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Superconnection currents and complex immersions

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Summary. Let $i: M' \to M$ be an immersion of complex manifolds, and let (ξ, v) be a complex of holomorphic Hermitian vector bundles on M which provides a projective resolution of the sheaf of sections of a holomorphic vector bundle η on M'. We study the convergence as $u \to +\infty$ of a class of superconnection currents ω_u introduced by Quillen, and we calculate the limit. Microlocal estimates of the speed of convergence are also given.

Contents

The purpose of this paper is to study various properties of superconnection currents in relation with complex immersions.

Let us recall that Quillen [Q] introduced superconnections on $\mathbb{Z}_2$ graded vector bundles to produce non trivial representatives of the Chern character of a difference bundle. In fact, let $E=E_+\oplus E_-$ be a $\mathbb{Z}_2$ graded Hermitian vector bundle on a manifold B and let L be a smooth section of $\mathrm{Hom}(E_+, E_-)$ which is invertible on an open set U of M. Quillen introduced in [Q] a family of differential forms $(\omega_u)_{u\in R_+}$ which are closed, represent in cohomology the Chern character of E_+-E_- and have the following properties:

- ω_0 is a usual Chern-Weil representative of $\mathrm{ch}(E_+-E_-)$.
- As $u\to+\infty$, ω_u decays exponentially fast on U, and tends to exhibit Gaussian like concentration near the closed set ${}^cU$.

In [Q], Quillen raised tha question of describing precisely the limit – if it exists – of the current ω_u as $u\to+\infty$.

In [MaQ], Mathai and Quillen applied Quillen's formalism to the case of a manifold M considered as the zero section of a Hermitian vector bundle F. If $y\in F$ acts by Clifford multiplication on the spinors of F, then M is exactly the set where this action is non invertible. In this case, Mathai and Quillen [MaQ, Theorem 4.5] produced various Gaussian shaped differential forms on the total space of F to represent canonically Thom forms on F.

In [B1], we used superconnections in an infinite dimensional context to give a heat equation proof of the families Index Theorem of Atiyah-Singer [AS].

In their proof of the families index Theorem, Berline and Vergne [BeV2] considered the case where the linear map $L\in\mathrm{Hom}(E_+, E_-)$ has a kernel and a cokernel which are vector bundles. They proved that as $u\to+\infty$, the Quillen forms ω_u converge to Chern character forms in Chern-Weil theory for the difference of the kernel and the cokernel.

In this paper, we consider an immersion of complex manifolds $i\colon M'\to M$, a complex vector bundle η on M', and a projective resolution of the sheaf $i_*\mathcal{O}_{M'}(\eta)$ by complex vector bundles $\xi_0, \dots, \xi_m$ on M, so that we have an exact sequence of sheaves

$$\mathcal{O}\to\mathcal{O}_M(\xi_m)\underset{v}{\longrightarrow}\dots\mathcal{O}_M(\xi_0)\underset{r}{\longrightarrow} i_*\mathcal{O}_{M'}(\eta)\to 0.$$

We equip $\xi_0, \dots, \xi_m$ with Hermitian metrics. We denote by v^* the adjoint of the chain map v, and we apply Quillen's superconnection formalism to the $\mathbb{Z}$ graded vector bundle $\xi=\bigoplus_0^m \xi_k$ equipped with the map $V=v+v^*$. We then prove in Theorem 3.2 that as $u\to+\infty$, the Quillen's superconnection currents ω_u have a limit ω_∞, which is a current concentrated on M', and we calculate the limit in terms of integrals of Gaussian shaped differential forms on the normal bundle N to M'. When the metrics on $\xi_0, \dots, \xi_m$ verify an assumption of compatibility with metrics on N and η – this is assumption (A) in Sect. 1c) – the localized forms are exactly the forms of Mathai and Quillen [MaQ]. In this case, the limit current ω_∞ can be explicitly calculated using Chern-Weil representatives of $Td^{-1}(N)\, ch(\eta)$. Part of the calculation is based on "extraordinary cancellations", which are a finite dimensional counterpart to the well known infinite dimensional analogue in index theory.

We here make several comments:

– There is an obvious C^∞ analogue of our result. Here, the local uniqueness of resolutions (see Serre [S, IV, Appendix 1], Eilenberg [E, Theorem 8]) makes that the complex (ξ, v) does not degenerate too fast near M'. In a C^∞ context, this should have to be introduced as a supplementary hypothesis.

– In earlier joint work with Gillet and Soulé [BGS1], we studied in detail Bott-Chern forms [BoC] associated with acyclic complexes of Hermitian vector bundles. In this case M' is empty, and ω_∞ vanishes. In [BGS1], we constructed a form $T(\xi)$ which is a solution of the equation.

$$\bar{\partial}\partial T(\xi) = -\omega_0 \tag{0.1}$$

and we proved that $T(\xi)$ was the "unique" solution of (0.1) in a restricted class of objects.

The obvious analogue of (0.1) is in general

$$\bar{\partial}\partial T(\xi) = \omega_\infty - \omega_0 . \tag{0.2}$$

In later work with Gillet and Soulé [BGS4], we will construct a current $T(\xi)$ which solves Eq. (0.2) by using the superconnection formalism. As in [BGS1], this current is obtained by a zêta function construction. To do this construction, we need to have a precise control of the speed of convergence of ω_u to ω_∞ (and we also study in this relation more general currents), the norm of $\omega_\infty - \omega_u$ (in the adequate Sobolev space) having to be dominated by $\frac{C}{\sqrt{u}}$. This is why the control of the speed of convergence plays such an important role.

– For later applications to intersection theory in a related joint work with Gillet and Soulé [BGS5], the wave front set of the considered currents plays an important role. Currents with a given wave front set have a natural topology (see Hörmander [H, Chap. VIII]). In this work, we prove that ω_u converges to ω_∞ in the class of currents whose wave front set is included in $N^*_{\mathbf{R}}$, and we also study related currents in the same way.

Our paper is organized as follows. In Sect. 1, we introduce the complex (ξ, v) on M, the vector bundle η on M', and we identify the homology of the complex $(\xi, v)_{|M'}$ with the Koszul complex $\Lambda N^* \otimes \eta$. Assumption (A) is also described.

In Sect. 2, we give a unified approach to the double transgression formulas of Bott-Chern [BoC], Donaldson [D] and [BGS1]. We essentially prove that the formulas of Bott-Chern [BoC] have an analogue for superconnections. From such formulas, we derive in particular the formulas of [BGS1], where Bott-Chern forms associated with acyclic complexes were constructed.

In Sect. 3, we prove our main result concerning the convergence and the speed of convergence of the currents ω_u. In Sect. 4, we establish similar results for other currents which were considered in Sect. 2. The results of Sect. 4 will be of essential importance in [BGS4, 5] to construct Bott-Chern singular currents.

In Sect. 5, we restrict the complex (ξ, v) considered in the previous sections to a submanifold M_1, which may have non transversal intersection with the manifold M'. In this more general case, under adequate assumptions on the various metrics, the explicit computation of the limit current uses a fundamental property of the Berezinian in supergeometry [M, p. 166].

Part of this paper is quite technical since we need precise estimates to control speeds of convergence, wave front sets, or more simply because we have to use dominated convergence. Also rather complicate algebraic identities are proved in the course of the proofs, which account for the simplicity of the final answers.

The results contained in this paper were anounced in [B2].

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I. Complex immersions and resolutions

In this Section, we introduce our basic datas which are:

- An immersion of complex manifolds $M' \to M$.
- A holomorphic vector bundle η on M'.
- A projective resolution of the sheaf of holomorphic sections of η by a chain complex of holomorphic vector bundles (ξ, v) on M.
- A submersion of complex manifolds $\pi\colon M \to B$ which restricts to a submersion $\pi\colon M' \to B$.

This last data is useful since we ultimately deal with integrals along the fibers of π. However the reader way well assume that B is reduced to a point.

Also in Sect. 1c, we introduce assumption (A), which is a compatibility assumption for metrics on the vector bundles ξ, η and on the normal bundle N to M' in M.

This Section is organized as follows. In (a) we introduce our main assumptions and notations. In (b), we relate the homology of $(\xi, v)_{|M'}$ to the Koszul complex of N. In (c), we introduce Hermitian metrics on our vector bundles. Finally in (d), we describe the holomorphic Hermitian connection on the homology of the complex $(\xi, v)_{|M'}$.

a. Assumptions and notations

Let $l, l', l_1, \ldots, l_n$ be nonnegative integers such that $l_j \leqq l$, $j = 1, \ldots, n$.

Let M be a compact connected complex manifold of complex dimension $l + l'$. Let $M' = \bigcup_1^n M'_j$ be a finite union of compact connected complex submanifolds of M such that if $j \neq j'$, $M'_j \cap M'_{j'} = \emptyset$. For $1 \leqq j \leqq n$, we assume that M'_j has complex dimension $l_j + l'$. Let i be the embedding $M' \to M$.

Let B be a compact connected complex manifold of complex dimension l'. Let $\pi: M \to B$ be a holomorphic submersion from M on B, whose fibers Z are compact connected complex submanifolds of M. The fibers Z have complex dimension l.

We assume that for $1 \leqq j \leqq n$, the restriction of π to M'_j is a holomorphic submersion from M'_j on B, whose fibers Y_j are compact connected complex submanifolds of M'_j. For $1 \leqq j \leqq n$, the fibers Y_j have complex dimension l_j.

Set $Y = \bigcup_1^n Y_j$. Clearly

$$Y_j = Z \cap M'_j; \qquad Y = Z \cap M'.$$

We still denote by i the embedding of fibers $Y \to Z$.

In the sequel, $T_R M$ denotes the real tangent bundle to M, and TM the (1,0) part of the bundle $T_C M = T_R M \otimes_R \mathbb{C}$. When necessary, we will also use the notation $T^{(1,0)} M$ instead of TM. We define $T_R M'$, TM', $T_R B$, TB in the same way.

$T_R Z$ denotes the real tangent bundle to the fiber Z, TZ the (1,0) part of $TZ = T_R Z \otimes_R \mathbb{C}$. When necessary, we still also write $T^{(1,0)} Z$ instead of TZ. We also define $T_R Y$, TY, $T^{(1,0)} Y$ in the same way. Of course we will use the notation $T_R Y_j$, TY_j, $T^{(1,0)} Y_j$ when necessary.

For $1 \leqq j \leqq n$, we denote by $N_{R,j}$ the real normal bundle to M'_j in M, and by N_j the (1,0) part of $N_{R,j} \otimes_R \mathbb{C}$. Then N_j has complex dimension $e_j = l - l_j$. We will often write N_R, N, e instead of $N_{R,j}$, N_j, e_j. Note that the restrictions of N_R, N to one given fiber Y are exactly the real and complex normal bundles to Y in Z.

We have the exact sequences of holomorphic vector bundles

$$\begin{array}{l} 0 \to TZ \to TM \to \pi^* TB \to 0 \\ 0 \to TY \to TM' \to \pi^* TB \to 0 \\ 0 \to TM' \to TM_{|M'} \to N \to 0 \\ 0 \to TY \to TZ_{|M'} \to N \to 0. \end{array} \tag{1.1}$$

Let η be a holomorphic vector bundle on the manifold M', let η_j be the restriction of η to M'_j. Of course the dimension of η_j depends in general on j. Let

$$(\xi, v): 0 \longrightarrow \xi_m \xrightarrow[v]{} \xi_{m-1} \xrightarrow[v]{} \dots \xrightarrow[v]{} \xi_0 \longrightarrow 0 \tag{1.2}$$

be a holomorphic chain complex of holomorphic vector bundles on M. Let r be a holomorphic restriction map $\xi_{0|M'} \to \eta$.

For $0 \leqq k \leqq m$, let $\mathcal{O}_M(\xi_k)$ be the sheaf of holomorphic sections of ξ_k on M. Similarly let $\mathcal{O}_{M'}(\eta)$ be the sheaf of holomorphic sections of η on M'. In the whole paper, we assume that the sequence of sheaves

$$0 \longrightarrow \mathcal{O}_M(\xi_m) \xrightarrow[v]{} \mathcal{O}_M(\xi_{m-1}) \longrightarrow \dots \xrightarrow[v]{} \mathcal{O}_M(\xi_0) \xrightarrow[r]{} i_* \mathcal{O}_{M'}(\eta) \longrightarrow 0 \tag{1.3}$$

is exact. In particular the complex (ξ, v) is acyclic on $M \backslash M'$.

b. The Koszul complex of N and the homology of the complex $(\xi, v)_{|M'}$

For $1 \leqq j \leqq n$, let N_j^* be the vector bundle on M'_j dual to N_j. We will often write N^* instead of N_j^*. Let $\Lambda(N_j^*) = \bigoplus_1^{e_j} \Lambda^q(N_j^*)$ be the exterior algebra of N_j^*. If $y \in N_j$, let i_y be the interior multiplication operator acting on $\Lambda(N_j^*)$.

Let ρ_j be the projection $N_j \to M'_j$. If E is a vector bundle on M'_j, we will still denote by E the vector bundle $\rho_j^*(E)$ on the total space of N_j.

M'_j can be embedded in N_j as the zero section of N_j. Then the Koszul complex on N_j $(\Lambda(N_j^*) \otimes \eta_j, i_y)$ provides a resolution of η_j. Equivalently, if k_j is the embedding $M'_j \to N_j$ then we have the exact sequence of sheaves on N_j

$$0 \to \mathcal{O}_{N_j}(\Lambda^{e_j}(N_j^*) \otimes \eta_j) \xrightarrow[i_y]{} \dots \xrightarrow[i_y]{} \mathcal{O}_{N_j}(\eta_j) \to k_{j*}\mathcal{O}_{M'_j}(\eta_j) \to 0. \tag{1.4}$$

Take $x \in M'_j$. There exists holomorphic coordinates $(z^1, \dots, z^l)$ on an open neighborhood U of x in M such that $M'_j \cap U$ is represented by $(z^1 = 0, \dots, z^{e_j} = 0)$. On $M'_j \cap U$, N_j^* is spanned by the forms $dz^1, \dots, dz^{e_j}$, which extend to the whole open set U. Therefore the vector bundle N_j^* on $M'_j \cap U$ extends into a holomorphic vector bundle on U, which we note $\tilde{N}_j^*$. Then $y = \sum_1^{e_j} z^k \frac{\partial}{\partial z^k}$ is a holomorphic section of $\tilde{N}_j$ (which is now the dual of $\tilde{N}_j^*$ on U) which exactly vanishes on $M'_j \cap U$.

Also if U is small enough, the holomorphic vector bundle $\eta_{j|M'_j \cap U}$ extends into a holomorphic vector bundle on U, which we note $\tilde{\eta}_j$.

Let $(\Lambda(\tilde{N}_j^*) \otimes \tilde{\eta}_j, i_y)$ denote the corresponding $\mathbb{Z}$ graded Koszul complex on U. By the local uniqueness of resolutions [S, Chap. IV, Appendix 1], [E, Theorem 8], we know that if U is small enough, there exists a holomorphic acyclic $\mathbb{Z}$ graded chain complex (A, a) on U such that on U, we have an isomorphism of holomorphic $\mathbb{Z}$ graded chain complexes

$$(\xi, v) \simeq (\Lambda(\tilde{N}_j^*) \otimes \tilde{\eta}_j, i_y) \oplus (A, a). \tag{1.5}$$

In particular in the local description (1.5) of the complex (ξ, v), the restriction map $r: \xi_{0|M'_j} \to \eta_j$ is given by

$$r: h \oplus a_0 \in \eta_j \oplus A_0 \to h \in \eta_j. \tag{1.6}$$

As before, we will often omit the subscripts j.

Definition 1.1. For $x \in M'$, let $F_{0,x}, F_{1,x}, \dots, F_{m,x}$ denote the homology groups of the $\mathbb{Z}$ graded chain complex $(\xi, v)_x$. Set

$$F_x = \bigoplus_0^m F_{k,x}. \tag{1.7}$$

Take now $x \in M'$ and choose a holomorphic trivialization of the chain complex (ξ, v) on an open neighborhood U of x in M. If $X \in (T_R M)_x$, we can define the derivative $\partial_X v(x)$ of the map $x \to v(x)$ in the direction X. Since $v^2=0$, we find that

$$\partial_X v v + v \partial_X v = 0. \tag{1.8}$$

From (1.8), we deduce that $\partial_X v$ acts naturally on the vector space F_x. If $\partial'_X v(x)$ is the derivative of v in the direction X with respect to another holomorphic trivialization of ξ, there exists $A_x(X)$ acting linearly on $\xi_{0,x}, \ldots, \xi_{m,x}$ such that

$$\partial'_X v(x) = \partial_X v(x) + [A_x(X), v(x)]. \tag{1.9}$$

From (1.9) we deduce that $\partial_X v(x)$ and $\partial'_X v(x)$ act in the same way on F_x.

We now denote by $\partial_X v(x)$ the action of any of the $\partial'_X v(x)$ on F_x, which is unambiguously defined by (1.9). In particular $\partial_X v(x)$ decreases by one the grading in F_x. Also since v is a holomorphic section of $\mathrm{End}(\xi)$, if $X \in T_x^{(0,1)} M$, then $\partial_X v(x) = 0$.

Theorem 1.2. *The vector spaces $F_{0,x}, \ldots, F_{m,x}$ are the fibers of smooth vector bundles $F_0, \ldots, F_m$, which inherit a canonical holomorphic structure from the holomorphic vector bundles $\xi_0, \ldots, \xi_m$.*

For any $x \in M'_j$, $X \in (T_{\mathbb{R}} M'_j)_x$, $\partial_X v(x) = 0$, so that the linear map $y \in N_{j,x} \to \partial_y v(x) \in \mathrm{End}\, F_x$ is well defined, and depends smoothly on x, y. Also for any $x \in M'_j$, $y \in N_{j,x}$

$$(\partial_y v(x))^2 = 0. \tag{1.10}$$

On the normal bundle N_j, the complex

$$(F, \partial_y v)\colon 0 \to F_m \xrightarrow[\partial_y v]{} F_{m-1} \cdots \xrightarrow[\partial_y v]{} F_0 \to 0$$

is a holomorphic $\mathbb{Z}$ graded chain complex, which is canonically isomorphic to the holomorphic $\mathbb{Z}$ graded Koszul complex $(\Lambda(N_j^*) \otimes \eta_j, i_y)$.

Proof. Take $x \in M'_j$, and let U be an open neighborhood of x in M taken as in (1.5). We also use the notations of (1.5). Since the complex (A, a) is acyclic, for any $x' \in M'_j \cap U$, we have the non canonical isomorphism

$$F_{k,x'} \simeq (\Lambda^k N_j^* \otimes \eta_j)_{x'}; \qquad 0 \leqq k \leqq m. \tag{1.11}$$

From (1.11), we deduce in particular that the vector spaces $\{F_{k,x'}\}_{0 \leqq k \leqq m}$ have constant dimension on M'_j. Since the F_k's are the homology groups of the holomorphic chain complex $(\xi, v)_{|M'_j}$ and have constant dimension, they are the fibers of smooth holomorphic vector bundles on M'_j. Tautologically, on $M'_j \cap U$, the identification (1.11) is an identification of smooth holomorphic vector bundles.

Using the coordinate system $(z^1, \ldots, z^l)$ on the open set U, we find that on $M'_j \cap U$, TM splits holomorphically into $TM = TM'_j \oplus N_j$. If $X \in TM$, and if X' is the component of X in N_j, we deduce from (1.5) that under the identification (1.11), $\partial_X v$ corresponds to $i_{X'}$. Therefore if $X \in TM'_j$, then $\partial_X v = 0$. So the map $y \in N_j \to \partial_y v \in \mathrm{End}\, F$ is well defined. Moreover we have the identity $(\partial_y v)^2 = 0$. Also $(\partial_y v)(x)$ depends holomorphically on x, y.

We then find that on $M'_j \cap U$, we have an identification of holomorphic $\mathbb{Z}$ graded chain complexes

$$(F, \partial_y v) \simeq (\Lambda N_j^* \otimes \eta_j, i_y); \qquad y \in N_j. \tag{1.12}$$

Since the description of (ξ, v) given in (1.5) is only local, the identification (1.12) is a priori noncanonical. We now prove it is indeed canonical.

Clearly the holomorphic map $r: \xi_{0|M'_j} \to \eta_j$ induces a holomorphic map $F_0 \to \eta_j$, which we still note r. From (1.6), it follows that $r: F_0 \to \eta_j$ is an isomorphism of holomorphic vector bundles on M'_j. We see from (1.5), (1.6) that in (1.12), the isomorphism $F_0 \simeq \eta_j$ on $M'_j \cap U$ is obtained by the map $r: F_0 \to \eta_j$, i.e. is canonical.

Using (1.12) we find that if $y_1, \dots y_{e_j}$ is a base of N_j, if $f \in \eta_j \simeq F_0$, there is a unique $\omega \in F_{e_j}$ such that

$$(\partial_{y_{e_j}} v \, \partial_{y_{e_j-1}} v \dots \partial_{y_1} v) \, \omega = f. \tag{1.13}$$

If $y^1, \dots, y^{e_j}$ is the base of N_j^* dual to $y_1, \dots, y_{e_j}$, we identify ω with $(y^1 \wedge y^2 \wedge \dots \wedge y^{e_j}) \otimes f$. So F_{e_j} is now canonically identified with $\Lambda^{e_j}(N_j^*) \otimes \eta_j$ and this identification is exactly the one in (1.12). Finally if $y'_1, \dots, y'_k \in N_j$, $k \leqq e_j$, we identify $(\partial_{y'_1} v \dots \partial_{y'_k} v) \, \omega$ with $(i_{y'_1} \dots i_{y'_k}) \, (y^1 \wedge \dots \wedge y^{e_j}) \otimes f \in \Lambda^{e_j-k}(N_j^*) \otimes \eta_j$. Still this identification is compatible with (1.12). We then find that the isomorphism in (1.12) is canonical. For different open sets U, the corresponding isomorphisms (1.12) patch together on M'_j.

Our Theorem is proved. □

Remark 1.3. Observe that for every fiber Z the sequence of sheaves

$$0 \to \mathcal{O}_Z(\xi_m) \to \mathcal{O}_Z(\xi_{m-1}) \to \dots \to \mathcal{O}_Z(\xi_0) \to i^* \mathcal{O}_Y(\eta) \to 0 \tag{1.14}$$

is still exact.

Remark 1.4. In previous work by Bismut-Gillet-Soulé [BGS1], the chain map v was supposed to increase the degree by 1, while here it decreases the degree in ξ by 1. We choose this convention to make the comparison with the Koszul complex easier. However this new convention leads to unescapable conflicts of signs with [BGS1].

c. A family of Hermitian metrics on (ξ, v) and the Koszul complex of N

Let $h^{\xi_0}, \dots, h^{\xi_m}$ be smooth Hermitian metrics on the vector bundles $\xi_0, \dots, \xi_m$. We equip the vector bundle $\xi = \bigoplus_{k=0}^{m} \xi_k$ with the metric $h^{\xi} = \bigoplus_{k=0}^{m} h^{\xi_k}$ which is the orthogonal sum of the metrics $h^{\xi_0}, \dots, h^{\xi_m}$.

