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Low degree unramified cohomology of generic diagonal hypersurfaces

J.-L. Colliot-Thélène and A. N. Skorobogatov

We prove that the i-th unramified cohomology group of the generic diagonal hypersurface in the projective space of dimension $n \geq i+1$ is trivial for $i \leq 3$.

1. Introduction

Let k be a field with separable closure k_{s} and absolute Galois group $\Gamma = \mathrm{Gal}(k_{\mathrm{s}}/k)$. Let μ be a finite commutative group k-scheme of order not divisible by $\mathrm{char}(k)$. The datum of such a group k-scheme μ is equivalent to the datum of the finite Γ-module $\mu(k_{\mathrm{s}})$ of order not divisible by $\mathrm{char}(k)$. For an integer $m \geq 2$ let μ_m be the group k-scheme of m-th roots of unity. If N is a positive integer not divisible by $\mathrm{char}(k)$ such that $N\mu = 0$, then $\mu(-1)$ denotes the commutative group k-scheme $\mathrm{Hom}_{k-\mathrm{gps}}(\mu_N, \mu)$. The Galois module $\mu(-1)(k_{\mathrm{s}})$ is $\mathrm{Hom}_{\mathbb{Z}}(\mu_N(k_{\mathrm{s}}), \mu(k_{\mathrm{s}}))$ with the natural Galois action.

Let X be a smooth integral variety over k. We denote by $X^{(n)}$ the set of points of X of codimension n. In this paper, the *unramified cohomology group* $\mathrm{H}^i_{\mathrm{nr}}(X, \mu)$, where i is a positive integer, is defined as the intersection of kernels of the residue maps

$$\partial_x \colon \mathrm{H}^i(k(X), \mu) \to \mathrm{H}^{i-1}(k(x), \mu(-1)),$$

for all $x \in X^{(1)}$. For equivalent definitions, see [CT95, Thm. 4.1.1]. Restriction to the generic point of X gives rise to a natural map

$$\mathrm{H}^i_{\text{ét}}(X, \mu) \to \mathrm{H}^i_{\mathrm{nr}}(X, \mu).$$

Purity for étale cohomology implies that it is an isomorphism for $i = 1$ and a surjection for $i = 2$, see [CT95, §3.4]. In the case $i = 2$ with $\mu = \mu_m$, where

m is not divisible by $\mathrm{char}(k)$, this gives a canonical isomorphism

$$\mathrm{Br}(X)[m] \xrightarrow{\sim} \mathrm{H}^2_{\mathrm{nr}}(X, \mu_m),$$

see [CT95, Prop. 4.2.1 (a), Prop. 4.2.3 (a)]. If X is a smooth, proper, and integral variety over k, then $\mathrm{H}^i_{\mathrm{nr}}(X, \mu)$ does not depend on the choice of X in its birational equivalence class, see [CT95, Prop. 4.1.5] and [R96, Remark (5.2), Cor. (12.10)].

Let $n \geq 2$ and let $K = k(a_1, \dots, a_n)$ be the field of rational functions in the variables $a_1, \dots, a_n$. Let $X_K \subset \mathbb{P}^n_K$ be the hypersurface with equation

$$x_0^d + a_1 x_1^d + \ldots + a_n x_n^d = 0,$$

where d is not divisible by $\mathrm{char}(k)$. In this paper, for $i = 1, 2, 3$ and $n \geq i + 1$, we prove that the natural map

$$\mathrm{H}^i(K, \mu) \to \mathrm{H}^i_{\mathrm{nr}}(X_K, \mu)$$

is an isomorphism, see Theorem 4.8. In the case when $i = 2$ and $\mu = \mu_m$ with $m \geq 2$, this gives that the natural map of Brauer groups $\mathrm{Br}(K) \to \mathrm{Br}(X_K)$ induces an isomorphism of subgroups of elements of order not divisible by $\mathrm{char}(k)$, see Corollary 4.9. In the case when k has characteristic zero, this result was obtained in [GS, Thm. 1.5] by a completely different method, using the topology of the Fermat surface as a complex manifold.

In this paper we use the formalism proposed by M. Rost in [R96] which applies *inter alia* to Galois cohomology [R96, Remarks (1.11), (2.5)]. We do not use the Gersten conjecture for étale cohomology [BO74].

Let us describe the structure of this note. In Section 2 we recall some basic facts about unramified cohomology including a functoriality property of the Bloch–Ogus complex with respect to faithfully flat morphisms with integral fibres. In Section 3 we show that for smooth complete intersections $X \subset \mathbb{P}^n_k$ there are canonical isomorphisms $\mathrm{H}^i(k, \mu) \xrightarrow{\sim} \mathrm{H}^i_{\mathrm{nr}}(X, \mu)$ for $i = 1, 2$ when $\dim(X) \geq i + 1$. Generic diagonal hypersurfaces are studied in Section 4. The easy proof of the main theorem in the case $i = 1$ is given in Section 4.1. This is used in the proof for $i = 2, 3$ in Section 4.3, after some preparations in Section 4.2. Finally, in Section 5 we use a similar idea to give a short proof of the triviality of the Brauer group of certain surfaces in $\mathbb{P}^3_{k(t)}$ defined by a pair of polynomials with coefficients in k. See Theorem 5.1, which was proved in [GS] in the case when $\mathrm{char}(k) = 0$.

Our proof in this note develops a geometric idea suggested by Mathieu Florence during the second author's talk at the seminar "Variétés rationnelles" in November 2022. The authors are very grateful to Mathieu Florence for his suggestion.

