T^*X.

In section 2, we give the Weitzenböck formula for the corresponding Laplacian, which turns out to be a hypoelliptic operator on T^*X.

In section 3, we show that the new Laplacian interpolates between classical Hodge theory and the geodesic flow.

In section 4, we give a self-adjointness property of the hypoelliptic Laplacian with respect to a Hermitian form of signature (∞, ∞).

In section 5, we summarize some of the results on the analysis of the new Laplacian obtained in [BL06] jointly with Lebeau.

In section 6, we state the main result we obtained in [BL06] saying that the Ray-Singer metric for the hypoelliptic Laplacian is the same as the Ray-Singer metric associated to the classical Laplacian.

Finally in section 7, we relate the above constructions to infinite dimensional versions of Chern-Gauss-Bonnet.

The construction of the hypoelliptic Laplacian was announced in [Bis04c, Bis04b, Bis04d]. It is detailed in [Bis05]. For a survey, we also refer to [Bis04a]. The analysis of the hypoelliptic Laplacian, and applications to analytic torsion are carried through in joint work with Lebeau [BL06].

1. A non standard Hodge theory

Let M be a smooth manifold. Let η be a nondegenerate bilinear form on TM. Let $\phi : TM \to T^*M$ be the morphism such that if $U, V \in TM$,

$$\eta(U,V) = \langle U, \phi V \rangle . \tag{1.1}$$

Let η^* the bilinear form on T^*M which corresponds to η by the morphism ϕ. Then η^* induces a nondegenerate bilinear form on $\Lambda^\cdot(T^*M)$. Let dv_M be a volume form on M. Let $\left(\Omega^\cdot(M), d^M\right)$ be the de Rham complex of smooth compactly supported differential forms on M. We equip $\Omega^\cdot(M)$ with the nondegenerate bilinear form,

$$\langle s, s' \rangle = \int_M \eta^*(s, s')\, dv_M. \tag{1.2}$$

Note that this bilinear form is in general neither symmetric nor antisymmetric.

Let $\overline{d}^M$ be the formal adjoint of d^M with respect to the bilinear form (1.2), so that if $s, s' \in \Omega^\cdot(M)$, then

$$\left\langle s, d^M s' \right\rangle = \left\langle \overline{d}^M s, s' \right\rangle . \tag{1.3}$$

Note that in general the formal adjoint of $\overline{d}^M$ in the sense of (1.3) is not equal to d^M.

Let X be a compact manifold of dimension n. Let $\pi : T^*X \to X$ be the cotangent bundle on X. Let $\theta = \langle p, dx\rangle$ be the canonical 1-form on T^*X. Let $\omega = d^{T^*X}\theta$ be the canonical symplectic form on T^*X. This is a nondegenerate bilinear form on TT^*X.

Let $\overline{d}^{T^*X}$ be the formal adjoint of d^{T^*X} with respect to the bilinear form $\langle\,\rangle$ on $\Omega^\cdot(T^*X)$, which is associated to ω and to the symplectic volume dv_{T^*X}.

It is easy to show that

$$\left[d^{T^*X}, \overline{d}^{T^*X}\right] = 0. \tag{1.4}$$

Observe that equation (1.4) is valid on any symplectic manifold. Indeed by using Darboux's theorem, equation (1.4) is just a reflection of the fact that $\omega(\xi, \xi) = 0$.

Equation (1.4) says that the Laplacian which is associated to the above bilinear form vanishes identically. Recall that our ultimate purpose is to produce a hypoelliptic Laplacian. The vanishing of our symplectic Laplacian simply indicates we have gone too far in the right direction.

Let us now explain in more detail the construction of the hypoelliptic Laplacian. Let g^{TX} be a metric on TX. We identify TX and T^*X by the metric g^{TX}. Let ∇^{TX} be the Levi-Civita connection on TX, and let R^{TX} be its curvature. The connection ∇^{TX} induces the splittings,

$$TT^*X = \pi^*(TX \oplus T^*X), \qquad T^*T^*X = \pi^*(T^*X \oplus TX). \tag{1.5}$$

From (1.5), we get the isomorphism,

$$\Lambda^\cdot(T^*T^*X) = \pi^*\left(\Lambda^\cdot(T^*X)\widehat{\otimes}\Lambda^\cdot(TX)\right). \tag{1.6}$$

We denote with a $\widehat{\ }$ the objects which refer to the second factor in the right-hand side in (1.6). Let $\nabla^{\Lambda^\cdot(T^*T^*X)}$ be the connection induced by ∇^{TX} on $\Lambda^\cdot(T^*T^*X)$.

Put

$$\phi = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}. \tag{1.7}$$

We identify ϕ with an automorphism of $TT^*X = TX \oplus T^*X$. The bilinear form η which is associated to ϕ as in (1.1) is given by

$$U, V \to \eta(U, V) = \langle \pi_*U, \pi_*V\rangle_{g^{TX}} + \omega(U, V). \tag{1.8}$$

Let $\langle\,\rangle_\phi$ be the associated nondegenerate bilinear form on $\Omega^\cdot(T^*X)$. Let $\overline{d}_\phi^{T^*X}$ the formal adjoint of d^{T^*X} with respect to $\langle\,\rangle_\phi$ and to the symplectic volume form dv_{T^*X}.

