Local index theory, eta invariants and holomorphic torsion : a survey

by

JEAN-MICHEL BISMUT

Département de Mathématique
Université Paris-Sud
URA D1169 du CNRS
Bâtiment 425
91405 ORSAY Cedex
FRANCE
bismut@topo.math.u-psud.fr

Abstract. The purpose of this paper is to review various results related to the local families index theorem, eta invariants, and the Ray-Singer holomorphic torsion.

Acknowledgments. The support of Institut Universitaire de France (IUF) is gratefully acknowledged.

Also the author is indebted to Jeff Cheeger and Paolo Piazza for very helpful comments.

The purpose of this paper is to survey various recent developments in local index theory, including applications to eta invariants, Quillen metrics on determinant bundles, and analytic torsion forms. In this refined index theory, the construction of secondary objects plays an important role. On the analytic side, these will be the $\widetilde{\eta}$ forms (which are extensions of eta invariants), analytic torsion forms (which extend the Ray-Singer torsion). These objects refine on the construction of the index bundle [**A**], [**AS2**]. On the geometric side, they include Chern-Simons forms [**ChSi**], Bott-Chern forms [**BoCh**] and Bott-Chern currents [**BGS4,5**], which refine on the classical cohomological objects of index theory like $\widehat{A}$, Td, ch. The ultimate purpose of the new theory is to relate the analytic secondary invariants to corresponding geometric secondary invariants.

This line of thought is most clearly illustrated by the Riemann-Roch formula in Arakelov geometry of Gillet-Soulé [**GS3,4**], which applies to arithmetic varieties. In its simplest form, it evaluates the artihmetic degree of a determinant bundle (whose evaluation involves Quillen metrics at places at infinity) in terms of integrals of arithmetic characteristic classes [**GS1,2**] (whose construction involves Bott-Chern classes and Bott-Chern currents at places at infinity).

Needless to say, we will always work here in the context of real or complex geometry. Still the idea that the above objects fit "naturally" in an algebraic context has been a powerful motivation for their development.

Let us now briefly discuss in more detail the content of this survey. As we said before, this paper is organized around the local index theorem [**P1,2**], [**Gi1,2**], [**ABoP**]. Let Z be a compact even dimensional Riemannian oriented spin manifold, let D^Z be a Dirac operator acting on smooth sections of the twisted spinors $S^{TZ} \otimes \xi$. The Atiyah-Singer index theorem [**AS1**] asserts that the index $\operatorname{Ind}(D_+^Z) \in \mathbf{Z}$ of D_+^Z (which is D^Z restricted to twisted positive spinors) is given by

$$\operatorname{Ind}(D_+^Z) = \int_Z \widehat{A}(TZ)\operatorname{ch}(\xi) \tag{0.1}$$

where the right-hand side is an integral of a characteristic class, which is a cohomology class.

Let $P_t(x,y)$ be the heat kernel for $\exp(-tD^{Z,2})$. The Mc-Kean Singer formula [**McKS**] says that for $T > 0$,

$$\operatorname{Ind}(D_+^Z) = \operatorname{Tr_s}\left[\exp(-tD^{Z,2})\right] = \int_Z \operatorname{Tr_s}\left[P_t(x,x)\right] dv_Z(x) \tag{0.2}$$

(in (0.2), $\operatorname{Tr_s}$ is our notation for a supertrace, which is a graded trace).

The local index theorem, conjectured in [**McKS**] and proved in [**P**], [**Gi1,2**], [**ABoP**], asserts that as $t \to 0$, we have "fantastic" cancellations in $\operatorname{Tr_s}[P_t(x,x)]$ (which means that as $t \to 0$, $\operatorname{Tr_s}[P_t(x,x]$ is non singular), and that

$$\operatorname{Tr_s}\left[P_t(x,x)\right] dv_X(x) \to \left\{\widehat{A}(TZ,\nabla^{TZ})\operatorname{ch}(\xi,\nabla^{\xi})\right\}^{\max}, \tag{0.3}$$

where in (0.3), the corresponding characteristic forms in Chern-Weil theory are calculated using the Levi-Civita connection ∇^{TZ} on TZ, and the given connection ∇^{ξ} on ξ. Of course, from (0.2), (0.3), we recover (0.1).

In the above form, the local index theorem was used in **[ABoP]** to give a new proof of the Atiyah-Singer index theorem.

In **[APS1]**, Atiyah-Patodi-Singer developed an index theory for manifolds with boundary. If Z is a compact even dimensional oriented spin manifold with boundary, the index problem on the Dirac operator D^Z on Z imposes global boundary conditions on ∂Z. The index formula of **[APS1]** takes the form

$$\operatorname{Ind}(D_+^Z) = \int_Z \widehat{A}(TZ, \nabla^{TZ}) \operatorname{ch}(\xi, \nabla^{\xi}) - \overline{\eta}^{D^{\partial Z}}(0) . \tag{0.4}$$

In (0.4), $\overline{\eta}^{D^{\partial Z}}(s)$ is a meromorphic function of S, which is calculated in terms of the spectrum of a Dirac operator $D^{\partial Z}$ on the boundary ∂Z . The quantity $\overline{\eta}^{D^{\partial Z}}(0)$ is called a reduced eta invariant. To establish (0.4), the local index theorem plays an essential role. In fact, the first term in the right-hand side of (0.4) is the integral of a closed differential form.

Formula (0.4) is quite important. In effect, $\int_Z \widehat{A}(TZ, \nabla^{TZ}) \operatorname{ch}(\xi, \nabla^{\xi})$ is a Chern-Simons invariant. On the other hand, $\overline{\eta}^{D^{\partial Z}}(0)$ is a global spectral invariant of ∂Z, which is a prototype of the analytic secondary invariants which will be considered later. Then (0.4) implies that mod($\mathbf{Z}$), $\overline{\eta}^{D^{\partial Z}}(0)$ is equal to a Chern-Simons invariant. This is a simple prototype of a refined index theorem. Such a theorem was formulated first in the context of differential characters by Cheeger and Simons **[CSi]**.

Section 1 is devoted to a short exposition of local index theory and eta invariants.

Let now $\pi : X \to B$ be a fibration with compact fibres as before. Let $(D_b^Z)_{b \in B}$ be the corresponding family of Dirac operators. In **[AS2]**, Atiyah and Singer have shown how to associate to this family an (analytic) index bundle $\operatorname{Ind}(D_+^Z) \in K^0(B)$. They also defined a topological index, and they proved a corresponding families index formula. When mapping $K^0(B)$ in $H(B, \mathbf{Q})$ by the Chern character ch, the formula of **[AS2]** takes the form

$$\operatorname{ch}(\operatorname{Ind}(D_+^Z)) = \pi_* \left[\widehat{A}(TZ) \operatorname{ch}(\xi)\right] \text{ in } H(B, \mathbf{Q}) . \tag{0.5}$$

The local index theorem of **[B2]** refines on the right hand-side of (0.5), by replacing it by an explicit geometrically constructed differential form $\pi_*[\widehat{A}(TZ, \nabla^{TZ}) \operatorname{ch}(\xi, \nabla^{\xi})]$. When $\operatorname{Ind}(D_+^Z)$ is a honest vector bundle, the theory of **[B2]** replaces the analytic index by an analytically constructed differential form $\operatorname{ch}(\ker D^Z, \nabla^{\ker D^Z, u})$. Then an essential by-product of **[B2]**, **[BeV]**, **[BeGeV]**, **[BC1]** is the construction of an explicit differential form $\widetilde{\eta}$ on B such that

$$d\widetilde{\eta} = \pi_* \left[\widehat{A}(TZ, \nabla^{TZ}) \operatorname{ch}(\xi, \nabla^{\xi})\right] - \operatorname{ch}(\ker D^Z, \nabla^{\ker D^Z, u}) . \tag{0.6}$$

Quillen's superconnections **[Q1]** are an important tool to obtain the above results. Superconnections provide a useful extension of Chern-Weil theory to $\mathbf{Z}_2$-graded (and possibly infinite dimensional) vector bundles. In **[B2]**, the form $\pi_*[\widehat{A}(TZ, \nabla^{TZ})\,\mathrm{ch}(\xi, \nabla^\xi)]$ is produced by refining the local index theorem (0.3) in a relative context, hence its name of a local families index theorem.

Equation (0.5) has been extended in **[BC2,3]**, **[MeP1,2]** to a families index theorem for Dirac operators on manifolds with boundary, by using either the techniques of Cheeger **[C1,2,3]** on manifolds with conical singularities, or the b-calculus of Melrose **[Me]**. The important concept of a spectral section **[MeP1,2]** has emerged in this context.

If (E, g^E) is a holomorphic Hermitian vector bundle on a complex manifold, E is naturally equipped with the holomorphic Hermitian connection ∇^E. We will denote by $\mathrm{Td}(E, g^E)$ the form $\mathrm{Td}(E, \nabla^E)$. It is a sum of forms of type (p, p).

If $\pi : X \to S$ is a holomorphic submersion with compact fibre Z, if TZ is equipped with a Hermitian metric g^{TZ}, and, if (ξ, g^ξ) is a holomorphic Hermitian vector bundle on X such that $R\pi_*\xi$ is locally free, a natural equation related to (0.6) is

$$\frac{\overline{\partial}\partial}{2i\pi}T = \mathrm{ch}(R\pi_*\xi, g^{R\pi_*\xi}) - \pi_*\left[\mathrm{Td}(TZ, g^{TZ})\,\mathrm{ch}(\xi, g^\xi)\right] , \tag{0.7}$$

where $g^{R\pi_*\xi}$ is the metric on $R\pi_*\xi$ obtained from g^{TZ}, g^ξ by Hodge theory along the fibres.

Assume that X is Kähler and let ω^X be the corresponding Kähler from. When g^{TZ} is obtained from ω^X by restriction to TZ, the forms $T(\omega^X, g^\xi)$ were constructed in **[BGS3]**, **[BK]** and were called analytic torsion forms, because in degree 0, $T^{(0)}(\omega^X, g^\xi)$ coincides with the Ray-Singer analytic torsion **[RS]** of the corresponding Dolbeault complex.

In fact of special interest are $\widetilde{\eta}^{(1)}$ and $T^{(0)}$. In **[BF1,2]**, **[BGS3]**, $\widetilde{\eta}^{(1)}$ appears as a connection form on the determinant line bundle $(\det\ker D^Z)^{-1}$, and $T^{(0)}(\omega^X, g^\xi)$ is the natural correction to the obvious Hodge metric on the line bundle $(\det R\pi_*\xi)^{-1}$ introduced by Quillen **[Q2]** to construct the Quillen metric on $(\det R\pi_*\xi)^{-1}$. When suitably interpreted, in degree 2, equations (0.6) and (0.7) appear as curvature theorems for natural connections on the line bundles $(\det\ker D^Z)^{-1}$ and $(\det R\pi_*\xi)^{-1}$. Of course, one of the points of **[BF1,2]**, **[BGS3]** is that such curvature theorems still hold without any assumption on $\ker D^Z$ or $R\pi_*\xi$.

Objects like $\widetilde{\eta}$ and $T(\omega^X, g^\xi)$ are secondary invariants which refine on the family index theorem of Atiyah-Singer **[AS2]** or on Riemann-Roch-Grothendieck. These last theorems are naturally functorial, in the sense they are compatible to the composition of maps. It is natural to ask whether $\widetilde{\eta}$ or $T(\omega^X, g^\xi)$ have related functorial properties.

A first simple question is to ask how $T(\omega^X, g^\xi)$ depends on ω^X, g^ξ. It was shown in **[BK]** that $T(\omega^X, g^\xi)$ depend on ω^X, g^ξ via obvious Bott-Chern classes **[BoC]**, **[BGS1]**.

Another question is related to the behaviour on $\widetilde{\eta}$ or $T(\omega^X, g^\xi)$ by the composition of two submersions. This question was solved in **[BC1]**, **[BeB]**, [**D**], [**Ma**], using the idea of adiabatic limits. Let us give an elementary application of this idea. In fact, if

$$0 \to L \to M \underset{j}{\to} N \to 0 \tag{0.8}$$

is an exact sequence of holomorphic vector bundles, and if g^M, g^N are Hermitian metrics on M, N for $\varepsilon > 0$, set

$$g_\varepsilon^M = g^M + \frac{1}{\varepsilon} j^* g^N . \tag{0.9}$$

Then if Q is any characteristic polynomial, one can easily show that if g^L is the metric on L induced by g^M,

$$Q(M, g_\varepsilon^M) \to Q(L, g^L) Q(N, g^N) . \tag{0.10}$$

Now (0.10) can be applied to the exact sequence

$$0 \to TZ \to TX \to \pi^* TS \to 0 .$$

Following a terminology introduced by physicists [**W**], studying geometric or spectral objects depending on a metric $g_\varepsilon^{TX} = g^{TX} + \frac{1}{\varepsilon}\pi^* g^{TS}$ as $\varepsilon \to 0$ is called passing to the adiabatic limits.

As was observed in [**BF2**], there is an analogue of (0.10) for the Levi-Civita connection of a fibered manifold.

On the other hand, the Leray spectral sequence for the de Rham or Dolbeault complexes of a fibered manifold makes the left-hand side of Riemann-Roch-Grothendieck compatible with the composition of submersions.

The behaviour of $\widetilde{\eta}$ and $T(\omega^X, g^\xi)$ under composition of submersions was obtained by adiabatic limit techniques. In [**BC1**], [**D**], [**BerB**], the case where the last submersion maps to a point was considered, and the results were expressed as results on the adiabatic limit of eta invariants, or on Quillen metrics. In [**Ma**], corresponding results were obtained for the composition of arbitrary holomorphic submersions. The above results also rely on an observation of Mazzeo-Melrose [**MazMe**] relating the adiabatic limit of the spectrum of the Dirac operator to the Leray spectral sequence.

Similar questions can be asked in the case of embeddings. We will explain the problem in the context of complex geometry. Let $i : Y \to X$ be an embedding of complex manifolds. Let η be a vector bundle on Y. If X is projective, there is a resolution of $i_*\eta$ by a complex (ξ, v) of holomorphic vector bundles on X. Then by definition, the direct image $i_!\eta \in K(X)$ is given by

$$i_!\eta = [\xi] \text{ in } K(X) , \tag{0.11}$$

the point being to show that $[\xi]$ does not depend on ξ. Now by Riemann-Roch-Grothendieck,

$$\mathrm{ch}(i_!\eta) = i_* \left(\mathrm{Td}^{-1}(N_{Y/X}) \, \mathrm{ch}(\eta) \right) \text{ in } H^{\mathrm{even}}(X, \mathbf{Q}) , \tag{0.12}$$

so that

$$\mathrm{ch}(\xi) = i_* \left(\mathrm{Td}^{-1}(N_{Y/X})\, \mathrm{ch}(\eta)\right) \text{ in } H^{\mathrm{even}}(X, \mathbf{Q})\,. \tag{0.13}$$

Let $g^{\xi}, g^{\eta}, g^{N_{Y/X}}$ be Hermitian metrics on $\xi, \eta, N_{Y/X}$. By analogy with (0.7) it is natural to ask whether one can refine (0.13). Namely one can ask for the existence of a current $T(\xi, g^{\xi})$ on X such that

$$\frac{\overline{\partial}\partial}{2i\pi} T(\xi, g^{\xi}) = \mathrm{Td}^{-1}(N_{Y/X}, g^{N_{Y/X}})\, \mathrm{ch}(\eta, g^{\eta})\delta_Y - \mathrm{ch}(\xi, g^{\xi})\,. \tag{0.14}$$

The current $T(\xi, g^{\xi})$ has been constructed in [**B3**], [**BGS4**]. Again it is natural to study the compatibility of the currents $T(\xi, g^{\xi})$ to the composition of embeddings. This has been done in detail in [**BGS5**].

Having now constructed objects $T(\omega^X, g^{\xi})$ and $T(\xi, g^{\xi})$ associated to a submersion or an embedding, which are compatible to the composition of submersions or of embeddings, the last obvious final step is to study the compatibility of these objects to the composition of an embedding and a submersion. This has been done in [**BL**] when the submersion maps to a point, and in [**B5,6**] in the general case. In [**BL**], the main result is formulated naturally in terms of Quillen metrics. In the proof of [**BL**], [**B5,6**], a mysterious secondary invariant associated to a short exact sequence of holomorphic Hermitian vector bundles appeared, whose construction was somewhat puzzling. A preliminary step for the proof of [**BL**], [**B5,6**] was the explicit evaluation of this class in [**B4**].

The most elaborate formula in [**B5,6**] expresses a combination of analytic torsion forms as a sum of integrals along the fibre of analytic torsion currents and of Bott-Chern classes. This indicates that the refined objects introduced above fit in a refined Riemann-Roch algebra. As explained before, Gillet and Soulé [**GS3**] have explained how, in the case of arithmetic varieties, these results can be used to prove a Riemann-Roch-Grothendieck formula in Arakelov geometry. They have proved such a formula in [**GS4**] for the first Chern class, and their proof for higher Chern classes is pending.

This paper is organized as follows. Section 1 is devoted to the local index theorem and the eta invariant. In Section 2, we review various results on the local families index theorem and the $\widetilde{\eta}$ forms. Finally, in Section 3, we consider analytic torsion forms and analytic torsion currents.

Part of the material contained in this survey already has been reviewed in [**B7**].

I. The local index theorem and the eta invariant.

In this Section, we review a few well-known results on the local index theorem for Dirac operators closed manifolds, on the index theorem for manifolds with boundary, and we also give related results on eta invariant.

This Section is organized as follows. In a), we state the local Atiyah-Singer index theorem for closed manifolds of Patodi **[P1,2]**, Gilkey **[Gi1,2]**, Atiyah-Bott-Patodi **[ABoP]**. In b), we state the Atiyah-Patodi-Singer index theorem on manifolds with boundary **[APS1]** and we introduce the associated eta invariant. In c), we give the formula for the signature of a manifold with boundary obtained by Atiyah-Patodi-Singer **[APS1]**. In d), we describe result by Cheeger **[C1,2]** and Chou **[Ch]** on the index theorem on manifolds with conical singularities. In e), we recall the result by Cheeger **[C3]** on the L_2 signature of such manifolds. Finally in f), we review briefly the approach by Melrose **[Me]** to the Atiyah-Patodi-Singer index theorem, using the b-calculus.

For a detailed approach to the local index theorem, we refer to Berline-Getzler-Vergne **[BeGeV]**.

a) The local Atiyah-Singer index theorem for Dirac operators on closed manifolds.

Let Z be an even dimensional compact oriented spin manifold. Let g^{TZ} be a Riemannian metric on TZ. Let $S^{TZ} = S^{TZ}_+ \oplus S^{TZ}_-$ be the Hermitian $\mathbf{Z}_2$-graded vector bundle of (TZ, g^{TZ}) spinors.

Let ∇^{TZ} be the Levi-Civita connection on (TZ, g^{TZ}). Let $\nabla^{S^{TZ}}$ be the connection induced by ∇^{TZ} on S^{TZ}. Let (ξ, g^ξ, ∇^ξ) be a Hermitian vector bundle equipped with a unitary connection. Let $\nabla^{S^{TZ}\otimes\xi}$ be the obvious connection on $S^{TZ} \otimes \xi$.

Let $c(TZ)$ be the bundle of Clifford algebras of (TZ, g^{TZ}). It is generated over $\mathbf{R}$ by 1, $X \in TZ$, and the commutation relations

$$XY + YX = -2\langle X, Y\rangle\,. \tag{1.1}$$

Then S^{TZ} is a $c(TZ)$-Clifford module. If $X \in TZ$, we denote by $c(X)$ the action of $X \in c(TZ)$ on S^{TZ}. Then $c(X)$ acts like $c(X) \otimes 1$ on $S^{TZ} \otimes \xi$.

Let D^Z be the Dirac operator associated to (g^{TZ}, ∇^ξ). If $e_1, \dots, e_n$ is an orthonormal basis of TZ,

$$D^Z = \sum_1^n c(e_i)\nabla^{S^{TZ}\otimes\xi}_{e_i}\,. \tag{1.2}$$

Then D^Z is an odd operator, i.e. it exchanges $C^\infty(Z, S^{TZ}_+ \otimes \xi)$ and $C^\infty(Z, S^{TZ}_- \otimes \xi)$. Also D^Z it is a first-order self-adjoint elliptic operator. Let $D^Z_\pm$ be the restriction of D^Z to $C^\infty(Z, S^{TZ}_\pm \otimes \xi)$, so that

$$D^Z = \begin{bmatrix} 0 & D^Z_- \\ D^Z_+ & 0 \end{bmatrix}. \tag{1.3}$$

Since D^Z_+ is elliptic, it is a Fredholm operator. By definition, the index $\operatorname{Ind}(D^Z_+) \in \mathbf{Z}$ is given by

$$\operatorname{Ind}(D^Z_+) = \dim\ker(D^Z_+) - \dim\ker(D^Z_-)\,. \tag{1.4}$$

Put

$$\begin{aligned}\widehat{A}(x) &= \frac{x/2}{\sinh(x/2)} \quad , \\ \mathrm{Td}(x) &= \frac{x}{1-e^{-x}}\,.\end{aligned} \tag{1.5}$$

We identify $\widehat{A}$ and Td with the corresponding multiplicative genera, the Hirzebruch genus and the Todd genus. Similarly the Chern character ch is the additive genus associated with the function $\exp(x)$.

If $\pi : F \to Z$ is a complex vector bundle, and if P is a real invariant polynomial, let $P(F) \in H^{\text{even}}(Z, \mathbf{R})$ be the corresponding characteristic class. If ∇^F is a connection on F, and $R^F = \nabla^{F,2}$ is its curvature, we denote by $P(F, \nabla^F)$ the closed even form $P\left(\frac{-R^F}{2i\pi}\right)$ on Z, which represents $P(F)$ in cohomology.

Then, by the Atiyah-Singer index theorem [**AS1**],

$$\mathrm{Ind}\,(D_+^Z) = \int_Z \widehat{A}(TZ)\,\mathrm{ch}(\xi)\,. \tag{1.6}$$

If $E = E_+ \oplus E_-$ is a $\mathbf{Z}_2$-graded vector space, let $\tau = \pm 1$ on $E_\pm$ define the $\mathbf{Z}_2$-grading. If $A \in \mathrm{End}\,(E)$, we define its supertrace $\mathrm{Tr_s}\,[A]$ by

$$\mathrm{Tr_s}\,[A] = \mathrm{Tr}[\tau A]\,. \tag{1.7}$$

The algebra $\mathrm{End}\,(E)$ is naturally $\mathbf{Z}_2$-graded, the even (resp. odd) elements commuting (resp. anticommuting) with τ. If $A, B \in \mathrm{End}\,(E)$, we define the supercommutator $[A, B]$ by the formula

$$[A,B] = AB - (-1)^{\deg A \deg B} BA. \tag{1.8}$$

By [**Q1**], if $A, B \in \mathrm{End}\,(E)$,

$$\mathrm{Tr_s}\,[[A,B]] = 0\,. \tag{1.9}$$

Observe that since D^Z is elliptic, for $t > 0$, $\exp(-tD^{Z,2})$ is trace class. We then have the formula of Mc Kean-Singer [**McKS**].

Proposition 1.1. *For $t > 0$,*

$$\mathrm{Ind}\,(D_+) = \mathrm{Tr_s}\left[\exp(-tD^{Z,2})\right]\,. \tag{1.10}$$

PROOF : By spectral theory,

$$\lim_{t\to+\infty} \mathrm{Tr_s}\left[\exp(-tD^{Z,2})\right] = \mathrm{Ind}\,(D_+^Z)\,. \tag{1.11}$$

Also we have the "Bianchi" identity

$$\left[D^Z, D^{Z,2}\right] = 0\,. \tag{1.12}$$

Using (1.9), (1.12), we get

$$\begin{aligned}\frac{\partial}{\partial t}\,\mathrm{Tr_s}\left[\exp(-tD^{Z,2})\right] &= -\,\mathrm{Tr_s}\left[D^{Z,2}\exp(-tD^{Z,2})\right] = \\ &\quad -\tfrac{1}{2}\,\mathrm{Tr_s}\left[\left[D^Z, D^Z\exp(-tD^{Z,2})\right]\right] = 0\,.\end{aligned} \tag{1.13}$$

Note that in the last steps of (1.13), one should express the various quantities in terms of smooth heat kernels to justify the use of (1.9). □

Let $P_t(x,y)$ be the smooth kernel of $\exp(-tD^{Z,2})$ with respect to the Riemannian volume dy . Then by (1.10),

$$\operatorname{Ind}(D_+^Z) = \int_Z \operatorname{Tr_s}[P_t(x,x)]\,dx\,. \tag{1.14}$$

Put $n = \dim Z$. By general results on elliptic differential operators, we know that for $x \in Z$, as $t \to 0$,

$$\operatorname{Tr_s}[P_t(x,x)] = \sum_{k=0}^{p} a_{-n/2+k}(x)t^{-n/2+k} + \mathcal{O}_x(t^{-n/2+p+1})\,, \tag{1.15}$$

and $\mathcal{O}_x(t^{-n/2+p+1})$ is uniform in $x \in X$. Moreover the $a_j(x)$ only depend on the complete symbol of D^Z near x. By (1.14), (1.15),

$$\begin{aligned} &\int_Z a_j(x)dx = 0 \qquad \text{for } j \neq 0\,, \\ &\int_Z a_0(x)dx = \operatorname{Ind}(D_+^Z)\,. \end{aligned} \tag{1.16}$$

Let ∇^{TZ} be the Levi-Civita connection on (TZ, g^{TZ}). In **[McKS]**, Mc Kean and Singer conjectured that "fantastic cancellations" should occur in (1.16) so that for the considered operator D^Z,

$$\begin{aligned} &a_j = 0\,, j < 0\,, \\ &a_0(x) = \left\{\widehat{A}(TZ, \nabla^{TZ})\operatorname{ch}(\xi, \nabla^{\xi})\right\}^{\max}\,. \end{aligned} \tag{1.17}$$

Needless to say, (1.16), (1.17) would imply the index formula (1.6).

Theorem 1.2. *Equation (1.17) holds.*

PROOF : There are two kinds of proofs of (1.17).
- The algebraic proofs of Patodi **[P1,2]**, Gilkey **[Gi1,2]**, Atiyah-Bott-Patodi **[ABoP]** describe explicitly the a_j $(j \leq 0)$ as polynomial functions of the metric g^{TZ}, the connection ∇^{ξ} and their derivatives. For $j < 0$, arguments of Gilkey show that there are no such polynomials other than 0. Also a_0 is shown to be a universal combination of certain Chern-Weil forms. One then only needs to verify the identity (1.17) for a_0 on sufficiently many examples, given by the $\mathbf{P}^n(\mathbf{C})$.
- The direct proofs of Getzler **[Ge1,2]**, Bismut **[B1]**, Berline-Vergne **[BeV]**. These proofs were stimulated by arguments by Alvarez-Gaumé **[Al]**, using functional integration, which suggested that there should be some explicit algebraic mechanism forcing the above local cancellations.

The proof of Getzler **[Ge2]** uses a powerful rescaling technique on the Clifford algebra. It is explained in detail in [**BeGeV**, Chapter 4]. Of particular importance is the Getzler operator **[Ge1,2]**, [**BeGeV**, Proposition 4.19], which appears when doing the Getzler rescaling on the Clifford algebra. This operator is an harmonic oscillator, and its heat kernel produces the genus $\widehat{A}$ in (1.17).

Theorem 1.2 is often called the local index theorem for Dirac operators. □

b) The Atiyah-Patodi-Singer index theorem.

Let Z be a compact manifold of dimension n with boundary $Y = \partial Z$. Let $\partial Z \times [0,1]$ be a tubular neighborhood of ∂Z in Z, u being the inward normal coordinate. Let g^{TZ} be a Riemannian metric on TZ, which is product near ∂Z, i.e.

$$g^{TZ} = g^{T\partial Z} + |du|^2 \,. \tag{1.18}$$

Let E, F be complex Hermitian vectors bundles on Z, which, near Y, are pull-backs of vector bundles on Y. Let $D : C^\infty(Z,E) \to C^\infty(Z,F)$ be an elliptic first order differential operator. Let $\sigma : E_{|Y} \to F_{|Y}$ be a bundle isometry. Let $A : C^\infty(Y, E_{|Y}) \to C^\infty(Y, F_{|Y})$ be an elliptic first order differential operator, which is self-adjoint with respect to the obvious L_2 Hermitian product. We assume that on $\partial Z \times [0,1]$,

$$D = \sigma\left(\frac{\partial}{\partial u} + A\right) . \tag{1.19}$$

Let $P_{\geq 0}$ (resp. $P_{<0}$) be the orthogonal projection operator on the direct sum of the eigenspaces of A associated to nonnegative (resp. negative) eigenvalues.

Let $C^\infty(Z,E,P)$ be the vector space of smooth sections of E over Z, such that

$$P_{\geq 0} f_{|Y} = 0 \,. \tag{1.20}$$

In **[APS1]**, Atiyah-Patodi-Singer showed that D defines a Fredholm operator. More precisely for $\ell \geq 0$, let $H^\ell(Z,E)$ be the ℓ^{th} Sobolev space of sections of E, and for $\ell \geq 1$, let $H^\ell(Z,E,P_{\geq 0})$ be the set of $s \in H^\ell(Z,E)$ such that $P_{\geq 0} s_{|Y} = 0$. Then in **[APS1]**, the authors proved that for any $\ell \geq 1$, $D : H^\ell(Z,E,P) \to H^{\ell-1}(Z,F)$ is Fredholm.