2. Functoriality of the Bloch–Ogus complex

For any smooth integral variety X over k and any $i \geq 2$ there is a complex

$$0 \longrightarrow \mathrm{H}^i(k(X), \mu) \xrightarrow{(\partial_x)} \bigoplus_{x \in X^{(1)}} \mathrm{H}^{i-1}(k(x), \mu(-1)) \xrightarrow{(\partial_y)} \bigoplus_{y \in X^{(2)}} \mathrm{H}^{i-2}(k(y), \mu(-2)),$$

which we call the *Bloch–Ogus complex.* The maps in this complex are defined in [R96, (2.1.0)]. (The map ∂_x is the residue defined for discrete valuation rings by Serre [S03], see also [CTS21, Def. 1.4.3].) The proof that the resulting sequence is a complex is given in [R96, Section 2]. If $y \in X^{(2)}$ is a regular point of the closure of $x \in X^{(1)}$, then the map

$$\partial_y \colon \mathrm{H}^{i-1}(k(x), \mu(-1)) \to \mathrm{H}^{i-2}(k(y), \mu(-2))$$

is the residue map for the local ring of y in the closure of x, which is a discrete valuation ring.

The unramified cohomology group $\mathrm{H}^i_{\mathrm{nr}}(X, \mu)$ is the homology group of this complex at the term $\mathrm{H}^i(k(X), \mu)$, i.e., the intersection of $\mathrm{Ker}(\partial_x)$ for all $x \in X^{(1)}$.

Let $p \colon X \to Y$ be a faithfully flat morphism of smooth integral k-varieties with integral fibres. By [R96, Section (3.5); Prop. (4.6)(2)], there is a chain map of complexes

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{H}^i(k(X), \mu) & \longrightarrow & \bigoplus_{v \in X^{(1)}} \mathrm{H}^{i-1}(k(x), \mu(-1)) & \longrightarrow & \bigoplus_{x \in X^{(2)}} \mathrm{H}^{i-2}(k(x), \mu(-2)) \\
& \uparrow & & \uparrow & & \uparrow \\
0 \longrightarrow & \mathrm{H}^i(k(Y), \mu) & \longrightarrow & \bigoplus_{y \in Y^{(1)}} \mathrm{H}^{i-1}(k(y), \mu(-1)) & \longrightarrow & \bigoplus_{y \in Y^{(2)}} \mathrm{H}^{i-2}(k(y), \mu(-2))
\end{array}$$

The middle vertical map is the natural one if $p(x) = y$, otherwise it is zero, and similarly for the right-hand vertical map.

The morphism $X \to Y$ is called an *affine bundle* if Zariski locally on Y, it is isomorphic to $Y \times_k \mathbb{A}^n \to Y$ with affine transition morphisms. In this case the vertical maps in the above diagram induce isomorphisms on the left-hand and middle homology groups, see [R96, Prop. (8.6)]. In particular, we have an isomorphism

$$\mathrm{H}^i_{\mathrm{nr}}(X,\mu) \cong \mathrm{H}^i_{\mathrm{nr}}(Y,\mu). \tag{1}$$

Combined with [R96, Cor. (12.10)], this implies that $\mathrm{H}^i_{\mathrm{nr}}(X,\mu)$ is a stable birational invariant of smooth and proper integral k-varieties.

3. Low degree unramified cohomology of complete intersections

For a variety X over a field k we write $X^{\mathrm{s}} = X \times_k k_{\mathrm{s}}$. By a k-group of multiplicative type we understand a group k-scheme M such that M^{s} is a group k_{s}-subscheme of $(\mathbb{G}_{m,k_{\mathrm{s}}})^n$, for some $n \geq 0$. Such a k-group M is smooth if and only if $\mathrm{char}(k)$ does not divide the order of the torsion subgroup of the finitely generated abelian group $\mathrm{Hom}_{k_{\mathrm{s}}-\mathrm{gps}}(M^{\mathrm{s}}, \mathbb{G}_{m,k_{\mathrm{s}}})$. A finite commutative group k-scheme of order not divisible by $\mathrm{char}(k)$ is a k-group of multiplicative type.

Proposition 3.1. *Let X be a smooth, projective, geometrically integral variety over a field k such that the natural map $\mathrm{Pic}(X) \to \mathrm{Pic}(X^{\mathrm{s}})$ is an isomorphism of finitely generated free abelian groups. Then for any smooth k-group of multiplicative type M the natural map*

$$\mathrm{H}^2(k, M) \to \mathrm{H}^2(k(X), M)$$

is injective.

Proof. We have a commutative diagram with exact rows and natural vertical maps

$$\begin{array}{ccccccccc} 0 & \longrightarrow & k_{\mathrm{s}}^{\times} & \longrightarrow & k_{\mathrm{s}}(X)^{\times} & \longrightarrow & \mathrm{Div}(X^{\mathrm{s}}) & \longrightarrow & \mathrm{Pic}(X^{\mathrm{s}}) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \uparrow{\scriptstyle\cong} & & \\ 0 & \longrightarrow & k^{\times} & \longrightarrow & k(X)^{\times} & \longrightarrow & \mathrm{Div}(X) & \longrightarrow & \mathrm{Pic}(X) & \longrightarrow & 0 \end{array} \tag{2}$$

The abelian group $\mathrm{Pic}(X)$ is free, so the homomorphism $\mathrm{Div}(X) \to \mathrm{Pic}(X)$ has a section. Then our assumption implies that the map of Γ-modules $\mathrm{Div}(X^{\mathrm{s}}) \to \mathrm{Pic}(X^{\mathrm{s}})$ has a section. By definition, the elementary obstruction $e(X) \in \mathrm{Ext}^2_k(\mathrm{Pic}(X^{\mathrm{s}}), k_{\mathrm{s}}^{\times})$ is the class of the 2-extension of Γ-modules given by the upper row of (2). Thus we have $e(X)=0$. The result now follows from [CTS87, Prop. 2.2.5]. □

For injectivity results for the map $\mathrm{H}^2(k,M)\to\mathrm{H}^2(k(X),M)$ in the case of integral, smooth k-varieties with a k-point see [CT95, Lemma 2.1.5] and [CT95, Thm. 3.8.1]. Note that the map $\mathrm{H}^2(k,\mathbb{G}_{m,k})\to\mathrm{H}^2(k(X),\mathbb{G}_{m,k})$ is not injective when X is a conic without a k-point.

