measure, J. Math. Kyoto Univ., **20** (1980), 263–289.
[79] ——, de Rham-Hodge-Kodaira's decomposition on abstract Wiener spaces and its application to infinite dimensional diffusion processes, preprint.
[80] ——, On a quasi everywhere existence of the local time of the 1-dimensional Brownian motion, to appear in Osaka J. Math.
[81] V. Skorohod, On a generalization of a stochastic integral, Theor. Prob. Appl. **20** (1975), 219–233.
[82] D. W. Stroock, The Malliavin calculus and its applications to second order parabolic differential operators, I, II, Math. System Theory, **14** (1981), 25–65, 141–171.
[83] ——, The Malliavin calculus, a functional analytic approach, J. Func. Anal., **44** (1981), 217–257.
[84] ——, The Malliavin calculus and its applications, in "Stochastic Integrals" ed. by D. Williams, Lect. Notes in Math., **851** (1981), 394–432, Springer-Verlag, Berlin.
[85] ——, Lectures on topics in stochastic differential equations, noted by S. Karmaker, Tata Inst. Fund. Res., 1982.
[86] ——, Some applications of stochastic calculus to partial differential equations, Ecole d'été de Probabilités de Saint Flour, ed. par P. L. Hennequin, Lect. Notes in Math., **976** (1983), 268–382, Springer-Verlag, Berlin.
[87] ——, Stochastic analysis and regularity properties of certain partial differential operators, Proc. ICM, 1983.
[88] D. W. Stroock and S. R. S. Varadhan, On the support of diffusion processes with applications to the strong maximum principle, Proc. Sixth Berkeley Symp. Math. Statist. Prob., III, 333–359, Univ. Calif. Press, Berkeley, 1972.
[89] H. Sugita, Sobolev spaces of Wiener functionals and Malliavin's calculus, to appear in J. Math. Kyoto Univ.
[90] M. Takeda, (r, p)-capacity on the Wiener space and properties of Brownian motion, preprint.
[91] S. Taniguchi, Malliavin's stochastic calculus of variations for manifold-valued Wiener functionals and is applications, Z. Wahr. verw. Geb., **65** (1983), 260–290.
[92] ——, The adjoint operator of weak derivative as Itô integrals, preprint.
[93] S. Watanabe, Malliavin's calculus in terms of generalized Wiener functionals, Theory and application of random fields, Proc. IFIP-WG 7/1 Working conf. at Bangalore, ed. by G. Kallianpur, Lect. Notes in Cont. and Inform. Sci., **49** (1983), 284–290, Springer-Verlag, Berlin.
[94] ——, Lectures on stochastic differential equations and Malliavin calculus, noted by M. Gopalan Nair and B. Rajeev, Tata Inst. Fund. Res., 1984.
[95] D. Williams, "To begin at the beginning . . .", in "Stochastic Integrals", ed. by D. Williams, Lect. Notes in Math., **851** (1981), 1–55, Springer-Verlag, Berlin.
[96] N. Wiener, Collected Works, Vol. 1, ed. by P. Masani, MIT Press, Cambridge.

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Jump Processes and Boundary Processes

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§ 0. Introduction

In this paper, we will present the results which we have recently obtained in our two papers [4] and [6]. Since the starting point of these two papers has been the development of the Malliavin calculus by Malliavin, we start by giving a brief history of this method. Consider the stochastic differential equation in Stratonovitch form

$$dx = X_0(x)dt + \sum_1^m X_i(x)\cdot dw^i\,, \qquad x(0) = x_0 \tag{0.1}$$

where $X_0, X_1, \cdots, X_m$ are smooth vector fields, and $w = (w^1, \cdots, w^m)$ is a Brownian motion. (0.1) defines a Markov continuous diffusion whose generator $\mathscr{L}$ is given by

$$\mathscr{L} = X_0 + \frac{1}{2}\sum_1^m X_i^2\,.$$

The smoothness of the transition probabilities for the diffusion (0.1)–which define the semi-group $e^{t\mathscr{L}}$–is usually studied via Hörmander's theorem on hypoelliptic second order differential operators [10].

In particular we know from Hörmander's theorem that under conditions on $X_0, X_1, \cdots, X_m$ and their Lie brackets, the operator $\partial/\partial t + \mathscr{L}$ is hypoelliptic, and that the transition probabilities are smooth.

In [20] and [21], Malliavin developed a purely probabilistic method of proof for the existence of smooth transition probabilities. The idea in [20] and [21] was to use the stochastic differential equation (0.1) itself to get a direct proof for smoothness. To do this, Malliavin showed that it was possible to integrate by parts on the Wiener space, and that a wide class of functionals of $x_.(\omega)$ given by (0.1) could be submitted to such a calculus of variations. To prove that integration by parts is indeed possible, Malliavin used the Ornstein-Uhlenbeck operator $\mathscr{A}$ which is an unbounded self-adjoint operator acting on the L_2 space (for the Wiener measure), and the corresponding infinite dimensional Ornstein-Uhlenbeck process whose generator is $\mathscr{A}$. Still using the Ornstein-Uhlenbeck operator, Shigekawa [24], Stroock [26], [27] and Ikeda-Watanabe [12] simplified and extended Malliavin's original approach. In particular the estimates which prove the smoothness of the transition probabilities were proved in Malliavin [21], Ikeda-Watanabe [12] and improved in Kusuoka-Stroock [19], [28] where the full Hörmander's theorem was in fact obtained.

In [3], we suggested a different approach to the Malliavin calculus, based on the quasi-invariance of the Wiener measure, which is expressed by the well-known Girsanov transformation. A formula of integration by parts was then obtained in [3], which was in fact deeply related to a result of Haussmann [9] on the representation of certain Fréchet differentiable functionals of the trajectory $x_.(\omega)$ as stochastic integrals with respect to the Brownian motion w.

In [4], we developed a calculus of variations on jump processes. Our motivation was:

a) To try to exploit the resources of the stochastic calculus on jump processes, and in particular the existence of a Girsanov transformation on jump processes (see Jacod [14]) in a framework where no clear-cut extension of the Ornstein-Uhlenbeck process exists.

b) To understand better how the calculus of variations on diffusions works, in particular in its relation to the classical Itô calculus and martingale theory.

c) To obtain specific analytic results on a class of jump processes.

In [4], the computations seem to be difficult. One of the key reasons (which may appear in Section 1) is that in spite of all the randomness of Brownian motion, all the α-variations of the Brownian motion are deterministic processes, which is not the case for jump processes. The calculus of variations developed in [4] consists in doing an "elementary" integration by parts at the level of each jump, so that in the end an infinite number of classical integration by parts has to be done. In the limiting Brownian motion case, many complications are smoothed out due to continuous stochastic integration, which makes the unpleasant variation terms disappear.

In Section 1 of this paper, we present another approach to the calculus of variations on jump processes, which is based on more elementary arguments than in [4], and does not rely on the Girsanov transformation on jump processes. Section 1 should make the reading of [4] easier although much of the technical work is done in [4]. The estimation techniques are briefly indicated.

In our later paper [6], we still focused on a special sort of jump processes, which are the boundary processes of continuous diffusions. Our motivations were:

a) To understand better the relation between certain pseudo-differential operators [30] and the stochastic calculus.

b) To exhibit the interplay between the continuous diffusion (and its continuous martingales) and the discontinuous boundary process (and its discontinuous martingales). In particular, we felt that the Itô theory of excursions (Itô-McKean [13], Ikeda-Watanabe [12]) could be a powerful analytic tool to study the boundary semi-groups.

c) To understand the possible interplay between the calculus of variations on the continuous diffusion and the calculus of variations on the jump boundary process.

d) To try to exhibit some "Hörmander-like" interaction between the "drift" and the Lévy kernel of the boundary processes or between two Lévy kernels.

e) To find degeneracy conditions on the continuous diffusion so that the boundary process would exhibit a slowly regularizing behavior which is typical of some jump processes.

Section 2 gives a simplified account of our results in [6]. Proofs are in general briefly indicated. The proofs exhibiting the interaction described in d) are given in detail. Relations with the techniques of enlargement of filtrations [15], [16], [17], and [38] are exhibited.

§ 1. The calculus of variations on jump processes

The purpose of this section is to present some of the methods and results which we obtained in [4] on the calculus of variations for jump processes (these results were announced in [5]).

Recall that in [3], we had given an approach to the Malliavin calculus using the quasi-invariance of the Brownian measure which is expressed through the Girsanov exponential formula.

In [4], our idea was to explore if the Girsanov transformation on jump processes (see Jacod [14]) could be the starting point for the development of another calculus of variations whose purpose would be to study the transition probability laws for Markov jumps processes. An integration by parts formula was proved in [4] using such arguments. In particular it appeared that such a formula could be obtained as the consequence of an infinite number of integration by parts in the Lévy kernel of the considered jump processes. In [4], we applied this technique to study the transition probabilities of a special class of pure jumps processes whose construction was elementary using auxiliary independent increment jump processes.

The Malliavin calculus of variations on diffusions is based on certain stochastic differential equations. The Brownian motion model is still important for two reasons:

a) It is essential to build explicitly the solutions of stochastic differential equations, which can then be submitted to the calculus of variations on the Brownian motion space.

b) The necessary estimates are obtained by reference to the Brownian motion model, [12], [19], [21], [28].

Of course, if the Malliavin calculus of variations is applied on the Brownian motion itself, it gives–not unexpectedly–essentially trivial results (for an illuminating discussion of this case, see Williams [36]).

As it appears in [4], this is not the case on jump processes, even when the calculus is applied to independent increment jump processes. Moreover although computations are elementary in their principle, the resulting formulas are extremely heavy to manipulate.

In this section, we will try to present the calculus of variations on independent increment jump processes in an elementary way, i.e. based on the most elementary aspects of their structure. Once this is done, the reader can at least have an intuition for how to study more complex Markov jump processes, constructed by means of such independent increment processes, as in [4]. Note that we will only briefly address the question of knowing what is the "right" formulation for expressing a Markov jump processes in terms of independent increment jump processes.

In a) an integration by parts formula is proved on the probability space of an independent increment jump process. In b) this formula is used to obtain an integration by parts formula on the semi-group of the considered process. In c) the estimates which are necessary to make such a formula valid are derived. In d), the application of such methods to more general processes is briefly considered along the lines of [4]. Finally e) is devoted to some geometrical considerations.

a) Integration by parts on independent increment jump processes

Let $g(z)$ be a function defined on $R^n/\{0\}$ with values in R^+ which has the following properties.

a) g is differentiable with a continuous derivative g_z.

b) g is such that

$$\int_{R^n}(|z|^2 \wedge 1)g(z)dz < +\infty . \tag{1.1}$$

Note that since g is ≥ 0, if $g(z)=0$, then also $g_z(z)=0$.

For $t \geq 0$, $\alpha \in R^n$, set

$$\psi_t(\alpha) = \exp\Big[t\int_{|z|\leq 1}(e^{-i\langle\alpha,z\rangle}-1+i\langle\alpha,z\rangle)g(z)dz + t\int_{|z|>1}(e^{-i\langle\alpha,z\rangle}-1)g(z)dz\Big] \tag{1.2}$$

$D(R^n)$ denotes the space of functions defined on R^+ with values in R^n which are right-continuous with left-hand limits. $\{\mathscr{F}_t\}_{t\geq 0}$ denotes the canonical filtration of $D(R^n)$ (here $\mathscr{F}_t = \mathscr{B}(z_s|s\leq t)$). $\{\mathscr{F}_t\}_{t\geq 0}$ will eventually be made right-continuous and complete as in Dellacherie-Meyer [8] without further mention. $\mathscr{P}$ is the predictable σ-field on $R^+\times D(R^n)$ [8], [14]. Let z_t be the independent increment right-continuous process whose characteristic function is given by (1.2). Let Π be the probability law of z. on $D(R^n)$. If $\lambda_t(\omega, z)$ is a function defined on $R^+\times\Omega\times R^n/\{0\}$ with values in R such that

a) λ_t is $\mathscr{P}\otimes\mathscr{B}(R^n/\{0\})$-measurable and

b) for any $t\geq 0$

$$E^\Pi\Big[\int_0^t ds\int_{R^n}|\lambda_s(\omega,z)|g(z)dz\Big] < +\infty , \tag{1.3}$$

(resp.

$$E^\Pi\Big[\int_0^t ds\int_{R^n}|\lambda_s(\omega,z)|^2 g(z)dz\Big] < +\infty\ ), \tag{1.3'}$$

then we denote by $S_{s\leq t}\lambda$, $S^c_{s\leq t}\lambda$ (resp. $S^c_{s\leq t}\lambda$) the right-continuous processes defined by

$$\begin{aligned} S_{s\leq t}\lambda &= \sum_{\substack{s\leq t\\ \Delta z_s\neq 0}}\lambda_s(\omega,\Delta z_s) ,\\ S^c_{s\leq t}\lambda &= S_{s\leq t}\lambda - \int_0^t ds\int_{R^n}\lambda_s(\omega,z)g(z)dz , \end{aligned} \tag{1.4}$$

(resp.

$$S^c_{s\leq t}\lambda = \lim_{\varepsilon\downarrow\downarrow 0} S^c_{s\leq t}(I_{|z|\geq\varepsilon}\lambda) \tag{1.4'}$$

where for $\varepsilon>0$, the right hand side of (1.4') is defined by (1.4), and the limit is taken in probability uniformly on every compact set in R^+). By the results in Jacod [14] and Ikeda-Watanabe [12], we know that under (1.3) (resp. (1.3')), $S^c_{s\leq t}\lambda$ is a martingale (resp. a square-integrable martingale).

We then have the first elementary result of integration by parts on $(D(R^n), \Pi)$, given in [5], [4].

Theorem 1.1. *Let $\Lambda_s(\omega, z)$ be a function defined on $R^+\times\Omega\times R^n/\{0\}$, with values in R^n which has the following properties:*

a) Λ is $\mathscr{P}\otimes\mathscr{B}(R^n/\{0\})$-measurable and bounded.

b) Λ is differentiable in the variable z and has a bounded differential Λ_z.

c) *There exist* ε *and* M *with* $0<\varepsilon<M$, *such that if* $|z|\le\varepsilon$ *or* $|z|\ge M$, $\Lambda_s(\omega,z)=0$. *Then for any* $f\in C_b^\infty(R^n)$, $T\ge 0$

$$E^{\Pi}[\langle d_z f(z_T), S_{s\le T}\Lambda\rangle]+E^{\Pi}\left[f(z_T)S^c_{s\le T}\frac{\operatorname{div}_z g(z)\Lambda(z)}{g(z)}\right]=0. \tag{1.5}$$

Proof. First observe that (1.5) makes sense. In fact:

a) Since for $|z|\le\varepsilon$, $\Lambda(z)=0$, Λ being bounded, we know that $S_{s\le T}\Lambda$ is integrable.

b) Observe that *a priori*, $\operatorname{div}_z g(z)\Lambda(z)/g(z)$ is not well defined when $g(z)=0$. However, it is elementary to prove that a.s., if s is such that $\Delta z_s\neq 0$, $g(\Delta z_s)\neq 0$. Now since

$$\frac{\operatorname{div}_z g(z)\Lambda(z)}{g(z)}=\left\langle\frac{g_z(z)}{g(z)},\ \Lambda(z)\right\rangle+\operatorname{div}_z\Lambda(z)$$

it is easy to see that

$$\int_{\varepsilon\le|z|\le M}\left|\frac{\operatorname{div}_z g(z)\Lambda(z)}{g(z)}\right|g(z)\,dz$$

is uniformly bounded. $S^c_{s\le T}(\operatorname{div}_z g(z)\Lambda(z))/g(z))$ is then a martingale.

We now give a short proof of (1.5). We first assume that

$$\int_{R^n}g(z)dz<+\infty. \tag{1.6}$$

(1.6) means that a.s., z_t has only finitely many jumps. Let $S_1, S_2,\cdots,S_n,\cdots$ be the increasing sequence of stopping times at which z jumps. It is well-known that $S_1, S_2-S_1,\cdots,S_n-S_{n-1},\cdots,\Delta z_{S_1},\Delta z_{S_2},\cdots,\Delta z_{S_n},\cdots$ are independent random variables and that moreover for every $i\in N$, the probability law of Δz_{S_i} is given by

$$\frac{g(z)dz}{\displaystyle\int_{R^n}g(z')dz'}. \tag{1.7}$$

Using (1.2), we know that

$$z_t=S_{s\le t}\Delta z_s-kt \tag{1.8}$$

where $k=\int_{|z|\le 1}zg(z)dz$. Finally it is easy to show that since Λ is $\mathscr{P}\otimes\mathscr{B}_{(R^n/\{0\})}$-measurable, then

(1.9) for $S_{n-1}<s\le S_n$, $\Lambda_s(\omega,z)=\Lambda^n_s(S_1,\cdots,S_{n-1},\Delta z_{S_1},\cdots,\Delta z_{S_{n-1}},z)$.

Let $E^{\Pi}[\cdot|S]$ be the conditional expectation operator given $S=(S_1,\cdots,S_n,\cdots)$. Let N_T be defined by

$$S_{N_T}\le T<S_{N_T+1} \tag{1.10}$$

Using (1.9) we obtain for $N_T\neq 0$.

$$\begin{aligned}&E^{\Pi}[\langle d_z f(z_T)\ \ S_{s\le T}\Lambda\rangle|S]\\&=\sum_1^{N_T}E^{\Pi}[\langle d_z f(z_T),\Lambda^n_{S_n}(S_1,\cdots,S_{n-1},\Delta z_{S_1},\cdots,\Delta z_{S_{n-1}},\Delta z_{S_n})\rangle|S]\\&=\frac{1}{\int g(z')\,dz'}\sum_{n=1}^{N_T}E^{\Pi}\Big[\int g(z)\,dz\Big\langle d_z f\Big(\sum_{\substack{n'=1\\n'\neq n}}^{N_T}\Delta z_{S_{n'}}+z-kT\Big),\\&\qquad\Lambda^n_{S_n}(S_1,\cdots,S_{n-1},\Delta z_{S_1},\cdots,\Delta z_{S_{n-1}},z)\Big\rangle g(z)dz|S\Big].\end{aligned} \tag{1.11}$$

Since $\Lambda(z)$ has compact support, it is feasible to integrate by parts in the variable z in each of the terms of the sum in the right hand side of (1.11), so that (1.11) is equal to

$$\begin{aligned}&-\frac{1}{\int g(z')dz'}\sum_{n=1}^{N_T}E^{\Pi}\Big[\int f\Big(\sum_{\substack{n'=1\\n'\neq n}}^{N_T}\Delta z_{S_{n'}}+z-kT\Big)\\&\qquad\times\frac{\operatorname{div}_z[g(z)\Lambda^n_{S_n}(S_1,\cdots,S_{n-1},\Delta z_{S_1},\cdots,\Delta z_{S_{n-1}},z)]}{g(z)}g(z)dz|S\Big].\end{aligned} \tag{1.12}$$

Observe that (1.12) is well defined *even* if g may be 0 at some points since $g(z)dz$ gives measure 0 to these points. Now clearly (1.12) is equal to

$$-E^{\Pi}\left[f(z_T)S_{s\le T}\frac{\operatorname{div}_{z=\Delta z_s}g(z)\Lambda_s(\omega,z)}{g(z)}\Big|S\right]. \tag{1.13}$$

This equality also holds if $N_T=0$. Using (1.11), (1.13) and integrating in the variables $S_1,\cdots,S_n,\cdots$, we get

$$E^{\Pi}[\langle d_z f(z_T), S_{s\le T}\Lambda\rangle]+E^{\Pi}\left[f(z_T)S_{s\le T}\frac{\operatorname{div}_z g(z)\Lambda(z)}{g(z)}\right]=0. \tag{1.14}$$

Now clearly since $\Lambda(z)$ has compact support

$$\int\frac{\operatorname{div}_z g(z)\Lambda(z)}{g(z)}g(z)dz=0 \tag{1.15}$$

so that

(1.16) $$S_{s\leq T}\frac{\operatorname{div}_z g(z)A(z)}{g(z)} = S^c_{s\leq T}\frac{\operatorname{div}_z g(z)A(z)}{g(z)}$$

(1.5) is proved when (1.6) is verified.