By definition, the index of D is given by

$$\operatorname{Ind}(D) = \dim\ker D - \dim\ \operatorname{coker} D \,. \tag{1.21}$$

Let D^* be the formal adjoint of D. We make D^* act on $C^\infty(Z,F,P_{<0})$ or on $H^\ell(Z,F,P_{<0})$. Then

$$\operatorname{Ind}(D) = \dim\ker D - \dim\ker D^* \,. \tag{1.22}$$

Let R_t, S_t be the heat kernels of $\exp(-tD^*D)$, $\exp(-tDD^*)$ acting on the smooth sections of E, F over the double of Z. If $x \in Z \setminus \partial Z$, as $t \to 0$, we have the asymptotic expansion

$$\operatorname{Tr}[(R_t - S_t)(x,x)] = \sum_{k=0}^{p} a_{-n/2+k}(x) t^{-n/2+k} + \mathcal{O}_x(t^{-n/2+p+1}) \,. \tag{1.23}$$

Observe that the $a_{-n/2+k}(x)$ vanish identically near ∂Z. Also the constant coefficient $a_0(x)$ in the expansion (1.24) vanishes if n is odd.

Let $\mathrm{Sp}(A)$ be the spectrum of the operator A. For $s \in \mathbf{C}$, $\mathrm{Re}(s) > \dim Y$, put

$$\eta^A(s) = \sum_{\substack{\lambda \in \mathrm{Sp}(A) \\ \lambda \neq 0}} \frac{\mathrm{sign}\,(\lambda)}{|\lambda|^s} . \tag{1.24}$$

Then $\eta^A(s)$ is holomorphic in s on its domain of definition, and moreover

$$\eta^A(s) = \frac{1}{\Gamma(\frac{s+1}{2})} \int_0^{+\infty} t^{(s-1)/2}\, \mathrm{Tr}\left[A e^{-tA^2}\right] dt . \tag{1.25}$$

As $t \to 0$,

$$\mathrm{Tr}\left[A e^{-tA^2}\right] = \sum_{k=0}^{p} b_{-(\frac{\dim Y}{2}+1)+k}\, t^{-(\frac{\dim Y}{2}+1)+k} . \tag{1.26}$$

From (1.26), we deduce that $\eta^A(s)$ extends to a meromorphic function of $s \in \mathbf{C}$, with simple poles. In particular 0 appears to be a simple pole of $\eta^A(s)$.

Now we state the index formula of Atiyah-Patodi-Singer [**APS1**, Theorem 3.10].

Theorem 1.3. *The function $\eta^A(s)$ is holomorphic at 0. Moreover*

$$\mathrm{Ind}\,(D) = \int_Z a_0(x)dx - \tfrac{1}{2}\left(\eta^A(0) + \dim\ker(A)\right) . \tag{1.27}$$

PROOF : The proof of both statements in [**APS1**] is briefly sketched. The proof can be divided into three main steps :

- The construction of a parametrix on $\partial Z \times [0, +\infty[$ for $\pm \frac{\partial}{\partial u} + A$

By using the spectral decomposition for A, one constructs a parametrix for $\pm\frac{\partial}{\partial u} + A$ with the required boundary conditions. The operator A is then replaced by a scalar $\lambda \in \mathbf{R}$ in $\mathrm{Sp}(A)$, and the point is to show that these finite dimensional parametrices patch into a global parametrix Q.

By patching these parametrices with an inner parametrix for D and D^*, Atiyah-Patodi-Singer construct a parametrix for D, which demonstrates the Fredholm property of D.

- The heat equation method on Z

By patching the heat kernel for DD^* and D^*D on $\partial Z \times [0, +\infty[$ (which is explicitly obtained by the functional calculus) with the heat kernel on the double of Z, one obtains a global heat kernel on Z, from which one obtains a Mc Kean-Singer formula for $\mathrm{Ind}\,(D)$. Namely for $t > 0$,

$$\mathrm{Ind}\,(D) = \mathrm{Tr}\left[\exp(-tD^*D)\right] - \mathrm{Tr}\left[\exp(-tDD^*)\right] . \tag{1.28}$$

Using the corresponding heat kernels, one gets

$$\operatorname{Ind}(D) = \int_Z (\operatorname{Tr}[P_t(x,x)] - \operatorname{Tr}[Q_t(x,x)])\,dt\,. \tag{1.29}$$

One then makes $t \to 0$ in (1.29). By taking the constant term in $\int_{Z\backslash \partial Z\times[0,1]}$ as $t \to 0$, one obtains the easy term $\int_{Z\backslash\partial Z\times[0,1]} a_0(x)dx$. The contribution of $\int_{\partial Z\times[0,1]}$ as $t \to 0$ is obtained from the explicit form of the heat kernel on $\partial Z \times [0,+\infty[$. In particular, in [**APS1**], the holomorphy at 0 of $\eta^A(0)$ is a consequence of the heat equation formula (1.29). The quantity $-\frac{1}{2}(\eta^A(0) + \dim\ker(A))$ is shown to be the contribution of $\int_{\partial Z\times[0,1]}$ to $\operatorname{Ind}(D)$. □

Let Z, g^{TZ} be as before. We assume that Z is even dimensional, oriented and spin. Let $S^{TZ} = S_+^{TZ} \oplus S_-^{TZ}$ be the Hermitian $\mathbf{Z}_2$-graded vector bundle of (TZ, g^{TZ}) spinors. Let (ξ, g^ξ, ∇^ξ) be a complex Hermitian vector bundle with unitary connection.

Let ∇^{TZ} be the Levi-Civita connection on (TZ, g^{TZ}). Then ∇^{TZ} lifts to a unitary connection on $S^{TZ} = S_+^{TZ} \oplus S_-^{TZ}$. Let $\nabla^{S^{TZ}\otimes\xi}$ be the obvious connection on $S^{TZ} \otimes \xi$.

We orient $T\partial Z$ so that if $e_1, \dots, e_{n-1}$ is an oriented basis of $T\partial Z$, $(e_1, \dots, e_{n-1},$
$\frac{\partial}{\partial u})$ is an oriented basis of $TZ_{|\partial Z}$.

Let $S^{T\partial Z}$ be the Hermitian vector bundle on ∂Z of the $(T\partial Z, g^{T\partial Z})$ spinors.Then $S^{T\partial Z} \simeq S^{TY}_{+|\partial Z} \simeq S^{TZ}_{-|\partial Z}$. The bundle $S^{T\partial Z} \otimes \xi_{|\partial Z}$ is naturally equipped with a connection $\nabla^{S^{T\partial Z}\otimes\xi_{|\partial Z}}$.

Let D^Z be the Dirac operator associated to the metric g^{TZ} and the connection ∇^ξ. We can write D^Z as a matrix operator,

$$D^Z = \begin{bmatrix} 0 & D_-^Z \\ D_+^Z & 0 \end{bmatrix}. \tag{1.30}$$

Also D^Z is an elliptic first order differential operator.

Let $D^{\partial Z}$ be the Dirac operator associated to $g^{T\partial Z}$ and ∇^ξ. Then $D^{\partial Z}$ acts on $C^\infty(\partial Z, S^{T\partial Z} \otimes \xi)$ as a self-adjoint first order elliptic operator.

To the operator $D = D_+^Z$, we can apply the general approach of Section 1 b), with A replaced by $D^{\partial Z}$. So D_+^Z is restricted to act on $C^\infty(Z, S_+^{TZ} \otimes \xi, P_{\geq 0})$ and D_-^Z on $C^\infty(Z, S_-^{TZ} \otimes \xi, P_{<0})$.

Let $\eta^{D^{\partial Z}}(s)$ be the eta function of $D^{\partial Z}$.

As an application of Theorem 1.3, Atiyah-Patodi-Singer [**APS1,** Theorem 4.2] obtain the following result.

Theorem 1.4. *The fonction $\eta^{D^{\partial Z}}(s)$ is holomorphic at 0. Moreover*

$$\operatorname{Ind}(D_+^Z) = \int_Z \widehat{A}(TZ, \nabla^{TY}) \operatorname{ch}(\xi, \nabla^\xi) - \tfrac{1}{2}(\eta^{D^{\partial Z}}(0) + \dim\ker(D^{\partial Z}))\,. \tag{1.31}$$

PROOF : By using the "fantastic cancellations" conjectured by Mc Kean-Singer in [**McKS**], it is shown in [**APS1**] that in (1.23),

$$a_0(x) = \left\{\widehat{A}(TZ, \nabla^{TZ}) \operatorname{ch}(\xi, \nabla^\xi)\right\}^{\max}. \tag{1.32}$$

Theorem 1.4 follows from Theorem 1.3. □

REMARK 1.5. The local index theorem, Theorem 1.2, has been used by [**ABoP**] as a tool to prove the Atiyah-Singer index theorem for Dirac operators, which by an argument of K-theory, is enough to prove the full Atiyah-Singer index theorem. However the "locality" of the index does not appear in the final answer. This is in dramatic contrast with Theorem 1.4, where $\widehat{A}(TZ, \nabla^{TZ}) \operatorname{ch}(\xi, \nabla^\xi)$ is viewed as a form, and not as a representative of a cohomology class.

c) The signature of manifolds with boundary.

Let Z be an oriented manifold with boundary of dimension $4k$. Let $H^\cdot(Z)$, $H^\cdot(Z, \partial Z)$ be the absolute and relative cohomology groups of Z. Then by Poincaré duality, $H^{2k}(Z)$ and $H^{2k}(Z, \partial Z)$ are Poincaré dual.

Let $\widehat{H}(Z)$ be the image of $H(Z, \partial Z)$ in $H(Z)$. Then $\widehat{H}^{2k}(Z)$ is naturally equipped with the symmetric intersection form

$$(\alpha, \beta) \in \widehat{H}^{2k}(Z) \to \int_Z \alpha \wedge \beta\,. \tag{1.33}$$

Let $\operatorname{sign}(Z)$ be the signature of this form.

Let (V, g^V) be a real euclidean vector space of odd dimension $2\ell - 1$. Let S^V be the vector space of (V, g^V) spinors.

The star operator $*^V$ induces an isomorphism $\Lambda^{\text{even}}(V^*) \simeq \Lambda^{\text{odd}}(V^*)$. Moreover

$$\Lambda^{\text{even}}(V^*) \simeq \Lambda^{\text{odd}}(V^*) \simeq S^V \otimes S^{V*}\,. \tag{1.34}$$

Since S^V and S^{V^*} are $c(V)$-Clifford modules, $\Lambda^{\text{even}}(V^*) \simeq \Lambda^{\text{odd}}(V^*)$ are left and right Clifford modules, and the corresponding Clifford actions commute. More precisely if $\alpha \in \Lambda^p(V^*)$, $e \in V$, if $e^* \in V^*$ corresponds to V by the metric, put

$$\begin{aligned} c(e)\alpha &= i^{\ell+p(p-1)}\left((-1)^V e^* \wedge *^V - *^V e^* \wedge\right)\alpha\,, \\ \widehat{c}(e)\alpha &= i^{\ell+p(p-1)}\left((-1)^V e^* \wedge *^V + *^V e^* \wedge\right)\alpha\,. \end{aligned} \tag{1.35}$$

Then $c(e), \widehat{c}(e)$ induce the left and right Clifford module structure on $\Lambda(V^*)$. They both preserve $\Lambda^{\text{even}}(V^*)$ and $\Lambda^{\text{odd}}(V^*)$. Let ρ be the one to one map from $\Lambda(V^*)$ into itself such that if $\alpha \in \Lambda^p(V^*)$,

$$\rho(\alpha) = i^{\ell+p(p-1)} *^V \alpha \in \Lambda^{2\ell-1-p}(V^*)\,. \tag{1.36}$$

Then ρ exchanges $\Lambda^{\text{even}}(V^*)$ and $\Lambda^{\text{odd}}(V^*)$ and moreover

$$\begin{aligned} c(e)\rho &= \rho c(e) \\ \widehat{c}(e)\rho &= \rho\widehat{c}(e)\,. \end{aligned} \tag{1.37}$$

Let Y be an oriented manifold of odd dimension $2\ell-1$. Let $\widetilde{D}^Y$ be the operator acting on $C^\infty(Y, \Lambda(T^*Y))$ such that if $\alpha \in C^\infty(Y, \Lambda^p(T^*Y))$,

$$\widetilde{D}^Y \alpha = i^{\ell+p(p-1)} \left((-1)^p d *^{TY} - *^{TY} d\right) \alpha\,. \tag{1.38}$$

In view of (1.35), it is clear that $\widetilde{D}^Y$ is a Dirac operator acting on $C^\infty(Y, \Lambda(T^*Y))$. Moreover it preserves $C^\infty(Y, \Lambda^{\text{even}}(T^*Y))$ and $C^\infty(Y, \Lambda^{\text{odd}}(T^*Y))$. By (1.36), it $\widetilde{D}^Y$ splits into two equivalent operators, $\widetilde{D}^Y = \widetilde{D}^{Y,\text{even}} \oplus \widetilde{D}^{Y,\text{odd}}$. Also one verifies easily that $\ker(\widetilde{D}^Y)$ consists of the harmonic forms, i.e.

$$\ker \widetilde{D}^Y \simeq H^\cdot(Y)\,. \tag{1.39}$$

Let $\eta^{\widetilde{D}^{Y,\text{even}}}(s)$ be the eta function of $\widetilde{D}^{Y,\text{even}}$.

Let $\mathcal{L}$ be the multiplicative genus associated to

$$\mathcal{L}(x) = \frac{x}{\tanh(x)}\,. \tag{1.40}$$

We then have the signature formula of [**APS1**, Theorem 4.14] for $\operatorname{sign}(Z)$.

Theorem 1.6. *The following identity holds,*

$$\operatorname{sign}(Z) = \int_Z \mathcal{L}(TZ, \nabla^{TZ}) - \eta^{\widetilde{D}^{\partial Z,\text{even}}}(0)\,. \tag{1.41}$$

PROOF : The proof of [**APS1**] consists in first using Theorem 1.4 with $\xi = S^{TZ}$. In a second step, the kernel of the corresponding Dirac operator D^Z is related to the kernel of the Dirac operator on $Z \bigcup_{\partial Z} \partial Z \times]-\infty, 0]$, which is Z extended by an infinite cylinder. $\square$

d) Manifolds with conical singularities : the index theorem of Cheeger and Chou.

Let Y be a smooth manifold equipped with a Riemannian metric g^{TY} on TY. Let $C(Y) = Y \times]0, +\infty[$ be the metric cone equipped with the metric $dr^2 + r^2 g^{TY}$. Then $C(Y)$ can be compactified into a honest cone with vertex δ. Let $C^{[0,\ell]}(Y)$ be the truncated cone $\{x = (r, y) \in C(Y), 0 \leq r \leq \ell\}$.

Let Z be a smooth manifold with boundary $Y = \partial Z$, taken as in Section 1 c). Let $\widehat{Z}_\ell = Z \bigcup_{Y \times \{\ell\}} C^{[0,\ell]}(\partial Z)$ be the manifold Z with the cone $C^{[0,\ell]}(\partial Z)$ attached. Let $g^{T\widehat{Z}_\ell}$ be a metric on $\widehat{Z}_\ell$, which coincides with the conical metric $dr^2 + \frac{r^2}{\ell^2} g^{T\partial Z}$ on $C^{[0,\ell]}(\partial Z)$.

Assume that Z is oriented, even dimensional and spin. Put $n = \dim Z$. Let $S^{T\widehat{Z}_\ell} = S_+^{T\widehat{Z}_\ell} \oplus S_-^{T\widehat{Z}_\ell}$ be the Hermitian vector bundle of $(T\widehat{Z}_\ell, g^{T\widehat{Z}_\ell})$ spinors. Let $S^{T\partial Z}$ be the vector bundle of $(T\partial Z, g^{T\partial Z})$ spinors. Then over $C^{[0,\ell]}(\partial Z)$, $S^{T\widehat{Z}_\ell} \simeq S^{T\partial Z} \oplus S^{T\partial Z}$.

Let (ξ, g^ξ, ∇^ξ) be a Hermitian vector bundle with unitary connection on Z, which is product near ∂Z. Then it extends to $\widehat{Z}_\ell$.

Let $D^{\widehat{Z}_\ell}$ be the formally self-adjoint Dirac operator on $\widehat{Z}^\ell$ associated to $(g^{T\widehat{Z}_\ell}, \nabla^\xi)$.

To simplify the exposition we will assume that $D^{\partial Z}$ is invertible. Let $\ell > 0$ be large enough so that for any $\lambda \in \mathrm{Sp}(D^{\partial Z}), \ell |\lambda| > \frac{1}{2}$.

Then by following ideas of Cheeger **[C1, 2]**, Chou showed in **[Ch]** that D_ℓ extends to a self-adjoint operator with domain the first Sobolev space $H^1(\widehat{Z}_\ell, S^{T\widehat{Z}_\ell} \otimes \xi)$. Moreover for $p \geq 1$, $D_+^{\widehat{Z}_\ell}$ is a Fredholm operator $H^p(\widehat{Z}_\ell, S_+^{\widehat{Z}_\ell} \otimes \xi) \to H^{p-1}(\widehat{Z}_\ell, S_-^{\widehat{Z}_\ell} \otimes \xi)$ and

$$\mathrm{Ind}\,(D_+^{\widehat{Z}_\ell}) = \dim\ker D_+^{\widehat{Z}_\ell} - \dim\ker D_-^{\widehat{Z}_\ell}\,. \tag{1.42}$$

Let D_+^Z be the Atiyah-Patodi-Singer operator considered in Section 1 e).

Proposition 1.7. *The following identity holds*

$$\mathrm{Ind}\,(D_+^Z) = \mathrm{Ind}\,(D_+^{\widehat{Z}_\ell})\,. \tag{1.43}$$

Proof : If s is a H^1 section of $S_+^{T\widehat{Z}_\ell} \otimes \xi$ such that $D_+^{\widehat{Z}_\ell} s = 0$, then on $C^{[0,\ell]}(\partial Z)$,

$$\left(\frac{\partial}{\partial r} + \frac{n-1}{2r}\right) s + \frac{\ell D^{\partial Z} s}{r} = 0\,. \tag{1.44}$$

Write s in the form

$$s = \sum_{\lambda \in \mathrm{Sp}(D^{\partial Z})} s_\lambda(r)\,,\; D^{\partial Z} s_\lambda = \lambda s_\lambda\,. \tag{1.45}$$

From (1.44), one deduces easily that since $\int_{[0,\ell]\times Z} |h|^2 r^{n-1} dr dx < +\infty$, if $\ell\lambda \geq \frac{1}{2}$, then $s_\lambda = 0$. Recall that if $\lambda \in \mathrm{Sp}(D^{\partial Z}), \ell|\lambda| > \frac{1}{2}$. So if $D^{\widehat{Z}_\ell} s = 0$, then $P_{\geq 0} s_{|\partial Z} = 0$.

We then find that $\ker D_+^Z \simeq \ker D_+^{\widehat{Z}_\ell}$. Similarly, one proves that $\ker D_-^Z \simeq \ker D_-^{\widehat{Z}_\ell}$. Our Proposition follows. □

REMARK 1.8. Proposition 1.7 asserts the remarkable fact that while Atiyah-Patodi-Singer conditions are global on ∂Z, the apparently "local" L_2 conditions of Cheeger and Chou imposes on the kernels and cokernels of $D^{\widehat{Z}_\ell}$ the global boundary conditions of Atiyah-Patodi-Singer. This is because any neighborhood of the vertex δ encodes the global geometry of the cross section.

Cheeger[**C1,2**] and Chou [**Cho**] apply the heat equation method to the index problem considered above. In particular the functional calculus over cones introduced by Cheeger allows an explicit evaluation of the heat kernel on $C(\partial Z)$. A direct computation shows that

$$\mathrm{Ind}\, D_+^{\widehat{Z}_\ell} = \int_A \widehat{A}(TZ, \nabla^{TZ})\, \mathrm{ch}(\xi, \nabla^\xi) - \tfrac{1}{2}\eta^{D^{\partial Z}}(0)\,. \tag{1.46}$$

In (1.46), $\frac{1}{2}\eta(0)$ appears as the contribution of the vertex δ to the index. Of course given Proposition 1.7, this result fits with Theorem 1.4.

e) The L_2 cohomology of manifolds with conical singularities.

Let $\widehat{Z} = \widehat{Z}_1$ be taken as before. Let d be the de Rham operator on $\widehat{Z}$, let d^* be the formal adjoint of d with respect to the metric $g^{T\widehat{Z}}$.

For simplicity, assume that $\widehat{Z}$ is even dimensional. In [**C3**], Cheeger calculated the L_2 cohomology of a class of manifolds with singularities, the riemannian pseudo-manifolds, whose simplest version is the above manifold $\widehat{Z}$. In this case the strong closures of d and d^* are adjoint to each other, and so the L^2 cohomology exhibits Poincaré duality. This remarkable fact raised the question of the connection between the L_2 cohomology and the intersection cohomology of the corresponding spaces of Goresky and Mc Pherson. For a history of the subject, we refer to [**Kl**].

For our $\widehat{Z}$ considered above, if $H_{(2)}(\widehat{Z})$ denote the L_2 cohomology of $\widehat{Z}$, by [**C1**], [**C2,** p. 132, 133],

$$\begin{aligned} H^i_{(2)}(Z) &= H^i(Z) \text{ for } i < \frac{\dim Z}{2}\,, \\ &= \text{ the image of } H^i(Z,\partial Z) \text{ in } H^i(Z) \text{ for } i = \frac{\dim Z}{2}\,, \\ &= H^i(Z,\partial Z) \text{ for } i > \frac{\dim Z}{2}\,. \end{aligned} \tag{1.47}$$

In particular if $\operatorname{sign}_2(Z)$ is the L_2 signature of $\widehat{Z}$,

$$\operatorname{sign}_2(Z) = \operatorname{sign}(Z)\,. \tag{1.48}$$

Put

$$D = d + d^*\,. \tag{1.49}$$

In this case, Cheeger showed that the operator D is essentially self-adjoint on its obvious domain, and that $\ker D \simeq H_{(2)}(\widehat{Z})$.

The L_2 signature $\operatorname{sign}_2(X)$ can be shown to be equal to $\operatorname{Ind}(D_+)$. By using a heat equation formula for the signature, Cheeger **[C2]** obtains the formula

$$\operatorname{sign}_2(\widehat{Z}) = \int_Z \mathcal{L}(TZ, \nabla^{TZ}) - \eta^{\widetilde{D}^{\partial Z,\,\mathrm{even}}}(0)\,. \tag{1.50}$$

In view of (1.48), it is natural that formula (1.50) coincides with the Atiyah-Patodi-Singer formula for $\operatorname{sign}(Z)$. In **[C2]**, Cheeger used (1.50) as a starting point for the construction of the $\mathcal{L}$-classes of pseudo-manifolds in terms of the eta invariants of the links.

The explicit computation of the L_2 signature for more general spaces is of considerable interest **[Mü2]**, **[St]**. Formula (1.50) for the L_2 signature of $\widehat{Z}$ appears as the prototype of such formulas.

When $\widehat{Z}$ is odd dimensional, there is an obstruction to Poincaré duality, which lies in the middle dimensional cohomology of ∂Z. When $\operatorname{sign}(\partial Z) = 0$, Cheeger **[C1]** has shown how to restore Poincaré duality by imposing $*$-invariant boundary conditions.

f) The *b*-calculus of Melrose.

To attack the index problem of Atiyah-Patodi-Singer from a different point of view, Melrose has developed a new machinery, the *b*-calculus. In **[Me]**, Melrose introduces the idea of a *b*-metric $\frac{dr^2}{r^2} + g^{TY}$ on the cylinder $]0, +\infty[\times Y$ (which differs from the conical metric $dr^2 + r^2 g^{TY}$ by the factor $\frac{1}{r^2}$).

Let us assume for simplicity that $D^{\partial Z}$ is invertible. If $\check{Z}$ is the manifold Z with the cylinder of $Y = \partial Z$ attached, Melrose considers the index problem for the Dirac operator associated to a corresponding *b*-metric. The operator $D^{\check{Z}}$ is still Fredholm, but the corresponding heat kernels are no longer trace class. Still they have a *b*-trace, i.e. a renormalized trace.

In **[Me]**, Melrose shows that in the appropriate context, if,

$$\alpha_t = b\text{ -Tr}\left[\exp(-tD_-^{\check{Z}} D_+^{\check{Z}})\right] - b\text{ -Tr}\left[\exp(-tD_+^{\check{Z}} D_-^{\check{Z}})\right]\,, \tag{1.51}$$

then

$$\begin{aligned} \lim_{t\to+\infty} \alpha_t &= \operatorname{Ind}(D_+^{\check{Z}})\,, \\ \lim_{t\to 0} \alpha_t &= \int_Z \widehat{A}(TZ, \nabla^{TZ})\,\mathrm{ch}(\xi, \nabla^{\xi})\,. \end{aligned} \tag{1.52}$$

The fundamental fact is that in (1.52), α_t is non constant. The eta invariant $\frac{\eta(0)}{2}$ then appears as a formula for $-\displaystyle\int_0^{+\infty} \frac{d\alpha}{dt} dt$.

II. The local families index theorem, adiabatic limits and the $\widetilde{\eta}$ form.

In this Section, we review various results connected with a refinement of the families index theorem of Atiyah-Singer [**AS2**], the local families index theorem [**B2**].

In a) and b), we briefly state the families index theorem of Atiyah-Singer [**AS2**], [**APS2**] in its cohomological form. In c) and d), we introduce one essential technical tool, the superconnections of Quillen [**Q1**], which provide a refinement of Chern-Weil theory. In e), when the fibres Z of the fibration $\pi : M \to B$ are even dimensional, we construct the corresponding Levi-Civita superconnection, and we state the local families index theorem of [**B2**]. Also we obtain an associated odd form $\widetilde{\eta}$ on B, which transgresses the families index theorem [**AS2**] at the level of differential forms. In f), we review the results of [**Q2**], [**BF1,2**] on determinant bundles. In g), we give the local families index theorem of [**BF2**] when the fibres Z are odd dimensional, and we construct an associated even $\widetilde{\eta}$ form on B. In h), we relate the eta invariant to the component of degree 0 of $\widetilde{\eta}$. In i), we state the holonomy theorem [**BF2**], [**C4**] in the form suggested by Witten [**W**]. In j), we give results of [**BC1**], [**D**] on the adiabatic limit of eta invariants. Finally in k), we state the families index theorem for families of manifolds with boundary [**BC2,3**], [**MeP1,2**].

a) The case where Z is even dimensional.

Let $\pi : X \to B$ be a submersion with compact fibre Z. Assume that the fibres are oriented and spin. Let g^{TZ} be a metric on the relative tangent bundle TZ, and let S^{TZ} be the vector bundle on Z of the (TZ, g^{TZ}) spinors.

Let $(\xi, g^{\xi}, \nabla^{\xi})$ be a Hermitian vector bundle on X, equipped with a unitary connection.

For $b \in B$, let D_b^Z be the Dirac operator acting on $C^{\infty}(Z_b, (S^{TZ} \otimes \xi)_{Z_b})$. Then $(D_b^Z)_{b \in S}$ is a family of elliptic self-adjoint operators.

Assume first that the fibres Z are even dimensional. Then $S^{TZ} = S_+^{TZ} \oplus S_-^{TZ}$. Also D_b^Z interchanges $C^{\infty}(Z_b, (S_+^{TZ} \otimes \xi)_{|Z_b})$ and $C^{\infty}(Z_b, (S_-^{TZ} \otimes \xi)_{|Z_b})$. Let $D_{+,b}^Z$ be the restriction of D_b^Z to $C^{\infty}(Z_b, (S_+^{TZ} \otimes \xi)_{|Z_b})$. Then $(D_{+,b}^Z)_{b \in B}$ is a family of Fredholm operators over B.

By Atiyah-Singer [**AS2**], the family $(D_{+,b})$ defines an element $\mathrm{Ind}\, D_+ \in K^0(B)$. When $\ker D_{\pm,b}^Z$ is of locally constant dimension,

$$\mathrm{Ind}\,(D_+^Z) = \left[\ker D_+^Z - \ker D_-^Z\right] \text{ in } K^0(B)\,. \tag{2.1}$$

In the general case [**AS2**], one can perturb the family (D_+^Z) by a family of fibrewise regularizing operator so as to represent $\mathrm{Ind}\,(D_+^Z)$ by an explicit difference bundle on B.

In [**AS2**], Atiyah and Singer have calculated the index of a general family of elliptic operators in terms of the principal symbol of these operators. Recall that the Chern character map $K^0(B) \otimes_{\mathbf{Z}} \mathbf{Q} \to H^{\mathrm{even}}(B, \mathbf{Q})$ is an identification of $\mathbf{Q}$-modules. In [**AS2**], Atiyah-Singer obtain the formula

$$\mathrm{ch}(\,\mathrm{Ind}\ D_+^Z) = \pi_* \left[\widehat{A}(TZ)\,\mathrm{ch}(\xi)\right] \text{ in } H^{\mathrm{even}}(B, \mathbf{Q})\,. \tag{2.2}$$

When B is reduced to a point, Ind $D_+ \in \mathbf{Z}$, and (2.2) reduces to the Atiyah-Singer index theorem [**AS1**],

$$\operatorname{Ind}(D_+^Z) = \int_Z \widehat{A}(TZ)\operatorname{ch}(\xi) \text{ in } \mathbf{Z}. \tag{2.3}$$

b) The case where Z is odd dimensional.