Clearly v is a smooth section of $\operatorname{End} \xi$. Let v^* be the adjoint of v with respect to the metric h. For $x \in M'$, we may develop the "Hodge theory" of the finite dimensional Hermitian complex $(\xi, v)_x$. Namely, for every $x \in M'$, $0 \leqq k \leqq m$, set

$$F'_{k,x} = \{f \in \xi_{k,x};\ v(x) f = 0;\ v^*(x) f = 0\}. \tag{1.15}$$

We then have a canonical identification of vector spaces

$$F_{k,x} \simeq F'_{k,x}; \qquad x \in M';\ 0 \leqq k \leqq m. \tag{1.16}$$

Now since the $F_{k,x}$ have constant dimension on each M'_j $(1 \leqq j \leqq n)$, the $F'_{k,x}$ also have constant dimension on each M'_j $(1 \leqq j \leqq n)$. Therefore $F'_{0,x}, \ldots, F'_{m,x}$ are the fibers of smooth vector bundles $F'_0, \ldots, F'_m$ on M'. One then easily verifies that the canonical identification (1.16) induces an identification of smooth vector bundles on M'.

As smooth vector subbundles of the vector bundles $\xi_{0|M'}, \ldots, \xi_{|M'}$, the vector bundles $F'_0, \ldots, F'_m$ on M' inherit smooth Hermitian metrics. Using Theorem 1.2 and (1.16), we then find that $F_0, \ldots, F_m$ are now holomorphic Hermitian vector bundles on M'. We denote by $h^{F_0}, \ldots, h^{F_m}$ the metrics on $F_0, \ldots, F_m$.

At this stage, we will often use the notation $F_0, \ldots, F_m$ instead of $F'_0, \ldots, F'_m$, and so we will consider $F_0, \ldots, F_m$ as vector subbundles of $\xi_0, \ldots, \xi_m$. We equip $F = \bigoplus_{k=0}^{k} F_k$ with the metric h^F which is the orthogonal sum of the metrics of the metrics $h^{F_0}, \ldots, h^{F_m}$.

Let $\nabla^{\xi} = \bigoplus_0^m \nabla^{\xi_k}$ denote the holomorphic Hermitian connection on $\xi = \bigoplus_0^m \xi_k$. Let $\nabla^{\xi'}$, $\nabla^{\xi''}$ be the holomorphic and antiholomorphic parts of ∇^{ξ}.

By proceeding as in Sect. 1b, we find that if $x \in M'$, $X \in (T_R M')_x$, and if $\partial_X v^*$ is the derivative of v^* in any given holomorphic trivialization of ξ near x, then $\partial_X v^*(x)$ acts naturally or F_x. Of course $\partial_X v^*(x)$ now depends on the metrics $h^{\xi_0}, \ldots, h^{\xi_m}$. Since $\nabla^{\xi''} v = 0$, then $\nabla^{\xi'} v^* = 0$. We then find that if $x \in M'$, $X \in T_x^{(1,0)} M$, then $\partial_X v^* = 0$.

Let f, g be smooth sections of ξ near $x \in M'$ which are such that $f_x, g_x \in F_x$. Since

$$\langle v f, g \rangle = \langle f, v^* g \rangle \tag{1.17}$$

and since at x, $vf = 0$, $vg = 0$, $v^* f = 0$, $v^* g = 0$, we deduce that if $x \in M'$, $X \in (T_R M)_x$

$$\langle (\partial_X v) f, g \rangle = \langle f, (\partial_X v^*) g \rangle. \tag{1.18}$$

So if $X \in (T_R M)_x$, $\partial_X v^*$ is the adjoint of $\partial_X v$. In particular if $X \in T_R M'$, we deduce from (1.18) that $\partial_X v^* = 0$.

Let $\bar{N}$ be the vector bundle conjugate to N. If $y \in N$, let $\bar{y}$ denote the conjugate element in $\bar{N}$. From the previous considerations and especially from (1.18), we deduce that if $y \in N$, $\partial_{\bar{y}} v^*(x)$ acts naturally on F_x as the adjoint of $\partial_y v(x)$.

Let now g^N and g^η be smooth Hermitian metrics on the vector bundles N and η. Then N^*, and more generally the exterior algebra $\Lambda(N^*)=\bigoplus_0^e \Lambda^p(N^*)$ are naturally equipped with smooth Hermitian metrics.

We equip $\Lambda(N^*)\otimes\eta$ with the product of the metric on $\Lambda(N^*)$ and of the metric g^η.

Definition 1.5. Given metrics g^N and g^η on N and η, we will say that the metrics $h^{\xi_0}, \dots, h^{\xi_m}$ on $\xi_0, \dots, \xi_m$ verify assumption (A) with respect to g^N and g^η if the identification of holomorphic chain complexes on N

$$(F, \partial_y v)\simeq(\Lambda(N^*)\otimes\eta, i_y) \tag{1.19}$$

also identifies the metrics.

Proposition 1.6. *Given smooth Hermitian metrics g^N and g^η on N and η, there exist smooth Hermitian metrics $h^{\xi_0}, \dots, h^{\xi_m}$ on $\xi_0, \dots, \xi_m$ which verify assumption (A) with respect to g^N and g^η.*

Proof. We first choose arbitrary smooth Hermitian metrics $h_{0|M'}, \dots, h_{m|M'}$ on $\xi_{0|M'}, \dots, \xi_{m|M'}$. We equip $\xi_{|M'}=\bigoplus_0^m \xi_{k|M'}$ with the orthogonal sum of the metrics $h_{k|M'}$. Let v^* denote the adjoint of v on M'.

For $0\leqq k\leqq m$, we define the vector bundle F'_k on M' by formula (1.15). For $0\leqq k\leqq m$, $x\in M'$ we then have the Hodge decomposition

$$\xi_{k,x}=v(\xi_{k+1})_x\oplus v^*(\xi_{k-1})_x\oplus F'_{k,x} \tag{1.20}$$

and the splitting in (1.20) is orthogonal with respect to the metric $h_{k|M'}$. Since $F'_0, \dots, F'_m$ are smooth vector bundles on M', one deduces easily that for $0\leqq k\leqq m$, $v(\xi_{k+1})_{|M'}$ and $v^*(\xi_{k-1})_{|M'}$ are smooth vector subbundles of $\xi_{k|M'}$.

We now modify the metric $h_{k|M'}$ on $\xi_{k|M'}$. Namely let $h'_{k|M'}$ be the metric on $\xi_{k|M'}$ which is defined by the following three properties:

- The splitting (1.20) is still orthogonal for the metric $h'_{k|M'}$.
- The restrictions of $h_{k|M'}$ and $h'_{k|M'}$ to $v(\xi_{k+1})\oplus v^*(\xi_{k-1})$ coincide.
- When F'_k is equipped with the metric induced by $h'_{k|M'}$, the canonical identification $F'_k\simeq\Lambda^k(N^*)\otimes\eta$ also identifies the metrics.

If we equip $\xi_{|M'}$ with the orthogonal sum of the metric $h'_{k|M'}$, we find that on M', v^* is still the adjoint of v. In particular (1.20) is now the Hodge decomposition of ξ_k with respect to the new metrics $h'_{0|M'}, \dots, h'_{m|M'}$.

By partition of unity, we extend the metrics $h'_{0|M'}, \dots, h'_{m|M'}$ into smooth Hermitian metrics $h^{\xi_0}, \dots, h^{\xi_m}$ on $\xi_0, \dots, \xi_m$. Tautologically, the metrics $h^{\xi_0}, \dots, h^{\xi_m}$ verify assumption (A) with respect to g^N and g^η. □

Remark 1.7. If N is equipped with a metric g^N, the metric g^N induces an isomorphism of smooth vector bundles $N^*\simeq\bar{N}$. If $y\in N$, the adjoint of the linear map i_y acting on $\Lambda(N^*)$ is given by

$$\omega\in\Lambda(N^*)\to\bar{y}\wedge\omega\in\Lambda(N^*). \tag{1.21}$$

d. The holomorphic Hermitian connection on **F**

Let $h^{\xi_0}, \ldots, h^{\xi_m}$ be smooth Hermitian metrics on $\xi_0, \ldots, \xi_m$. We equip $\xi = \bigoplus_{k=0}^{m} \xi_k$ with the metric h^ξ which is the orthogonal sum of the metrics $h^{\xi_0}, \ldots, h^{\xi_m}$.

By (1.15), (1.16), the holomorphic vector bundles $F, F_0, \ldots, F_m$ are now equipped with Hermitian metrics, and $F_0, \ldots, F_m$ are mutually orthogonal in F.

For $x \in M'$, P_x denotes the orthogonal projection operator from ξ_x on F_x. Let ∇^ξ, ∇^F be the holomorphic Hermitian connections on the vector bundles ξ, F.

Proposition 1.8. *If f is a smooth section of F on M', then*

$$\nabla^F f = P \nabla^\xi f. \tag{1.22}$$

Proof. We will prove that the connection $P\nabla^\xi$ on F is holomorphic. Since this connection preserves the metric of F, it will necessarily coincide with ∇^F.

Let f be a locally defined holomorphic section of the vector bundle F. Since $F = \mathrm{Ker}(v)/\mathrm{Im}(v)$, if j is the canonical map $\mathrm{Ker}(v) \to F$, there is a locally defined holomorphic section k of $\mathrm{Ker}(v)$ such that $f = jh$. Since F is identified with the vector bundle of "harmonic" elements for the Hodge theory of the complex (ξ, v), we then find that $f = Ph$.

Remember that $vh = 0$. Using the Hodge decomposition of h, we find that there exists a locally defined smooth section h' of ξ such that

$$h = vh' + f. \tag{1.23}$$

Since $\nabla^{\xi''} h = 0$, we deduce from (1.23) that

$$v\nabla^{\xi''} h' + \nabla^{\xi''} f = 0 \tag{1.24}$$

and so

$$P\nabla^{\xi''} f = 0. \tag{1.25}$$

We have thus shown that the connection on F $P\nabla^\xi$ is holomorphic. Our Proposition is proved. □

II. Double transgression formulas for superconnections

The purpose of this Section is to give a unified treatment of the double transgression formulas of Bismut-Gillet-Soulé [BGS1], which extended formulas obtained earlier by Bott-Chern [BoC] and Donaldson [D].

Given a complex (ξ, v) of holomorphic Hermitian vector bundles

$$(\xi, v)\colon 0 \to \xi_m \xrightarrow[v]{} \xi_{m-1} \cdots \xrightarrow[v]{} \xi_0 \to 0$$

we consider the corresponding Quillen's superconnections forms [Q] as a function of the metrics $h^{\xi_0}, \dots, h^{\xi_m}$ on $\xi_0, \dots, \xi_m$, and we establish variation formulas for these forms. These formulas include the formulas of Bott-Chern [BoC] and Donaldson [D], and also the formulas of Bismut-Gillet-Soul'e [BGS1, Sect. 1c] (where the number operator of the complex (ξ, v) played a prominent role) as special cases.

This Section is organized as follows. In (a), we briefly recall the superconnection formalism of Quillen [Q]. In (b), we extend results of [BoC] and [D] to superconnections. In (c), we reobtain the double transgression formulas of [BGS1, Sect. 1c] as a special case of the results of Section 2b.

a. The superconnection formalism

Our assumptions and notations are the same as in Sect. 1c. In particular we assume that the vector bundles $\xi_0, \dots, \xi_m$ are equipped with Hermitian metrics $h^{\xi_0}, \dots, h^{\xi_m}$. v^* still denotes the adjoint of v. Set

$$\xi_+ = \bigoplus_{k\,\text{even}} \xi_k, \qquad \xi_- = \bigoplus_{k\,\text{odd}} \xi_k$$

so that $\xi = \xi_+ \oplus \xi_-$. Remember that $\xi = \bigoplus_{k=0}^{m} \xi_k$ is equipped with the orthogonal sum of the metrics $h^{\xi_0}, \dots, h^{\xi_m}$. In particular ξ_+ and ξ_- are orthogonal in ξ.

We now briefly describe the formalism of Quillen [Q] in this framework. The vector bundle $\xi = \xi_+ \oplus \xi_-$ is $\mathbb{Z}_2$ graded. Let τ be the involution in End ξ which defines the grading, i.e. $\tau = \pm 1$ on $\xi_\pm$. The algebra End ξ is naturally $\mathbb{Z}_2$ graded, the even (resp. odd) elements in End ξ commuting (resp. anticommuting) with τ. If $A \in \operatorname{End} \xi$, we define its supertrace $\operatorname{Tr}_s[A]$ by the formula

$$\operatorname{Tr}_s[A] = \operatorname{Tr}[\tau A]. \tag{2.1}$$

Tr_s extends into a linear map from the $\mathbb{Z}_2$ graded algebra $\Lambda(T_R^* M) \hat{\otimes} \operatorname{End} \xi$ into $\Lambda(T_R^* M)$ with the convention that if $\omega \in \Lambda(T_R^* M)$, $A \in \operatorname{End} \xi$

$$\operatorname{Tr}_s[\omega A] = \omega \operatorname{Tr}_s[A].$$

If $\alpha, \alpha' \in \Lambda(T_R^* M) \hat{\otimes} \operatorname{End} \xi$, let $[\alpha, \alpha']$ be the supercommutator of α and α', i.e.

$$[\alpha, \alpha'] = \alpha\alpha' - (-1)^{\deg\alpha \deg\alpha'} \alpha'\alpha. \tag{2.2}$$

Then by [Q], the supertrace Tr_s vanishes on supercommutators. Set

$$V = v + v^*. \tag{2.3}$$

Then v, v^*, V are odd sections of End ξ. Let $\nabla^\xi = \bigoplus_{k=0}^{m} \nabla^{\xi_k}$ be the holomorphic Hermitian connection on $\xi = \bigoplus_{k=0}^{m} \xi_k$. Set

$$A = \nabla^\xi + V; \qquad A' = \nabla^{\xi'} + v^*; \qquad A'' = \nabla^{\xi''} + v. \tag{2.4}$$

Then A is a superconnection on ξ in the sense of Quillen [Q]. Its curvature A^2 is a smooth even section of $\Lambda(T_R^* M) \hat{\otimes} \operatorname{End} \xi$. Also by [BGS1, Prop. 1.6]

$$\begin{gathered} A'^2 = A''^2 = 0, \\ [A', A^2] = [A'', A^2] = 0, \\ A^2 = [A', A'']. \end{gathered} \tag{2.5}$$

b. Double transgression formulas for superconnections

For $0 \leqq i \leqq m$, let $\mathcal{M}_i$ be the set of smooth Hermitian metrics on the vector bundle ξ_i. Set $\mathcal{M} = \prod_0^m \mathcal{M}_i$. Now v^*, V and ∇^ξ depend explicitly on the metrics $h^\xi = (h^{\xi_0}, \dots, h^{\xi_m}) \in \mathcal{M}$.

Let P^M be the vector space of smooth sections of $\Lambda(T_R^* M)$ which are sums of differential forms of type (p, p) $(0 \leqq p \leqq l + l')$. If $\partial, \bar{\partial}$ are the standard exterior differentiation operators acting on smooth forms on M, let $P^{M,0}$ be the vector space of the $\omega \in P^M$ which can be written in the forms $\omega = \partial \eta + \bar{\partial} \eta'$, where η, η' are smooth forms on M. We use the notation $P^{M'}$, $P^{M',0}$, P^B, $P^{B,0}$, ... for the corresponding objects constructed on the manifolds M', B....

If E is a vector bundle on M with connection ∇ and curvature ∇^2, we define the normalized Chern character $\overline{\operatorname{ch}}(E)$ to be the cohomology class represented by the form $\operatorname{Tr}[\exp(-\nabla^2)]$.

By Quillen [Q] and by [BGS1], Theorem 1.9], the smooth differential form $\operatorname{Tr}_s[\exp(-A^2)]$ lies in P^M, is closed and represents the normalized Chern character $\overline{\operatorname{ch}}(\xi_0 - \xi_1 + \dots + (-1)^m \xi_m)$.

The form $\operatorname{Tr}_s[\exp(-A^2)]$ on M depends smoothly on $h^\xi = (h^{\xi_0}, \dots, h^{\xi_m}) \in \mathcal{M}$. From now on, we consider $\operatorname{Tr}_s[\exp(-A^2)]$ as a smooth form on $M \times \mathcal{M}$, which is of course of partial degree zero in the Grassmann variable on $\mathcal{M}$.

If $h_j \in \mathcal{M}_j$, $x \in M$, let $\mathcal{B}_x^{h_j}$ be the set of endomorphisms of $\xi_{j,x}$ which are self-adjoint with respect to the metric h_j. As in [BGS1, Sect. 1e], we identify the tangent space $T_{h_j} \mathcal{M}_j$ with the set of smooth sections of $\mathcal{B}^{h_j}$ on M. In fact if $k_j \in \operatorname{End}(\xi_j, \xi_j^*)$ is an infinitesimal deformation of h_j in $\mathcal{M}_j$, then $h_j^{-1} k_j$ is the corresponding element in $\mathcal{B}^{h_j}$.

Let $d^{\mathcal{M}}$ be the exterior differentiation operator on $\mathcal{M}$. If $h^\xi = (h^{\xi_0}, \dots, h^{\xi_m}) \in \mathcal{M}$, if $k = (k^{\xi_0}, \dots, k^{\xi_m})$ is an infinitesimal deformation of h^ξ *in* $\mathcal{M}$, then

$((h^{\xi_0})^{-1} k^{\xi_0}, \ldots, (h^{\xi_m})^{-1} k^{\xi_m})$ is a smooth section of $\mathscr{B}^h = \prod_0^m \mathscr{B}^{h_j}$. If $d^{\mathcal{M}} h^{\xi_i} = k^{\xi_i}$ $(0 \leqq i \leqq m)$, we will note this element by $(h^\xi)^{-1} d^{\mathcal{M}} h^\xi$. $(h^\xi)^{-1} d^{\mathcal{M}} h^\xi$ will be considered as a one form on $\mathcal{M}$ with values in $\mathscr{B}$.

For $h^\xi = (h^{\xi_0}, \ldots, h^{\xi_m}) \in \mathcal{M}$, let ω be the connection form associated with the corresponding holomorphic Hermitian connection ∇^ξ. Then $d^{\mathcal{M}} \omega$ is a two form which is the equivariant representation of a two form γ on $M \times \mathcal{M}$ with values in End ξ (where ξ is now considered as a vector bundle on $M \times \mathcal{M}$). Of course γ preserves the splitting $\xi = \bigoplus_0^m \xi_k$. Note that γ is of complex type (1, 0) in the Grassmann variables in $T_R^* M$.

The operators $d^{\mathcal{M}}, \partial, \bar\partial$ act naturally on smooth forms on $M \times \mathcal{M}$ as partial exterior differentiation operators.

Note that in the sequel, we will use the superconnection formalism on $M \times \mathcal{M}$.

We now generalize a result of Bott and Chern [BoC, 3.28], [BGS1, Theorem 1.24] concerning usual connections on a Hermitian vector bundle to superconnections.

Theorem 2.1. *The following identities hold*

$$\begin{aligned} d^{\mathcal{M}} \operatorname{Tr}_s[\exp(-A^2)] &= \bar\partial \operatorname{Tr}_s[(\gamma - [v^*, (h^\xi)^{-1} d^{\mathcal{M}} h^\xi]) \exp(-A^2)] \\ \operatorname{Tr}_s[(\gamma - [v^*, (h^\xi)^{-1} d^{\mathcal{M}} h^\xi]) \exp(-A^2)] &= -\partial \operatorname{Tr}_s[((h^\xi)^{-1} d^{\mathcal{M}} h^\xi) \exp(-A^2)]. \end{aligned} \tag{2.6}$$

In particular,

$$d^{\mathcal{M}} \operatorname{Tr}_s[\exp(-A^2)] = -\bar\partial\partial \operatorname{Tr}_s[((h^\xi)^{-1} d^{\mathcal{M}} h^\xi) \exp(-A^2)]. \tag{2.7}$$

Proof. We proceed as in [BGS1, Theorem 1.24]. The Hermitian vector bundle ξ on $M \times \mathcal{M}$ is naturally equipped with a connection $\tilde\nabla^\xi$ which restricts to the holomorphic Hermitian connection ∇^ξ associated with $h^\xi \in \mathcal{M}$ on $M \times \{h^\xi\}$, and is trivial on $\{0\} \times T\mathcal{M} \subset T(M \times \mathcal{M})$. $\tilde\nabla^\xi + V$ is now a superconnection on the vector bundle ξ on $M \times \mathcal{M}$. One verifies easily that

$$(\tilde\nabla^\xi + V)^2 = (\nabla^\xi + V)^2 + \gamma - [v^*, (h^\xi)^{-1} d^{\mathcal{M}} h^\xi] \tag{2.8}$$

(note that the $-$sign in $-[v^*, (h^\xi)^{-1} d^{\mathcal{M}} h^\xi]$ comes from the fact that $(h^\xi)^{-1} d^{\mathcal{M}} h^\xi$ is odd).