Let ρ be a C^∞ function defined on $\boldsymbol{R}$ with values in [0, 1] which is equal to 1 for $|t|\geq 1$, and to 0 for $|t|\leq 1/2$. Let z'_t, z''_t be the independent increment processes associated to the Lévy measures $\rho(|z|/\varepsilon)g(z)dz$ and $(1-\rho(|z|/\varepsilon))g(z)dz$, and let Π', Π'' be their corresponding probability laws. On $(D(\boldsymbol{R}^n)\times D(\boldsymbol{R}^n), \Pi'\times\Pi'')$, the law of the process z_t given by

$$z_t = z'_t + z''_t$$

is exactly Π. Now by (1.1), we have

(1.17) $$\int \rho\left(\frac{|z|}{\varepsilon}\right)g(z)dz < +\infty\,.$$

Moreover $\Pi'\otimes\Pi''$ a.s., z' and z'' do not have the same jump times. Using (1.17), and reasoning on z' as previously for each fixed z'', it is not hard to obtain

(1.18) $$\begin{aligned} &E^{\Pi'\otimes\Pi''}[\langle d_z f(z_T), S_{s\leq T}A(\omega(z), \Delta z'_s)\rangle] \\ &+ E^{\Pi'\otimes\Pi''}\left[f(z_T)S_{s\leq T}\operatorname{div}_{z=\Delta z'_s}\frac{g(z)A(\omega(z), z)}{g(z)}\right] = 0\,. \end{aligned}$$

Here $A(\omega(z), \cdot)$ is written so as to indicate that $A(\omega, .)$ is evaluated on the trajectory of z. Now all the $|\Delta z''_s|$ are $<\varepsilon$. Since $A(z)=0$ if $|z|\leq\varepsilon$, we may as well replace $\Delta z'_s$ by Δz in (1.18). Using (1.15) again, we obtain (1.5) in the general case.

Remark 1. The integration by parts formula has been obtained very simply by considering at each stage finitely many random variables. We wrote $S^c_{s\leq T}(\operatorname{div}_z g(z)A(z)/g(z))$ instead of $S_{s\leq T}(\operatorname{div}_z g(z)A(z)/g(z))$ for aesthetic reasons, since for $f=1$, it is satisfactory that $0=0$ obtains "obviously" in (1.5). However the Markov property of z forces the appearance of a martingale term in an integration by parts formula. It is a striking feature of the stochastic calculus of variations that since the analysis is done on the path space, the dynamic of the path is reflected in the integration by parts formula.

We will now extend Theorem 1.1 to more general A.

Theorem 1.2. *Let $A_s(\omega, z)$ be a function defined on $\boldsymbol{R}^+\times\Omega\times\boldsymbol{R}^n/\{0\}$ with values in $\boldsymbol{R}^n$, which has the following properties*:

a) *It is $\mathscr{P}\otimes\mathscr{B}(\boldsymbol{R}^n/\{0\})$-measurable.*

b) *A is differentiable in the variable z, with a bounded differential A_z.*

c) *A measurable function λ: $\boldsymbol{R}^n/\{0\}\to\boldsymbol{R}^+$ exists such that*

(1.19) $$\int_{|z|\leq 1}\lambda(z)g(z)dz < +\infty\,,$$

(resp.

(1.19′) $$\int_{|z|\leq 1}\lambda^2(z)g(z)dz < +\infty)$$

and for $|z|\leq 1$,

(1.20) $$|A_s(\omega, z)|\leq\lambda(z)|z|\,.$$

d) *The function*

(1.21) $$\int_{R^n/\{0\}}|\operatorname{grad}_z g(z)A(z)|dz\,,$$

(resp.

(1.21′) $$\int_{R^n/\{0\}} I_{g(z)\neq 0}\frac{|\operatorname{grad}_z g(z)A(z)|^2}{g(z)}dz)$$

is uniformly bounded.

Then for any $f\in C^\infty_b(\boldsymbol{R}^n)$, $T\geq 0$, (1.5) still holds.

Proof. (1.5) still makes sense. In fact, if (1.19) holds, then, if $|z|\leq 1$, $|A_s(\omega, z)|\leq\lambda(z)$, and so using (1.1) and the boundedness of A, we see that $\int|A_s(\omega, z)|g(z)dz$ is uniformly bounded. Similarly if (1.19′) holds, by using (1.1) and Schwarz's inequality, $\int|A_s(\omega, z)|g(z)dz$ is still bounded, and so $S_{s\leq T}A$ exists and is integrable. Similarly, if (1.21) (resp. (1.21′) holds, $S^c_{s\leq T}(\operatorname{div}_z g(z)A(z)/g(z))$ is a martingale (resp. a square-integrable martingale).

We now prove (1.5). Assume $M>0$ exists such that if $|z|\geq M$, $A(z)=0$. Take ρ as in the proof of Theorem 1.1. For $0<\varepsilon<M$, set

(1.22) $$A^\varepsilon_s(\omega, z) = \rho\left(\frac{|z|}{\varepsilon}\right)A_s(\omega, z)\,.$$

(1.5) applies for A^ε. Now we make $\varepsilon\to 0$. Clearly

(1.23) $$E^\Pi\left[\int_0^T ds\int|A_s(\omega, z) - A^\varepsilon_s(\omega, z)|g(z)dz\right]\to 0$$

and so

(1.24) $$S_{s\le T}A^\varepsilon \to S_{s\le T}A \quad \text{in } L_1(\Pi).$$

Assume that (1.19) and (1.21) hold. Then

(1.25) $$\begin{aligned} &E^\Pi \int_0^T ds \int |\mathrm{div}_z gA^\varepsilon - \mathrm{div}_z gA|\,dz \\ &\le E^\Pi \int_0^T ds \int_{|z|\le\varepsilon} \left(|\mathrm{div}_z gA| + \frac{1}{\varepsilon}\left|\rho'\left(\frac{|z|}{\varepsilon}\right)\right| |A(z)|g(z)\right)dz.\end{aligned}$$

Using (1.21), the first term in the right hand side of (1.25) is easily dealt with. Moreover $|\rho'|$ is bounded. Using (1.20) we find

(1.26) $$E^\Pi \int_0^T ds \left[\int_{|z|\le\varepsilon} \frac{1}{\varepsilon}\left|\rho'\left(\frac{z}{\varepsilon}\right)\right| |A(z)|g(z)dz\right] \le C\int_{|z|\le\varepsilon} \lambda(z)g(z)dz.$$

Using (1.19), we see that the right hand side of (1.26) tends to 0 as $\varepsilon\to 0$. So we find that

(1.27) $$\begin{aligned} &S_{s\le T}\frac{\mathrm{div}_z gA^\varepsilon}{g} \to S_{s\le T}\frac{\mathrm{div}_z gA}{g} \quad \text{in } L_1(\Pi), \\ &0 = \int_0^T ds \int \mathrm{div}_z gA^\varepsilon\, ds \to \int_0^T ds \int \mathrm{div}_z gA\, ds \quad \text{in } L_1(\Pi).\end{aligned}$$

As a by-product of (1.27), we also obtain that

(1.28) $$S_{s\le T}\frac{\mathrm{div}_z gA}{g} = S^c \frac{\mathrm{div}_z gA}{g},$$

and so (1.5) holds.

Under (1.19′), (1.21′), we have

(1.29) $$\begin{aligned} &E^\Pi\left[\left|S^c_{s\le T}\frac{\mathrm{div}_z gA}{g} - S^c_{s\le T}\frac{\mathrm{div}_z gA^\varepsilon}{g}\right|^2\right] \\ &= E^\Pi\left[\int_0^T ds \int \left|\frac{\mathrm{div}_z gA}{g} - \frac{\mathrm{div}_z gA^\varepsilon}{g}\right|^2 g(z)ds\right] \\ &\le CE^\Pi\left[\int_0^T ds \int_{|z|\le\varepsilon}\left[\left|\frac{\mathrm{div}_z gA}{g}\right|^2 \right.\right. \\ &\qquad \left.\left. + \frac{1}{\varepsilon^2}\rho'^2\left(\frac{|z|}{\varepsilon}\right)|A(z)|^2\right]g(z)dz\right].\end{aligned}$$

Using (1.21′), the first term in the right hand side of (1.29) is again controled. From (1.20), we get

(1.30) $$E^\Pi\left[\int_0^T ds \int_{|z|\le\varepsilon} \frac{1}{\varepsilon^2}\rho'^2\left(\frac{|z|}{\varepsilon}\right)|A(z)|^2 g(z)dz\right] \le C\int_{|z|\le\varepsilon} \lambda^2(z)g(z)dz.$$

So by (1.19′), we see that the right hand side of (1.30) tends to 0 as $\varepsilon\to 0$. It follows that as $\varepsilon\to 0$

(1.31) $$S^c_{s\le T}\frac{\mathrm{div}_z gA^\varepsilon}{g} \to S^c_{s\le T}\frac{\mathrm{div}_z gA}{g} \quad \text{in } L_2(\Pi).$$

(1.5) still holds. It is then easy to get rid of the conditions that if $|z| \ge M$, $A(z) = 0$.

Remark 2. In general, if (1.19′) and (1.21′) hold, the equality (1.28) does not make sense any more, since the sum $S_{s\le T}(\mathrm{div}_z gA/g)$ is not well defined. Also observe that since A_z is bounded, $A(0) = \lim_{z\to 0} A(z)$ exists. If (1.19) or (1.19′) hold, $\int |A(z)|g(z)dz$ is bounded. So if

$$\int g(z)dz = +\infty,$$

it is clear that $A(0)=0$. Moreover if $\mathrm{grad}_z gA/g$ is bounded, (1.21) implies (1.21′).

Observe that by (1.1), we may take in (1.19′)

$$\lambda(z) = z.$$

In this case the constraint (1.20) reads

(1.32) $$|A(z)| \le |z|^2.$$

This is no accident, since (1.32) shows that the perturbation $S_{s\le T}A$ can at most be of the order of the quadratic variation of the process z_t.

Remark 3. If $n = 1$, it is easy to prove that

(1.33) $$\frac{\Delta}{2} = \lim_{\gamma\uparrow 2}(2-\gamma)\,\mathrm{p.p.}\frac{dz}{|z|^{1+\gamma}} \quad \text{in } \mathscr{D}'(R).$$

Let u_t be a bounded predictable process. Let $z\in R\to \nu(z)$ be a non-negative C^∞ function with compact support such that for $|z|\le 1$, $\nu(z) = z^2$. For $0<\gamma<2$, set

$$g_\gamma(z) = \frac{2-\gamma}{|z|^{1+\gamma}}.$$

Let Π^γ be the probability law of the process $z.$ associated to g_γ. Set

$$A_s(\omega, z) = u_s \nu(z) .$$

If $\lambda(z) = C|z|$, (1.19′) holds, and moreover for C large enough (1.20) holds. It is trivial to check that (1.21′) also holds. For any $f \in C_b^\infty(\mathbf{R})$, we have

$$(1.34) \quad E^{\Pi^\gamma}[f'(z_T) S_{s\le T} u_s \nu(\Delta z_s)] + E^{\Pi^\gamma}\Big[f(z_T) S^c_{s\le T} u_s\Big(\Big[\frac{\nu}{z^{1+\gamma}}\Big]'(z) z^{1+\gamma}\Big)\Big] = 0 .$$

Now make $\gamma \uparrow\uparrow 2$ in (1.34). Using (1.33), it may be proved that $\{\Pi^\gamma\}$ is tight on $\mathscr{D}(\mathbf{R})$ (endowed with the Skorokhod topology) and moreover that as $\gamma \to 2$, Π^γ converges to the Brownian measure P on $\mathscr{C}(\mathbf{R})$. We will take a *formal* limit in (1.34) without too many justifications. Observe that if for $0 \le s \le T$, $|\Delta z_s| \le 1$, then $S_{s\le T} u_s \nu(\Delta z_s) = \int_0^T u_s\, d[z, z]_s$. Now as $\gamma \to 2$, the jumps of Π^γ become "smaller and smaller".

The quadratic variation of z_t for P being equal to t, as $\gamma \uparrow\uparrow 2$, the first term in (1.5) "tends" to $E^P\Big[f'(z_t) \int_0^T u\, ds\Big]$. Moreover for $|z| \le 1$, we have

$$\Big(\frac{\nu}{z^{1+\gamma}}\Big)'(z) z^{1+\gamma} = (1-\gamma) z ,$$

so that $S^c_{s\le T} u_s[\nu/z^{1+\gamma}]'(z) z^{1+\gamma}$ is the compensated sum of jumps, which, when $|\Delta z_s| \le 1$, are exactly $u_s \Delta z_s(1-\gamma)$. It is then reasonable to expect that as $\gamma \uparrow\uparrow 2$, the limit of $S^c_{s\le T} u_s[\nu/z^{1+\gamma}]'(z) z^{1+\gamma}$ will be the Itô integral $-\int_0^T u \delta z$. So as $\gamma \uparrow\uparrow 2$, (1.34) becomes formally

$$(1.35) \quad E^P\Big[f'(z_T) \int_0^T u ds\Big] - E^P\Big[f(z_T) \int_0^T u \delta z\Big] = 0 .$$

It is gratifying that (1.35) holds, and this is shown by a rigorous argument given in [3], [9]. (1.5) then appears to be the natural extension of integration by parts on elementary functionals of Brownian motion to jump processes.

b) Integration by parts on the semi-group of an independent increment process

We will now extend Theorem 1.2 in order to obtain a formula of integration by parts on the semi-group of z_t, i.e. we will prove that for adequately chosen $T > 0$ we have that for $f \in C_b^\infty(\mathbf{R}^n)$, and $1 \le i \le n$

$$(1.36) \quad E^{\Pi}\Big[\frac{\partial f}{\partial z_i}(z_T)\Big] + E^{\Pi}[f(z_T) D_T^i] = 0 .$$

As previously pointed out, formula (1.36) itself is not very interesting, since in this case, we know the characteristic function $\psi_T(\alpha)$ explicitly. However the method to obtain (1.36) is closely related to our work [4], where the calculus of variations is still performed on independent increment jump processes, but the considered functionals are much more complicated. $\lambda(z)$ is a measurable function $\mathbf{R}^n/\{0\} \to \mathbf{R}^+$ such that

$$(1.37) \quad \int_{|z|\le 1} \lambda^2(z) g(z) dz < +\infty .$$

$\nu(z)$ is a function defined on $\mathbf{R}^n/\{0\}$ with values in $\mathbf{R}^+$ having the following properties.

a) $\nu(z)$ is bounded and differentiable, and has a bounded differential ν_z.

b) $C > 0$ exists such that if $|z| \le 1$

$$(1.38) \quad \nu|z| \le C\lambda(z)|z| .$$

c) The following inequality holds

$$(1.39) \quad \int \frac{|\mathrm{grad}_z\, \nu g|^2}{g} dz < +\infty .$$

Choose ρ as in the proof of Theorem 1.1, and for $\eta > 0$ set

$$(1.40) \quad \rho_\eta(t) = \rho(t/\eta) .$$

We now have

Theorem 1.3. *For any $T > 0$, $\eta > 0$, $f \in C_b^\infty(\mathbf{R}^n)$, and i, $(1 \le i \le n)$,*

$$(1.41) \quad \begin{aligned} &E^{\Pi}\Big[\rho_\eta(S_{s\le T}\nu) \frac{\partial f}{\partial z_i}(z_T)\Big] - E^{\Pi}\Big[\frac{\rho_\eta(S_{s\le T}\nu) f(z_T)}{(S_{s\le T}\nu)^2} S_{s\le T}\Big(\frac{\partial \nu}{\partial z_i}\nu\Big)\Big] \\ &+ E^{\Pi}\Big[\frac{\rho'_\eta(S_{s\le T}\nu)}{S_{s\le T}\nu} S_{s\le T}\Big(\frac{\partial \nu}{\partial z_i}\nu\Big) f(z_T)\Big] \\ &+ E^{\Pi}\Big[\frac{\rho_\eta(S_{s\le T}\nu)}{S_{s\le T}\nu} f(z_T)\, S^c_{s\le T} \frac{\partial/\partial z_i(g\nu)}{g}\Big] = 0 . \end{aligned}$$

Proof. (1.41) makes sense. First note that $\rho_\eta(S_{s\le T}\nu)$ and $\rho'_\eta(S_{s\le T}\nu)$ are 0 if $S_{s\le T}\nu \le \eta/2$, so that $\rho_\eta(S_{s\le T}\nu)/S_{s\le T}\nu$, $\rho_\eta(S_{s\le T}\nu)/(S_{s\le T}\nu)^2$, $\rho'_\eta(S_{s\le T}\nu)/S_{s\le T}\nu$ are bounded. Since $\partial\nu/\partial z_i$ is bounded, $S_{s\le T}(\partial\nu/\partial z_i)\nu$ exists and is integrable for the same reason as $S_{s\le T}\nu$ in the proof of Theorem 1.2.