If Z is odd dimensional, $(D^Z)_{s\in S}$ is a family of self-adjoint operators. By [**AS3**], [**APS2,** Section 3], it defines an element $\operatorname{Ind}(D) \in K^1(B)$.

Again there is a Chern character map ch : $K^1(B) \otimes_{\mathbf{Z}} \mathbf{Q} \to H^{\text{odd}}(B, \mathbf{Q})$ which is compatible with Bott periodicity. Then the obvious analogue of (2.2) still holds.

c) Superconnections : the $\mathbf{Z}_2$-graded case.

Let A be a $\mathbf{Z}_2$-graded algebra. If $a, b \in A$, we define the supercommutator $[a,b]$ by

$$[a,b] = ab - (-1)^{\deg a \deg b} ba. \tag{2.4}$$

Let $\pi : E = E_+ \oplus E_- \to B$ be a complex $\mathbf{Z}_2$-graded vector bundle. As we saw in Section 1 a), the bundle of algebras End (E) is $\mathbf{Z}_2$-graded.

Consider the $\mathbf{Z}_2$-graded bundle of algebras $\Lambda(T^*B)\widehat{\otimes}$ End (E). We extend the supertrace $\operatorname{Tr_s}$: End $(E) \to \mathbf{C}$ defined in (1.7) to a map : $\Lambda(T^*B)\widehat{\otimes}$ End $(E) \to \Lambda(T^*B)$, by the formula

$$\operatorname{Tr_s}[\omega A] = \omega \operatorname{Tr_s}(A), \ \omega \in \Lambda(T^*B), A \in \operatorname{End}(E). \tag{2.5}$$

Using (1.9), one verifies that $\operatorname{Tr_s}$ still vanishes on supercommutators in $\Lambda(T^*_{\mathbf{R}}B)\widehat{\otimes}$ End (E).

The vector bundle $\Lambda(T^*B)\widehat{\otimes}E$ is naturally $\mathbf{Z}_2$-graded.

DEFINITION 2.1. A superconnection A is an odd differential operator acting on $C^\infty(B, \Lambda(T^*B)\widehat{\otimes}E)$ such that if $\omega \in C^\infty(B, \Lambda(T^*B))$, $s \in C^\infty(B, E)$,

$$A(\omega s) = d\omega s + (-1)^{\deg \omega}\omega A s. \tag{2.6}$$

Let $\nabla^E = \nabla^{E_+} \oplus \nabla^{E_-}$ be a connection on E preserving the splitting $E = E_+ \oplus E_-$. Then if $S = A - \nabla^E$, S is a smooth section of $(\Lambda(T^*S)\widehat{\otimes} \operatorname{End}(E))^{\text{odd}}$. Conversely, any superconnection can be written in the form $A = \nabla^E + S$.

DEFINITION 2.2. The curvature of the superconnection A is the operator A^2.

Since A is odd, one verifies easily that A^2 is a tensor, so that $A^2 \in C^\infty(B, (\Lambda(T^*B)\widehat{\otimes} \operatorname{End}(E))^{\text{even}})$.

One has the Bianchi identity

$$\left[A, A^2\right] = 0. \tag{2.7}$$

Let φ be the endomorphism of $\Lambda(T^*S)^{\text{even}} : \omega \to (2i\pi)^{-\deg \omega/2}\omega$.

DEFINITION 2.3. Let $\mathrm{ch}(E,A)$ be the even form

$$\mathrm{ch}(E,A) = \varphi \operatorname{Tr_s}\left[\exp(-A^2)\right] . \tag{2.8}$$

Now we have the result of Quillen **[Q1]**.

Theorem 2.4. *The even form* $\mathrm{ch}(E,A)$ *is closed and its cohomology class* $[\mathrm{ch}(E,A)]$ *is given by*

$$[\mathrm{ch}(E,A)] = \mathrm{ch}(E) . \tag{2.9}$$

PROOF : Using the Bianchi identity (2.7), and the vanishing of $\operatorname{Tr_s}$ on supercommutators, we get

$$d \operatorname{Tr_s}\left[\exp(-A^2)\right] = \operatorname{Tr_s}\left[\left[A, \exp(-A^2)\right]\right] = 0 . \tag{2.10}$$

Therefore the form $\mathrm{ch}(E,A)$ is closed. By universality, we find that $[\mathrm{ch}(E,A)]$ does not depend on A. Taking $A = \nabla^E$ as before, we get (2.9). □

EXAMPLE. Let $\pi : E \to B$ be a real oriented even dimensional spin vector bundle. Let g^E be an Euclidean metric on V. Let $S^E = S_+^E \oplus S_-^E$ be the vector bundle of (E, g^E) spinors.

If $Y \in E$, let $c(Y)$ denote the Clifford action of Y on S^E. Then $\sqrt{-1}c(Y)$ is a self-adjoint odd endomorphism of S^E.

Let ∇^E be an Euclidean connection on E, let ∇^{S^E} be its lift to S^E. Put

$$A = \pi^* \nabla^{S^E} + \sqrt{-1} c(Y) . \tag{2.11}$$

Then A is a superconnection on $\pi^* S^E$.

Clearly

$$A^2 = \nabla^2 + \sqrt{-1}\left[\nabla^{S^E}, c(Y)\right] + |Y|^2 . \tag{2.12}$$

The form $\mathrm{ch}(E,A)$ on the total space of E is gaussian-shaped along the fibres of E . Therefore $\mathrm{ch}(E,A)$ represents a cohomology class in $H_c^{\mathrm{even}}(E, \mathbf{Q})$, the cohomology with compact support in E. By Mathai-Quillen [**MQ,** Theorem 4.5 and 4.10],

$$\mathrm{ch}(E,A) = (-1)^{\dim E/2} \pi^* \widehat{A}^{-1}(E, \nabla^E) \omega , \tag{2.13}$$

where ω is a closed form on E representing the Thom class of E.

Note that in (2.13), one cannot make $c(Y) = 0$, if we want to produce a class in $H_c^{\mathrm{even}}(V, \mathbf{Q})$.

Let $\pi : E = E_+ \oplus E_- \to B$ be a $\mathbf{Z}_2$-graded complex vector bundle. Let $g^E = g^{E_+} \oplus g^{E_-}$ be a Hermitian metric on E such that E_+ and E_- are orthogonal in E. Let $\nabla^E = \nabla^{E_+} \oplus \nabla^{E_-}$ be a split unitary connection on $E = E_+ \oplus E_-$.

Let $V \in C^\infty(S, \mathrm{End}^{\mathrm{odd}}(E))$ be self-adjoint. For $t > 0$, let A_t be the superconnection

$$A_t = \nabla^E + \sqrt{t}V \,. \tag{2.14}$$

We extend φ to a map $\Lambda(T^*B) \otimes_{\mathbf{R}} \mathbf{C} \to \Lambda(T^*B) \otimes_{\mathbf{R}} \mathbf{C}$.

DEFINITION 2.5. Put

$$\begin{aligned} \alpha_t &= \varphi \operatorname{Tr_s} \left[\exp(-A_t^2)\right] , \\ \beta_t &= \frac{1}{(2i\pi)^{1/2}} \varphi \operatorname{Tr_s} \left[\frac{\partial A_t}{\partial t} \exp(-A_t^2)\right] . \end{aligned} \tag{2.15}$$

One verifies easily that, the forms α_t, β_t are real. Also by Theorem 2.4, α_t is closed, and $[\alpha_t] = \mathrm{ch}(E)$. By an obvious extension of Chern-Simons theory to superconnections,

$$\frac{\partial}{\partial t}\alpha_t = -d\beta_t \,. \tag{2.16}$$

Clearly

$$\alpha_0 = \mathrm{ch}(E, \nabla^E) \,. \tag{2.17}$$

Assume that $\ker V$ has locally constant dimension, i.e. $\ker V$ is a $\mathbf{Z}_2$-graded smooth vector subbundle of E. Then

$$[E] = [\ker V] \text{ in } K^0(B) \,. \tag{2.18}$$

Let $\nabla^{\ker V}$ be the orthogonal projection of ∇^E on $\ker V$.

Now we state a result by Berline-Vergne [**BeV,** Theorem 1.9], [**BeGeV,** Theorems 9.2 and 9.7].

Theorem 2.6. *As $t \to +\infty$,*

$$\begin{aligned} \alpha_t &= \mathrm{ch}(\ker V, \nabla^{\ker V}) + \mathcal{O}\left(\frac{1}{\sqrt{t}}\right) , \\ \beta_t &= \mathcal{O}\left(\frac{1}{t^{3/2}}\right) . \end{aligned} \tag{2.19}$$

DEFINITION 2.7. Put

$$\widehat{\eta} = \int_0^{+\infty} \beta_t dt \,. \tag{2.20}$$

Theorem 2.8. *The form $\widehat{\eta}$ is real and odd, and moreover*

$$d\widehat{\eta} = \mathrm{ch}(E, \nabla^E) - \mathrm{ch}(\ker V, \nabla^{\ker V}). \tag{2.21}$$

PROOF : This follows from (2.16), (2.17), (2.19). □

DEFINITION 2.9. We will say that (E, g^E, ∇^E, V) splits if $E = \ker V \oplus \mathrm{Im}\ V$, if the connection ∇^E preserves $\ker D$ and $\mathrm{Im}\ V$, and V is a unitary odd section of $\mathrm{End}(\mathrm{Im}\ V)$ preserving $\nabla^{\mathrm{Im}\ V}$.

Now we state a result characterizing the form $\widehat{\eta}$ uniquely. This is an obvious analogue of corresponding results for Bott-Chern classes in **[BGS1]**.

Theorem 2.10. *There exists a unique way to associate to (E, g^E, ∇^E, V) an odd form $\widehat{\eta}$ in $C^\infty(B, \Lambda^{\mathrm{odd}}(T^*B))/dC^\infty(B, \Lambda^{\mathrm{even}}(T^*B))$ having the following three properties*

a) *$\widehat{\eta}$ is functorial.*

b) *If (E, g^E, ∇^E, V) splits, then $\widehat{\eta} = 0$.*

c) *The following identity holds*

$$d\widehat{\eta} = \mathrm{ch}(E, \nabla^E) - \mathrm{ch}(\ker V, \nabla^{\ker V}). \tag{2.22}$$

PROOF : If (E, g^E, ∇^E, V) splits, then

$$A_t^2 = \nabla^{E,2} + tP^{\mathrm{Im}\ V} \tag{2.23}$$

and so

$$\beta_t = 0. \tag{2.24}$$

It follows that if $\widehat{\eta}$ is taken as in (2.20),

$$\widehat{\eta} = 0. \tag{2.25}$$

So the form $\widehat{\eta}$ of Definition 2.9 has properties a), b), c). To establish uniqueness, one verifies easily that over $S \times [0,1]$, one can deform (E, g^E, ∇^E, V) at $s = 0$ into a split object at $s = 1$. Let $(\widetilde{E}, g^{\widetilde{E}}, \nabla^{\widetilde{E}}, \widetilde{V})$ be the corresponding object on $S \times [0,1]$. Then if $\widehat{\eta}$ is taken as in our Theorem,

$$\widehat{\eta} = \int_{[0,1]} \left(\mathrm{ch}\, \widetilde{E}, \nabla^{\widetilde{E}}) - \mathrm{ch}(\ker \widetilde{V}, \nabla^{\ker \widetilde{V}})\right) \text{ modulo coboundaries}, \tag{2.26}$$

which characterizes the class of $\widehat{\eta}$ uniquely. □

d) Superconnections : the odd case.

Let now $\pi : E \to B$ be a Hermitian vector bundle on B.

Let σ be an odd variable such that $\sigma^2 = 1$. Then $E \otimes \mathbf{C}(\sigma)$ is a $\mathbf{Z}_2$-graded vector bundle, and $\mathrm{End}\,(E) \otimes \mathbf{C}(\sigma)$ is a $\mathbf{Z}_2$-graded algebra. Let Tr^σ be the functional from $\Lambda(T^*B)\widehat{\otimes}(\mathrm{End}\,(E) \otimes \mathbf{C}(\sigma))$ into $\Lambda(T^*B)$ such that if $\omega \in A(T^*B)$, $A \in \mathrm{End}\,(E)$,

$$\begin{aligned} &\mathrm{Tr}^\sigma(\omega A) = 0\,. \\ &\mathrm{Tr}^\sigma(\omega A\sigma) = \omega\ \mathrm{Tr}(A)\,. \end{aligned} \tag{2.27}$$

Again Tr^σ vanishes on supercommutators.

DEFINITION 2.11. A superconnection is an odd differential operator acting on $C^\infty(B, \Lambda(T^*B))\widehat{\otimes}(E \otimes \mathbf{C}(\sigma))$ such that (2.6) still holds

Let A^2 be the curvature of A. Then $A^2 \in C^\infty(B, (\Lambda(T^*B)\widehat{\otimes}(\mathrm{End}\,(E)\otimes \mathbf{C}(\sigma))^{\mathrm{even}})$.

Now we have the result of Quillen [**Q1**, Section 5].

Proposition 2.12. *The odd form*

$$\alpha = \mathrm{Tr}^\sigma\left[\exp(-A^2)\right] \tag{2.28}$$

is closed and exact.

PROOF : By proceeding as in the proof of Theorem 2.4, we see that α is closed. Moreover, we find that $[\alpha]$ does not depend on A. Also if $A = \nabla^E$, $\alpha = 0$. The proof of our Proposition is complete. □

Let $\pi : (E, g^E) \to B$ be a complex Hermitian vector bundle. Let ∇^E be a unitary connection on (E, g^E).

Let $V \in C^\infty(B, \mathrm{End}\,(E))$ be self-adjoint. For $t > 0$, let A_t be the superconnection

$$A_t = \nabla^E + \sqrt{t}V\sigma\,. \tag{2.29}$$

DEFINITION 2.13. Put

$$\begin{aligned} \alpha_t &= (2i)^{1/2}\varphi\,\mathrm{Tr}^\sigma\left[\exp(-A_t^2)\right]\,, \\ \beta_t &= \frac{1}{\sqrt{\pi}}\varphi\,\mathrm{Tr}^\sigma\left[\frac{\partial A_t}{\partial t}\exp(-A_t^2)\right]\,, \end{aligned} \tag{2.30}$$

As before, the forms α_t, β_t are real and by Proposition 2.12, the form α_t is closed and exact. Again as in (2.16),

$$\frac{\partial \alpha_t}{\partial t} = -d\beta_t\,. \tag{2.31}$$

Clearly

$$\alpha_0 = 0\,. \tag{2.32}$$

Assume that $\ker(V)$ is of locally constant dimension, so that $\ker V$ is a vector subbundle of E. Let $\nabla^{\ker(V)}$ be the orthogonal projection of ∇^E on $\ker V$.

By [**BeV**, Theorem 1.9], [**BeGeV**, Theorems 9.2 and 9.7], we have

Theorem 2.14. *As $t \to +\infty$,*

$$\alpha_t = \mathcal{O}\left(\frac{1}{\sqrt{t}}\right), \quad \beta_t = \mathcal{O}\left(\frac{1}{t^{3/2}}\right). \tag{2.33}$$

DEFINITION 2.15. Put

$$\widehat{\eta} = \int_0^{+\infty} \beta_t dt. \tag{2.34}$$

Theorem 2.16. *The even form $\widehat{\eta}$ is real and closed.*

PROOF : By (2.32), (2.33)

$$\alpha_0 = 0, \, \alpha_\infty = 0. \tag{2.35}$$

Using (2.31), we find that

$$d\widehat{\eta} = 0. \tag{2.36}$$

□

Let $E_{>0}$ ($E_{<0}$) be the direct sum of the eigenspaces of E associated to positive (resp. negative) eigenvalues of V. Put $E_0 = \ker V$. Then E splits orthogonally as

$$E = E_{>0} \oplus E_{<0} \oplus E_0. \tag{2.37}$$

The following result was proved in [**BC**, Theorem 2.43].

Theorem 2.17. *The cohomology class $[\widehat{\eta}]$ of $\widehat{\eta}$ does not depend on ∇^E. More precisely*

$$[\widehat{\eta}] = \tfrac{1}{2}\left(\mathrm{ch}(E_{>0}) - \mathrm{ch}(E_{<0})\right). \tag{2.38}$$

PROOF : Using Theorem 2.16 and the universality of $\widehat{\eta}$, it is clear that $[\widehat{\eta}]$ does not depend on the metric g^E, the connection ∇^E or V as long as the splitting is kept fixed and orthogonal.

By deforming V into $\frac{V}{|V|}$ (which acts like 0 on E_0), we may and will assume that V is $+1$ on $E_{>0}$, -1 on $E_{<0}$ and 0 on E_0, and suppose that ∇^E is a split connection $\nabla^E = \nabla^{E_{>0}} \oplus \nabla^{E_{<0}} \oplus \nabla^{E_0}$. Then we get

$$\beta_t = \frac{1}{\sqrt{\pi}}\varphi \operatorname{Tr}^\sigma\left[\frac{V\sigma}{2\sqrt{t}} \exp\left(-\left(\nabla^{E,2} + t 1_{E_{>0}\oplus E_{<0}}\right)\right)\right] \tag{2.39}$$

so that

$$\widehat{\eta} = \tfrac{1}{2}\varphi \operatorname{Tr}^\sigma\left[V\sigma \exp(-\nabla^{E,2})\right] = \tfrac{1}{2}\left(\mathrm{ch}(E_{>0}, \nabla^{E_{>0}}) - \mathrm{ch}(E_{<0}, \nabla^{E_{<0}})\right) \tag{2.40}$$

from which we get (2.38). □

REMARK 2.18. Observe that

$$(2.41)\qquad \widehat{\eta}=\frac{1}{2\sqrt{\pi}}\int_0^{+\infty} t^{-1/2}\varphi\,\mathrm{Tr}^\sigma\left[V\sigma\exp\left(-\left(\nabla^E+\sqrt{t}V\sigma\right)^2\right)\right]dt\,.$$

In particular

$$(2.42)\qquad \widehat{\eta}^{(0)}=\frac{1}{2\sqrt{\pi}}\int_0^{+\infty} t^{-1/2}\,\mathrm{Tr}\left[V\exp\left(-tV^2\right)\right]dt\,.$$

We discover that the expression for $\widehat{\eta}^{(0)}$ is formally exactly the same as the one in (1.25) for $\frac{1}{2}\eta^A(0)$. By (2.38) or by (2.42),

$$(2.43)\qquad \widehat{\eta}^{(0)}=\tfrac{1}{2}\left(\mathrm{rk}(E_{>0})-\ \mathrm{rk}(E_{<0})\right)\,.$$

Of course in (1.25), $\frac{1}{2}\eta^A(0)$ is a renormalized version of (2.42). This formal analogy will be very important in the sequel.

e) The local families index theorem : the even dimensional case.

We now make the same assumptions as in Section 2a). For $b\in B$, put

$$(2.44)\qquad H_{\pm,b}=C^\infty(Z_b,(S^{TZ}\otimes\xi)_{|Z_b})\,.$$

Then $H=H_+\oplus H_-$ is an infinite dimensional $\mathbf{Z}_2$-graded vector bundle on B.

Let T^HX be a smooth vector bundle of TX such that $TX=T^HX\oplus TZ$.

We claim that (T^HX,g^{TZ}) determines a canonical Euclidean connection ∇^{TZ} on (TZ,g^{TZ}) [**B2,** Theorem 1.9]. We give two descriptions of ∇^{TZ}. In fact, if g^{TX} is a metric on TX such that T^HX is orthogonal to TZ and that g^{TZ} is the restriction of g^{TX} to TZ, if ∇^{TX} is the Levi-Civita connection on (TX,g^{TX}), then

$$(2.45)\qquad \nabla^{TZ}=P^{TZ}\nabla^{TX}\,.$$

Another description of ∇^{TZ} is as follows [**BC,** Section 4] :

- Fibrewise, ∇^{TZ} is the Levi-Civita connection of (Z,g^{TZ}).
- If $U\in TB$, if $U^H\in T^HX$ lifts U, if V is a smooth section of TZ,

$$(2.46)\qquad \nabla^{TZ}_{U^H}V=\left[U^H,V\right]+\tfrac{1}{2}\left(L_{U^H}g^{TZ}\right)V\,.$$

Let g^{TB} be a Riemannian metric on g^{TB}. Let ∇^{TB} be the Levi-Civita connection on (TB,g^{TB}). Then ∇^{TB} lifts to a connection ∇^{T^HX} on T^HX.

Let T be the torsion of the connection of the connection $\nabla^{T^HX}\oplus\nabla^{TZ}$ on $TX=T^HX\oplus TZ$. Then by [**B2,** Theorem 1.9], T does not depend on g^{TB}. More precisely

- T vanishes on $TZ\times TZ$.

- If $U, V \in TB$,

$$T(U^H, V^H) = -P^{TZ}\left[U^H, V^H\right] . \tag{2.47}$$

- If $U \in TB$, $V \in TZ$

$$T(U^H, V) = \tfrac{1}{2} L_{U^H} g^{TZ} V . \tag{2.48}$$

Let ∇'^{TX} be the Levi-Civita connection on $(TX, \pi^* g^{TB} \oplus g^{TZ})$. Put

$$S = \nabla'^{TX} - \nabla^{TX} . \tag{2.49}$$

Since ∇'^{TX} is torsion free, if U, V, $W \in TX$,

$$2\langle S(U)V, W\rangle + \langle T(U,V), W\rangle + \langle T(W,U), V\rangle - \langle T(V,W), U\rangle . \tag{2.50}$$

Then by (2.50), the tensor $\langle S(.).,.\rangle$ does not depend on g^{TB}.

The connection ∇^{TZ} lifts to a unitary connection $\nabla^{S^{TZ}} = \nabla^{S_+^{TZ}} \oplus \nabla^{S_-^{TZ}}$ on $S^{TZ} = S_+^{TZ} \oplus S_-^{TZ}$. Let $\nabla^{S^{TZ} \otimes \xi}$ be the obvious connection on $S^{TZ} \otimes \xi$.

Let dv_Z be the Riemannian volume form along the fibres Z. If U is a smooth section of TB, the Lie derivative operator L_{U^H} acts on tensors along the fibre Z. Put

$$\operatorname{div}_Z(U) = \frac{L_{U^H} dv_Z(x)}{dv_Z(x)} . \tag{2.51}$$

Then one verifies easily that $\operatorname{div}_Z(U)$ is a tensor.

In the sequel, we identify smooth sections of H on B to smooth sections of $S^{TZ} \otimes \xi$ on X.

DEFINITION 2.19. Let ∇^H be the connection on H such that if $U \in TS$, if h is a smooth section of H on S,

$$\nabla_U^H s = \nabla_{U^H}^{S^{TZ} \otimes \xi} s . \tag{2.52}$$

Put

$$\nabla_U^{H,u} = \nabla_U^H + \tfrac{1}{2} \operatorname{div}_Z(U) . \tag{2.53}$$

Then one verifies easily that $\nabla^{H,u}$ is a unitary connection on H, preserving H_+ and H_-. Moreover the curvatures of ∇^H and $\nabla^{H,u}$ take their values in first order differential operators acting along the fibres Z.

Let $f_1, \ldots, f_m$ be a basis of TB, let $f^1, \ldots, f^m$ be the dual basis of T^*B. Set

$$c(T) = \tfrac{1}{2} f^\alpha f^\beta c(T(f_\alpha^H, f_\beta^H)) . \tag{2.54}$$

Then $c(T) \in (\Lambda(T^*B) \widehat{\otimes} \operatorname{End}(H))^{\mathrm{odd}}$.

DEFINITION 2.20. For $t > 0$, let A_t be the superconnection on H,

$$A_t = \nabla^{H,u} + \sqrt{t}D^Z - \frac{c(T)}{4\sqrt{t}}. \tag{2.55}$$

Observe that A_t^2 is fibrewise elliptic, so that $\exp(-A_t^2)$ is fibrewise trace class.

DEFINITION 2.21. Put

$$\begin{aligned} \alpha_t &= \varphi \ \mathrm{Tr_s}\left[\exp(-A_t^2)\right], \\ \beta_t &= \frac{1}{(2i\pi)^{1/2}}\varphi \ \mathrm{Tr_s}\left[\frac{\partial A}{\partial t}\exp(-A_t^2)\right]. \end{aligned} \tag{2.56}$$

Now we have the local families index theorem of [**B2**, Theorems 3.4, 4.12 and 4.16].

Theorem 2.22. *The forms α_t and β_t are real and the form α_t is closed. The cohomology class $[\alpha_t]$ of α_t is constant and*

$$[\alpha_t] = \mathrm{ch}(\mathrm{Ind}\ D^Z) \ \text{in}\ H^{\mathrm{even}}(B, \mathbf{Q}). \tag{2.57}$$

Also

$$\frac{\partial \alpha}{\partial t} = -d\beta_t. \tag{2.58}$$

Finally as $t \to 0$,

$$\begin{aligned} \alpha_t &= \pi_*\left(\widehat{A}(TZ, \nabla^{TZ})\,\mathrm{ch}(\xi, \nabla^{\xi})\right) + \mathcal{O}(t), \\ \beta_t &= \mathcal{O}(1). \end{aligned} \tag{2.59}$$

REMARK 2.23. The fact that α_t is closed follows from the arguments in Theorem 2.4. The most difficult result is (2.59). In fact the main point of [**B2**] was to produce the "right" superconnection A_t such that a result like (2.59) would hold.

Observe that, in general, (2.59) does not hold for the "simpler" superconnection $\nabla^{H,u} + \sqrt{t}D^Z$.

Needless to say, from [**B2**, Theorem 2.22], we recover the cohomological form of the families index theorem of Atiyah-Singer [**AS2**] given in (2.2). Still our proof [**B2**] is completely local on the basis B, hence the fact Theorem 2.22 is called a local families index theorem. Still to prove (2.59), we use the fibrewise heat kernel for $\exp(-A_t^2)$, and prove a corresponding convergence result which is local on the fibre Z.

Let g^{TB} be a metric on TB. For $\varepsilon > 0$, let g_ε^{TX} be the metric on $TX = T^H X \oplus TZ$,

$$g_\varepsilon^{TX} = \frac{1}{\varepsilon}\pi^* g^{TB} \oplus g^{TZ}. \tag{2.60}$$

Assume that X and B are compact and that B is even dimensional, oriented and spin. Then X is also even dimensional, oriented and spin. Let S^{TB} be the vector bundle of (TB, g^{TB}) spinors. For $\varepsilon > 0$, let D_ε^X be the Dirac operator acting on $C^\infty(X, S^{TX} \otimes \xi)$, associated to $(g_\varepsilon^{TX}, \nabla^\xi)$.

Let $P_t^\varepsilon(x, x')$ $(x, x' \in X)$ be the smooth heat kernel for $\exp(-tD_\varepsilon^{X,2})$ associated to the volume $dv(x')$. In view of Theorems 1.2 and 2.22, it is natural to ask whether, given $t > 0$, as $\varepsilon \to 0$, $\mathrm{Tr_s}\left[P_t^\varepsilon(x, x)\right]$ has a limit.

Let $Q_t(x,x')$ be the smooth fibrewise kernel of $\exp(-A_t^2)$. Following a terminology introduced by physicists [**W**], the idea of studying the limit of certain quantities as $\varepsilon \to 0$ is called passing to the adiabatic limit.

A first result in that direction is as follows [**B2**, Theorem 5.3].

Theorem 2.24. *For $t > 0$,*

$$\lim_{\varepsilon \to 0} \mathrm{Tr_s}\,[P_t^\varepsilon(x,x)] = \left\{\varphi\ \mathrm{Tr_s}\,[Q_t(x,x)]\,\widehat{A}(TB,\nabla^{TB})\right\}^{\max(B)}. \tag{2.61}$$

PROOF : In [**B2**], the idea is to view (2.61) as a consequence of the local index theorem over B with coefficients in the infinite dimensional $\mathbf{Z}_2$-graded vector bundle H.

Put

$$D^H = \sum c(f_\alpha)\left(\nabla_{f_\alpha^H}^{\pi^* S^{TB}\widehat{\otimes} S^{TZ}\otimes\xi} + \tfrac{1}{2}\,\mathrm{div}(f_\alpha^H)\right). \tag{2.62}$$

Then a simple computation [**BC**, eq. (4.26)] shows that

$$D_\varepsilon^X = D^Z + \sqrt{\varepsilon} D^H - \varepsilon \tfrac{1}{8} c(f_\alpha)c(f_\beta)c(T(f_\alpha^H, f_\beta^H)). \tag{2.63}$$

At least formally, (2.61) follows easily from local index theoretic techniques over B. □

Let ∇'^{TX}_ε be the Levi-Civita connection on (TX, g_ε^{TX}). Put

$$S_\varepsilon = \nabla'^{TX}_\varepsilon - \nabla^{TX}. \tag{2.64}$$

Then by (2.50), we find that

$$\begin{aligned} P^{TZ} S_\varepsilon &= P^{TZ} S, \\ P^{T^H X} S_\varepsilon &= \varepsilon P^{T^H X} S. \end{aligned} \tag{2.65}$$

From (2.64), (2.65), we find that as $\varepsilon \to 0$, the connection ∇_ε^{TX} has a limit. More precisely,

$$\nabla'^{TX}_\varepsilon \to \nabla^{TX} + P^{TZ} S. \tag{2.66}$$

From (2.66), we find [**BF2**, eq. (3.196)] that as $\varepsilon \to 0$,

$$\widehat{A}(TX, \nabla'^{TX}_\varepsilon) \to \widehat{A}(TZ, \nabla^{TZ})\pi^*\widehat{A}(TB, \nabla^{TB}). \tag{2.67}$$

Now while $\widehat{A}(TX, \nabla'^{TX}_\varepsilon)\,\mathrm{ch}(\xi, \nabla^\xi)$ appears naturally when applying the local index Theorem 1.2 to the operator D_ε^X over X, $\widehat{A}(TZ, \nabla^{TZ})\,\mathrm{ch}(\xi, \nabla^\xi)$ appears naturally in the local families index theorem stated in Theorem 2.22.