Let $\mathcal{H} : T^*X \to \mathbf{R}$ be a smooth function. Let $Y^{\mathcal{H}}$ be the corresponding Hamiltonian vector field, so that

$$d^{T^*X}\mathcal{H} + i_{Y^{\mathcal{H}}}\omega = 0. \tag{1.9}$$

Set

$$\langle s, s' \rangle_{\phi,\mathcal{H}} = \int_{T^*X} \eta^* \left(s, s'\right) e^{-2\mathcal{H}} dv_{T^*X}. \tag{1.10}$$

Put

$$d_{\mathcal{H}}^{T^*X} = e^{-\mathcal{H}} d^{T^*X} e^{\mathcal{H}}, \qquad \overline{d}_{\phi,\mathcal{H}}^{T^*X} = e^{\mathcal{H}} \overline{d}_{\phi}^{T^*X} e^{-\mathcal{H}}. \tag{1.11}$$

Then $\overline{d}_{\phi,2\mathcal{H}}^{T^*X}$ is the formal adjoint of d^{T^*X} with respect to $\langle\,\rangle_{\phi,\mathcal{H}}$, and $\overline{d}_{\phi,\mathcal{H}}^{T^*X}$ is the formal adjoint of $d_{\mathcal{H}}^{T^*X}$ with respect to $\langle\,\rangle_{\phi}$.

Set

$$A_{\phi,\mathcal{H}} = \frac{1}{2}\left(\overline{d}_{\phi,2\mathcal{H}}^{T^*X} + d^{T^*X}\right), \qquad \mathfrak{A}_{\phi,\mathcal{H}} = \frac{1}{2}\left(\overline{d}_{\phi,\mathcal{H}}^{T^*X} + d_{\mathcal{H}}^{T^*X}\right). \tag{1.12}$$

Clearly,

$$\mathfrak{A}_{\phi,\mathcal{H}} = e^{-\mathcal{H}} A_{\phi,\mathcal{H}} e^{\mathcal{H}}. \tag{1.13}$$

If Z is a vector field on T^*X, let L_Z be the corresponding Lie derivative operator acting on $\Omega^{\cdot}\left(T^*X\right)$.

More generally, let $\left(F, \nabla^F\right)$ be a complex flat vector bundle on X, and let g^F be a non necessarily flat Hermitian metric on F. Let $\left(\Omega^{\cdot}\left(T^*X, \pi^*F\right), d^{T^*X}\right)$ be the de Rham complex of smooth compactly supported forms on T^*X with coefficients in F. The operator L_Z still acts naturally on $\Omega^{\cdot}\left(T^*X, \pi^*F\right)$. Set

$$\omega\left(\nabla^F, g^F\right) = \left(g^F\right)^{-1} \nabla^F g^F. \tag{1.14}$$

The 1-form $\omega\left(\nabla^F, g^F\right)$ takes values in self-adjoints endomorphisms of F.

Also there is an obvious extension of the bilinear form in (1.10) to a skew-linear form on $\Omega^{\cdot}\left(T^*X, \pi^*F\right)$, in which the metric g^F is incorporated in the obvious way. It is then possible to extend the above constructions, and still obtain operators like the ones in (1.11)-(1.13), which now act on $\Omega^{\cdot}\left(T^*X, \pi^*F\right)$. In the sequel, we will deal with this more general situation.

2. The Weitzenböck formula for the hypoelliptic Laplacian

Let $e_1, \ldots, e_n$ be an orthonormal basis of TX, let $e^1, \ldots, e^n$ be the corresponding dual basis of T^*X. Let $\widehat{e}_1, \ldots, \widehat{e}_n$ and $\widehat{e}^1, \ldots, \widehat{e}^n$ be other copies of these bases.

Then $e_1, \ldots, e_n, \widehat{e}^1, \ldots, \widehat{e}^n$ is a basis of TT^*X, and $e^1, \ldots, e^n, \widehat{e}_1, \ldots, \widehat{e}_n$ is the dual basis of T^*T^*X. Set

$$\widehat{\nabla^V \mathcal{H}} = \nabla_{\widehat{e}^i} \mathcal{H} \widehat{e}^i. \tag{2.1}$$

We give the Weitzenböck formula established in [Bis05, Theorem 3.3].