To prove (1.41), the easiest way is to go back to the assumptions in the proof of Theorem 1.1. Namely, we assume that ε, M, $0<\varepsilon<M$, exist such that if $|z|\le\varepsilon$, or $|z|\ge M$, $\nu(z)=0$. We also temporarily assume that (1.6) holds. Using the notations in the proof of Theorem 1.1, we have if $N_T\neq 0$,

$$
\begin{aligned}
E^{\Pi}\left[\rho_\eta(S_{s\le T}\nu)\frac{\partial f}{\partial z_i}(z_T)\middle| S\right] &= E^{\Pi}\left[\frac{\rho_\eta(S_{s\le T}\nu)}{S_{s\le T}\nu}\frac{\partial f}{\partial z_i}(z_T)S_{s\le T}\nu\middle| S\right]\\
&=\sum_1^{N_T}E^{\Pi}\left[\frac{\rho_\eta(S_{s\le T}\nu)}{S_{s\le T}\nu}\frac{\partial f}{\partial z_i}(z_T)\nu(\Delta z_{S_n})\middle| S\right].
\end{aligned}
\tag{1.42}
$$

Set for $n\le N_T$

$$
K_n=\sum_{\substack{n'\neq n\\ n'=1}}^{N_T}\nu(\Delta z_{S_{n'}}),\qquad z_n=z_T-\Delta z_{S_n}.
\tag{1.43}
$$

(1.42) is then equal to

$$
\begin{aligned}
&\frac{1}{\int g(z')dz'}\sum_{n=1}^{N_T}E^{\Pi}\left[\int g(z)dz\,\frac{\rho_\eta(K_n+\nu(z))}{K_n+\nu(z)}\frac{\partial f}{\partial z_i}(z_n+z)\nu(z)\middle| S\right].\\
&=-\frac{1}{\int g(z')dz'}\sum_{n=1}^{N_T}E^{\Pi}\left[\int g(z)dzf(z_n+z)\right.\\
&\quad\times\left\{\frac{(\partial/\partial z_i(g\nu))}{g}(z)\frac{\rho_\eta(K_n+\nu(z))}{K_n+\nu(z)}\right.\\
&\quad+\left.\left.\left(\frac{\rho_\eta'(K_n+\nu(z))}{K_n+\nu(z)}-\frac{\rho_\eta(K_n+\nu(z))}{(K_n+\nu(z))^2}\right)\left(\frac{\partial\nu}{\partial z_i}\nu\right)(z)\right\}\middle| S\right].
\end{aligned}
\tag{1.44}
$$

If $N_T=0$, $\rho_\eta(S_{s\le T}\nu)=0$, so that equality between (1.42) and (1.44) still holds (if $N_T=0$, $\sum_1^{N_T}\cdots$ is taken to be 0). By integrating (1.44) in all variables, we find easily that (1.41) holds.

Assumption (1.6) is released by using the same argument as we used in the proof of Theorem 1.1. The support condition on ν is released as in the proofs of Theorems 1.1 and 1.2, using in particular (1.37), (1.38) and (1.39).

We now make $\eta\to 0$ in (1.41). To obtain (1.36), we need that

$$
\rho_\eta(S_{s\le T}\nu)\to 1\qquad\text{a.s.}
\tag{1.45}
$$

and so we need that $S_{s\le T}\nu>0$ a.s. A necessary condition is that

$$
\int g(z)dz=+\infty.
\tag{1.46}
$$

(for (1.36) to hold (1.46) is needed since otherwise a Dirac mass is left in the law of z_T!).

From Theorem 1.3, we get

Theorem 1.4. *If $T>0$ and ν are such that $1/S_{s\le T}\nu$ is in a given $L_p(\Pi)$ with $p>2$, then for any $f\in C_b^\infty(\boldsymbol{R}^n)$,* (1.41) *still holds with ρ_η replaced by* 1. *For any $k\in\boldsymbol{R}^+$, $t\ge kT$, $|\alpha|^k\psi_t(\alpha)$ is a bounded function. For any $\ell\in\boldsymbol{N}$, $t>(\ell+n)T$, the probability law of z_t is given by $q_t(y)dy$, where $q_t(\cdot)\in C_b^\ell(\boldsymbol{R}^n)$. In particular, if for any $T>0$, $1/S_{s\le T}\nu$ is in a given $L_p(\Pi)$ with $p>2$, then for any $t>0$, the law of z_t is given by $q_t(\cdot)dy$, where $q_t(\cdot)\in C_b^\infty(\boldsymbol{R}^n)$.*

Proof. We make $\eta\to 0$ in (1.41). Clearly (1.46) holds so that the first term in the left hand side of (1.41) is taken care of. It is not hard to prove, using (1.38) and the boundedness of $(\partial\nu/\partial z_i)\nu$ that $S_{s\le T}(\partial\nu/\partial z_i)\nu$ is in all the $L_q(\Pi)$ $(1\le q<+\infty)$. $S_{s\le T}(\partial\nu/\partial z_i)\nu/(S_{s\le T}\nu)^2$ is then in $L_1(\Pi)$.

Moreover

$$
\rho_\eta'(t)=\frac{1}{\eta}\rho'\left(\frac{t}{\eta}\right).
$$

For $t\ge\eta$, $\rho_\eta'(t)=0$, so that if $t>0$, $\rho_\eta'(t)\to 0$ as $\eta\to 0$. Clearly

$$
|\rho_\eta'(t)|\le\frac{C}{t}.
\tag{1.47}
$$

Finally $S_{s\le T}^c(\partial(g\nu)/\partial z_i)/g$is in $L_2(\Pi)$. Using (1.47) we see that the dominated convergence theorem applies in the three last terms of the left hand side of (1.41).

Using (1.41) with $f(z)=e^{-i\langle\alpha,z\rangle}$, we see that for any $i(1\le i\le n)$, $|\alpha^i|\psi_T(\alpha)$ is a bounded function. Since $\psi_t(\alpha)=[\psi_T(\alpha)]^{t/T}$, the result on function ψ_t follows. The results on the law of z_t are then standard results on Fourier transform.

Remark 4. A result of Tucker [31] states that if $\int g(z)dz=+\infty$, for any $t>0$, the law of z_t has a density with respect to the Lebesgue measure. Although the proof in [31] is probabilistic, this result admits an easy analytic proof by differentiating $\psi_t(\alpha)$.

Remark 5. We use the same assumptions and notations as in Remark 3. In particular, assume temporarily that $n=1$. As we shall see later (in Theorem 1.6) if ν is C^∞ with compact support and is such that $\nu(z)=z^2$ for $|z|\le 1$, then for every $\gamma<2$, the conditions of Theorem 1.4 are verified. Now when all the jumps of z are in size ≤ 1, then $S_{s\leq T}\nu = [z, z]_T$. As $\gamma\uparrow\uparrow 2$, it is reasonable to replace $S_{s\leq T}\nu$ by T. Moreover $(\partial\nu/\partial z)\nu = 2z^3$. The 3-variation of z for the Brownian measure P is 0, so for P, we may *formally* cancel $S_{s\leq T}\frac{\partial\nu}{\partial z}\nu$. For $|z|\leq 1$, $\frac{\partial}{\partial z}(g_\gamma\nu)\Big/g_\gamma = (1-\gamma)z$. As $\gamma\uparrow\uparrow 2$, we may expect $S^c_{s\leq T}\left(\frac{\partial}{\partial z}g_\gamma\nu\Big/g\right)$ to become $-z_T$. Taking the formal limit in (1.41) (with $\rho_\gamma = 1$), we get that for $T>0, f\in C_b^\infty(R)$,

$$E^P\left[\frac{\partial f}{\partial z}(z_T)\right] - E^P\left[f(z_T)\frac{z_T}{T}\right] = 0 . \tag{1.48}$$

(1.48) is of course trivially true.

Remark. 6. As should be expected, formula (1.41) with ρ_γ replaced by 1 can be directly obtained by non trivial manipulations on the characteristic functions (see [4]).

c) Some estimates on independent increment processes

We now are left to find sufficient condition under which the assumptions of Theorem 1.4 are verified. As we shall see, the effect of the calculus of variations is to transfer an estimate on a Fourier transform to an estimate on a Laplace transform.

Let η_ν be the non-negative measure on $]0,+\infty[$ which is the image measure of $g(z)dz$ by the mapping $z\to\nu(z)$. Since $\int\nu g(z)dz < +\infty$, we have $\int x d\eta_\nu(x) < +\infty$. More generally, let m be a non-negative σ-finite measure on $]0,+\infty[$, such that $\int x\wedge 1\, dm(x) < +\infty$. For $\beta\geq 0$, $\alpha\in R$, set

$$\tau_m(\beta) = \int(e^{-\beta x}-1)dm(x)\,, \quad M_m(\alpha) = \int(\cos\alpha x - 1)dm(x)\,, \tag{1.49}$$
$$\chi_{m,T}(\beta) = \exp\{T\tau_m(\beta)\}\,.$$

It is clear that

$$\chi_{\eta_\nu,T}(\beta) = E^\Pi[\exp(-\beta S_{s\leq T}\nu)]\,. \tag{1.50}$$

Clearly, for $p\geq 1$

$$E^\Pi\left[\left|\frac{1}{S_{s\leq T}\nu}\right|^p\right] = \frac{1}{\Gamma(p)}\int_0^{+\infty}\beta^{p-1}\chi_{\eta_\nu,T}(\beta)\,d\beta\,. \tag{1.51}$$

For (1.51) to be finite for at least one $p>2$, it suffices that $\varepsilon>0$,

$C\geq 0$ exist such that for $\beta\in R^+$

$$\chi_{\eta_\nu,T}(\beta)\leq\frac{C}{\beta^{2+\varepsilon}} \tag{1.52}$$

or equivalently, that for β large enough

$$T\tau_{\eta_\nu}(\beta)\leq -(2+\varepsilon)\operatorname{Log}\beta\,. \tag{1.53}$$

We will now sufficient give conditions on a measure m on $]0,+\infty[$ so that $\tau_m(\beta)$ behaves adequately as $\beta\to+\infty$. The following is proved in [4].

Theorem 1.5. *The following two conditions are equivalent*

a) As $x\to 0$

$$m(]x,+\infty[)\sim C\operatorname{Log}\frac{1}{x}\,.$$

b) As $\beta\to+\infty$

$$\tau_m(\beta)\sim -C\operatorname{Log}\beta\,.$$

A sufficient condition for a)–b) *to hold is that*

c) As $|\alpha|\to+\infty$, $M_m(\alpha)\sim -C\operatorname{Log}|\alpha|$.

Either of the two conditions

d) $\displaystyle\varliminf_{x\to 0}\frac{m]x,+\infty[}{\log 1/x} = C$, *or*

e) $\displaystyle\varliminf_{|\alpha|\to+\infty}\left(-\frac{M_m(\alpha)}{\operatorname{Log}|\alpha|}\right) = C$,

implies

f) $\displaystyle\varlimsup_{\beta\to+\infty}\frac{\tau_m(\beta)}{\operatorname{Log}\beta}\leq -C$.

Proof. The proof relies on standard Abelian and Tauberian techniques. For the full proof, see [4].

Corollary. Assume that D $(0<D\leq+\infty)$, exists such that

$$\varliminf_{x\to 0}\frac{\eta_\nu]x,+\infty[}{\operatorname{Log} 1/x} = D\,. \tag{1.54}$$

Then for $T>2/D$, $1/S_{s\leq T}\nu$ belongs to one given $L_p(\Pi)$ with $p>2$.

Proof. This is obvious using (1.54) and the implication d) $\Rightarrow$ f) in Theorem 1.5.

Remark 7. In general the function λ appearing in Section 1 b) is a bounded function, so that for $C>0$ $\nu(z) \le C|z|$. If (1.54) holds, then also

$$\varliminf_{h\to 0} \frac{\int I_{|z|\ge h} g(z)dz}{\operatorname{Log} 1/h} \ge D . \tag{1.55}$$

If $n=1$, it is crucial to remark in Theorem 1.5 that in general a) or b) do *not* imply c), and d) or f) do not imply e). Otherwise using (1.55), we would find that

$$\varliminf_{|\alpha|\to+\infty} - \frac{\int (\cos \alpha x - 1)g(x)dx}{\operatorname{Log} |\alpha|} \ge D \tag{1.56}$$

and so for $\varepsilon > 0$, as $\alpha \to +\infty$

$$|\psi_T(\alpha)| \le \frac{C_\varepsilon}{|\alpha|^{T(D-\varepsilon)}} , \tag{1.57}$$

which would make the conclusions of Theorem 1.4 trivially true without any calculus of variations! The counter example is as follows. Let m be given by

$$m = \sum \delta_{1/2^n} .$$

As $x \to 0$, $m]x, +\infty[\sim \dfrac{\operatorname{Log} 1/x}{\operatorname{Log} 2}$. Moreover for $k \in N$

$$- M_m(2\pi 2^k) = \sum_n (1 - \cos 2\pi 2^{k-n}) = \sum_{n=k+1}^{\infty} (1 - \cos 2\pi 2^{k-n})$$
$$= \sum_1^{+\infty} (1 - \cos 2\pi 2^{-n}) = - M_m(2\pi) ,$$

and so $|M_m(\alpha)|$ does not tend to $+\infty$ as $|\alpha| \to +\infty$.

In this case, the probability law whose Fourier transform is

$$\exp\left\{\int_0^{+\infty} (e^{-i\alpha x} - 1)\, dm(x)\right\}$$

is singular with respect to the Lebesgue measure. So if $n=1$, at the critical logarithmic concentration, the calculus of variations gives non trivial results for independent increment jump processes.

In a private communication, Prof. H. Delange has shown us how to construct a function $g \ge 0$ and C^∞ on $]0, +\infty[$ such that as $x \to 0$

$$\int_x^{+\infty} g(y)dy \sim \operatorname{Log} \frac{1}{x} \tag{1.58}$$

and that if

$$M(\alpha) = \int_0^{+\infty} (\cos \alpha x - 1)g(x)dx , \tag{1.59}$$

$|M(\alpha)|$ has an arbitrarily slow growth at infinity. It is then clear that even when g is C^∞, (1.58) is not enough to imply a logarithmic behavior of (1.59) as $|\alpha| \to +\infty$.

Note also the following result of [4].

Theorem 1.6. *Let γ be such that $0 < \gamma < 1$. Then the following conditions are equivalent*:

a) *As $x \to 0+$, $m(]x, +\infty[) \sim C/x^\gamma$.*
b) *As $|\alpha| \to +\infty$, $M_m(\alpha) \sim - C\Gamma(1-\gamma)(\sin \pi/2(1-\gamma))|\alpha|^\gamma$.*
c) *As $\beta \to +\infty$, $\tau_m(\beta) \sim - C\Gamma(1-\gamma)\beta^\gamma$.*

If the equivalent conditions of Theorem 1.6 are verified, then for $T>0$

$$\psi_T(\alpha) = \exp\left\{T\int (e^{-i\alpha x} - 1)dm(x)\right\}$$

is such that for any $n \in N$, $|\alpha|^n \psi_T(\alpha)$ is bounded. So $\psi_T(\alpha)$ is the Fourier transform of a probability measure which has C^∞ density with respect to the Lebesgue measure. In this case, it is the concentration of m which determines the regularity of the corresponding probability.

d) The calculus of variations on general jump processes

As we have already said, the previous method is not devised to be applied to independent increment jumps processes, but can be used on processes which are constructed by using independent increment jump processes as an instrument, in the same way as continuous diffusions are constructed using Brownian motion, which is plugged into a stochastic differential equation.

In [4], we have treated the case where the process x_t with values in $\boldsymbol{R}^n$ is given by the solution of

$$s_t = x_0 + \int_0^+ X_0(x_s)ds + y_t^1 + \cdots + y_t^q \tag{1.60}$$

where $y^1, \cdots, y^q$ are (mutually independent) independent increment jump processes, where the probability law is modified by using the Girsanov transformation on jump processes [14].

To treat this case, the previously developed calculus of variations must be applied to functionals of $y^1, \cdots, y^q$ which are much more complex than those which we previously considered. Namely, functionals of process x, which involve jump martingales constructed by means of x, must be submitted to the calculus of variations.

In the same way as in the calculus of variations on diffusions, a random flow $\varphi_t: x_0 \to x_t$ must be considered and lifted to various bundles, and these lifts must also be submitted to the calculus. In [4], we have treated the case where the jump process is "elliptic", i.e. the support of the jump measures for arbitrarily small jumps spans the whole space $\boldsymbol{R}^n$.

Let us note that in the proof of Theorem 1.4, as soon as we are able to control the differential of order 1 of the law of z and so prove that $|\alpha|\psi_T(\alpha)$ is bounded, the boundedness of $|\alpha|^n\psi_t(\alpha)$ as long as $t \geq nT$ is obvious. For the case of the process x given by (1.60), no such argument exists *a priori*. However, it is possible to mimic the previous argument by using a step by step integration by parts procedure, i.e. to make a variation of the processes $y^1, \cdots, y^q$ first on $[0, T]$, then on $[T, 2T]$, $[2T, 3T]$, $\cdots$, $[(n-1)T, nT]$, so that at each step, a control is obtained for the differentials of higher and higher order of the law of x_{nT}. This procedure is fully developped in [4], and avoids the iteration of the calculus on the same interval $[0, T]$ which would require:

more differentiability on the Lévy kernels.
frightening computations.

Note that this procedure can also be applied to ordinary diffusions whose generator is everywhere hypoelliptic, and can be localized using the localization procedure of Stroock [26].

e) Some geometrical considerations

The structure of equation (1.60) is not completely satisfactory. In fact it makes full use of the vector space structure of $\boldsymbol{R}^n$ since the various jumps are "added" to each other. In particular, there is no interplay between the jumps of $y^1, \cdots, y^q$ which would be similar to the interaction of vector fields by the bracketing in ordinary continuous diffusions.

A natural idea would be to replace (1.60) by more general stochastic differential equations with jumps (see Jacod [14]). However technical difficulties do arise, essentially because contrary to (0.1), such general equations do not define flows of diffeomorphisms i.e. trajectories starting from different points may collide. As will appear on an example later, it seems that for general jump processes, the Lévy kernel gives an analytically useless description of the process (except in the case of independent increment processes). Namely, it is very hard to describe explicitly how does a Lévy kernel $M(x, dy)$ vary with x.

Although we now know how to describe in a geometrically invariant way a much larger class of jump processes, we will concentrate on the boundary processes of certain continuous diffusions, where, hopefully, our point will clearly appear.

§ 2. The calculus of boundary processes

In this section we report on some results which we have obtained in our forthcoming paper [6] on the calculus of boundary processes.

Assume that z is a reflecting Brownian motion on $[0, +\infty[$ ([13], p. 40, [12], p. 119), L its standard local time at 0, $w = (w^1, \cdots, w^m)$ a Brownian motion independent of z. Consider the stochastic differential equation in Stratonovitch form

$$dx = X_0(x, z)dt + \sum_1^m X_i(x, z) \cdot dw^i + D(x)dL \qquad (2.1)$$
$$x(0) = x_0$$

where $X_0, \cdots, X_m, D$ are smooth vector fields. A drift $b(x, z)$ is introduced on z using a Girsanov transformation. If A_t is the right-continuous inverse of L, we study in [6] the transition probabilities of the Markov process (A_t, x_{A_t}). Of course (A_t, x_{A_t}) is a jump process, whose jumps correspond to the excursions of z out of 0. In [6], we use the fact that the Lévy kernel of the jump process is itself the image of the excursion measure of (z, w) corresponding to the excursions of z out of 0 through the solutions of a stochastic differential equation.