Lemma 3.2. *Let $X\subset\mathbb{P}^n_k$ be a complete intersection. Let μ be a finite commutative group k-scheme of order not divisible by* $\mathrm{char}(k)$*.*

(a) *If* $\dim(X)\geq 2$*, then the natural map* $\mathrm{H}^1(k,\mu)\to\mathrm{H}^1_{\text{ét}}(X,\mu)$ *is an isomorphism.*

(b) *If* $\dim(X)\geq 3$*, then the natural map* $\mathrm{H}^2_{\text{ét}}(\mathbb{P}^n_k,\mu)\to\mathrm{H}^2_{\text{ét}}(X,\mu)$ *is an isomorphism.*

Proof. A combination of the weak Lefschetz theorem with Poincaré duality gives that the map $\mathrm{H}^i_{\text{ét}}(\mathbb{P}^n_{k_s},\mu)\to\mathrm{H}^i_{\text{ét}}(X^{\mathrm{s}},\mu)$ is an isomorphism for $i<\dim(X)$, see [K04, Cor. B.6]. In particular, if $\dim(X)\geq 2$, then $\mathrm{H}^1_{\text{ét}}(X^{\mathrm{s}},\mu)=0$. Then the spectral sequence

$$E_2^{p,q}=\mathrm{H}^p(k,\mathrm{H}^q_{\text{ét}}(X^{\mathrm{s}},\mu))\Rightarrow\mathrm{H}^{p+q}_{\text{ét}}(X,\mu)$$

implies the first claim.

If $\dim(X)\geq 3$, then $\mathrm{H}^2_{\text{ét}}(\mathbb{P}^n_{k_s},\mu)\to\mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}},\mu)$ is an isomorphism of Γ-modules. The above spectral sequence gives rise to the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{H}^2(k,\mu) & \longrightarrow & \mathrm{H}^2_{\text{ét}}(X,\mu) & \longrightarrow & \mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}},\mu)^{\Gamma} & \longrightarrow & \mathrm{H}^3(k,\mu) \\
 & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow & & \uparrow{\scriptstyle \cong} & & \uparrow{\scriptstyle \mathrm{id}} \\
0 & \longrightarrow & \mathrm{H}^2(k,\mu) & \longrightarrow & \mathrm{H}^2_{\text{ét}}(\mathbb{P}^n_k,\mu) & \longrightarrow & \mathrm{H}^2_{\text{ét}}(\mathbb{P}^n_{k_s},\mu)^{\Gamma} & \longrightarrow & \mathrm{H}^3(k,\mu)
\end{array}$$

By the 5-lemma we deduce that $\mathrm{H}^2_{\text{ét}}(\mathbb{P}^n_k,\mu)\to\mathrm{H}^2_{\text{ét}}(X,\mu)$ is an isomorphism. □

Proposition 3.3. *Let $X\subset\mathbb{P}^n_k$ be a smooth complete intersection of dimension* $\dim(X)\geq 3$*. Let μ be a finite commutative group k-scheme of order not divisible by* $\mathrm{char}(k)$*. Then the natural map*

$$\mathrm{H}^2(k,\mu)\to\mathrm{H}^2_{\mathrm{nr}}(X,\mu)$$

is an isomorphism.

Proof. The map $\mathbb{Z} \cong \mathrm{Pic}(\mathbb{P}^n_{k_s}) \to \mathrm{Pic}(X^{\mathrm{s}})$ is an isomorphism by [H70, Ch. IV, Cor. 3.2], hence $\mathrm{Pic}(X) \to \mathrm{Pic}(X^{\mathrm{s}})$ is an isomorphism. By Proposition 3.1 it is thus enough to prove that the map $\mathrm{H}^2(k,\mu) \to \mathrm{H}^2_{\mathrm{nr}}(X,\mu)$ is surjective.

Choose an affine subspace $\mathbb{A}^n_k \subset \mathbb{P}^n_k$ such that $X \cap \mathbb{A}^n_k \neq \emptyset$. Our map is the composition of maps in the top row of the following natural commutative diagram:

$$\begin{array}{ccccccc}
\mathrm{H}^2(k,\mu) & \longrightarrow & \mathrm{H}^2_{\text{ét}}(\mathbb{P}^n_k,\mu) & \xrightarrow{\cong} & \mathrm{H}^2_{\text{ét}}(X,\mu) & \longrightarrow & \mathrm{H}^2_{\mathrm{nr}}(X,\mu) \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \cap} \\
\mathrm{H}^2(k,\mu) & \xrightarrow{\cong} & \mathrm{H}^2_{\text{ét}}(\mathbb{A}^n_k,\mu) & \longrightarrow & \mathrm{H}^2_{\text{ét}}(X\cap\mathbb{A}^n_k,\mu) & \longrightarrow & \mathrm{H}^2(k(X),\mu)
\end{array}$$

In the top row, the middle map is an isomorphism by Lemma 3.2 (b), and the right-hand map is surjective, as was recalled in the introduction. Thus any $a \in \mathrm{H}^2_{\mathrm{nr}}(X,\mu)$ can be lifted to an element $b \in \mathrm{H}^2_{\text{ét}}(\mathbb{P}^n_k,\mu)$. The image of b in $\mathrm{H}^2_{\text{ét}}(\mathbb{A}^n_k,\mu)$ comes from a unique element $c \in \mathrm{H}^2(k,\mu)$. The commutativity of the diagram gives that the image of c in $\mathrm{H}^2(k(X),\mu)$ is equal to the image of a. But the right-hand vertical map is injective, hence c is a desired lifting of a to $\mathrm{H}^2(k,\mu)$. □