By Quillen [Q], the form $\operatorname{Tr}_s[\exp(-(\tilde\nabla^\xi + V)^2]$ is closed on $M \times \mathcal{M}$. Using (2.8) and Duhamel's formula, we get

$$\begin{aligned} &\operatorname{Tr}_s[\exp(-(\tilde\nabla^\xi + V)^2] \\ &\quad = \operatorname{Tr}_s[\exp(-A^2)] - \operatorname{Tr}_s[(\gamma - [v^*, (h^\xi)^{-1} d^{\mathcal{M}} h^\xi]) \exp(-A^2)] + C \end{aligned} \tag{2.9}$$

where C is of degree $\geqq 2$ in the Grassmann variables in $T^*\mathcal{M}$. Since the form $\operatorname{Tr}_s[\exp(-(\tilde\nabla^\xi + V)^2]$ is closed on $M \times \mathcal{M}$, we deduce from (2.8) that if d is the exterior differentiation operator on M, then

$$d^{\mathcal{M}} \operatorname{Tr}_s[\exp(-A^2)] = d \operatorname{Tr}_s[(\gamma - [v^*, (h^\xi)^{-1} d^{\mathcal{M}} h^\xi]) \exp(-A^2)]. \tag{2.10}$$

Remember that γ is of type (1, 0) in the Grassmann variables in $T_R^* M$. Using the fact that the left hand side of (2.10) is a one form on $\mathscr{M}$ with values in P^M, and the degree counting argument of [BGS1, Prop. 1.8], we deduce from (2.10) that

$$\begin{aligned} d^{\mathscr{M}} \operatorname{Tr}_s[\exp(-A^2)] &= \bar{\partial} \operatorname{Tr}_s[(\gamma - [v^*, (h^\xi)^{-1} d^{\mathscr{M}} h^\xi]) \exp(-A^2)] \\ \partial \operatorname{Tr}_s[(\gamma - [v^*, (h^\xi)^{-1} d^{\mathscr{M}} h^\xi]) \exp(-A^2)] &= 0. \end{aligned} \tag{2.11}$$

Using (2.5), we find that

$$\partial \operatorname{Tr}_s[(h^\xi)^{-1} d^{\mathscr{M}} h^\xi \exp(-A^2)] = \operatorname{Tr}_s[[A', (h^\xi)^{-1} d^{\mathscr{M}} h^\xi] \exp(-A^2)]. \tag{2.12}$$

By [BGS1, Prop. 1.23], we get

$$[\nabla^{\xi\prime}, (h^\xi)^{-1} d^{\mathscr{M}} h^\xi] = -\gamma. \tag{2.13}$$

From (2.12), (2.13), we find that

$$\partial \operatorname{Tr}_s[(h^\xi)^{-1} d^{\mathscr{M}} h^\xi \exp(-A^2)] = -\operatorname{Tr}_s[(\gamma - [v^*, (h^\xi)^{-1} d^{\mathscr{M}} h^\xi]) \exp(-A^2)] \tag{2.14}$$

(2.6) follows from (2.11) and (2.14). (2.7) is a consequence of (2.6). □

We now extend to superconnections the result established in [BGS1, Theorem 1.25] for usual connections. Let z be an odd Grassmann variable (so that $z^2=0$). Then z anticommutes with $(\Lambda(T_R^* M) \hat{\otimes} \operatorname{End} \xi)^{\mathrm{odd}}$.

Theorem 2.2. *The following identity holds*

$$\begin{aligned} d^{\mathscr{M}} \operatorname{Tr}_s[\exp(-A^2 + z(h^\xi)^{-1} d^{\mathscr{M}} h^\xi)] &= \tfrac{1}{2} \partial \operatorname{Tr}_s[[A'', (h^\xi)^{-1} d^{\mathscr{M}} h^\xi] \exp(-A^2 + z(h^\xi)^{-1} d^{\mathscr{M}} h^\xi)] \\ &\quad - \tfrac{1}{2} \bar{\partial} \operatorname{Tr}_s[[A', (h^\xi)^{-1} d^{\mathscr{M}} h^\xi] \exp(-A^2 + z(h^\xi)^{-1} d^{\mathscr{M}} h^\xi)]. \end{aligned} \tag{2.15}$$

Proof. We proceed as in [BGS1, Theorem 1.25]. Set $\theta = (h^\xi)^{-1} d^{\mathscr{M}} h^\xi$. Then

$$d^{\mathscr{M}} \operatorname{Tr}_s[\exp(-A^2 + z\theta)] = \operatorname{Tr}_s[[d^{\mathscr{M}}, -A^2 + z\theta] \exp(-A^2 + z\theta)]. \tag{2.16}$$

By [BGS1, Prop. 1.23] and by (2.8)

$$[d^{\mathscr{M}}, A''] = 0; \qquad [d^{\mathscr{M}}, A'] = -[A', \theta]; \qquad d^{\mathscr{M}} \theta = -\theta^2. \tag{2.17}$$

From (2.5), (2.16), (2.17), we deduce that

$$d^{\mathscr{M}} \operatorname{Tr}_s[\exp(-A^2 + z\theta)] = \operatorname{Tr}_s[(-[A'', [A', \theta]] + z\theta^2) \exp(-A^2 + z\theta)]. \tag{2.18}$$

If $\alpha, \beta \in \Lambda(T_R^* M)$, set $(\alpha + z\beta)^z = z\beta$. Using (2.5), (2.6), and Duhamel's formula, we have

$$\begin{aligned} \partial \operatorname{Tr}_s[[A'', \theta] \exp(-A^2 + z\theta)] &= \operatorname{Tr}_s[[A', [A'', \theta]] \exp(-A^2 + z\theta)] \\ &\quad + \operatorname{Tr}_s[[A'', \theta] \exp(-A^2 - z[A', \theta])]^z \\ \bar{\partial} \operatorname{Tr}_s[[A', \theta] \exp(-A^2 + z\theta)] &= \operatorname{Tr}_s[[A'', [A', \theta]] \exp(-A^2 + z\theta)] \\ &\quad + \operatorname{Tr}_s[[A', \theta] \exp(-A^2 - z[A'', \theta])]^z. \end{aligned} \tag{2.19}$$

Using the fact that Tr_s vanishes on supercommutators, we find easily that the last terms in the right hand sides of both formulas in (2.19) coincide. Moreover by (2.5), we have

$$[A', [A'', \theta]] = [A^2, \theta] - [A'', [A', \theta]]. \tag{2.20}$$

Also

$$\begin{aligned} &\tfrac{1}{2} \operatorname{Tr}_s[[A^2, \theta] \exp(-A^2 + z\theta)] \\ &\quad = \tfrac{1}{2} \operatorname{Tr}_s[[\theta, -A^2 + z\theta] \exp(-A^2 + z\theta)] + z \operatorname{Tr}_s[\theta^2 \exp(-A^2 + z\theta)] \\ &\quad = z \operatorname{Tr}_s[\theta^2 \exp(-A^2 + z\theta)]. \end{aligned} \tag{2.21}$$

(2.15) follows from (2.18)–(2.21). □

c. *Number operator and Bott-Chern classes*

We now will show to derive the transgression formulas of [BGS1, Sect. 1b] from Theorem 2.1.

Definition 2.3. N_H denotes the operator in $\operatorname{End} \xi$ which maps $f \in \xi_k$ into $kf \in \xi_k$.

Let $h^{\xi_0}, \ldots, h^{\xi_m}$ be smooth Hermitian metrics on $\xi_0, \ldots, \xi_m$. We equip $\xi = \bigoplus_0^m \xi_k$ with the metric h^ξ which is the orthogonal sum of the metrics $h^{\xi_0}, \ldots, h^{\xi_m}$. Let v^* be the adjoint of v.

Let $\nabla^\xi = \bigoplus_{k=0}^m \nabla^{\xi_k}$ be the holomorphic Hermitian connection on $\xi = \bigoplus_{k=0}^m \xi_k$. Set $V = v + v^*$. For $u \geq 0$, set

$$\begin{aligned} A_u &= \nabla^\xi + \sqrt{u}\, V \\ A'_u &= \nabla^{\xi'} + \sqrt{u}\, v^* \\ A''_u &= \nabla^{\xi''} + \sqrt{u}\, v. \end{aligned} \tag{2.22}$$

Then A_u is a superconnection. Moreover, as in (2.5), we have

$$A_u'^2=0; \qquad A_u''^2=0$$
$$A_u^2=[A_u', A_u''] \tag{2.23}$$
$$[A_u', A_u^2]=[A_u'', A_u^2]=0.$$

We now prove a slight modification of [BGS1, Theorem 1.15].

Theorem 2.4. *For any $u \geqq 0$, the smooth forms* $\mathrm{Tr}_s[\exp(-A_u^2)]$ *and* $\mathrm{Tr}_s[N_H \exp(-A_u^2)]$ *lie in P^M. Moreover for any $u>0$, the following identities hold*

$$\begin{aligned}\frac{\partial}{\partial u}\mathrm{Tr}_s[\exp(-A_u^2)] &= -\bar{\partial}\,\mathrm{Tr}_s\left[\frac{v^*}{\sqrt{u}}\exp(-A_u^2)\right]\\ &= -\partial\,\mathrm{Tr}_s\left[\frac{v}{\sqrt{u}}\exp(-A_u^2)\right]\\ \mathrm{Tr}_s\left[\frac{v^*}{\sqrt{u}}\exp(-A_u^2)\right] &= -\partial\,\mathrm{Tr}_s\left[\frac{N_H}{u}\exp(-A_u^2)\right]\\ \mathrm{Tr}_s\left[\frac{v}{\sqrt{u}}\exp(-A_u^2)\right] &= \bar{\partial}\,\mathrm{Tr}_s\left[\frac{N_H}{u}\exp(-A_u^2)\right].\end{aligned} \tag{2.24}$$

In particular for $u>0$

$$\frac{\partial}{\partial u}\mathrm{Tr}_s[\exp(-A_u^2)]=\frac{1}{u}\,\bar{\partial}\partial\,\mathrm{Tr}_s[N_H\exp(-A_u^2)]. \tag{2.25}$$

Proof. For $u>0$, we temporarily replace the metrics $h^\xi=(h^{\xi_0}, \dots, h^{\xi_m})$ by the metrics $h_u^\xi=\left(h^{\xi_0}, \frac{h^{\xi_1}}{u}, \dots, \frac{h^{\xi_m}}{u^m}\right)$. The adjoint of v with respect to this new family of metrics h_u^ξ is now uv^*.

Let T_u be the linear map in $\mathrm{End}\,\xi$ which is such that if $f\in\xi_k$, then $T_u f=u^{k/2}f$. One verifies that

$$T_u^{-1}\, v\, T_u=\sqrt{u}\,v; \qquad T_u^{-1}\, v^*\, T_u=\frac{v^*}{\sqrt{u}}. \tag{2.26}$$

Clearly

$$\begin{aligned}(h_u^\xi)^{-1}\frac{\partial}{\partial u}h_u^\xi &= -\frac{N_H}{u}\\ \left[uv^*, (h_u^\xi)^{-1}\frac{\partial h_u^\xi}{\partial u}\right] &= v^*.\end{aligned} \tag{2.27}$$

Also if we equip ξ with the orthogonal sum of the metrics $h^{\xi_0}, \frac{h^{\xi_1}}{u}, \ldots, \frac{h^{\xi_m}}{u^m}$, ∇^ξ is still the corresponding holomorphic Hermitian connection. By Theorem 2.1, and taking signs into account, we get

$$\frac{\partial}{\partial u}\operatorname{Tr}_s[\exp(-(\nabla^\xi+v+uv^*)^2)] = -\bar{\partial}\operatorname{Tr}_s[v^*\exp(-(\nabla^\xi+v+uv^*)^2)]$$

$$\operatorname{Tr}_s[v^*\exp(-(\nabla^\xi+v+uv^*)^2)] = -\partial\operatorname{Tr}_s\left[\frac{N_H}{u}\exp(-(\nabla^\xi+v+uv^*)^2)\right] \quad (2.28)$$

$$\frac{\partial}{\partial u}\operatorname{Tr}_s[\exp(-(\nabla^\xi+v+uv^*)^2)] = \frac{1}{u}\bar{\partial}\partial\operatorname{Tr}_s[N_H\exp(-(\nabla^\xi+v+uv^*)^2)].$$

Using (2.26) and (2.28), we find that half of the equations in (2.24) and also Eq. (2.25) have been proved. Echanging the roles of ∂ and $\bar{\partial}$, and of $\nabla^{\xi'}$ and $\nabla^{\xi''}$, it is easy to derive the second half of Eqs. (2.24). Our Theorem is proved. □

Remark 2.5. We have thus shown that the various double transgression formulas of [BGS1] can be derived from the double transgression formula for superconnections given in Theorem 2.1.

III. Convergence of the superconnection Chern character currents

The purpose of this Section is to prove that as $u \to +\infty$, the Chern character forms $\operatorname{Tr}_s[\exp(-A_u^2)]$ converge to a current concentrated on M', which we explicitly calculate. As was already pointed out in the introduction, for later purposes, we need to obtain a precise control of the speed of convergence at a microlocal level. This in part explains some of the technical difficulties which appear in the proofs.

Our main result is stated in Theorem 3.2. Its proof essentially consists in expressing $\operatorname{Tr}_s[\exp(-A_u^2)]$ as a contour integral, and in controlling the integrand as $u \to +\infty$. The simple form of the limiting current follows from complicate algebraic manipulations on connections.

This Section is organized as follows. In (a), we describe our main assumption and notations. In (b), we establish a simple coercitivity result on the map $(\partial_Y V)^2$. In (c), we briefly recall some technical results on wave front sets of currents, and on the natural topology of the set of currents which have a given wave front set. We refer to Hörmander [H, Chap. VIII], for more details. In (d), we state our main result, whose proof occupies Sect. 3e–j.

a. Assumptions and notations

We make the same assumptions as in Sect. 1, and we use the notations of Sects. 1 and 2.

Let $h^{\xi_0}, \ldots, h^{\xi_m}$ be smooth Hermitian metrics on the vector bundles $\xi_0, \ldots, \xi_m$. We equip $\xi = \bigoplus_0^m \xi_k$ with the metric h^ξ which is the orthogonal sum of the metrics $h^{\xi_0}, \ldots, h^{\xi_m}$. Let v^* be the adjoint of v.

Let $\nabla^\xi = \bigoplus_{k=0}^m \nabla^{\xi_k}$ be the holomorphic Hermitian connection on $\xi = \bigoplus_{k=0}^m \xi_k$. For $u \geqq 0$, let A_u be the superconnection on ξ

$$A_u = \nabla^\xi + \sqrt{u}\, V. \tag{3.1}$$

By the results of Sect. 1c), the vector bundles $F_0, \ldots, F_m$ are naturally equipped with Hermitian metrics $h^{F_0}, \ldots, h^{F_m}$ and are orthogonal in $F = \bigoplus_{k=0}^m F_k$. Also if $y \in N$, $\partial_{\bar{y}} v^*$ is the adjoint of $\partial_y v$. If $Y = y + \bar{y} \in N_R$, set

$$\partial_Y V = \partial_y v + \partial_{\bar{y}} v^*. \tag{3.2}$$

Then $\partial_Y V$ is a self-adjoint operator acting on F.

Let $\nabla^F = \bigoplus_{k=0}^m \nabla^{F_k}$ be the holomorphic Hermitian connection on $F = \bigoplus_{k=0}^m F_k$. F lifts naturally to a vector bundle on N, which we still note F. In particular ∇^F still denotes the holomorphic Hermitian connection on the lifted bundle. Set

$$F_+ = \bigoplus_{k \text{ even}} F_k; \qquad F_- = \bigoplus_{k \text{ odd}} F_k.$$

Note that the vector bundle $F = F_+ \oplus F_-$ on N is $\mathbb{Z}_2$ graded, and that we can apply the superconnection formalism to F. If $Y \in N_R$, $\partial_Y V$ is odd in End F. Let B be the superconnection on the vector bundle F on N

$$B = \nabla^F + \partial_Y V.$$

Let g^N, g^η be smooth Hermitian metrics on the vector bundles N, η, let ∇^N, ∇^η be the associated holomorphic Hermitian connections.

b. Coercitivity of the map $(\partial_Y V)^2$

We here establish a simple coercitivity result on the linear map $(\partial_Y V)^2$.

Proposition 3.1. *There exists a constant $c > 0$ such that for any $x \in M'$, $Y \in N_{R,x}$, $f \in F_x$*

$$|\partial_Y V(x) f|^2 \geqq c\, |Y|^2_{N_R} |f|^2_F. \tag{3.3}$$

Proof. Clearly we only need to prove the Proposition when $|Y|_{N_R} = 1$. If $X = y + \bar{y}$, $|Y| = 1$, the complex $(F, \partial_y v) \simeq (\Lambda N^* \otimes \eta, i_y)$ is acyclic. By finite dimensional Hodge theory, we find that $\partial_Y V = \partial_y v + (\partial_y v)^*$ is a self-adjoint invertible operator.

Since the sphere bundle S_{N_R} of N_R is compact, we deduce that there exists $c>0$ which is a lower bound for the smallest eigenvalues of $(\partial_Y V)^2_{Y\in S_{N_R}}$. The proof of the Proposition is completed. □

In particular, we deduce from Prop. 3.1 that the form on N $\mathrm{Tr}_s[\exp(-B^2)]$ decays faster than $\exp(-C|y|^2_N)$ (for one $C>0$) when $|y|_N\to+\infty$.

c. *Wave front sets*

The wave front set of a distribution or of a current is defined in Hörmander [H, Chap. VIII]. Let us just recall that if γ is a current on M, its wave front set $WF(\gamma)$ is a closed conic subset of $T^*_R M\setminus\{0\}$. If p is the projection $T^*_R M\to M$, then the singular support of γ is exactly $p(WF(\gamma))$, i.e. γ is smooth on the open set $M p(WF(\gamma))$.

Let $\mathscr{D}'(M)$ be the set of currents on M. Let $\mathscr{D}'_{N^*_R}(M)$ be the set of currents on M whose wave front set is included in N^*_R. In particular currents in $\mathscr{D}'_{N^*_R}(M)$ are smooth on $M\setminus M'$.

Then by Hörmander [H, p. 262], $\mathscr{D}'_{N^*_R}(M)$ can be naturally equipped with a family of semi-norms. Namely let U be an open set in M which is holomorphically equivalent to an open ball in $\mathbb{C}^{l+l'}\simeq R^{2(l+l')}$. Over U, we identify $T^*_R M$ with $U\times R^{2(l+l')}$. Let Γ be a closed cone in $R^{2(l+l')}$ such that on $U\cap M'$, $\Gamma\cap N^*_R=\{0\}$. Let ϕ be a smooth current with compact support in U, and let m be an integer. $\wedge$ denotes Fourier transform on $\mathbb{C}^{l+l'}$.

If $\omega\in\mathscr{D}'_{N^*_R}(M)$, set

$$p_{U,\Gamma,\phi,m}(\omega)=\sup_{\xi\in\Gamma}|\xi|^m|\widehat{\phi\omega}(\xi)|. \tag{3.4}$$

We will say that a sequence of currents $\omega_n\in\mathscr{D}'_{N^*_R}(M)$ converges to $\omega\in\mathscr{D}'_{N^*_R}(M)$ if:

- $\omega_n\to\omega$ in $\mathscr{D}'(M)$.
- If U, Γ, ϕ, m are taken as in (3.4), then

$$\lim_{n\to+\infty} p_{U,\Gamma,\phi,m}(\omega_n-\omega)=0.$$

Let $\delta_{M'}$ be the current of integration on the oriented manifold M', so that if ω is a smooth form on M

$$\int_M \omega\delta_{M'}=\int_{M'}\omega. \tag{3.5}$$

d. *Convergence of superconnection currents on M*

For $k\in N$, let $C^k(M)$ be the set of sections of $\Lambda(T^*_R M)$ which are k-times continuously differentiable. Then for any $k\in N$, $C^k(M)$ is a Banach space. Let $\|\ \|_{C^k(M)}$ be a norm on $C^k(M)$. We define $C^k(B)$, $\|\ \|_{C^k(B)}$ in a similar way.

In the sequel, $\int_Z$ or $\int_Y$ denotes integration of differential forms along the fibers of $M \xrightarrow[\pi]{} B$ or of $M' \xrightarrow[\pi]{} B$. Similarly $\int_N$ denotes integration along the fiber of $N \xrightarrow[\rho]{} M'$.

Let R^N be the curvature of the holomorphic Hermitian connection ∇^N on the vector bundle N. Let $(\nabla^\eta)^2$ be the curvature of the holomorphic Hermitian connection ∇^η on the vector bundle η.

Let Td denote the ad-invariant Todd polynomial defined on (p, p) matrices such that if the diagonal matrix C has diagonal entries $x_1, \ldots, x_p$, then

$$\mathrm{Td}(C) = \prod_1^p \frac{x_i}{1 - e^{-x_i}}.$$

Let Td^{-1} be the inverse of Td, so that if C is taken as before

$$\mathrm{Td}^{-1}(C) = \prod_1^p \frac{(1 - e^{-x_i})}{x_i}.$$

We now prove the main result of this paper.

Theorem 3.2. *Let μ be a smooth differential form on the manifold M. For $u \geqq 0$, let $\theta_u(\mu)$ be the smooth differential form on B*

$$\theta_u(\mu) = \int_Z \mu \operatorname{Tr}_s[\exp(-A_u^2)] - \int_Y i^* \mu \int_N \operatorname{Tr}_s[\exp(-B^2)]. \tag{3.6}$$

Then for any $k \in N$, there exists $C_k > 0$ such that for any smooth differential form μ on M and any $u \geqq 1$

$$\|\theta_u(\mu)\|_{C^k(B)} \leqq \frac{C_k}{\sqrt{u}} \|\mu\|_{C^{k+1}(M)}. \tag{3.7}$$

Also as $u \to +\infty$

$$\operatorname{Tr}_s[\exp(-A_u^2)] \to \int_N \operatorname{Tr}_s[\exp(-B^2)]\, \delta_{M'} \quad \text{in } \mathscr{D}'_{N^*_{\mathbb{R}}}(M). \tag{3.8}$$

If U, Γ, ϕ, m are taken as in (3.4), *there exists $C' > 0$ such that for $u \geqq 1$*

$$p_{U, \Gamma, \phi, m}\Big(\operatorname{Tr}_s[\exp(-A_u^2)] - \int_N \operatorname{Tr}_s[\exp(-B^2)]\, \delta_{M'}\Big) \leqq \frac{C'}{\sqrt{u}}. \tag{3.9}$$

If the metrics $h^{\xi_0}, \ldots, h^{\xi_m}$ on the vector bundles $\xi_0, \ldots, \xi_m$ verify assumption (*A*) *with respect to the metrics g^N and g^η on N and η, then we have the equality of differential forms on M'*

$$\int_N \operatorname{Tr}_s[\exp(-B^2)] = (2i\pi)^{\dim N}\, \mathrm{Td}^{-1}(-R^N) \operatorname{Tr}_s[\exp(-(\nabla^\eta)^2)]. \tag{3.10}$$

Proof. Assume first that the compact support of μ is included in $M \backslash M'$. Remember that the linear self-adjoint map $V \in \operatorname{End} \xi$ is invertible on $M \backslash M'$. Using Duhamel's formula, one then finds easily that

$$\int_Z \mu \operatorname{Tr}_s[\exp(-A_u^2)] \to 0 \quad \text{uniformly together with its derivatives.} \tag{3.11}$$

More precisely, if K is a compact subset of $M \backslash M'$, for any $k \in N$, there exists $C>0$ such that if the support of μ is included in K, for $u \geqq 1$, the $C^k(B)$ norm of $\int_Z \mu \operatorname{Tr}_s[\exp(-A_u^2)]$ is dominated by $\exp(-Cu)\|\mu\|_{C_k(M)}$.

Take now $x_0 \in M'$. Let $z=(z^1, \ldots, z^{l+l'})$ be a holomorphic system of coordinates on an open neighborhood of x_0 in M such that locally, M' is the vector subspace $(z^1=0, \ldots, z^e=0)$.

By our assumptions on π, the holomorphic map $z=(z^1, \ldots, z^{l+l'}) \to (\pi z, z^1, \ldots, z^e)$ is a submersion near x_0. Therefore there exists an open neighborhood $\mathscr{V}$ of x_0 in M' and $\varepsilon>0$ such that if $\mathscr{B}_\varepsilon$ is the open ball of center 0 and radius ε in $\mathbb{C}^{e_j}$, then $U=\mathscr{V} \times \mathscr{B}_\varepsilon$ is an open neighborhood of x_0 in M, and moreover if $(x, y) \in \mathscr{V} \times \mathscr{B}_\varepsilon$, $\pi(x, y)=\pi x$.