We will essentially focus on some aspects of our work [6], and insist on some possible connections with other recent developments in probability.

In a) the main notations and assumptions are given. In b) a stochastic flow is associated to the considered stochastic differential equation. In c) the Girsanov transformation is briefly introduced. In d) the boundary process is defined. In e) a partial calculus of variations on w is presented along the lines of Bismut-Michel [7]. The key quadratic form $C_t^{x_0}$ is introduced as in [20], [21]. In f) some simple considerations relating the calculus of variations to the method of enlargement of filtrations (see Jeulin [15], Jeulin-Yor [16], [17], Yor [38]) are developed. In particular the "non differentiability" of local time L with respect to any natural differential structure on the space $C(\boldsymbol{R}^+; \boldsymbol{R}^+)$ associated to z forces us in [4] to use the calculus of variations on jump processes to study the component A_t in (A_t, x_{A_t}). This is briefly done in g). In h), the key problem of the a.s. invertibility of $C_{A_t}^{x_0}$ is studied. As in [20], we know *a priori* from g) that if this is the case, the law of (A_t, x_{A_t}) has a density with respect to the Lebesgue measure. Sufficient conditions under which this is the case are proved. The consequences are interesting since they show that the Lévy kernel of the process (A_t, x_{A_t}) and the vector field D may interact through some sort of Lie bracketing which is precisely expressed through true Lie brackets of $(X_0, X_1, \cdots, X_m, \partial/\partial z, D)$.

In i), j) the so called "localizable" and "non localizable" cases for regularity of the transitions probabilities for the boundary semi-group are considered. In particular, in the non localizable case, conditions are given under which the boundary semi-group is slowly regularizing.

In k), z is now a standard Brownian motion, and x is driven by the vector fields $(X_0, X_1, \cdots, X_m)$ when $z > 0$, by $(X'_0, X'_1, \cdots, X'_m)$ when $z < 0$, and D for $z = 0$. Existence of densities for the transitions probabilities of the boundary process is proved under conditions which still exhibit interactions between all the considered vector fields. In l), regularity results for the two-sided case are briefly presented.

a) Assumptions and notations

m is a positive integer. Ω (resp. Ω') is the space $C(\boldsymbol{R}^+; \boldsymbol{R}^m)$ (resp. $C(\boldsymbol{R}^+; \boldsymbol{R}^+)$). The trajectory of $\omega \in \Omega$ (resp. $\omega' \in \Omega'$) is written as $w_t = (w_t^1, \cdots, w_t^m)$ (resp. z). The σ-field $\mathscr{F}_t$ (resp. $\mathscr{F}'_t$) in Ω (resp. Ω') is defined by $\mathscr{F}_t = \mathscr{B}(w_s | s \leq t)$ (resp. $\mathscr{F}'_t = \mathscr{B}(z_s | s \leq t)$). $\bar{\Omega}$ is the space $\Omega \times \Omega'$, endowed with the filtration $\{\bar{\mathscr{F}}_t\}_{t \geq 0}$, where $\bar{\mathscr{F}}_t = \mathscr{F}_t \otimes \mathscr{F}'_t$.

All filtrations will be made right-continuous and complete as in Dellacherie-Meyer [8], without further mention. P is the Brownian measure on Ω, such that $P[w_0 = 0] = 1$. For $z \in \boldsymbol{R}^+$, P'_z is the probability measure on Ω' associated to the reflecting Brownian motion on $[0, +\infty[$, starting at z, i.e. $P'_z[z_0 = z] = 1$. For notational convenience we will write P' instead of P'_0.

On (Ω', P'_z), L_t denotes the local time at 0 of z. By [12], p. 120, we know that $B_t = z_t - z - L_t$ is a Brownian martingale with $B_0 = 0$. We also know that if $z_0 = 0$

$$L_t = \sup_{s \leq t}(-B_s)\,. \tag{2.2}$$

A_t is the right continuous inverse of L, i.e.

$$A_t = \inf\{A \geq 0; L_A > t\}\,. \tag{2.3}$$

d is a positive integer. $Y = (x, z)$ is the standard element in $\boldsymbol{R}^{d+1}$, with $x \in \boldsymbol{R}^d$ and $z \in \boldsymbol{R}$. $\boldsymbol{R}^d$ will be identified to $\boldsymbol{R}^d \times \{0\}$. $X_0(x, z), \cdots, X_m(x, z)$ are $m + 1$ vectors fields defined on $\boldsymbol{R}^{d+1}$ with values in $\boldsymbol{R}^d$, whose components are in $C_b^\infty(\boldsymbol{R}^{d+1})$. $D(x)$ is a vector field defined on $\boldsymbol{R}^d$ with values in $\boldsymbol{R}^d$ whose components are in $C_b^\infty(\boldsymbol{R}^d)$. $b(x, z)$ is a function defined on $\boldsymbol{R}^{d+1}$, with values in $\boldsymbol{R}$, which is in $C_b^\infty(\boldsymbol{R}^{d+1})$. If X_t is a continuous semi-martingale, dX denotes its differential in the sense of Stratonovitch, and δX its differential in the sence of Itô [22]. If h is a C^∞ diffeomorphism of $\boldsymbol{R}^d$ onto $\boldsymbol{R}^d$, and if $K(x)$ is a tensor field on $\boldsymbol{R}^d$, $(h^{*-1}K)(x)$ denotes the pull-back of $K(h(x))$ to x through the differential $\partial h/\partial x(x)$ (see [1]). If $Y(x)$ is a vector field, we see that

$$(h^{*-1}Y)(x) = \left[\frac{\partial h(x)}{\partial x}\right]^{-1} Y(h(x))\,.$$

b) The reflecting process and its flow

We now build a reflected process as in Ikeda-Watanabe [12], p. 203. Take $(x_0, z_0) \in \boldsymbol{R}^d \times \boldsymbol{R}^+$. On $(\bar{\Omega}, P \otimes P'_{z_0})$ consider the stochastic differential equation

$$\begin{aligned} dx &= X_0(x, z)dt + D(x)\cdot dL + X_i(x, z)\cdot dw^i \\ x(0) &= x_0 \end{aligned} \tag{2.4}$$

(the summation sign $\sum_{i=1}^m$ is systematically omitted). The equivalent Itô form of (2.4) is

$$\begin{aligned} dx &= \left(X_0(x, z) + \frac{1}{2}\frac{\partial X_i}{\partial x} X_i(x, z)\right)dt + D(x)\cdot dL + X_i(x, z)\cdot \delta w^i \\ x(0) &= x_0\,. \end{aligned} \tag{2.5}$$

Theorem 2.1. *There exists a mapping defined on $\bar{\Omega} \times \boldsymbol{R}^+ \times \boldsymbol{R}^d$ with values in $\boldsymbol{R}^d$ $(\bar{\omega}, t, x) \to \phi_t(\bar{\omega}, x)$ having the following properties*:

a) *For every $(t, x) \in \boldsymbol{R}^+ \times \boldsymbol{R}^d$, $\bar{\omega} \to \phi_t(\bar{\omega}, x)$ is measurable, and for every $\bar{\omega} \in \bar{\Omega}$, $(t, x) \to \phi_t(\bar{\omega}, x)$ is continuous.*

b) *For any $\bar{\omega} \in \bar{\Omega}$, $\phi_0(\bar{\omega}, \cdot)$ is the identity mapping of $\boldsymbol{R}^d$.*

c) *For any $\bar{\omega} \in \bar{\Omega}$, $t \to \phi_t(\bar{\omega}, \cdot)$ is a family of C^∞ diffeomorphisms of $\boldsymbol{R}^d$ onto $\boldsymbol{R}^d$, which depends continuously on $t \in \boldsymbol{R}^+$ for the topology of uniform convergence of C^∞ functions and their derivatives on the compact sets of $\boldsymbol{R}^d$.*

d) *For any $z_0 \in \boldsymbol{R}^+$, on $(\bar{\Omega}, P \otimes P'_{z_0})$, for any $x_0 \in \boldsymbol{R}^d$, $\phi_t(\bar{\omega}, x_0)$ is the essentially unique solution of equation* (2.4).

e) *For any $z_0 \in \boldsymbol{R}^+$, any compact set K in $\boldsymbol{R}^+ \times \boldsymbol{R}^d$, any multi-index m, for any $n \in \boldsymbol{N}$ and any $p \geq 1$, the random variables*

$$\mathbf{1}_{L_t \leq n} \sup_{(t,x) \in K} \left|\frac{\partial^m \phi_t}{\partial x^m}(\bar{\omega}, x)\right|, \qquad \mathbf{1}_{L_t \leq n} \sup_{(t,x) \in K} \left|\left[\frac{\partial \phi_t}{\partial x}(\bar{\omega}, x)\right]^{-1}\right| \tag{2.6}$$

are in $L_p(\bar{\Omega}, P \otimes P'_{z_0})$, and their norms in $L_p(\bar{\Omega}, P \otimes P'_{z_0})$ may be bounded independently of $z_0 \in \boldsymbol{R}^+$.

On $(\bar{\Omega}, P\otimes P'_{z_0})$, $\phi\cdot(\bar{\omega}, \cdot)$ *is essentially uniquely defined by properties* a) and d).

Proof. The proof is elementary using the results in Bismut [1] and Kunita [18] on stochastic flows.

Remark 1. It follows from Bismut [1] and Kunita [18] that the usual rules of variations of parameters on ordinary differential equations can be extended to stochastic differential equations. For example, for $x_0 \in R^d$, $Z_t = \partial\phi_t/\partial x(\bar{\omega}, x_0)$ and $Z'_t = [\partial\phi_t/\partial x(\bar{\omega}, x_0)]^{-1}$ are the solutions of the stochastic differential equations

$$\begin{aligned} dZ &= \frac{\partial X_0}{\partial x}(x, z)Zdt + \frac{\partial D}{\partial x}(x)ZdL + \frac{\partial X_i}{\partial x}(x, z)Z\cdot dw^i \\ Z(0) &= I \\ dZ' &= -Z'\frac{\partial X_0}{\partial x}(x, z)dt - Z'\frac{\partial D}{\partial x}(x)dL - Z'\frac{\partial X_i}{\partial x}(x, z)\cdot dw^i \\ Z'(0) &= I\,. \end{aligned} \tag{2.7}$$

In (2.7) x_t is of course the process $\phi_t(\bar{\omega}, x_0)$. We will use these facts without further mention.

Remark 2. The situation considered here is very similar to the situation studied in Bismut-Michel [7]. As in [7], z_t and (x_t, z_t) are Markov processes. The analogy will be better illustrated in the sequel.

c) The Girsanov transformation

Take $(x_0, z_0) \in R^d\times R^+$.

Definition 2.2. On $(\bar{\Omega}, P\times P'_{z_0})$, if x_t is the process $\phi_t(\bar{\omega}, x_0)$, M_t is the positive continuous $\mathscr{F}_t$-martingale.

$$M_t = \exp\left\{\int_0^t b(x_s, z_s)\delta B_s - \frac{1}{2}\int_0^t b^2(x_s, z_s)ds\right\}. \tag{2.8}$$

Since b is bounded, it is easy to see that M_t is in all the $L^p(\bar{\Omega}, P\otimes P'_{z_0})$ $(1 \le p < +\infty)$. Proceeding as in Ikeda-Watanabe [12], we now define a new probability measure on $\bar{\Omega}$.

Definition 2.3. $Q_{(x_0,z_0)}$ is the probability measure on $\bar{\Omega}$ whose density with respect to $P\otimes P'_{z_0}$ on $\mathscr{F}_t$ is M_t, i.e.

$$\left.\frac{dQ_{(x_0,z_0)}}{d(P\otimes P'_{z_0})}\right|_{\mathscr{F}_t} = M_t \tag{2.9}$$

By the fundamental property of the Girsanov transformation, under $Q_{(x_0,z_0)}$

$$B'_t = B_t - \int_0^t b(x_s, z_s)ds \tag{2.10}$$

is a Brownian martingale, and $(w^1_t, \cdots, w^m_t, B'_t)$ is a multidimensional Brownian martingale.

Remark 3. Let $\mathscr{L}$ be the second-order differential operator

$$\mathscr{L} = X_0 + b\frac{\partial}{\partial z} + \frac{1}{2}X_i^2 + \frac{1}{2}\frac{\partial^2}{\partial z^2}\,. \tag{2.11}$$

It is easily proved that if $\bar{\mathscr{L}}$ is any second order differential operator on R^{d+1} written in Hörmander's form [10]

$$\bar{\mathscr{L}} = \bar{X}_0 + \frac{1}{2}\sum_1^{m'}\bar{X}_i^2 \tag{2.12}$$

which is non characteristic at $\bar{x} \in \partial D$ (where ∂D is a smooth hypersurface of R^{d+1}), then on a neighborhood of $\bar{x}$, there is a chart (x, z) such that ∂D is exactly given by $(z = 0)$, and $\bar{\mathscr{L}}$ can be written in the form (2.11) (see [6] 1, g)).

Remark 4. It is crucial to observe that if T is a $\{\mathscr{F}_t\}_{t\ge 0}$ stopping time, $P\otimes P'_{z_0}$ and $Q_{(x_0,z_0)}$ are in general not equivalent on $\mathscr{F}_T$, so that possibly $E^{P\otimes P'_{z_0}}[M_T] < 1$. In particular although $L_\infty = +\infty$ $P\otimes P'_{z_0}$ a.s., it may be that $L_\infty < +\infty$ $Q_{(x_0,z_0)}$ a.s. This is the case in particular if $b=\delta$, (where δ is a positive constant). For connections with the enlargement of filtrations and the Föllmer measure, see [6].

d) The boundary process

We now define the boundary process which is the object of our study. Δ is a cemetery point, so that $(R^+\times R^d)\cup\{\Delta\}$ is the state space of the boundary process.

Definition 2.4. D is the set of functions (a_t, y_t) defined on R^+ with values in $(R^+\times R^d)\cup\{\Delta\}$ which are right-continuous with left hand limits such that if ζ is the function defined on D by

$$\zeta = \inf\{t\ge 0; (a_t, y_t) = \Delta\}$$

then if $\zeta < +\infty$, for $s \ge \zeta$, $(a_s, y_s) = \Delta$.

D is of course endowed with the Skorokhod topology so that it is a Polish space.

Definition 2.5. Let $(a_0, x_0) \in \boldsymbol{R}^+ \times \boldsymbol{R}^d$. On $(\bar{\Omega}, Q_{(x_0, 0)})$, the boundary process (a_t, y_t) with values in $(\boldsymbol{R}^+ \times \boldsymbol{R}^d) \cup \{\Delta\}$ is defined by

$$(2.13) \qquad (a_t, y_t) = \begin{cases} (a_0 + A_t, \varphi_{A_t}(\bar{\omega}, x_0)), & t < L_\infty, \\ \Delta, & t \geq L_\infty. \end{cases}$$

Note that on $(\bar{\Omega}, Q_{(x_0, 0)})$, $A_0 = 0$, so that $(a(0), y(0)) = (a_0, y_0)$.

Since L_t is an additive functional of the strong Markov process $(\varphi_t(\bar{\omega}, x_0), z_t)$, and since for any $t > 0$, $z_{A_t} = 0$ on $(A_t < +\infty)$, it is easy to see that (a_t, y_t) is a strong Markov process.

Definition 2.6. Take (a_0, x_0) as in Definition 2.5. $R_{(a_0, x_0)}$ is the probability law on D of the process (a_t, y_t) under $Q_{(x_0, 0)}$.

The system of probability measures $\{R_{(a_0, x_0)}\}$ on D defines a strong Markov process, which is the object of our study. More precisely, we shall study the smoothness of the probability laws of (a_t, y_t) $(t > 0)$.

Of course (a_t, x_t) is a jump process where jumps are obtained through the excursions of z out of 0 (see [6]). It should be pointed out that in this case, the Lévy kernel of this process is itself obtained as the image of the excursion measure of (w, z)–which is carried by an infinite dimensional space–through the solution of a stochastic differential equation. In particular the possibility of using the $\{\mathscr{F}_t\}_{t\geq 0}$ stochastic calculus in the natural time t is a truly extraordinary possibility since the time of the process (A_t, x_{A_t}) is in fact the local time L_t; during the jumps, which occur in 0 local time, we may still use the Itô calculus.

The arch-typical example is the case where $d = 1$, $m = 1$, $X_1 = 1$, $D = 0$. If $x(0) = 0$, the probability law of x_{A_t} under $P \otimes P'$ is the Cauchy law

$$(2.14) \qquad \frac{t\,dx}{\pi(t^2 + x^2)}.$$

Now assume that $X_1(x)$ is a one dimensional vector field which is in $C_0^\infty(R)$, > 0 on $]-\infty, 1[$ and which is equal to $1 - x$ on a neighborhood of 1. Clearly, if $x(0) = 0$, the law of x_{A_t} is the image law of (2.14) through the mapping $s \to y_s$, where y_s is the solution of the differential equation

$$\frac{dy}{ds} = X_1(y), \qquad y(0) = 0.$$

i.e.

$$(2.15) \qquad \frac{\mathrm{I}_{y<1}\, t\,dy}{\pi X_1(y)\left[t^2 + \left[\int_0^y \frac{dx}{X_1(x)}\right]^2\right]}.$$

Since for $y < 1$, $y \to 1$, $\int_0^y dx/X_1(x) \sim -\operatorname{Log}(1 - y)$, it is trivial to see that the left-hand limit at $y = 1$ of (2.15) is $+\infty$. The law of x_{A_t} is not smooth. Such a phenomenon does not happen if (2.14) is replaced by a Gaussian law, and so, the explicit form of the law of A_t ([13], p. 25) shows that the law of (A_t, x_{A_t}) is still smooth.

In this example, the non smoothness of the law of x_{A_t} comes from the integration on the possibly large values of A_t. The introduction of the supplementary component A_t has the effect of smoothing out the considered probability law.

Remark 5. Note that for $p \geq 2$, if for a given (z, L), Z, Z' are given by (2.7), an obvious application of Itô's formula and Gronwall's lemma to the processes $|Z_s|^p$ and $|Z'_s|^p$ shows that

$$(2.16) \qquad E^P[|Z_s|^p] \leq C e^{C'(s+L_s)}, \qquad E^P[|Z'_s|^p] \leq C e^{C'(s+L_s)}$$

(where C, C' are fixed > 0 constants). It is then feasible to take $s = A_t(\omega')$ in (2.16). However, since for $t > 0$, $E^{P'}[e^{C'A_t}] = +\infty$, we find no adequate bound for $E^P[|Z_{A_t}|^p]$ and $E^P[|Z'_{A_t}|^p]$. This is another explanation of the necessity of introducing the component A_t, strongly connected with the result in (2.15).