Theorem 2.1. *The following identities hold,*

$$\begin{aligned}A^2_{\phi,\mathcal{H}} &= \frac{1}{4}\left(-\Delta^V - \frac{1}{2}\left\langle R^{TX}\left(e_i,e_j\right)e_k,e_l\right\rangle e^i e^j i_{\widehat{e}^k} i_{\widehat{e}^l} + 2L_{\widehat{\nabla^V\mathcal{H}}}\right)\\ &- \frac{1}{2}\left(L_{Y^{\mathcal{H}}} + \frac{1}{2}e^i i_{\widehat{e}^j}\nabla^F_{e_i}\omega\left(\nabla^F,g^F\right)\left(e_j\right) + \frac{1}{2}\omega\left(\nabla^F,g^F\right)\left(e_i\right)\nabla_{\widehat{e}^i}\right),\\ \mathfrak{A}^2_{\phi,\mathcal{H}} &= \frac{1}{4}\Bigg(-\Delta^V - \frac{1}{2}\left\langle R^{TX}\left(e_i,e_j\right)e_k,e_l\right\rangle e^i e^j i_{\widehat{e}^k} i_{\widehat{e}^l} + \left|\nabla^V\mathcal{H}\right|^2\\ &\qquad - \Delta^V\mathcal{H} + 2\nabla_{\widehat{e}^i}\nabla_{\widehat{e}^j}\mathcal{H}\widehat{e}_i i_{\widehat{e}^j} + 2\nabla_{\widehat{e}^i}\nabla_{e_j}\mathcal{H}e^j i_{\widehat{e}^i}\Bigg)\\ &- \frac{1}{2}\Bigg(L_{Y^{\mathcal{H}}} + \frac{1}{2}\omega\left(\nabla^F,g^F\right)\left(Y^{\mathcal{H}}\right) + \frac{1}{2}e^i i_{\widehat{e}^j}\nabla^F_{e_i}\omega\left(\nabla^F,g^F\right)\left(e_j\right)\\ &\qquad + \frac{1}{2}\omega\left(\nabla^F,g^F\right)\left(e_i\right)\nabla_{\widehat{e}^i}\Bigg).\end{aligned} \tag{2.2}$$

Given $c \in \mathbf{R}$, set

$$\mathcal{H} = \frac{|p|^2}{2}, \qquad \mathcal{H}^c = c\frac{|p|^2}{2}. \tag{2.3}$$

When $c \in \mathbf{R}^*$, put $c = \pm 1/b^2, b > 0$. We state a result which was established in [Bis05, Theorem 3.4].

Theorem 2.2. *The following identity holds,*

$$L_{Y^{\mathcal{H}^c}} = \nabla^{\Lambda^{\cdot}(T^*T^*X)\otimes F}_{Y^{\mathcal{H}^c}} + c\widehat{e}_i i_{e_i} + c\left\langle R^{TX}\left(p,e_i\right)p,e_j\right\rangle e^i i_{\widehat{e}^j}. \tag{2.4}$$

Moreover,

$$\begin{aligned}A^2_{\phi,\mathcal{H}^c} &= \frac{1}{4}\left(-\Delta^V + 2cL_{\widehat{p}} - \frac{1}{2}\left\langle R^{TX}\left(e_i,e_j\right)e_k,e_l\right\rangle e^i e^j i_{\widehat{e}^k} i_{\widehat{e}^l}\right)\\ &- \frac{1}{2}\left(L_{Y^{\mathcal{H}^c}} + \frac{1}{2}e^i i_{\widehat{e}^j}\nabla^F_{e_i}\omega\left(\nabla^F,g^F\right)\left(e_j\right) + \frac{1}{2}\omega\left(\nabla^F,g^F\right)\left(e_i\right)\nabla_{\widehat{e}^i}\right),\\ \mathfrak{A}^2_{\phi,\mathcal{H}^c} &= \frac{1}{4}\left(-\Delta^V + c^2|p|^2 + c\left(2\widehat{e}_i i_{\widehat{e}^i} - n\right) - \frac{1}{2}\left\langle R^{TX}\left(e_i,e_j\right)e_k,e_l\right\rangle e^i e^j i_{\widehat{e}^k} i_{\widehat{e}^l}\right)\\ &- \frac{1}{2}\Bigg(L_{Y^{\mathcal{H}^c}} + \frac{1}{2}\omega\left(\nabla^F,g^F\right)\left(Y^{\mathcal{H}^c}\right) + \frac{1}{2}e^i i_{\widehat{e}^j}\nabla^F_{e_i}\omega\left(\nabla^F,g^F\right)\left(e_j\right)\\ &\qquad + \frac{1}{2}\omega\left(\nabla^F,g^F\right)\left(e_i\right)\nabla_{\widehat{e}^i}\Bigg).\end{aligned} \tag{2.5}$$

For $c \in \mathbf{R}^*$, the operators $\frac{\partial}{\partial u} - A^2_{\phi,\mathcal{H}^c}$, $\frac{\partial}{\partial u} - \mathfrak{A}^2_{\phi,\mathcal{H}^c}$ are hypoelliptic.

Proof. Observe here that the result of hypoellipticity follows from a well-known result by Hörmander [Hör67]. $\square$

Observe that the operators $A^2_{\phi,\mathcal{H}^c}, \mathfrak{A}^2_{\phi,\mathcal{H}^c}$ are not elliptic and not self-adjoint.