We now will prove Theorem 3.2 when μ has compact support included in U. By partition of unity, we will thus obtain our Theorem in full generality. Of course in the course of the proof, we may take $\mathscr{V}$ and $\varepsilon>0$ as small as needed.

Let σ_u be the map $(x, y) \in \mathscr{V} \times \mathscr{B}_{\varepsilon\sqrt{u}} \to \sigma_u(x, y)=\left(x, \frac{y}{\sqrt{u}}\right)$. Clearly since the support of μ is included in U

$$\int_Z \mu \operatorname{Tr}_s[\exp(-A_u^2)] = \int_{(Y \cap \mathscr{V}) \times \mathscr{B}_{\varepsilon\sqrt{u}}} (\sigma_u^* \mu)(\sigma_u^* \operatorname{Tr}_s[\exp(-A_u^2)]). \tag{3.12}$$

We now will evaluate the limit of the right hand side of (3.12) as $u \to +\infty$. This will be done in five main steps.

– We prove in Prop. 3.3 that the self-adjoint operators $V^2(x, y)$ are coercive enough near M'. This will be useful when using dominated convergence.

– We construct a trivialization of ξ on $\mathscr{V} \times \mathscr{B}_\varepsilon$. For well chosen $\lambda \in \mathbb{C}^*$, we calculate in Prop. 3.4 an asymptotic formula for $\left(\lambda I_\xi - \sqrt{u}\, V\left(\frac{x, y}{\sqrt{u}}\right)\right)^{-1}$.

– In Prop. 3.5, we calculate the non diagonal piece of the connection ∇^ξ with respect to a non trivial splitting $\xi=\xi^+ \oplus \xi^-$ of the vector bundle ξ, and in Prop. 3.7 we calculate second covariant derivatives of V on M'.

– In a fourth step, we obtain a formula for the right hand side of (3.12) is terms of certain contour integrals. Using the first three steps, the proof of the first part of our Theorem will be completed.

– Finally, we obtain formula (3.10) as in Mathai-Quillen [MaQ].

The proof of (3.7)–(3.9) is technically complicated in particular because we need to prove estimates related to dominated convergence. Still its principle is quite simple.

The proof of Theorem 3.2 occupies Sect. 3e) to 3j) of this paper. We suggest the reader to look first at Sect. 3h) and Remark 3.9 to get acquainted with the non technical aspects of the proof.

e. Coercitivity of the map $V^2(x, y)$

In the sequel, $|\ |$ denotes an arbitrary norm on $\mathbb{C}^{e_j}$.

Proposition 3.3. *If $\mathscr{V}$ and $\varepsilon>0$ are small enough, there exists $C>0$ such that if $(x, y)\in\mathscr{V}\times\mathscr{B}_\varepsilon$, then*

$$V^2(x, y)\geqq C|y|^2. \tag{3.13}$$

Proof. With the notations of (1.5), we know that if $\mathscr{V}$ and ε are small enough, then

$$(\xi, v)\simeq(\Lambda\tilde{N}^*\otimes\tilde{\eta}, i_y)\oplus(A, a). \tag{3.14}$$

We can equip the complex in the right-hand side of (3.14) with a metric $\tilde{h}$ which is such that

- The $(A_k)_{0\leqq k\leqq m}$ are mutually orthogonal.
- The splitting (3.14) of the complex (ξ, v) is orthogonal, i.e. $\Lambda\tilde{N}^*\otimes\tilde{\eta}$ and A are orthogonal in ξ.
- The metric on the complex $(\Lambda\tilde{N}^*\otimes\tilde{\eta}, i_y)$ comes from metrics $g^{\tilde{N}}, \tilde{g}^\eta$ on $\tilde{N}$, $\tilde{\eta}$.

Let $\tilde{i}_y^*$ be the adjoint of i_y with respect to the metric $g^{\tilde{N}}$. If $\bar{y}\in\bar{\tilde{N}}$ is identified to an element of $\tilde{N}^*$ by the metric $g^{\tilde{N}}$, then $\tilde{i}_y^*=\bar{y}\wedge$. Therefore

$$(i_y+\tilde{i}_y^*)^2=|y|^2_{g^{\tilde{N}}}. \tag{3.15}$$

Let $\tilde{v}^*$ be the adjoint of v with respect to the metric $\tilde{h}$. Set $\tilde{V}=v+\tilde{v}^*$. From (3.15), we find that there exists $C>0$ such that if $(x, y)\in\mathscr{V}\times\mathscr{B}_\varepsilon$, then

$$\tilde{V}^2(x, y)\geqq C|y|^2. \tag{3.16}$$

We will deduce (3.13) from (3.16) by proceeding as in Bismut-Bost [BBos, Prop. 8.1]. We fix $(x, y)\in\mathscr{V}\times\mathscr{B}_\varepsilon$, $y\neq 0$. All our calculations will be done at (x, y) and the notation (x, y) will be omitted. Clearly V^2 and $\tilde{V}^2$ preserve $\operatorname{Ker}(v)$. By Hodge theory, the lowest eigenvalues of V^2 and $\tilde{V}^2$ can be calculated by restricting these operators to $\operatorname{Ker}(v)$. So take $f\in\operatorname{Ker}(v)\cap\xi_k$. Set

$$g=\tilde{v}^*(\tilde{V}^2)^{-1}f. \tag{3.17}$$

Then $f=vg$. By Hodge theory, we can write g in the form

$$g=v\alpha+v^*\beta. \tag{3.18}$$

From (3.18), we get

$$f=vv^*\beta. \tag{3.19}$$

Since $vf=0$, we deduce from (3.19) that

$$v^*\beta=v^*(V^2)^{-1}f. \tag{3.20}$$

Let $\| \|_h$, $\| \|_{\tilde{h}}$ denote the norms on ξ for the metrics h, $\tilde{h}$. From (3.18), it is clear that

$$\|g\|_{\tilde{h}}^2 \geqq \|v^*\beta\|_{\tilde{h}}^2 = \langle (V^2)^{-1} f, f\rangle_h. \tag{3.21}$$

If $\mathscr{V}$ and $\varepsilon>0$ are small enough, there exists constants $C>0$, $C'>0$ such that on $\mathscr{V}\times\mathscr{B}_\varepsilon$

$$C'\| \|_{\tilde{h}}^2 \leqq \| \|_h^2 \leqq C\| \|_{\tilde{h}}^2. \tag{3.22}$$

From (3.17), we have

$$\|g\|_{\tilde{h}}^2 = \langle (\tilde{V}^2)^{-1} f, f\rangle_{\tilde{h}}. \tag{3.23}$$

From (3.21)–(3.23), we conclude that

$$\langle (V^2)^{-1} f, f\rangle_h \leqq C\langle (\tilde{V}^2)^{-1} f, f\rangle_{\tilde{h}}. \tag{3.24}$$

From (3.22), (3.24), it is clear that if λ, $\tilde{\lambda}$ are the lowest eigenvalues of V^2, $\tilde{V}^2$, there is $c>0$, which is uniform on $\mathscr{V}\times\mathscr{B}_\varepsilon$, such that

$$\lambda \geqq c\tilde{\lambda}. \tag{3.25}$$

(3.13) follows from (3.16), (3.25). □

f. A trivialization of ξ

By Theorem 1.2, since M' is compact, there exists $b>0$ such that if $x\in M'$, the self-adjoint nonnegative operator $V^2(x)$ has no eigenvalue in the interval $]0, 2b]$. Therefore if $\varepsilon>0$ is small enough, if $x\in\mathscr{V}$, $|y|\leqq\varepsilon$, b is not an eigenvalue of the operator $V^2(x,y)$.

For $0\leqq k\leqq m$, let $\xi^-_{k,(x,y)}$ (resp. $\xi^+_{k,(x,y)}$) be the direct sum of the eigenspaces of the restriction of $V^2(x,y)$ to $\xi_{k,(x,y)}$ corresponding to eigenvalues which are strictly smaller (resp. larger) than b. Set

$$\begin{aligned} \xi^\pm_{+,(x,y)} &= \bigoplus_{k \text{ even}} \xi^\pm_{k,(x,y)} \\ \xi^\pm_{-,(x,y)} &= \bigoplus_{k \text{ odd}} \xi^\pm_{k,(x,y)} \\ \xi^\pm_{(x,y)} &= \xi^\pm_{+,(x,y)} \oplus \xi^\pm_{-,(x,y)}. \end{aligned} \tag{3.26}$$

Then $\xi^\pm_k$, $\xi^\pm_\pm$, $\xi^\pm$ are smooth vector bundles on $\mathscr{V}\times\mathscr{B}_\varepsilon$. Also on $\mathscr{V}\times\mathscr{B}_\varepsilon$

$$\begin{aligned} \xi_k &= \xi^+_k \oplus \xi^-_k \\ \xi_\pm &= \xi^+_\pm \oplus \xi^-_\pm. \end{aligned} \tag{3.27}$$

Moreover the various splittings in (3.26) and (3.27) are orthogonal splittings.

Note that the restriction to $\mathscr{V}$ of the $\mathbb{Z}$ graded vector bundle ξ^- coincides with the $\mathbb{Z}$ graded vector bundle F. Therefore the restriction of ξ^+ to $\mathscr{V}$ is exactly the subbundle $F^\perp$ of $\xi_{|\mathscr{V}}$ orthogonal to F. Let $P^\pm$ be the orthogonal projection operator from ξ on $\xi^\pm$. Note that on $\mathscr{V} \simeq \mathscr{V} \times \{0\}$, P^- coincides with the orthogonal projection operator P from ξ on F considered in Prop. 1.8.

Let ∇^{ξ^+}, ∇^{ξ^-} be the connections on ξ^+, ξ^-

$$\nabla^{\xi^+} = P^+ \nabla^\xi, \qquad \nabla^{\xi^-} = P^- \nabla^\xi.$$

Let $\tilde{\nabla}^\xi = \nabla^{\xi^+} \oplus \nabla^{\xi^-}$ be the direct sum of the connections ∇^{ξ^+} and ∇^{ξ^-} on $\xi = \xi^+ \oplus \xi^-$.

For $x \in \mathscr{V}$, $|y| \leqq \varepsilon$, we identify the fiber $\xi_{(x,y)}$ with the fiber $\xi_{(x,0)} = \xi_x$ by parallel transport along the line $s \in [0, 1] \to x + sy$ with respect to the unitary connection $\tilde{\nabla}^\xi$. In particular the linear map $V(x, y)$ now acts as a self-adjoint operator on the fiber ξ_x and preserves the splitting $\xi_x = \xi_x^+ \oplus \xi_x^-$.

Let $|V|$ be the nonnegative square root of the self-adjoint nonnegative operator V^2. By Prop. 3.3 it is clear that if $\mathscr{V}$ and $\varepsilon > 0$ are small enough, there exists $C_0 > 0$ such that if $x \in \mathscr{V}$, $|y| \leqq \varepsilon$, we have the inequality of quadratic forms

$$|V(x, y)| \geqq C_0 |y|. \tag{3.28}$$

Observe that if $x \in \mathscr{V}$, $y \in N_x$, $Y = y + \bar{y} \in N_{\mathbb{R},x}$

$$\begin{aligned} \partial_Y V(x) &= P \nabla_Y^\xi V(x) P \\ &= P \tilde{\nabla}_Y^\xi V(x) P. \end{aligned} \tag{3.29}$$

As in (1.9), the key point is that if $A \in \operatorname{End} \xi_x$, then $P[A, V] P = 0$. Using (3.3), we find that by modifying the positive constant C_0 if necessary, then if $x \in \mathscr{V}$, $y \in N_x$, $Y = y + \bar{y}$

$$|P \nabla_Y^\xi V(x) P| \geqq C_0 |y|. \tag{3.30}$$

Let I_ξ, I_F be the identity maps on ξ and F. The orthogonal projection operator Q from $\xi_{|M'}$ to $F^\perp$ is given by $Q = I_\xi - P$. Of course since on $\mathscr{V}$, $\xi^+ = F^\perp$, $\xi^- = F$, we have the identities $P = P^-$, $Q = P^+$.

For $y \in \mathbb{C}^e$, $y \neq 0$, let Γ_y be the following oriented contour in $\mathbb{C}$

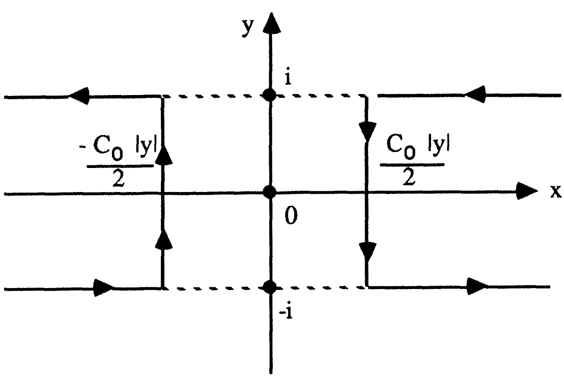

Fig. 1

Take now $y \in \mathbb{C}^e$, $y \neq 0$, and $u > 0$ such that $|y| \leqq \varepsilon \sqrt{u}$. Set $Y = y + \bar{y}$. From (3.28), (3.30) we find that if $\lambda \in \Gamma_y$ of if $|\mathrm{Im}\, \lambda| = 1$, the linear maps

$$\begin{aligned} &\lambda I_{\xi_x} - \sqrt{u}\, V\left(x, \frac{y}{\sqrt{u}}\right) \in \mathrm{End}\, \xi_x \\ &\lambda I_{F_x} - P \nabla_Y^{\xi} V(x) P \in \mathrm{End}\, F_x \end{aligned} \tag{3.31}$$

are invertible.

For $x \in M'$, let $(V^+)^{-1}(x) \in \mathrm{End}\, \xi_x$ be the linear map acting on ξ_x which is 0 on F_x, and coincides with the inverse of the restriction of $V(x)_{|F_x^\perp}$ on $F_x^\perp$.

Proposition 3.4. *For $u > 0$, $x \in \mathscr{V}$, $y \in \mathbb{C}^e$ such that $y \neq 0$, $|y| \leqq \varepsilon \sqrt{u}$, if $\lambda \in \Gamma_y$ or if $|\mathrm{Im}\, \lambda| = 1$, if $Y = y + \bar{y}$, let $A(u, x, u, \lambda) \in \mathrm{End}\, \xi_x$ be defined by the equation*

$$\begin{aligned} \left(\lambda I_\xi - \sqrt{u}\, V\left(x, \frac{y}{\sqrt{u}}\right)\right)^{-1} &= P(\lambda I_F - P \nabla_Y^{\xi} V(x) P)^{-1} P \\ &+ \frac{1}{\sqrt{u}} \{\tfrac{1}{2} P(\lambda I_F - P \nabla_Y^{\xi} V(x) P)^{-1} P \tilde{\nabla}_Y^{\xi} \tilde{\nabla}_Y^{\xi} V(x) P \\ &\cdot (\lambda I_F - P \nabla_Y^{\xi} V(x) P)^{-1} P - Q (V^+)^{-1}(x) Q\} + A(u, x, y, \lambda). \end{aligned} \tag{3.32}$$

For $\varepsilon > 0$ small enough, there exists a constant $C > 0$ such that if u, x, y, λ are taken as before, then

$$\|A(u, x, y, \lambda)\| \leqq \frac{C}{u} (|y| + |y|^4 + |\lambda| + |\lambda|^3). \tag{3.33}$$

Proof. Let $I_{\xi^\pm}$ be the identity map of $\xi^\pm$, and let $V^\pm$ be the restriction of V to $\xi^\pm$. Remember that since the connection $\tilde{\nabla}^{\xi}$ preserves the splitting $\xi = \xi^+ \oplus \xi^-$, for $x \in \mathscr{V}$, $|y| \leqq \varepsilon$, $\xi^\pm_{(x,y)}$ is identified with $\xi^\pm_x$.

Let $A^+(u, x, y, \lambda)$ be defined by the equation

$$\left(\lambda I_{\xi^+} - \sqrt{u}\, V^+\left(x, \frac{y}{\sqrt{u}}\right)\right)^{-1} = -\frac{1}{\sqrt{u}} (V^+)^{-1}(x, 0) + A^+(u, x, y, \lambda). \tag{3.34}$$

Then as $u \to +\infty$, $A_+(u, x, y, \lambda) = O\left(\frac{1}{u}\right)$. We now estimate $\|A^+(u, x, y, \lambda\|$ more precisely. In what follows, the constants $C > 0$ may vary from line to line. We have

$$\left(\lambda I_{\xi^+} - \sqrt{u}\, V^+\left(x, \frac{y}{\sqrt{u}}\right)\right)^{-1} = -\frac{(V^+)^{-1}\left(x, \frac{y}{\sqrt{u}}\right)}{\sqrt{u}} \left(I_{\xi^+} - \frac{\lambda}{\sqrt{u}} (V^+)^{-1}\left(x, \frac{y}{\sqrt{u}}\right)\right)^{-1}. \tag{3.35}$$

By finite increments, we have the inequalities

$$\left\| \left(I_{\xi^+} - \frac{\lambda}{\sqrt{u}} (V^+)^{-1} \left(x, \frac{y}{\sqrt{u}} \right) \right)^{-1} - I_{\xi^+} \right\| \leqq \frac{\lambda}{\sqrt{u}} \left\| (V^+)^{-1} \left(x, \frac{y}{\sqrt{u}} \right) \right\| \sup_{c \in [0,1]} \left\| \left(I_{\xi^+} - \frac{c\lambda}{\sqrt{u}} (V^+)^{-1} \left(x, \frac{y}{\sqrt{u}} \right) \right)^{-1} \right\|^2 . \tag{3.36}$$

If $\lambda \in \mathbb{C}^*$, one has the obvious

$$\inf_{d \in R} |1 - d\lambda|^2 = \frac{|\mathrm{Im}\,\lambda|^2}{|\lambda|^2}. \tag{3.37}$$

Therefore if $|\mathrm{Im}\,\lambda| = 1$, we deduce from (3.36), (3.37) that

$$\left\| \left(I_{\xi^+} - \frac{\lambda}{\sqrt{u}} (V^+)^{-1} \left(x, \frac{y}{\sqrt{u}} \right) \right)^{-1} - I_{\xi^+} \right\| \leqq C \frac{|\lambda|^3}{\sqrt{u}}. \tag{3.38}$$

Also by finite increments, we get

$$\left\| \left(V^+ \left(x, \frac{y}{\sqrt{u}} \right) \right)^{-1} - (V^+(x, 0))^{-1} \right\| \leqq \frac{C|y|}{\sqrt{u}}. \tag{3.39}$$

From (3.35)–(3.39), we find that if $x \in \mathscr{V}$, $|y| \leqq \varepsilon\sqrt{u}$, $|\mathrm{Im}\,\lambda| = 1$

$$\|A^+(u, x, y, \lambda)\| \leqq \frac{C}{u} (|y| + |\lambda|^3). \tag{3.40}$$

Similarly if $\lambda \in \Gamma_y$, $|\mathrm{Im}\,\lambda| \neq 1$, then $|\mathrm{Re}\,\lambda| = \frac{C_0 |y|}{2}$. Therefore if $c \in [0, 1]$, we have the inequality of quadratic forms

$$\left| \frac{c}{\sqrt{u}} \mathrm{Re}(\lambda) (V^+)^{-1} \left(x, \frac{y}{\sqrt{u}} \right) \right| \leqq C' \frac{|y|}{\sqrt{u}}. \tag{3.41}$$

For $\varepsilon > 0$ small enough, we deduce from (3.41) that since $|y| \leqq \varepsilon\sqrt{u}$

$$\left\| \left(I_{\xi^+} - \frac{c\lambda}{\sqrt{u}} (V^+)^{-1} \left(x, \frac{y}{\sqrt{u}} \right) \right)^{-1} \right\| \leqq 2. \tag{3.42}$$

From (3.35), (3.36), (3.39), (3.42), we find that if $\varepsilon > 0$ is small enough, if $x \in \mathscr{V}$, $|y| \leqq \varepsilon\sqrt{u}$, $\lambda \in \Gamma_y$, $|\mathrm{Im}\,\lambda| \neq 1$

$$\|A^+(u, x, y, \lambda)\| \leqq \frac{C}{u} (|\lambda| + |y|). \tag{3.43}$$

Remember that $V^-(x)=0$. Therefore if $A^-(u,x,y,\lambda)$ is defined by the equation

$$\left(\lambda I_{\xi^-}-\sqrt{u}\,V^-\left(x,\frac{y}{\sqrt{u}}\right)\right)^{-1}=(\lambda I_{\xi^-}-\tilde{V}_Y^{\xi}V^-(x))^{-1}+\frac{1}{2\sqrt{u}}(\lambda I_{\xi^-}-\tilde{V}_Y^{\xi}V^-(x))^{-1}$$
$$\cdot\tilde{V}_Y^{\xi}\tilde{V}_Y^{\xi}V^-(x)(\lambda I_{\xi^-}-\tilde{V}_Y^{\xi}V^-(x))^{-1}+A^-(u,x,y,\lambda) \tag{3.44}$$

then as $u\to+\infty$, $A^-(u,x,y,\lambda)=O\left(\frac{1}{u}\right)$. We now estimate $A^-(u,x,y,\lambda)$ more precisely. If $B(u,x,y)$ is defined by the equation

$$\sqrt{u}\,V^-\left(x,\frac{y}{\sqrt{u}}\right)=\tilde{V}_Y^{\xi}V^-(x)+\frac{1}{2\sqrt{u}}\tilde{V}_Y^{\xi}\tilde{V}_Y^{\xi}V^-(x)+B(u,x,y) \tag{3.45}$$

then if $x\in\mathscr{V}$, $|y|\leqq\varepsilon\sqrt{u}$

$$\|B(u,x,y)\|\leqq\frac{C|y|^3}{u}. \tag{3.46}$$

In particular if $|y|\leqq\varepsilon\sqrt{u}$, then

$$\left\|\sqrt{u}\,V^-\left(x,\frac{y}{\sqrt{u}}\right)-\tilde{V}_Y^{\xi}V^-(x)\right\|\leqq C\varepsilon|y|. \tag{3.47}$$

Let $D(u,\lambda,x,y)$ be defined by the equation

$$\left(\lambda I_{\xi^-}-\sqrt{u}\,V^-\left(x,\frac{y}{\sqrt{u}}\right)\right)^{-1}=(\lambda I_{\xi^-}-\tilde{V}_Y^{\xi}V^-(x))^{-1}+(\lambda I_{\xi^-}-\tilde{V}_Y^{\xi}V^-(x))^{-1}$$
$$\cdot\left(\sqrt{u}\,V^-\left(x,\frac{y}{\sqrt{u}}\right)-\tilde{V}_Y^{\xi}V^-(x)\right)(\lambda I_{\xi^-}-\tilde{V}_Y^{\xi}V^-(x))^{-1}+D(u,\lambda,x,y). \tag{3.48}$$