4. Generic diagonal hypersurfaces

Let Π_1 (respectively, Π_2) be the projective space with homogeneous coordinates $x_0,\ldots,x_n$ (respectively, $t_0,\ldots,t_n$). Write $K = k(\Pi_2)$. Let $X \subset \Pi_1 \times \Pi_2$ be the smooth hypersurface

$$t_0x_0^d + \ldots + t_nx_n^d = 0, \tag{3}$$

where d is coprime to the characteristic exponent of k. Let p be the projection $X \to \Pi_1$, and let f be the projection $X \to \Pi_2$. The generic fibre X_K of f is a smooth diagonal hypersurface of degree d in the projective space $(\Pi_1)_K \cong \mathbb{P}^n_K$.

Lemma 4.1. *With notation as above, the following statements hold.*

(i) *The fibres of f above the codimension 1 points of Π_2 are integral if $n \geq 2$ and geometrically integral if $n \geq 3$.*

(ii) *The fibres of f above the codimension 2 points of Π_2 are integral if $n \geq 3$ and geometrically integral if $n \geq 4$.*

Proof. One only needs to check this for the singular fibres, which are the fibres above the generic points of the projective subspaces given by $t_i = 0$ or by $t_i = t_j = 0$. □

4.1. Unramified cohomology in degree 1

Lemma 4.2. *Let $f\colon X \to Y$ be a proper and flat morphism of smooth and geometrically integral varieties over a field k. Write $K = k(Y)$ and let X_K be the generic fibre of f. Assume that the fibres of f above the points of Y of codimension 1 are integral and X_K is geometrically integral. Let $m \geq 2$ be an integer. Then the map $f^*\colon \mathrm{Pic}(Y)/m \to \mathrm{Pic}(X)/m$ is injective if and only if $\mathrm{Pic}(X)[m] \to \mathrm{Pic}(X_K)[m]$ is surjective.*

Proof. In our situation we have an exact sequence

$$0 \to \mathrm{Pic}(Y) \xrightarrow{f^*} \mathrm{Pic}(X) \to \mathrm{Pic}(X_K) \to 0. \tag{4}$$

Exactness at $\mathrm{Pic}(X_K)$: since f is proper and flat, and X is smooth, the Zariski closure in X of a codimension 1 point of X_K has codimension 1 in X. On a regular variety, any Weil divisor is a Cartier divisor. Exactness at $\mathrm{Pic}(X)$: if $D \in \mathrm{Div}(X)$ restricts to a principal divisor on X_K, then D is the sum of a principal divisor in X and a divisor D' supported on a finite union of irreducible codimension 1 subvarieties of X whose generic points are not in X_K. Since f is flat and proper, hence surjective, and the fibres $f^{-1}(y)$, for $y \in Y^{(1)}$, are integral, f induces a bijection between the points $x \in X^{(1)}$ which are not in X_K and the points $y \in Y^{(1)}$. For such a pair (x, y) with $y = f(x)$, the inverse image of the divisor on Y defined by y is the divisor on X defined by x, with multiplicity one. Thus $D' \in f^*\mathrm{Div}(Y)$. Exactness at $\mathrm{Pic}(Y)$: if $D \in \mathrm{Div}(Y)$ is such that $f^*D = \mathrm{div}_X(\phi)$, where $\phi \in k(X)^\times$, then the restriction of ϕ to X_K is a regular function. Since X_K is proper over K and integral, ϕ is contained in the algebraic closure of K in $K(X)$, which is K itself because X_K is geometrically integral, see [P17, Prop. 2.2.22]. Thus we have $\phi \in K^\times$. Then $D - \mathrm{div}_Y(\phi) \in \mathrm{Div}(Y)$ goes to zero in $\mathrm{Div}(X)$. Since the map f is proper and flat, it is surjective, hence $D = \mathrm{div}_Y(\phi)$ is a principal divisor in Y.

From (4) we get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Pic}(Y) & \xrightarrow{f^*} & \mathrm{Pic}(X) & \longrightarrow & \mathrm{Pic}(X_K) & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle [m]} & & \uparrow{\scriptstyle [m]} & & \uparrow{\scriptstyle [m]} & & \\ 0 & \longrightarrow & \mathrm{Pic}(Y) & \xrightarrow{f^*} & \mathrm{Pic}(X) & \longrightarrow & \mathrm{Pic}(X_K) & \longrightarrow & 0 \end{array}$$

Applying the snake lemma to this diagram, we prove the lemma. □

Proposition 4.3. *Let $m \geq 2$ be an integer. Let k be a field of characteristic exponent coprime to m. Let $f\colon X \to Y$ be a proper and flat morphism of smooth and geometrically integral varieties over k such that*

(i) *the fibres of f above the codimension 1 points of Y are integral and the generic fibre X_K, where $K = k(Y)$, is geometrically integral;*

(ii) $\mathrm{Pic}(X)[m] = 0$*;*

(iii) $f^*\colon \mathrm{Pic}(Y)/m \to \mathrm{Pic}(X)/m$ *is injective.*

Then $\mathrm{H}^1(K, \mu_m) \to \mathrm{H}^1_{\text{ét}}(X_K, \mu_m)$ is an isomorphism.