Given Theorem 2.24, the "local in the fibre" version of the local families index Theorem of **[B2]** stated in Theorem 2.22 is just the assertion that the following diagram commutes

$$
\begin{tikzcd}
\operatorname{Tr_s}\left[P_t^\varepsilon(x,x)\right] \arrow[r, "t\to 0"'] \arrow[d, "\varepsilon\to 0"] & \begin{array}{c}\{\widehat{A}(TX,\nabla'^{TX}_\varepsilon)\\ \operatorname{ch}(\xi,\nabla^\xi\}^{\max(X)}\end{array} \arrow[d, "\varepsilon\to 0"] \\
\begin{array}{c}\{\varphi \operatorname{Tr_s}\left[Q_t(x,x)\right]\\ \widehat{A}(TB,\nabla^{TB})\}^{\max(B)}\end{array} \arrow[r, "t\to 0"'] & \begin{array}{c}\{\{\widehat{A}(TZ,\nabla^{TZ})\operatorname{ch}(\xi,\nabla^\xi)\}^{\max(Z)}\\ \widehat{A}(TB,\nabla^{TB})\}^{\max(B)}\end{array}
\end{tikzcd}
\tag{2.68}
$$

Needless to say, the explicit form for A_t in (2.55) was found by trying to make the above diagram commute by brute force. The comparison of formulas (2.55) and (2.63) for A_t and D_ε^X provides overwhelming evidence that A_t is the "right" superconnection.

Finally observe that given Theorems 1.2 and 2.24, and also (2.66), a proof of Theorem 2.22 can be given, which makes the commutativity of the above diagram a tautology, by showing that the convergence as $t \to 0$ in the upper row is uniform in $\varepsilon \in]0,1]$.

Assume that $\ker D^Z$ is of locally constant dimension. Then $\ker(D^Z)$ is a smooth subbundle of H. Let $\nabla^{\ker D^Z,u}$ be the orthogonal projection of $\nabla^{H,u}$ on $\ker D^Z$.

The we have the following result of Berline-Getzler-Vergne [**BeGeV**, Theorems 9.19 and 9.23], which extends Theorem 2.6 to an infinite dimensional situation.

Theorem 2.25. *As $t \to +\infty$,*

$$
\begin{aligned}
\alpha_t &= \operatorname{ch}(\ker D^Z, \nabla^{\ker D^Z,u}) + \mathcal{O}\left(\frac{1}{\sqrt{t}}\right), \\
\beta_t &= \mathcal{O}\left(\frac{1}{t^{3/2}}\right).
\end{aligned}
\tag{2.69}
$$

DEFINITION 2.26. Let $\widetilde{\eta}$ be the odd form on B

$$
\widetilde{\eta} = \int_0^{+\infty} \beta_t \, dt. \tag{2.70}
$$

By (2.58), (2.59), (2.69) we get the following result.

Theorem 2.27. *The odd smooth form $\widetilde{\eta}$ on B is such that*

$$
d\widetilde{\eta} = \pi_*\left[\widehat{A}(TZ,\nabla^{TZ})\operatorname{ch}(\xi,\nabla^\xi)\right] - \operatorname{ch}(\ker D^Z, \nabla^{\ker D^Z,u}). \tag{2.71}
$$

REMARK 2.28. From equation (2.71), one deduces easily how, modulo coboundaries, $\widetilde{\eta}$ depends on $(T^H X, g^{TZ}, g^\xi)$. This is because any two sets of such data can be deformed into each other.

f) The determinant bundle.

We make the same assumptions as in Section 2 e).

Complex lines form a group under the $\otimes$ operation. In particular, if λ is a complex line, let λ^{-1} be the dual line, so that $\lambda \otimes \lambda^{-1} = \mathbf{C}$, the canonical complex line.

If E is a complex vector space, put

$$\det E = \Lambda^{\max} E \,. \tag{2.72}$$

If $E = E_+ \oplus E_-$ is a $\mathbf{Z}_2$-graded vector space, set

$$\det E = \det E_+ \otimes (\det E_-)^{-1} \,. \tag{2.73}$$

DEFINITION 2.29. For $b \in B$, set

$$\lambda_b = \left(\det \ker D_b^Z\right)^{-1} \,. \tag{2.74}$$

Then in **[Q2]**, Quillen has shown how to glue the λ_b's into a honest line bundle λ, even though, in general, the dimension of $\ker D^Z_{\pm,b}$ is not locally constant. The idea is as follows. For $a > 0$, let U_a be the open set

$$U_a = \left\{ b \in B, a \notin \mathrm{Sp}(D_b^{Z,2}) \right\} \,. \tag{2.75}$$

Let $H^{[0,a]}$ be the direct sum of the eigenspaces of $D_b^{Z,2}$ for eigenvalues $\mu \leq a$. Put

$$\lambda^{[0,a]} = \left(\det H^{[0,a]}\right)^{-1} \,. \tag{2.76}$$

Then $\lambda^{[0,a]}$ is a smooth line bundle on U_a.

Given $0 < a < a'$, let $H^{[a,a']}$ be the direct sum of eigenspace of $D_b^{Z,2}$ for eigenvalues $\mu \in [a, a']$. Set

$$\lambda^{[a,a']} = (\det H^{[a,a']})^{-1} \,. \tag{2.77}$$

Then $\lambda^{[a,a']}$ has a canonical nonzero section $\det D_+^{Z,[a,a']}$, which is smooth on $U^a \cap U^{a'}$. Also on $U^a \cap U^{a'}$,

$$\lambda^{[0,a']} \simeq \lambda^{[0,a]} \otimes \lambda^{[a,a']} \,. \tag{2.78}$$

Since $\lambda^{[a,a']}$ is canonically trivialized, on $U^a \cap U^{a'}$

$$\lambda^{[0,a']} \simeq \lambda^{[0,a]} \,. \tag{2.79}$$

DEFINITION 2.30. The inverse determinant bundle λ is the line bundle which restricts to $\lambda^{[0,a]}$ on U^a.

By (2.79), we find that for any $b \in B$,

$$\lambda_b \simeq \left(\det \ker D_b^Z\right)^{-1} . \tag{2.80}$$

By the Atiyah-Singer family index theorem [**AS2**] (see eq. (2.2)),

$$c_1(\lambda) = -\pi_* \left[\widehat{A}(TZ)\operatorname{ch}(\xi)\right]^{(2)} . \tag{2.81}$$

In [**Q2**], [**BF1,2**], Quillen and Bismut and Freed have shown how to equip λ with a smooth metric $\| \;\; \|_\lambda$ and a unitary connection ∇^λ such that

$$c_1(\lambda, \nabla^\lambda) = -\pi_* \left[\widehat{A}(TZ, \nabla^{TZ})\operatorname{ch}(\xi, \nabla^\xi)\right]^{(2)} . \tag{2.82}$$

Here we will concentrate on the construction of the imaginary part of the connection ∇^λ [**BF1,2**]. Assume first that $\ker D^Z$ is of locally constant dimension. Then $(\det \ker D^Z)^{-1}$ is a smooth line bundle and $\lambda \simeq (\det \ker D^Z)^{-1}$.

The connection $\nabla^{\ker D^Z,u}$ induces a connection ${}^1\nabla^\lambda$ on λ. Put

$${}^2\nabla = {}^1\nabla + +2i\pi\widetilde{\eta}^{(1)} . \tag{2.83}$$

Clearly

$$c_1(\lambda, {}^1\nabla^\lambda) = -\left[\operatorname{ch}(\ker D^Z, \nabla^{\ker D^Z,u})\right]^{(2)} . \tag{2.84}$$

From (2.84), we deduce that

$$c_1(\lambda, {}^2\nabla^\lambda) = -\left[\operatorname{ch}(\ker D^Z, \nabla^{\ker D^Z,u})\right]^{(2)} - d\widetilde{\eta}^{(1)} . \tag{2.85}$$

By (2.71), (2.85), we find that

$$c_1(\lambda, {}^2\nabla^\lambda) = -\pi_* \left[\widehat{A}(TZ, \nabla^{TZ})\operatorname{ch}(\xi, \nabla^\xi)\right]^{(2)} . \tag{2.86}$$

The connection ∇^λ in [**BF1,2**] of differs from ${}^1\nabla^\lambda$ by an exact real form, so that (2.82) follows from (2.86).

The remarkable fact is that even if $\ker D_\pm^Z$ is not of locally constant dimension, in [**BF1,2**], it is possible to define the connection ${}^2\nabla^\lambda$ by formulas similar to (2.83). The idea is to construct over U^a a connection $\nabla^{\lambda^{[0,a]}}$ by a suitable modification of (2.83), and to establish that the connections $\nabla^{\lambda^{[0,a]}}$, suitably modified, define a connection ∇^λ on λ, for which (2.82) still holds.

g) The local families index theorem : the odd case.

Now we assume that the fibres Z are odd dimensional. Put

$$H_b = C^\infty\left(Z_b, (S^{TZ} \otimes \xi)_{|Z_b}\right) . \tag{2.87}$$

Then (D^Z) is a family of self-adjoint operators acting on H.

Take σ as in Section 2 d).

DEFINITION 2.31. For $t > 0$, put

$$A_t = \nabla^{H,u} + \sqrt{T} D^Z \sigma - \frac{c(T)\sigma}{4\sqrt{t}}. \tag{2.88}$$

Again, $\exp(-A_t^2)$ is fibrewise trace class.

DEFINITION 2.32. Put

$$\begin{aligned} \alpha_t &= (2i)^{1/2} \varphi \operatorname{Tr}^\sigma \left[\exp(-A_t^2)\right], \\ \beta_t &= \frac{1}{\sqrt{\pi}} \varphi \operatorname{Tr}^\sigma \left[\frac{\partial A_t}{\partial t} \exp(-A_t^2)\right]. \end{aligned} \tag{2.89}$$

Now we state a result of [**BF2,** Theorem 2.10].

Theorem 2.33. *The forms α_t and β_t are real, and the form α_t is closed. The cohomology class $[\alpha_t]$ of α_t is constant, and*

$$[\alpha_t] = \operatorname{ch}(D^Z) \ \textit{in} \ H^{\mathrm{odd}}(B, \mathbf{Q}). \tag{2.90}$$

Also

$$\frac{\partial \alpha}{\partial t} = -d\beta_t. \tag{2.91}$$

Finally as $t \to 0$

$$\begin{aligned} \alpha_t &= \pi_* \left[\widehat{A}(TZ, \nabla^{TZ}) \operatorname{ch}(\xi, \nabla^\xi)\right] + \mathcal{O}(t), \\ \beta_t &= \mathcal{O}(1). \end{aligned} \tag{2.92}$$

Assume now that $\ker D^Z$ is of locally constant dimension. Then $\ker D^Z$ is a vector bundle on B.

We have the obvious analogue of Theorems 2.14 and 1.25.

Theorem 2.34. *As $t \to +\infty$,*

$$\begin{aligned} \alpha_t &= \mathcal{O}\left(\frac{1}{\sqrt{t}}\right), \\ \beta_t &= \mathcal{O}\left(\frac{1}{t^{3/2}}\right). \end{aligned} \tag{2.93}$$

DEFINITION 2.35. Let $\widetilde{\eta}$ be the even form

$$\widetilde{\eta} = \int_0^{+\infty} \beta_t \, dt. \tag{2.94}$$

Theorem 2.36. *The even form $\widetilde{\eta}$ is such that*

$$d\widetilde{\eta} = \pi_* \left[\widehat{A}(TZ, \nabla^{TZ}) \operatorname{ch}(\xi, \nabla^{\xi})\right] . \tag{2.95}$$

PROOF : This follows from (2.91)-(2.93). □

h) The odd local families index theorem and the eta invariant.

Observe that

$$\beta_t^{(0)} = \frac{1}{2\sqrt{\pi}} \operatorname{Tr}\left[\frac{D^Z}{\sqrt{t}} \exp(-tD^{Z,2})\right] . \tag{2.96}$$

By (2.92), as $t \to 0$,

$$\beta_t^{(0)} = \mathcal{O}(1) . \tag{2.97}$$

Now by (1.25), (2.97) guarantees that the eta function $\eta^{D^Z}(s)$ is holomorphic at 0. Note that this result extends Theorem 1.4 to the case where Z does not necessarily bound. This result is also a consequence of [**APS3,** Theorem 4.5].

By (2.94), (2.96),

$$\widetilde{\eta}^{(0)} = \frac{1}{2\sqrt{\pi}} \int_0^{+\infty} t^{-1/2} \operatorname{Tr}\left[D^Z \exp(-tD^{Z,2})\right] dt . \tag{2.98}$$

Using (1.25), (2.97), (2.98), we get

$$\widetilde{\eta}^{(0)} = \tfrac{1}{2}\eta^{D^Z}(0) . \tag{2.99}$$

Moreover in degree 1, (2.95) specializes to

$$d\widetilde{\eta}^{(0)} = \pi_* \left[\widehat{A}(TZ, \nabla^{TZ}) \operatorname{ch}(\xi, \nabla^{\xi})\right]^{(1)} . \tag{2.100}$$

In view of (2.99), formula (2.100) gives a local expression for $d\frac{1}{2}\eta(0)$, which, when the fibres Z bound, can also be derived from the index theorem of Atiyah-Patodi-Singer [**APS1**].

When B is a point (i.e. in the case of a single fibre), the condition that $\ker(D^Z)$ is of locally constant dimension is empty. However in general, this condition is non empty, since it implies that the family (D^Z) is trivial in $K^1(B)$.

Set

$$\overline{\eta}^{D^Z}(s) = \tfrac{1}{2}\left(\eta^{D^Z}(s) + \dim\ker D^Z\right) . \tag{2.101}$$

Then $\overline{\eta}^{D^Z}(0)$ is called the reduced eta invariant of D^Z. In [**APS3,** Section 2], $\overline{\eta}^{D^Z}(0)$ is shown to define a smooth function with values in $\mathbf{R}/\mathbf{Z}$, and the general form of (2.100) is

$$d\overline{\eta}^{D^Z}(0) = \pi_* \left[\widehat{A}(TZ, \nabla^{TZ}) \operatorname{ch}(\xi, \nabla^{\xi})\right]^{(1)} . \tag{2.102}$$

i) The holonomy Theorem.

Now we use the assumptions and notation of Sections 2 e) and 2 f).

Let $s \in S_1 \to c_s \in B$ be an oriented smooth curve to B. In **[W]**, Witten raised the question of calculating the holonomy of a connection ∇^λ on λ in terms of the eta invariant of the odd dimensional oriented compact spin manifold $M = \pi^{-1}(C)$.

Let g^{TX} be a Riemannian metric on TX, let g^{TB} be a metric on TB. Put

$$g_\varepsilon^{TX} = g^{TX} + \frac{1}{\varepsilon}\pi^* g^{TB} . \tag{2.103}$$

Let g_ε^{TM} be the metric on TM induced by g_ε^{TX} on TM.

We equip S_1 with the non trivial spin structure. Then since TZ is spin, TM inherits an obvious spin structure. Let D_ε^M be the Dirac operator on M associated to g_ε^{TM}, ∇^ξ.

Let $\tau_1^0 \in S_1$ be the parallel transport with respect to the connection ∇^λ along $s \in S_1 \to c_s$.

Then we have the following result by Bismut-Freed [**BF2,** Theorem 3.16] and Cheeger [**C4**].

Theorem 2.37. *The limit as $\varepsilon \to 0$ of $\overline{\eta}^{D_\varepsilon^M}(0) \in \mathbf{R}/\mathbf{Z}$ exists, and moreover*

$$\tau_1^0 = \exp(-2i\pi \lim_{\varepsilon\to 0} \overline{\eta}^{D_\varepsilon^M}(0)) . \tag{2.104}$$

PROOF : Assume first that c bounds Δ in B. Then by (2.82)

$$\tau_1^0 = \exp\left(-2i\pi \int_{\pi^{-1}(\Delta)} \widehat{A}(TZ, \nabla^{TZ})\, \mathrm{ch}(\xi, \nabla^\xi)\right) . \tag{2.105}$$

On the other hand, by the index Theorem of Atiyah-Patodi-Singer [**APS1**] (see Theorem 1.3),

$$\overline{\eta}^{D_\varepsilon^M}(0) = \int_{\pi^{-1}(\Delta)} \widehat{A}(TM, g_\varepsilon^{TM})\, \mathrm{ch}(\xi, \nabla^\xi) \text{ in } \mathbf{R}/\mathbf{Z} . \tag{2.106}$$

As in (2.67), one verifies easily that as $\varepsilon \to 0$,

$$\widehat{A}(TM, g_\varepsilon^{TM}) \to \widehat{A}(TZ, \nabla^{TZ}) . \tag{2.107}$$

From (2.106), (2.107), we find that as $\varepsilon \to 0$,

$$\overline{\eta}^{D_\varepsilon^M}(0) \to \int_{\pi^{-1}(\Delta)} \widehat{A}(TZ, \nabla^{TZ})\, \mathrm{ch}(\xi, \nabla^\xi) \text{ in } \mathbf{R}/\mathbf{Z} . \tag{2.108}$$

By (2.105)-(2.108), we get (2.104) in this case. In general, using (2.102) and (2.107), one finds easily that $\lim_{\varepsilon\to 0} \overline{\eta}^{D_\varepsilon^M}(0) \in \mathbf{R}/\mathbf{Z}$ exists. The main point of

[**BF2**], [**C4**] is to extend (2.104) when c does not bound in B. Then if for $b \in C$, D_b^Z is invertible, a direct study of the formula (2.98) for $\overline{\eta}^{D_\varepsilon^M}(0)$ by the methods used in the proof of (2.68) shows that as $\varepsilon \to 0$,

$$\overline{\eta}^{D_\varepsilon^M}(0) \to \int_c \widetilde{\eta}, \tag{2.109}$$

from which (2.104) follows easily. When $\ker D^Z$ is nonzero and not even a vector bundle over c, a non trivial perturbation argument shows that (2.104) still holds. □

REMARK 2.38. Theorem 2.37 is one of the motivations for studying adiabatic limits of eta invariants, when instead of the circle c, the base of the fibration is arbitrary.

j) Adiabatic limits of eta invariants.

Assume first that B is an odd dimensional compact oriented spin Riemannian manifold. Let $E = E_+ \oplus E_-$ be a $\mathbf{Z}_2$-graded Hermitian vector bundle as in Section 2 c), and let $V \in \operatorname{End}(E)$ be a self-adjoint section of $\operatorname{End}^{\text{odd}}(E)$, such that $\ker V$ is of locally constant dimension. Let $V_\pm$ be the restriction of V to $E_\pm$. In the sequel, the assumptions of Section 2 c) will be in force. In particular $\nabla^E = \nabla^{E_+} \oplus \nabla^{E_-}$ is a split unitary connection on $E = E_+ \oplus E_-$, and $\nabla^{\ker V}$ is the orthogonal projection of ∇^E on $\ker V$. Also the odd form $\widehat{\eta}$ was defined in Definition 2.7.

Let $D^{B,E_\pm}$, $D^{B,\ker V_\pm}$ be the Dirac operators associated to the above data, acting on smooth actions over B of $S^{TB} \otimes E_\pm$, $S^{TB} \otimes \ker V_\pm$. Let $\overline{\eta}^{D^B, E_\pm}(0)$, $\overline{\eta}^{D^{B,\ker V_\pm}}(0)$ be the corresponding reduced eta invariants.

Theorem 2.39. *The following identity holds*

$$\begin{aligned} &\overline{\eta}^{D^{B,E_+}}(0) - \overline{\eta}^{D^B, E_-}(0) \\ &\quad = \overline{\eta}^{D^B, \ker V_+}(0) - \overline{\eta}^{D^B,\ker V_-}(0) + \int_B \widehat{A}(TB, \nabla^{TB})\widehat{\eta} \ \textit{in}\ \mathbf{R}/\mathbf{Z}. \end{aligned} \tag{2.110}$$

PROOF : In view of (2.21), (2.102), it is clear that both sides of (2.110) vary in the same way. By a simple deformation argument, we may as well assume that $E_+ = E_{0,+} \oplus F$, $E_- = E_{0,-} \oplus F$, V is the identity on F and vanishes on $E_{0,+} \oplus E_{0,-}$, $\nabla^{E_+} = \nabla^{E_{0,+}} \oplus \nabla^F$, $\nabla^{E_-} = \nabla^{E_{0,-}} \oplus \nabla^F$. In this situation, by Theorem 2.10, $\widehat{\eta}$ vanishes, and (2.110) is a trivial identity. □

Assume now that B is instead even dimensional, that E is a Hermitian vector bundle, that V is a self-adjoint section of $\operatorname{End}(E)$ such that $\ker V$ is a vector bundle, and ∇^E is a unitary connection on E. We use the notation of Section 2 d).

Let $D^{B,E}$, $D^{B,\ker V}$ be the Dirac operators acting on smooth sections of $C^\infty(B, S^{TB} \otimes E)$, $C^\infty(B, S^{TB} \otimes \ker V)$. Clearly

$$\begin{aligned} \overline{\eta}^{D^{B,E}}(0) &= \tfrac{1}{2} \operatorname{Ind}(D^{B,E}) \ \text{in}\ \mathbf{R}/\mathbf{Z} \\ \overline{\eta}^{D^{B,\ker V}}(0) &= \tfrac{1}{2} \operatorname{Ind}(D^{B,\ker V}) \ \text{in}\ \mathbf{R}/\mathbf{Z}. \end{aligned} \tag{2.111}$$

Theorem 2.40. *The following identities holds*

$$\bar{\eta}^{D^{B,E}}(0) = \bar{\eta}^{D^{B,\ker V}}(0) + \int_B \widehat{A}(TB, \nabla^{TB})\widehat{\eta} \text{ in } \mathbf{R}/\mathbf{Z}. \tag{2.112}$$

PROOF : By Theorem 2.17 and the Atiyah-Singer index theorem [**AS1**],

$$\int_B \widehat{A}(TB, \nabla^{TB})\widehat{\eta} = \tfrac{1}{2}\left(\operatorname{Ind}(D_+^{B,E_{>0}}) - \operatorname{Ind}(D_+^{B,E_{<0}})\right), \tag{2.113}$$

and so

$$\int_B \widehat{A}(TB, \nabla^{TB})\widehat{\eta} = \tfrac{1}{2}\left(\operatorname{Ind}(D_+^{B,E_{>0}}) + \operatorname{Ind}(D_+^{B,E_{<0}})\right) \text{ in } \mathbf{R}/\mathbf{Z}. \tag{2.114}$$

By (2.111), (2.114), we get (2.112). □

Now we make again the same assumptions and use the same notation as in Section 2 c). Also we suppose that B is compact, oriented, spin and odd dimensional.

Let $\tau = \pm 1$ on $E_\pm$. Then $D'^B = \tau D^B + V$ is a self-adjoint elliptic operator acting on $C^\infty(B, S^{TB} \otimes E)$. Let $\bar{\eta}^{D'^B}(0)$ be the reduced eta invariant associated to D'^B. By a simple deformation argument, one finds that

$$\bar{\eta}^{D'^B}(0) = \bar{\eta}^{D^{B,E_+}}(0) - \bar{\eta}^{D^{B,E_-}}(0) \text{ in } \mathbf{R}/\mathbf{Z}. \tag{2.115}$$

Now for $\varepsilon > 0$, we replace g^{TB} by $\frac{g^{TB}}{\varepsilon}$. Let D'^B_ε be the corresponding Dirac operator. Then by (2.102), $\bar{\eta}^{D'^B_\varepsilon}(0)$ remains constant in $\mathbf{R}/\mathbf{Z}$. Now, we give a refinement of Theorem 2.39, established in [**BC1**, Theorem 2.28].

Theorem 2.41. *If* $\ker V = \{0\}$, *then the limit as* $\varepsilon \to 0$ *of* $\bar{\eta}^{D'^B_\varepsilon}(0)$ *exists in* $\mathbf{R}$, *and moreover*

$$\lim_{\varepsilon \to 0} \bar{\eta}^{D'^B_\varepsilon}(0) = \int_B \widehat{A}(TB, \nabla^{TB})\widehat{\eta}. \tag{2.116}$$

PROOF : The main point in the proof of [**BC1**] is to show that

$$\lim_{\varepsilon \to 0} \frac{1}{2\sqrt{\pi t}} \operatorname{Tr}\left[D'^B_\varepsilon \exp(-tD'^{B,2}_\varepsilon)\right] = \int_B \widehat{A}(TB, \nabla^{TB})\beta_t, \tag{2.117}$$

which in turn follows from local index theory techniques. □

If B is even dimensional, an obvious analogue of Theorem 2.41 holds.

Now we make the same assumptions as in Sections 2 a) and 2 e). Suppose that X and B are compact, that X is odd dimensional and that B is oriented and spin. Then X is also oriented and spin. Let g^{TZ}, g^{TB} be metrics on TZ, TB. Put

$$g_\varepsilon^{TX} = \frac{1}{\varepsilon}\pi^* g^{TB} \oplus g^{TZ}. \tag{2.118}$$

Let D_ε^X be the Dirac operator on (X, g_ε^{TX}) as in Section 2 e). Using the variation formula for eta invariants, one verifies easily that as $\varepsilon \to 0$, $\overline{\eta}^{D_\varepsilon^X}(0)$ converges in $\mathbf{R}/\mathbf{Z}$.

Now we state the main result of [**BC1**, Theorems 4.35 and 4.95].

Theorem 2.42. *If* $\ker D^Z = \{0\}$, *as* $\varepsilon \to 0$, $\overline{\eta}^{D_\varepsilon^X}(0)$ *converges in* $\mathbf{R}$, *and moreover*

$$\lim_{\varepsilon\to 0} \overline{\eta}^{D_\varepsilon^X}(0) = \int_B \widehat{A}(TB, \nabla^{TB})\widetilde{\eta}. \tag{2.119}$$

PROOF : Formally, the proof of Theorem 2.42 is closely related to the proof of Theorem 2.41. In fact, (2.119) is an infinite dimensional version of (2.116), as should be clear from formula (2.63). The proof has three main steps:

- One proves that as $\varepsilon \to 0$,

$$\frac{1}{\sqrt{\pi}} \operatorname{Tr}\left[\frac{D_\varepsilon^X}{2\sqrt{t}} \exp(-tD_\varepsilon^{X,2})\right] \to \int_B \widehat{A}(TB, \nabla^{TB})\beta_t. \tag{2.120}$$

- One controls the lowest eigenvalue of D_ε^X as $\varepsilon \to 0$.
- One uses a version of finite propagation speed to control the integrand in (2.98) uniformly in ε as $t \to +\infty$. □

REMARK 2.43. In [**D**], Dai has given a very interesting extension of Theorem 2.42 to the case where $\ker D^Z$ is not necessarily zero.

Dai's result apply in particular to the case where B is odd dimensional, and D_ε^X is the signature operator of [**APS1**] associated to the metric g_ε^{TX}. In this case, there is no spectral flow, so that $\lim_{\varepsilon\to 0} \overline{\eta}^{D_\varepsilon^X}(0)$ exists in $\mathbf{R}$. Using results of Mazzeo and Melrose [**MazMe**] relating small eigenvalues of $D_\varepsilon^{X,2}$ to the Leray spectral of the fibration, Dai obtains a formula extending (2.119) by adding to the right-hand side of (2.119) the reduced eta invariant of a signature operator on B (twisted by the cohomology of the fibres) and a sum of half integers. These integers are the "signatures" of the Leray spectral sequence (E_r, d_r), for $r \geq 3$.

Adiabatic limits of eta invariants appeared naturally in the context of the solution by Atiyah-Donnelly-Singer [**ADS**] of the Hirzebruch conjecture [**Hir**] on the signature of Hilbert modular varieties. The fibrations which appear in this context are fibrations by tori over a torus basis. The calculation of [**ADS**] was recovered in the context of $\widetilde{\eta}$ forms in [**BC5**]. For a L_2 approach to the same problem, we refer to [**Mül**].

k) The families index theorem for manifolds with boundary.

Let now X be a manifold with boundary, let B be a manifold, and let $\pi : X \to B$ be a fibration, whose fibres Z are smooth compact manifolds with boundary.

We assume the fibres Z to be even dimensional, oriented and spin. Let g^{TZ} be a metric on TZ, which is fibrewise product near ∂Z. Let $T^H X$ be a horizontal vector subbundle of TX, such that $T^H X_{|\partial X} \subset T\partial X$. Put $T^H \partial X = T^H X_{|\partial X}$. Then $T^H \partial X$ is a horizontal subbundle of $T\partial X$.

Let (ξ, g^ξ, ∇^ξ) be a Hermitian vector bundle on X with unitary connection.

Assume first that Z is even dimensional. For $b \in B$, let D_b^Z be the Dirac operator with the Atiyah-Patodi Singer boundary conditions on ∂Z.

In order that the index bundle $\operatorname{Ind}(D_+^Z)$ to be well-defined, it is crucial that the family of boundary Dirac operators $D^{\partial Z}$ does not have spectral flow. So we first assume that

$$\ker D^{\partial Z} = 0\,. \tag{2.121}$$

In this case, (D_+^Z) is a family of Fredholm operators and its index $\operatorname{Ind}(D_+^Z) \in K^0(B)$ is well-defined.

