Norms of Weierstrass-sections

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Gerd Faltings

Max Planck Institute for Mathematics

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Weierstrass points play an important role in diophantine geometry. Recall their definition: Assume C is a compact Riemann surface, $\mathcal{L}$ a linebundle on C, and $f_1....f_r \in \Gamma(C, \mathcal{L})$ global sections. Then $W(f_1, ..., f_r) \in \Gamma(C, \mathcal{L}^{\otimes r} \otimes \omega_C^{\otimes r(r-1)/2})$ is the global section which in a local coordinate z and a local trivialisation of $\mathcal{L}$ is equal to the Wronskian of $f_1, ..., f_r$, that is the determinant of the matrix $1/j! df_i/dz^j$. If the f_i are linearly independant then $W(f_1, ..., f_r)$ does not vanish. This follows because for exponents $e_1 < e_2 < ... < e_r$ the Wronskian of the powers z^{e_i} does not vanish identically. For example for the canonical bundle the Weierstrass-sections give a map from $\wedge^g(\Gamma(C, \omega_C) \to \Gamma(C, \omega_C^{g(g+1)/2})$ which is used to prove the positivity of the relative ω in semistable families which are not isotrivial. The archimedean analogue would be that the Weierstrass-section has norm ≤ 1.

Here we plan to give estimates for the archimedean norm of $W(f_1, ..., f_r)$ as well as a non-archimean analogue, that is its divisibility for a semistable curve over a discrete valuation ring.

An intrinsic definition

A more intrinsic view of $W(f_1, ..., f_r)$ is obtained by considering the jet-bundle $J_r(\mathcal{L})$ defined by $J_r(\mathcal{L}) = pr_{1,*}(pr_2^*(\mathcal{L})/I_\Delta^r)$, where I_Δ denotes the ideal of the diagonal in $C \times C$, and pr_i the projections from $C \times C$ to C. $J_r(\mathcal{L}$ admits a filtration with subquotients $\mathcal{L} \otimes \omega_C^{\otimes i}$, and $W(f_1, ..., f_r)$ denotes the determinant of the sections of $J_r(\mathcal{L})$ defined by $pr_2^*(f_i))$. In positive characteristic $W(f_1, ..., f_r)$ may vanish identically, but a variant with a higher power of ω_C is still nonzero: For some big r (for example $r > deg(\mathcal{L}) + 1)$ the space spanned by $f_1, ..., f_r$ injects in $\Gamma(C, J_r(\mathcal{L}))$. The filtration on $J_r(\mathcal{L})$ defined by powers of I_Δ induces a filtration on the space spanned by $f_1, ..., f_r$. If the nontrivial jumps of this induced filtration occur in degrees $e_1, ...e_r$ then our construction gives a canonical section of $\mathcal{L}^{\otimes r} \otimes \omega_C^{e_1 + ... + e_r}$. This also works for semistable curves.

Estimates in the hyperbolic metric

We assume that C is hyperbolic, so its genus $g > 1$. We choose a point $x \in C$ and write $C = \mathbb{D}/\Gamma$ as a quotient of the unit disc under a discrete cocompact torsionfree subgroup $\Gamma \subset PSU(1,1)$. This defines a coordinate z on $\mathbb{D}$ which gives a local coordinate (with the same name) z near x. It is welldefined up to multiplication with a constant of absolute value one. The hyperbolic metric on $\mathbb{D}$ is given by the Kăhler form $-2idz \wedge d\bar{z}/(1-|z|^2)^2$. It is normalised such that the hyperbolic volume of C is $4\pi(g-1)$.

Linear forms

The holomorphic differentials $\alpha \in \Gamma(C, \omega_C)$ induce Γ-invariant holomorphic differentials $\alpha(z)dz$ with $i/2 \int_{\mathbb{D}/\Gamma} |\alpha(z)|^2 dz \wedge d\bar{z} < \infty$. The coefficient of $z^j dz$ in $\alpha(z) = \sum_j a_j z^j dz$ is given by $a_j = i/2 \int_{\mathbb{D}} \alpha \wedge \bar{\alpha}_j$ for some form $\alpha_j \in \Gamma(C, \omega_C)$. Then up to a factor $dz^{g(g+1)/2}$ the value of the Weierstrass-section at x is the determinant of the matrix defined by integrating holomorphic forms on C against the $\bar{\alpha}_j$. So its normsquare (in some metric on ω_C) is the product of the square of the norm of $dz^{g(g+1)/2}$ with the determinant of the matrix with entries $i/2 \int_C \alpha_j \wedge \bar{\alpha}_k$. If we change the local coordinate z by a factor of absolute value one this does not change.

