Fusive Loop-Spin structures
Conference in honor of Jean-Michel Bismut

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Ultimate aim

- We would really like to study the Dirac-Ramond operator on the loop space of a string manifold.
- This is defined by (formal) analogy with the spin Dirac operator.
- Ultimately one would like to obtain Witten's genus (in elliptic cohomology) via an appropriate index theorem.
- The basic principle (open to refutation) is that it is easier to work on the loop space rather than on the manifold.
- The loop space is infinite-dimensional but then so is the string structure.
- In the loop setting fusion conditions are important – the objects are like holonomy.

Main result

- The first problem is to define the Dirac-Ramond operator; this involves the differential geometry of the loop space.
- The first part of THAT problem is to establish

Fusive spin structures on loop space ('loop-spin structures')

$$\updownarrow$$

String structures on manifold

which is what I will discuss here.

- At the end I will indicate how we hope to proceed further and at least define the Dirac-Ramond operator.

Whitehead tower for $O(n)$

- I will take $n \geq 5$ throughout, to avoid quibbling.
- For a (Lie) group, Whitehead showed the existence of successive topological groups (well-defined only up to homotopy) killing the lowest remaining homotopy group.
- For $O(n)$:

$$
\begin{array}{ccccccccc}
 & & & & \mathbb{Z}_2 & & K(\mathbb{Z},2) & & \dots \\
 & & & & \downarrow & & \downarrow & & \\
\mathrm{O}(n) & \longleftarrow & \mathrm{SO}(n) & \twoheadleftarrow & \mathrm{Spin}(n) & \twoheadleftarrow & \mathrm{String}(n) & \twoheadleftarrow & \dots \\
 & & & & & & & & \\
\Pi_0 = \mathbb{Z}_2 & & \Pi_1 = \mathbb{Z}_2 & & \Pi_3 = \mathbb{Z} & & \Pi_7 = \mathbb{Z} & & \dots
\end{array}
$$

- The first three groups are (realized as) standard Lie groups.
- The String group cannot be, but there are recent constructions of models as an infinite-dimensional Lie group which can be refined to a Lie 2-group (Nikolaus and Waldorf).

Refinement of the frame bundle

- A choice of Riemann metric on a finite dimensional, compact, connected manifold M reduces the structure group of the tangent bundle to $\mathrm{O}(n)$, with orthonormal frame bundle F_{O}.
- There are the successive principal bundle lifting problems:-

$$
\begin{array}{ccccl}
\mathrm{String}(n) & - & F_{\mathrm{St}} & & \exists \text{ iff } 0 = \frac{1}{2}p_1 \in H^4(M;\mathbb{Z}) \text{ then } H^3(M;\mathbb{Z}) \text{ torsor} \\
\downarrow & & \downarrow & & \\
\mathrm{Spin}(n) & \text{---} & F & & \exists \text{ iff } 0 = w_2 \in H^2(M;\mathbb{Z}_2) \text{ then } H^1(M;\mathbb{Z}_2) \text{ torsor} \\
\downarrow & & \downarrow & & \\
\mathrm{SO}(n) & \text{---} & F_{\mathrm{SO}} & & \exists \text{ iff } 0 = w_1 \in H^1(M;\mathbb{Z}_2) \text{ then } \mathbb{Z}_2 \text{ torsor} \\
\downarrow & & \downarrow & & \\
\mathrm{O}(n) & \text{---} & F_{\mathrm{O}} & & \\
 & & \downarrow & & \\
 & & M. & &
\end{array}
$$

Orientation and spin

- An orientation on M can be identified with a continuous (hence smooth) function on the frame bundle

$$o : F_{\mathrm{O}} \longrightarrow \mathbb{Z}_2$$

which takes both values on each fibre.

- Once oriented, a spin structure on M can be identified with

$$\begin{array}{cc} F & \text{Spin frame bundle} \\ \big\downarrow & \\ F_{\mathrm{SO}} & \text{Oriented frame bundle} \end{array}$$

as a $\mathbb{Z}_2$-bundle which restricts to each fibre to represent a fixed generator of $H^1(\mathrm{Spin}; \mathbb{Z}_2)$.