By finite increments, we find that

$$\|D(u,\lambda,x,y)\|\leqq\sup_{0\leqq c\leqq 1}\left\{\left\|\left(\lambda I_{\xi^-}-c\,\tilde{V}_Y^{\xi}V^-(x)-(1-c)\sqrt{u}\,V^-\left(x,\frac{y}{\sqrt{u}}\right)\right)^{-1}\right\|^3\right.$$
$$\left.\cdot\left\|\sqrt{u}\,V^-\left(x,\frac{y}{\sqrt{u}}\right)-\tilde{V}_Y^{\xi}V^-(x)\right\|^2\right\}. \tag{3.49}$$

If $|\mathrm{Im}\,\lambda|=1$, using (3.45), (3.46), (3.49) and the fact that $\tilde{V}_Y^{\xi}V^-(x)$ and $V^-\left(x,\frac{y}{\sqrt{u}}\right)$ are self-adjoint in End ξ_x^-, we get

$$\|D(u,\lambda,x,y)\|\leqq\frac{C}{u}|y|^4. \tag{3.50}$$

From (3.46), (3.50), we deduce that if $x \in \mathscr{V}$, $|y| \leqq \varepsilon \sqrt{u}$, $|\mathrm{Im}\,\lambda| = 1$, then

$$\|A^-(u, x, y, \lambda)\| \leqq \frac{C}{u} (|y|^4 + |y|^3). \tag{3.51}$$

If $y \neq 0$, if $\lambda \in \Gamma_y$, and if $|\mathrm{Im}\,\lambda| \neq 1$, then $|\mathrm{Re}\,\lambda| = \frac{C_0 |y|}{2}$. Using (3.47), if $c \in [0, 1]$, we get

$$\left\| (\mathrm{Re}\,\lambda) I_{\xi^-} - c \tilde{\nabla}_Y^\xi V^-(x) - (1-c)\sqrt{u}\, V^-\left(x, \frac{y}{\sqrt{u}}\right) + \tilde{\nabla}_Y^\xi V^-(x) \right\| \leqq \left(\frac{C_0}{2} + C\varepsilon\right) |y|. \tag{3.52}$$

Clearly $\tilde{\nabla}_Y^\xi V^\pm(x)$ maps $\xi_x^\pm$ into itself. Since ξ_x^- is identified with F_x, $\tilde{\nabla}_Y^\xi V^-(x)$ acts on F_x like $\partial_Y V(x)$. From (3.29), (3.30), (3.52), we deduce that

$$\begin{aligned} &\left\| \left((\mathrm{Re}\,\lambda) I_{\xi^-} - c \tilde{\nabla}_Y^\xi V^-(x) - (1-c)\sqrt{u}\, V^-\left(x, \frac{y}{\sqrt{u}}\right) + \tilde{\nabla}_Y^\xi V^-(x) \right) (\tilde{\nabla}_Y^\xi V^-(x))^{-1} \right\| \\ &\leqq \left(\frac{\frac{C_0}{2} + C\varepsilon}{C_0 |y|} \right) |y| = \frac{1}{2} + \frac{C\varepsilon}{C_0}. \end{aligned} \tag{3.53}$$

Since $\sqrt{u}\, V^-\left(x, \frac{y}{\sqrt{u}}\right)$ and $\tilde{\nabla}_Y^\xi V^-(x)$ are self-adjoint operators, we find that

$$\begin{aligned} &\left\| \left(\lambda I_{\xi^-} - c \tilde{\nabla}_Y^\xi V^-(x) - (1-c)\sqrt{u}\, V^-\left(x, \frac{y}{\sqrt{u}}\right) \right)^{-1} \right\| \\ &\leqq \left\| \left((\mathrm{Re}\,\lambda) I_{\xi^-} - c \tilde{\nabla}_Y^\xi V^-(x) - (1-c)\sqrt{u}\, V^-\left(x, \frac{y}{\sqrt{u}}\right) \right)^{-1} \right\|. \end{aligned} \tag{3.54}$$

By taking $\varepsilon > 0$ small enough so that $\frac{1}{2} + \frac{C\varepsilon}{C_0} < 1$, we deduce from (3.53), (3.54) that

$$\left\| \left(\lambda I_{\xi^-} - c \tilde{\nabla}_Y^\xi V^-(x) - (1-c)\sqrt{u}\, V^-\left(x, \frac{y}{\sqrt{u}}\right) \right)^{-1} \right\| \leqq C \,\|(\tilde{\nabla}_Y^\xi V^-(x))^{-1}\|. \tag{3.55}$$

From, (3.29), (3.30), (3.46), (3.49), (3.55), we find that if $x \in \mathscr{V}$, $y \neq 0$, $|y| \leqq \varepsilon \sqrt{u}$, if $\lambda \in \Gamma_y$, $|\mathrm{Im}\,\lambda| \neq 1$, then

$$\|D(u, \lambda, x, y)\| \leqq C \frac{|y|}{u}. \tag{3.56}$$

From (3.29), (3.30), (3.45), (3.49) and (3.55), (3.56), we deduce that if $x \in \mathscr{V}$

$$\|A^-(u, \lambda, x, y)\| \leq C \frac{|y|}{u}. \tag{3.57}$$

Clearly

$$A(u, x, y, \lambda) = A^+(u, x, y, \lambda) + A^-(u, x, y, \lambda). \tag{3.58}$$

Then (3.33) follows from (3.40), (3.43), (3.51), (3.57). □

g. The non diagonal part of the connection ∇^ξ and the covariant derivatives of V

Remember that ∇^F is the holomorphic Hermitian connection on the vector bundle F. By Prop. 1.8, we know that on $\mathscr{V}$, ∇^F coincides with the restriction of $\tilde{\nabla}^\xi$ to F. Also remember that our computations are done in the graded algebra $\Lambda(T_R^* M) \hat{\otimes} \operatorname{End} \xi$.

We now prove an extension of a result of Berline-Vergne [BeV2, Lemma 1.17]. Set

$$S = \nabla^\xi - \tilde{\nabla}^\xi. \tag{3.59}$$

S is a one form on $\mathscr{V} \times \mathscr{B}_\varepsilon$ taking values in skew-adjoint elements in $\operatorname{End} \xi$. Of course S preserves the $\mathbb{Z}$ grading of ξ and interchanges ξ^+ and ξ^-.

Proposition 3.5. *If $x \in M'$, $X \in (T_R M')_x$, then $\nabla_X^\xi V(x)$ maps F into $F^\perp$. With respect to the splitting $\xi_{|M'} = F \oplus F^\perp$, if $x \in \mathscr{V}$, $U \in (T_R M)_x$, the matrix of $S_x(U)$ is given by*

$$S_x(U) = \begin{pmatrix} 0 & P(\nabla_U^\xi V) Q (V^+)^{-1} \\ -(V^+)^{-1} Q (\nabla_U^\xi V) P & 0 \end{pmatrix}. \tag{3.60}$$

In particular the curvature $(\nabla^F)^2$ of the connection ∇^F is given by the formula

$$(\nabla^F)^2 = P(\nabla^\xi)^2 P - P(\nabla^\xi V (V^+)^{-2} \nabla^\xi V) P. \tag{3.61}$$

Proof. If f is a smooth section of F, then $Vf = 0$, and so if $X \in T_R M'$

$$(\nabla_X^\xi V) f + V \nabla_X^\xi f = 0. \tag{3.62}$$

From (3.62), we deduce that $(\nabla_X^\xi V) f \in F^\perp$. Also on $\mathscr{V} \times \mathscr{B}_\varepsilon$, if $U \in T_R M$

$$\nabla_U^\xi V = \tilde{\nabla}_U^\xi V + S(U) V - V S(U). \tag{3.63}$$

On $\mathscr{V}$, $\tilde{\nabla}_U^\xi V$ preserves the splitting $\xi_{|V} = F \oplus F^\perp$. Therefore

$$P \nabla_U^\xi V Q = P S(U) V Q$$

$$Q \nabla_U^\xi V P = -Q V S(U) P. \tag{3.64}$$

From (3.64), we deduce (3.60). Taking anticommutation rules into account, (3.61) is a standard consequence of (3.60). □

Remark 3.6. When $U \in T_R M'$, (3.60) was already proved in Berline-Vergne [BeV2, Lemma 1.17], as well as the identity (3.61).

One forms on $\mathscr{V} \times \mathscr{B}_\varepsilon$ are sums of one forms on $\mathscr{V}$ and of one forms on $\mathscr{B}_\varepsilon$. We denote by H the set of one forms of the first kind, and by $H^\perp$ the set one forms of the second kind. Two forms on $\mathscr{V} \times \mathscr{B}_\varepsilon$ are sums of forms of type (H, H), $(H, H^\perp)$ or $(H^\perp, H^\perp)$.

Let $(x^1, \ldots, x^{l_j+l'})$ be a holomorphic system of coordinates on $\mathscr{V}$. By definition

$$\nabla_H^\xi V(x, y) = \sum \left(dx^\alpha \nabla^\xi_{\frac{\partial}{\partial x^\alpha}} V(x, y) + d\bar{x}^\alpha \nabla^\xi_{\frac{\partial}{\partial \bar{x}^\alpha}} V(x, y)\right).$$

Using the identification $\xi_{(x,y)} \simeq \xi_x$, we find that $\nabla_H^\xi V(x, y)$ lies in $(\Lambda^1(T_R^* M') \hat{\otimes} \operatorname{End} \xi)_x$. We now identify $Y \in R^{2e_j}$ with the vector field $(x, Y) \to (0, Y)$ on $\mathscr{V} \times R^{2e_j}$. Then for any α, $\left[Y, \frac{\partial}{\partial x^\alpha}\right] = 0$, $\left[Y, \frac{\partial}{\partial \bar{x}^\alpha}\right] = 0$. $\nabla_H^F(P \nabla_Y^\xi V(x) P)$ is a well-defined one form on $\mathscr{V}$ taking values in $\operatorname{End} F$.

Remember that ∇^ξ and $\tilde{\nabla}^\xi$ act as differential operators on smooth sections of $\Lambda(T_R^* M) \hat{\otimes} \operatorname{End} \xi$.

Proposition 3.7. *For any $x \in \mathscr{V}$, $Y \in R^{2e_j}$*

$$P \tilde{\nabla}_Y^\xi (\nabla_H^\xi V)(x) P = \nabla_H^F (P \nabla_Y^\xi V(x) P). \tag{3.65}$$

Proof. Clearly

$$\tilde{\nabla}_Y^\xi (\nabla_H^\xi V)(x) = \nabla_Y^\xi (\nabla_H^\xi V)(x) - [S(Y), \nabla_H^\xi V](x). \tag{3.66}$$

Also since $(\nabla^\xi)^2$ is the curvature of ∇^ξ, and since the vector field Y commutes with the vector fields $\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial \bar{x}^\alpha}$

$$\nabla_Y^\xi (\nabla_H^\xi V)(x) = \nabla_H^\xi \nabla_Y^\xi V(x) + [(\nabla^\xi)^2 (Y, H), V](x). \tag{3.67}$$

By Prop. 3.5, $\nabla_H^\xi V(x)$ interchanges F and $F^\perp$. Using (3.66), (3.67), we find that

$$P \tilde{\nabla}_Y^\xi (\nabla_H^\xi V)(x) P = P \{\nabla_H^\xi \nabla_Y^\xi V(x) - S(Y) \nabla_H^\xi V(x) + \nabla_H^\xi V(x) S(Y)\} P. \tag{3.68}$$

On the other hand, we know by (3.29) that

$$P \nabla_Y^\xi V(x) P = \tilde{\nabla}_Y^\xi V(x) P \tag{3.69}$$

and so

$$P \nabla_Y^\xi V(x) P = (\nabla_Y^\xi V(x) + V(x) S(Y)) P. \tag{3.70}$$

Therefore if f is a smooth section of F on $\mathscr{V}$, using Prop. 1.8, we find that

$$(\nabla_H^F (P \nabla_Y^\xi V P)) f = P \{\nabla_H^\xi (\nabla_Y^\xi V + V S(Y)) f) - (\nabla_Y^\xi V)(\nabla_H^\xi f - S(H) f)\}. \tag{3.71}$$

and so

$$\nabla_H^\xi (P \nabla_Y^F V P) = P (\nabla_H^\xi \nabla_Y^\xi V + (\nabla_H^\xi V) S(Y) + (\nabla_Y^\xi V) S(H)) P. \tag{3.72}$$

Using Prop. 3.5, we find that since $\nabla_H^\xi V(x)$ maps F_x into $F_x^\perp$, then

$$P(S(Y)\nabla_H^\xi V(x))P = P(\nabla_Y^\xi V)(x)(V^+)^{-1}(x)(\nabla_H^\xi V(x))P$$
$$P((\nabla_Y^\xi V)(x)\,S(H))P = -P(\nabla_Y^\xi V(x))(V^+)^{-1}(x)(\nabla_H^\xi V(x))P. \tag{3.73}$$

From (3.68), (3.72), (3.73), we deduce (3.65). □

h. Contour integrals and superconnection forms

Let Δ be the oriented contour in $\mathbb{C}$

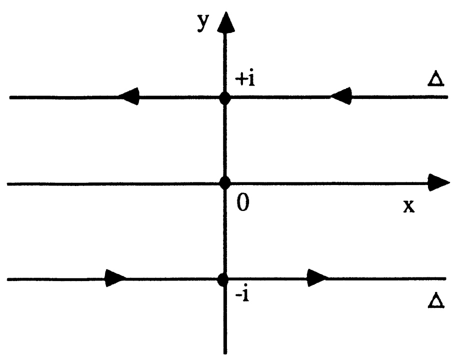

Fig. 2

Let A be a (n, n) matrix, and let $\mathrm{Sp}(A)$ be its spectrum. We assume that if $\mu \in \mathrm{Sp}(A)$, then $|\mathrm{Im}\,\mu| < 1$. Let I be the (n, n) identity matrix. We then have the identity

$$\exp(-A^2) = \frac{1}{2\pi i} \int_\Delta \frac{\exp(-\lambda^2)}{\lambda I - A}\, d\lambda. \tag{3.74}$$

In particular if A is a self-adjoint (n, n) matrix, and if B is a (n, n) matrix such that $\|B\| < 1$, then

$$\exp(-(A+B)^2) = \frac{1}{2\pi i} \int_\Delta \frac{\exp(-\lambda^2)}{\lambda I - A - B}\, d\lambda. \tag{3.75}$$

From (3.75), we deduce that the Taylor expansion of the function $B \to \exp(-(A+B))^2$ at $B=0$ is given by the series

$$\frac{1}{2\pi i} \int_\Delta \exp(-\lambda^2)[(\lambda I - A)^{-1} + (\lambda I - A)^{-1} B (\lambda I - A)^{-1} + \dots]\, d\lambda. \tag{3.76}$$

Remember that ∇^ξ acts as a first order differential operator on smooth sections of $\Lambda(T_R^* M) \hat{\otimes} \xi$ on M, so that if η, f are smooth sections of $\Lambda(T_R^* M)$, ξ respectively, then

$$\nabla^\xi(\eta f) = d\eta f + (-1)^{\deg \eta} \eta \nabla^\xi f. \tag{3.77}$$

The curvature $(\nabla^\xi)^2$ of ∇^ξ – which is a smooth section of $\Lambda^2(T_R^* M) \hat{\otimes} \operatorname{End} \xi$ – is the square of the differential operator ∇^ξ.

If $\lambda \in \Delta$, we can write the formal expansion

$$\frac{1}{\lambda I_\xi - \nabla^\xi - \sqrt{u} V} = (\lambda I_\xi - \sqrt{u} V)^{-1} + (\lambda I_\xi - \sqrt{u} V)^{-1} \nabla^\xi (\lambda I_\xi - \sqrt{u} V)^{-1} + \dots. \tag{3.78}$$

Note that ∇^ξ is of degree 1 in the Grassman variables in $\Lambda(T_R^* M)$, and so the expansion (3.78) in contains only a finite number of terms. Since $(\nabla^\xi)^2$ is a differential operator of degree 0, (3.78) is a differential operator of degree at most 1.

From now on, we consider (3.78) as an identity which defines the left-hand side of (3.78). We then claim that

$$\exp(-(\nabla^\xi + \sqrt{u} V)^2) = \frac{1}{2\pi i} \int_\Delta \frac{\exp(-\lambda^2)}{\lambda I_\xi - \nabla^\xi - \sqrt{u} V} d\lambda. \tag{3.79}$$

Of course (3.79) should be considered as an identity of formal power series in $\Lambda(T_R^* M) \hat{\otimes} \operatorname{End} \xi$. The fact that (3.79) holds is then a simple consequence of (3.75)–(3.78).

Still the left hand-side of (3.74) is a differential operator of degree 0, while a priori the right hand side of (3.79) is a differential operator of degree 1. We now briefly explain why the first order part of the operator $(\lambda I_\xi - \nabla^\xi - \sqrt{u} V)^{-1}$ is killed by integration in (3.79). Note that the supercommutator $[\nabla^\xi, V]$ lives in $\Lambda^1(T_R^* M) \hat{\otimes} \operatorname{End} \xi$. One has the easy formula

$$\begin{aligned} &((\lambda I_\xi - \sqrt{u} V)^{-1} + (\lambda^2 I_\xi - u V^2)^{-1} \nabla^\xi)(\lambda I_\xi - \nabla^\xi - \sqrt{u} V) \\ &\quad = I_\xi - (\lambda^2 I_\xi - u V^2)^{-1}((\nabla^\xi)^2 + \sqrt{u}[\nabla^\xi, V]). \end{aligned} \tag{3.80}$$

Equivalently, we have the equality

$$\begin{aligned} (\lambda I_\xi - \nabla^\xi - \sqrt{u} V)^{-1} &= \{I_\xi - (\lambda^2 I_\xi - u V^2)^{-1}((\nabla^\xi)^2 + \sqrt{u}[\nabla^\xi, V])\}^{-1}((\lambda I_\xi - \sqrt{u} V)^{-1} \\ &\quad + (\lambda^2 I_\xi - u V^2)^{-1} \nabla^\xi). \end{aligned} \tag{3.81}$$

(3.81) expresses $(\lambda I_\xi - \nabla^\xi - \sqrt{u} V)^{-1}$ as the sum of two differential operators of respective degree 0 and 1. Now observe that the function

$$\lambda \to \exp(-\lambda^2)\{I_\xi - (\lambda^2 I_\xi - u V^2)^{-1}((\nabla^\xi)^2 + \sqrt{u}[\nabla^\xi, V])\}^{-1}(\lambda^2 I_\xi - u V^2)^{-1} \tag{3.82}$$

is even and so its integral on Δ vanishes. From (3.79), (3.81), we also obtain

$$\exp(-(\nabla^\xi+\sqrt{u}\,V)^2)=\frac{1}{2\pi i}\int_\Delta (\exp(-\lambda^2)\{I_\xi-(\lambda^2 I_\xi-u V^2)^{-1}$$

$$\cdot((\nabla^\xi)^2+\sqrt{u}[\nabla^\xi, V])\}^{-1}(\lambda I_\xi-\sqrt{u}\,V)^{-1}\,d\lambda. \quad (3.83)$$

Of course, if we expand $\{(I_\xi-(\lambda^2 I_\xi-u V^2)^{-1}((\nabla^\xi)^2+\sqrt{u}[\nabla^\xi, V])\}^{-1}$ as a formal power series, (3.83) can be interpreted as the Taylor expansion of $\exp(-(u V^2+\sqrt{u}[\nabla^\xi, V]+(\nabla^\xi)^2))$. In particular $\exp(-u V^2)$ is calculated by a contour integral on the parabole Δ^2 which is the image of Δ by the map $\lambda\to\lambda^2$.

Also remember that by (3.28), for $|y|\leqq\varepsilon\sqrt{u}$, then

$$\sqrt{u}\left|V\left(x, \frac{y}{\sqrt{u}}\right)\right|\geqq C_0\,|y|. \quad (3.84)$$

Therefore if $y\neq 0$, $|y|\leqq\varepsilon\sqrt{u}$, the spectrum of the self-adjoint operator $\sqrt{u}\;V\left(x, \frac{y}{\sqrt{u}}\right)$ is contained in the interior of the contour Γ_y. So if $x\in\mathscr{V}$, $|y|\leqq\varepsilon\sqrt{u}$, $y\neq 0$, at (x, y), in (3.79), (3.83), the contour Δ can be replaced by the contour Γ_y.

i. Proof of the convergence

From now on, $[\nabla^\xi, V]$ (which is a smooth section of $\Lambda^1(T_R^* M)\hat{\otimes}\operatorname{End}\xi$) will be denoted $\nabla^\xi V$. We take $\mathscr{V}$ and $\varepsilon>0$ small enough so that the inequalities in (3.33) hold. We want to evaluate

$$\lim_{u\to+\infty}\int_{\mathscr{V}\times\mathscr{B}_{\varepsilon\sqrt{u}}}(\sigma_u^*\,\mu)\,\sigma_u^*\operatorname{Tr}_s[\exp(-(\nabla^\xi+\sqrt{u}\,V)^2)].$$

The vector bundle $\sigma_u^*\,\xi$ on $\mathscr{V}\times\mathscr{B}_{\varepsilon\sqrt{u}}$ is naturally equipped with the metric $\sigma_u^*\,h^\xi$, the connection $\sigma_u^*\,\nabla^\xi=\nabla_u^\xi$ and with the linear map $\sigma_u^*\,V=V\left(x, \frac{y}{\sqrt{u}}\right)$ so that for $|y|\leqq\varepsilon\sqrt{u}$

$$\sigma_u^*\operatorname{Tr}_s[\exp(-(\nabla^\xi+\sqrt{u}\,V)^2)]=\operatorname{Tr}_s\left[\exp-\left(\nabla_u^\xi+\sqrt{u}\;V\left(x, \frac{y}{\sqrt{u}}\right)\right)^2\right]. \quad (3.85)$$

Remember that if σ_∞ is the projection $(x, y)\in\mathscr{V}\times\mathscr{B}_\varepsilon\to\sigma_\infty(x, y)=x$, we constructed in Sect. 3 e) an explicit identification of $\xi_{|\mathscr{V}\times\mathscr{B}_\varepsilon}$ with $\sigma_\infty^*\,\xi$ (which identifies the $\mathbb{Z}$ grading and the metrics). We then consider ∇_u^ξ as a connection on $\sigma_\infty^*\,\xi$, and $V\left(x, \frac{y}{\sqrt{u}}\right)$ as an element of $\operatorname{End}\xi_x$.