However, assume that $b = \delta$ (where δ is a constant $\neq 0$). In this case

$$(2.17) \qquad M_{A_t} = \exp\left\{-\delta t - \frac{1}{2}\delta^2 A_t\right\}$$

For a given $p \geq 2$, if $\delta^2 \geq 2C'$, using (2.16), we find that

$$(2.18) \qquad E^{P\otimes P'}[M_{A_t}|Z_{A_t}|^p] < +\infty, \qquad E^{P\otimes P'}[M_{A_t}|Z'_{A_t}|^p] < +\infty.$$

As we shall later see, if (2.18) holds for sufficiently large p, under adequate conditions, the law of x_{A_t} will be smooth ,and so the introduction of A_t is not needed. The full explanation of such a phenomenon appears in [6].

e) The calculus of variations on w

As in Bismut-Michel [7], it is possible to make a variation of the Brownian motion w, by using a perturbation of w by a suitable drift. We first define the key covariance matrix, which extends Malliavin [20] and [21].

Definition 2.7. Take $x_0 \in R^d$. On $(\bar{\Omega}, P \otimes P')$, $C_t^{x_0}$ is the process of linear mappings of $T^*_{x_0} R^d$ into $T_{x_0} R^d$ given by

$$p \in T^*_{x_0}(R^d) \to C_t^{x_0} p = \sum_{i=1}^{m} \int_0^t \langle \varphi_s^{*-1} X_i(x_0, z_s), p \rangle (\varphi_s^{*-1} X_i)(x_0, z_s) ds . \tag{2.19}$$

$C_t^{x_0}$ defines a nonnegative quadratic form on $T^*_{x_0} R^d$

$$p \to \langle C_t^{x_0} p, p \rangle .$$

Clearly, as a quadratic form, C_t increases with t.

We now have the key results.

Theorem 2.8. *If $x_0 \in R^d$, $t' > 0$ are such that for any $T \geq 0$, $\mathrm{I}_{A_{t'} \leq T} |[C_{A_{t'}}^{x_0}]^{-1}|$ is in all the $L_p(P \otimes P')(1 \leq p < +\infty)$, then for any multi-index m, any $t \geq t'$ there exists a random variable B_t^m such that*

a) *for any $T > 0$, and any $p \geq 1$, $\mathrm{I}_{A_t \leq T} B_t^m$ is in $L_p(P \otimes P')$.*

b) *for any $f \in C_c^\infty(R \times R^d)$, if x_s is the process $\varphi_s(\bar{\omega}, x_0)$, then*

$$E^{P \otimes P'}\left[M_{A_t} \frac{\partial^m f}{\partial x^m}(A_t, x_{A_t})\right] = E^{P \otimes P'}[f(A_t, x_{A_t}) B_t^m] . \tag{2.20}$$

Proof. For the complete proof, see [6]. The partial calculus of variations is very close to Bismut-Michel [7].

Remark 6. Because of Remark 5, it is essential that f has compact support. However, by using (2.18), this assumption may be released, so that smoothness of the law of x_{A_t} will follow (the component A_t is no longer needed).

Theorem 2.9. *Assume that $t' > 0$ is such that*

a) *for every $x \in R^d$, $C_{A_{t'}}^x$ is $P \otimes P'$ a.s. invertible.*

b) *for every $T \geq 0$, there is $q > 2$ such that for any $x \in R^d$,*

$$\mathrm{I}_{A_{t'} \leq T} |[C_{A_{t'}}^x]^{-1}|$$

is in $L_q(\bar{\Omega}, P \otimes P')$ with a norm which is bounded independently of $x \in R^d$.

Then for any $x_0 \in R^d$, any multiindex m, any $t \geq |m| t'$, on $(\bar{\Omega}, P \otimes P')$ there exists a random variable D_t^m having the following properties:

a) *For any $T \geq 0$, $\mathrm{I}_{A_t \leq T} D_t^m$ is $P \otimes P'$ integrable.*

b) *For any $f \in C_c^\infty(R \times R^d)$, if x_s is the process $\varphi_s(\bar{\omega}, x_0)$,*

$$E^{P \otimes P'}\left[M_{A_t} \frac{\partial^m f}{\partial x^m}(A_t, x_{A_t})\right] = E^{P \otimes P'}[f(A_t, x_{A_t}) D_t^m] . \tag{2.21}$$

Proof. See [6]. We use there the previous step by step integration by parts procedure of [4] briefly described in Section 1 d).

Remark 7. In [6], the two sets of assumptions in Theorem 2.8 and 2.9 are labelled respectively "localizable" and "non-localizable" for reasons which will later clearly appear. In particular, Theorem 2.9 is useful to study slowly regularizing semi-groups.

Remark 8. At this stage, we know how to control the differentials of the law of (A_t, x_{A_t}) under $Q_{(x_0, 0)}$ in the variable x (corresponding to x_{A_t}). We now describe the situation for A_t.

f) Enlargement of filtrations and the calculus of variations.

We must now try to control the differentials of the law of (A_t, x_{A_t}) under $Q_{(x_0, 0)}$ in the variable a (corresponding to A). It is known ([13], p. 25) that under P', the law of A_t is given by

$$\frac{I_{s \geq 0}}{\sqrt{2\pi s^3}} t \exp\left\{\frac{-t^2}{2s}\right\} ds . \tag{2.22}$$

In principle, although (2.22) shows us that the law of A_t is smooth, we cannot conclude anything on the joint law of (A_t, x_{A_t}). In principle, we would need an argument to prove that in a certain sense, $x_.$ depends in a "differentiable" way on A_t.

A natural idea is to find out what are the conditional laws of the considered processes *given* A_t, or in a different terminology, to enlarge the filtration $\mathscr{F}_t$ so as to contain at time 0 the random variable A_t (for the general technique of enlargement of filtrations, see Jeulin [15], Jeulin-Yor [16]–[17], Yor [38]).

Now since A_t is exactly the first time where B_s hits $-t$, it is known that conditionally on A_t, for $0 \leq s \leq A_t$, $\beta_s = B_s + t$ is a Bes(3) process starting at t and conditioned on $\beta_{A_t} = 0$ (for Bessel processes, see [13]). Using the scaling properties of Bessel processes, we find easily that if

$$\bar{B}_s = \frac{1}{\sqrt{A_t}} B_{sA_t} \tag{2.23}$$

conditionally on A_t, for $0 \leq s \leq 1$, the process

$$\bar{\beta}_s = \bar{B}_s + \frac{t}{\sqrt{A_t}} \tag{2.24}$$

is a Bes (3) process starting at $t/\sqrt{A_t}$ and conditioned on $\bar{\beta}_1 = 0$. Set

$$\bar{L}_s = \frac{1}{\sqrt{A_t}} L_{sA_t} . \tag{2.25}$$

Clearly

$$\bar L_s = \sup_{s'\le s}(-\bar B_{s'}) = \sup_{s'\le s}(-\bar\beta_{s'}) + \frac{t}{\sqrt{A_t}}\,. \tag{2.26}$$

If $\bar w$ is the m-dimensional process

$$\bar w_s = \frac{1}{\sqrt{A_t}}\, w_{sA_t}\,, \tag{2.27}$$

$\bar w$ is a Brownian motion independent of A_t, B. Conditionally on A_t, $x'_s = x_{sA_t}$ is then a solution of

$$\begin{aligned} dx' = {} & A_t X_0(x', \sqrt{A_t}(\bar B_s + \bar L_s))ds + D(x')\cdot\sqrt{A_t}\,d\bar L_s \\ & + \sqrt{A_t}\,X_i(x', \sqrt{A_t}(\bar B_s + \bar L_s))d\bar w^i\,. \end{aligned} \tag{2.28}$$

The idea is then to vary A_t in (2.28). It is easy to consider an auxiliary probability space so that $\bar\beta$ is made to depend differentiably on A_t (recall that $\bar\beta$ starts at $t/\sqrt{A_t}$). In fact if w' is a three dimensional Brownian motion (chosen independently of $\bar w$, A_t), then if e_1 is a unit vector in R^3, if $\bar\beta_s^{A_t}$ is defined by

$$\bar\beta_s^{A_t} = \left\|(1-s)\frac{te_1}{\sqrt{A_t}} + w'_s - sw'_1\right\| \tag{2.29}$$

clearly for every t, $\bar\beta_s^{A_t}$ $(0 \le s \le 1)$ is a Bes (3) bridge starting at $t/\sqrt{A_t}$ at time 0, with $\bar\beta_1^{A_t} = 0$. However if

$$\bar B_s^{A_t} = \bar\beta_s^{A_t} - \frac{t}{\sqrt{A_t}}\,, \qquad \bar L_s^{A_t} = \sup_{s'\le s}(-\bar B_{s'}^{A_t}) \tag{2.30}$$

there is no way that $\bar L^{A_t}$ depends differentiably on A_t. So even rewriting (2.28) in the form

$$\begin{aligned} dx' = {} & A_t X_0(x', \sqrt{A_t}(\bar B_s^{A_t} + \bar L_s^{A_t}))ds + D(x')\cdot\sqrt{A_t}\,d\bar L_s^{A_t} \\ & + \sqrt{A_t}\,X_i(x', \sqrt{A_t}(\bar B_s^{A_t} + \bar L_s^{A_t}))\cdot d\bar w^i \end{aligned} \tag{2.31}$$

on an adequately enlarged probability space, we find no way of differentiating x' in the variable A_t.

It is however essential to note that if instead of studying the probability laws of boundary processes, we had studied the harmonic measures associated to the hitting distributions of certain diffusions, then local time would play *no role any more* and the previous enlargement would be sufficient.

As the previous example should make clear, there is *not only* one calculus of variations on a given probability space–say the Wiener space–but as many as needed. Random variables like hitting times are certainly not differentiable functions of the trajectory and so cannot be submitted to the usual Malliavin calculus. However, by adequate conditioning—or equivalently, by enlargement of filtrations—we may make them become adequately "differentiable". Of course, as made clear in (2.24) and (2.29), this newly acquired differentiability is clearly related to the probability law itself (e.g. the scaling property of Brownian motion).

g) The A_t component

As made clear in f), the difficulty for varying A_t adequately is that L is not a differentiable function of B. We must then try other variations which will leave L untouched.

Let S be the last exit time of 0 by B before hitting t. Using the results of Williams [35], (see Jeulin [15]), we tried to do variations on B before S. Unfortunately S is much smaller than A_t, so that its probability law is non smooth. There is no way that a variation of S produces the desired result on A_t. A variation of $A_t - S$ would still involve L (since L increases after S). All excursions of z out of 0 contribute then to the regularity of the law of A_t. In an earlier version of [6], and following Itô and Ikeda-Watanabe [12], under $P\otimes P'$, we considered the Poisson point process of the excursions of z out of 0. For $s \in R$, the excursion e_s of z is empty if $A_{s-} = A_s$, and given by

$$e_s(u) = z_{A_s+u}\,, \qquad 0 \le u \le A_s - A_{s-}$$

if $A_{s-} < A_s$. Now the characteristic measure n^+ of the point process e_s is known by Itô-McKean [13] p. 75–81, Ikeda-Watanabe [12] p. 123 and p. 224. Namely let B be the law of the Bes (3) process $\bar r$ starting at 0 and conditioned on $\bar r_1 = 0$. Then n^+ is the image of the measure on $R^+ \times \Omega'$

$$I_{t\ge 0}\frac{dt}{\sqrt{2\pi t^3}} \otimes dB(\bar r) \tag{2.32}$$

by the mapping

$$(\sigma, \bar r) \to e \qquad \text{where} \quad e(u) = \sqrt{\sigma}\,\bar r_{u/\sigma}\,, \quad 0 \le u \le \sigma$$

Of course σ represents the length of the excursion and $\bar r$ is the normalized excursion so as to have length 1. (2.32) also reflects the known fact that under P', $A_.$ is a stable process whose characteristic measure is exactly

$$1_{t\geq 0}\frac{dt}{\sqrt{2\pi t^3}}\ .$$

It is of course possible to enlarge the point process so as to include w also.

In an earlier version of [6], the variation of the component $A_.$ was obtained through the following steps:

a) The excursions of z are renormalized so as to have length 1. If $A_{s-}<A_s$, the process $\bar{x}_s(u)=\varphi_{A_{s-}+(A_s-A_{s-})u}(\bar{\omega},x_0)$ is a solution for $0\leq u\leq 1$ of

$$d_u\bar{x}_s(u)=(A_s-A_{s-})X_0(\bar{x}_s(u),\sqrt{(A_s-A_{s-})}\bar{r}_u)du + \sqrt{A_s-A_{s-}}X_i(\bar{x}_s(u),\sqrt{(A_s-A_{s-})}\bar{r}_u)\cdot d\bar{w}_u^i \tag{2.33}$$

(where $\bar{w}$ is a Brownian motion which is the adequate renormalization of w). The Girsanov density is submitted to the same treatment.

b) The calculus of variations on jump processes of Section 1 is applied on the stable process $A_.$, leaving "everything else" unchanged, i.e. the renormalized processes $(\bar{r},\bar{w})$ are kept unchanged.

Of course the variation of x must be computed. Differentials in x_{A_t} appear, which are controlled using the techniques of Section 2, d).

More recently, we succeeded controlling A_t by using directly some more elaborate transformations on the trajectories of z.

Theorem 2.10. *Under the assumptions of Theorem* 2.8, *for any $n\in N$, any $t\geq t'$, there exists a random variable $\bar{D}_t^n$ such that*

a) *for any $T>0$, and any $p\geq 1$, $1_{A_t\leq T}\bar{D}_t^n$ is in $L_p(P\otimes P')$ $(1\leq p<+\infty)$,*

b) *for any $f\in C_c^\infty(R\times R^d)$, if x_s is the process $\varphi_s(\bar{\omega},x_0)$,*

$$E^{P\otimes P'}\left[M_{A_t}\frac{\partial^n f}{\partial a^n}(A_t,x_{A_t})\right]=E^{P\otimes P'}[f(A_t,x_{A_t})\bar{D}_t^n]\ . \tag{2.34}$$

The law of (A_t,x_{A_t}) under $Q_{(x_0,0)}$ is given by $p_t(a,y)dady$ where $p_t(a,y)\in C^\infty(R\times R^d)$.

Theorem 2.11. *Under the assumptions of Theorem* 2.9, *for any $n\in N$, and any $t\geq nt'$, there exists a random variable $\bar{D}_t^n$ such that*

a) *for any $T\geq 0$, $I_{A_t\leq T}\bar{D}_t^n$ is $P\otimes P'$ integrable.*

b) *for any $f\in C_c^\infty(R\times R^d)$, if x_s is the process $\varphi_s(\bar{\omega},x_0)$,*

$$E^{P\otimes P'}\left[M_{A_t}\frac{\partial^n f}{\partial a^n}(A_t,x_{A_t})\right]=E^{P\otimes P'}[f(A_t,x_{A_t})\bar{D}_t^n]\ . \tag{2.35}$$

If $k\in N$, $t\geq(k+d+2)t'$, the law of (A_t,x_{A_t}) under $Q_{(x_0,0)}$ is given by $p_t(a,y)dady$, where $p_t(a,y)\in C^k(R\times R^d)$.

We finally have a natural extension of Malliavin [20].

Theorem 2.12. *If $t\in R^+$ is such that $C_{A_t}^{x_0}$ is $P\otimes P'$ a.s. invertible, then the law of (A_t,x_{A_t}) under $Q_{(x_0,0)}$ is of the type $p_t(a,y)dady$.*

Proof. In [6], we use mollifiers as in [20].

h) Invertibility of $C_{A_t}^{x_0}$

We first give conditions under which $C_{A_t}^{x_0}$ is a.s. invertible. As pointed out in Theorem 2.12, this will imply that the law of (A_t,x_{A_t}) has densities with respect to the Lebesgue measure.

Definition 2.13. For $\ell\in N$, E_ℓ, F_ℓ are the families of vector fields in R^n defined by

$$\begin{aligned}&E_1=(X_1,X_2,\cdots,X_m);\ F_1=\{0\}\,,\\&E_{\ell+1}=[(X_0,X_1,\cdots,X_m,\partial/\partial z),E_\ell];\\&F_{\ell+1}=[D,E_\ell]\cup[(D,X_1,\cdots,X_m),F_\ell]\,.\end{aligned} \tag{2.36}$$

In (2.36) we use the notation [,] to indicate that Lie brackets should be taken between the vector fields of each considered family of vectors fields.