Let $\widetilde{\eta}$ be the even form on B constructed in Definition 2.35, which is attached to the family $D^{\partial Z}$.

The following result is proved in Bismut-Cheeger [**BC3,** Theorem 6.11].

Theorem 2.44. *The following identity holds*

$$\operatorname{ch}(\operatorname{Ind} D_+^Z) = \pi_* \left[\widehat{A}(TZ, \nabla^{TZ}) \operatorname{ch}(\xi, \nabla^\xi)\right] - \widetilde{\eta} \ \ \textit{in} \ \ H^{\text{even}}(B, \mathbf{Q})\,. \tag{2.122}$$

PROOF : The basic idea in [**BC2,3**] is to replace the manifolds with boundary Z by the manifolds with conical singularity $\widehat{Z}_\ell = Z \cup_{\partial Z} C^{[0,\ell]}(\partial Z)$. Then we equip the fibres $\widehat{Z}$ with a family of metric of conical type. By proceeding as in Proposition 1.7, for ℓ large enough, the Atiyah-Patodi-Singer family (D_+^Z) and the family of L_2 Dirac operators $(D_+^{\widehat{Z}_\ell})$ have the same index. To the family $\widehat{Z}_\ell$, one attaches a natural Levi-Civita superconnection A_t^ℓ, to which the techniques of [**B2**] are formally applied. Note here that the advantage of using $\widehat{Z}_\ell$ is that the Atiyah-Patodi-Singer boundary conditions only appear in implicit form.

Observe that equation (2.95) explains why the right-hand side of (2.122) is closed. □

In [**MeP1**], Melrose and Piazza have extended Theorem 2.44 in a fundamental way. In fact, they observe that even if $\ker D^{\partial Z}$ is non trivial, by the family index Theorem of Atiyah-Singer [**AS2**], the family $(D^{\partial Z}) \in K^1(B)$ is trivial. They show that if B is compact, the triviality of the family $(D^{\partial Z})$ is equivalent to the existence of a spectral section P, i.e. a smooth family of self-adjoint projections $P_b : C^\infty(\partial Z_b, (S^{T\partial Z} \otimes \xi)_{|Z_b}) \to C^\infty(\partial Z_b, (S^{T\partial Z} \otimes \xi)_{|Z_b})$, such that there is $R > 0$ for which for any $b \in B$,

$$D^{\partial Z} u = \lambda u\,, \lambda > R\,, \ \text{then } Pu = u \qquad \lambda < -R\,, \ \text{then } Pu = 0\,. \tag{2.123}$$

Then Melrose and Piazza [**MeP1**] prove that if for every $b \in B$, the Atiyah-Patodi-Singer projection $P_{\geq 0,b}$ is replaced by P_b, the family of Dirac operators $D^{Z,P}_{+,b}$ associated to the boundary conditions attached to P has a honest index bundle $\operatorname{Ind}(D^{Z,P}_+)$. They construct an even form $\widetilde{\eta}^P$ on B, formally similar to the form $\widetilde{\eta}$ in (2.94). However in Melrose-Piazza's construction, the term $\sqrt{t}D^{\partial Z}$ is replaced by a more complicate expression $\sqrt{t}D'^{\partial Z}_t$, where $D'^{\partial Z}_t$ interpolates between $D^{\partial Z}$ for $t \ll 1$ and a suitable perturbation $D^{\partial Z} + A_P$ (with A_P depending on P) for $t \gg 1$. The family A_P is smoothing and such that $D^{\partial Z} + A_P$ is invertible. It can be seen as providing an explicit trivialization of the zero class $(D^{\partial Z}) \in K^1(B)$. Modulo exact forms, the form $\widetilde{\eta}^P$ only depends on P and not on the particular choice of A_P.

Then Melrose and Piazza [**MeP1**] prove :

Theorem 2.45. *The following identity holds*

$$\operatorname{ch}(\operatorname{Ind}(D^{Z,P}_+)) = \pi_*\left[\widehat{A}(TZ, \nabla^{TZ})\operatorname{ch}(\xi, \nabla^\xi)\right] - \widetilde{\eta}^P \quad \text{in} \quad H^{\text{even}}(B, \mathbf{Q})\,. \tag{2.124}$$

Besides in [**MeP1**], Melrose and Piazza compare the forms $\widetilde{\eta}^P$ for different choices of P.

When the fibres Z are odd dimensional and the family $D^{\partial Z}$ is invertible, Bismut-Cheeger [**BC4,** Section 6] conjectured a formula like (2.124) for a family of self-adjoint Dirac operators D^Z. This conjecture has been proved and extended by Melrose-Piazza [**MeP2**]. They adapted the idea of a spectral section in this new context, produced a superconnection whose Chern character forms are shown to represent the index, and established the corresponding index formula.

III. Analytic torsion forms and analytic torsion currents.

The purpose of this Section is to review the properties of the analytic torsion forms of [**BGS2**] and [**BK**] , and of the analytic torsion currents of [**B2**], [**BGS4,5**].

As explained in the introduction, analytic torsion forms are naturally associated to a family of Hermitian Dolbeault complexes. Analytic torsion currents are associated to an embedding $i : Y \to X$ and a resolution of a holomorphic Hermitian vector bundle η on Y by a holomorphic complex of Hermitian vector bundles on X. Analytic torsion forms and analytic torsion currents are secondary objects which refine the Riemann-Roch-Grothendieck theorem for submersions and immersions at the level of differential forms or currents.

This Section in organized as follows. In a), we construct the torsion forms associated to a holomorphic Hermitian complex of vector bundles [**BGS1**], and we relate them to the secondary classes of Bott-Chern [**BoCh**]. In b), we consider a holomorphic submersion $\pi : X \to S$, and a holomorphic Hermitian vector bundle ξ on X. When this fibration is Kähler (in a sense to be described), we show that the Levi-Civita superconnection of Definition 2.20 "respects" the holomorphic structure of the problem. When $R\pi_*\xi$ is locally free, we construct analytic torsion forms on S, which refine on the $\widetilde{\eta}$ forms of Definition 2.26.

In c), we introduce the Quillen metrics on the inverse of the determinant of the cohomology $\lambda(\xi) = (\det R\pi_*\xi)^{-1}$. The construction of the Quillen metric only involves the component of degree 0 of the above analytic torsion forms. Then we state the curvature theorem of [**BGS1,3**] for the Quillen metric on $\lambda(\xi)$.

In d), we describe the results of [**BerB**] and [**Ma**] on the compatibility of the analytic torsion forms to the composition of submersions.

In e), we construct the analytic torsion currents of [**B2**], [**BGS4,5**]. In f), we show that these currents are compatible to the composition of immersions.

In g) and h), we describe the results of [**BL**] and [**B5,6**] on the compatibility of the analytic torsion forms and analytic torsion currents to the composition of an immersion and a submersion.

In i), we give a short introduction to the proof of the main result in [**BL**]. In j), we develop a simple but crucial technical tool in [**BL**], the Hodge theory of the resolution of a point.

In k), we explain the construction in [**B4**] of the analytic torsion forms associated to a short exact sequence of holomorphic vector bundles, which plays an important role in the proof of the main result in [**BL**] and [**B5**]. In particular the evaluation of [**B4**] produces the genus R of Gillet and Soulé [**GS3**] in the final formula.

a) The torsion forms of a holomorphic Hermitian complex.

Let S be a complex manifold. Let

$$(E,v) : 0 \to E_m \underset{v}{\to} \dots \underset{v}{\to} E_0 \to 0 \tag{3.1}$$

be a holomorphic complex of vector bundles on S . Put

$$E_+ = \bigoplus_{i \text{ even}} E_i \quad , \quad E_- = \bigoplus_{i \text{ odd}} E_i \, . \tag{3.2}$$

Then $E = E_+ \oplus E_-$ is $\mathbf{Z}_2$-graded.

Let $g^E = \bigoplus_{i=0}^{m} g^{E_i}$ be a Hermitian metric on E. Let $\nabla^E = \bigoplus_{i=0}^{m} \nabla^{E_i}$ be the holomorphic Hermitian connection on $E = \bigoplus_{i=0}^{m} E_i$. Let v^* be the adjoint of v. Put

$$V = v + v^* . \tag{3.3}$$

Then V is a self-adjoint section of $\operatorname{End}^{\mathrm{odd}}(E)$. For $t > 0$, put

$$\begin{aligned} A''_t &= \nabla^{E''} + \sqrt{t}v\,, \\ A'_t &= \nabla^{E'} + \sqrt{t}v^*\,, \\ A_t &= \nabla^{E} + \sqrt{t}V\,. \end{aligned} \tag{3.4}$$

Clearly

$$A_t = A''_t + A'_t\,. \tag{3.5}$$

Also A_t is a superconnection of the kind we already met in (2.14). Let N be the number operator on E, i.e. N acts by multiplication by k on E_k.

The following result is established in [**BGS1**, Proposition 1.6].

Proposition 3.1. *The following identities hold*

$$\begin{aligned} &A''^2_t = 0 \ , \ A'^2_t = 0\,, \\ &A^2_t = [A''_t, A'_t]\ , \\ &[A''_t, A^2_t] = 0 \ , \ [A'_t, A^2_t] = 0\,, \\ &\frac{\partial A''_t}{\partial t} = \frac{1}{2t}[A''_t, N]\ , \\ &\frac{\partial A'_t}{\partial t} = \frac{-1}{2t}[A'_t, N]\ . \end{aligned} \tag{3.6}$$

Proof : Since (E, v) is a holomorphic complex,

$$A''^2_t = 0\,. \tag{3.7}$$

Also

$$\frac{\partial A''_t}{\partial t} = \frac{v}{2\sqrt{t}} = \frac{1}{2t}[A''_t, N]\ . \tag{3.8}$$

The other identities in (3.6) follow easily from analogues of (3.7), (3.8). □

DEFINITION 3.2. For $t > 0$, put

$$\begin{aligned} \alpha_t &= \varphi \operatorname{Tr_s}\left[\exp(-A_t^2)\right], \\ \beta_t &= \frac{1}{(2i\pi)^{1/2}} \varphi \operatorname{Tr_s}\left[\frac{\partial A_t}{\partial t} \exp(-A_t^2)\right], \\ \gamma_t &= \varphi \operatorname{Tr_s}\left[N \exp(-A_t^2)\right]. \end{aligned} \tag{3.9}$$

Observe that in our context, the forms α_t and β_t were already introduced in Definition 2.5.

Let P^S be the set of smooth real forms on S which are sums of forms of type (p,p). Let $P^{S,0}$ be the subspace of the $\alpha \in P^S$ such that $\alpha = \overline{\partial}\beta + \partial\gamma$, with β and γ smooth.

Now we have the result of [**BGS1,** Theorem 1.15].

Theorem 3.3. *The forms α_t and γ_t lie in P^S. Moreover*

$$\begin{aligned} \frac{\partial \alpha_t}{\partial t} &= -d\beta_t, \\ \beta_t &= \frac{1}{2i\pi} \frac{(\overline{\partial} - \partial)\gamma_t}{2t}. \end{aligned} \tag{3.10}$$

In particular

$$\frac{\partial \alpha_t}{\partial t} = \frac{\overline{\partial}\partial}{2i\pi} \frac{\gamma_t}{t}. \tag{3.11}$$

PROOF : We only prove part of Theorem 3.3. The first identity in (3.10) was already established in (2.16). Using (3.6), we obtain the second identity in (3.10). □

Assume now that the homology $H(E,v)$ is of locally constant dimension. Then $H(E,v)$ is a holomorphic $\mathbf{Z}$-graded holomorphic vector bundle on S. Clearly

$$H(E,v) \simeq \ker V. \tag{3.12}$$

Let $g^{H(E,v)}$ be the metric on $H(E,v)$ induces by g^E via (3.12).

One verifies easily that $\nabla^{\ker V} = P^{\ker V} \nabla^E$ is the holomorphic Hermitian connection on $(H(E,v), g^{H(E,v)})$. Put

$$\operatorname{ch}'(E, g^E) = \sum_{i=0}^{m} (-1)^i i \operatorname{ch}(E_i, g^{E_i}). \tag{3.13}$$

Then by an analogue of Theorem 2.6, as $t \to +\infty$,

$$\gamma_t = \operatorname{ch}'(H(E,v), g^{H(E,v)}) + \mathcal{O}\left(\frac{1}{\sqrt{t}}\right). \tag{3.14}$$

DEFINITION 3.4. For $s \in \mathbf{C}$, $0 < \mathrm{Re}(s) < \frac{1}{2}$, put

$$R(E, g^E)(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} (\gamma_t - \gamma_\infty) dt . \tag{3.15}$$

By (3.14), $R(E, g^E)(s)$ is a holomorphic function of s, which extends holomorphically to $s = 0$.

DEFINITION 3.5. Set

$$T(E, g^E) = \frac{\partial}{\partial s} R(E, g^E)(s)|_{s=0} . \tag{3.16}$$

Recall that the odd form $\widehat{\eta}$ was defined in Definition 2.7. Now we have the result of [**BGS1,** Theorem 1.17].

Theorem 3.6. *The form $T(E, g^E)$ lies in P^S. Moreover*

$$\begin{aligned} &\frac{\overline{\partial}\partial}{2i\pi} T(E, g^E) = \mathrm{ch}(H(E, v), g^{H(E,v)}) - \mathrm{ch}(E, g^E) , \\ &\frac{1}{2i\pi} (\overline{\partial} - \partial) \frac{T}{2}(E, g^E) = \widehat{\eta} . \end{aligned} \tag{3.17}$$

PROOF : These identities follow easily from Theorems 2.6 and 3.3. □

DEFINITION 3.7. We will say that $((E, v), g^E)$ is split, if $E_i = F_i \oplus F_{i-1} \oplus H_i, v|_{E_i}$ vanishes on $F_i \oplus H_i$ and is the identity on F_{i-1}, and the above splitting is orthogonal with respect to g^{E_i}.

Now we state a result of [**BGS1,** Corollary 1.30].

Theorem 3.7. *There is a unique way to associate to $((E, v), g^E)$, with $H(E, v)$ of locally constant dimension, a class $T(E, v) \in P^S / P^{S,0}$ such that*

a) *$T(E, g^E)$ is functorial.*

b) *If (E, g^E) is split, $T(E, g^E) = 0$.*

c) *The following identity holds,*

$$\frac{\overline{\partial}\partial}{2i\pi} T(E, g^E) = \mathrm{ch}(H(E, v), g^{H(E,v)}) - \mathrm{ch}(E, g^E) . \tag{3.18}$$

PROOF : Existence is almost obvious by the above construction. As to uniqueness, observe that over $S \times \mathbf{P}^1$, one constructs easily a complex $(\widetilde{E}, \widetilde{v})$ and a metric $g^{\widetilde{E}}$ such that

$$((\widetilde{E}, v), g^{\widetilde{E}})_{S\times\{0\}} = ((E, v), g^E), \quad ((\widetilde{E}, v), g^{\widetilde{E}})_{S\times\{0\}} \text{ is split}. \tag{3.19}$$

Using the obvious equation

$$\frac{\overline{\partial}\partial}{2i\pi} \log |z|^2 = \delta_{\{0\}} - \delta_{\{\infty\}}, \tag{3.20}$$

we get

$$T(E, g^E) = \int_{\mathbf{P}^1} \log |z|^2 \left(\mathrm{ch}\left(H(\widetilde{E}, \widetilde{v}), g^{H(\widetilde{E},\widetilde{v})} \right) - \mathrm{ch}(\widetilde{E}, g^{\widetilde{E}}) \right) \quad \text{in} \quad P^S/P^{S,0}, \tag{3.21}$$

which guarantees uniqueness. □

REMARK 3.8. Observe that

$$R(E, g^E)^{(0)}(s) = \mathrm{Tr_s}\left[N[V^2]^{-s} \right], \tag{3.22}$$

so that

$$T(E, g^E)^{(0)} = - \mathrm{Tr_s}\left[N \log(V^2) \right]. \tag{3.23}$$

By (3.18), we get

$$c_1(\det E, g^E) = c_1(\det H(E,v), g^{\det H(E,v)}) - \frac{\overline{\partial}\partial}{2i\pi} T(E, g^E)^{(0)}. \tag{3.24}$$

The interpretation of (3.24) is easy. In fact there is a canonical holomorphic isomorphism **[KMu]**

$$\det E \simeq \det H(E,v). \tag{3.25}$$

Let us briefly describe this isomorphism. First assume that $H(E,v) = \{0\}$, i.e. (E,v) is acyclic. Then (3.25) says that $\det E$ has a canonical non zero section $\tau(E,v)$. To construct $\tau(E,v)$, we choose $\omega_m \in \det E_m$, $\omega_m \neq 0$, $\omega_{m-1} \in \Lambda^{\dim E_{m-1} - \dim E_m} E_{m-1}$ such that $v\omega_m \wedge \omega_{m-1} \in \det E_{m-1}$ is non zero, $\omega_{m-2} \in \Lambda^{\dim E_{m-2} - \dim E_{m-1} + \dim E_m} E_{m-2}$ such that $v\omega_{m-1} \wedge \omega_{m-2} \in \det E_{m-2}$ is non zero... . These choices are possible because (E,v) is acyclic. Then

$$\tau(E,v) = \left(\omega_m \otimes (v\omega_m \wedge \omega_{m-1})^{-1} \otimes (v\omega_{m-1} \wedge \omega_{m-2}) \otimes \dots \right)^{(-1)^m} \tag{3.26}$$

does not depend on the above choices.

When $H(E,v)$ is non zero, the construction of the canonical isomorphism (3.25) is similar.

Then one verifies easily that

$$g^{\det E} = g^{\det H(E,v)} \exp\{T(E, g^E)^{(0)}\}, \tag{3.27}$$

from which (3.24) follows immediately.

The class of forms $T(E,g^E) \in P^S/P^{S,0}$ appears as a prototype of Bott-Chern classes [**BoCh**], [**BGS1**]. Let us give a construction of these classes in the simplest case.

Let $\rho : F \to S$ be a holomorphic vector bundle. Let g^F, g'^F be two Hermitian metrics on F. Let Q be a characteristic polynomial. The following result in established in [**BGS1**, Theorem 1.29].

Theorem 3.8. *There exists a unique way to assign to (F, g^F, g'^F) a class $\widetilde{Q}(F, g^F, g'^F) \in P^S/P^{S,0}$ such that*

a) *$\widetilde{Q}(F, g^F, g'^F)$ is functorial.*

b) *If $g^F = g'^F$, $\widetilde{Q}(F, g^F, g'^F) = 0$.*

c) *The following identity holds*

$$\frac{\overline{\partial}\partial}{2i\pi}\widetilde{Q}(F, g^F, g'^F) = Q(F, g'^F) - Q(F, g^F). \tag{3.28}$$

PROOF : We just outline a construction of $\widetilde{Q}(F, g^F, g'^F)$ [**BGS1**]. Extend F to a vector bundle $\widetilde{F}$ on $S \times \mathbf{P}^1$. Let $g^{\widetilde{F}}$ be a metric on $\widetilde{F}$ such that $g^{\widetilde{F}}_{S\times\{0\}} = g^F$, $g^{\widetilde{F}}_{S\times\{\infty\}} = g'^F$. Put

$$\widetilde{Q}(F, g^F, g'^F) = -\int_{\mathbf{P}^1} \log(|z|^2) Q(\widetilde{F}, g^{\widetilde{F}}). \tag{3.29}$$

Then by (3.20), (3.28) holds. □

Let $g^E = \bigoplus_{i=0}^{m} g^{E_i}$, $g'^E = \bigoplus_{i=0}^{m} g'^{E_i}$ be two set of Hermitian metrics on E. Let $g^{H(E,v)}$, $g'^{H(E,v)}$ be the corresponding metrics on $H(E,v)$.

Theorem 3.9. *The following identity holds*

$$\begin{aligned} T(E, g'^E) - T(E, g^E) = {} & \widetilde{\mathrm{ch}}(H(E,v), g^{H(E,v)}, g'^{H(E,v)}) \\ & - \widetilde{\mathrm{ch}}(E, g^E, g'^E) \quad \textit{in} \quad P^S/P^{S,0}. \end{aligned} \tag{3.30}$$

PROOF : Using Theorem 3.7, our Theorem is a straightforward consequence of the uniqueness of Bott-Chern classes stated in Theorem 3.8. □

b) The Levi-Civita superconnection of a Kähler fibration and the analytic torsion forms.

Let $\pi : X \to S$ be a holomorphic submersion with compact fibre Z. Let ξ be a holomorphic vector bundle on X. Let $R\pi_*\xi$ be the direct image of ξ.

Let ω^X be a real closed $(1,1)$-form on X, such that the restriction of ω^X to TZ is the Kähler form ω^{TZ} of a Hermitian metric g^{TZ} on $TZ = TX/S$. If $J^{T_{\mathbf{R}}Z}$ is the complex structure of $T_{\mathbf{R}}Z$, if $U, V \in T_{\mathbf{R}}Z$, $\omega^{TZ}(U,V) =$

$\langle U, J^{T_{\mathbf{R}}Z} V\rangle$. Let g^{ξ} be a Hermitian metric on ξ, let ∇^{ξ} be the holomorphic Hermitian connection on (ξ, g^{ξ}). Let $T^H X$ be the orthogonal bundle to TZ with respect to ω^X.

Let $(\Omega(Z, \xi_{|Z}), \overline{\partial}^Z)$ be the family of relative Dolbeault complex along the fibres Z. We equip $\Omega(Z, \xi_{|Z})$ with the L_2 metric attached to g^{TZ}, g^{ξ},

$$\langle s, s'\rangle = \int_Z \langle s, s'\rangle \frac{dv_Z}{(2\pi)^{\dim Z}}\,. \tag{3.31}$$

Let $\overline{\partial}^{*Z}$ be the formal adjoint of $\overline{\partial}^Z$ with respect to (3.31). Set

$$D^Z = \overline{\partial}^Z + \overline{\partial}^{Z*}\,. \tag{3.32}$$

Then by **[Hi]**, $\sqrt{2}D^Z$ is a family of standard Dirac operators along the fibre Z. The only minor difference is that the fibres Z only have spin^c structure.

To the data $(g^{TZ}, T^H X)$ we can associate the objects constructed in Section 2 e).

The following result is proved in [**BGS2,** Theorem 1.7].

Theorem 3.10. *The connection $\nabla^{T_{\mathbf{R}}Z}$ on $T_{\mathbf{R}}Z$ preserves the complex structure of $T_{\mathbf{R}}Z$. It induces the holomorphic Hermitian connection on (TZ, g^{TZ}).*

As a 2-form, T is of complex type $(1,1)$.

Let $\nabla^{\Lambda(T^{*(0,1)}Z)\otimes\xi}$ be the connection induced by ∇^{TZ}, ∇^{ξ} on $\Lambda(T^{*(0,1)}Z)\otimes \xi$.

If $U \in TZ$, let $U^H \in T^H X$ be the horizontal lift of U.

DEFINITION 3.11. If $U \in T_{\mathbf{R}}S$, if s is a smooth section of $\Omega(Z, \xi_{|Z})$ over S, put

$$\nabla_U^{\Omega(Z,\xi_{|Z})} s = \nabla_{U^H}^{\Lambda(T^{*(0,1)}Z)\otimes\xi} s\,. \tag{3.33}$$

The following result is established in [**BGS2,** Theorem 1.14].

Theorem 3.12. *The connection $\nabla^{\Omega(Z,\xi_{|Z})}$ on $\Omega(Z, \xi_{|Z})$ preserves the Hermitian product (3.31) on $\Omega(Z, \xi_{|Z})$. Its curvature is of complex type* $(1,1)$. *Also*

$$\left[\nabla^{\Omega(Z,\xi_{|Z})''}, \overline{\partial}^Z\right] = 0\ ,\quad \left[\nabla^{\Omega(Z,\xi_{|Z})'}, \overline{\partial}^{Z*}\right] = 0\,. \tag{3.34}$$

Amazingly enough, $(\Omega(Z, \xi_{|Z}), \overline{\partial}^Z)$ appears to be a “holomorphic“ Hermitian vector bundle compact over S. By (3.34), we find that

$$\left(\nabla^{\Omega(Z,\xi_{|Z})''} + \overline{\partial}^Z\right)^2 = 0\ ,\quad \left(\nabla^{\Omega(Z,\xi_{|Z})'} + \overline{\partial}^{Z*}\right)^2 = 0\,. \tag{3.35}$$

The explanation for (3.35) given in [**BGS2,** Theorem 2.8] is that using the smooth identification $\Lambda(T^{*(0,1)}X) \simeq \Lambda(T^{*(0,1)}Z)\widehat{\otimes}\pi^*\Lambda(T^{*(0,1)}S)$, $\nabla^{\Omega(Z,\xi_{|Z})''} + \overline{\partial}^Z$ is exactly the full Dolbeault operator $\overline{\partial}^X$ acting on $\Omega(X, \xi)$.

Recall that $\Lambda(T^{*(0,1)}X)\otimes\xi$ is a $c(T_{\mathbf{R}}Z)$ Clifford module. Namely if $X\in TZ$, let $X^*\in T^{*(0,1)}Z$ correspond to X by the metric g^{TZ}. Then if $X\in TZ$, $Y\in\overline{TZ}$, put

$$c(X)=\sqrt{2}X^*\wedge\ ,\quad c(Y)=-\sqrt{2}i_Y\,. \tag{3.36}$$

Extend c to a linear map $T_{\mathbf{R}}Z\otimes_{\mathbf{R}}\mathbf{C}\to\mathrm{End}\,((\Lambda(T^{*(0,1)}Z)\otimes\xi)$. Then if $X,X'\in T_{\mathbf{R}}Z\otimes_{\mathbf{R}}C$,

$$c(X)c(X')+c(X')c(X)=-2\,\langle X,X'\rangle_{g^{TZ}}\,. \tag{3.37}$$

Let (f_α) be a basis of $T_{\mathbf{R}}S$, let (f^α) be the dual basis of $T^*_{\mathbf{R}}S$. Put

$$\begin{aligned}c(T^{(1,0)})&=\tfrac{1}{2}f^\alpha f^\beta c(T^{(1,0)}(f^H_\alpha,f^H_\beta))\,,\\ c(T^{0,1)})&=\tfrac{1}{2}f^\alpha f^\beta c(T^{(0,1)}(f^H_\alpha,f^H_\beta))\,.\end{aligned}\tag{3.38}$$

With the notation in (2.54),

$$c(T)=c(T^{(1,0)})+c(T^{(0,1)})\,. \tag{3.39}$$

DEFINITION 3.13. For $t>0$, put

$$\begin{aligned}B''_t&=\sqrt{t}\ \overline{\partial}^Z+\nabla^{\Omega(Z,\xi_{|Z})''}-\frac{c(T^{(1,0)})}{2\sqrt{2t}}\,,\\ B'_t&=\sqrt{t}\ \overline{\partial}^{Z*}+\nabla^{\Omega(Z,\xi_{|Z})'}-\frac{c(T^{(0,1)})}{2\sqrt{2t}}\,,\\ B_t&=B''_t+B'_t\,.\end{aligned}\tag{3.40}$$

Then by (2.55) and Theorem 3.12, for $t>0$, B_t is exactly the superconnection $A_{\frac{t}{2}}$ in the sense of **[B1]**, i.e. B_t is a Levi-Civita superconnection.

Put

$$\omega^X=\omega^{TZ}+\omega^H\,. \tag{3.41}$$

In particular $\omega^H\in\pi^*\Lambda^2(T^*_{\mathbf{R}}S)$ is the restriction of ω^X to $T^H_{\mathbf{R}}X=\pi^*T_{\mathbf{R}}S$.

Let N_V be the number operator of $\Omega(Z,\xi_{|Z})$, i.e. N_V acts by multiplication by k on $\Omega^k(Z,\xi_{|Z})$.

DEFINITION 3.14. For $t>0$, put

$$N_t=N_V+\frac{i\omega^H}{t}\,. \tag{3.42}$$

Then $N_t\in\left(\Lambda(T^*_{\mathbf{R}}S)\widehat{\otimes}\ \mathrm{End}\,(\Omega(Z,\xi_{|Z}))\right)^{\mathrm{even}}$.

The following result is proved in [**BGS2**, Theorem 2.6].

Theorem 3.15. *The following identities hold,*

$$
\begin{aligned}
&B''^2_t = 0 \ , \ B'^2_t = 0 \,, \\
&B^2_t = [B''_t, B'_t] \ , \\
&[B''_t, B^2_t] = 0 \ , \ [B'_t, B^2_t] = 0 \,, \\
&\frac{\partial B''_t}{\partial t} = \frac{-1}{2t}[B''_t, N_t] \ , \\
&\frac{\partial B'_t}{\partial t} = \frac{1}{2t}[B'_t, N_t] \ .
\end{aligned}
\tag{3.43}
$$

REMARK 3.16. The identities in (3.43) are remarkable. They guarantee that the Levi-Civita superconnection B_t also has natural holomorphic properties, i.e. it splits as $B_t = B''_t + B'_t$. Besides, by comparing (3.43) with (3.6), N_t appears as the right "number operator" associated to B_t.

DEFINITION 3.17. For $t > 0$, set

$$
\begin{aligned}
\alpha_t &= \varphi \ \mathrm{Tr_s}\left[\exp(-B^2_t)\right] \ , \\
\beta_t &= \left(\frac{1}{2i\pi}\right)^{1/2} \varphi \ \mathrm{Tr_s}\left[\frac{\partial B_t}{\partial t}\exp(-B^2_t)\right] \ , \\
\gamma_t &= \varphi \ \mathrm{Tr_s}\left[N_t \exp(-B^2_t)\right] \ .
\end{aligned}
\tag{3.44}
$$

Now we state a result taken from [**BGS3**, Theorems 2.9 and 2.16].