- The obstruction classes then arise from transgression.
- The spin-frame bundle is denote simply F since it appears most often.

Spin Dirac (prototype for Dirac-Ramond)

- The Dirac operator is an important object on a spin manifold.
- The Spin representation, coming from the Clifford algebra, induces the spinor bundle, S; $\mathbb{Z}_2$-graded if n is even.
- The Dirac operator is given by the Levi-Civita connection and the Clifford action of T^*M on S :

$$\eth = \mathrm{cl} \circ \nabla_S : \mathcal{C}^\infty(M; S) \longrightarrow \mathcal{C}^\infty(M; S)$$
$$\mathrm{ind}(\eth) = \hat{A} \ (n \text{ even}).$$

- One consequence of the Atiyah-Singer index theorem is the identification of the index of $\eth$, with the $\hat{A}$ genus of the manifold, explaining the integrality of $\hat{A}$ for a spin manifold.
- In odd dimensions the Spin Dirac operator has applications to problems of the existence of metrics of constant curvature – and many others besides.

Loop manifold

- The loop space of a manifold is the space of smooth maps from the circle into M.

$$\mathcal{L}M = \mathcal{C}^\infty(\mathrm{U}(1); M).$$

- This is a particularly nice Fréchet manifold, it is paracompact.
- $\mathcal{L}M$ has many Hilbert completions, in particular the energy space $\mathcal{L}^E M = H^1(\mathrm{U}(1); M)$.
- The tangent space to $\mathcal{L}M$ at a loop is naturally the space of sections of the pulled-back tangent bundle

$$T_\lambda \mathcal{L}M = \mathcal{C}^\infty(\mathrm{U}(1); \lambda^* TM), \ \lambda \in \mathcal{L}M.$$

- If M is oriented this is associated to the corresponding loop principal bundle

$$\begin{array}{ccc} \mathcal{L}\,\mathrm{SO} & \text{———} & \mathcal{L}F_{\mathrm{SO}} \\ & & \downarrow \\ & & \mathcal{L}M. \end{array}$$

Spin and loop-orientation

- Since $\Pi_1(\mathrm{SO}) = \mathbb{Z}_2$, $\mathcal{L}\,\mathrm{SO}$ has two components, the identity component is naturally $\mathcal{L}\,\mathrm{Spin}$ and one can ask whether $\mathcal{L}F_{\mathrm{SO}}$ has a reduction to a principal $\mathcal{L}\,\mathrm{Spin}$ bundle.
- In the 80's Atiyah observed that the existence of such a 'loop-orientation', a continuous map

$$u : \mathcal{L}F_{\mathrm{SO}} \longrightarrow \mathbb{Z}_2 \tag{1}$$

taking both signs on each fibre, follows from the existence of a spin structure on M

Spin structure (on M) $\Longrightarrow$ Loop-orientation on $\mathcal{L}M$

- This can be understood as holonomy. The spin structure is a $\mathbb{Z}_2$ bundle over F_{SO} and its holonomy around a loop in F_{SO} is a map (1).
- The converse implication is in general false, although shown by McLaughlin to be true if M is simply connected.

Fusion of paths

- The relationship between spin structures and loop-orientations was clarified by Stolz and Teichner (2005) in terms of fusion.
- The path space $\mathcal{I}M = \mathcal{C}^\infty([0, 2\pi]; M)$ is a fibre bundle over the end-point evaluation maps

$$\mathcal{I}M \xrightarrow{\text{ev}(0),\text{ev}(2\pi)} M^2$$

- The fibre product $\mathcal{I}^{[2]}M$ is the space of pairs with the same endpoints and there is a 'fusion' map

$$\psi : \mathcal{I}^{[2]}M \longrightarrow \mathcal{L}^E M$$

obtained by following the first path, then the reverse of the second and reparameterizing to the circle – it is defined on energy paths.