3. An interpolation property

Let $r : T^*X \to T^*X$ be the map $(x,p) \to (x,-p)$. Set

(3.1)
$$\begin{aligned}\mathfrak{a}_\pm &= \frac{1}{2}\left(-\Delta^V \pm 2L_{\widehat{p}} - \frac{1}{2}\left\langle R^{TX}\left(e_i,e_j\right)e_k,e_l\right\rangle e^ie^j i_{\widehat{e}^k} i_{\widehat{e}^l}\right),\\ \mathfrak{b}_\pm &= -\left(\pm L_{Y^\mathcal{H}} + \frac{1}{2}e^i i_{\widehat{e}^j}\nabla^F_{e_i}\omega\left(\nabla^F,g^F\right)\left(e_j\right) + \frac{1}{2}\omega\left(\nabla^F,g^F\right)\left(e_i\right)\nabla_{\widehat{e}^i}\right).\end{aligned}$$

Then $\mathfrak{a}_\pm$ commutes with r^*, and that $\mathfrak{b}_\pm$ anticommutes with r^*.

For $a \in \mathbf{R}$, let $r_a : T^*X \to T^*X$ be the dilation $(x,p) \to (x,ap)$, so that $r = r_{-1}$. For $c = \pm 1/b^2$, set

$$A_{\phi_b,\pm\mathcal{H}} = r_b^* A_{\phi,\mathcal{H}^c} r_b^{*-1}. \tag{3.2}$$

By (2.5), we get

$$2A^2_{\phi_b,\pm\mathcal{H}} = \frac{1}{b^2}\mathfrak{a}_\pm + \frac{1}{b}\mathfrak{b}_\pm. \tag{3.3}$$

Let $o(TX)$ be the orientation bundle of TX. Let Φ^{T^*X} be the Thom form on T^*X of Mathai-Quillen [MQ86] which is associated to the metric g^{TX} and to the connection ∇^{TX}. The form Φ^{T^*X} is a closed form of degree n with coefficients in $o(TX)$, such that $\pi_*\Phi^{T^*X} = 1$. It is normalized in such a way that

$$\Phi^{T^*X} = \exp\left(-\left|p\right|^2 + \dots\right). \tag{3.4}$$

In (3.4), ... designates an explicit complicate expression involving curvature. As is suggested by (3.4), the form Φ^{T^*X} restricts to a Gaussian form along the fibre.

One verifies easily that the operators $\mathfrak{a}_\pm$ are semisimple. The kernel of $\mathfrak{a}_+$ is generated by the function 1, and the corresponding projector $Q^{T^*X}_+$ on this kernel is given by $\alpha \to \pi_*\left(\alpha \wedge \Phi^{T^*X}\right)$. The kernel of $\mathfrak{a}_-$ is generated by Φ^{T^*X}, and the corresponding projector $Q^{T^*X}_-$ is given by $\alpha \to (\pi_*\alpha) \wedge \Phi^{T^*X}$.

Let d^X be the de Rham operator acting on $\Omega^\cdot(X,F)$ in the $+$ case or on $\Omega^\cdot(X, F\otimes o(TX))$ in the $-$ case, and let d^{X*} be its formal adjoint with respect to the standard L^2 Hermitian product. Let $\square^X = \left[d^X, d^{X*}\right]$ denote the corresponding Hodge Laplacian.

The following result is established in [Bis05, Theorem 3.13]

Theorem 3.1. *The following identity holds,*

$$-Q^{T^*X}_\pm \mathfrak{b}_\pm \mathfrak{a}_\pm^{-1} \mathfrak{b}_\pm Q^{T^*X}_\pm = \frac{1}{2}\square^X. \tag{3.5}$$

Observe that a formula similar to (3.5) plays a key role in the paper by Bismut et Lebeau [BL91], where the Hodge theory of a compact complex manifold is deformed into the Hodge theory of a submanifold. Identities (3.3) and (3.5) indicate that the matrix structure of the operator in (3.3) is essentially similar to the one in [BL91].

Also observe that in degree 0, equation (3.5) is equivalent to

$$\int_{T^*X} \nabla_p \nabla_p e^{-|p|^2} \frac{dp}{\pi^{n/2}} = \frac{1}{2}\Delta^X, \tag{3.6}$$

which itself is equivalent to

$$\sum_1^n \nabla_{e_i}^2 = \Delta^X. \tag{3.7}$$

The contribution of $\mathfrak{a}_{\pm}^{-1}$to equation (3.6) is in fact equal to 1.

In [BL06], equation (3.5) provides one of the key algebraic results from which one shows that in the proper sense, when $c \to \pm\infty$, the resolvent of a suitably conjugate version of the operator $2A^2_{\phi,\mathcal{H}^c}$ converges to the resolvent of $\frac{1}{2}\square^X$. The relevant conjugation is described in [Bis05] and in [BL06].