Theorem 3.18. *The forms $\alpha_t, \beta_t, \gamma_t$ are real. The forms α_t and γ_t lie in P^S. The cohomology class of α_t is constant, and*

$$
[\alpha_t] = \mathrm{ch}(R\pi_*\xi) \quad in \quad H^{\mathrm{even}}(S, \mathbf{R}) \,. \tag{3.45}
$$

Also,

$$
\begin{aligned}
\frac{\partial \alpha_t}{\partial t} &= -d\beta_t, \\
\beta_t &= -\frac{1}{2i\pi}(\overline{\partial} - \partial)\frac{\gamma_t}{2t} \,.
\end{aligned}
\tag{3.46}
$$

In particular,

$$
\frac{\partial \alpha_t}{\partial t} = -\frac{\overline{\partial}\partial}{2i\pi}\frac{\gamma_t}{t} \,. \tag{3.47}
$$

Finally as $t \to 0$,

$$
\begin{aligned}
\alpha_t &= \pi_*\left[\mathrm{Td}(TZ, g^{TZ})\,\mathrm{ch}(\xi, g^{\xi})\right] + \mathcal{O}(t), \\
\gamma_t &= \frac{C_{-1}}{t} + C_0 + \mathcal{O}(t) \ , \ C_{-1}, C_0 \in P_S \,.
\end{aligned}
\tag{3.48}
$$

PROOF : We just sketch the proof of part of Theorem 3.18. Equation (3.45) follows from Theorem 2.22. Equation (3.46) follows from (3.43) as in (3.10). The first equation in (3.48) follows from Theorem 2.22. The second equation in (3.48) is proved in **[BGS2]** by local index theoretic techniques. □

Now we assume that $R\pi_*\xi$ is locally free. So $R\pi_*\xi$ is a holomorphic $\mathbf{Z}$-graded vector bundle on S, and moreover $(R\pi_*\xi)_s \simeq H(Z_s, \xi_{|Z_s})$. Since $H(Z, \xi_{|Z}) \simeq \ker D^Z$, $R\pi_*\xi$ inherits a smooth metric $g^{R\pi_*\xi}$.

Theorem 3.19. *As $t \to +\infty$,*

$$\begin{aligned} \alpha_t &= \mathrm{ch}(R\pi_*\xi, g^{R\pi_*\xi}) + \mathcal{O}\left(\frac{1}{\sqrt{t}}\right), \\ \gamma_t &= \mathrm{ch}'(R\pi_*\xi, g^{T\pi_*\xi}) + \mathcal{O}\left(\frac{1}{\sqrt{t}}\right). \end{aligned} \tag{3.49}$$

PROOF : With the notation of Theorem 2.25, using **[BGS3,** Theorem 3.11] (which relies on (3.35)), one shows easily that $\nabla^{\ker D^Z, u}$ is just the holomorphic Hermitian connection on $(R\pi_*\xi, g^{R\pi_*\xi})$. Theorem 3.19 is then an obvious modification of Theorem 2.25. □

DEFINITION 3.20. For $s \in \mathbf{C}$, $0 < \mathrm{Re}(s) < \frac{1}{2}$, put

$$R(\omega^X, g^\xi) = -\frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1}(\gamma_t - \gamma_\infty)dt. \tag{3.50}$$

By (3.48), (3.49), $R(\omega^X, g^\xi)$ is a holomorphic function of s, which extend holomorphically near $s = 0$.

DEFINITION 3.21. Set

$$T(\omega^X, g^\xi) = \frac{\partial}{\partial s} R(\omega^X, g^\xi)(0). \tag{3.51}$$

Recall that the form $\widetilde{\eta}$ was defined in Definition 2.26. Then we have the result of **[BGS2,** Theorem 2.20], **[BK,** Theorem 3.9].

Theorem 3.22. *The form $T(\omega^X, g^\xi)$ lies in P^S. Moreover*

$$\begin{aligned} &\frac{\overline{\partial}\partial}{2i\pi} T(\omega^X, g^\xi) = \mathrm{ch}(R\pi_*\xi, g^{R\pi_*\xi}) - \pi_*\left[\mathrm{Td}(TZ, g^{TZ})\,\mathrm{ch}(\xi, g^\xi)\right], \\ &\frac{1}{2i\pi}(\overline{\partial} - \partial)\frac{T}{2}(E, g^E) = \widetilde{\eta}. \end{aligned} \tag{3.52}$$

PROOF : Equation (3.52) follows from Theorems 3.18 and 3.19. □

In Remark 2.28, we observed that the dependence of $\widetilde{\eta}$ (modulo coboundaries) on the various geometric data is quite explicit. Also in Theorem 3.9, we found that the dependence of $T(E, g^E) \in P^S/P^{S,0}$ on the metric g^E can be explicitly given in terms of Bott-Chern classes.

It is then natural to ask how $T(\omega^X, g^\xi)$ depends on (ω^X, g^ξ). In fact if only g^ξ is made to vary, equation (3.52) and the methods of **[BGS1]** used in the proof of Theorem 3.7 provide the answer immediately. However if ω^X also varies, the answer certainly does not rely on the methods of Theorem 3.7. In fact, for Theorem 3.22 to hold, it is crucial for ω^X to be closed. So in order to calculate $T(\omega'^X, g^\xi) - T(\omega^X, g^\xi)$ using (3.52), a necessary condition would be, for example, that the fibres Z have the same volume for ω^X and ω'^X.

Let (ω'^X, g'^ξ) be taken as before. The following "anomaly formulas" were established in **[BK,** Theorem 3.10], extending earlier work in degree 0 **[BGS3,** Theorem 1.23].

Theorem 3.23. *The following identity holds*

$$(3.53)\quad T(\omega'^X, g'^\xi) - T(\omega^X, g^\xi) = \widetilde{\mathrm{ch}}\left(R\pi_*\xi, g^{R\pi_*\xi}, g'^{R\pi_*\xi}\right)$$
$$-\pi_*\left[\widetilde{\mathrm{Td}}(TZ, g^{TZ}, g'^{TZ})\,\mathrm{ch}(\xi, g^\xi) + \mathrm{Td}(TZ, g'^{TZ})\widetilde{\mathrm{ch}}(\xi, g^\xi, g'^\xi)\right] \quad \textit{in } P^S/P^{S,0}.$$

In particular, $T(\omega^X, g^\xi) \in P^S/P^{S,0}$ *only depends on* (g^{TZ}, g^ξ).

The last statement in Theorem 3.23 is if particular importance. It says that, as should be the case, the class of $T(\omega^X, g^\xi)$ in $P^S/P^{S,0}$ only depends on the geometric data which appear in the right-hand side of the first equation in (3.52).

c) Quillen metrics.

Assume first that S is a point. Let g^{TZ}, g^ξ be the Hermitian metrics on TZ, ξ.

DEFINITION 3.24. Put

$$(3.54)\qquad \theta(s) = -\operatorname{Tr_s}\left[N_V\left[D^{Z,2}\right]^{-s}\right].$$

Then $\theta(s)$ is a linear combination of the zeta functions of the Laplacian $D^{Z,2}$ acting on forms in $\Omega(Z, \xi_{|Z})$ of degree $0, 1 \dots, \dim Z$.

Put

$$(3.55)\qquad \lambda(\xi) = (\det H(Z, \xi_{|Z}))^{-1}.$$

Then $\lambda(\xi)$ is a complex line. The metric $g^{H(Z,\xi_{|Z})}$ induces a metric $|\ \ |_{\lambda(\xi)}$ on $\lambda(\xi)$.

In **[Q2]**, Quillen introduced the following metrics on $\lambda(\xi)$.

DEFINITION 3.25. The Quillen metric $\| \quad \|_{\lambda(\xi)}$ on $\lambda(\xi)$ is given by

$$\| \quad \|_{\lambda(\xi)} = | \quad |_{\lambda(\xi)} \exp\left\{-\tfrac{1}{2}\frac{\partial\theta}{\partial s}(0)\right\}. \tag{3.56}$$

The underlying motivation for formula (3.56) is equation (3.27). In fact (3.56) is a way of making sense of the metric $g^{(\det \Omega(Z,\xi_{|Z}))^{-1}}$, which $(\det \Omega(Z,\xi_{|Z}))^{-1}$ does not exist. The quantity $\frac{\partial\theta}{\partial s}(0)$ is called the Ray-Singer analytic torsion [**RS**].

Let now $\pi : X \to S$ be a holomorphic sumersion with compact fibre Z. Let ξ be a holomorphic vector bundle on X.

By a construction due to Grothendieck-Knudsen-Munford [**KMu**], there is a canonically defined holomorphic line bundle $\lambda(\xi)$ on S, called the inverse of the determinant of the direct image $R\pi_*\xi$. In particular if $s \in S$, there is a canonical isomorphism

$$\lambda(\xi)_s \simeq \left(\det H(Z_s, \xi_{|Z_s})\right)^{-1}. \tag{3.57}$$

Needless to say, if $R\pi_*\xi$ is locally free,

$$\lambda(\xi) = (\det R\pi_*\xi)^{-1}. \tag{3.58}$$

In the general case, we will still use the notation $\lambda(\xi) = (\det R\pi_*\xi)^{-1}$.

Let g^{TZ}, g^{ξ} be arbitrary Hermitian metrics on TZ, ξ. Then by the construction in Definition 3.25 and using (3.57), the fibre $\lambda(\xi)_s$ can be equipped with the Quillen metric $\| \cdot \|_{\lambda(\xi)_s}$.

A first result on Quillen metrics is as follows [**BGS3,** Theorem 3.14].

Theorem 3.26. *The Quillen metric is a smooth metric on $\lambda(\xi)$.*

PROOF : If $R\pi_*\xi$ is locally free, $| \quad |_{\lambda(\xi)}$ and $\| \quad \|_{\lambda(\xi)}$ are smooth. The remarkable fact is that in the general case, $\| \quad \|_{\lambda(\xi)}$ is still smooth. However formula (3.27) partly explains the smoothness of $\| \quad \|_{\lambda(\xi)}$. □

DEFINITION 3.27. We will say that $\pi : X \to S$ is locally Kähler if there is a covering of S by open sets U such that $\pi^{-1}(U)$ is Kähler.

We now state the result of [**BGS3,** Theorem 1.27].

Theorem 3.28. *Assume that $\pi : X \to S$ is locally Kähler and that g^{TZ} is fibrewise Kähler. Then*

$$c_1(\lambda(\xi), \| \quad \|_{\lambda(\xi)}) = -\pi_* \left[\mathrm{Td}(TZ, g^{TZ})\,\mathrm{ch}(\xi, g^{\xi})\right]^{(2)}. \tag{3.59}$$

PROOF : Clearly, we can assume that X is Kähler. Let g^{TX} be a Kähler metric on TX, with Kähler form ω^X, and assume first that g^{TZ} is the metric on TZ induced by g^{TX}. By (3.50), (3.54),

$$R(\omega^X, g^\xi)^{(0)}(s) = \theta(s)\,, \tag{3.60}$$

and so

$$T(\omega^X, g^\xi)^{(0)} = \frac{\partial \theta}{\partial s}(0)\,. \tag{3.61}$$

Suppose that $R\pi_*\xi$ is locally free. By (3.52), we get

$$\frac{\bar\partial \partial}{2i\pi} T(\omega^X, g^\xi)^{(0)} = -c_1(\lambda(\xi), |\quad|_{\lambda(\xi)}) - \pi_*\left[\mathrm{Td}(TZ, g^{TZ})\,\mathrm{ch}(\xi, g^\xi)\right]^{(2)}\,. \tag{3.62}$$

From (3.62), we get (3.59). If $R\pi_*\xi$ is not locally free, more work is needed to establish (3.59) **[BGS3]**.

Suppose now that g'^{TZ} is a metric on TZ which is only fibrewise Kähler. Then by (3.53),

$$\log\left(\frac{\|\quad\|'_{\lambda(\xi)}}{\|\quad\|_{\lambda(\xi)}}\right)^2 = \int_Z \widetilde{\mathrm{Td}}(TZ, g^{TZ}, g'^{TZ})\,\mathrm{ch}(\xi, g^\xi)\,. \tag{3.63}$$

From (3.59) (established for g^{TZ}) and (3.63), we get (3.59) for g'^{TZ}. $\square$

d) Adiabatic limits of Quillen metrics, analytic torsion forms, and composition of submersions.

Let $\pi : X \to S$ be a submersion of compact complex manifolds. Let ξ be holomorphic vector bundle on X. We assume that $R\pi_*\xi$ is locally free.

Let g^{TX} be a Kähler metric on X, let ω^X be the corresponding Kähler form. Let g^{TS} be a Kähler metric on S. Let g^ξ be a Hermitian metric on ξ.

Put

$$\begin{aligned} \lambda &= \left(\det H(X, \xi_{|X})\right)^{-1}\,, \\ \lambda' &= \otimes\left(\det H(S, R^i\pi_*\xi)\right)^{(-1)^{i+1}}\,. \end{aligned} \tag{3.64}$$

Let

$$\Omega(X,\xi) = F^0\Omega(X,\xi) \supset F^1\Omega(X,\xi) \supset \ldots \supset F^{\dim S+1}\Omega(X,\xi) = 0 \tag{3.65}$$

be the obvious filtration by the partial degree in $\Lambda(T^{*(0,1)}S)$ of the Dolbeault complex $\Omega(X,\xi)$. Let (E_r, d_r) be the associated spectral sequence. Then

$$E_2^{(p,q)} = H^p(S, R^q\pi_*\xi)\,. \tag{3.66}$$

By (3.66), it follows that the lines λ and λ' are canonically isomorphic.

We can equip the line λ with the Quillen metric associated to g^{TX}, g^{ξ}, and the line λ' with the Quillen metric associated to $g^{TS}, g^{R\pi_*\xi}$.

Consider the exact sequence

$$0 \to TZ \to TX \to \pi^* TS \to 0\,. \tag{3.67}$$

Let $\widetilde{\mathrm{Td}}(TX, TS, g^{TX}, g^{TS}) \in P^X/P^{X,0}$ be the Bott-Chern class [**BoCh**], [**BGS1**] such that

$$\frac{\bar\partial\partial}{2i\pi}\widetilde{\mathrm{Td}}(TX, TS, g^{TX}, g^{TS}) = \mathrm{Td}(TX, g^{TX}) - \mathrm{Td}(TZ, g^{TZ})\pi^* \,\mathrm{Td}(TS, g^{TS})\,. \tag{3.68}$$

The following result is established in [**BerB,** Theorem 3.1].

Theorem 3.29. *The following identity holds*

$$\log\left(\frac{\|\ \ \|_\lambda}{\|\ \ \|_{\lambda'}}\right)^2 = -\int_S \mathrm{Td}(TS, g^{TS}) T(\omega^X, g^\xi) + \int_X \widetilde{\mathrm{Td}}(TX, TS, g^{TX}, g^{TS})\,\mathrm{ch}(\xi, g^\xi)\,. \tag{3.69}$$

We identify $\frac{\mathrm{Td}'}{\mathrm{Td}}(x)$ to the corresponding additive genus. By definition, the genus Td' is the product of the additive genus $\frac{\mathrm{Td}'}{\mathrm{Td}}$ and the multiplicative genus Td.

In [**BerB,** Theorem 3.2], it is shown that (3.69) is essentially equivalent to the following statement. For $\varepsilon > 0$, put

$$g_\varepsilon^{TX} = g^{TX} + \frac{1}{\varepsilon}\pi^* g^{TS}\,. \tag{3.70}$$

Let $\|\ \ \|_{\lambda,\varepsilon}$ be the Quillen metric on λ associated to $(g_\varepsilon^{TX}, g^\xi)$.

Theorem 3.30. *As $\varepsilon \to 0$,*

$$\log\left(\frac{\|\ \ \|^2_{\lambda,\varepsilon}}{\|\ \ \|^2_{\lambda}}\right) - \int_X \pi^*\,\mathrm{Td}'(TS)\,\mathrm{Td}(TZ)\,\mathrm{ch}(\xi)\log(\varepsilon) \to -\int_S \mathrm{Td}(TS, g^{TS}) T(\omega^X, g^\xi) + \log\left(\frac{\|\ \ \|_{\lambda'}}{\|\ \ \|_{\lambda}}\right)^2\,. \tag{3.71}$$

REMARK 3.31. The proof of Theorems 3.29 and 3.30 is a combination of the adiabatic limit techniques of Bismut-Cheeger [**BC1**], and of the Leray spectral arguments of Mazzeo-Melrose [**MazMe**] and Dai [**D**].

Recently, Ma [**Ma**] has established an extension of Theorem 3.29 for the higher analytic torsion forms $T(\omega^X, g^\xi)$. Namely let

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "\pi_{Z/Y}"'] & W \arrow[d, "\pi_{W/V}"'] \arrow[dr, "\pi_{W/S}"] & \\ Y \arrow[r] & V \arrow[r, "\pi_{V/S}"'] & S \end{tikzcd} \tag{3.72}$$

be a commutative diagram of holomorphic submersions, with compact fibres Z and Y. Let ξ be a holomorphic vector bundle on W. Assume that $R\pi_{W/S*}\xi$, $R\pi_{W/V*}\xi$ and $R\pi_{V/S*}R\pi_{W/V*}\xi$ are locally free.

Let g^{ξ} be a Hermitian metric on ξ. Let ω^W, ω^V be $(1,1)$ closed forms on W, V having the properties described in Section 3 b). Then by proceeding as in Section 3 b), the three direct images vector bundles described above inherit Hermitian metrics.

Let $T_{W/V}(\omega^W, g^{\xi}), T_{W/S}(\omega^W, g^{\xi}), T_{V/S}(\omega^V, g^{R\pi_{W/V*}\xi})$ be the analytic torsion forms on V and S associated to $\pi_{W/V}, \pi_{W/S}, \pi_{V/S}$, and the given metrics.

A problem which arises in **[Ma]** is the adequate definition of

$$\alpha = \widetilde{\mathrm{ch}}\left(R\pi_{W/S*}\xi, R\pi_{V/S*}R\pi_{W/V*}\xi, g^{R\pi_{W/S*}\xi}, g^{R\pi_{W/S*}R\pi_{W/V*}\xi}\right)\,, \tag{3.73}$$

so that

$$\frac{\overline{\partial}\partial}{2i\pi}\alpha = \mathrm{ch}\left(R\pi_{V/S*}R\pi_{W/V*}\xi, g^{R\pi_{V/S*}R\pi_{W/V*}\xi}\right) - \mathrm{ch}\left(R\pi_{W/S*}\xi, g^{R\pi_{W/S*}\xi}\right)\,. \tag{3.74}$$

In fact there is a spectral sequence E_r of sheaves on S such that $E_2 = R\pi_{W/S*}R\pi_{W/V*}\xi$, which abuts to $R\pi_{W/S*}\xi$.

Under an adequate assumption of ampleness, this spectral sequence is trivial, so that the definition of α is easy. If the E_r are locally free, there is also a natural definition of α. In general, if W and V are projective, a definition of α is given in **[Ma]**.

Then Ma's result is as follows.

Theorem 3.32. *The following identity holds*

$$\begin{aligned} T_{W/S}(\omega^W, g^{\xi}) &= T_{V/S}(\omega^V, g^{R\pi_{W/V*}\xi}) \\ &\quad + \pi_{W/S*}\left[\mathrm{Td}(TY, g^{TY}) T_{W/V}(\omega^W, g^{\xi})\right] \\ &\quad + \alpha - \pi_{W/S*}\left[\widetilde{\mathrm{Td}}(TZ, TY, g^{TZ}, g^{TY})\,\mathrm{ch}(\xi, g^{\xi})\right] \ \textit{in}\ P^S/P^{S,0}\,. \end{aligned} \tag{3.75}$$

e) Analytic torsion currents.

Let $i : Y \to X$ be an embedding of complex manifolds. Let η be a holomorphic vector bundle on Y. Let

$$(\xi, v) : 0 \to \xi_m \underset{v}{\to} \xi_{m-1} \cdots \underset{v}{\to} \xi_0 \to 0 \tag{3.76}$$

be a holomorphic complex of vector bundles on X, which, together with a holomorphic restriction map $r : \xi_{0|Y} \to \eta$, provides a resolution of the sheaf $i_*\mathcal{O}_Y(\eta)$, i.e. we have an exact sequence of sheaves

$$0 \to \mathcal{O}_X(\xi_m) \underset{v}{\to} \ldots \to \mathcal{O}_X(\xi_0) \to i_*\mathcal{O}_Y(\eta) \to 0\,. \tag{3.77}$$

In particular the complex (ξ, v) is acyclic on $X \setminus Y$. If $y \in Y$, $U \in TX_y$, let $\partial_U v(y)$ be the derivative of v in any holomorphic trivialization of (ξ, v)

near y. Then by [**B3**, Theorem 1.2], $\partial_U v(y)$ acts on $H((\xi,v)_y)$, the action only depends on the image $z \in N_{Y/X,y}$ of U, and will be denoted by $\partial_z v(y)$. Let π be the projection $N_{Y/X} \to Y$. Then by [**B3**, Theorem 1.2], there is a canonical isomorphism of holomorphic complexes on $N_{Y/X}$,

$$(\pi^* H(\xi,v)_{|Y}), \partial_z v) \simeq \left(\pi^*(\Lambda N^*_{Y/X} \otimes \eta), i_z\right) , \tag{3.78}$$

where in the right-hand side of (3.78), appears a Koszul complex. Let $g^\xi = \bigoplus_{i=0}^m g^{\xi_i}$ be a Hermitian metric on $\xi = \bigoplus_{i=0}^m \xi_i$. Let $g^{N_{Y/X}}, g^\eta$ be Hermitian metrics on $N_{Y/X}, \eta$.

Put

$$V = v + v^* . \tag{3.79}$$

By finite dimensional Hodge theory,

$$H(\xi,v)_{|Y}) \simeq \ker V_{|Y} . \tag{3.80}$$

As a subbundle of $\xi_{|Y}$, $\ker V_{|Y}$ inherits a Hermitian metric. Let $g^{H(\xi,v)_{|Y}}$ be the corresponding metric on $H(\xi,v)_{|Y}$.

DEFINITION 3.33. We will say that $g^{\xi_0},\dots,g^{\xi_m}$ verify assumption (A) with respect to $g^{N_{Y/X}}, g^\eta$ if the identification (3.78) is an isometry.

By [**B3**, Proposition 1.6], given metrics $g^{N_{Y/X}}, g^\eta$, there exist metrics $g^{\xi_0},\dots,g^{\xi_m}$ such that (A) is verified.

In the sequel we assume that (A) holds.

Let ∇^ξ be the holomorphic Hermitian connection on ξ. For $t>0$, put

$$A_t = \nabla^\xi + \sqrt{t} V . \tag{3.81}$$

Let N_H be the number operator of (ξ,v).

DEFINITION 3.34. For $t \geq 0$, put

$$\begin{aligned} \alpha_t &= \varphi \ \mathrm{Tr_s}\left[\exp(-A_t^2)\right] , \\ \gamma_t &= \varphi \ \mathrm{Tr_s}\left[N_H \exp(-A_t^2)\right] . \end{aligned} \tag{3.82}$$

Of course, equations (3.10), (3.11) are still valid. Now we give a result of [**B3**, Theorems 3.2 and 4.3], which replaces Theorem 2.6 and (3.14) in this new situation. Let δ_Y be the current of integration on Y.

Theorem 3.35. *As $t \to +\infty$,*

$$\begin{aligned} \alpha_t &= \mathrm{Td}^{-1}(N_{Y/X}, g^{N_{Y/X}}) \,\mathrm{ch}(\eta, g^\eta)\delta_Y + \mathcal{O}\left(\frac{1}{\sqrt{t}}\right) , \\ \gamma_t &= -((\mathrm{Td})^{-1})'(N_{Y/X}, g^{N_{Y/X}}) \,\mathrm{ch}(\eta, g^\eta)\delta_Y + \mathcal{O}\left(\frac{1}{\sqrt{t}}\right) . \end{aligned} \tag{3.83}$$

REMARK 3.36. In(3.83), $\mathcal{O}\left(\frac{1}{\sqrt{t}}\right)$ is taken in the adequate Sobolev space of currents. Also, the convergence is shown to be microlocal in the set of currents whose wave front set is conormal to Y.

By Theorem 2.4 and by (3.83), we see that

$$\mathrm{ch}(\xi) = \mathrm{Td}^{-1}(N_{Y/X})\,\mathrm{ch}(\eta)\delta_Y \quad \text{in} \quad H^{\text{even}}(X, \mathbf{Q})\,. \tag{3.84}$$

This is exactly the content of Riemann-Roch-Grothendieck for immersions, which says that

$$\mathrm{ch}(i_*\eta) = \mathrm{Td}^{-1}(N_{Y/X})\,\mathrm{ch}(\eta)\delta_Y \quad \text{in} \quad H^{\text{even}}(X, \mathbf{Q})\,. \tag{3.85}$$

Using Theorem 3.35, we can now imitate Definition 3.5 and construct a current $T(\xi, g^\xi)$ on X by formulas (3.15), (3.16). The following result is proved in [**BGS4,** Theorem 2.5].

Theorem 3.37. *The current $T(\xi, g^\xi)$ is a sum of currents of type (p,p), and its wave front set lies in $N^*_{Y/X,\mathbf{R}}$. Moreover*

$$\frac{\bar\partial\partial}{2i\pi} T(\xi, g^\xi) = \mathrm{Td}^{-1}(N_{Y/X}, g^{N_{Y/X}})\,\mathrm{ch}(\eta, g^\eta)\delta_Y - \mathrm{ch}(\xi, g^\xi)\,. \tag{3.86}$$

Let P^X_Y be the set of currents on X, which are sums of currents of type (p,p), whose wave front set lies in $N^*_{Y/X,\mathbf{R}}$. We define $P^{X,0}_Y$ as in Section 3 a), with the adequate condition on the wave front set of β, γ.

The following extension of Theorem 3.9 is established in [**BGS5,** Theorem 2.5].

Theorem 3.38. *Let $(g^\xi, g^{N_{Y/X}}, g^\eta)$ and $(g'^\xi, g'^{N_{Y/X}}, g'^\eta)$ be triples of metrics verifying condition (A). Then*

$$\begin{aligned} T(\xi, g'^\xi) - T(\xi, g^\xi) = \Big(&\widetilde{\mathrm{Td}}^{-1}(N_{Y/X}, g^{N_{Y/X}}, g'^{N_{Y/X}})\,\mathrm{ch}(\eta, g^\eta) \\ &+ \mathrm{Td}^{-1}(N_{Y/X}, g'^{N_{Y/X}})\widetilde{\mathrm{ch}}(\eta, g^\eta, g'^\eta)\Big)\delta_Y \\ &- \widetilde{\mathrm{ch}}(\xi, g^\xi, g'^\xi)) \quad \text{in} \quad P^X_Y / P^{X,0}_Y\,. \end{aligned} \tag{3.87}$$

PROOF : The proof of Theorem 3.38 is essentially the same as the proof of Theorem 3.9. □

f) Compatibility of the currents $T(\xi, g^\xi)$ to the composition of immersions.

Let $i : Y \to X$, $i' : Y' \to X$ be two complex submanifolds of X intersecting transversally. In particular $\dim Y + \dim Y' \geq \dim X$. Let η, η' be holomorphic vector bundles on Y, Y', let (ξ, v), (ξ', v') be two holomorphic complexes of vector bundles on X which provide resolutions of $i_*\eta$, $i'_*\eta'$. Then one verifies easily that if $Y'' = Y \cap Y'$, if $i'' : Y'' \to X$ is the corresponding embedding, then $(\xi\widehat{\otimes}\xi', v+v')$ provides a resolution of $i''_*(\eta_{|Y''}\otimes\eta'_{|Y''})$.

Then we have the diagram

$$\begin{tikzcd} Y'' \arrow[r] \arrow[d] \arrow[dr] & Y \arrow[d] \\ Y' \arrow[r] & X \end{tikzcd} \tag{3.88}$$

Let $(g^{N_{Y/X}}, g^{\eta}, g^{\xi})$ and $(g^{N_{Y'/X}}, g^{\eta'}, g^{\xi'})$ be Hermitian metrics verifying (A). Then we equip $N_{Y''/X} = N_{Y/X|Y''}\oplus N_{Y'/X|Y''}$ with the metric $g^{N_{Y''/X}} = g^{N_{Y/X|Y''}}\oplus g^{N_{Y'/X|Y''}}$. One verifies easily that $(g^{N_{Y''/Y}}, g^{\eta_{|Y''}}\otimes g^{\eta'_{|Y''}}, g^{\xi''})$ verifies (A).

Let $P^X_{Y\cup Y'}, P^{X,0}_{Y\cup Y'}$ be the obvious analogues of $P^X_Y, P^{X,0}_Y$ when replacing Y by $Y\cup Y'$.

The following result is established in [**BGS5,** Theorem 2.7].

Theorem 3.39. *The following identity holds*

$$\begin{aligned} T(\xi\widehat{\otimes}\xi', g^{\xi\otimes\xi'}) &= T(\xi, g^{\xi})\,\mathrm{ch}(\xi', g^{\xi'}) + \\ &\mathrm{Td}^{-1}(N_{Y/X}, g^{N_{Y/X}})\,\mathrm{ch}(\eta, g^{\eta}) T(\xi', g^{\xi'})\delta_Y \ \ \textit{in } P^X_{Y\cup Y'}/P^{X,0}_{Y\cup Y'}\,. \end{aligned} \tag{3.89}$$

g) Complex immersions and Quillen metrics.