Squareintegrals

The α_j depend on x. We claim that their squareintegral for the hyperbolic metric is given by

$$i/2 \int_C \alpha_j \wedge \bar{\alpha}_j = 2g(j+1).$$

We first show this for $j = 0$: The hyperbolic norm of dz at the origin is $2^{-1/2}$. If β_j runs through an orthonormal basis of $\Gamma(C, \omega_C)$ then the value of β_j at x is

$$i/2 \int_C \beta_j \wedge \bar{\alpha}_0.$$

If we take the sums of the squares of the hyperbolic norms we get $\|\alpha_0\|^2$. On the other hand the squareintegrals of the norms of β_j are 1, thus the result for $j = 0$.

Continuation

For higher j's use that the space of holomorphic differentials $\alpha = \sum_n a_n z^n dz$ forms a topologically irreducible representation of the group $G = PSU(1,1)$. The linear forms a_j span an irreducible Harish-Chandra module in the dual space. The group G operates on the slightly bigger space of differentials holomorphic in a neighbourhood of the closure of $\mathbb{D}$, that is the series $\sum_n b_n z^n dz$ with $|b_n| R^n$ bounded for some $R > 1$. The pairing with $\Gamma(\mathbb{D}, \omega_{\mathbb{D}})$ is formally given by integration but this may not converge. For a given $x \in C$ the restriction to Γ-invariant forms maps our Harish-Chandra module to the dual space (that is to the complex conjugate) of $\Gamma(C, \omega_C)$, and extends to the slightly bigger topological module. Thus the inner product on this space of differentials induces a Γ-invariant inner product on the topological model. If we integrate over G/Γ we obtain a G-invariant inner product which (because of irreducibility) must be a multiple of usual square-integration. As the α_j are induced by $\beta_j = (j+1)/\pi z^j dz$ the integrals of the squarenorms of α_j are proportional to the squarenorms of the β_j, that is to $j+1$.

From traces to determinants

The hyperbolic squarenorm at x of the Weierstrass-section is equal to the product of the $g(g+1)$'st power of the norm of dz with the determinant of the inner products of α_j's. The first factor is $2^{-g(g+1)/2}$. For the second we first replace α_j by $\alpha_j/(j+1)^{1/2}$ and then estimate the determinant of the g-th power of the trace divided by g (inequality between geometric and arithmetic mean). Thus the squaranorm of $W(x)$ is bounded above by

$$2^{-g(g+1)/2} g! g^{-g} (\sum_j \|\alpha_j\|^2/(j+1))^g .$$

The result

If we form its g'th root and integrate over C (with the hyperbolic volumeform) we get as result:

Theorem

$$\int_C |W|^{2/g} \leq 2^{-(g-1)/2}(g!)^{1/g}.$$

The dependance on the hyperbolic norm goes away if we integrate $|W|^{4/g(g+1)}$ because this is naturally a density. The resulting upper bound is

$$(g!)^{2/g(g+1)}(2\pi(g-1))^{(g-1)/(g+1)}.$$

p-adic theory

Assume that C is a semistable curve over a discrete valuation ring V, with smooth generic fibre C_η, and special fibre C_s. We assume that the residue-field k of V is algebraically closed. Burnol observed that for reducible s the Weierstrass section becomes divisible by a power of the uniformiser π of V. The structure of irreducible components of C_s is described by a graph $\mathcal{G}$ whose vertices $v \in V$ label irreducible components C_v of C_s and whose oriented edges $e \in E$ label double-points. For each edge the completed local ring of C in the corresponding double point is isomorphic to $V[[u, v]]/(uv - \pi^{r_e})$ for some integer $r_e \geq 1$. We let $X = H_1(\mathcal{G}, \mathbb{Z})$ denote its first homology. X is a subgroup of $\mathbb{Z}^E$ and consists of sequences n_e such that for each vertex v the sum $\pm n_e = 0$. The sum is over edges starting or ending in e, and the signs are given by the orientation. Furthermore we need the symmetric bilinear form on X (and on $\mathbb{Z}^E$) defined by $b(m_e, n_e) = \sum_E r_e m_e n_e$. If we replace C by a regular semistable model (all $r_e = 1$) we replace each edge e by a chain of r_e edges. This does change neither the homology $H_1(\mathcal{G}, \mathbb{Z})$ nor the bilinear form b.

