Volume estimates, Chow invariants and moduli of Kähler-Einstein metrics

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Background

(X, L) a compact polarised manifold of complex dimension n.

When does X admit a constant scalar curvature Kähler metric in the class $c_1(L)$?

Special case: $L = K_X^{\pm 1}$. Then *CSC* is equivalent to Kähler-Einstein: $\mathrm{Ricci} = \mp\omega$.
$L = K_X$—-negative Ricci curvature. Always exist (Aubin, Yau).
$L = K_X^{-1}$—positive Ricci curvature (X Fano). Do not always exist.

Yau's conjecture Existence $\Leftrightarrow$ "stability" of X.

Similar conjectures for constant scalar curvature. Various candidate definitions of “stability”. (In particular, recent work of Szekelyhidi, see later.)

Chen, Donaldson, Sun (2012). Proof of Yau's conjecture for Fano manifolds.

Theorem 0 X (Fano) admits Kähler-Einstein metric iff stable.

(The direction $KE \Leftarrow$ stable, in the sharp form here, is due to Berman.)

In this lecture we discuss a different, related, result.

Let Σ be a quasi-projective variety parametrising a family of Fano manifolds X_σ with $\mathrm{Aut}(X_\sigma)$ finite.
Let $\Sigma' \subset \Sigma$ be the subset defined by the existence of a KE metric. It is well-known that Σ' is open in the C^∞ topology.

Theorem 1 Σ' is Zariski-open in Σ.

Theorem 1 has been proved by Y.Odaka (arxiv 1211.4833) using Theorem 0. We want to explain a different proof.

Remarks

- The arguments of [Chen, Donaldson, Sun 2012] make heavy use of pluripotential theory, metrics with cone singularities, ... We avoid these here.
- The method here is related to another approach to a version of Theorem 0, but an interesting difficulty arises, which we want to explain.
- Many of the arguments apply to constant scalar curvature metrics.

Theorem 2 Let Σ parametrise a family of polarised manifolds with $\mathrm{Aut}(X_\sigma)$ finite. Let $\Sigma' \subset \Sigma$ be the subset such that X_σ admits a CSC metric ω_σ. *Suppose* there is a C such that for all $\sigma \in \Sigma'$ we have

- $|\mathrm{Ricci}| \leq C$;
- $\mathrm{Diam}(X_\sigma, \omega_\sigma) \leq C$.

Then Σ' is Zariski open.

This has only theoretical interest for the moment, since the only case when one knows the hypotheses apply is the Fano case.

We need preliminaries in:
I. Algebraic geometry
II. “Hermitian projective geometry”
III. Differential geometry and asymptotics.

I. Algebraic geometry.
Write Chow for the Chow variety of n-dimensional varieties of degree d in $\mathbf{P}^N$.

"stability" is related to the orbit structure of the $SL(N+1)$ action on Chow.

Given $V \subset \mathbf{P}^N$ we define the incidence variety $I_V \subset \mathrm{Gr}(N-n, N+1)$. It is a hypersurface cut out by a polynomial F in $s^d\Lambda^{N-n}(\mathbf{C}^{N+1})$. In this way we get an embedding

$$\mathrm{Chow} \to \mathbf{P}(s^d\Lambda^{N-n}(\mathbf{C}^{N+1})).$$

V is "Chow stable" if the map

$$g \mapsto g(F) \quad SL(N+1) \to s^d\Lambda^{N-n}(\mathbf{C}^{N+1}),$$

is proper.

Numerical criterion

Let $g_t \in SL(N+1)$ be a meromorphic function of $t \in \Delta \subset \mathbf{C}$. Define an integer $\mathrm{Ch}(g_t)$ to be the largest order of a pole of the components of $g_t(F)$ at $t = 0$.

Chow stability is equivalent to saying that $\mathrm{Ch}(g_t) > 0$ for all such maps.

There is a formulation in terms of families

$$\mathcal{V} \subset \mathbf{P}^N \times \Delta \quad \pi : \mathcal{V} \to \Delta$$

such that $\pi^{-1}(t) \cong V$ for $t \neq 0$.