Assume now that X and Y are compact. Let g^{TX}, g^{TY} be Kähler metrics on TX, TY. Let $g^{N_{Y/X}}, g^{\eta}$ be Hermitian metrics on $N_{Y/X}, \eta$, let $g^{\xi} = \bigoplus_{i=0}^{m} g^{\xi_i}$ be a Hermitian metric on $\xi = \bigoplus_{i=0}^{m}\xi_i$ which verifies (A) with respect to $g^{N_{Y/X}}, g^{\eta}$.

Put

$$\begin{aligned} \lambda(\eta) &= (\det H(Y,\eta))^{-1}\,, \\ \lambda(\xi_i) &= (\det H(X,\xi_i))^{-1}\,, \\ \lambda(\xi) &= \bigotimes_{i=0}^{m} (\lambda(\xi_i))^{(-1)^i}\,. \end{aligned} \tag{3.90}$$

We claim that there is a canonical isomorphism

$$\lambda(\xi) \simeq \lambda(\eta)\,. \tag{3.91}$$

In fact let $H(X,\xi)$ be the hypercohomology of the sheaf $\mathcal{O}_X(\xi)$. Namely if δ is the natural Čech coboundary, we consider the complex $(\mathcal{O}_X(\xi), \delta+v)$. Needless to say, δ and v are graded so that

$$(\delta+v)^2 = \delta v + v\delta = 0\,. \tag{3.92}$$

Also if $N_{\check{\mathrm{C}}\mathrm{ech}}$ is the natural Čech number operator, we grade $(\mathcal{O}_X(\xi), \delta + v)$ by $N_{\check{\mathrm{C}}\mathrm{ech}} - N_H$ so that $\delta + v$ increase the total degree by 1. Then $r : (\mathcal{O}_X(\xi), \delta + v) \to (\mathcal{O}_Y(\eta), \delta)$ is a quasi-isomorphism, so that

$$H(X, \xi) \simeq H(Y, \eta) . \tag{3.93}$$

Now there is a spectral sequence whose E_1 is given by

$$E_1^{(p,q)} = H^q(X, \xi_{-p}) . \tag{3.94}$$

By (3.94), we get

$$(\det H(X, \xi))^{-1} \simeq \bigotimes_p \left(\left(\det E_1^{(p,\cdot)} \right)^{-1} \right)^{(-1)^p} , \tag{3.95}$$

which is equivalent to

$$(\det H(X, \xi))^{-1} \simeq \bigotimes (\lambda(\xi_i))^{(-1)^i} . \tag{3.96}$$

By (3.93), (3.96) we get (3.91).

Now $\lambda(\xi)$ and $\lambda(\eta)$ are equipped with Quillen metrics $\| \ \|_{\lambda(\xi)}$ and $\| \ \|_{\lambda(\eta)}$. It is natural to compare these metrics. This question was first varied by Gillet and Soulé [**GS3**] in their program to prove a Riemann-Roch-Grothendieck formula in Arakelov geometry.

In fact if A is the ring of integers of a number field k, if $X \to \mathrm{Spec}(A)$ is an arithmetic variety, if ξ is an algebraic vector bundle on X, then $H(X, \xi)$ is A-module. If $\lambda(\xi) = (\det H(X, \xi))^{-1}$, then if $\lambda(\xi)$ is equipped with a metric at places at infinity, $\lambda(\xi)$ has an Arakelov degree $\deg \lambda(\xi)$. The idea in [**GS3**] is to precisely equip $\lambda(\xi)$ with Quillen metrics at the places at infinity.

In [**GS3**], Gillet and Soulé gave a conjectural formula for $\deg \lambda(\xi)$ in terms of arithmetic characteristic classes. Still, when calculating the ratio of two Quillen metrics on the same algebraic object, only the contributions at infinity remain, so that [**GS3**] suggests a comparison formula which should be valid for any complex Kähler manifold.

Now we describe a result of Bismut-Lebeau [**BL,** Theorem 0.1] where the conjectured comparison formula was established.

First we introduce the Gillet-Soulé series R [**GS3**]. Let $\zeta(s) = \sum_1^{+\infty} \frac{1}{n^s}$ be the Riemann zeta function.

DEFINITION 3.40. Let $R(x)$ be the power series

$$R(x) = \sum_{\substack{n \geq 1 \\ n \text{ odd}}} \left(2 \frac{\zeta'(-n)}{\zeta(-n)} + \sum_1^n \frac{1}{j} \right) \zeta(-n) \frac{x^n}{n!} . \tag{3.97}$$

We identify $R(x)$ with the additive genus $\sum_1^q R(x_i)$.

Let $\widetilde{\mathrm{Td}}(TX_{|Y}, g^{TY}, g^{TY_{|Y}}, g^{N_{Y/X}}) \in P^Y/P^{Y,0}$ be the Bott-Chern class, such

$$(3.98)\quad \frac{\overline{\partial}\partial}{2i\pi}\widetilde{\mathrm{Td}}(TX_{|Y}, g^{TY}, g^{TX_{|Y}}, g^{N_{Y/X}}) = \mathrm{Td}(TX_{|Y}, g^{TX_{|Y}}) - \mathrm{Td}(TY, g^{TY})\,\mathrm{Td}(N_{Y/X}, g^{N_{Y/X}})\,.$$

Then the result of Bismut-Lebeau [**BL**, Theorem 0.1] is as follows.

Theorem 3.41. *The following identity holds*

$$(3.99)\quad \log\left(\frac{\|\ \|_{\lambda(\xi)}}{\|\ \|_{\lambda(\eta)}}\right)^2 = -\int_X \mathrm{Td}(TX, g^{TX})T(\xi, g^\xi) + \int_Y \frac{\widetilde{\mathrm{Td}}(TX_{|Y}, g^{TY}, g^{TX_{|Y}}, g^{N_{Y/X}})}{\mathrm{Td}(N_{Y/X}, g^{N_{Y/X}})}\,\mathrm{ch}(\eta, g^\eta) - \int_X \mathrm{Td}(TX)R(TX)\,\mathrm{ch}(\xi) + \int_Y \mathrm{Td}(TY)R(TY)\,\mathrm{ch}(\eta)\,.$$

PROOF : Some details on the proof of Theorem 3.41 are given in Section 3 i)- 3 k). □

h) Analytic torsion forms, analytic torsion currents and the composition of an immersion and a submersion.

Let now $i : W \to V$ be an embedding of complex manifolds, let $\pi_V : V \to S$ be a holomorphic submersion with compact fibre X, which restricts to a submersion $\pi_W : W \to S$ with compact fibre Y. Then we have the diagram

$$(3.100)\quad \begin{array}{ccccc} Y & \longrightarrow & W & & \\ {\scriptstyle i}\downarrow & & {\scriptstyle i}\downarrow & \searrow^{\pi_W} & \\ X & \longrightarrow & V & \xrightarrow{\pi_V} & S \end{array}$$

Let η be a holomorphic vector bundle on W, let (ξ, v) be a holomorphic complex of vector bundles of X which resolves $i_*\eta$. Of course, fibrewise, the situation is the same as in Section 3 g). Equivalently, the case where S is a point is just the case considered in Section 3 g).

Assume that $R\pi_{W*}\eta$ is locally free. Then $R\pi_{W*}\eta \simeq H(Y, \eta_{|Y})$. By (3.93), $R\pi_{V*}\xi$ is also locally free.

Let ω^V (resp. ω^W) be a $(1,1)$ closed form on V (resp. W). Let $g^\xi = \bigoplus_{i=0}^m g^{\xi_i}$, $g^{N_{Y/X}}$, g^η be Hermitian metrics on $\xi = \bigoplus_{i=0}^m \xi_i$, $N_{Y/X}$, η, which verify assumption (A) (keeping in mind that $N_{Y/X} = N_{W/V}$).

Let $T(\omega^W, g^\eta) \in P^S$ be the analytic torsion forms constructed in Section 3 b). They verify the equation

(3.101)
$$\frac{\overline{\partial}\partial}{2i\pi} T(\omega^W, g^\eta) = \mathrm{ch}\left(H(Y, \eta_{|Y}), g^{H(Y,\eta_{|Y})}\right) - \pi_{W*}\left[\mathrm{Td}(TY, g^{TY})\,\mathrm{ch}(\eta, g^\eta)\right] .$$

Similarly, to the family of double complexes $(\Omega(X, \xi_{|X}), \overline{\partial}^X + v)$, we can associate analytic torsion forms $T(\omega^V, g^\xi) \in P^S$, which verify the equation

(3.102)
$$\frac{\overline{\partial}\partial}{2i\pi} T(\omega^V, g^\xi) = \mathrm{ch}(H(X, \xi_{|X}), g^{H(X,\xi_{|X})}) - \pi_{V*}\left[\mathrm{Td}(TX, g^{TX})\,\mathrm{ch}(\xi, g^\xi)\right] .$$

Since $H(X, \xi_{|X}) \simeq H(Y, \eta_{|Y})$, the Bott-Chern class $\widetilde{\mathrm{ch}}(H(Y, \eta_{|Y}), g^{H(X,\xi_{|X})}, g^{H(Y,\eta_{|Y})})$ is well-defined.

The main result proved in [**B5**, Theorem 0.1], which extends Theorem 3.41 to the relative situation, is as follows.

Theorem 3.42. *The following identity holds*

$$\begin{aligned}(3.103)\quad & \widetilde{\mathrm{ch}}(H^{(Y,\eta_{|Y})}, g^{H(X,\xi|X)}, g^{H(Y,\eta|Y)}) - T(\omega^W, g^\eta) + T(\omega^V, g^\xi) \\ & - \int_X \mathrm{Td}(TX, g^{TX}) T(\xi, g^\xi) + \int_Y \frac{\widetilde{\mathrm{Td}}(TX|_W, g^{TY}, g^{TX|_W}, g^{N_{Y/X}})}{\mathrm{Td}(N_{Y/X}, g^{N_{Y/X}})}\,\mathrm{ch}(\eta, g^\eta) \\ & - \int_X \mathrm{Td}(TX) R(TX)\,\mathrm{ch}(\xi) + \int_Y \mathrm{Td}(TY) R(TY)\,\mathrm{ch}(\eta) = 0 \quad \text{in} \quad P^S/P^{S,0} .\end{aligned}$$

Assume now that for $j > 0$, $R^j\pi_{V*}\xi_k = 0$, $(0 \le k \le m)$, $R^j\pi_{W*}\eta = 0$. Then we have a holomorphic complex of vector bundles K on S,

(3.104).
$$K : 0 \to H^0(X, \xi_m) \to H^0(X, \xi_{m-1}) \ldots \to H^0(X, \xi_0) \to H^0(X, \xi_{|X}) \to 0$$

Let $\widetilde{\mathrm{ch}}(K, g^K) \in P^S/P^{S,0}$ be the Bott-Chern class such that

$$\begin{aligned}(3.105)\quad \frac{\overline{\partial}\partial}{2i\pi}\widetilde{\mathrm{ch}}(K, g^K) = {} & \mathrm{ch}\left(H^0(X, \xi_{|X}), g^{H^0(X,\xi_{|X})}\right) \\ & - \sum_{i=0}^m (-1)^i\,\mathrm{ch}\left(H^0(X, \xi_{i|X}), g^{H^0(X,\xi_{i|X})}\right) .\end{aligned}$$

The following result is proved in [**B5**, Theorem 0.2].

Theorem 3.43. *The following identity holds*

$$(3.106)\quad T(\omega^V, g^\xi) - \sum_{i=0}^m (-1)^i T(\omega^V, g^{\xi_i}) - \widetilde{\mathrm{ch}}(K, g^K) = 0 \quad \text{in} \quad P^S/P^{S,0} .$$

REMARK 3.44. Needless to say, Theorems 3.39, 3.41-3.43 are compatible to each other.

i) A sketch of the proof of Theorem 3.41.

Using the anomaly formulas of Theorem 3.23 as in (3.63), and also Theorem 3.38, to establish Theorem 3.41, we may and will assume that $g^{TY}, g^{N_{Y/X}}$ are the metrics induced by g^{TX} on $TY, N_{Y/X}$.

Put

$$E = C^\infty(X, \Lambda(T^{*(0,1)}X)\widehat{\otimes}\xi)\,. \tag{3.107}$$

We define the total **Z**-grading on E by the operator $N_V - N_H$. Then $\overline{\partial}^X + v$ acts on E and

$$(\overline{\partial}^X + v)^2 = 0\,. \tag{3.108}$$

Therefore $(E, \overline{\partial}^X + v)$ is a **Z**-graded complex, whose hypercohomology $H(E, \overline{\partial}^X + v)$ is finite dimensional. Dolbeault's theory shows that

$$H(E, \overline{\partial}^X + v) \simeq H(X, \xi)\,. \tag{3.109}$$

Set

$$F = C^\infty(Y, \Lambda(T^{*(0,1)}Y)\widehat{\otimes}\eta)\,. \tag{3.110}$$

The restriction map $r : \xi_{0|Y} \to \eta$ extends to a map of complexes $r : (E, \overline{\partial}^X + v) \to (F, \overline{\partial}^Y)$. By [**BL,** Theorem 1.7], it induces the canonical identification $H(X, \xi) \simeq H(Y, \eta)$.

Put

$$\widetilde{\lambda}(\xi) = (\det H(X, \xi))^{-1}\,. \tag{3.111}$$

Then by (3.93), (3.96)

$$\widetilde{\lambda}(\xi) \simeq \lambda(\xi) \simeq \lambda(\eta)\,. \tag{3.112}$$

Moreover by imitating Definition 3.25, we can equip $\widetilde{\lambda}(\xi)$ with a Quillen metric $\|\ \|_{\widetilde{\lambda}(\xi)}$. A first step in the proof of Theorem 3.41 is the simple fact, established in [**BL,** Theorem 2.1], that

$$\|\ \|_{\widetilde{\lambda}(\xi)} = \|\ \|_{\lambda(\xi)}\,. \tag{3.113}$$

To establish Theorem 3.41, we must then compute $\operatorname{Log}\left(\dfrac{\|\ \|_{\widetilde{\lambda}(\xi)}}{\|\ \|_{\lambda(\eta)}}\right)^2$.

Put

$$D^X = \overline{\partial}^X + \overline{\partial}^{X*}, \quad V = v + v^*. \tag{3.114}$$

For $T \geq 0$, it is clear that

$$(\overline{\partial}^X + Tv)^2 = 0. \tag{3.115}$$

Also for $T > 0$

$$H^{\cdot}(E, \overline{\partial}^X + Tv) \simeq H^{\cdot}(E, \overline{\partial}^X + v). \tag{3.116}$$

For $T \geq 0$, put

$$A_T = D^X + TV. \tag{3.117}$$

Then by Hodge theory,

$$\ker A_T = H^{\cdot}(E, \overline{\partial}^X + Tv), \tag{3.118}$$

so that for $T > 0$,

$$\ker(A_T) \simeq H^{\cdot}(E, \overline{\partial}^X + Tv) \simeq H^{\cdot}(F, \overline{\partial}^Y). \tag{3.119}$$

Set

$$D^Y = \overline{\partial}^Y + \overline{\partial}^{Y*}. \tag{3.120}$$

Then

$$\ker D^Y \simeq H(Y, \eta). \tag{3.121}$$

Now we describe in some detail a few difficulties which appear in the evaluation of $\operatorname{Log}\left(\dfrac{\| \ \|_{\widetilde{\lambda}(\xi)}}{\| \ \|_{\lambda(\eta)}}\right)^2$ given in [**BL**]. We will here take a toy object to describe these difficulties.

Let $\chi(\xi), \chi(\eta)$ be the Euler characteristics of $\mathcal{O}_X(\xi), \mathcal{O}_Y(\eta)$. Then, by the Mc Kean-Singer formula [**McKS**], for $t > 0$, $T \geq 0$,

$$\begin{aligned} \chi(\xi) &= \operatorname{Tr_s}\left[\exp(-(tA_T)^2\right], \\ \chi(\eta) &= \operatorname{Tr_s}\left[\exp(-(tD_Y)^2\right]. \end{aligned} \tag{3.122}$$

Of course

$$\chi(\xi) = \chi(\eta). \tag{3.123}$$

Let $P_t^T(x,x')$ $(x, x' \in X)$, $Q_t(y,y')$ $(y, y' \in Y)$ be the smooth kernels for $\exp(-(tA_T)^2)$, $\exp(-(tD^Y)^2)$ with respect to $dv_X(x')$, $dv_Y(y')$. Here $\operatorname{Tr_s}\left[P_t^T(x,x)\right] dv_X(x)$, $\operatorname{Tr_s}\left[Q_t(y,y)\right] dv_Y(y)$ will be be considered as currents on X.

In [**BL,** Sections 9 and 13], the following result is proved.

Theorem 3.45. *For any $t > 0$,*

$$\lim_{T \to +\infty} \mathrm{Tr_s}\left[P_t^T(x,x)\right] dv_X(x) = \mathrm{Tr_s}\left[Q_t(y,y)\right] dv_Y(y)\,. \tag{3.124}$$

Also, by the local index theorem given in Theorem 1.2,

$$\begin{aligned} &\lim_{t \to 0} \mathrm{Tr_s}\left[P_t^T(x,x)\right] dv_X(x) = \left\{\mathrm{Td}(TX, g^{TX})\,\mathrm{ch}(\xi, g^\xi)\right\}^{\max}\,, \\ &\lim_{t \to 0} \mathrm{Tr_s}\left[Q_t(y,y)\right] dv_Y(y) = \left\{\mathrm{Td}(TY, g^{TY})\,\mathrm{ch}(\eta, g^\eta)\right\}^{\max}\,. \end{aligned} \tag{3.125}$$

We then have the non commutative diagram

$$\begin{array}{ccc} \mathrm{Tr_s}\,[P_t^T(x,x)]dv_X(x) & \xrightarrow[t\to 0]{} & \{\mathrm{Td}(TX, g^{TX})\,\mathrm{ch}(\xi, g^\xi)\}^{\max} \\ \Big\downarrow {\scriptstyle T\to+\infty} & & \\ \mathrm{Tr_s}\,[Q_t(y,y)]dv_Y(y)\delta_Y & \xrightarrow[t\to 0]{} & \{\mathrm{Td}(TY, g^{TY})\,\mathrm{ch}(\eta, g^\eta)\}^{\max}\delta_Y \end{array} \tag{3.126}$$

Needless to say, (3.126) fits with (3.122), (3.123), because of (3.86).

Observe that

$$tA_{\frac{T}{t}} = tD^X + TV\,. \tag{3.127}$$

We use the notation in Section 3 e). A simple application of local index techniques shows that

$$\lim_{t \to 0} \mathrm{Tr_s}\left[P_t^{T/t}(x,x)\right] dv_X(x) = \left\{\mathrm{Td}(TX, g^{TX})\alpha_{T^2}\right\}^{\max}\,. \tag{3.128}$$

In view of (3.83), (3.124), (3.128), we have the new diagram

$$\begin{array}{ccc} \mathrm{Tr_s}\,[P_t^{T/t}(x,x)]dv_X(x) & \xrightarrow[t\to 0]{} & \{\mathrm{Td}(TX, g^{TX})\alpha_{T^2}\}^{\max} \\ \Bigg\downarrow {\scriptstyle T\to+\infty} & & \Big\downarrow {\scriptstyle T\to+\infty} \\ & & \left\{\frac{\mathrm{Td}(TX, g^{TX})}{\mathrm{Td}(N_{Y/X}, g^{N_{Y/X}})}\,\mathrm{ch}(\eta, g^\eta)\right\}^{\max}\delta_Y \\ \mathrm{Tr_s}\,[Q_t(y,y)]dv_Y(y)\delta_Y & \xrightarrow[t\to 0]{} & \{\mathrm{Td}(TY, g^{TY})\,\mathrm{ch}(\eta, g^\eta)\}^{\max}\delta_Y \end{array} \tag{3.129}$$

The whole point is now to find how to close the gaps in the diagrams (3.126), (3.129) at the level of currents (the gap is of course 0 in cohomology).

Clearly

$$tA_{T/t^2} = tD^X + \frac{T}{t}V\,. \tag{3.130}$$

Consider the exact sequence of holomorphic Hermitian vector bundles on Y

$$0 \to TY \to TX_{|Y} \to N_{Y/X} \to 0 \tag{3.131}$$

and more generally any short exact sequence as in (3.131). In [**B4**], for $T > 0$, a form $\delta_T \in P^Y$ is constructed, such that

$$\begin{aligned} \lim_{T\to 0} \delta_T &= \frac{\mathrm{Td}(TX_{|Y}, g^{TX_{|Y}})}{\mathrm{Td}(N_{Y/X}, g^{N_{Y/X}})}, \\ \lim_{T\to +\infty} \delta_T &= \mathrm{Td}(TY, g^{TY}). \end{aligned} \tag{3.132}$$

Then, in [**BL,** Section 12], it is shown that for $T > 0$,

$$\lim_{t\to 0} \mathrm{Tr_s}\left[P_t^{T/t^2}(x,x)\right] dv_X(x) = \{\delta_T \,\mathrm{ch}(\eta, g^\eta)\}^{\max} \delta_Y . \tag{3.133}$$

Then we have the new diagram

$$\begin{CD} \mathrm{Tr_s}\,[P_t^0(x,x)]dv_X(x) @. \frac{\mathrm{Td}(TX,g^{TX})}{\mathrm{Td}(N_{Y/X},g^{N_{Y/X}})}\,\mathrm{ch}(\eta,g^\eta)\delta_Y \\ @AA{T\to 0}A @AA{T\to 0}A \\ \mathrm{Tr_s}\,[P_t^{T/t^2}(x,x)]dv_X(x) @>>{t\to 0}> \{\delta_T\,\mathrm{ch}(\eta,g^\eta)\}^{\max}\delta_Y \\ @VV{T\to +\infty}V @VV{T\to +\infty}V \\ \mathrm{Tr_s}\,[Q_t(y,y)]dv_Y(y)\delta_Y @>>{t\to 0}> \{\mathrm{Td}(TY,g^{TY})\,\mathrm{ch}(\eta,g^\eta)\}^{\max}\delta_Y \end{CD} \tag{3.134}$$

Let P_∞^T be the orthogonal projection operator on $\ker(A_T)$. Then P_∞^T is given by a smooth kernel $P_\infty^T(x,x')$ on X. Similarly let Q_∞ be the orthogonal projection operator on $\ker(D_Y)$, and let $Q_\infty(y,y')$ be the corresponding kernel on Y.

Then by [**BL,** Section 10], we have the diagram

$$\begin{CD} \mathrm{Tr_s}\,[P_t^T(x,x)]dv_X(x) @>>{t\to +\infty}> \mathrm{Tr_s}\,[P_\infty^T(x,x)]dv_X(x) \\ @VV{T\to +\infty}V @VV{T\to +\infty}V \\ \mathrm{Tr_s}\,[Q_t(y,y)]dv_Y(y)\delta_Y @>>{t\to +\infty}> \mathrm{Tr_s}\,[Q_\infty(y,y)]dv_Y(y)\delta_Y \end{CD} \tag{3.135}$$

Part of the proof of Theorem 3.41 [**BL**] consists in putting together the diagrams (3.126), (3.129), (3.134) and (3.135).

j) The Hodge theory of the resolution of the point.

In the proof in **[BL]** of Theorem 3.41, or in the construction of the form δ_T in **[B4]**, the following toy model of an embedding $i : Y \to X$ appears naturally.

Let $V_{\mathbf{R}}$ be a real even dimensional vector space, let J be a complex structure on $V_{\mathbf{R}}$. Let V be the corresponding complex vector space, so that $V_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C} = V \oplus \overline{V}$. Put $n = \dim V$. Recall that if $z \in V$, z represents $Z = v + \overline{v} \in V_{\mathbf{R}}$.

Let i be the embedding $\{0\} \to V$. If $z \in V$, let i_z be the interior multiplication by z. Let

(3.136)

$$(\Lambda V^*, \sqrt{-1} i_z) : 0 \to \Lambda^n V^* \xrightarrow[\sqrt{-1}i_z]{} \Lambda^{n-1} V^* \to \dots \xrightarrow[\sqrt{-1}i_z]{} \Lambda^0(V^*) = \mathbf{C} \to 0$$

be the obvious Koszul complex. Let r be the restriction map $\alpha \in \Lambda^0(V^*)_{|0} \to \alpha \in \mathbf{C}$. Then by **[GrH,** p. 688], the complex $(\Lambda V^*, \sqrt{-1} i_z)$ provides a resolution of the sheaf $i_* \mathbf{C}$.

Let $(C^\infty(V, \Lambda \overline{V}^* \widehat{\otimes} \Lambda V^*), \overline{\partial}^V)$ be the Dolbeault complex over V of smooth sections of $\Lambda \overline{V}^* \widehat{\otimes} \Lambda V^*$.

Let N_V, N_H be the number operator of $\Lambda \overline{V}^*$, ΛV^*. The $\mathbf{Z}$-grading of the complex $(C^\infty(V, \Lambda \overline{V}^* \otimes \Lambda V^*), \overline{\partial}^V + \sqrt{-1} i_y)$ is given by $N_V - N_H$.

Let $r : C^\infty(V, \Lambda \overline{V}^* \otimes \Lambda V^*) \to \mathbf{C}$ be such that if $\alpha \in C^\infty(V, \Lambda^p(\overline{V}^*))$, $\beta \in C^\infty(V, \Lambda^q(V^*))$, then

$$\begin{aligned} r(\alpha \widehat{\otimes} \beta) &= 0 \text{ if } p + q > 0\,, \\ &= \alpha_0 \beta_0 \text{ if } p = 0, q = 0\,. \end{aligned} \tag{3.137}$$

Observe that $\Lambda V_R^* \otimes_{\mathbf{R}} \mathbf{C} = \Lambda \overline{V}^* \otimes \Lambda V^*$, and that r is nothing else than the restriction of a smooth form on V to $\{0\}$.

Let $\mathbf{C}$ be the trivial complex, equipped with the chain map $\overline{\partial}^{\{0\}} = 0$. By the arguments of **[BL,** Theorem 1.7], the chain map

$$r : (C^\infty(V, \Lambda \overline{V}^* \widehat{\otimes} \Lambda V^*), \overline{\partial}^V + \sqrt{-1} i_z) \to (\mathbf{C}, \overline{\partial}^{\{0\}}) \tag{3.138}$$

is a quasi-isomorphism. In particular if $\mathcal{H}$ is the hypercohomology of $(C^\infty(V, \Lambda \overline{V}^* \widehat{\otimes} \Lambda V^*), \overline{\partial}^V + \sqrt{-1} i_z)$,

$$\begin{aligned} \mathcal{H}^p &= 0 \text{ if } p \neq 0\,, \\ &= \mathbf{C} \text{ if } p = 0\,, \end{aligned} \tag{3.139}$$

and $r = \mathcal{H} \to \mathbf{C}$ identify canonically $\mathcal{H}$ with $\mathbf{C} = H^0(\{0\}, \mathbf{C})$.

Let now g^V be a Hermitian metric on V. Let $\langle\, , \rangle_{\Lambda(\overline{V}^*) \widehat{\otimes} \Lambda(V^*)}$ be the corresponding Hermitian product on $\Lambda(\overline{V}^*) \widehat{\otimes} \Lambda(V^*)$ and let dv_V be the associated volume form on V. Then we equip $C_c^\infty(V, \Lambda \overline{V}^* \otimes \Lambda V^*)$ with the Hermitian product

$$\langle s, s' \rangle = \left(\frac{1}{2\pi}\right)^{\dim V} \int_{V_R} \langle \alpha, \beta \rangle_{\Lambda(\overline{V}^*) \widehat{\otimes} \Lambda(V^*)}\, dv_V\,. \tag{3.140}$$

Let $\overline{\partial}^{V^*}$ be the formal adjoint of $\overline{\partial}^V$. Then $\overline{\partial}^{V^*} - \sqrt{-1}i_z^*$ is the formal adjoint of $\overline{\partial}^V + \sqrt{-1}i_z$.

Let θ be the Kähler form of $V_{\mathbf{R}}$, i.e.

$$\theta(X,Y) = \langle X, JY\rangle_{V_{\mathbf{R}}} . \tag{3.141}$$

Let $L = \theta\wedge$, and let Λ be the adjoint of L. Put

$$S = -(L + \Lambda) . \tag{3.142}$$

By [**B4,** Proposition 1.4],

$$(\overline{\partial}^V + \sqrt{-1}i_z + \overline{\partial}^{V^*} - \sqrt{-1}i_z^*)^2 = -\frac{\Delta^V}{2} + \frac{|Z|^2}{2} + S . \tag{3.143}$$

Then the Laplacian (3.143) is an harmonic oscillator.

The following elementary result is proved in [**B4,** Theorem 1.6].

Theorem 3.46. *Let $\beta \in C^\infty(V, \Lambda(\overline{V}^*)\widehat{\otimes}\Lambda(V^*))$ be given by*

$$\beta = \exp\left(\theta - \frac{|Z|^2}{2}\right) . \tag{3.144}$$

Then β has total degree 0, and moreover

$$\|\beta\|_{L_2} = 1 . \tag{3.145}$$

Also

$$\begin{aligned} (\overline{\partial}^V + \sqrt{-1}i_z)\beta &= 0 , \\ (\overline{\partial}^{V^*} - \sqrt{-1}i_z^*)\beta &= 0 . \end{aligned} \tag{3.146}$$

Moreover β spans the 1-dimensional kernel of $(\overline{\partial}^V + \sqrt{-1}i_z + \overline{\partial}^{V^*} - \sqrt{-1}i_z^*)^2$. Finally

$$r\beta = 1 \tag{3.147}$$

i.e. β represents canonically $1 \in H^0(\{0\}, \mathbf{C})$ in $C^\infty(V, \Lambda(\overline{V}^*)\widehat{\otimes}\Lambda(V^*))$.