Proof. The Kummer sequence gives rise to an exact sequence

$$0 \to K^\times/K^{\times m} \to \mathrm{H}^1_{\text{ét}}(X_K, \mu_m) \to \mathrm{Pic}(X_K)[m] \to 0.$$

By Lemma 4.2 we have $\mathrm{Pic}(X_K)[m] = 0$. □

Theorem 4.4. *Let μ be a finite commutative group k-scheme of order not divisible by* $\mathrm{char}(k)$*. Let $n \geq 2$. Let Π_1, Π_2, X, $K = k(\Pi_2)$ be as above. Then the map $\mathrm{H}^1(K, \mu) \to \mathrm{H}^1_{\text{ét}}(X_K, \mu)$ is an isomorphism.*

Proof. Let us first prove the statement for $\mu = \mu_m$ with m not divisible by $\mathrm{char}(k)$. We check the assumptions of Proposition 4.3 for $f\colon X \to \Pi_2$. Since all fibres of f have the same dimension, f is flat by miracle flatness. By Lemma 4.1, assumption (i) is satisfied. The projection $p\colon X \to \Pi_1$ is a projective bundle over Π_1. Therefore we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Pic}(\Pi_1) & \longrightarrow & \mathrm{Pic}(X) & \longrightarrow & \mathrm{Pic}(\mathbb{P}^{n-1}_{k(\Pi_1)}) & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow & & \uparrow{\scriptstyle \cong} & & \\ 0 & \longrightarrow & \mathrm{Pic}(\Pi_1) & \longrightarrow & \mathrm{Pic}(\Pi_1 \times \Pi_2) & \longrightarrow & \mathrm{Pic}((\Pi_2)_{k(\Pi_1)}) & \longrightarrow & 0 \end{array}$$

The right-hand vertical map is induced by the inclusion of a projective hyperplane in a projective space, so it is an isomorphism. Hence (ii) holds and the restriction map $\mathrm{Pic}(\Pi_1 \times \Pi_2) \to \mathrm{Pic}(X)$ is an isomorphism. It follows that $\mathrm{Pic}(\Pi_2) \to \mathrm{Pic}(X)$ is split injective, hence (iii) holds.

Let E/k be a finite Galois extension, with Galois group G, such that $\mu_E = \mu \times_k E$ is isomorphic to a finite product of groups $\mu_{m,E}$ where m is coprime to $\mathrm{char}(k)$. Let L be the compositum of the linearly disjoint field extensions K/k and E/k. We have $\mu(E) = \mu(L) = \mathrm{H}^0_{\text{ét}}(X_L, \mu)$. The Hochschild–Serre spectral sequence gives rise to the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{H}^1(G,\mu(L)) & \longrightarrow & \mathrm{H}^1_{\text{ét}}(X_K,\mu) & \longrightarrow & \mathrm{H}^1_{\text{ét}}(X_L,\mu)^G & \longrightarrow & \mathrm{H}^2(G,\mu(L)) \\
 & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow & & \uparrow{\scriptstyle \cong} & & \uparrow{\scriptstyle \mathrm{id}} \\
0 & \longrightarrow & \mathrm{H}^1(G,\mu(L)) & \longrightarrow & \mathrm{H}^1(K,\mu) & \longrightarrow & \mathrm{H}^1(L,\mu)^G & \longrightarrow & \mathrm{H}^2(G,\mu(L))
\end{array}$$

Since the result is already proved for μ_m, all vertical maps, except possibly the map $\mathrm{H}^1(K,\mu) \to \mathrm{H}^1_{\text{ét}}(X_K,\mu)$, are isomorphisms. Hence so is this map. □

Remark 4.5. The geometric argument based on the projective bundle structure of $X \subset \Pi_1 \times \Pi_2$ over Π_1 in the proof of Theorem 4.4 is needed only in the case $n = 2$, that is, when the hypersurface $X_K \subset \mathbb{P}^2_K$ is a smooth curve of degree d. When $n \geq 3$ and $X \subset \mathbb{P}^n_K$ is an *arbitrary* smooth hypersurface, we have $\mathrm{H}^1(K,\mu) \cong \mathrm{H}^1(X_K,\mu)$ by Lemma 3.2 (a).

4.2. Basic diagram

We now assume that $n \geq 3$ and $i \geq 2$, keeping the assumption that μ is a finite commutative group k-scheme of order not divisible by $\mathrm{char}(k)$. Recall the Bloch–Ogus complex from Section 2:

$$\mathrm{H}^i(k(X),\mu) \xrightarrow{(\partial_x)} \bigoplus_{x \in X^{(1)}} \mathrm{H}^{i-1}(k(x),\mu(-1)) \to \bigoplus_{x \in X^{(2)}} \mathrm{H}^{i-2}(k(x),\mu(-2)).$$

Since the fibres $X_y = f^{-1}(y)$ above $y \in \Pi_2^{(1)}$ are integral (which holds for $n \geq 2$, see Lemma 4.1) we obtain a complex

$$\mathrm{H}^i_{\mathrm{nr}}(X_K,\mu) \xrightarrow{(\partial_y)} \bigoplus_{y \in \Pi_2^{(1)}} \mathrm{H}^{i-1}(k(X_y),\mu(-1)) \to \bigoplus_{x \in X^{(2)}} \mathrm{H}^{i-2}(k(x),\mu(-2)).$$

To simplify notation, in what follows we do not write the coefficients of cohomology groups. One should bear in mind that there is a change of twist when the codimension of points increases.