Determinant of cohomology

It is wellknown that the Weierstrass-section is related to the determinant of cohomology. If we choose a V-point Q of C and a basis $\alpha_1, ..., \alpha_g$ of the regular differentials $\Gamma(C, \omega_C) = \Gamma(C, \omega_C(Q))$ then for each S-point P of C (S any V-scheme) disjoint from Q the induced map

$$\mathcal{O}_S^g \to \Gamma(C_S, \omega_C(Q)/\omega_C(Q - gP))$$

has as determinant the Weierstrass-section (at P). On the other hand this determinant can be identified with the canonical section (the thetafunction) of the inverse of the determinant of cohomology of $\omega_C(Q - gP)$ (This line-bundle has degree $g - 1$ and thus vanishing Euler-characteristic). If Q is not disjoint from P we need a slight modification. Namely the image of the Weierstrass-section $W \in \Gamma(C, \omega_C^{g(g+1)/2})$ in $\Gamma(C, \omega_C^{g(g+1)/2}(gQ))$ can be identified with the canonical section of the inverse of the determinant of cohomology of $\omega_C(Q - gP)$.

Néron-model

We fix one line-bundle $\mathcal{M}$ of relative degree $g-1$ on C, and try to estimate the π-power dividing the determinant of cohomology of $\mathcal{L} \otimes \mathcal{M}$, for a linebundle $\mathcal{L}$ of degree zero. Such $\mathcal{L}$'s are parametrised by the Néron-model of the Jacobian $J(C_\eta)$. The formal completion of this Néron-model can be described as a quotient $G = \tilde{G}/\iota(X)$. Here $\tilde{G}$ is an extension

$$0 \to T \to \tilde{G} \to A \to 0,$$

with A an abelian variety and T the torus with charactergroup X, and it parametrises line-bundles on C (or its formal completion) whose restriction to each componet C_v has degree zero. ι is a map

$$\iota : X \to \tilde{G}(K)$$

which up tis the sum of a map into $\tilde{G}(V)$ and the map into $T(K)$ described by b (or better π^b).

Degrees

Linebundles of total degree zero on C have a degree vector $deg(\mathcal{L}|C_v) \in \mathbb{Z}^V$ which lies in the kernel of the projection onto $H_0(\mathcal{G}, \mathbb{Z}) = \mathbb{Z}$, that is in the image of $\mathbb{Z}^E$. Thus the degree-vectors lie in $\mathbb{Z}^E/X$. If C is regular the degree of $\mathcal{O}(C_v)$ is the sum (over all edges connecting to v, with sign depending on orientation) $\sum \pm e$. To form the Néronmodel we have to divide by these. If we do this on $\mathbb{Z}^E$ we obtain as quotient the dual X^t, and thus the connected components are parametrised by $X^t/b(X)$ as also follows from the description as rigid quotient.

Relation Néron - determinant of cohomology

The (inverse of) the determinant of cohomology of $\mathcal{L} \otimes \mathcal{M}$ is a line-bundle on the Picard-functor which satisfies the theorem of the cube but unfortunately is not invariant under tensoring with $\mathcal{O}(C_v)$'s. Thus it does not descend to the Néron-model. In fact we have:

a) The determinant of cohomology remains invariant if we replace $\mathcal{L} \otimes \mathcal{M}$ by $\omega_C \otimes \mathcal{L}^{-1} \otimes \mathcal{M}^{-1}$, that is $\mathcal{L}$ by $\omega_C \otimes \mathcal{L}^{-2} \otimes \mathcal{M}^{-1}$.

b) If we replace $\mathcal{L}$ by $\mathcal{L}(-C_v)$ the inverse of the determinant of cohomology is changed by the divisor of π to the power $deg(\mathcal{L} \otimes \mathcal{M}|C_v) + 1 - g_v$, where g_v denotes the genus of C_v.

c) For a generic linebundle $\mathcal{L}$ with $deg(\mathcal{L} \otimes \mathcal{M}|C_v) = g_v - 1 + n_v^+$ the cohomology vanishes (and thus its determinant becomes a unit). Here n_v^+ denotes the number of edges starting in v.