Theorem 2.14. *If x_0 is such that $\bigcup_1^{+\infty}(E_\ell\cup F_\ell)(x_0,0)$ spans R^d, then for any $t>0$, $C_t^{x_0}$ is invertible $P\otimes P'$ a.s..*

Proof. Let U_s be the vector space in $T_{x_0}(R^d)$ spanned by $(\varphi_s^{*-1}X_i)(x_0)$ $(1\leq i\leq n)$ and V_t the vector space spanned by $\bigcup_{s\leq t}(U_s)$. We define V_t^+ by

$$V_t^+=\bigcap_{s>t}V_s\,. \tag{2.37}$$

By the zero-one law, we know that $P\otimes P'$ a.s., V_0^+ is a fixed space, not depending on $\bar{\omega}$. Let us assume that $V_0^+\neq T_{x_0}(R^d)$. If S is the $\{\bar{\mathscr{F}}_t\}_{t\geq 0}$ stopping time

$$S=\inf\{t>0;\ V_t\neq V_0^+\}\,. \tag{2.38}$$

then S is >0 a.s.. Let f be a non-zero element in $T_{x_0}^*(R^d)$ orthogonal to V_0^+. Then

$$\langle f,(\varphi_s^{*-1}X_i)(x_0)\rangle=0 \qquad \text{for}\quad s\leqslant S\,. \tag{2.39}$$

Now from [1]-Theorem IV.1.1, we know that

$$\varphi_t^{*-1}X_i = X_i + \int_0^t \varphi_s^{*-1}[X_0, X_i]ds + \int_0^t \varphi_s^{*-1}\Big[D + \frac{\partial}{\partial z}, X_i\Big]\cdot dL$$
$$(2.40) \qquad + \int_0^t \varphi_s^{*-1}[X_j, X_i]\cdot dw^j + \int_0^t \varphi_s^{*-1}\Big[\frac{\partial}{\partial z}, X_i\Big]\cdot dB$$

or equivalently

$$\varphi_t^{*-1}X_i = X_i + \int_0^t \varphi_s^{*-1}\Big([X_0, X_i] + \frac{1}{2}[X_j, [X_j, X_i]]$$
$$(2.41) \qquad + \frac{1}{2}\Big[\frac{\partial}{\partial z}, \Big[\frac{\partial}{\partial z}, X_i\Big]\Big]\Big)ds + \int_0^t \varphi_s^{*-1}\Big[D + \frac{\partial}{\partial z}, X_i\Big]\cdot dL$$
$$+ \int_0^t \varphi_s^{*-1}[X_j, X_i]\cdot \delta w^j + \int_0^t \varphi_s^{*-1}\Big[\frac{\partial}{\partial z}, X_i\Big]\cdot \delta B .$$

Now (2.41) gives the Itô-Meyer decomposition of the $\{\mathscr{F}_t\}_{t\geq 0}$ semi-martingale $\langle f, \varphi_s^{*-1}X_i\rangle$, which is 0 for $s \leq S$. By canceling the martingale terms, we find that for $s \leq S$

$$(2.42) \qquad \int_0^s \langle f, \varphi_u^{*-1}[X_j, X_i]\rangle \delta w^j = 0 , \qquad 1 \leqslant i, j \leqslant m ,$$
$$\int_0^s \Big\langle f, \varphi_u^{*-1}\Big[\frac{\partial}{\partial z}, X_i\Big]\Big\rangle \delta B = 0 .$$

An elementary reasoning on the quadratic variation of the local martingales (2.42) and the continuity of the processes $\varphi_s^{*-1}[X_j, X_i]$, $\varphi_s^{*-1}[\partial/\partial z, X_i]$ (see [3]-Theorem 5.2) show that $P \otimes P'$ a.s., for $s \leq S$

$$(2.43) \qquad \langle f, \varphi_s^{*-1}[X_j, X_i]\rangle = 0 , \qquad 1 \leqslant i, j \leqslant m ,$$
$$\Big\langle f, \varphi_s^{*-1}\Big[\frac{\partial}{\partial z}, X_i\Big]\Big\rangle = 0 .$$

Reapplying (2.41) on (2.42), we find that for $s \leq S$

$$(2.44) \qquad \langle f, \varphi_s^{*-1}[X_j, [X_j, X_i]]\rangle = 0 ,$$
$$\Big\langle f, \varphi_s^{*-1}\Big[\frac{\partial}{\partial z}, \Big[\frac{\partial}{\partial z}, X_i\Big]\Big]\Big\rangle = 0 .$$

We now cancel the bounded variation process in the Meyer decomposition of $\langle f, \varphi_s^{*-1}X_i\rangle$ $(s \leq S)$, i.e. using (2.41)–(2.43), we get for $s \leq S$

$$(2.45) \qquad \Big\langle f, \int_0^s \varphi_u^{*-1}[X_0, X_i]du\Big\rangle + \Big\langle f, \int_0^s \varphi_u^{*-1}[D, X_i]dL\Big\rangle = 0 .$$

Since $P \otimes P'$ a.s., the support of the measure dL is exactly the closed set

$(z_s = 0)$ which is negligible for the Lebesgue measure ([13], p. 44), from (2.45) we deduce that for $s \leq S$

$$(2.46) \qquad \Big\langle f, \int_0^s \varphi_u^{*-1}[X_0, X_i]du\Big\rangle = 0 ,$$
$$\Big\langle f, \int_0^s \varphi_u^{*-1}[D, X_i]dL\Big\rangle = 0$$

and so using the continuity of $\varphi_u^{*-1}[X_0, X_i]$, $\varphi_u^{*-1}[D, X_i]$ and the support property of dL, we get from (2.46)

$$(2.47) \qquad \langle f, \varphi_s^{*-1}[X_0, X_i]\rangle = 0 \qquad s \leq S ,$$
$$\langle f, \varphi_s^{*-1}[D, X_i]\rangle = 0 \qquad \text{on} \quad (z_s = 0) \cap [0, S[.$$

By iteration of the previous procedure on (2.43) and on the first line in (2.47) we find that for any $\ell \in N$, if $Y_1, \cdots, Y_\ell$ are taken among $(X_0, X_1, \cdots, X_m, \partial/\partial z)$, then $P \otimes P'$ a.s., for $1 \leq i \leq m$

$$(2.48) \qquad \langle f, \varphi_s^{*-1}[Y_\ell, [Y_{\ell-1}, \cdots, [Y_1, X_i]]\cdot\cdot]\rangle = 0 \qquad s \leqslant S ,$$
$$\langle f, \varphi_s^{*-1}[D, [Y_{\ell-1}, \cdots, [Y_1, X_i]]\cdot\cdot]\rangle = 0 \qquad \text{on } (z_s = 0) \cap [0, S[$$

so that in particular at $s = 0$, we get

$$(2.49) \qquad \langle f, [Y_\ell, [Y_{\ell-1}, \cdots, [Y_1, X_i]]\cdots] (x_0, 0)\rangle = 0 , \qquad 1 \leqslant i \leqslant m ,$$
$$\langle f, [D, [Y_{\ell-1}, \cdots, [Y, X_i]]\cdots] (x_0, 0)\rangle = 0 , \qquad 1 \leqslant i \leqslant m ,$$

and so f is orthogonal to $((\bigcup_1^{+\infty} E_\ell) \cup [D, \bigcup_1^{+\infty} E_\ell])(x_0, 0)$.

We will now exploit the second line of (2.47). Let $H(x)$ be a C^∞ vector field defined on $\boldsymbol{R}^d$ with values in $\boldsymbol{R}^d$ be such that

$$(2.50) \qquad \langle f, \varphi_s^{*-1} H\rangle = 0 \qquad \text{on} \quad (z_s = 0) \cap [0, S[.$$

This is the case for $H = X_i(x, 0)$ or $H = [D, X_i] (x, 0)$. We claim that

$$(2.51) \qquad \langle f, \varphi_s^{*-1}[D, H]\rangle = 0 \qquad \text{on} \quad (z_s = 0) \cap [0, S[,$$
$$\langle f, \varphi_s^{*-1}[X_j, H]\rangle = 0 \qquad \text{on} \quad (z_s = 0) \cap [0, S[, 1 \leq j \leq m .$$

Note that in (2.51), we may as well assume that $[D, H] = [D, H] (x)$, $[X_j, H] = [X_j, H] (x, 0)$, so that the previous procedure can be iterated. We have

$$(2.52) \qquad (\varphi_t^{*-1}H) = H(x_0) + \int_0^t \varphi_s^{*-1}\Big([X_0, H] + \frac{1}{2}[X_j, [X_j, H]]\Big)ds$$
$$+ \int_0^t \varphi_s^{*-1}[D, H]dL + \int_0^{+} \varphi_s^{*-1}[X_j, H]\cdot \delta w^j$$

(it is crucial at this stage H does *not* depend on z so that no stochastic integral $\int_0^t \cdots \delta z$ appears).

Now it is proved by Ikeda-Watanabe in [12] p. 306–307 that if g is a $\{\mathscr{F}_t\}_{t\geq 0}$ optional right-continuous process with left-hand limits then for any $t \geq 0$, for $1 \leq j \leq m$

$$\lim_{\varepsilon \downarrow\downarrow 0} \sum_{A_u - A_{u-} > \varepsilon} \int_{A_{u-}\wedge t}^{A_u \wedge t} g\delta w^j = \int_0^t g\delta w^j \tag{2.53}$$

where the limit in the left hand side of (2.53) is taken in probability. We will omit $\lim_{\varepsilon\downarrow\downarrow 0}$ in what follows. Clearly, using (2.50), we have that for any $t \geq 0$

$$\sum \int_{A_{u-}\wedge t \wedge S}^{A_u \wedge t \wedge S} \delta \langle f, \varphi_s^{*-1} H\rangle = \langle f, \varphi_{t\wedge S}^{*-1} H\rangle . \tag{2.54}$$

Applying (2.53) on (2.52), we find that a.s., for any $t \geq 0$,

$$\begin{aligned} \langle f, \varphi_{t\wedge S}^{*-1} H\rangle = \int_0^{t\wedge S} \left\langle f, \varphi_s^{*-1}\left([X_0, H] + \frac{1}{2}[X_j, [X_j, H]]\right)\right\rangle ds \\ + \int_0^{t\wedge S} \langle f, \varphi_s^{*-1}[X_j, H]\rangle \delta w^j \end{aligned} \tag{2.55}$$

(the integrals $\int_0^t \cdots ds$ raise no special difficulty since $(z = 0)$ is dt-negligible). Since both sides of (2.55) are continuous, (2.55) holds a.s. for any t. Comparing with (2.52), we find that

$$\left\langle f, \int_0^{t\wedge S} \varphi_s^{*-1}[D, H] dL\right\rangle = 0 \tag{2.56}$$

and so

$$\langle f, \varphi_s^{*-1}[D, H]\rangle = 0 \qquad \text{on} \quad (z_s = 0) \cap [0, S[\tag{2.57}$$

We now prove the second line in (2.51). Set

$$\begin{aligned} &\rho_s = z_s^2 + \langle f, (\varphi_s^{*-1} H)(x_0, 0)\rangle^2 , \\ &K_s = \left\langle f, \varphi_s^{*-1}\left([X_0, H] + \frac{1}{2}[X_j, [X_j, H]]\right)(x_0, 0)\right\rangle , \\ &M_s = \sum_{j=1}^m \langle f, \varphi_s^{*-1}[X_j, H](x_0, 0)\rangle^2 . \end{aligned} \tag{2.58}$$

Using (2.50) it is clear that

$$\rho_s = 0 \qquad \text{on} \quad (z_s = 0) \cap [0, S[. \tag{2.59}$$

Since dL is supported by $(z = 0)$, we find from (2.52) that

$$\begin{aligned} d\rho = (1 + 2\langle f, \varphi_s^{*-1} H\rangle K_s + M_s) ds \\ + 2z_s \delta B_s + 2\langle f, \varphi_s^{*-1} H\rangle \langle f, \varphi_s^{*-1}[X_j, H]\rangle \delta w^j . \end{aligned} \tag{2.60}$$

Since a.s., for a.e. t $z_t \neq 0$, for a.e. s

$$\beta_s = \frac{\langle f, \varphi_s^{*-1} H\rangle^2 M_s + z_s^2}{\rho_s} \tag{2.61}$$

is well defined and $\leq 1 + M_s$. The process

$$\gamma_s = \int_0^s \beta_u \, du \tag{2.62}$$

is then well defined and strictly increasing. Set

$$\delta_t = \inf\{\delta \geq 0 ; \gamma_\delta > t\} . \tag{2.63}$$

For $t < \gamma_S$, set

$$\Gamma_t = \int_0^{\delta_t} \frac{\langle f, \varphi_s^{*-1} H\rangle \langle f, \varphi_s^{*-1}[X_j, H]\rangle \delta w^j + z_s \delta B_s}{\rho_s^{1/2}} . \tag{2.64}$$

Of course Γ_t is a $\{\mathscr{F}_{\delta_t}\}_{t\geq 0}$ martingale, and moreover $d\langle \Gamma_t, \Gamma_t\rangle = dt$, so that Γ is Brownian motion. If $\rho_t' = \rho_{\delta_t}$, we have

$$d\rho_t' = \frac{1 + 2\langle f, \varphi_{\delta_t}^{*-1} H\rangle K_{\delta_t} + M_{\delta_t}}{\beta_{\delta_t}} dt + 2\rho_t'^{1/2} \delta\Gamma_t . \tag{2.65}$$

Assume that $M_0 \neq 0$. By suitably renormalizing f, we may assume that $M_0 = 1$. Let η be a positive constant such that

$$\eta \leq \frac{1}{4} . \tag{2.66}$$

Since $\langle f, H(x_0)\rangle = 0$, we may suppose that $S > 0$ has been taken small enough so that for $0 \leq s \leq S$

$$1 - \eta \leq M_s \leq 1 + \eta , \qquad 2|\langle f, (\varphi_s^{*-1} H)(x_0)\rangle| K_s \leq \eta . \tag{2.67}$$

Now for $0 \leq s \leq S$, we have for $z_s \neq 0$

$$\frac{1 + 2\langle f, \varphi_s^{*-1}H\rangle K_s + M_s}{\beta_s} - 1$$

$$(2.68) \qquad = \frac{\langle f, \varphi_s^{*-1}H\rangle^2 (2\langle f, \varphi_s^{*-1}H\rangle K + 1) + z^2(2\langle f, \varphi_s^{*-1}H\rangle K + M_s)}{\langle f, \varphi_s^{*-1}H\rangle^2 M_s + z_s^2}$$

$$\geq \frac{1-\eta}{1+\eta} = C > 0 .$$

Consider the stochastic differential equation

$$(2.69) \qquad \begin{aligned} dr'_t &= (1 + C)dt + 2r_t'^{1/2}\delta\Gamma_t \\ r_0 &= 0 . \end{aligned}$$

By a result of Yamada (see Ikeda-Watanabe [12], p. 168), we know that (2.69) has a unique strong solution. Using (2.68) and the comparison theorem of Yamada [37] (Ikeda-Watanabe [12], p. 352), we find that

$$(2.70) \qquad \rho'_t \geq r'_t \quad \text{for} \quad t < \gamma_S .$$

In particular by using (2.59), we find that

$$(2.71) \qquad r'_t = 0 \quad \text{on} \quad [0, \gamma_S[\cap (z_{\delta_t} = 0) .$$

Now for $0 \leq s \leq S$

$$(2.72) \qquad (1 - \eta) \leq \beta_s \leq (1 + \eta) .$$

Recall that the scale s and speed measure m of the square r_t of a Bes (d) process $(0 < d < 2)$ are given by

$$(2.73) \qquad s = r^{1-d/2} , \qquad dm = \frac{1}{2-d} r^{d/2-1}dr$$

so that

$$(2.74) \qquad dm = \frac{2s^{2(d-1)/(2-d)}ds}{(2-d)^2} .$$

Using the results in Itô-McKean [13], p. 224–226, we know that if $r_0 = 0$, then a.s., for any $\varepsilon > 0$, the Hausdorff-Besicovitch dimension of $(r_t = 0, 0 \leq t \leq \varepsilon)$ is the constant $1 - d/2$. Using this result with $d = 1$, we find that a.s. the dimension of $(z_t = 0, 0 \leq t < S)$ is $1/2$. Because of (2.72), it is clear that a.s. the dimension of $(z_{\delta_t} = 0, t < \gamma_S)$ is $1/2$.

Now (2.69) shows that for $0 \leq t \leq \gamma_S$, r'_t is the square of a Bes $(1 + C)$ process and moreover $0 < C < 1$. We then know that a.s., the dimension of $(r'_t = 0, 0 \leq t < \gamma_S)$ is $(1 - C)/2 < 1/2$. This is a contradiction to (2.71). So we find that $M_0 = 0$.

Assume that the second line in (2.51) does not hold. By using the optional selection Theorem [8]-IV-84, we can find a $\{\mathscr{F}_t\}_{t\geq 0}$ stopping time T such that $(P \otimes P')(T < +\infty) > 0$, and moreover if $T < +\infty$, then $T < S$, $z_T = 0$, $M_T \neq 0$.

Now using the strong Markov property of (w, z), the whole reasoning can be restarted after time T (instead than after time 0); we still arrive at a contradiction. The second line in (2.51) holds. By iterating the whole procedure on (2.51) we find in particular that f is orthogonal to $\bigcup_1^{+\infty}(E_\ell \cup F_\ell)(x_0, 0)$ (by taking $s = 0$ in (2.51)). By the assumption in the Theorem f is 0. This is a contradiction to $S > 0$.

Remark 9. Instead of using the dimension properties of the set of zeros of a Bes²(d) process r_t $(0 < d < 2)$ starting at 0, we could as well use the fact (Itô-McKean [13], p. 226) that if L^d is the local time of r at 0, and A^d is its inverse, then A^d is a stable process with exponent $1 - d/2$, so that its characteristic measure will be proportional to $I_{x\geq 0}(dx/x^{2-d/2})$). By proceeding as in Itô-McKean [13], p. 43, this shows that if $N_\varepsilon^d(t)$ is the total length of the intervals in $(r > 0) \cap [0, t]$ whose length is $\leq \varepsilon$, then a.s., for any $t > 0$, as $\varepsilon \downarrow 0$

$$(2.75) \qquad N_\varepsilon^d(t) \sim C_d \varepsilon^{d/2} L^d(t)$$

where C_d is > 0. Using (2.75), a contradiction is easily obtained from the assumption $M_0 \neq 0$ in the proof of Theorem 2.14.

Remark 10. Theorem 2.14 exhibits clearly that the excursions of the process (x, z) out of $z = 0$ can interact with the process when it stays on $z = 0$ through D, so that the probability law of (A_t, x_{A_t}) has densities, although the Lévy kernel of this process may be degenerate.

i) Regularity of the boundary semi-group: the localizable case

We now give sufficient conditions under which the assumptions of Theorem 2.10 are verified.

Definition 2.15. For $\ell \in N$, the function $k^\ell(x, z)$ is defined by

$$(2.76) \qquad k^\ell(x, z) = \inf_{f \in R^d, \|f\|=1} \sum_{j=1}^{\ell} \sum_{Y \in E_j} \langle f, Y(x, z)\rangle^2 .$$

We then have

Theorem 2.16. *If $x_0 \in R^d$ is such that for a given $\ell \in N$, $\theta > 0$*

$$(2.77) \qquad \lim_{z>0, z\to 0} z \operatorname{Log} \inf_{|x-x_0|<\theta} k^\ell(x, z) = 0$$

then for any $t>0$, $T\geq 0$, $I_{A_t\leq T}|[C_{A_t}^{x_0}]^{-1}|$ *is in all the* $L_p(P\otimes P')$ $(1\leq p<+\infty)$.

Proof. The proof in [6] uses the estimates of Kusuoka-Stroock [19], [28] on standard hypoelliptic diffusions as well as the key estimate that if for $\gamma>0$, T_1^γ is the stopping time

$$T_1^\gamma=\inf\{t>0;\ z_t=\gamma\} \tag{2.78}$$

then

$$P'[T_1^\gamma\geq t\gamma\sqrt{2}]\leq \exp\left\{-\frac{t\sqrt{2}}{8\gamma}\right\}. \tag{2.79}$$

j) Regularity of the boundary semi-group: the non-localizable case

We now give conditions under which the assumptions of Theorem 2.11 are verified.

For $\ell\in N$, define $m_\ell=6\times 20^{\ell-1}$.