Since this is a complex, the image of ∂_y is unramified over the smooth locus of X_y. If X_y is smooth we write $X'_y = X_y$. In the opposite case, X_y is the projective cone over the hyperplane section of X given by some $t_i = 0$, and then we denote by X'_y this hyperplane section, which is geometrically integral and smooth since $n \geq 3$. In this case, the smooth locus $X_{y,\mathrm{sm}} \subset X_y$ is an affine bundle over X'_y, so we have $\mathrm{H}^{i-1}_{\mathrm{nr}}(X_{y,\mathrm{sm}}) \cong \mathrm{H}^{i-1}_{\mathrm{nr}}(X'_y)$ by (1). Thus $\mathrm{Im}(\partial_y)$ is contained in $\mathrm{H}^{i-1}_{\mathrm{nr}}(X'_y)$. Since the fibres X_y above $y \in \Pi_2^{(2)}$ are integral (note that they need not be geometrically integral if $n = 3$), from the diagram in Section 2 we obtain a commutative diagram of complexes

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{H}^i_{\mathrm{nr}}(X_K)/\mathrm{H}^i(k) & \longrightarrow & \bigoplus_{y\in\Pi_2^{(1)}} \mathrm{H}^{i-1}_{\mathrm{nr}}(X'_y) & \longrightarrow & \bigoplus_{y\in\Pi_2^{(2)}} \mathrm{H}^{i-2}(k(X_y)) \\
& \uparrow & & \uparrow & & \uparrow \\
0 \longrightarrow & \mathrm{H}^i(K)/\mathrm{H}^i(k) & \longrightarrow & \bigoplus_{y\in\Pi_2^{(1)}} \mathrm{H}^{i-1}(k(y)) & \longrightarrow & \bigoplus_{y\in\Pi_2^{(2)}} \mathrm{H}^{i-2}(k(y))
\end{array}$$

where the vertical maps are induced by f. Note that since X is a projective bundle over the projective space Π_1, the map $\mathrm{H}^i(k) \to \mathrm{H}^i(k(X))$ is injective. So is the map $\mathrm{H}^i(k) \to \mathrm{H}^i(K) = \mathrm{H}^i(k(\Pi_2))$.

Let $Y = \mathbb{A}^n_k \subset \Pi_2$ be the affine space given by $t_0 \neq 0$. From the previous diagram we then get a commutative diagram of complexes

(5)
$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{H}^i_{\mathrm{nr}}(X_K)/\mathrm{H}^i(k) & \longrightarrow & \bigoplus_{y\in Y^{(1)}} \mathrm{H}^{i-1}_{\mathrm{nr}}(X'_y) & \longrightarrow & \bigoplus_{y\in Y^{(2)}} \mathrm{H}^{i-2}(k(X_y)) \\
& \uparrow & & \uparrow & & \uparrow \\
0 \longrightarrow & \mathrm{H}^i(K)/\mathrm{H}^i(k) & \longrightarrow & \bigoplus_{y\in Y^{(1)}} \mathrm{H}^{i-1}(k(y)) & \longrightarrow & \bigoplus_{y\in Y^{(2)}} \mathrm{H}^{i-2}(k(y))
\end{array}$$

Since $Y \cong \mathbb{A}^n_k$, the bottom complex is exact by [R96, Prop. 8.6].

The homology group of the top complex at the first term is $\mathrm{H}^i_{\mathrm{nr}}(X_Y)/\mathrm{H}^i(k)$, where $X_Y = f^{-1}(Y) \subset X$. Let us show that this group is zero. The fibres of $p\colon X \to \Pi_1$ are hyperplanes in Π_2. The map $p\colon X_Y \to U$ is an affine bundle, and $p(X_Y) = U$, where $U = \mathbb{P}^n_k \setminus \{(1:0:\ldots:0)\}$. By (1) the map $p^*\colon \mathrm{H}^i_{\mathrm{nr}}(U) \to \mathrm{H}^i_{\mathrm{nr}}(X_Y)$ is an isomorphism. Since U is the complement to a k-point in $\Pi_1 \cong \mathbb{P}^n_k$, and $n \geq 2$, we have

$$\mathrm{H}^i(k,\mu) \cong \mathrm{H}^i_{\mathrm{nr}}(\Pi_1,\mu) \cong \mathrm{H}^i_{\mathrm{nr}}(U,\mu).$$

The following lemma is proved by a straightforward diagram chase.

Lemma 4.6. *Suppose that we have a commutative diagram of abelian groups*

$$\begin{array}{ccccccc} & & A & \overset{i}{\hookrightarrow} & B & \overset{j}{\longrightarrow} & C \\ & & {\scriptstyle a}\uparrow & & {\scriptstyle b}\uparrow{\scriptstyle \cong} & & {\scriptstyle c}\uparrow \\ 0 & \longrightarrow & D & \longrightarrow & E & \longrightarrow & F \end{array}$$

where i is injective, b is an isomorphism, c is injective, the top row is a complex, and the bottom row is exact. Then a is an isomorphism.

From Lemma 4.6 we conclude:

Proposition 4.7. *With notation as above, if the middle vertical map in diagram* (5) *is an isomorphism and the right-hand vertical map is injective, then*

$$f^*\colon \mathrm{H}^i(K,\mu) \to \mathrm{H}^i_{\mathrm{nr}}(X_K,\mu)$$

is an isomorphism.

4.3. Unramified cohomology in degrees 2 and 3

The main result of this paper is the following

Theorem 4.8. *Let Π_1 (respectively, Π_2) be the projective space with homogeneous coordinates $x_0,\dots,x_n$ (respectively, $t_0,\dots,t_n$). Write $K = k(\Pi_2)$. Let $X \subset \Pi_1 \times \Pi_2$ be the hypersurface*

$$t_0x_0^d + \ldots + t_nx_n^d = 0. \tag{6}$$

where d is coprime to the characteristic exponent of k. Let $f\colon X \to \Pi_2$ be the natural projection, and let X_K be the generic fibre of f. Let μ be a finite commutative group k-scheme of order not divisible by $\mathrm{char}(k)$.