Clearly for $u \geqq \frac{1}{\varepsilon^2}$

$$\int_{\mathscr{V} \times \mathscr{B}_\varepsilon \sqrt{u}} (\sigma_u^* \mu) \operatorname{Tr}_s \left[\exp -\left(\nabla_u^\xi + \sqrt{u}\, V\left(x, \frac{y}{\sqrt{u}}\right)\right)^2\right]$$

$$= \int_{\mathscr{V} \times \mathscr{B}_1} (\sigma_u^* \mu) \operatorname{Tr}_s \left[\exp -\left(\nabla_u^\xi + \sqrt{u}\, V\left(x, \frac{y}{\sqrt{u}}\right)\right)^2\right]$$

$$+ \int_{\mathscr{V} \times (\mathscr{B}_{\varepsilon\sqrt{u}} \setminus \mathscr{B}_1)} (\sigma_u^* \mu) \operatorname{Tr}_s \left[\exp\left(-\left(\nabla_u^\xi + \sqrt{u}\, V\left(x, \frac{y}{\sqrt{u}}\right)\right)\right)^2\right]. \tag{3.86}$$

Set

$$I(u, \lambda) = \left\{\left(I_\xi - \left(\lambda^2 I_\xi - u\, V^2\left(x, \frac{y}{\sqrt{u}}\right)\right)\right)^{-1} \left((\nabla_u^\xi)^2 + \sqrt{u}\, \nabla_u^\xi\, V\left(x, \frac{y}{\sqrt{u}}\right)\right)\right\}^{-1}$$

$$\cdot \left(\lambda I_\xi - \sqrt{u}\, V\left(x, \frac{y}{\sqrt{u}}\right)\right)^{-1}. \tag{3.87}$$

Clearly

$$I(u, \lambda) = \sum_{n \geqq 0} \left\{\left(\lambda^2 I_\xi - u\, V^2\left(x, \frac{y}{\sqrt{u}}\right)\right)^{-1} \left((\nabla_u^\xi)^2 + \sqrt{u}\, \nabla_u^\xi\, V\left(x, \frac{y}{\sqrt{u}}\right)\right)\right\}^n$$

$$\cdot \left(\lambda I_\xi - \sqrt{u}\, V\left(x, \frac{y}{\sqrt{u}}\right)\right)^{-1}. \tag{3.88}$$

Using (3.83), we know that

$$\exp\left(-\left(\nabla^\xi + \sqrt{u}\, V\left(x, \frac{y}{\sqrt{u}}\right)\right)^2\right) = \frac{1}{2\pi i} \int_\Delta \exp(-\lambda^2)\, I(u, \lambda)\, d\lambda. \tag{3.89}$$

The connection ∇_u^ξ on ξ obviously converges to the connection ∇_∞^ξ which is the pull-back by σ_∞ of the connection $\nabla_{|\mathscr{V}}^\xi$ on the vector bundle $\xi_{|\mathscr{V}}$. This means that in a given trivialization of ξ, the connection forms of the connections ∇_u^ξ converge uniformly to the connection form of the connection ∇_∞^ξ together with their derivatives. In particular if $E(u, x, y)$ is defined by the equation

$$(\nabla_u^\xi)^2(x, y) = (\nabla_\infty^\xi)^2(x) + E(u, x, y) \tag{3.90}$$

then

$$\|E(u, x, y)\| \leqq \frac{C}{\sqrt{u}} (1 + |y|). \tag{3.91}$$

Clearly if $F(u, x, y)$ is defined by the equation

$$\sqrt{u}\, \nabla_u^\xi\, V\left(x, \frac{y}{\sqrt{u}}\right) = \sqrt{u}\, \nabla_H^\xi\, \mathrm{V}(x) + \tilde{\nabla}_Y^\xi\, \nabla_H^\xi\, V(x) + \nabla_{H^\perp}^\xi\, V(x) + F(u, x, y) \tag{3.92}$$

then

$$\|F(u, x, y)\| \leqq \frac{C}{\sqrt{u}} (|y| + |y|^2). \tag{3.93}$$

Remember that by Prop. 3.5, $\nabla_H^\xi V(x)$ maps F_x into $F_x^\perp$. By Prop. 3.4, we find that if $\lambda\in\Delta$, or if $\lambda\in\Gamma_y$, as $u\to+\infty$

$$\left(\lambda I_\xi+\sqrt{u}\,V\left(x,\frac{y}{\sqrt{u}}\right)\right)^{-1}\sqrt{u}\,\nabla_H^\xi V(x)\left(\lambda I_\xi-\sqrt{u}\,V\left(x,\frac{y}{\sqrt{u}}\right)\right)^{-1}$$
$$=(V^+)^{-1}(x)(\nabla_H^\xi V(x))\,P(\lambda I_F-P\nabla_Y^\xi V(x)P)^{-1}P$$
$$-P(\lambda I_F-P\nabla_Y^\xi V(x)P)^{-1}P(\nabla_H^\xi V(x))(V^+)^{-1}+O\left(\frac{1}{\sqrt{u}}\right). \tag{3.94}$$

Using Prop. 3.4 and (3.94), we find that if $\lambda\in\Delta$, or if $\lambda\in\Gamma_y$, as $\mu\to+\infty$

$$\left(\lambda^2 I_\xi-u V^2\left(x,\frac{y}{\sqrt{u}}\right)\right)^{-1}\sqrt{u}\,\nabla_H^\xi V(x)\left(\lambda^2 I_\xi-u V^2\left(x,\frac{y}{\sqrt{u}}\right)\right)^{-1}=O\left(\frac{1}{\sqrt{u}}\right)$$
$$\cdot\left(\lambda^2 I_\xi-u V^2\left(x,\frac{y}{\sqrt{u}}\right)\right)^{-1}\sqrt{u}\,\nabla_H^\xi V(x)\left(\lambda^2 I_\xi-u V^2\left(x,\frac{y}{\sqrt{u}}\right)\right)^{-1}\sqrt{u}\,\nabla_H^\xi V(x)$$
$$\cdot\left(\lambda I_\xi-\sqrt{u}\,V\left(x,\frac{y}{\sqrt{u}}\right)\right)^{-1}$$
$$=-P(\lambda^2 I_F-(P\nabla_Y^\xi V(x)P)^2)^{-1}P\cdot(\nabla_H^\xi V(x))(V^+)^{-2}(x))(\nabla_H^\xi V(x))$$
$$\cdot P(\lambda I_F-P\nabla_Y^\xi V(x)P)^{-1}P+O\left(\frac{1}{\sqrt{u}}\right). \tag{3.95}$$

By Prop. 3.7, we know that

$$P\,\tilde{\nabla}_Y^\xi(\nabla_H^\xi V)(x)=\nabla_H^F(P\nabla_Y^\xi V(x)P). \tag{3.96}$$

From Prop. 3.4, and from (3.95), we find that if $\lambda\in\Delta$ of if $\lambda\in\Gamma_y$, as $u\to+\infty$

$$\left(\lambda I_{\xi_x}+\sqrt{u}\,V\left(x,\frac{y}{\sqrt{u}}\right)\right)^{-1}\tilde{\nabla}_Y^\xi(\nabla_H^\xi V)(x)\left(\lambda I_{\xi_x}-\sqrt{u}\,V\left(x,\frac{y}{\sqrt{u}}\right)\right)^{-1}$$
$$=P(\lambda I_{F_x}+P\nabla_Y^\xi V(x)P)^{-1}\nabla_H^F(P\nabla_Y^\xi V(x)P)$$
$$\cdot(\lambda I_{F_x}-P\nabla_Y^\xi V(x)P)^{-1}P+O\left(\frac{1}{\sqrt{u}}\right). \tag{3.97}$$

Also by definition

$$P\nabla_{H^\perp}^\xi V(x)P=P\nabla_{H^\perp}^F(P\nabla_Y^\xi V(x)P).$$

Using Prop. 3.4, we then find that if $\lambda\in\Delta$ of if $\lambda\in\Gamma_y$, as $u\to+\infty$

$$\left(\lambda I_{\xi_x}+\sqrt{u}\,V\left(x,\frac{y}{\sqrt{u}}\right)\right)^{-1}\nabla_{H^\perp}^\xi V(x)\left(\lambda I_{\xi_x}-\sqrt{u}\,V\left(x,\frac{y}{\sqrt{u}}\right)\right)^{-1}$$
$$=P(\lambda I_{F_x}+P\nabla_Y^\xi V(x)P)^{-1}\nabla_{H^\perp}^F(P\nabla_Y^\xi V(x)P)$$
$$\cdot(\lambda I_{F_x}-P\nabla_Y^\xi V(x)P)^{-1}P+O\left(\frac{1}{\sqrt{u}}\right). \tag{3.98}$$

From (3.32), (3.90), (3.92), (3.94), we find that for $\lambda \in \Delta$, as $u \to +\infty$, the forms $I(u, \lambda)$ converge pointwise to a form $I(\infty, \lambda)$ on $\mathscr{V} \times R^{2e_J}$.

Moreover in view of (3.94), it is clear that the terms $\sqrt{u}\, \nabla_H^\xi V$ survive in the limit either when grouped by pair as in the second line of (3.94), or at the very end of the sum by (3.93). By formula (3.61) for $(\nabla^F)^2$, and by (3.89), (3.93)–(3.98) we get the explicit formula

$$I(\infty, \lambda) = P[\sum_{n \geqq 0} \{(\lambda^2 I_F - (P \nabla_Y^\xi V P)^2)^{-1} ((\nabla^F)^2 + \nabla^F P \nabla_Y^\xi V P)\}^n]$$
$$\cdot((\lambda I_F - P \nabla_Y^\xi V P)^{-1} P - (\lambda^2 I_F - (P \nabla_Y^\xi V P)^2) P (\nabla_H^\xi V)(V^+)^{-1} Q). \tag{3.99}$$

Equivalently

$$I(\infty, \lambda) = P\{I_F - (\lambda^2 I_F - (P \nabla_Y^\xi V P)^2)^{-1} ((\nabla^F)^2 + \nabla^F P \nabla_Y^\xi V P))\}^{-1}$$
$$\cdot((\lambda I_F - P \nabla_Y^\xi V P)^{-1} - (\lambda^2 I_F - (P \nabla_Y^\xi V P)^2) P \nabla_H^\xi V (V^+)^{-1} Q). \tag{3.100}$$

Set

$$J(\lambda) = P\{I_F - (\lambda^2 I_F - (P \nabla_Y^\xi V P)^2)^{-1} ((\nabla^F)^2 + \nabla^F P \nabla_Y^\xi V P)\}^{-1}$$
$$\cdot(\lambda I_F - P \nabla_Y^\xi V P)^{-1}. \tag{3.101}$$

Using the fact that the second term in the right hand side of (3.100) is an even function of λ, we find that

$$\frac{1}{2\pi i} \int_\Delta \exp(-\lambda^2)\, I(\infty, \lambda)\, d\lambda = \frac{1}{2\pi i} \int_\Delta \exp(-\lambda^2)\, J(\lambda)\, d\lambda. \tag{3.102}$$

Using (3.83), we also get

$$\exp(-(\nabla^F + P \nabla_Y^\xi V P)^2) = \frac{1}{2\pi i} \int_\Delta \exp(-\lambda^2)\, J(\lambda)\, d\lambda. \tag{3.103}$$

For $\lambda \in \Delta$, note the estimate

$$\left\| \left(\lambda I_{\xi_x} - \sqrt{u}\, V\left(x, \frac{y}{\sqrt{u}}\right)\right)^{-1} \right\| \leqq 1$$
$$\|(\lambda I_{F_x} - P \nabla_Y^\xi V(x) P)^{-1}\| \leqq 1. \tag{3.104}$$

Using (3.33), (3.93), (3.104), we deduce easily that if $x \in \mathscr{V}$, $|y| \leqq 1$, $\lambda \in \Delta$, the norms of the various $O\left(\frac{1}{\sqrt{u}}\right)$ in (3.94)–(3.98), which of course are functions of u, x, y, λ, can be dominated by

$$\frac{C}{\sqrt{u}} (1 + |\lambda|^3). \tag{3.105}$$

If $G(u, x, y)$ is the form defined by the relation

$$(\sigma_u^* \mu)(x, y) = (\sigma_\infty^* i^* \mu)(x, y) + G(x, y, u) \tag{3.106}$$

then

$$|G(x, y, u)| \leqq \frac{C}{\sqrt{u}} \|\mu\|_{C^1(M)} (1 + |y|). \tag{3.107}$$

Also if $\lambda \in \Delta$, then

$$|\exp(-\lambda^2)| = \exp(-|\operatorname{Re} \lambda|^2 + 1) \tag{3.108}$$

and so for any $p \in N$, $|\lambda|^p \exp(-\lambda^2)|$ is integrable on the contour Δ. Using (3.89) (3.103)–(3.108) we find that as $u \to +\infty$

$$\left| \int_{\mathscr{V} \times \mathscr{B}_1} (\sigma_u^* \mu) \operatorname{Tr}_s \left[\exp\left(-\left(\nabla_u^\xi + \sqrt{u}\, V\left(x, \frac{y}{\sqrt{u}}\right)\right)^2 \right) \right] - \int_{\mathscr{V}} i^* \mu \int_{|y| \leqq 1} \operatorname{Tr}_s [\exp(-(\nabla^F + P \nabla_Y^\xi V P)^2)] \right| \leqq \frac{C}{\sqrt{u}} \|\mu\|_{C^1(M)}. \tag{3.109}$$

We now consider the second integral in the right hand side of (3.86). For a given $u > 0$, we only consider those $y \in \mathbb{C}^{e_j}$ such that $1 \leqq |y| \leqq \varepsilon \sqrt{u}$. As pointed our after Eq. (3.84), instead of Eq. (3.89), we now use the identity

$$\exp\left(-\left(\nabla_u^\xi + \sqrt{u}\, V\left(x, \frac{y}{\sqrt{u}}\right)\right)^2 \right) = \frac{1}{2\pi i} \int_{\Gamma_y} \exp(-\lambda^2)\, I(u, \lambda)\, d\lambda. \tag{3.110}$$

If $y \in \mathscr{B}_{\varepsilon\sqrt{u}} \backslash \mathscr{B}_1$, $\lambda \in \Gamma_y$, using (3.28), (3.30), we get

$$\begin{aligned} &\left\| \left(\lambda I_{\xi_x} - \sqrt{u}\, V\left(x, \frac{y}{\sqrt{u}}\right)\right)^{-1} \right\| \leqq \sup\left(\frac{2}{C_0}, 1\right) \\ &\|(\lambda I_{F_x} - P \nabla_Y^\xi V(x) P)^{-1}\| \leqq \sup\left(\frac{2}{C_0}, 1\right). \end{aligned} \tag{3.111}$$

Using (3.33), (3.90), (3.92), we now find that there exists $p \in N$, which is fixed once and for all, such that if $x \in \mathscr{V}$, $y \in \mathscr{B}_{\varepsilon\sqrt{u}} \backslash \mathscr{B}_1$, $\lambda \in \Gamma_y$, the norms of the terms $O\left(\frac{1}{\sqrt{u}}\right)$ in (3.93)–(3.98) are dominated by

$$\frac{c}{\sqrt{u}} (|y|^p + |\lambda|^p). \tag{3.112}$$

Also if $\lambda \in \Gamma_y$

$$\begin{aligned} |\exp(-\lambda^2)| &\leqq e \exp(-(\operatorname{Re} \lambda)^2) \\ &\leqq e \exp\left(\frac{-(\operatorname{Re} \lambda)^2}{2} - \frac{C_0^2}{4} |y|^2 \right). \end{aligned} \tag{3.113}$$

Using (3.110), (3.99)–(3.103) (with Δ replaced by Γ_y) and (3.111)–(3.113) we find that as $u \to +\infty$

$$\left| \int_{\mathscr{V} \times \mathscr{B}_{\varepsilon\sqrt{u}} \setminus \mathscr{B}_1} (\sigma_u^* \mu) \operatorname{Tr}_s \left[\exp\left(-\left(\nabla_u^\xi + \sqrt{u}\, V\left(x, \frac{y}{\sqrt{u}}\right)\right)^2 \right)\right] - \int_{\mathscr{V}} i^* \mu \int_{1 \leqq |y| \leqq \varepsilon\sqrt{u}} \operatorname{Tr}_s[\exp(-(\nabla^F + P \nabla_Y^\xi V P)^2)] \right| \leqq \frac{C}{\sqrt{u}} \|\mu\|_{C^1(M)}. \quad (3.114)$$

On the other hand, by Prop. 3.1, it is clear that

$$\left| \int_{\mathscr{V}} i^* \mu \int_{\varepsilon\sqrt{u} \leqq |y| < +\infty} \operatorname{Tr}_s[\exp(-(\nabla^F + P \nabla_Y^\xi V P)^2)] \right| \leqq \frac{C}{\sqrt{u}} \|\mu\|_{C^1(M)}. \quad (3.115)$$

From (3.12), (3.86), (3.109), (3.114), (3.115), we find that

$$\left| \int_Z \mu \operatorname{Tr}_s \exp(-A_u^2) - \int_Y i^* \mu \int_N \operatorname{Tr}_s[\exp(-B^2)] \right| \leqq \frac{C}{\sqrt{u}} \|\mu\|_{C^1(M)}. \quad (3.116)$$

Using the bound after (3.11), (3.112) and also partition of unity, we thus obtain (3.7) for $k=0$.

We now briefly explain how to obtain (3.7) for arbitrary $k \in N$. By using the bound after (3.11) and partition of unity, we may and we will assume that μ verifies the same support condition as in (3.12).

Let $x=(x_1, x_2)$ be a holomorphic system of coordinates on V such that $\pi(x_1, x_2)=x_1$. Then by our choice of the coordinate y, we know that $\pi(x_1, x_2, y)=x_1$. Therefore any smooth real vector field X_1 on B lifts naturally into the vector field $(X_1, 0, 0)$ in the coordinate system (x_1, x_2, y). We still note by X_1 this vector field. We then will study the behavior as $u \to +\infty$ of

$$(L_{X_1})^k \int_{\mathscr{V} \times \mathscr{B}_{\varepsilon\sqrt{u}}} (\sigma_u^* \mu) \operatorname{Tr}_s \left[\exp\left(-\left(\nabla_u^\xi + \sqrt{u}\, V\left(x, \frac{y}{\sqrt{u}}\right)\right)^2 \right)\right]. \quad (3.117)$$

Observe that if λ is taken as in Prop. 3.4, then

$$\tilde{\nabla}_{X_1}^\xi \left(\lambda I_\xi - \sqrt{u}\, V\left(x, \frac{y}{\sqrt{u}}\right)\right)^{-1} = \left(\lambda I_\xi - \sqrt{u}\, V\left(x, \frac{y}{\sqrt{u}}\right)\right)^{-1} \sqrt{u} (\tilde{\nabla}_{X_1}^\xi V)\left(x, \frac{y}{\sqrt{u}}\right) \left(\lambda I_\xi - \sqrt{u}\, V\left(x, \frac{y}{\sqrt{u}}\right)\right)^{-1}. \quad (3.118)$$

Now V and $\tilde{\nabla}^\xi$ preserve the splitting $\xi = \xi^- \oplus \xi^+$ and, on M', V vanishes on $\xi^- = F$ and maps $\xi^+ = F^\perp$ into itself. Therefore $(\tilde{\nabla}_{X_1}^\xi V)(x), \ldots, (\tilde{\nabla}_{X_1}^\xi)^k V(x) \ldots$ vanish on F and map $F^\perp$ into itself.

In particular, using Prop. 3.4, we find that as $u \to +\infty$, (3.118) has an asymptotic expansion similar to (3.32), which also starts with a constant term.

Similarly, since $\nabla_H^\xi V(x)$ maps F into $F^\perp$, this is also the case for $\tilde{\nabla}_{X_1}^\xi \nabla_H^\xi V(x), \ldots, (\tilde{\nabla}_{X_1}^\xi)^k \nabla_H^\xi V(x)$. Using (3.12), (3.86), (3.89), (3.110), (3.118) and by proceeding as before, we deduce easily that for any $k \in N$, the form $L^k_{X_1} \int_Z \mu \operatorname{Tr}_s[\exp(-A_u^2)]$ has a limit as $u \to +\infty$, and that the norm in $C^0(B)$ of the difference with the limit can be dominated by $\frac{C_k}{\sqrt{u}} \|\mu\|_{C^{k+1}(M)}$. Since this is true for any X_1 and any $k \in N$, we obtain (3.7) in full generality. From (3.7), it follows in particular that as $u \to +\infty$

$$\operatorname{Tr}_s[\exp(-A_u^2)] \to \int_N \operatorname{Tr}_s[\exp(-B^2)]\, \delta_{M'} \quad \text{in } \mathscr{D}'(M). \tag{3.119}$$

Clearly on compact subsets of $M \backslash M'$, as $u \to +\infty$, the forms $\operatorname{Tr}_s[\exp(-A_u^2)]$ converge uniformly to 0 together with their derivatives faster than $\exp(-Cu)$ (with $C>0$). So to establish (3.8), (3.9), we may and we will assume that U is a small open neighborhood of $x_0 \in M'_j$ $(1 \leqq j \leqq n)$ of the form $\mathscr{V} \times \mathscr{B}_\varepsilon$ chosen as before. Let μ be a smooth differential form on U. We now use the notations in (3.12).

Let $\|\mu\|$, $\left\|\frac{\partial \mu}{\partial y}\right\|$ be the sup of the norms of μ and of the partial derivative $\frac{\partial \mu}{\partial y}$ on U. Clearly if $|y| \leqq \varepsilon \sqrt{u}$

$$|(\sigma_u^* \mu) - (\sigma_\infty^* \mu)(x, y)| \leqq \frac{\|\mu\| + \left\|\frac{\partial \mu}{\partial y}\right\|}{\sqrt{u}} (1 + |y|). \tag{3.120}$$

Recall that $\dim M'_j = l_j + l'$. Let $\mathscr{A}$ be a partial differential operator with constant coefficients on $\mathbb{C}^{l_j + l'}$. In the sequel, $\mathscr{A}$ will be considered as acting on the variable $x \in \mathscr{V}$. We now will apply the previous results to the fibration $\mathscr{V} \times \mathscr{B}_\varepsilon \to \mathscr{V}$ with fiber $\mathscr{B}_\varepsilon$.