Then the invariant is given by the formula

$$\mathrm{Ch}(\mathcal{V}) = \frac{1}{N+1} c_1(\pi_*(\mathcal{L})) - \frac{1}{(n+1)!} \pi_*(c_1(\mathcal{L})^{n+1}),$$

where $\mathcal{L}$ is the hyperplane bundle and the formula is interpreted using the trivialisation over Δ^*.
Now replace $\mathcal{L}$ by $\mathcal{L}^k$ to define $\mathrm{Ch}_k(\mathcal{V})$.
The *Futaki invariant* is

$$\mathrm{Fut}(\mathcal{V}) = \lim_{k \to \infty} \mathrm{Ch}_k(\mathcal{V}).$$

II "Hermitian projective geometry"

Consider $V \subset \mathbf{P}^N$, where $\mathbf{P}^N$ has a fixed Fubini-Study metric. We define $M(V) \in \mathbf{su}(N+1)$ by

$$M(V)_{\alpha\beta} = \left(i \int_V \frac{z_\alpha \overline{z_\beta}}{|z|^2} d\mu_{FS} \right)_{\mathrm{Trace-free}} .$$

(**Significance**: M is a "moment map" for the action of $SU(N+1)$ on the Chow variety.)

Suppose $\pi : \mathcal{V} \to \Delta$ corresponds to $g_t = L(t) t^A R(t)$ with L, R, L^{-1}, R^{-1} holomorphic across $t = 0$ and A hermitian. Then we have

$$\mathrm{Ch}(\mathcal{V}) \leq \langle M(V_0), iA \rangle. \qquad (*)$$

If $L = R = 1$ we have a $\mathbf{C}^*$-equivariant family $\mathcal{V}$. This is the situation usually considered in the literature. In this case equality holds in (*). The inequality in the general case is related to the “Hilbert-Mumford Theorem”. For our purposes we do not want to restrict to $\mathbf{C}^*$-equivariant families.

III. Differential geometry.

Suppose that ω is a constant scalar curvature metric on X and the hypotheses of Theorem 2 apply. We consider the embedding

$$T_k : X \to \mathbf{P}^{N_k} = \mathbf{P}(H^0(X, L^k)^*),$$

where $H^0(X, L^k)$ is given a hermitian metric from the L^2 norm on sections.

Proposition A (Main estimate)

$$\|M(T_k(X)\|_1 \leq Ck^{-2} \log k,$$

where

$$\|\mathrm{diag}(\lambda_i)\|_1 = \sum |\lambda_i|.$$

Proposition B If X is the generic fibre of a degeneration $\mathcal{X} \to \Delta$ then $\mathrm{Fut}(\mathcal{X}) > 0$.

We will come back to these.

Main point

Suppose $\sigma_i \in \Sigma'$ and $\sigma_i \to \sigma_\infty$. Thus $X_i = X_{\sigma_i}$ have CSC metrics ω_i. We have $T_k(X_i) \in \mathrm{Chow}_k$. For each fixed k we can suppose that there is a limit $W_k \in \mathrm{Chow}_k$.

Problems

1. W_k might vary with k.
2. W_k might not lie in the closure of the orbit of X_{σ_∞} due to possible "splitting of orbits".

To handle (1): we can suppose that (X_i, ω_i) have a Gromov-Hausdorff limit Z. The results of Donaldson and Sun (2012) imply that Z has a natural algebraic structure and for some k_0 we have $W_k \cong Z$ for $k = mk_0$, all $m \geq 1$. There is no loss of generality in supposing that $k_0 = 1$
(Generalisations of these facts about Gromov-Hausdorff limits are also fundamental in the proof of Theorem 0.)

To handle (ii): There is a Zariski open subset $\Sigma_0 \subset \Sigma$ such that for $\sigma \in \Sigma_0$:

- The orbit of $[X_\sigma]$ in the Chow variety has maximal degree.
- Any degeneration $\mathcal{X}$ of X_σ can be deformed to a degeneration of X_τ for τ close to σ.

Remark In an analogous discussion for rank 2 bundles over a curve , degenerations correspond to line sub-bundles. Then the analogue of Σ_0 is defined by bundles which have “generic sub-bundles”.