Theorem 2.17. *Assume that for a given* $\ell\in N$, *there exists* $C>0$ *such that*

$$\lim_{z>0,z\to 0} z\operatorname{Log}\inf_{x\in R^d} k^\ell(x,z)=-C. \tag{2.80}$$

For any $t>16\sqrt{2}m_\ell C$, $T\geq 0$, *there exists* $q>2$ *such that for any* $x_0\in R^d$, $I_{A_t\leq T}|[C_{A_t}^{x_0}]^{-1}|$ *is in* $L_q(P\otimes P')$ *with a norm bounded independently of* x_0.

For any $t>0$, *the law of* (A_t, x_{A_t}) *is given by* $p_t(a,y)dady$ *and* $p_t(a,y)$ *is such that*

a) *it is* C^∞ *on* $]0,+\infty[\times R^d$,

b) *if* $t>(k+d+2)16\sqrt{2}m_\ell C$, $p_t(a,y)\in C^k(R\times R^d)$.

Proof. See [6].

Remark 11. The condition (2.77) is a local one, while (2.80) is a global condition, which justifies the terminology which we have used. Moreover, in [6], we show how instead of assuming that (2.77) is verified on a neighborhood of the starting point x_0, it may be verified on a neighborhood of the final point y as well (for such a problem on standard diffusions see Stroock [26]).

Remark 12. The conditions of Theorem 2.17 give exactly the analytical conditions under which the boundary semi-group is slowly regularizing in the sense of [4] (also see Theorem 1.4). Moreover it is shown in [6] that under conditions like (2.80), the generalized symbol of the generator of the boundary process exhibits a logarithmic behavior (see [6], Section 6, Remark 5).

The conditions of Theorem 2.17 are minimal. In fact consider the stochastic differential equation

$$dx=\exp\left\{-\frac{1}{2z_s}\right\}dw_s^1, \qquad x(0)=0. \tag{2.81}$$

Conditionally on z, the law of x_{A_t} is clearly a centered Gaussian whose variance is

$$C_{A_t}=\int_0^{A_t}\exp\left\{-\frac{1}{z_s}\right\}ds. \tag{2.82}$$

Now, if n^+ and σ are defined as in Section 2, g),

$$\begin{aligned} n^+\left[\int_0^\sigma e^{-1/z_s}ds\geq\alpha\right]&\leq C+n^+\left[\int_0^\sigma e^{-1/z_s}ds\geq\alpha;\ \sigma\leq 1\right]\\ &\leq C+n^+\left[\sup_{0\leq s\leq\sigma} z_s\geq\frac{1}{\operatorname{Log}1/\alpha}\right]\\ &=C+\operatorname{Log}1/\alpha \end{aligned} \tag{2.83}$$

(the last equality in (2.83) is classical [12]). From (2.83), we see that

$$\begin{aligned} \tau(\beta)&=\int\left[\exp\left(-\beta\int_0^\sigma e^{-1/z_s}ds\right)-1\right]dn^+\\ &\geq C-\beta\int_0^{+\infty}e^{-\beta\alpha}\left(C+\operatorname{Log}\frac{1}{\alpha}\right)d\alpha\geq C-C'\operatorname{Log}\beta \end{aligned} \tag{2.84}$$

and so

$$E^{P'}\left[\exp\left\{-\beta\int_0^{A_t}e^{-1/z_s}ds\right\}\right]=\exp t\tau(\beta)\geq\frac{D}{\beta^{C't}}. \tag{2.85}$$

Since

$$E^{P'}\left[\frac{1}{C_{A_t}^{1/2}}\right]=\frac{1}{\Gamma(1/2)}\int_0^{+\infty}\beta^{-1/2}E^{P'}[e^{-\beta C_{A_t}}]d\beta, \tag{2.86}$$

we see from (2.85) that for t small enough, (2.86) is $+\infty$. Now the law of x_{A_t} under $P\otimes P''$ is $h_t(x)dx$, where

$$h_t(x)=\int\frac{1}{\sqrt{2\pi C_{A_t}}}\exp\left\{\frac{-x^2}{2C_{A_t}}\right\}dP'. \tag{2.87}$$

We then find that for t small enough, $h_t(0) = +\infty$. h_t is not even continuous at 0. It should be pointed out that this result has nothing to do with the fact that we are considering x_{A_t} instead of (A_t, x_{A_t}), since differentiation in x as in (2.7) is irrelevant here.

k) Two sided boundary processes: existence of a density

Let $X'_0(x, z), \cdots, X'_m(x, z)$ be another family of vector fields having the same properties as $X_0, \cdots, X_m$. z now denotes a standard (i.e. non reflecting) Brownian motion. The corresponding probability space is still written as Ω' and the probability law of z' is written as P'_{z_0}. L_t is the local time at 0 of $|z|$ (i.e. L is twice the standard local time at 0 of z), so that

$$|z_t| = |z_0| + \int_0^t \operatorname{sgn} z_s \,\delta z_s + L_t . \tag{2.88}$$

z_t^+, z_t^- are defined by

$$z_t^+ = z_t \vee 0 , \qquad z_t^- = z_t \wedge 0 .$$

A_t is still the inverse of L as in (2.3). Consider the stochastic differential equation

$$\begin{aligned} dx &= I_{z>0}[X_0(x, z)dt + X_i(x, z)\cdot dw^i] \\ &\quad + I_{z<0}[X'_0(x, z)dt + X'_i(x, z)\cdot dw^i] + D(x)dL , \\ x(0) &= x_0 . \end{aligned} \tag{2.89}$$

A flow $\varphi_t(\bar\omega, x_0)$ can be associated to (2.89) in the same way as in Theorem 2.1. If $b'(x, z)$ is a function having the same properties as $b(x, z)$, the Girsanov exponential is changed into

$$\begin{aligned} M_t = \exp\Big[\int_0^t I_{z_s>0}\Big(b(x_s, z_s)\delta z_s - \frac{1}{2}b^2(x_s, z_s)ds\Big) \\ + \int_0^t I_{z_s<0}\Big(b'(x_s, z_s)\delta z_s - \frac{1}{2}b'^2(x_s, z_s)ds\Big)\Big] . \end{aligned} \tag{2.90}$$

$Q_{(x_0, z_0)}$ is defined formally as in (2.9). We still want to study the boundary process (A_t, x_{A_t}) under $Q_{(x_0, 0)}$. The process $C_t^{x_0}$ is now replaced by $C_t'^{x_0}$ which is given by

$$\begin{aligned} p \to C_t'^{x_0}p = \sum_{i=1}^m \int_0^t I_{z>0}\langle(\varphi_s^{*-1}X_i)(x_0, z_s), p\rangle(\varphi_s^{*-1}X_i)(x_0, z_s)ds \\ + \sum_{i=1}^m \int_0^t I_{z<0}\langle(\varphi_s^{*-1}X'_i)(x_0, z_s), p\rangle(\varphi_s^{*-1}X'_i)(x_0, z_s)ds . \end{aligned} \tag{2.91}$$

In [6], the analogues of Theorems 2.8–2.12 are proved to be still valid. So we first study the a.s. invertibility of $C_t'^{x_0}$.

Definition 2.18. For $\ell \in N$, E_ℓ, E'_ℓ, F_ℓ are the family of vector fields defined by

$$\begin{aligned} &E_1 = (X_1, \cdots, X_m), \quad E'_1 = (X'_1, \cdots, X'_m), \quad F_1 = \{0\}, \\ &E_{\ell+1} = \Big[\Big(X_0, X_1, \cdots, X_m, \frac{\partial}{\partial z}\Big), E_\ell\Big], \\ &E'_{\ell+1} = \Big[\Big(X'_0, X'_1, \cdots, X'_m, \frac{\partial}{\partial z}\Big), E'_\ell\Big], \\ &F_{\ell+1} = [(X'_1, \cdots, X'_m, D), E_\ell] \cup [(X_1, \cdots, X_m, D), E'_\ell] \\ &\qquad \cup [(X_1, \cdots, X_m, X'_1, \cdots, X'_m, D), F_\ell] . \end{aligned} \tag{2.92}$$

Theorem 2.19. *If $\bigcup_{\ell=1}^{+\infty}(E_\ell \cup E'_\ell \cup F_\ell)(x_0, 0)$ spans R^d, then $P \otimes P'$ a.s., for any $t > 0$, $C_t'^{x_0}$ is invertible.*

Proof. U_s is the vector space spanned by $I_{z_s>0}(\varphi_s^{*-1}X_i)(x_0)$ $(1 \le i \le m)$ and $I_{z_s<0}(\varphi_s^{*-1}X'_i(x_0))$ $(1 \le i \le m)$. V_t is the vector space spanned by $\bigcup_{s\le t} U_s$ and V_{t+} is defined by

$$V_{t+} = \bigcap_{s>t} V_s . \tag{2.93}$$

We then proceed as in the proof of Theorem 2.14. Namely assume that V_{0+} (which is a non random vector space) is $\neq T_{x_0}(R^d)$. Then if S is the stopping time

$$S = \inf\{t > 0;\ V_t \neq V_{0+}\} . \tag{2.94}$$

S is positive a.s. Let f be a non-zero element of $T^*_{x_0}(R^d)$ orthogonal to V_0^+. Then

$$\begin{aligned} \langle f, (\varphi_u^{*-1}X_i)(x_0)\rangle &= 0 \quad \text{on} \quad (z_u > 0)\cap[0, S] , \\ \langle f, (\varphi_u^{*-1}X'_i)(x_0)\rangle &= 0 \quad \text{on} \quad (z_u < 0)\cap[0, S] . \end{aligned} \tag{2.95}$$

Using the optional selection Theorem [8]–IV–84, it is easily proved that $(z = 0)$ is included in both closures of $(z > 0)$ and $(z < 0)$. (2.95) can be replaced by

$$\begin{aligned} \langle f, (\varphi_u^{*-1}X_i)(x_0)\rangle &= 0 \quad \text{on} \quad (z_u \ge 0)\cap[0, S[, \\ \langle f, (\varphi_u^{*-1}X'_i)(x_0)\rangle &= 0 \quad \text{on} \quad (z_u \le 0)\cap[0, S[. \end{aligned} \tag{2.96}$$

From (2.95), we find that

$$z_t^+ \langle f, \varphi_u^{*-1}(X_i)(x_0)\rangle = 0 \qquad \text{on } [0, S]. \tag{2.97}$$

From Itô-Tanaka's formula, we know that

$$\begin{aligned} z_t^+\varphi_t^{*-1}X_i(x_0) = \int_0^t z_s^+\varphi_s^{*-1}\Big([X_0, X_i] + \frac{1}{2}[X_j, [X_j, X_i]] \\ + \frac{1}{2}\Big[\frac{\partial}{\partial z}, \Big[\frac{\partial}{\partial z}, X_i\Big]\Big]\Big)ds + \int_0^t z_s^+\varphi_s^{*-1}[X_j, X_i]\delta w^j \\ + \int_0^t \mathrm{I}_{z>0}\varphi_s^{*-1}\Big(X_i + z^+\Big[\frac{\partial}{\partial z}, X_i\Big]\Big)\delta z \\ + \int_0^t \mathrm{I}_{z>0}\varphi_s^{*-1}\Big[\frac{\partial}{\partial z}, X_i\Big]ds, \end{aligned} \tag{2.98}$$

(there is no integral $\int_0^t \cdots dL$ because the support of dL is $(z = 0)$, and of (2.96)). From (2.97), we find easily that for $1 \leq j \leq m$

$$\begin{aligned} &\langle f, \varphi_s^{*-1}[X_j, X_i]\rangle = 0 \qquad \text{on } (z_s > 0)\cap[0, S], \\ &\Big\langle f, \varphi_s^{*-1}\Big[\frac{\partial}{\partial z}, X_i\Big]\Big\rangle = 0 \qquad \text{on } (z_s > 0)\cap[0, S]. \end{aligned} \tag{2.99}$$

By iteration, using (2.99) again as in (2.44), and reasoning as in (2.45), we find that for $0 \leq j \leq m$.

$$\begin{aligned} &\langle f, \varphi_s^{*-1}[X_j, X_i]\rangle = 0 \qquad \text{on } (z_s \geq 0)\cap[0, S[, \\ &\Big\langle f, \varphi_s^{*-1}\Big[\frac{\partial}{\partial z}, X_i\Big]\Big\rangle = 0 \qquad \text{on } (z_s \geq 0)\cap[0, S[. \end{aligned} \tag{2.100}$$

We now will use the following result in Ikeda-Watanabe [12], p. 307. Namely if g is $\{\mathscr{F}_t\}_{t\geq 0}$ predictable right-continuous process with left hand limits then for any $t \geq 0$

$$\begin{aligned} &\lim_{\varepsilon\downarrow\downarrow 0}\Big[\sum_{A_u-A_{u-}>\varepsilon}\int_{A_{u-}\wedge t}^{A_u\wedge t} \mathrm{I}_{z>0}g\delta w^i\Big] = \int_0^t \mathrm{I}_{z>0}g\delta w^i \\ &\lim_{\varepsilon\downarrow\downarrow 0}\Big[\sum_{A_u-A_{u-}>\varepsilon}\int_{A_{u-}\wedge t}^{A_u\wedge t} \mathrm{I}_{z<0}g\delta w^i\Big] = \int_0^t \mathrm{I}_{z<0}g\delta w^i \end{aligned} \tag{2.101}$$

where the limits as $\varepsilon\downarrow\downarrow 0$ are taken in probability.

In [12] such a result is proved in the case of a reflecting Brownian motion (see (2.53)). The proof of [12] can be mimicked so that (2.101) holds. Also note that if c_t, κ_t are the processes

$$c_t = \int_0^t \mathrm{I}_{z>0}ds, \qquad \kappa_t = \inf\{\kappa, c_\kappa > t\}, \tag{2.102}$$

then by [12], p. 123, z_{κ_t} is a reflecting Brownian motion, and moreover $\int_0^{\kappa_t} \mathrm{I}_{z>0}\delta w^i$ $(1 \leq i \leq m)$ are also Brownian motions independent of z_{κ_t}. Let $H(x)$ be a C^∞ vector field defined on $\boldsymbol{R}^n$ with values in $\boldsymbol{R}^n$. Assume that

$$\langle f, \varphi_t^{*-1}H\rangle = 0 \qquad \text{on } (z_t = 0)\cap[0, S[. \tag{2.103}$$

We claim that

$$\begin{aligned} &\langle f, \varphi_t^{*-1}[D, H]\rangle = 0 \qquad \text{on } (z_t = 0)\cap[0, S[, \\ &\langle f, \varphi_t^{*-1}[X_j, H]\rangle = 0 \qquad \text{on } (z_t = 0)\cap[0, S[, \quad 1 \leq j \leq m, \\ &\langle f, \varphi_t^{*-1}[X'_j, H]\rangle = 0 \qquad \text{on } (z_t = 0)\cap[0, S[, \quad 1 \leq j \leq m. \end{aligned} \tag{2.104}$$

We first prove the first line of (2.104). We have

$$\begin{aligned} \varphi_t^{*-1}H = H(x_0) + \int_0^t \mathrm{I}_{z>0}\varphi_u^{*-1}\Big([X_0, H] + \frac{1}{2}[X_j, [X_j, H]]\Big)du \\ + \int_0^t \mathrm{I}_{z<0}\varphi_u^{*-1}\Big([X'_0, H] + \frac{1}{2}[X'_j, [X'_j, H]]\Big)du \\ + \int_0^t \varphi_u^{*-1}[D, H]dL + \int_0^t \mathrm{I}_{z>0}\varphi_u^{*-1}[X_j, H]\delta w^j \\ + \int_0^t \mathrm{I}_{z<0}\varphi_u^{*-1}[X'_j, H]\delta w^j. \end{aligned} \tag{2.105}$$

Let G_t, G'_t be the $\{\mathscr{F}_t\}_{t\geq 0}$ predictable processes as in [8]–IV–90

$$G_t = \overline{\lim_{s\uparrow\uparrow t}}\, \mathrm{I}_{z_s>0}, \qquad G'_t = \overline{\lim_{s\uparrow\uparrow t}}\, \mathrm{I}_{z_s<0}. \tag{2.106}$$

We claim that for any $t \geq 0$

$$\sum \int_{A_{u-}\wedge t\wedge S}^{A_u\wedge t\wedge S} \mathrm{I}_{z_u>0}\delta\langle f, \varphi_u^{*-1}H\rangle = G_{t\wedge S}\langle f, \varphi_{t\wedge S}^{*-1}H\rangle. \tag{2.107}$$

In fact

if $t < S$, if $z_t > 0$, using (2.103), the sum is $\langle f, \varphi_t^{*-1}H\rangle$,
$(z_t = 0)$ is negligible, and if $z_t < 0$, the sum is 0.
if $t \geq S$, if $z_S > 0$, the sum is $\langle f, \varphi_S^{*-1}H\rangle$.

If $z_S = 0$, and if S is a left cluster point of $(z = 0)$, the sum is still 0, and moreover by (2.103), $\langle f, \varphi_S^{*-1}H\rangle = 0$. If $z_S = 0$ and z is positive on a left neighborhood of S, the sum is $\langle f, \varphi_S^{*-1}H\rangle$, and $G_S = 1$. If $z_S = 0$ and if z is negative on a left neighborhood of S, both sides of (2.107) are

0. Finally if $z_S < 0$, the left hand side of (2.107) is 0 and $G_S = 0$.

Let E_t and E'_t be the processes

$$(2.108)\quad \begin{aligned} E_t &= \int_0^t \mathrm{I}_{z>0}\Big(\Big\langle f, \varphi_u^{*-1}\Big([X_0, H] + \frac{1}{2}[X_j, [X_j, H]]\Big)\Big\rangle\Big)du \\ &\quad + \int_0^t \mathrm{I}_{z>0}\langle f, \varphi_u^{*-1}[X_j, H]\rangle \delta w^j , \\ E'_t &= \int_0^t \mathrm{I}_{z<0}\Big(\Big\langle f, \varphi_u^{*-1}\Big([X'_0, H] + \frac{1}{2}[X'_j, [X'_j, H]]\Big)\Big\rangle\Big)du \\ &\quad + \int_0^t \mathrm{I}_{z<0}\langle f, \varphi_u^{*-1}[X'_j, H]\rangle \delta w^j \end{aligned}$$

respectively. By using the first line in (2.101) as well as (2.107), we find that for any $t \geq 0$, a.s.

$$(2.109)\quad E_{t\wedge S} = G_{t\wedge S}\langle f, \varphi_{t\wedge S}^{*-1}H\rangle .$$

Now the process $G_{t\wedge S}\langle f, \varphi_{t\wedge S}^{*-1}H\rangle$ is continuous. This is clear if $t < S$, by using (2.103). If S is a left cluster point of $(z = 0)$, $\langle f, \varphi_S^{*-1}X_i\rangle = 0$ and continuity at S still holds, while if S is isolated on the left from $(z = 0)$, G will be continuous at S. From (2.109), we find that a.s.

$$(2.110)\quad E_t = G_t\langle f, \varphi_t^{*-1}H\rangle \qquad \text{on} \quad [0, S] .$$

Similarly

$$(2.111)\quad E'_t = G'_t\langle f, \varphi_t^{*-1}H\rangle \qquad \text{on} \quad [0, S] .$$

We claim that for $t \leq S$

$$(2.112)\quad (G_t + G'_t)\langle f, \varphi_t^{*-1}H\rangle = \langle f, \varphi_t^{*-1}H\rangle .$$

We only need to prove (2.112) if $z_t = 0$. If $t < S$, $\langle f, \varphi_t^{*-1}H\rangle = 0$ and (2.112) is true. If $t = S$, and S is a cluster point on the left of $(z = 0)$, the same reasoning applies. If S is not a cluster point on the left of $(z = 0)$, $G_S + G'_S = 1$, and (2.112) still holds. From (2.110)–(2.112), we see that

$$(2.113)\quad \langle f, \varphi_t^{*-1}H\rangle = E_t + E'_t \qquad \text{on} \quad [0, S] .$$

Comparing with (2.105), we find that

$$(2.114)\quad \Big\langle f, \int_0^t \varphi_u^{*-1}[D, H]dL\Big\rangle = 0 \qquad \text{on} \quad [0, S]$$

so that

$$(2.115)\quad \langle f, \varphi_t^{*-1}[D, H]\rangle = 0 \qquad \text{on} \quad (z = 0)\cap [0, S[.$$

The first line in (2.104) has been proved.