Fusive loop-orientations

- From the triple fibre product of paths there are three such fusion maps defined from the (simplicial) projections

$$\pi_{ij} : \mathcal{I}^{[3]} M \longrightarrow \mathcal{I}^{[2]} M, \;\; ij = 12, 23, 13,$$
$$\psi_{ij} = \psi \circ \pi_{ij} : \mathcal{I}^{[3]} M \longrightarrow \mathcal{L}^{E} M.$$

- The holonomy of a $\mathbb{Z}_2$ bundle – such as F over F_{SO} – satisfies the fusion condition giving a fusive loop-orientation

$$\psi_{12}^{*} u \cdot \psi_{23}^{*} u = \psi_{13}^{*} u \text{ on } \mathcal{I}^{[3]} F_{\mathrm{SO}}$$

since traversing a 'there-and-back' path does nothing.

- Stolz and Teichner show that there is a 1-1 correpondence

Spin structures on $M \longleftrightarrow$ Fusive loop-orientations on $\mathcal{L}M$

Regression of holonomy

- The reverse, regression, map is worth understanding in this simple setting.
- Take the path fibration with trivial $\mathbb{Z}_2$ factor thought of as a bundle. A loop-orientation gives a map $u : \mathcal{I}^{[2]} F_{\mathsf{SO}} \longrightarrow \mathbb{Z}_2$ and hence a relation

$$\mathcal{I}F_{\mathsf{SO}} \times \mathbb{Z}_2 \ni (\lambda, \sigma) \sim (\lambda', \sigma') \in \mathcal{I}F_{\mathsf{SO}} \times \mathbb{Z}_2$$
$$\Longleftrightarrow (\lambda, \lambda') \in \mathcal{I}^{[2]} F_{\mathsf{SO}},\ \sigma' = u(\sigma, \sigma')\sigma.$$

- The fusion condition is precisely the requirement that this be an equivalence relation, giving a $\mathbb{Z}_2$ bundle over F_{SO}^2 as quotient.
- This is actually a 'simplicial' bundle – really made from a bundle over F_{SO} as the tensor product of the bundle pulled back to the two factors and inverted on one side. This is the spin structure.

Fusive Čech cohomology

- There is a *transgression* map to the loop space in cohomology

$$\begin{array}{ccc} H^k(\mathrm{U}(1) \times \mathcal{L}M; \mathbb{Z}) & \xleftarrow{\mathrm{ev}^*} & H^k(M; \mathbb{Z}) \\ \Big\downarrow \int_{\mathrm{U}(1)} & \swarrow_{\mathrm{Tg}} & \\ H^{k-1}(\mathcal{L}M; \mathbb{Z}) & & \end{array}$$

- In general this map is neither injective nor surjective but by adding fusion conditions it can be 'corrected' to an isomorphism.

Theorem (Kottke-M.)

Fusive Čech cohomology, with values in $\mathrm{U}(1)$, *can be defined over* $\mathcal{L}M$ *giving a regression isomorphism and commutative diagram for each* $k \geq 1$

$$\begin{array}{ccc} H^{k+1}(M; \mathbb{Z}) & \underset{\simeq}{\xleftarrow{\mathrm{Rg}}} & H^k_{\mathrm{fus}}(\mathcal{L}M) \\ & \searrow^{\mathrm{Tg}} & \Big\downarrow \\ & & H^k(\mathcal{L}M; \mathbb{Z}) \end{array}$$

String structures

- Back to the question of string structures assuming M is spin. So we are looking for covers of F by a principal bundle with structure group String .
- Redden showed (in the topological category)

$$\begin{array}{c}\text{String structures/Compatible equivalence of principal bundles} \\ \updownarrow \\ \mathsf{C}(F) = \{\alpha \in H^3(F;\mathbb{Z}); \alpha\big|_{\mathrm{fib}(F)} = \beta\},\ H^3(\mathrm{Spin};\mathbb{Z}) = \mathbb{Z}\cdot\beta \end{array} \tag{2}$$

- By a transgression argument

$$\mathsf{C}(F) \neq \emptyset \iff 0 = \frac{1}{2}p_1 \in H^4(M;\mathbb{Z})$$

where $\frac{1}{2}p_1$ is the first spin-Pontryagin class.