Suppose again that $F = \mathbf{R}$. Let $N^V = \sum_1^n \widehat{e}_i i_{\widehat{e}^i}$ be the vertical number operator, i.e. the operator which counts the vertical degree of forms in $\Omega^{\cdot}(T^*X, \pi^*F)$. We have the identity of [Bis05, eq. (3.79)],

$$\begin{aligned} r_{b^2}^* 2\mathfrak{A}^2_{\phi,\mathcal{H}^c} r_{b^2}^{*-1} = \frac{1}{2}\left(-c^2\Delta^V + |p|^2 - cn\right) + cN^V \\ - \frac{c^2}{4}\left\langle R^{TX}(e_i, e_j) e_k, e_l\right\rangle e^i e^j i_{\widehat{e}^k} i_{\widehat{e}^l} \mp L_{Y^{\mathcal{H}}}, \end{aligned} \tag{3.8}$$

so that as $b \to +\infty$,

$$r_{b^2}^* 2\mathfrak{A}^2_{\phi,\mathcal{H}^c} r_{b^2}^{*-1} \simeq \frac{1}{2}|p|^2 \mp L_{Y^{\mathcal{H}}}. \tag{3.9}$$

In the right-hand side of (3.9), there is essentially the Lie derivative operator $\mp L_{Y^{\mathcal{H}}}$.

This should convince the reader that as when $b \to +\infty$, the trace of the heat kernel $\exp\left(-tA^2_{\phi,\mathcal{H}^c}\right)$ should localize near the closed geodesics in X.

From the above, we find that up to scaling, $2A^2_{\phi,\mathcal{H}^c}$ interpolates in a proper sense between the Hodge Laplacian and the geodesic flow.

4. A self-adjointness property

The operator $A^2_{\phi,\mathcal{H}^c}$ is certainly not self-adjoint in the classical sense. However it is shown in [Bis05] that it is self-adjoint with respect to a non-degenerate Hermitian form of signature (∞,∞), which we now describe.

Let g^{T^*X} be the metric on the fibres of T^*X which is dual to g^{TX}. Let $\mathfrak{g}^{TT^*X}$ be the Riemannian metric on T^*X whose matrix with respect to the splitting $TT^*X = \pi^* (TX \oplus T^*X)$ is given by

$$\mathfrak{g}^{TT^*X} = \begin{pmatrix} g^{TX} & 1|_{T^*X} \\ 1|_{TX} & 2g^{T^*X} \end{pmatrix}. \tag{4.1}$$

The volume form attached to $\mathfrak{g}^{TT^*X}$ is the symplectic volume form dv_{T^*X}.

Let F be the $\mathfrak{g}^{TT^*X}$ isometric involution of TT^*X whose matrix with respect to the above splitting is given by

$$F = \begin{pmatrix} 1|_{TX} & 2g^{T^*X} \\ 0 & -1|_{T^*X} \end{pmatrix}. \tag{4.2}$$

Then F acts like $\widetilde{F}^{-1}$ on $\Lambda^{\cdot} (T^*T^*X)$.

Let $\langle\,\rangle_{\mathfrak{g}^{\Omega^{\cdot}(T^*X,\pi^*F)}}$ be the Hermitian product on $\Omega^{\cdot} (T^*X, \pi^*F)$ associated to the metrics $\mathfrak{g}^{TT^*X}, g^F$.

Let u be the isometric involution of $\Omega^{\cdot} (T^*X, \pi^*F)$,

$$us(x,p) = Fs(x,-p). \tag{4.3}$$

Let $\mathfrak{h}^{\Omega^{\cdot}(T^*X,\pi^*F)}$ be the Hermitian form on $\Omega^{\cdot} (T^*X, \pi^*F)$,

$$\left\langle s, s' \right\rangle_{\mathfrak{h}^{\Omega^{\cdot}(T^*X,\pi^*F)}} = \left\langle us, s' \right\rangle_{\mathfrak{g}^{\Omega^{\cdot}(T^*X,\pi^*F)}}. \tag{4.4}$$

Note here that a Hermitian form has the same properties as a Hermitian product, except for positivity.

If $\mathcal{H}$ is a r-invariant smooth function on T^*X, if $s, s' \in \Omega^{\cdot} (T^*X, \pi^*F)$, set

$$\left\langle s, s' \right\rangle_{\mathfrak{h}_{\mathcal{H}}^{\Omega^{\cdot}(T^*X,\pi^*F)}} = \left\langle e^{-2\mathcal{H}} s, s' \right\rangle_{\mathfrak{h}^{\Omega^{\cdot}(T^*X,\pi^*F)}}. \tag{4.5}$$

Note that since $\mathcal{H}$ is r-invariant, $\mathfrak{h}_{\mathcal{H}}^{\Omega^{\cdot}(T^*X,\pi^*F)}$ is still a Hermitian form. The Hermitian forms in (4.4), (4.5) have signature (∞, ∞).