From (3.105), (3.108), (3.112), (3.113), (3.118) and from the considerations which follow (3.118), we find that if $\mathscr{V}$ are small enough, there exist $C>0$, $C'>0$ such that if $x \in \mathscr{V}$, $u \geqq 1$, $y \in \mathbb{C}^{e_j}$, $|y| \leqq \varepsilon \sqrt{u}$, then

$$|\mathscr{A} \sigma_u^* \operatorname{Tr}_s[\exp(-A_u^2)] - \mathscr{A} \operatorname{Tr}_s[\exp(-B^2)]| \leqq \frac{C}{\sqrt{u}} \exp(-C' |y|^2). \tag{3.121}$$

From (3.120), (3.121) and from the fact that the form $\operatorname{Tr}_s(-B^2))$ and its derivatives decay as $|y| \to +\infty$ faster that $\exp(-C'' |y|^2)$ (with $C''>0$), we get

$$\left| \int_U \mu \mathscr{A} \operatorname{Tr}_s[\exp(-A_u^2)] - \int_{\mathscr{V} \times \mathbb{C}^{e_j}} i^* \mu \mathscr{A} \operatorname{Tr}_s[\exp(-B^2)] \right|$$

$$\leqq \frac{C}{\sqrt{u}} \left(\|\mu\| + \left\|\frac{\partial \mu}{\partial y}\right\| \right). \tag{3.122}$$

Let $\hat{x} \in \mathbb{C}^{l_j + l'}$ and $\hat{y} \in \mathbb{C}^{e_j}$ be the variable conjugate to x and y. Set

$$z = (x, y); \qquad \xi = (\hat{x}, \hat{y}). \tag{3.123}$$

We will denote by $\langle z, \xi\rangle$ the real scalar product of z and ξ. Let ϕ be a smooth current with support in U. Take $\alpha>0$. Let Γ^α be the cone

$$\Gamma^\alpha=\{\xi=(\hat{x}, \hat{y})\in\mathbb{C}^{l'+l};\ |\hat{y}|\leqq\alpha|\hat{x}|\}. \tag{3.124}$$

Take $m\in N$. We will prove that there exists $C>0$ such that if $\xi\in\Gamma^\alpha$, $u\geqq 1$

$$|\xi|^{2m-1}\left|\int_M e^{i\langle z,\xi\rangle}\phi(z)\operatorname{Tr}_s[\exp(-A_u^2)]-\int_{M'} e^{i\langle x,\hat{x}\rangle} i^*\phi\int_N \operatorname{Tr}_s[\exp(-B^2)]\right|\leqq\frac{C}{\sqrt{u}}. \tag{3.125}$$

Let P_m be the differential operator with constant coefficient in the variable x such that

$$|\hat{x}|^{2m} e^{i\langle z,\xi\rangle}=P_m(e^{i\langle z,\xi\rangle}).$$

Integrating by parts, we get

$$\begin{aligned}
&|\hat{x}|^{2m}\left|\int_M e^{i\langle z,\xi\rangle}\phi\operatorname{Tr}_s[\exp(-A_u^2)]-\int_{M'} e^{i\langle x,\hat{x}\rangle} i^*\phi\int_N \operatorname{Tr}_s[\exp(-B^2)]\right|\\
&\quad=\Big|\int_M e^{i\langle z,\xi\rangle} P_m(\phi\operatorname{Tr}_s[\exp(-A_u^2)])\\
&\qquad-\int_{M'\times Ce_j} e^{i\langle x,\hat{x}\rangle} P_m(i^*\phi\operatorname{Tr}_s[\exp(-B^2)])\Big|.
\end{aligned} \tag{3.126}$$

Using (3.122) and the fact that the first derivative of the function $e^{i\langle z,\xi\rangle}$ in the variable y is bounded by $|\hat{y}|$, we get

$$\begin{aligned}
&|\hat{x}|^{2m}\left|\int_M e^{i\langle z,\xi\rangle}\phi\operatorname{Tr}_s[\exp(-A_u^2)]-\int_{M'} e^{i\langle x,\hat{x}\rangle} i^*\phi\int_N \operatorname{Tr}_s[\exp(-B^2)]\right|\\
&\quad\leqq\frac{C}{\sqrt{u}}(1+|\hat{y}|).
\end{aligned} \tag{3.127}$$

If $\xi=(\hat{x}, \hat{y})\in\Gamma^\alpha$, then $|\hat{y}|\leqq\alpha|\hat{x}|$. From (3.127), we immediately deduce (3.125). From (3.125), we obtain (3.8) and (3.9). □

Remark 3.8. In view of results in Hörmander [H, Theorems 8.2.12 and 8.2.13], (3.7) is in part a consequence of (3.8), (3.9).

Remark 3.9. Note that by (3.78), (3.79), if μ is taken as in (3.12), then,

$$\begin{aligned}
\int_Z \mu\operatorname{Tr}_s[\exp(-A_u^2)]=&\int_Z\frac{1}{2\pi i}\Sigma\int_\Delta \exp(-\lambda^2)(\sigma_u^*\mu)\operatorname{Tr}_s\left\{\left(\lambda I_\xi-\sqrt{u}\,V\left(x,\frac{y}{\sqrt{u}}\right)\right)^{-1}\right.\\
&\left.\cdot\nabla_u^\xi\left(\lambda I_\xi-\sqrt{u}\,V\left(x,\frac{y}{\sqrt{u}}\right)\right)^{-1}\cdots\nabla_u^\xi\left(\lambda I_\xi-\sqrt{u}\,V\left(x,\frac{y}{\sqrt{u}}\right)\right)^{-1}\right\}d\lambda.
\end{aligned} \tag{3.128}$$

As explained after (3.79), the integrand in the integral $\int_\Delta$ is considered as a differential operator. Now by Prop. 3.4,

$$\lim_{u\to+\infty}\left(\lambda I_\xi-\sqrt{u}\,V\left(x,\frac{y}{\sqrt{u}}\right)\right)^{-1}=P_x(\lambda I_{F_x}-P\nabla_Y^\xi V(x)P)^{-1}P_x. \tag{3.129}$$

By Prop. 1.8, we know that on M'

$$\nabla^F=P\nabla^\xi P. \tag{3.130}$$

Using (3.129), (3.130), if we take formally the limit in (3.129) without caring about delicate matters of convergence, we find that as $u\to+\infty$

$$\int_Z \mu\operatorname{Tr}_s[\exp(-A_u^2)]\to\int_Y i^*\mu\frac{1}{2\pi i}\int_N\Sigma\int_\Delta\exp(-\lambda^2)$$
$$\cdot\operatorname{Tr}_s\{(\lambda I_F-P\nabla_Y^\xi V(x)P)^{-1}\nabla^F\dots\nabla^F(\lambda I_F-P\nabla_Y^\xi V(x)P)^{-1}\}\,d\lambda. \tag{3.131}$$

Equivalently, if we use (3.78), (3.79) again, we get

$$\int_Z \mu\operatorname{Tr}_s[\exp(-A_u^2)]\to\int_Y i^*\mu\int_N\operatorname{Tr}_s[\exp(-B^2)] \tag{3.132}$$

which is the right answer stated in (3.7).

Our proof of (3.7) has precisely consisted in making sense of the arguments in (3.128)–(3.132).

j. Proof of the Mathai-Quillen identity

Under assumption (A), we have the identity of holomorphic Hermitian chain complexes on $N(F,\partial_y v)=(\Lambda N^*\otimes\eta, i_y)$. In particular $\partial_y v^*=\bar y\wedge$, and

$$\int_N\operatorname{Tr}_s[\exp(-B^2)]$$
$$=\operatorname{Tr}[\exp(-(\nabla^\eta)^2)]\int_N\operatorname{Tr}_s[\exp(-(\nabla^{\Lambda N^*}+i_y+\bar y\wedge)^2)]. \tag{3.133}$$

To prove (3.10) we then can assume that η is the trivial line bundle $\mathbb{C}$, so that $(F,\partial_y v)=(\Lambda N^{*,i_y})$. (3.10) is then a result of Mathai-Quillen [MaQ, Theorem 4.5], whose proof is reproduced here for later purposes.

Remember that we identify $\bar N$ with N^* by the metric of N. The algebra $\Lambda(N^*)$ is a N_R Clifford module. Namely if $U\in N$, $V\in\bar N$, set

$$c(U)=-i\sqrt{2}\,i_U;\qquad c(V)=-i\sqrt{2}\,V\wedge.$$

Then if $X, X'\in N\oplus\bar N=N_R\otimes\mathbb{C}$

$$c(X)\,c(X')+c(X')\,c(X)=-2\langle X, X'\rangle.$$

If $Y=y+\bar{y}$, then

$$\partial_y v+\partial_{\bar{y}} v^* = \frac{i c(Y)}{\sqrt{2}}. \tag{3.134}$$

The connection ∇^N splits the tangent bundle $T_R N$ into a horizontal part and a vertical part identified with N_R. If $Y \in N_R$, $X \in T N_R$, let $D_X Y$ be the vertical component of X. If (h_α) is a base of $T_R N$, if (h^α) is the dual base in $T_R^* N$, set

$$c(D Y) = -\Sigma h^\alpha c(D_{h_\alpha} Y).$$

Then

$$\left(\nabla^F + \frac{i c(Y)}{\sqrt{2}}\right)^2 = (\nabla^F)^2 - \frac{i c(D Y)}{\sqrt{2}} + \frac{|Y|^2}{2}. \tag{3.135}$$

Under assumption (A), $(\nabla^F)^2$ is the action on $\Lambda(N^*)$ of the curvature tensor $(\nabla^N)^2$ of the connection ∇^N on N. Let $e_1, \ldots, e_{\dim N}$ be a complex orthonormal base of N, let $\bar{e}_1, \ldots, \bar{e}_{\dim N}$ be the conjugate base of $\bar{N}$. We find easily that under assumption (A)

$$(\nabla^F)^2 = \tfrac{1}{2}\langle (\nabla^N)^2 e_i, \bar{e}_j\rangle\, c(\bar{e}_i)\, c(e_j). \tag{3.136}$$

We now temporarily replace $(\nabla^N)^2$ by any skew-adjoint endomorphism A of N. As in [MaQ, Theorem 4.5], we temporarily that A is invertible. The action A' induced by A on $\Lambda(N^*)$ is given by the analogue of (3.136)

$$A' = \tfrac{1}{2}\langle A e_i, \bar{e}_j\rangle\, c(\bar{e}_i)\, c(e_j). \tag{3.137}$$

We now proceed as in Mathai-Quillen [MaQ, Lemma 2.12]. Observe that if $X \in N_R$

$$[c(A^{-1} D Y), c(X)] = 2\langle A^{-1} D Y, X\rangle \in T_R^* N. \tag{3.138}$$

From (3.134), we deduce

$$[c(A^{-1} D Y), [c(A^{-1} D Y), c(X)]] = 0 \tag{3.139}$$

and so

$$e^{\frac{i c(A^{-1} D Y)}{\sqrt{2}}}\, c(X)\, e^{-\frac{i c(A^{-1} D Y)}{\sqrt{2}}} = c(X) + \frac{2i}{\sqrt{2}}\langle A^{-1} D Y, X\rangle. \tag{3.140}$$

From (3.137), (3.140), we get

$$A' - \frac{i c(D Y)}{\sqrt{2}} = e^{\frac{i c(A^{-1} D Y)}{\sqrt{2}}}\, A'\, e^{-\frac{i c(A^{-1} D Y)}{\sqrt{2}}} - \frac{1}{2}\langle A^{-1} D Y, D Y\rangle. \tag{3.141}$$

From (3.137) and from the fact that Tr_s vanishes on supercommutators [Q], we obtain

$$\mathrm{Tr}_s\left[\exp\left(-A'+\frac{ic(DY)}{\sqrt{2}}\right)\right]=\mathrm{Tr}_s[\exp(-A')]\exp\{\tfrac{1}{2}\langle A^{-1}DY, DY\rangle\}. \tag{3.142}$$

Classically

$$\mathrm{Tr}_s[\exp(-A')]=\det(I-\exp A). \tag{3.143}$$

Therefore from (3.138), (3.139), we get

$$\int_N \mathrm{Tr}_s\left[\exp\left(-A'+\frac{ic(DY)}{\sqrt{2}}-\frac{|Y|^2}{2}\right)\right]$$
$$=\det(I-\exp A)\int_N \exp\left\{-\frac{|Y|^2}{2}-\frac{1}{2}\langle DY, A^{-1}DY\rangle\right\}. \tag{3.144}$$

With the canonical orientation of N_R, the Pfaffian of $(-A^{-1})$ is given by

$$Pf(-A^{-1})=\frac{i^{\dim N}}{\det(-A)}. \tag{3.145}$$

Using (3.144), (3.145), we get

$$\int_N \mathrm{Tr}_s\left[\exp\left(-A'+\frac{ic(DY)}{\sqrt{2}}-\frac{|Y|^2}{2}\right)\right]=(2\pi i)^{\dim N}\mathrm{Td}^{-1}(-A). \tag{3.146}$$

Then by (3.135), (3.146), if $R^N=(\nabla^N)^2$, we obtain

$$\int_N \mathrm{Tr}_s[\exp(-B^2)]=(2\pi i)^{\dim N}\mathrm{Td}^{-1}(-R^N). \tag{3.147}$$

The proof of (3.10), and of Theorem 3.2 is completed. □

Remark 3.10. More generally, we could easily prove that for any $k\in N$, as $u\to+\infty$, $\theta_u(\mu)$ has an asymptotic expansion

$$\theta_u(\mu)=\sum_1^k \frac{\theta^j(\mu)}{u^{j/2}}+O\left(\frac{1}{u^{\frac{k+1}{2}}}\right) \tag{3.148}$$

where the $\theta^j(\mu)$ are forms in P^B and $O\left(\frac{1}{u^{\frac{k+1}{2}}}\right)$ is uniform on B. Of course the $\theta^j(\mu)$ are obtained by pairing μ with currents supported by M'.

Also, if K is any smooth section of $\mathrm{End}(\xi)$, the analogue of (3.6)–(3.9) would still hold for the forms $\mathrm{Tr}_s[K\exp(-A_u^2)]$, whose limit as $u\to+\infty$ will then be the current $\int_N \mathrm{Tr}_s[PKP\exp(-B^2)]\,\delta_{M'}$. In fact the arguments in the proof of Theorem 3.2 can be reproduced verbatim to treat this more general result, of which a special case will be needed in Theorem 4.3.

IV. Number operator and superconnection currents

In Sect. 3, we studied the behavior as $u \to +\infty$ of the current $\mathrm{Tr}_s[\exp(-A_u^2)]$. Here we study the corresponding behavior of the currents which appear in the transgression formulas of [BGS1, Theorem 1.15]. These formulas were proved again in Theorem 2.4.

This study is essential for the construction of singular Bott-Chern currents which extend the smooth Bott-Chern forms of [BGS1, Sect. 1c] to non acyclic complexes. The construction of such currents is done in a joint work with Gillet and Soulé [BGS4], and their main properties are studied in [BGS5]. In particular the microlocal properties of our currents will be constantly used in these two works.

This Section is organized as follows. In (a), we study the behavior as $u \to +\infty$ of the currents $\mathrm{Tr}_s[\sqrt{u}\, V \exp(-A_u^2)]$ and in (b), we study the currents $\mathrm{Tr}_s[N_H \exp(-A_u^2)]$.

a. *The currents* $\mathrm{Tr}_s[\sqrt{u}\, V \exp(-A_u^2)]$

In the sequel, our assumptions and notations are the same as in Sect. 3.

Theorem 4.1. *For any $k \in N$, there exists $C_k > 0$ such that for any smooth differential form μ on M, for any $u \geqq 1$*

$$\Big\| \int_Z \mu \, \mathrm{Tr}_s[\sqrt{u}\, V \exp(-A_u^2)] \Big\|_{C^k(B)} \leqq \frac{C_k}{\sqrt{u}} \|\mu\|_{C^{k+1}(M)}. \tag{4.1}$$

As $u \to +\infty$

$$\mathrm{Tr}_s[\sqrt{u}\, V \exp(-A_u^2)] \to 0 \quad \text{in } \mathscr{D}'_{N_{\mathbf{R}}^*}(M). \tag{4.2}$$

If U, Γ, ϕ, m are taken as in (3.4), *there exists $C > 0$ such that*

$$p_{U,\Gamma,\phi,m}(\mathrm{Tr}_s[\sqrt{u}\, V \exp(-A_u^2)] \leqq \frac{C}{\sqrt{u}}. \tag{4.3}$$

Proof. For $(x, a) \in M \times \mathbb{C}^*$, set $\tilde{\pi}(x, a) = (\pi x, a) \in B \times C^*$. Then the holomorphic map $\tilde{\pi}$ has essentially the same properties as π (except that $B \times C^*$ is not compact). Let j be the embedding $M' \times \mathbb{C}^* \to M \times \mathbb{C}^*$. The vector bundles $\xi_0, \ldots, \xi_m$ (resp. η) extend naturally to $M \times \mathbb{C}$ (resp. $M' \times \mathbb{C}$). Then on $M \times \mathbb{C}^*$, we have the exact sequence of sheaves

$$0 \to \mathcal{O}_{M \times \mathbb{C}^*}(\xi_m) \xrightarrow[av]{} \mathcal{O}_{M \times \mathbb{C}^*}(\xi_{m-1}) \ldots \xrightarrow[av]{} \mathcal{O}_{M \times \mathbb{C}^*}(\xi_0) \xrightarrow[r]{} j_* \mathcal{O}_{M' \times \mathbb{C}^*}(\eta) \to 0.$$

The natural holomorphic Hermitian connection on the vector bundle $\xi = \bigoplus_0^m \xi_k$ on $M \times \mathbb{C}^*$ is given by $\nabla^{\xi} + da \dfrac{\partial}{\partial a} + d\bar{a} \dfrac{\partial}{\partial \bar{a}}$.

We now restrict $a \in C^*$ to vary in the open set $O = \{a \in \mathbb{C}^*, |a-1| < \frac{1}{2}\}$. By Theorem 3.2, we know that

$$\lim_{u \to +\infty} \int_Z \mu \operatorname{Tr}_s \left[\exp\left(-\left(\nabla^\xi + da \frac{\partial}{\partial a} + d\bar{a} \frac{\partial}{\partial \bar{a}} + \sqrt{u}(a v + \bar{a} v^*)\right)^2\right)\right]$$

$$= \int_Y i^* \mu \int_N \operatorname{Tr}_s \left[\exp\left(-\left(\nabla^F + da \frac{\partial}{\partial a} + d\bar{a} \frac{\partial}{\partial \bar{a}} + a \partial_y v + \bar{a} \partial_{\bar{y}} v^*\right)^2\right)\right]. \tag{4.4}$$

More precisely we know that on $B \times O$, for any $k \in N$, $u \geqq 1$, the norm in $C^k(B \times O)$ of the difference between the expressions appearing in both sides of (4.4) is dominated by $\frac{C_k}{\sqrt{u}} \|\mu\|_{C^{k+1}(M)}$. The right hand side of (4.4) is a differential form on $B \times \mathbb{C}^*$. Using Duhamel's formula, we find that for $u \geqq 0$, there exists a form γ_u on B such that on $B \times \{1\}$

$$\int_Z \mu \operatorname{Tr}_s \left[\exp\left(-\left(\nabla^\xi + da \frac{\partial}{\partial a} + d\bar{a} \frac{\partial}{\partial \bar{a}} + \sqrt{u}(a v + \bar{a} v^*)\right)^2\right)\right]$$

$$= \int_Z \mu \operatorname{Tr}_s[\exp(-A_u^2)] - \int_Z \mu \, da \operatorname{Tr}_s[\sqrt{u} v \exp(-A_u^2)]$$

$$- \int_Z \mu \, d\bar{a} \operatorname{Tr}_s[\sqrt{u} v^* \exp(-A_u^2)] + da \, d\bar{a} \, \gamma_u. \tag{4.5}$$

Similarly there exists a form γ on B such that on $B \times \{1\}$

$$\int_Y i^* \mu \int_N \operatorname{Tr}_s \left[\exp\left(-\left(\nabla^F + da \frac{\partial}{\partial a} + d\bar{a} \frac{\partial}{\partial \bar{a}} + a \partial_y v + \bar{a} \partial_{\bar{y}} v^*\right)^2\right)\right]$$

$$= \int_Y i^* \mu \int_N \operatorname{Tr}_s[\exp(-B^2)] - \int_Y i^* \mu \, da \int_N \operatorname{Tr}_s[\partial_y v \exp(-B^2)]$$

$$- \int_Y i^* \mu \, d\bar{a} \int_N \operatorname{Tr}_s[\partial_{\bar{y}} v^* \exp(-B^2)] + da \, d\bar{a} \, \gamma. \tag{4.6}$$

Now if $y \in N$ is considered as a vector field on the total space of N, clearly $i_y (\nabla^F)^2 = 0$. Also

$$B^2 = (\nabla^F)^2 + \nabla^F (P \nabla_Y^\xi \vee P) - + (P \nabla_Y^\xi V P)^2 \tag{4.7}$$

and so

$$i_y (B^2) = \partial_y v. \tag{4.8}$$

The operator i_y is a derivation of the $\mathbb{Z}_2$ graded algebra $\Lambda(T^* N_R) \hat{\otimes} \operatorname{End} F$. Using Duhamel's formula, and the fact that the supertrace Tr_s vanishes on supercommutators, we deduce that

$$i_y \operatorname{Tr}_s[\exp(-B^2)] = -\operatorname{Tr}_s[\partial_y v \exp(-B^2)]. \tag{4.9}$$

From (4.9), we find that the form $\mathrm{Tr}_s[\partial_y v \exp(-B^2)]$ has no component of maximal degree in the direction of the vector fibers N, and so

$$\int_N \mathrm{Tr}_s[\partial_y v \exp(-B^2)]=0. \tag{4.10}$$

Similarly

$$i_{\bar{y}} \mathrm{Tr}_s[\exp(-B^2)] = -\mathrm{Tr}_s[\partial_{\bar{y}} v^* \exp(-B^2)] \tag{4.11}$$

and so

$$\int_N \mathrm{Tr}_s[\partial_{\bar{y}} v^* \exp(-B^2)]=0. \tag{4.12}$$

(4.1) now follows from the statement after (4.4) and from (4.5), (4.12). (4.2) and (4.3) follow from Theorem 3.2. □

b. The currents $\mathrm{Tr}_s[N_H \exp(-A_u^2)]$

Remember that the operator $N_H \in \mathrm{End}\,\xi$ was defined in Definition 2.3. It maps $f \in \xi_k$ into $kf \in \xi_k$. We still note by N_H the corresponding operator acting on the Z graded vector bundle $F = \bigoplus_0^m F_k$.

The definition of the Todd polynomial Td and of its inverse Td^{-1} was recalled in Sect. 3c). We now define two other polynomials Td′ and $(\mathrm{Td}^{-1})'$.

Definition 4.2. Let Td′ and $(\mathrm{Td}^{-1})'$ be the ad-invariant polynomials defined on (p, p) matrices such that if the diagonal matrix C has diagonal elements $x_1, \ldots, x_p$, then

$$\begin{aligned} \mathrm{Td}'(C) &= \frac{\partial}{\partial b}\left\{\prod_1^p \left(\frac{x_i+b}{1-e^{-(x_i+b)}}\right)\right\}_{b=0} \\ (\mathrm{Td}^{-1})'(C) &= \frac{\partial}{\partial b}\left\{\prod_1^p \left(\frac{1-e^{-(x_i+b)}}{x_i+b}\right)\right\}_{b=0}. \end{aligned} \tag{4.13}$$

Note that

$$\frac{\mathrm{Td}'(C)}{\mathrm{Td}(C)} = -\frac{(\mathrm{Td}^{-1})'(C)}{\mathrm{Td}^{-1}(C)}.$$

Observe that the polynomial Td′ appears explicitly in Bismut-Gillet-Soulé [BGS2, Theorem 2.16] in the context of Bott-Chern forms associated with direct images.