Central extension of $\mathcal{L}\,\mathsf{Spin}$

- The loop group of Spin has central extensions

$$\mathsf{U}(1) \longrightarrow E\mathcal{L}\,\mathsf{Spin} \longrightarrow \mathcal{L}\,\mathsf{Spin}\,.$$

- Each corresponds to a circle bundle over $\mathcal{L}\,\mathsf{Spin}$ which was shown by Waldorf to have the fusion property

$$\psi_{12}^{*}E \otimes \psi_{23}^{*}E \simeq \psi_{13}^{*}E \text{ over } \mathcal{I}^{[3]}\,\mathsf{Spin} \qquad (3)$$

  with a corresponding associativity condition over $\mathcal{I}^{[4]}\,\mathsf{Spin}$.
- We want the basic extension associated to the class in $H^2_{\mathsf{fus}}(\mathsf{Spin})$ which regresses to a generator of $H^3(\mathsf{Spin};\mathbb{Z}) = \mathbb{Z}$.
- For the pointed groups this gives part of the Whitehead tower

$$\dot{\mathcal{L}}\,\mathsf{SO} \longleftarrow \dot{\mathcal{L}}\,\mathsf{Spin} \longleftarrow E\dot{\mathcal{L}}\,\mathsf{Spin} \twoheadleftarrow \cdots$$

$$\Pi_0 = \mathbb{Z}_2 \qquad \Pi_2 = \mathbb{Z} \qquad \Pi_6 = \mathbb{Z}$$

- The Lie algebra was constructed by Kac and Moody. The full group was discussed by Segal, and as a Fréchet manifold is the determinant bundle from the Toeplitz algebra.

Toeplitz extension

- The $N \times N$ matrix Toeplitz alegra sits inside the compression of the pseudodifferential operators on the circle to the Hardy space $H\mathcal{C}^\infty(\mathrm{U}(1);\mathbb{C}^N)$:

$$H\mathcal{C}^\infty(\mathrm{U}(1);M(N))H + H\Psi^{-\infty}(\mathrm{U}(1);\mathbb{C}^N)H = \Psi_{\mathrm{To}}(\mathrm{U}(1);\mathbb{C}^N)$$
$$\subset H\Psi^0(\mathrm{U}(1);\mathbb{C}^N)H \subset \Psi^0(\mathrm{U}(1);\mathbb{C}^N) \ni H$$

- For an invertible matrix loop $\lambda \in \mathcal{C}^\infty(\mathrm{U}(1);\mathrm{GL}(N))$

$$\mathrm{ind}(H\lambda H) = -\mathrm{winding\ no}(\det(\lambda))$$

- Since Spin is simply connected there is a ‘big’ group of invertible unitary extensions in terms of the spin representation

$$\mathcal{G} = \{A \in \Psi_{\mathrm{To}}(\mathrm{U}(1);\mathbb{C}^N),\ A^* = A,\ \sigma(A) \in \mathcal{L}\,\mathrm{Spin}\}.$$

Toeplitz extension cont.

- The kernel of the symbol map consists of the unitary perturbations of the identity by smoothing operators

$$U_H^{-\infty}(\mathrm{U}(1);\mathbb{C}^N) \subset \mathcal{G}.$$

- The subgroup of Fredholm determinant one is normal

$$U_H^{-\infty}(\mathrm{U}(1);\mathbb{C}^N)_{\det=1} \subset U_H^{-\infty}(\mathrm{U}(1);\mathbb{C}^N) \subset \mathcal{G}.$$

- The quotient is the basic central extension of $\mathcal{L}\,\mathrm{Spin}$:

$$\mathrm{U}(1) = U_H^{-\infty}(\mathrm{U}(1);\mathbb{C}^N)\big/U_H^{-\infty}(\mathrm{U}(1);\mathbb{C}^N)_{\det=1} \longrightarrow E\mathcal{L}\,\mathrm{Spin} = \mathcal{G}\big/U_H^{-\infty}(\mathrm{U}(1);\mathbb{C}^N)_{\det=1} \longrightarrow \mathcal{L}\,\mathrm{Spin}\,.$$

- The regularized trace gives a connection on this circle bundle with the Kac-Moody cocycle given by the residue trace.