Now we state a result established in [Bis05, Theorem 2.21].

Theorem 4.1. *If $\mathcal{H}$ is r-invariant, then $A_{\phi,\mathcal{H}}$ is $\mathfrak{h}_{\mathcal{H}}^{\Omega^{\cdot}(T^*X,\pi^*F)}$ self-adjoint, and $\mathfrak{A}_{\phi,\mathcal{H}}$ is $\mathfrak{h}^{\Omega^{\cdot}(T^*X,\pi^*F)}$ self-adjoint.*

Of course, Theorem 2.1 applies to the operators associated to $\mathcal{H} = \mathcal{H}^c$ which were considered in section 2. Its implications are discussed in [Bis05, section 1] and [BL06].

5. The analysis of the hypoelliptic Laplacian

Now we briefly describe some results on the analysis of the operator $\mathfrak{A}^2_{\phi,\mathcal{H}^c}$ which are established in [BL06]. One of the key results is that $\mathfrak{A}^2_{\phi,\mathcal{H}^c}$ has compact resolvent, that its spectrum is discrete, and that the corresponding characteristic subspaces are finite dimensional and included in the Schwartz space $\mathcal{S}^{\cdot} (T^*X, \pi^*F)$.

Of special interest from the point of view of Hodge theory is the characteristic subspace $\mathcal{S}^{\cdot} (T^*X, \pi^*F)_0$ attached to the eigenvalue 0. The spectral projection provides a natural supplementary subspace $\mathcal{S}^{\cdot}(T^*X, \pi^*F)_*$ to $\mathcal{S}^{\cdot}(T^*X, \pi^*F)_0$.

Let $\mathfrak{H}^{\cdot}(X,F)$ denote the ordinary cohomology of T^*X when $c > 0$, and the compactly supported cohomology of T^*X for $c < 0$. For $c > 0$, $\mathfrak{H}^{\cdot}(X,F) = H^{\cdot}(X,F)$, and for $c < 0$, $\mathfrak{H}^{\cdot}(X,F) = H^{\cdot - n}(X, F \otimes o(TX))$, this last identification being the Thom isomorphism.

In [BL06], it is shown that the complex $\left(\mathcal{S}^{\cdot}(T^*X, \pi^*F)_*, d^{T^*X}_{\mathcal{H}^c}\right)$ is acyclic, and that the cohomology of $\left(\mathcal{S}^{\cdot}(T^*X, \pi^*F)_0, d^{T^*X}_{\mathcal{H}^c}\right)$ is just $\mathfrak{H}^{\cdot}(X,F)$.

We will say that $b > 0$ is of Hodge type if all the classical consequences of Hodge theory hold for the hypoelliptic Laplacian $\mathfrak{A}^2_{\phi,\mathcal{H}^c}$, which means in particular that $d^{T^*X}_{\mathcal{H}^c}$ vanishes on $\mathcal{S}^{\cdot}(T^*X, \pi^*F)_0$.

In [BL06], it is shown that for $b > 0$ small enough, b is of Hodge type, and also that the set of $b > 0$ which are not of Hodge type is discrete. The proof relies in particular on the fact that classical Hodge theory is... of Hodge type, and moreover that being of Hodge type is an open property.

Finally it is shown in [BL06] that, as explained in section 3, the resolvent of a suitably conjugate version of $A^2_{\phi,\mathcal{H}^c}$ converges in the strongest possible sense to the resolvent of $\square^X/4$, and also that the corresponding heat kernels converge in a very strong sense.

6. The hypoelliptic Laplacian and analytic torsion

Set

$$\lambda(F) = \det H^{\cdot}(X,F). \tag{6.1}$$

Put

$$\begin{aligned} \lambda = & \lambda(F) \text{ if } c > 0, \\ & (\lambda(F \otimes o(TX)))^{(-1)^n} \text{ if } c < 0. \end{aligned} \tag{6.2}$$

The line λ can be equipped with the classical Ray-Singer metric $\| \|^2_{\lambda,0}$, which one obtains via the Ray-Singer analytic torsion for the Hodge Laplacian $\square^X$.

On the other hand, for $b > 0$, one can define a generalized metric $\| \lambda\|^2_{\lambda,b}$ on λ, which is obtained via the analytic torsion or $A^2_{\phi,\mathcal{H}^c}$. Its construction also involves the Hermitian form $\mathfrak{h}^{\Omega^{\cdot}(T^*X,\pi^*F)}_{\mathcal{H}^c}$. Contrary to an usual metric, this generalized metric has a sign. When the sign is positive, it is a usual metric.

The main result established in [BL06] is as follows.

Theorem 6.1. *For $b > 0$, we have the identity,*

$$\| \|^2_{\lambda,b} = \| \|^2_{\lambda,0}. \tag{6.3}$$

The proof of Theorem 6.1 is difficult. Besides the functional analytic machine which is needed to handle the hypoelliptic Laplacian properly, one also needs to develop a local index theory for this operator. It is rather easy to show that the generalized metric $\| \|^2_{\lambda,b}$ does not depend on $b > 0$. Showing equality in (6.3) is much harder. One has to take full advantage of the convergence of resolvents which was described in sections 3 and 5.