To prove Theorem 1 we need to show that if $\sigma_i \in \Sigma'$ and $\sigma_i \to \sigma_\infty$ with $\sigma_\infty \in \Sigma_0$ then $\sigma_\infty \in \Sigma'$. If not, we get a non-trivial degeneration $\mathcal{X}$ of X_{σ_∞} with central fibre W. For each power k we can represent this with a generator A_k and one shows that the operator norm is bounded by $\|A_k\|_{\mathrm{op}} \leq ck$. By Proposition A we get

$$\mathrm{Ch}_k(\mathcal{X}) \leq \langle M(W), A_k \rangle \leq Ckk^{-2} \log k = Ck^{-1} \log k,$$

so

$$\mathrm{Fut}(\mathcal{X}) \leq 0.$$

On the other hand, since $\sigma_\infty \in \Sigma_0$, the degeneration $\mathcal{X}$ can be deformed to a degeneration of X_i for large i. The Futaki invariant is deformation invariant, so by Proposition B we have $\mathrm{Fut}(\mathcal{X}) > 0$, which is a contradiction.

Discussion of Proposition B

When $\mathcal{X}$ has a $\mathbf{C}^*$ action, this is a result of Stoppa. The proof in the general case is a variant of Stoppa's. One approach is to use a much more general result of Szekelyhidi. He defines a notion of stability based on filtrations of the ring $\bigoplus_k H^0(X, L^k)$ and a degeneration $\mathcal{X}$ defines a filtration.

Discussion of Proposition A

The "density of states" function is

$$\rho_k = \sum |s_\alpha|^2,$$

where (s_α) is any orthonormal basis of sections of L^k.

The Tian-Zelditch-Lu expansion For any *fixed* metric we have an asymptotic expansion as $k \to \infty$

$$\rho_k \sim 1 + a_1 k^{-1} + a_2 k^{-2} + \dots,$$

where $a_1 = S/2$.

Note. Replacing L by L^k corresponds to scaling lengths by $\sqrt{k}$.

Suppose $A = \mathrm{diag}(\lambda_\alpha)$ with respect to a basis s_α and that $\mathrm{Tr}(A) = 0$.
Then

$$\langle M(T_k(X)), A\rangle = \int_X H d\mu_{FS},$$

where

$$H = \rho^{-1} \sum \lambda_\alpha |s_\alpha|^2.$$

For simplicity, suppose that the scalar curvature is zero.
Since $0 = \sum \lambda_\alpha = \int_X H\rho\omega^n$ we need to show that

$$\int_X H(\rho\omega^n - d\mu_{FS}) \leq Ck^{-2} \log k \|H\|_{L^\infty}$$

i.e. that

$$\int_X |\rho\omega^n - d\mu_{FS}| \leq Ck^{-2} \log k. \quad (**)$$

We can write

$$\rho\omega^n - d\mu_{FS} = \rho\omega^n - (\omega + k^{-1}\partial\overline{\partial} \log \rho)^n.$$

The volume estimate

For $r > 0$ define $\Omega_r \subset X$ by

$$\Omega_r = \{x \in X : |\mathrm{Riem}| \leq r^{-2} \text{ on } B_r(x)\}.$$

In other words, scaling $B_r(x)$ to unit size we get a ball with uniformly bounded geometry. Write $\rho_k = 1 + \eta_k$.

Uniform asymptotic estimates: If $r \geq k^{-1/2}$ then on Ω_r we have

$$|\nabla^j \eta| \leq C k^{-2} r^{-4-j}.$$

This expresses the “locality” in the analysis of the asymptotic expansion.

Proposition C (Chen-Donaldson/Cheeger-Naber)
We have

$$\mathrm{Vol}(X \setminus \Omega_r) \leq Cr^4.$$

One ingredient in the proof is the L^2 bound on Riem coming from Chern-Weil theory.
The estimate (**) follows by elementary arguments.
(For example the region in X where the uniform asymptotic estimate gives no information has volume $\mathrm{Vol}(X \setminus \Omega_{k^{-1/2}}) \leq Ck^{-2}$.)