(i) *If $n \geq 3$, then $f^*\colon \mathrm{H}^2(K,\mu) \to \mathrm{H}^2_{\mathrm{nr}}(X_K,\mu)$ is an isomorphism.*
(ii) *If $n \geq 4$, then $f^*\colon \mathrm{H}^3(K,\mu) \to \mathrm{H}^3_{\mathrm{nr}}(X_K,\mu)$ is an isomorphism.*

Proof. (i) Consider diagram (5) for $i = 2$. Then the middle vertical map of the diagram is an isomorphism. This follows from Theorem 4.4 when X_y is singular, which happens exactly when the codimension 1 point y is given by $t_i = 0$ for some $i = 1,\dots,n$. (Note that if $n = 3$ we need Theorem 4.4 in the case $n = 2$.) If X_y is smooth, the isomorphism follows from Lemma 3.2 (a). By Lemma 4.1, each fibre X_y above a codimension 2 point y is integral, hence the right hand vertical map is injective. By Proposition 4.7, this proves (i).

(ii) Consider diagram (5) for $i = 3$. For $y \in Y^{(1)}$ such that X_y is singular, the vertical map $\mathrm{H}^2(k(y)), \mu(-1)) \to \mathrm{H}^2_{\mathrm{nr}}(X'_y, \mu(-1))$ is an isomorphism by (i). For $y \in Y^{(1)}$ such that X_y is smooth, the map $\mathrm{H}^2(k(y), \mu(-1)) \to \mathrm{H}^2_{\mathrm{nr}}(X_y, \mu(-1))$ is an isomorphism by Proposition 3.3. For $y \in \Pi_2^{(2)}$ the fibre X_y is geometrically integral over $k(y)$ by Lemma 4.1, hence $k(y)$ is separably closed in $k(X_y)$. Thus the restriction map $\mathrm{H}^1(k(y), \mu(-2)) \to \mathrm{H}^1(k(X_y), \mu(-2))$ is injective, so the right-hand vertical map in the diagram is injective. By Proposition 4.7, this proves (ii). □

Corollary 4.9. *For $n \geq 3$, the map $\mathrm{Br}(K) \to \mathrm{Br}(X_K)$ induces an isomorphism of subgroups of elements of order not divisible by $\mathrm{char}(k)$.*

Proof. This follows from Theorem 4.8 (i) by taking $\mu = \mu_m$ for each integer m not divisible by $\mathrm{char}(k)$. □

Remark 4.10. Only the case $n = 3$ of this corollary requires the above proof. For $n \geq 4$ and any smooth hypersurface in $\mathbb{P}^n$, we have the general Proposition 3.3.

5. Pairs of polynomials

In this section we give a short elementary proof that the Brauer group of the surface given by the equation (7) below over the field of rational functions $K = k(\tau)$, where $\tau = \lambda/\mu$, is naturally isomorphic to $\mathrm{Br}(K)$ away from p-primary torsion if $\mathrm{char}(k) = p$. The motivation for this comes from the recent paper [GS], where the same result was proved in the case when $\mathrm{char}(k) = 0$ (combine [GS, Thm. 1.1 (i)] and [GS, Thm. 1.4]).

Theorem 5.1. *Let k be a field. Let d be a positive integer. Let $f(x, y)$ and $g(z, t)$ be products of d pairwise non-proportional linear forms. Let $X \subset \mathbb{P}^1_k \times_k \mathbb{P}^3_k$ be the hypersurface given by*

$$\lambda f(x, y) = \mu g(z, t), \tag{7}$$

where $(\lambda : \mu)$ are homogeneous coordinates in $\mathbb{P}^1_k$ and $(x : y : z : t)$ are homogeneous coordinates in $\mathbb{P}^3_k$. Let $K = k(\mathbb{P}^1_k)$ and let X_K be the generic fibre of *the projection* $f\colon X \to \mathbb{P}^1_k$. *Then the natural map* $\mathrm{Br}(K) \to \mathrm{Br}(X_K)$ *induces an isomorphism of subgroups of elements of order not divisible by* $\mathrm{char}(k)$.

Proof. The singular locus X_{sing} is contained in the union of fibres of f above $\lambda = 0$ and $\mu = 0$. The fibre above $\mu = 0$ is given by $f(x, y) = 0$. It is a union of d planes in $\mathbb{P}^3_k$ through the line $x = y = 0$. The intersection of X_{sing} with the fibre above $\mu = 0$ is the zero-dimensional scheme given by $x = y = g(z, t) = 0$. The situation above $\lambda = 0$ is entirely similar. Let $Y = X \setminus X_{\mathrm{sing}}$ be the smooth locus of X/k. The projection $p\colon X \to \mathbb{P}^3_k$ is a birational morphism which restricts to an isomorphism $Y_V \xrightarrow{\sim} V$ on the complement V to the curve in $\mathbb{P}^3_k$ given by $f(x, y) = g(z, t) = 0$. We have

$$\mathrm{Br}(k) \cong \mathrm{Br}(\mathbb{P}^3_k) \cong \mathrm{Br}(V) \cong \mathrm{Br}(Y_V),$$

where the first isomorphism is by [CTS21, Thm. 6.1.3] and the second one is by purity for the Brauer group [CTS21, Thm. 3.7.6]. Since $Y(k) \neq \emptyset$, we have $\mathrm{Br}(k) \subset \mathrm{Br}(Y) \subset \mathrm{Br}(Y_V)$ where the second inclusion is by [CTS21, Thm. 3.5.5]. We conclude that $\mathrm{Br}(Y) \cong \mathrm{Br}(k)$.

Let $m \geq 2$ be an integer not divisible by $\mathrm{char}(k)$. If a closed fibre $X_M = f^{-1}(M)$ is smooth, then X_M is a smooth surface in $\mathbb{P}^3_{k(M)}$, thus we have

$$\mathrm{H}^1_{\text{ét}}(X_M, \mathbb{Z}/m) \cong \mathrm{H}^1(k(M), \mathbb{Z}/m) \tag{8}$$

by Lemma 3.2 (a). The smooth locus of the fibre of f above $\mu = 0$ is a disjoint union of d affine planes $\mathbb{A}^2_k$. We have

$$\mathrm{H}^1_{\text{ét}}(\mathbb{A}^2_k, \mathbb{Z}/m) \cong \mathrm{H}^1(k, \mathbb{Z}/m) \tag{9}$$

since $\mathrm{char}(k)$ does not divide m.