Now from (2.103)–(2.110) it is clear that

$$(2.116)\quad E_t = 0 \qquad \text{on} \quad (z_t = 0)\cap [0, S[.$$

Set

$$(2.117)\quad \begin{aligned} \rho_s &= z_s^2 + E_s^2 , \\ K_s &= \Big\langle f, \varphi_s^{*-1}\Big([X_0, H] + \frac{1}{2}[X_j, [X_j, H]]\Big)\Big\rangle , \\ M_s &= \sum_{j=1}^m \langle f, \varphi_s^{*-1}[X_j, H](x_0\ 0)\rangle^2 . \end{aligned}$$

From (2.116), we see that

$$(2.118)\quad \rho_t = 0 \qquad \text{on} \quad (z_t = 0) \cap [0, S[.$$

c_t, κ_t have been defined in (2.102). Set

$$(2.119)\quad \begin{aligned} &\bar z_t = z_{\kappa_t},\quad \bar w_t^i = \int_0^{\kappa_t} \mathrm{I}_{z>0}\delta w^i, \quad \bar B_t = \int_0^{\kappa_t} \mathrm{I}_{z>0}\delta z , \\ &\bar\rho_t = \rho_{\kappa_t}, \quad \bar K_t = K_{\kappa_t}, \quad \bar M_t = M_{\kappa_t}, \quad \bar E_t = E_{\kappa_t} . \end{aligned}$$

We know that $\bar z$ is a reflecting Brownian motion, and that $(\bar w^1, \cdots, \bar w^m, \bar B)$ is a $\{\mathscr{F}_{\kappa_t}\}_{t\geq 0}$ Brownian martingale. Moreover c_S is a $\{\mathscr{F}_{\kappa_t}\}_{t\geq 0}$ stopping time. Using (2.118), we have

$$\bar\rho_t = 0 \qquad \text{on} \quad (\bar z_t = 0)\cap [0, c_S[.$$

Moreover using standard results on semi-martingales we know that $\bar E_t$, and hence $\bar\rho_t$ is a continuous process. Using (2.108), we find that

$$(2.120)\quad \bar E_t = \int_0^t \bar K_s\, ds + \int_0^t \langle f, \varphi_{\kappa_s}^{*-1}[X_j, H]\rangle \delta\bar w^j .$$

Assume that $M_0 \neq 0$. By renormalizing f, we may assume that $M_0 = 1$. Let η be a positive constant such that $\eta \leq 1/4$. Since $E_0 = 0$, we may suppose that $S > 0$ has been chosen to be small enough so that if $s \leq S$

$$(2.121)\quad 1 - \eta \leq M_s \leq 1 + \eta , \qquad 2|E_s K_s| \leq \eta$$

so that if $t < c_S$

(2.122) $$1 - \eta \leq \overline{M}_t \leq 1 + \eta\,, \qquad 2|\overline{E}_t \overline{K}_t| \leq \eta\,.$$

Obviously, we have

(2.123) $$\begin{aligned} d\overline{\rho}_t &= (1 + 2\overline{E}_t\overline{K}_t + \overline{M}_t)dt \\ &\quad + 2\overline{z}\,\delta\overline{B} + 2\overline{E}_t\langle f, \varphi_{\overline{x}_t}^{*-1}[X_j, H]\rangle \delta\overline{w}^j\,. \end{aligned}$$

At this stage, we are exactly back to the situation described in (2.60) (except that we are working with a larger filtration that the canonical filtration of $(\overline{w}, \overline{z})$, but this is irrelevant). We then find that $M_0 \neq 0$ is a contradiction to (2.103). We obtain the second line of (2.104) using the optional selection Theorem as in the proof of Theorem 2.14.

The third line in (2.104) is proved in the same way.

Of course, using (2.96), we may take H to be equal to $X_i(x, 0)$, $X'_i(x, 0)$ $(1 \leq i \leq m)$ or to any bracket appearing by iteration like in (2.100). We find in particular that f is orthogonal to

$$\bigcup_\ell (E_\ell \cup E'_\ell \cup F_\ell)(x_0, 0),$$

and so $f = 0$. This is a contradiction to $S > 0$.

Remark 13. Since under the assumptions of Theorem 2.19, the boundary semi-group has densities, we see that unexpected interaction may occur at the boundary between the two sides ($z > 0$ and $z < 0$) of the process (x_s, z_s) as well as with the vector field D. This is a clear example that Levy kernels do interact.

l) The two sided boundary process: regularity of the semi-group

The assumptions to get regularity for the boundary semi-group are in general much stronger than for one-sided processes. The following counter example is developed in [6]. Consider first the stochastic differential equations

(2.124) $$dx = \mathrm{I}_{t\leq T}\, dw_t^1\,, \qquad x(0) = x\,.$$

which can be put in the equivalent form

(2.125) $$\begin{aligned} dx &= I_{h\leq T} dw_t^1\,, & x(0) &= 0 \\ dh &= dt\,, & h(0) &= 0\,. \end{aligned}$$

The calculus of variations applies to the component x_{A_t} in (2.125). In fact conditionally on z, the law of x_{A_t} is a centered Gaussian, whose variance is $C_{A_t \wedge T}$ where $C_s = s$. By [13], p. 26, the law of A_t is

$$\mathrm{I}_{s\geq 0} \frac{t}{\sqrt{2\pi s^3}} e^{-t^2/2s} ds\,.$$

One finds that $1/\sqrt{C_{A_t\wedge T}}$ is in all L_p, and the law of x_{A_t} is proved to be smooth. Consider now the system

(2.126) $$\begin{aligned} dx &= \mathrm{I}_{z>0}\mathrm{I}_{h\leq T}\, dw_t^1\,, & x(0) &= x_0 \\ dh &= \mathrm{I}_{z>0}dt + \mathrm{I}_{z<0}dt\,, & h(0) &= 0\,. \end{aligned}$$

(2.126) appears as a two-sided perturbation of (2.125). We claim that now the law of x_{A_t} is not smooth. In fact conditionally on z, the law of x_{A_t} is a centered Gaussian whose variance is $C'_{A_t\wedge T}$, where

(2.127) $$C'_s = \int_0^s \mathrm{I}_{z>0} ds\,.$$

Clearly, $C'_{A_t\wedge T} \leq C'_T$. By Lévy's Arcsine law ([13] p. 57), the law of C'_T is

(2.128) $$\mathrm{I}_{0\leq s\leq T} \frac{ds}{\pi[s(T-s)]^{1/2}}\,.$$

and so

$$\int \frac{1}{\sqrt{C'_{A_t\wedge T}}} dP' \geq \int \frac{1}{\sqrt{C'_T}} dP' = +\infty\,.$$

The law of x_{A_t} is given by a density which is $+\infty$ at $x = 0$!

This counterexample strongly indicates that, say, negative excursions can destroy the regularity which is given by the positive excursions, since these negative excursions are "pushing" x too fast from the regularizing region in $(z > 0)$.

In the two-sided case there is in general no "localizable" result, except when both sides are equally regularizing. We prove however a regularity result in [6]. Recall that the families of vector fields E_ℓ, E'_ℓ have been defined in Definition 2.18.

Definition 2.20. For $\ell \in N$, $\chi^\ell(x)$ is the function defined by

(2.129) $$\chi^\ell(x) = \inf_{\|f\|=1} \sum_{n=1}^{\ell} \Big(\sum_{Y\in E_n} \langle f, Y(x, 0)\rangle^2 + \sum_{Y'\in E'_n} \langle f, Y'(x, 0)\rangle^2 \Big)\,.$$

We then have

Theorem 2.21. *Assume that $\ell \in N$, $\eta > 0$ exist so that for any $x \in R^d$,*

(2.130) $$\chi^\ell(x) \geq \eta$$

Then for any $x_0 \in \boldsymbol{R}^d$, *any* $T \geq 0$, $I_{A_t \leq T} |[C'^{x_0}_{A_t}]^{-1}|$ *is in all the* $L_p(P \otimes P')$.

Proof. Note that (2.130) is a global assumption on $\boldsymbol{R}^d$. (2.130) indicates in particular that $\bigcup_1^q (E_n \cup E'_n)(x, 0)$ spans $\boldsymbol{R}^d$ for each x. The techniques of estimations of Theorem 2.16 based on classical stochastic calculus do not work any more. Apparently, no single side ($z > 0$) or ($z < 0$) is enough by itself to get the desired result. Assume for instance that $\sum_{n=1}^q \langle f, Y(x_0\ 0)\rangle^2$ is large enough. If T_1^γ is still defined by (2.78) (z is now a standard Brownian motion), we now only have the estimate

$$P'[T_1^\gamma \geq t\gamma\sqrt{2}] \leq \frac{C\gamma^{1/2}}{(t\sqrt{2})^{1/2}}. \tag{2.131}$$

In the estimation process we will have to take γ so that (2.131) is small, but in fact γ has to be so small that z_t comes back to 0 before C'^{x_0} has become large enough. z can then go to the region ($z < 0$) where again the same phenomenon can happen. We end up trying to follow Brownian motion in an endless run.

In [6] we chose another route. Namely, we know the excursion law of z out of 0. We end up estimating the contribution of each excursion to regularity, use the fact that positive and negative excursions have equal (or better said strictly positive) weights, and then use the exponential martingales for ponit processes (some of which appear in Section 1) to get the necessary estimates.

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References

[1] J. M. Bismut, Mécanique aléatoire, Lecture Notes in Math. **866**, Springer Berlin, 1981.
[2] ——, A generalized formula of Ito and some other properties of stochastic flows, Z. Wahrsch, **55** (1981), 331–350.
[3] ——, Martingales, the Malliavin calculus and hypoellipticity under general Hörmander's conditions, Z. Wahrsch., **56** (1981), 469–505.
[4] ——, Calcul des variations stochastique et processus de sauts, Z. Wahrsch., **63** (1983), 147–235.
[5] ——, Calcul des variations sur les processus de sauts, C.R.A.S. 293, Série I (1981) 565–568.
[6] ——, The calculus of boundary processes, to appear in Ann. E.N.S. (1984).
[7] J. M. Bismut and D. Michel, Diffusions conditionnelles. J. Funct. Anal, Part

I:44 (1981), 174–211. Part II: **45** (1982), 274–292.
[8] C. Dellacherie and P. A. Meyer Probabilités et potentiels, Chap. I–IV. Paris, Hermann 1975, Chap. V-VIII. Paris, Hermann 1980.
[9] U. Haussmann, On the integral representation of Ito processes, Stochastics **3** (1979), 17–27.
[10] L. Hörmander, Hypoelliptic second order differential equations, Acta Math., **117** (1967), 147–171.
[11] N. Ikeda, On the construction of two-dimensional diffusion processes satisfying Wentzell's boundary conditions and its application to boundary value problems, Mem. Coll. Sci. Univ. Kyoto Math., **33** (1961), 367–427.
[12] N. Ikeda and S. Watanabe, Stochastic differential equations and diffusion processes, Amsterdam, North Holland, 1981.
[13] K. Itô and H. P. McKean, Diffusion processes and their sample paths, Grundlehren Math., Wissenschaften, Band **125**, Berlin, Springer 1974.
[14] J. Jacod, Calcul stochastique et problème des martingales, Lecture Notes in Math., **714** (1979), Springer, Berlin.
[15] T. Jeulin, Semi-martingales et grossissement d'une filtration, Lecture Notes in Math. **833** (1980), Springer, Berlin.
[16] T. Jeulin and M. Yor, Sur les distributions de certaines fonctionnelles du mouvement brownien, Séminaire de Probabilités XV, 210–226, Lecture Notes in Math., **850** (1981), Springer, Berlin.
[17] ——, Grossissement d'une filtration et semi-martingales: formules, explicites, Séminaire de Probabilités XII, 78–97, Lecture Notes in Math., **649** (1978), Berlin, Springer.
[18] H. Kunita, On the decomposition of solutions of stochastic differential equations, In "Stochastic Integrals", D. Williams ed., 213–255, Lecture Notes in Math., **851** (1981), Berlin, Springer.
[19] S. Kusuoka and D. Stroock, Applications of the Malliavin calculus, Part I, Proceeding of the Taniguchi international symposium on Stochastic Analysis, Katata and Kyoto, ed. by K. Itô, 1982, Kinokuniya, 1984.
[20] P. Malliavin, Stochastic calculus of variations and hypoelliptic operators, Proceedings of the Conference on Stochastic Differential equations of Kyoto (1976), ed. by K. Itô 155–263, Kinokuniya and Wiley New-York, 1978.
[21] ——, C^k hypoellipticity with degeneracy, "Stochastic Analysis", 199–214, ed. by A. Friedman and M. Pinsky, Acad. Press New York, 1978.
[22] P. A. Meyer, Un cours sur les intégrales stochastiques, Séminaire de Probabilités X, 245–400, Lecture Notes in Math, **511** (1973), Springer, Berlin.
[23] L. C. G. Rogers, Williams characterization of the Brownian excursion law: proof and applications. Séminaire de Probabilités XV, 227–250, Lecture Notes in Math., **850** (1981), Springer, Berlin.
[24] I. Shigekawa, Derivatives of Wiener functionals and absolute continuity of induced measures, J. Math. Kyoto Univ., **20** (1980), 263–289.
[25] D. Stroock, Diffusion processes associated with Lévy generators, Z. Wahrsch., **32**, (1975), 209–244.
[26] ——, The Malliavin calculus and its applications to second-order parabolic differential equations, Math. Syst. Theory, Part I, **14** (1981), 25–65, Part II:14 (1981), 141–171.
[27] ——, The Malliavin calculus: a functional analytic approach, J. Funct. Anal., **44** (1981), 212–257.
[28] ——, Some applications of stochastic calculus to partial differential equations, Ecole d'été de Probabilité de Saint Flour, ed. par P. L. Hennequin, Lect. Notes in Math., **976** (1983), 268–382, Springer, Berlin.
[29] D. Stroock and S. R. S. Varadhan, Diffusion processes with boundary conditions, Comm. Pure Appl. Math. **24** (1971), 147–225.
[30] F. Treves, Introduction to Pseudodifferential Operators and Fourier Integral Operators, Vol. 1. Plenum Press New York, 1981.

[31] H. G. Tucker, Absolute continuity of infinitely divisible distributions, Pacific J. Math., **12** (1962), 1125–1129.
[32] S. Watanabe, On stochastic differential equations for multidimensional diffusion processes with boundary conditions, J. Math. Kyoto Univ., Part I, **11** (1971), 169–180, Part **II**, **11** (1971), 545–551.
[33] ——, Excursion point process of diffusion and stochastic integral, Proceedings of the International Conference on Stochastic differential equations of Kyoto (1976), ed. by K. Itô 437–461, Kinokuniya Tokyo, and Wiley New-York, 1978.
[34] D. Williams, Path decomposition and continuity of local time for one-dimensional diffusions, Proc. London Math. Society Ser. **3**, **28** (1974), 738–768.
[35] ——, Diffusions, Markov processes and Martingales, Vol. 1: Foundations, Wiley New-York, 1979.
[36] ——, To begin at the beginning. In "Stochastic Integrals", ed. by D. Williams 1–55. Lecture Notes in Math. **851** (1981), Springer, Berlin.
[37] T. Yamada, On a comparison theorem for solutions of stochastic differential equations and its applications, J. Math. Kyoto Univ., **13** (1973), 497–512.
[38] Yor M., Grossissement d'une filtration et semi-martingales: Théorèmes généraux, Séminaire de Probabilités XII, 61–69. Lecture Notes in Math. **649** (1978), Springer, Berlin.

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Diffusive Behavior of a Random Walk in a Random Medium

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§ 0. Introduction

Consider a conducting medium whose conductivity is an ergodic stationary random function of position. Let ε be the ratio of a characteristic length of variations of the conductivity to that of the linear dimensions of the medium. In the limit $\varepsilon \to 0$, the random medium behaves like a homogeneous deterministic conducting medium with an effective conductivity that can be computed in principle [4].

In this paper we consider a similar problem for a discrete medium, a d-dimensional lattice. A result analogous to the one obtained in [4] was recently given by Kühnemann [2]. Our objective here is to show that the estimates needed for the compactness, Nash's estimates [3] in the lattice case, can be obtained in a simple manner.

In Section 1 we formulate the problem, in Section 2 we give Nash's estimate in detail and in Section 3 we outline the rest of the proof which is analogous to the one in [4].

§ 1. Formulation

We establish first the notation which will be used in the following: We will denote by $\boldsymbol{Z}_a^d$ the simple cubic d-dimensional lattice of span a and by $\boldsymbol{B}_a^d$ the corresponding set of bonds. We will always think of $\boldsymbol{Z}_a^d$ as imbedded in $\boldsymbol{R}^d$ with coordinates which are integer multiples of a, with respect to a cartesian system defined by the unit vectors $\{e_i\}_{i=1,\dots,d}$.

On the Hilbert space

$$l^2(\boldsymbol{Z}_a^d) \equiv l^2(\boldsymbol{Z}_a^d, \boldsymbol{R}) \equiv \{f: \boldsymbol{Z}_a^d \longrightarrow \boldsymbol{R} \mid \sum_{x\in \boldsymbol{Z}_a^d} |f(x)|^2 < \infty\}$$

equipped with the inner product

$$(f, g) = \sum_{x\in \boldsymbol{Z}_a^d} f(x)g(x),$$

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