REMARK 3.47. Several remarks are in order here. First note that θ, as a $(1,1)$ form, has total degree 0, so that indeed β is of total degree 0. Also observe that $\|\beta\|_{L_2} = 1$ and $r\beta = 1$, so that $1 \to \beta$ is an isometry.

Now we go back to the formalism of Section 3 h).

Let $\pi' : N_{Y/X} \to Y$ be the obvious projection. Consider the embedding $i' : Y \to N_{Y/X}$ (where Y is identified with the zero section of $N_{Y/X}$).

A fundamental fact, established in [**BL,** Section 10] is that for $T \to +\infty$, $\ker(A_T) \simeq H^\bullet(E, \overline{\partial}^X + Tv) \simeq H^\bullet(F, \overline{\partial}^Y)$ can be asymptotically described as follows. Put

$$D^Y = \overline{\partial}^Y + \overline{\partial}^{Y*} . \tag{3.148}$$

Then

$$\ker(D^Y) \simeq H(Y, \eta) . \tag{3.149}$$

Take $\alpha \in \ker(D^Y) \simeq H(Y, \eta)$. Then by [**BL,** Section 10], the element $\gamma_T \in \ker(A_T) \simeq H(X, \xi)$ canonically identified with α can be described asymptotically in a tubular neighborhood of Y by

$$\gamma_T \simeq \pi'^* \beta \exp\left(\theta_{N_{Y/X}} - \frac{T\,|Z|^2}{2}\right) . \tag{3.150}$$

Of course γ_T can be viewed locally as a smooth section of $\Lambda(T^{*(0,1)}X)\widehat{\otimes}\xi$ because of (3.80).

k) The forms δ_T : A toy model for the analytic torsion forms.

Let Y be a complex manifold. Let

$$0 \to L \underset{i}{\to} M \underset{j}{\to} N \to 0 \tag{3.151}$$

be a short exact sequence of holomorphic vector bundle on Y. Let g^M be a Hermitian metric on M, let g^L be the induced metric on L. By identifying N to $L^\perp$, let g^N be the metric induced by g^M on N.

Let $\mathcal{L}, \mathcal{M}$ be the total spaces of L, M. Then we have the diagram

$$\begin{tikzcd} L \arrow[r] \arrow[d, "i"] & \mathcal{L} \arrow[d, "i"] \arrow[dr, "\pi_{\mathcal{L}/S}"] & \\ M \arrow[r] & \mathcal{M} \arrow[r, "\pi_{\mathcal{M}/S}"'] & S \end{tikzcd} \tag{3.152}$$

Also on $\mathcal{M}$, the Koszul complex $\left(\pi_M^* \Lambda N^*, \sqrt{-1} i_{j(z)}\right)$ is a resolution of the constant sheaf $i_* \mathbf{C}$.

Let $\nabla^L, \nabla^M, \nabla^N$ be the holomorphic Hermitian connections on L, M, N, and let R^M, R^L, R^N be their curvatures. Then the connections ∇^L, ∇^M define horizontal subbundles $T^H\mathcal{L}$, $T^H\mathcal{M}$. Put

$$\begin{aligned} \omega^{\mathcal{L}} &= \frac{i\overline{\partial}\partial\,|z|_L^2}{2} , \\ \omega^{\mathcal{M}} &= \frac{i\overline{\partial}\partial\,|z|_M^2}{2} . \end{aligned} \tag{3.153}$$

Clearly

$$\omega^{\mathcal{L}} = i^* \omega^{\mathcal{M}} . \tag{3.154}$$

Also $\omega^{\mathcal{L}}, \omega^{\mathcal{M}}$ induce the tautological Kähler forms along the fibres L, M. Moreover one verifies easily that $T^H\mathcal{L}, T^H\mathcal{M}$ are exactly the orthogonal bundles to L, M with respect to $\omega^{\mathcal{L}}, \omega^{\mathcal{M}}$.

Then we are in a situation formally similar to the one we met in Section 3 g). We will then construct the associated Levi-Civita superconnection. In our context, if $U, V \in T_{\mathbf{R}}S$,

$$\begin{aligned} T^{\mathcal{L}}(U^H, V^H) &= R^L(U,V)Z , \\ T^{\mathcal{M}}(U^H, V^H) &= R^M(U,V)Z . \end{aligned} \tag{3.155}$$

Let E, F be the bundles on Y of smooth sections of $\Lambda(\overline{M}^*)\widehat{\otimes}\Lambda(N^*), \Lambda(\overline{N}^*)$ along the fibres M, L. Let ∇^E, ∇^F be the connections on E, F constructed as in Definition 3.11.

DEFINITION 3.48. For $T > 0$, let $\mathcal{B}_T^{\mathcal{M}}$ be the Levi-Civita superconnection on E,

(3.156)

$$\mathcal{B}_T^M = \left(\overline{\partial}^M + \overline{\partial}^{M*}\right) + \sqrt{T}\left(\sqrt{-1}i_{j(z)} + (\sqrt{-1}i_{j(z)})^*\right) + \nabla^E - \frac{c(R^M Z)}{2\sqrt{2}} ,$$

Similarly by making $T = 0$ in (3.156), we can construct the superconnection $\mathcal{B}^N$.

Recall that $\Lambda(\overline{M}^*), \Lambda(N^*)$ are $c(M_R), c(N_R)$ Clifford modules. Let $c, \widehat{c}$ denote the corresponding Clifford actions.

We have the C^∞ identification $M = L \oplus N$. Put

$$A = \nabla^M - \nabla^L \oplus \nabla^N . \tag{3.157}$$

Then A exchanges $L_{\mathbf{R}}$ and $N_{\mathbf{R}}$.

Let $P^L : M \to L, P^N : M \to N$ be the orthogonal projection operators. If (f_α) is a basis of $T_{\mathbf{R}}S$, put

$$\widehat{c}(AP^L Z) = -\Sigma f^\alpha \widehat{c}(A(f_\alpha)P^L Z) . \tag{3.158}$$

Let $e_1, \dots, e_{2n}$ be an orthonormal basis of $N_{\mathbf{R}}$. Put

$$S = \frac{\sqrt{-1}}{2} \sum_1^{2n} c(e_i)\widehat{c}(e_i) . \tag{3.159}$$

Then by [**B4,** Theorem 3.10] (or by the more general curvature identity of [**B2,** Theorem 3.5], if $e_1, \dots, e_{2m}$ is an orthonormal basis of $M_{\mathbf{R}}$,

$$\begin{aligned} \mathcal{B}_T^{\mathcal{M},2} = &-\tfrac{1}{2}\sum_1^{2m} \left(\nabla_{e_i} + \tfrac{1}{2}\langle R^M Z, e_i\rangle\right)^2 \\ &+ \frac{T\left|P^N Z\right|^2}{2} + \sqrt{T}S + \frac{\sqrt{-T}}{\sqrt{2}}\widehat{c}(AP^L Z) \\ &+ \tfrac{1}{2}\operatorname{Tr}\left[R^M\right] + R^{\Lambda(N^*)} . \end{aligned} \tag{3.160}$$

In particular,

$$(3.161) \qquad \mathcal{B}_0^{\mathcal{M},2} = -\tfrac{1}{2}\sum_1^m \left(\nabla_{e_i} + \tfrac{1}{2}\left\langle R^M Z, e_i\right\rangle\right)^2 + \tfrac{1}{2}\operatorname{Tr}\left[R^M\right] + R^{\Lambda(N^*)}.$$

With some surprise, we see that $\mathcal{B}_0^{\mathcal{M},2}$ is nothing else than the Getzler operator **[Ge1,2]** [**BeGeV,** Proposition 4.19] in local index theory. The fact that the Getzler operator (obtained by rescaling the square of the Dirac operator) is itself a square is a surprise. Another surprising feature of $\mathcal{B}_T^{\mathcal{M},2}$ is that its matrix part only acts on $\Lambda(\overline{N}^*)\widehat{\otimes}\Lambda(N^*)$ and not on $\Lambda(\overline{M}^*)\widehat{\otimes}\Lambda(N^*)$.

For $T > 0$, $\mathcal{B}_T^2$ is essentially a perturbation of the harmonic oscillator in (3.143). For $T > 0$, let $S_T(Z,Z')$ $(Z, Z' \in M_{\mathbf{R}})$ be the smooth kernel of the operator $\exp\left(-\mathcal{B}_T^{\mathcal{M},2}\right)$ with respect to $\dfrac{dv_M(Z')}{(2\pi)^{\dim M}}$. Then $S_T(Z,Z') \in \Lambda(T_{\mathbf{R}}^*Y)\widehat{\otimes} \operatorname{End}(\Lambda(\overline{N}^*)\widehat{\otimes}\Lambda(N^*))$. Then a simple fact proved in [**B4,** Theorem 4.2] is that $Z \to \operatorname{Tr_s}[S_T(Z,Z)]$ only depends on $j(Z) \in N_{\mathbf{R}}$. Also one verifies easily that if $Z \in N_{\mathbf{R}}$,

$$(3.162) \qquad |S_T(Z,Z)| \le c(T)\exp(-C(T)\,|Z|^2)\,.$$

The operator $\exp(-\mathcal{B}_T^2)$ is in general not trace class. Still we can define a generalized supertrace as follows.

DEFINITION 3.49. Set

$$(3.163) \qquad \operatorname{Tr_s}\left[\exp\left(-\mathcal{B}_T^{\mathcal{M},2}\right)\right] = \int_{N_{\mathbf{R}}} \operatorname{Tr_s}\left[S_T(Z,Z)\right]\frac{dv_N(Z)}{(2\pi)^{\dim N}}\,.$$

Put

$$(3.164) \qquad \delta_T = \Phi \operatorname{Tr_s}\left[\exp\left(-\mathcal{B}_T^{\mathcal{M},2}\right)\right]\,.$$

Then we have the result of [**B4,** Theorems 4.8 and 7.7].

Theorem 3.50. *The forms δ_T lie in P^B, they are closed, and their cohomology class does not depend on T. Moreover as $T \to 0$,*

$$(3.165) \qquad \delta_T = \frac{\operatorname{Td}(M,g^M)}{\operatorname{Td}(N,g^N)} + \mathcal{O}(T)\,,$$

and as $T \to +\infty$

$$(3.166) \qquad \delta_T = \operatorname{Td}(L,g^L) + \mathcal{O}\left(\frac{1}{\sqrt{T}}\right)\,.$$

Let N_H be the number operator of $\Lambda(N^*)$. Set

$$(3.167) \qquad \varepsilon_T = \Phi \operatorname{Tr_s}\left[N_H \exp(-\mathcal{B}_T^2)\right]$$

(where again, the right-hand side of (3.167) is a generalized supertrace).
By [**B4**, Theorem 4.6]

$$\frac{\partial}{\partial T}\delta_T = \frac{\overline{\partial}\partial}{2i\pi}\frac{\varepsilon_T}{T}. \tag{3.168}$$

Also by [**B4**, Theorem 7.7], as $T \to +\infty$,

$$\varepsilon_T = \frac{\dim N}{2}\operatorname{Td}(L, g^L) + \mathcal{O}\left(\frac{1}{\sqrt{T}}\right). \tag{3.169}$$

DEFINITION 3.51. For $s \in \mathbf{C}, 0 < \operatorname{Re}(s) < 1/2$, put

$$\mathbf{B}(s) = \frac{1}{\Gamma(s)}\int_0^{+\infty} T^{s-1}(\varepsilon_T - \varepsilon_\infty)dT. \tag{3.170}$$

Then $\mathbf{B}(s)$ extends holomorphically near $s = 0$.

DEFINITION 3.52. Put

$$\mathbf{B}(L, M, g^M) = \frac{\partial}{\partial s}\mathbf{B}(s)_{|s=0}. \tag{3.171}$$

Clearly $\mathbf{B}(L, M, g^M)$ is a generalized analytic torsion form on Y.

The following result is established in [**B4**, Theorem 8.3].

Theorem 3.53. *The form* $\mathbf{B}(L, M, g^M)$ *lies in* P^Y. *Moreover*

$$\frac{\overline{\partial}\partial}{2i\pi}\mathbf{B}(L, M, g^M) = \operatorname{Td}(L, g^L) - \frac{\operatorname{Td}(M, g^M)}{\operatorname{Td}(N, g^N)}. \tag{3.172}$$

The construction of $\mathbf{B}(L, M, g^M)$ is functorial. In view of (3.172), a natural question is to evaluate

$$\mathbf{B}(L, M, g^M) + \operatorname{Td}^{-1}(N, g^N)\widetilde{\operatorname{Td}}(L, M, g^M). \tag{3.173}$$

in $P^Y/P^{Y,0}$. Using equation (3.172), it is enough to calculate (3.173) in a split situation.

DEFINITION 3.54. Let $D(x)$ be the formal power series

$$D(x) = \sum_{\substack{n\geq 1\\ n \text{ odd}}}\left(\Gamma'(1) + \frac{2\zeta'(-n)}{\zeta(-n)} + \sum_1^n \frac{1}{j}\right)\zeta(-n)\frac{x^n}{n!}. \tag{3.174}$$

We identify $D(x)$ to the corresponding additive genus. Let $\mathrm{Td}(L), D(N)$ be the classes of $\mathrm{Td}(L, g^L)$, $\mathrm{Td}(N, g^N)$ in $P^Y/P^{Y,0}$. Clearly they do not depend on g^L, g^N.

Then we have the result of [**B4**, Theorem 8.5].

Theorem 3.55. *The following identity holds*

(3.175)

$$\mathbf{B}(L, M, g^M) = -\,\mathrm{Td}^{-1}(N, g^N)\widetilde{\mathrm{Td}}(L, M, g^M) + \mathrm{Td}(L)D(N) \text{ in } P^Y/P^{Y,0}.$$

PROOF : As explained in (3.173), it is enough to evaluate $\mathbf{B}(L, M, g^M)$ in the case where the exact sequence splits holomorphically and metrically.

Let $\varphi(T, x)$ be the function

$$\varphi(T,x) = \frac{4}{T}\sinh\left(\frac{x+\sqrt{x^2+4T}}{4}\right)\sinh\left(\frac{-x+\sqrt{x^2+4T}}{4}\right). \tag{3.176}$$

□

We identify $\frac{\partial\varphi/\partial x}{\varphi}(T, x)$, as a function of x with the corresponding additive genus. Then in [**B4**, eq. (8.26)-(8.28)], it is shown by explicit computation that in the split case, for $T > 0$

$$\varepsilon_T - \varepsilon_\infty = -\,\mathrm{Td}(L, g^L)\frac{\partial\varphi/\partial x}{\varphi}(T, N, g^N). \tag{3.177}$$

Set

$$\begin{aligned} C(s,x) &= -\frac{1}{\Gamma(s)}\int_0^{+\infty} T^{s-1}\frac{\partial\varphi/\partial x}{\varphi}(T,x)\,dT, \\ D(x) &= \frac{\partial C}{\partial s}(0,x). \end{aligned} \tag{3.178}$$

We identify $D(x)$ with the corresponding additive genus. Then by (3.170), (3.173), (3.178), when (3.151) splits,

$$\mathbf{B}(L, M, g^M) = \mathrm{Td}(L, g^L)D(N, g^N). \tag{3.179}$$

Now we have the easy expressions for $\varphi(T, x)$

$$\varphi(T,x) = \prod_{k=1}^{+\infty}\left(1+\frac{ix}{2k\pi}+\frac{T}{4k^2\pi^2}\right)\left(1-\frac{ix}{2k\pi}+\frac{T}{4k^2\pi^2}\right), \tag{3.180}$$

which makes the computation of $\frac{\partial\varphi/\partial x}{\varphi}(T, x)$ quite pleasing.

In [**B4**, Appendix], Bismut and Soulé obtain the expression of $D(x)$ given in (3.174) by using (3.180) and the functional equation for $\zeta(s)$.

By (3.97), (3.174),

$$D(x) = R(x) + \Gamma'(1)\frac{\widehat{A}'}{\widehat{A}}(x).$$

In [**BL**], the term related to $\Gamma'(1)\frac{\widehat{A}'}{\widehat{A}}(x)$ disappears in the final result, because it is killed by a corresponding term in $T(\xi, g^\xi)$.

REMARK 3.56. In [**GS3**], Gillet and Soulé have obtained the genus R by evaluating the analytic torsion of $\mathbf{P}_n(\mathbf{C})$ equipped with the Fubini study metric, and by calculating the degree over $\mathbf{P}_n(\mathbf{Z})$ of their Todd genus $\widehat{\mathrm{Td}}$. They obtained the R-genus as a defect in their conjectured Riemann-Roch formula in Arakelov theory.

It has in fact been made clear in the work of Bost [**Bos**] and Roessler [**Ro**] that the evaluation of the analytic torsion of $\mathbf{P}_n(\mathbf{C})$ can be obtained as a consequence of [**BL**]. Using the results of [**BGS5**], the formula of [**GS3**] for $\mathbf{P}_n(\mathbf{Z})$ can then be also obtained as a consequence of [**BL**].

The amazing almost coincidence of the genera R of [**GS3**] and D of [**B4**] is then explained.

References

[A] Atiyah, M.F. : K-Theory. New-York-Amsterdam: Benjamin 1967.

[ADS] Atiyah, M.F., Donnelly, H., Singer, I.M. : Eta invariants, signature defects of cusps and values of L functions. Ann. of Math. **18**, 131-177 (1983).

[Al] Alvarez-Gaumé, L. : Supersymmetry and the Atiyah-Singer index theorem. Comm. Math. Phys. **90**, 161-173 (1983).

[ABoP] Atiyah, M.F., Bott, R., Patodi, V.K. : On the heat equation and the index theorem. Invent. Math. **19**, 279-330 (1973).

[APS1] Atiyah, M.F., Patodi, V.K., Singer, I.M. : Spectral asymmetry and Riemannian geometry, I. Proc. Cambridge Philos. Soc. **77**, 43-69 (1975).

[APS2] Atiyah, M.F., Patodi, V.K., Singer, I.M. : Spectral asymmetry and Riemannian geometry, III. Proc. Cambridge Philos. Soc. **79**, 71-99 (1976).

[AS1] Atiyah, M.F., Singer, I.M. : The index of elliptic operators. III, Ann. of Math. **87**, 546-604 (1968).

[AS2] Atiyah, M.F., Singer, I.M. : The index of elliptic operators. IV, Ann. of Math. **93**, 119-138 (1971).

[AS3] Atiyah, M.F., Singer, I.M. : Index Theory for skew-adjoint Fredholm operators. Publ. Math. IHES, **37**, 5-26 (1969).

[BeV] Berline, M., Vergne, M. : A proof of Bismut local index theorem for a family of Dirac operators. Topology **26**, 435-463 (1987).

[BeGeV] Berline, N., Getzler E., Vergne, M. : Heat kernels and the Dirac operator. Grundl. der Math. Wiss. Band 298. Berlin-Heidelberg-New-York: Springer 1992.

[BerB] Berthomieu A., Bismut, J.-M. : Quillen metrics and higher analytic torsion forms. J. reine angew. Math. **457**, 85-184 (1994).

[B1] Bismut, J.-M. : The Atiyah-Singer index theorem : A probabilistic approach. I. The index theorem. J. Funct. Anal. **57**, 56-99 (1984).

[B2] Bismut, J.-M. : The index Theorem for families of Dirac operators : two heat equation proofs. Invent. Math. **83**, 91-151 (1986).

[B3] Bismut, J.-M. : Superconnection currents and complex immersions. Invent. Math. **99**, 59-113 (1990).

[B4] Bismut, J.-M. : Koszul complexes, harmonic oscillators and the Todd class. J.A.M.S. **3**, 159-256 (1990).

[B5] Bismut, J.-M. : Holomorphic families of immersions and higher analytic torsion forms. Astérisque **244**. Paris: SMF 1997.

[B6] Bismut, J.-M. : Familles d'immersions et formes de torsion analytique en degré supérieur. C.R. Acad. Sci. Paris, Série 1, **320**, 969-974 (1995).

[B7] Bismut, J.-M. : From Quillen metrics to Ray-Singer metrics. Some aspects of the Ray-Singer analytic torsion. In Topological methods in modern mathematics. L. Goldberg and A. Phillips eds. p. 273-324. Houston : Publish or Perish 1993.

[BC1] Bismut, J.-M., Cheeger, J. : η-invariants and their adiabatic limits. J.A.M.S. **2**, 33-70 (1989).

[BC2] Bismut, J.-M., Cheeger, J. : Families index for manifolds with boundary, superconnections and cones, I. J. F. Anal. **89**, 313-363 (1990).

[BC3] Bismut, J.-M., Cheeger, J. : Families index for manifolds with boundary, superconnections and cones, II. J. F. Anal. **90**, 306-354 (1990).

[BC4] Bismut, J.-M., Cheeger, J. : Remarks on the index theorem for families of Dirac operators on manifolds with boundary. In Differential Geometry. F. Cheeger et al. eds, 59-83, Harlow Longman (1991).

[BC5] Bismut, J.-M., Cheeger, J. : Transgressed Euler classes of $SL(2n, \mathbf{Z})$ vector bundles, adiabatic limits of eta invariants and special values of L functions. Ann. Scient. Ec. Norm. Sup. **25**, 335-391 (1992).

[BF1] Bismut, J.-M., Freed D.S. : The analysis of elliptic families I, Metrics and connections on determinants bundles. Comm. Math. Phys. **106**, 159-176 (1986).

[BF2] Bismut, J.-M., Freed D.S. : The analysis of elliptic families II, Dirac operators, eta invariants and the holonomy theorem. Comm. Math. Phys. **107**, 103-163 (1986).

[BGS1] Bismut, J.-M., Gillet, H., Soulé, C. : Analytic torsion and holomorphic determinant bundles, I. Comm. Math. Phys. **115**, 49-78 (1988).

[BGS2] Bismut, J.-M., Gillet, H., Soulé, C. : Analytic torsion and holomorphic determinant bundles, II. Comm. Math. Phys. **115**, 79-126 (1988).

[BGS3] Bismut, J.-M., Gillet, H., Soulé, C. : Analytic torsion and holomorphic determinant bundles, III. Comm. Math. Phys. **115**, 301-351 (1988).

[BGS4] Bismut, J.-M., Gillet, H., Soulé, C. : Bott-Chern currents and complex immersions. Duke Math. Journal **60**, 255-284 (1990).

[BGS5] Bismut, J.-M., Gillet, H., Soulé, C. : Complex immersions and Arakelov geometry. The Grothendieck Festschrift, P. Cartier and al. ed. Vol. I, pp. 249-331. Progress in Math. n° 86. Boston : Birkhaüser 1990.

[BK] Bismut, J.-M., Köhler, K. : Higher analytic torsion forms for direct images and anomaly formulas, J. Alg. Geom. **1**, 647-684 (1992).

[BL] Bismut, J.-M., Lebeau, G. : Complex immersions and Quillen metrics, Publ. Math. IHES, **74**, 1-297 (1991).

[Bos] Bost, J.-M. : Analytic torsion of projective spaces and compatibility with immersions of Quillen metrics. I.M.R.N. **8**, 427-435 (1998).

[BoCh] Bott, R., Chern, S.S. : Hermitian vector bundles and the equidistribution of the zeroes of their holomorphic sections. Acta Math. **114**, 71-112 (1965).

[C1] Cheeger, J. : On the spectral geometry of spaces with cone-like singularities. Proc. Math. Acad. Sci. USA **76**, 2103-2166 (1979).

[C2] Cheeger, J. : Spectral geometry of singular Riemannian spaces. J. Diff. Geom. **18**, 575-657 (1983).

[C3] Cheeger, J. : On the Hodge theory of Riemannian pseudomanifolds. In Proceedings of Symposia in Pure Math., vol. 36, 91-146, Am. Math. Soc. Providence R.I. 1980.

[C4] Cheeger, J. : η-invariants, the adiabatic approximation and conical singularities. J. Diff. Geom. **26**, 175-221 (1987).

[CSi] Cheeger, J., Simons, J. : Differential characters and geometric invariants. In Geometry and Topology, J. Alexander and J. Harer ed., Lect. Notes in Math. n° 1167, p. 50-80. Berlin-Heidelberg-New-York: Springer 1985.

[ChS] Chern, S.S., Simons, J. : Characteristic forms and geometric invariants. Ann. of Math. **99**, 48-69 (1974).

[Cho] Chou, A.W. : The Dirac operator on spaces with conical singularities and positive scalar curvature. Trans. Am. Math. Soc. **289**, 1-40 (1985).

[D] Dai, X. : Adiabatic limits, non multiplicativity of signature and Leray spectral sequence, J.A.M.S. **4**, 265-321 (1991).

[Ge1] Getzler, E. : Pseudodifferential operator on supermanifolds and the Atiyah-Singer index theorem. Comm. Math. Phys. **92**, 163-178 (1983).

[Ge2] Getzler, E. : A short proof of the Atiyah-Singer Index Theorem, Topology **25**, 111-117 (1986).

[Gi1] Gilkey, P. : Curvature and the eigenvalues of the Laplacian for elliptic complexes. Adv. Math. **10**, 344-382 (1973).

[Gi2] Gilkey, P. : Invariance theory, the heat equation and the Atiyah-Singer index theorem. Washington : Publish or Perish 1984.

[GS1] Gillet, H., Soulé, C. : Arithmetic Intersection Theory. Publ. Math. IHES, **72**, 93-174 (1990).

[GS2] Gillet, H., Soulé, C. : Characteristic classes for algebraic vector bundles with Hermitian metrics. Ann. Math. I. **131**, 163-203 (1990). II. **131**, 205-238 (1990).

[GS3] Gillet, H., Soulé, C. : Analytic torsion and the arithmetic Todd genus. Topology, **30**, 21-54 (1991).

[GS4] Gillet, H., Soulé., C.: An arithmetic Riemann-Roch theorem. Inv. Math., **110**, 473-543 (1992).

[GrH] Griffiths, P., Harris, J. : Principles of algebraic geometry, Wiley, New York 1978.

[Hir] Hirzebruch, F. : Hilbert modular surfaces. Enseign. Math. 183-281 (1973).

[Hi] Hitchin, N.J. : Harmonic spinors. Adv. in Math. **14**, 1-55 (1974).

[Kl] Kleiman, S. L. : The development of intersection homology theory. A century of Mathematics in America. Part II. History of Mathematics 2, 543-585. Providence : A.M.S. 1989.

[KMu] Knudsen, F.F., Mumford, D. : The projectivity of the moduli space of stable curves, I : Preliminaries on "det" and "div". Math. Scand. **39**, 19-55 (1976).

[Ma] Ma, X. : Formes de torsion analytique et familles de submersion. C.R.A.S. **324** (Série I), 205-210 (1997).

[MazMe] Mazzeo, R., Melrose, R. : The adiabatic limit, Hodge cohomology and Leray's spectral sequence of a fibration. J. Diff. Geom. **31**, 185-213 (1990).

[McKS] McKean, H., Singer, I.M. : Curvature and the eigenforms of the Laplacian. J. Diff. Geom. **1**, 43-69 (1967).

[Me] Melrose, R. : The Atiyah-Patodi-singer index theorem. Wellesley: Peters 1993.

[MeP1] Melrose, R., Piazza, P. : Families of Dirac operators, boundaries and the b-calculus. To appear in J. of Diff. Geom.

[MeP2] Melrose, R., Piazza, P. : An index theorem for families of Dirac operators on odd dimensional manifolds with boundary. To appear in J. of Diff. Geom.

[MQ] Mathai, V., Quillen, D. : Superconnections, Thom classes, and equivariant differential forms. Topology **25**, 85-110 (1986).

[Mü1] Müller, W. : Signature defects of cusps of Hilbert modular varieties and values of L series at $s = 1$. J. Diff. Geom. **20**, 55-119 (1984).

[Mü2] Müller, W. : Manifolds with cusps of rank 1, spectral theory and L_2 index theorem. Lecture Notes in Math. n° 1244. Berlin-Heidelberg-New-York: Springer 1987.

[P1] Patodi, V.K. : Curvature and the eigenforms of the Laplace operator. J. Diff. Geom. **5**, 233-249 (1971).

[P2] Patodi, V.K. : An analytic proof of Riemann Roch Hirzebruch theorem for Kähler manifolds. J. Diff. Geom. **5**, 251-283 (1971).

[Q1] Quillen, D. : Superconnections and the Chern character. Topology **24**, 89-95 (1985).

[Q2] Quillen, D. : Determinants of Cauchy-Riemann operators over a Riemann surface. Funct. Anal. Appl. **14**, 31-34 (1985).

[RS] Ray, D.B., Singer, I.M. : Analytic torsion for complex manifolds. Ann. of Math. **98**, 154-177 (1973).

[Ro] Roessler, D. : An Adams-Riemann-Roch theorem in Arakelov geometry. To appear in Duke Math. J. (1999).

[St] Stern, M. : L^2 index theorems on locally symmetric spaces. Inventiones Math. **96**, 231-282 (1989).

[W] Witten, E. : Global anomalies in string theory. In Proceedings of the Symposium on anomalies, Geometry and Topology of Chicago (1985). W.A. Bardeen, A.R. White Eds, pp. 61-99. Singapore : World Scientific 1985.