Loop-spin structures

- A loop-spin structure is the loop analogue of a spin structure.
- It is a lifting of the principal $\mathcal{L}\,\mathrm{Spin}$ bundle $\mathcal{L}F$ over $\mathcal{L}M$ for a spin manifold M to a principal $E\mathcal{L}\,\mathrm{Spin}$ bundle.
- As such it is a circle bundle T over $\mathcal{L}F$ with a twisted action of $\mathcal{L}\,\mathrm{Spin}$ so that

$$\gamma^* T \simeq T \otimes E_\gamma \text{ over } \mathcal{L}F$$

  with an associativity condition.
- Following Waldorf we demand that T satisfy a fusion condition consistent with the fusion property of E

$$\psi_{12}^* T \otimes \psi_{23}^* T \simeq \psi_{13}^* T.$$

  with associativity.
- We also impose a strong smoothness condition we call ‘litheness’.

Loop-spin and string structures

- McLaughlin, showed that there is a (non-fusion) loop-spin structure if and only if $\frac{1}{2}p_1 = 0$, provided M is 2-connected.
- Waldorf showed that there is a fusive (topological) loop-spin structure if and only if $\frac{1}{2}p_1 = 0$.

Theorem (Kottke-M.)

Fusive loop-spin structures up to fusion-preserving isomorphism are in 1-1 correspondence with $\mathrm{C}(F)$ *and hence with string structures up to equivalence.*

- It is highly desireable to show reparameterizaton equivariance, corresponding to the action of the oriented diffeomorphism group, $\mathrm{Dff}^+(\mathrm{U}(1))$, of the circle on loops.
- In particular Witten's genus is formally identified with the U(1)-equivariant index of the Dirac-Ramond operator, corresponding to the rotation of loops.
- Brylinski has suggested that equivariance under $\mathrm{Dff}^+(\mathrm{U}(1))$ should play a role in the index, through a realization of some form of elliptic cohomology.

Lithe smoothness

- As mentioned at the beginning, the loop space is a very special Fréchet manifold.
- In general smoothness of functions on a Fréchet manifold is rather weak (weakened further as ‘convenient’ smoothness) because the derivative at a point is ‘only’ an element of the dual of the tangent space, which is the model Fréchet space.
- For $\mathcal{L}M$, the model is $\mathcal{C}^\infty(\mathsf{U}(1); \mathbb{R}^n) = \mathcal{C}^\infty(\mathsf{U}(1))^n$.
- The dual of this is a space of distribution(al densities) on the circle, but it contains $\mathcal{C}^\infty(\mathsf{U}(1))^n$ as a subspace.
- The coordinate transformations are such that this subspace is preserved, so well-defined on the loop manifold. Having (successive) derivatives in such subspaces is ‘litheness’.
- We construct lithe bundles and functions since it is essential for the subsequence analytic steps to have as much regularity as possible!

Proof of main theorem

- Passage from a loop-spin structure to a 3-class on F is by regression of a circle bundle to a bundle gerbe in the sense of Murray.
- Conversely a 3-class in $\mathrm{C}(F)$ corresponds to a $\mathrm{PU}(H)$ bundle over F the holonomy of which is a circle bundle D over $\mathcal{L}F$.
- Over fibre loops D is identified with the central extension.

$$\begin{array}{ccc} D & & E \\ \downarrow & & \downarrow \\ \mathcal{L}_{\mathrm{fib}}F & = & F \times \dot{\mathcal{L}}\,\mathrm{Spin} \end{array}$$

- General loops are ‘blipped’ to special loops to construct the twisted $\mathcal{L}\,\mathrm{Spin}$ action on D using fusion

Loop-spinor bundle

- Segal has constructed the 'spin' representation of $E\mathcal{L}\,\mathrm{Spin}$ and such 'positive energy representations' have been classified.
- Once again a quite smooth version of this bundle can be constructed using the Toeplitz algebra.
- Thus, there is a smooth infinite-dimensional bundle over $\mathcal{L}M$ associated to a string structure on M.
- The Dirac-Ramond operator should act on (appropriately smooth) sections of this bundle.
- By analogy with the spin Dirac operator a Levi-Civita compatible connection is needed, and can be constructed.
- Finally, the analogue of a 'Clifford action' of $T^*_\gamma\mathcal{L}M$ on the loop-spinor bundle is needed, this is under active construction.
- This should give a well-defined Dirac-Ramond operator which is then open to analysis.

Happy Birthday Jean-Michel!