A cubical bundle

To get a cubical linebundle on the Néron-model we modify the determinant of cohomology as follows: The degree-vectors of $\mathcal{M}$ and $\omega_C \otimes \otimes \mathcal{M}^{-1}$ differ by the image of a linear combination $\alpha = \sum_e m_e e$ where all coefficients m_e are odd integers. Such a representation is unique modulo $2X$. This holds because the parity of $deg(\omega_C | C_v)$ is the number of edges connecting to v. Also the degree of $\mathcal{L}$ is the image of an element $\beta = \sum_e n_e e$, well determined modulo X.
Then for such a representation modify the determinant of cohomology by the π-power with exponent one eigth the norm squared of the projection of $\alpha - 2\beta$ to $X^{\perp}$. This is a rational number but its denominator is bounded so we get a line-bundle over the extension to a finite ramified extension V' of V (we only change the basering but keep the Néron-model, that is we do not pass to the model over V' which has more components).

A cubical bundle 2

One checks that the result is invariant under tensoring with $\mathcal{O}(C_v)$'s. Namely if we replace $\mathcal{L}$ by $\mathcal{L}(-C_v)$ we get new representatives for the degree-vectors by substracting ± 2 from n_e, for e an edge starting or ending at v. Then the sum $\sum_e (n_e^2 - 1)/8$ changes by the sum over edges connecting to v of

$$(-\pm n_e + 1)/2 = (-deg(\omega_C \otimes \mathcal{L}^{-2} \otimes \mathcal{M}^{-2} | C_v) + \sum_e 1)/2$$
$$= -(g_v - 1) - deg(\mathcal{L} \otimes \mathcal{M} | C_v).$$

(one needs to change β by the image of v in $\mathbb{Z}^E$ which lies in $X^\perp$). Also replacing $\mathcal{L} \otimes \mathcal{M}$ by $\omega_C \mathcal{L}^{-1} \otimes \mathcal{M}^{-1}$ gives the same. The inverse determinant of cohomology defines (over K) a global section of this bundle (a theta-divisor) which is symmetric in the sense that it is invariant if we replace $\mathcal{L}$ by $\omega_C \otimes \mathcal{M}^{-2} \otimes \mathcal{L}^{-1}$. On the generic fibre our bundle coincides with the theta-bundle giving the polarisation, thus differs from this theta-bundle by a divisor supported in the special fibre. Because both bundles have cubical structures the coefficients this divisor are constant.

Uniformisation

The polarisation on the Néron-model is given by a thetafunction which is rigid analytic defined by a sum

$$\sum_{\mu \in X} a(\mu)\mu,$$

where $a(\mu)$ is a multiplicatively quadratic function (or better section of a linebundle on A) of μ whose quadratic term is $\pi^{b(\mu,\mu)/2}$. On the component parametrised by $\rho \in X^t$ its π-valuation in a generic point is, up to a constant independant of ρ, given by the minimum of $b(\rho - \alpha/2 + \mu)/2$ for $\mu \in X$. This follows because the quadratic term in the valuation of a is $b/2$ and the thetadivisor is symmetric around $\alpha/2$. To get the π-adic valuation on the original determinant of cohomology we have to add one eigth of the norm square of the projection to $X^\perp$ of $\alpha - 2\beta$. The result is the minimum (over $\mu \in X$) of the normsquare of $\alpha - 2\beta - 2\mu$. Furthermore the unknown constant is determined by c) above and we get:

The result

Theorem

The degree-vectors of $\mathcal{L} \otimes \mathcal{M}$ and $\omega_C \otimes \mathcal{L}^{-1} \otimes \mathcal{M}^{-1}$ differ by the image of a linear combination $\sum_e m_e e$ where all coefficients m_e are odd integers. Such a representation is unique modulo $2X$. Then the π-power is the minimum over all such representations of $\sum_e (m_e^2 - 1)/8$.

Proof.

Namely this is true up to a constant which can be determined by property c). One also can check directly that it satisfies a) (change the sign of the m_e) and c) (one can chose $m_e = \pm 1$). $\square$

Remarks

Remarks a) If C is not a regular semistable model one has to change the sum to $\sum_e r_e(n_e^2 - 1)/8$.
b) The divisibility by powers of π of the Weierstrass-section is due to the fact that ω_C and $\mathcal{O}(2gP - 2Q)$ have quite different degrees on various components.
c) A similar reasoning applies to other linebundles (instead of ω_C).
d) The bound need not be optimal as $\omega_C(Q - gP)$ is not a generic linebundle with given degree-vector. For example it depends on the choice of Q. The most canonical choice is if Q lies in the same component as P, but this gives the worst estimate.