In fact equality in (6.3) should not be taken for granted. Indeed the small time asymptotics of the heat kernels associated to elliptic or hypoelliptic operators are very different. On a priori grounds, one could expect a term measuring the transition from the hypoelliptic regime to the elliptic one. In fact such a term appears when one considers the equivariant version of the above metrics.

7. The hypoelliptic Laplacian and Chern-Gauss-Bonnet

Let $\left(E, g^E, \nabla^E\right)$ be a real Euclidean vector bundle of dimension n on a manifold M, which is equipped with a metric preserving connection. Let Φ^E be the Mathai-Quillen Thom form [MQ86] associated to $\left(g^E, \nabla^E\right)$. The Mathai-Quillen Thom form, which is a form of degree n, will be normalized in such a way that if p is the generic element of E,

$$\Phi^E = \exp\Big(-\left|p\right|^2/2 + \dots\Big). \tag{7.1}$$

Note that the normalization in (7.1) is different from the one which is used in (3.4).

Let s be a smooth section of E on M. Then $s^*\Phi^E$ is a closed n-form on M, whose cohomology class does not depend on s. For $T > 0$, set

$$a_T = (Ts)^* \, \Phi^E. \tag{7.2}$$

Then a_T is a family of closed n-forms, which lie in the same cohomology class. The form a_0 is just the Chern form $e\left(E, \nabla^E\right) = \operatorname{Pf}\left[R^E/2\pi\right]$ which appears in Chern's version of Chern-Gauss-Bonnet [Che44]. By (7.1), we get

$$a_T = \exp\Big(-T^2\left|s\right|^2/2 + \dots\Big). \tag{7.3}$$

Equation (7.3) indicates that when $T \to +\infty$, the current a_T localizes near the vanishing locus Y of s. If the section s is generic, then Y is a submanifold of M. One can establish that when $T \to +\infty$, a_T converges as a current to an explicit current localized on Y.

The strategy used by Chern [Che44] to prove Chern-Gauss-Bonnet is closely related to the above argument. Indeed he constructs directly a transgressed version of the Thom form, from which the Gauss-Bonnet theorem follows by an argument essentially similar to the one outlined above.

Physicists have taught us that some version of the Chern-Gauss-Bonnet theorem still holds in infinite dimensions, thereby establishing a connection between an often mathematically ill-defined functional integral and its localisation on the zero set of the section of some infinite dimensional vector bundle, which is directly accessible to mathematical understanding.

We will illustrate this point in the context of the Witten deformation of classical Hodge theory, and later explain the relevance of Chern's point of view to the hypoelliptic Laplacian.

Indeed let X be a Riemannian manifold as above. We take here F to be just $\mathbf{R}$ equipped with its canonical metric. Let $f : X \to \mathbf{R}$ be a smooth function. In [Wit82], Witten proposed a deformation of Hodge theory associated to the function f. Given $T \in \mathbf{R}$, the idea is to replace the de Rham operator d^X by the twisted version $d^X_T = e^{-Tf} d^X e^{Tf}$, and to form the corresponding Laplacian $\square^X_T$.

Observe the following simple three points:

- For $T = 0$, $\square^X_T = \square^X$.
- The operator $\square^X_T$ is still a second order elliptic self-adjoint nonnegative operator.
- The Hodge theorem still holds for $\square^X_T$, i.e. $\ker \square^X_T$ still represents $H^{\cdot}(X, \mathbf{R})$.

Assume that f is a Morse function. In [Wit82], Witten showed that when $T \to +\infty$, most of the eigenvalues of $\square^X_T$ tend to $+\infty$, except a finite family of them which are either 0 or are exponentially small. Moreover the finite dimensional complex $\left(F^{\cdot}_T, d^X_T\right)$ of eigenforms associated to small eigenvalues localizes near the critical points of f, the forms of degree i localizing near the critical points of index i, from which the Morse inequalities immediately follow.

We will not focus here on the refinements suggested by Witten concerning the explicit description of the complex $\left(F^{\cdot}_T, d^X_T\right)$ in terms of the Morse-Smale complex associated to the gradient field $-\nabla f$. The main point we want to make is that $\square^X_T$ provides an interpolation between Hodge theory and Morse theory.

When f is Morse, the gradient field ∇f is a generic section of TX. The corresponding forms a_T as in (7.2) interpolate between the Chern form $\mathrm{Pf}\left[\frac{R^X}{2\pi}\right]$ and a signed sum of Dirac masses concentrated at the critical points of f.

We will briefly explain how the fact that $\square^X_T$ interpolates between classical Hodge theory and Morse theory can be interpreted as a consequence of the same localization principle on the loop space LX of X, which is the set of smooth maps $s \in S^1 \to x_s \in X$.