Theorem 4.3. *Let μ be a smooth differential form on the manifold M. For $u \geqq 0$, let $\eta_u(\mu)$ be the smooth differential form on B*

$$\eta_u(\mu) = \int_Z \mu\, \mathrm{Tr}_s[N_H \exp(-A_u^2)] - \int_Y i^* \mu \int_N \mathrm{Tr}_s[N_H \exp(-B^2)]. \tag{4.14}$$

Then for any $k \in N$, there exists $C_k > 0$ such that for any smooth differential for μ on M, and any $u \geqq 1$

$$\|\eta_u(\mu)\|_{C^k(B)} \leqq \frac{C_k}{\sqrt{u}} \|\mu\|_{C^{k+1}(M)}. \tag{4.15}$$

Also as $u \to +\infty$,

$$\mathrm{Tr}_s[N_H \exp(-A_u^2)] \to \int_N \mathrm{Tr}_s[N_H \exp(-B^2)]\, \delta_{M'} \quad \text{in } \mathscr{D}'_{N_{\mathbb{R}}^*}(M). \tag{4.16}$$

If U, Γ, ϕ, m are taken as in (3.4), *there exists $C' > 0$ such that*

$$p_{U,\Gamma,\phi,m}\Big(\mathrm{Tr}_s[N_H \exp(-A_u^2)] - \int_N \mathrm{Tr}_s[N_H \exp(-B^2)]\, \delta_{M'}\Big) \leqq \frac{C}{\sqrt{u}}. \tag{4.17}$$

The smooth differential form on M' $\int_N \mathrm{Tr}_s[N_H \exp(-B^2)]$ is closed. If the metrics $h^{\xi_0}, \ldots, h^{\xi_m}$ on the vector bundles $\xi_0, \ldots, \xi_m$ verify assumption (*A*) *with respect to the metrics g^N, g^η on N, η, then we have the equality of differential forms on M'*

$$\int_N \mathrm{Tr}_s[N_H \exp(-B^2)] = -(2i\pi)^{\dim N} (\mathrm{Td}^{-1})'(-R^N)\, \mathrm{Tr}[\exp(-(\nabla^\eta)^2)]. \tag{4.18}$$

Proof. As was pointed out in Remark 3.10, the proof of (4.14)–(4.17) is strictly similar to the proof of (3.7)–(3.9) in Theorem 3.2.

By Theorem 2.4, we know that on the total space of the manifold N

$$\begin{aligned} \partial \mathrm{Tr}_s[N_H \exp(-B^2)] &= -\mathrm{Tr}_s[\partial_{\bar{y}} v^* \exp(-B^2)] \\ \bar{\partial} \mathrm{Tr}_s[N_H \exp(-B^2)] &= \mathrm{Tr}_s[\partial_y v \exp(-B^2)]. \end{aligned} \tag{4.19}$$

On the other hand, we also have

$$\begin{aligned} \partial \int_N \mathrm{Tr}_s[N_H \exp(-B^2)] &= \int_N \partial \mathrm{Tr}_s[N_H \exp(-B^2)] \\ \bar{\partial} \int_N \mathrm{Tr}_s[N_H \exp(-B^2)] &= \int_N \bar{\partial} \mathrm{Tr}_s[N_H \exp(-B^2)]. \end{aligned} \tag{4.20}$$

From (4.10), (4.12), (4.19), we deduce that the form $\int_N \mathrm{Tr}_s[N_H \exp(-B^2)]$ is closed.

We now prove (4.18). We take $e_1, \ldots, e_{\dim N}$ as in (3.136) and we use the notations of Sect. 3j). One finds easily that

$$N_H = -\tfrac{1}{2} \langle I_N e_i, \bar{e}_j \rangle\, c(\bar{e}_i)\, c(e_j). \tag{4.21}$$

Also using (3.135), we find that

$$\mathrm{Tr}_s[N_H \exp(-B^2)] = \frac{\partial}{\partial b} \mathrm{Tr}_s \left[\exp\left(-\left((\nabla^F)^2 - b N_H - \frac{i c(DY)}{\sqrt{2}} + \frac{|Y|^2}{2}\right)\right)\right]_{b=0}. \tag{4.22}$$

From (3.146), (4.21), (4.22), we get

$$\int_N \operatorname{Tr}_s\left[\exp\left(-\left((\nabla^F)^2 - b\,N_H - \frac{i\,c(DY)}{\sqrt{2}} + \frac{|Y|^2}{2}\right)\right)\right]$$
$$= (2i\pi)^{\dim N}\,\mathrm{Td}^{-1}(-R^N - b\,I_N)\operatorname{Tr}[\exp(-(\nabla^\eta)^2)]. \tag{4.23}$$

(4.18) follows from (4.22), (4.23). □

Remark 4.4. Let $\mathring{Y}$ be the real vector field on N

$$\mathring{Y} = \sqrt{-1}\,(y - \bar{y}).$$

$\mathring{Y}$ generates the group of transformations $y \in N \to e^{it}\,y \in N$. Then one verifies easily that for any $b \in R$

$$(d + b\,i_{\mathring{Y}})\operatorname{Tr}_s[\exp(-B^2 + b\sqrt{-1}\,N_H)] = 0. \tag{4.24}$$

Using (4.9), (4.11) and (4.24), (4.19) follows. In fact $-\sqrt{-1}\,N_H$ is the moment map associated to the action of $\mathring{Y}$ on $\Lambda(N^*)$ in the sense of [ABo], [BeV1]. (4.24) can then be viewed as a special case of a result of Berline-Vergne [BeV1].

V. Convergence of superconnection currents on submanifolds: the non transversal case

We here study the convergence of superconnection currents associated with a complex of vectors bundles which is still acyclic out of a submanifold, but which does not provide a resolution of the sheaf of sections of a vector bundle on the submanifolds. To do this, we essentially restrict the currents considered in Sects. 3 and 4 to a submanifold which does not intersect M' transversally.

Although the general strategy of the computations is the same as in Sect. 3, the explicit formulas which extend the formula of Mathai-Quillen [MaQ, Theorem 4.5] are very interesting. Their calculation makes explicit use of basic properties of the Berezinian [M, p. 166] in supergeometry.

This section is organized as follows. In (a), we introduce our assumptions. In (b) we study the convergence of superconnection currents.

a. Assumptions and notations

We make the same assumptions as in Sect. 1 a), and we use the same notations.

Let M_1 be a complex submanifold of the manifold M. We assume that $M'_1 = M' \cap M_1$ is a complex manifold and that $TM'_1 = TM' \cap TM_1$. We also assume that the restriction of π to M_1 (resp. to M'_1) is a holomorphic submersion from M_1 (resp. M'_1) to B. Let Z_1 (resp. Y_1) be the fibers of the submersion $\pi: M_1 \to B$ (resp. $\pi: M'_1 \to B$).

Let N_1 be the normal bundle to M'_1 in M_1. Let $\tilde{N}_1$ be the holomorphic vector bundle on M'_1

$$\tilde{N}_1 = \frac{TM}{TM_1 + TM'}. \tag{5.1}$$

Then we have the exact sequence of holomorphic vector bundles on M'_1

$$0 \to N_1 \to N \to \tilde{N}_1 \to 0. \tag{5.2}$$

If the vector bundle N is equipped with a smooth Hermitian metric g^N, we equip N_1 with the induced metric. We also identify $\tilde{N}_1$ to the orthogonal $N_1^{\perp}$ to N_1 in N, and so we equip $\tilde{N}_1$ with the metric $g^{\tilde{N}_1}$ induced by the metric g^N. Let R^{N_1}, R^N, $R^{\tilde{N}}$ be the curvatures of the corresponding holomorphic Hermitian connections ∇^{N_1}, ∇^N, $\nabla^{\tilde{N}_1}$ on N_1, N, $\tilde{N}_1$.

Recall that M_1 is transversal to M' if $TM = TM_1 + TM'$, or equivalently if $\tilde{N}_1 = \{0\}$, so that $N \simeq N_1$. Here we will be essentially interested in the case where M_1 is not transversal to M'.

Let i_1 be the embedding $M'_1 \to M_1$. Let j be the embedding $M_1 \to M$. Consider the complex of sheaves

$$0 \to \mathcal{O}_{M_1}(\xi_m) \xrightarrow[v]{} \mathcal{O}_{M_1}(\xi_{m-1}) \to \dots \xrightarrow[v]{} \mathcal{O}_{M_1}(\xi_0) \xrightarrow[r]{} i_{1*}\mathcal{O}_{M'_1}(\eta) \to 0. \tag{5.3}$$

Using the local description (1.5) of the complex (ξ, v) near M' in terms of the Koszul complex of N, we deduce easily that the complex of sheaves (5.3) is exact if and only if M_1 is transversal to M'.

b. Convergence of superconnection currents on submanifolds

We now will prove a generalization of Theorem 3.2.

$\int_{Z_1}$, $\int_{Y_1}$ denotes integral along the fiber of differential forms along $\pi = M_1 \to B$, π: $M'_1 \to B$.

Theorem 5.1. *Let μ be a smooth differential form on the manifold M_1. For $u \geqq 0$, let $\theta_{1,u}(\mu)$ be the smooth differential form on B*

$$\theta_{1,u}(\mu) = \int_{Z_1} \mu j^* \operatorname{Tr}_s[\exp(-A_u^2)] - \int_{Y_1} i_1^* \mu \int_{N_1} \operatorname{Tr}_s[\exp(-B^2)]. \tag{5.4}$$

Then for any $k \in N$, there exists $C_k > 0$ such that for $u \geqq 1$

$$\|\theta_{1,u}(\mu)\|_{C^k(B)} \leqq \frac{C_k}{\sqrt{u}} \|\mu\|_{C^{k+1}(M_1)}. \tag{5.5}$$

Also as $u \to +\infty$

$$j^* \operatorname{Tr}_s[\exp(-A_u^2)] \to \int_{N_1} \operatorname{Tr}_s[\exp(-B^2)]\, \delta_{M'_1} \quad \textit{in } \mathcal{D}'_{N_{1\mathbb{R}}^*}(M_1). \tag{5.6}$$

The obvious analogue of (3.9) *still holds. If the metrics* $h^{\xi_0}, \ldots, h^{\xi_m}$ *on the vector bundles* $\xi_0, \ldots, \xi_m$ *verify assumption* (*A*) *with respect to the metrics* g^N *and* g^η *on* N *and* η, *then we have the identity of differential forms on* M'_1

$$\int_{N_1} \operatorname{Tr}_s[\exp(-B^2)] = (2i\pi)^{\dim N_1} (\operatorname{Tr}_s[\exp(-(\nabla^\eta)^2)] \operatorname{Td}^{-1}(-R^N) \det(-R^{\tilde{N}_1}). \tag{5.7}$$

Proof. The first part of Theorem 5.1 can be proved by the same method as the first part of Theorem 3.2, simply replacing $\int_Z \mu \operatorname{Tr}_s[\exp(-A_u^2)]$ by $\int_{Z_1} \mu \operatorname{Tr}_s[\exp(-A_u^2)]$ at every stage.

In particular the coercitivity conditions (3.3) and (3.13) still hold under over new assumptions. So we now concentrate on the proof of the identity (5.4).

We use the same notations as in Sect. 3j, where the identity (3.10) of Mathai-Quillen [MaQ] was proved.

Our starting point is the identity (3.144). In particular we recall that in (3.144), DY is calculated with respect to the connection ∇^N. If A is a skew-adjoint endomorphism of N, which we temporarily assume to be invertible, we deduce from (3.144) that if A' is given by (3.137), then

$$\int_{N_1} \operatorname{Tr}_s\left[\exp\left(-A' + \frac{ic(DY)}{\sqrt{2}} - \frac{|Y|^2}{2}\right)\right]$$
$$= \det(I - \exp A) \int_{N_1} \exp\left\{\frac{-|Y|^2}{2} - \frac{1}{2}\langle DY, A^{-1} DY\rangle\right\}. \tag{5.8}$$

We now evaluate the Gaussian integral in the right hand side of (5.8) by the method of Berezin integration.

Let P be the orthogonal projection operator from N on N_1. Clearly we have the identity of connections on N_1

$$\nabla^{N_1} = P \nabla^N P. \tag{5.9}$$

Then $Q = I - P$ is the orthogonal projection operator from N on $N_1^\perp$. Remember that we identify $N_1^\perp$ and $\tilde{N}_1$ as C^∞ vector bundles on M'_1. Also $\tilde{N}_1$ is exactly the unique homology group of the chain complex $0 \to N_1 \to N \to 0$. By Prop. 1.8, we get the identity of connection on $\tilde{N}_1$

$$\nabla^{\tilde{N}_1} = Q \nabla^N Q. \tag{5.10}$$

Let $\nabla' = \nabla^{N_1} \oplus \nabla^{\tilde{N}_1}$ be the orthogonal sum of the connections ∇^{N_1} and $\nabla^{\tilde{N}_1}$ on $N \simeq N_1 \oplus N_1^\perp$. Set

$$S = \nabla - \nabla'. \tag{5.11}$$

Then S is a one form on M'_1 with values in skew-adjoint elements of $\operatorname{End} N$ which interchange N_1 and $N_1^\perp$.

We now perform a reduction of squares in the two form $\langle DY, A^{-1}DY\rangle$. If $Y\in N_R$, $X\in TN_R$, let $D'_X Y$ be the vertical component of X in N_R with respect to the connection ∇'. Clearly if $Y\in N_{1R}$, if we restrict the two form $\langle DY, A^{-1}DY\rangle$ to vectors in TN_{1R} – which is the case in (5.8) –, we obtain

$$\begin{aligned}\langle DY, A^{-1}DY\rangle &= \langle D'Y+SY, A^{-1}(D'Y+SY)\rangle\\ &= \langle D'Y, A^{-1}D'Y\rangle + 2\langle D'Y, PA^{-1}QSY\rangle\\ &\quad -\langle Y, SQA^{-1}QSY\rangle.\end{aligned} \tag{5.12}$$

Set

$$B=PA^{-1}P.$$

Assume temporily that B is also invertible. From (5.11), (5.12) we obtain

$$\begin{aligned}\langle DY, A^{-1}DY\rangle &= \langle D'Y+B^{-1}PA^{-1}QSY, B(D'Y+B^{-1}PA^{-1}QSY)\rangle\\ &\quad +\langle PA^{-1}QSY, B^{-1}PA^{-1}QSY\rangle - \langle Y, SQA^{-1}QSY\rangle\\ &= \langle D'Y+B^{-1}PA^{-1}QSY, B(D'Y+B^{-1}PA^{-1}QSY)\rangle\\ &\quad -\langle Y, PSQ(A^{-1}-A^{-1}PB^{-1}PA^{-1})QSPY\rangle.\end{aligned} \tag{5.13}$$

Let ϕ be the two form on M'_1 with values in $\operatorname{End} N_1$

$$\begin{aligned}U, V\in T_R M'_1 \to \phi(U,V) &= -PS(U)Q(A^{-1}-A^{-1}PB^{-1}PA^{-1})QS(V)P\\ &\quad + PS(V)Q(A^{-1}-A^{-1}PB^{-1}PA^{-1})QS(U)P.\end{aligned}$$

Observe that

$$\phi^*(U,V)=\phi(U,V).$$

Therefore if $\operatorname{End}^s(N_1)$ is the set of self-adjoint operators in $\operatorname{End}(N_1)$, then

$$\phi\in \Lambda^2(T_R^* M'_1)\otimes \operatorname{End}^s(N_1).$$

Using (5.13), and noting that integration along N_1 saturates all the Grassmann variables $D'Y$, we obtain

$$\begin{aligned}&\det(I-\exp A)\int_{N_1}\exp\left\{\frac{-|Y|^2}{2}-\frac{1}{2}\langle DY, A^{-1}DY\rangle\right\}\\ &\quad =(2i\pi)^{\dim N_1}\det(I-\exp A)\det(-B)\det{}^{-1}\\ &\qquad \cdot(I_{N_1}-PSQ(A^{-1}-A^{-1}PB^{-1}PA^{-1})QSP).\end{aligned} \tag{5.14}$$

We may express A^{-1} in matrix form with respect to the splitting $N=N_1\oplus N_1^{\perp}$. We get

$$A^{-1}=\begin{bmatrix} B & E'_1\\ E_1 & B'\end{bmatrix}. \tag{5.15}$$

Then

$$Q(A^{-1}-A^{-1}PB^{-1}PA^{-1})Q=B'-E_1B^{-1}E'_1 \tag{5.16}$$

and so if QAQ is invertible

$$Q(A^{-1}-A^{-1}PB^{-1}PA^{-1})Q=(QAQ)^{-1}. \tag{5.17}$$

On the other hand, one has the trivial relation

$$(\det A)(\det B)=\det(QAQ). \tag{5.18}$$

So from (5.14), (5.17), we get

$$\det(I-\exp A)\int_{N_1}\exp\left\{\frac{-|Y|^2}{2}-\frac{1}{2}\langle DY, A^{-1}DY\rangle\right\}$$
$$=(2i\pi)^{\dim N_1}\operatorname{Td}^{-1}(-A)\det(-QAQ)\det{}^{-1}(I_{N_1}-PS(QAQ)^{-1}SP). \tag{5.19}$$

Set

$$C=\begin{bmatrix} I_{N_1} & PS(.)Q \\ QS(.)P & QAQ \end{bmatrix}. \tag{5.20}$$

The algebra $\operatorname{End} N$ is naturally $\mathbb{Z}_2$ graded, the even (resp. odd) elements preserving (resp. exchanging) N_1 and $N_1^\perp$. C is then even $\Lambda(T_R^* M_1')\hat{\otimes}\operatorname{End} N$.

Let $\operatorname{Ber} C\in\Lambda(T_R^* M_1')$ be the Berezinian of C [M, p 166]. By definition, we have

$$\operatorname{Ber} C=\det(I_{N_1}-PSQ(QAQ)^{-1}QSP)\det((QAQ)^{-1}). \tag{5.22}$$

On the other hand using a formula in [M, p 167] – which expresses in an essential way the fact the Berezinian is a homomorphism – we also know that

$$\operatorname{Ber} C=\det((QAQ-QSPSQ)^{-1}). \tag{5.22}$$

From (5.18)–(5.22), we obtain

$$\det(I-\exp A)\int_{N_1}\exp\left\{\frac{-|Y|^2}{2}-\frac{1}{2}\langle DY, A^{-1}DY\rangle\right\}$$
$$=(2i\pi)^{\dim N_1}\operatorname{Td}^{-1}(-A)\det(-QAQ+QS^2Q). \tag{5.23}$$

Note that in (5.22), all the invertibility conditions can be released. From (5.6), (5.22), we find that

$$\int_{N_1}\operatorname{Tr}_s[\exp(-B^2)]=(2i\pi)^{\dim N_1}\operatorname{Td}^{-1}(-R^N)\det(-Q(R^N-S^2)Q) \tag{5.24}$$

Now one verifies easily that

$$R^{\tilde{N}_1}=Q(R^N-S^2)Q. \tag{5.25}$$

(5.4) follows from (5.23)–(5.25). □

Remark 5.2: Equation (5.7) can be rewritten in the form

$$\int_{N_1}\operatorname{Tr}_s[\exp(-B^2)]$$
$$=(2i\pi)^{\dim N_1}\left[\frac{\operatorname{Td}(-R^N)}{\operatorname{Td}(-R^{\tilde{N}_1})}\right]^{-1}\det(I-\exp(R^{\tilde{N}_1}))\operatorname{Tr}[\exp(-(\nabla^\eta)^2)]. \tag{5.26}$$

Equation (5.26) is especially interesting in view of the fact that by [SGA6, p 431]

$$\operatorname{Tor}(\mathcal{O}_M(\xi), \mathcal{O}_{M_1}) = \mathcal{O}_{M'_1}(\Lambda \tilde{N}_1^* \otimes j^* \eta). \tag{5.27}$$

The obvious extension of Theorem 4.1 still holds here, simply replacing in formula (4.1) $\int\limits_Z$ by $\int\limits_{Z_1}$. The proof is of course exactly the same.

We now will prove an extension of Theorem 4.3.

Definition 5.3. Let det′ be the ad-invariant polnomial on (p, p) matrices such that if C is a (p, p) matrix

$$\det'(C) = \frac{\partial}{\partial b}\{\det(C + b\, I)\}. \tag{5.28}$$

Theorem 5. *Let μ be a smooth differential form on the manifold M_1. For $u \geqq 0$, let $\eta_{1,u}(\mu)$ be the smooth differential form on B*

$$\eta_{1,u}(\mu) = \int\limits_{Z_1} \mu j^* \operatorname{Tr}_s[N_H \exp(-A_u^2)] - \int\limits_{Y_1} i_1^* \mu \int\limits_{N_1} \operatorname{Tr}_s[N_H \exp(-B^2)]. \tag{5.29}$$

Then for any $k \in N$, there exists $C_k > 0$ such that

$$\|\eta_{1,u}(\mu)\|_{C^k(B)} \leqq \frac{C_k}{\sqrt{u}} \|\mu\|_{C^{k+1}(M_1)}. \tag{5.30}$$

As $u \to +\infty$

$$j^* \operatorname{Tr}_s[N_H \exp(-A_u^2)] \to \int\limits_{N_1} \operatorname{Tr}_s[N_H \exp(-B^2)]\, \delta_{M'_1} \quad \textit{in } \mathcal{D}'_{N_1^* R}(M_1). \tag{5.31}$$

Also the obvious analogue of (4.17) *still holds in this case.*

The smooth differential form on M'_1 $\int\limits_{N_1} \operatorname{Tr}_s[N_H \exp(-B^2)]$ is closed. If the metrics $h^{\xi_0}, \ldots, h^{\xi_m}$ on the vector bundles $\xi_0, \ldots, \xi_m$ verify assumption (A) with respect to the metrics g^N and g^η on N and η, then we have the equality of differential forms on M'_1

$$\int\limits_{N_1} \operatorname{Tr}_s[N_H \exp(-B^2)] = -(2i\pi)^{\dim N_1}\{(\operatorname{Td}^{-1})'(-R^N)\det(-R^{\tilde{N}_1})$$
$$+ \operatorname{Td}^{-1}(-R^N)\det'(-R^{\tilde{N}_1})\} \operatorname{Tr}[\exp(-(\nabla^\eta)^2)]. \tag{5.32}$$

Proof. The proof of the first two parts of Theorem 5.3 is identical to the proof of Theorem 4.3. Using (4.21), (4.22), and also (5.23), (5.25) instead of (3.146), we obtain (5.32). The Theorem is proved. □

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