Without loss of generality we can write

$$f(x, y) = c\prod_{i=1}^{d}(x - \xi_i y), \qquad g(z, t) = c'\prod_{j=1}^{d}(z - \rho_j t),$$

where $c, c' \in k^\times$ and $\xi_i, \rho_j \in k$ for $i, j = 1, \dots, d$. We note that for each pair (i, j) the map $s_{ij}\colon (\lambda : \mu) \to \big((\lambda : \mu), (\xi_i : 1 : \rho_j : 1)\big)$ is a section of the morphism $f\colon X \to \mathbb{P}^1_k$.

Each section s_{ij} gives a K-point of X_K. Thus the natural map $\mathrm{Br}(K) \to \mathrm{Br}(X_K)$ is injective.

Let $\alpha \in \mathrm{Br}(X_K)[m]$. Evaluating α at the K-point of X_K given by $s_{1,1}$ gives an element $\beta \in \mathrm{Br}(K)[m]$. We replace α by $\alpha - \beta$.

Note that each section $s_{ij}(\mathbb{P}^1_k)$ meets every closed fibre of f at a smooth point. The new element $\alpha \in \mathrm{Br}(X_K)[m]$ has trivial residue on the irreducible component of the smooth locus of every fibre of f that $s_{1,1}(\mathbb{P}^1_k)$ intersects. Indeed, by (8) and (9) this residue is constant, but specialises to zero at the intersection point with $s_{1,1}(\mathbb{P}^1_k)$. In particular, α has trivial residues at the smooth fibres of f, as well as at the affine plane given by $x - \xi_1 y = 0$ in the fibre $\mu = 0$ and the affine plane given by $z - \rho_1 t = 0$ in the fibre $\lambda = 0$.

We now evaluate α at the K-point of X_K given by $s_{1,j}$, where $j = 2, \ldots, d$. The result is an element of $\mathrm{Br}(K)$ which is unramified everywhere except possibly at the k-point of $\mathbb{P}^1_k$ given by $\lambda = 0$. By Faddeev reciprocity [GS17, Thm. 6.9.1], the residue at that point must be zero, too. This implies that α is unramified at the smooth locus of the fibre at $\lambda = 0$. A similar argument using sections $s_{i,1}$ for $i = 2, \ldots, d$ shows that α is unramified at the smooth locus of the fibre at $\mu = 0$.

We see that the residue of α at every codimension 1 point of Y is zero. By the purity for the Brauer group, α belongs to $\mathrm{Br}(Y)$. We have proved earlier that the natural map $\mathrm{Br}(k) \to \mathrm{Br}(Y)$ is an isomorphism, hence $\alpha \in \mathrm{Br}(k)$. It follows that $\mathrm{Br}(K)[m] \to \mathrm{Br}(X_K)[m]$ is an isomorphism. □

References

[BO74] S. Bloch and A. Ogus. Gersten's conjecture and the homology of schemes. Ann. Sci. École Norm. Sup. (4) **7** (1974), 181–201 (1975).

[CT95] J.-L. Colliot-Thélène. Birational invariants, purity and the Gersten conjecture. *K-theory and algebraic geometry: connections with quadratic forms and division algebras* (Santa Barbara, 1992), Proc. Sympos. Pure Math. **58**, Part 1, Amer. Math. Soc., 1995, pp. 1–64.

[CTS87] J.-L. Colliot-Thélène and J.-J. Sansuc. La descente sur les variétés rationnelles, II. *Duke Math. J.* **54** (1987), 375–492.

[CTS21] J.-L. Colliot-Thélène and A.N. Skorobogatov. *The Brauer–Grothendieck group.* Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, Band 71, Springer, 2021.

[GS17] P. Gille and T. Szamuely. *Central simple algebras and Galois cohomology.* 2nd ed. Cambridge Studies in Advanced Mathematics **165**, Cambridge University Press, 2017.

[GS] D. Gvirtz-Chen and A.N. Skorobogatov. Surfaces defined by pairs of polynomials. `arXiv:2305.08632`

[H70] R. Hartshorne. *Ample subvarieties of algebraic varieties.* Lecture Notes in Mathematics **156**, Springer-Verlag, 1970.

[K04] N.M. Katz. Applications of the Weak Lefschetz Theorem. Appendix to: B. Poonen and J.F. Voloch. Random Diophantine equations. *Arithmetic of higher-dimensional algebraic varieties* (Palo Alto, 2002), Progress in Mathematics **226**, 175–184, Birkhäuser, 2004.

[P17] B. Poonen. *Rational points on varieties.* Graduate Studies in Mathematics **186**. American Mathematical Society, 2017.

[R96] M. Rost. Chow groups with coefficients. *Doc. Math.* **1** (1996), 319–393.

[S03] J.-P. Serre. Cohomological invariants, Witt invariants, and trace forms. Notes by S. Garibaldi. *Cohomological invariants in Galois cohomology*, University Lecture Series **28**, American Mathematical Society, 2003, pp. 1–100.

Laboratoire de Mathématiques d'Orsay
Université Paris-Saclay, CNRS
91405, Orsay, France
E-mail address: jean-louis.colliot-thelene@universite-paris-saclay.fr

Department of Mathematics, Imperial College London
South Kensington Campus, SW7 2BZ, England, UK
and Institute for the Information Transmission Problems
Russian Academy of Sciences, Moscow 127994, Russia
E-mail address: a.skorobogatov@imperial.ac.uk

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