We start from observations of Atiyah and Witten [Ati85]. Note that LX is a Riemannian manifold, which inherits its Riemannian metric g^{TLX} from the metric g^{TX}. Also S^1 acts isometrically on LX , so that if $t \in S^1, x \in LX$, $k_t x_{\cdot} = x_{\cdot + t}$. The generator of this action is the Killing vector field $K(x) = \dot{x}$. The manifold X sits inside LX as the zero set of K.

The function f lifts to the S^1-invariant function F on LX,

$$F(x) = \int_{S^1} f(x_s)\, ds. \tag{7.4}$$

By the McKean-Singer formula [MS67], we find that if $\chi(X)$ is the Euler characteristic of X, then

$$\chi(X) = \mathrm{Tr_s}\left[\exp\left(-\square^X_T/2\right)\right]. \tag{7.5}$$

Using functional integration, and more specifically the theory of Brownian motion, we can rewrite the right-hand side of (7.5) in the form,

$$\mathrm{Tr_s}\left[\exp\left(-\square^X_T/2\right)\right] = \int_{L^0X} d\mu_T. \tag{7.6}$$

In (7.6), μ_T is a signed measure on L^0X, the set of continuous loops in X, which is S^1-invariant. The fact that μ_T is carried by L^0X and not by LX is a well-known pathology associate with functional integration.

By using arguments developed first by Atiyah and Witten in [Ati85] and later pursued in [Bis85, Bis86], one can transform the well defined integral in the right-hand side of (7.6) into an ill defined integral of a current on LX. More specifically, we rewrite (7.6) in the form,

$$\mathrm{Tr_s}\left[\exp\left(-\square^X_T/2\right)\right] = \int_{LX} \alpha \wedge (T\nabla F)^* \,\Phi^{TLX}. \tag{7.7}$$

Note that we have replaced L^0X by LX for notational expediency. Let us briefly describe the two forms which appear in the right-hand side of (7.7). First they are both closed with respect to the operator $d + i_K$, which is the equivariant version of the de Rham operator. Since $L_K = (d + i_K)^2$, these forms are also S^1-invariant. The vanishing under $d + i_K$ is called supersymmetry in the physics literature.

Let $E(x) = \frac{1}{2}\int_{S^1} |\dot{x}|^2 ds$ be the energy functional on LX. The form α takes the form

$$\alpha = \exp(-E + \omega). \tag{7.8}$$

The form ω is a closed 2-form which we will not describe more precisely.

The form Φ^{TLX} is the equivariant Thom form for TLX equipped with the metric g^{TLX}, the Levi-Civita connection ∇^{TLX} and the action of K. In view of (7.1), (7.7), (7.8), we get

$$\mathrm{Tr_s}\left[\exp\left(-\square^X_T/2\right)\right] = \int_{LX} \exp\left(-\frac{1}{2}\int_{S^1} |\dot{x}|^2 ds - \frac{T^2}{2}\int_{S^1} |\nabla f(x_s)|^2 ds + \dots\right). \tag{7.9}$$

The point about (7.9) is that for $T = 0$, we get a classical Brownian integral which is known to be connected with the Hodge Laplacian $\square^X/2$. For $T \to +\infty$, the integral (7.9) should localize on $\nabla f = 0$.

The above picture gives us a geometric understanding of the localization of the heat kernels on the diagonal near the critical points of f, of which the standard localization of the form a_T associated with ∇f appears as a semiclassical limit, when scaling the metric g^{TX} by a factor $1/t$ and making $t \to 0$.

Now LX carries many natural S^1 functionals like the energy E or more generally any functional

$$I(x) = \int_{S^1} L(x, \dot{x})\, ds, \tag{7.10}$$

where L is a classical Lagrangian. Of course when $L(x, \dot{x}) = \frac{1}{2} |\dot{x}|^2$, then $I = E$. The idea is then to replace F by E in (7.7). More precisely consider a path integral of the type

$$\int_{LX} \alpha \wedge (T\nabla E)^* \Phi^{TLX}. \tag{7.11}$$

One can ask whether there is a new Hodge theory which would extend (7.7) to an expression of the type (7.11).

This is exactly what the the hypoelliptic Laplacian $2A^2_{\phi,\mathcal{H}^c}$ does, with $c = \pm 1/b^2, T = b^2$. Indeed in this case equation (7.9) is replaced by

$$\operatorname{Tr_s}\left[\exp\left(-2A^2_{\phi,\mathcal{H}^c}\right)\right] = \int_{LX} \exp\left(-\frac{1}{2}\int_{S^1} |\dot{x}|^2\, ds - \frac{T^2}{2}\int_{S^1} |\ddot{x}|^2\, ds + \dots\right). \tag{7.12}$$

For $T = 0$, we should recover the classical Hodge theory for $\square^X/2$, and for $T \to +\infty$, the integral in (7.12) should localize on closed geodesics.

The results which were described in the previous sections come as close as possible to fulfil this dream.

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