ARITHMETIC ON SOME SINGULAR CUBIC HYPERSURFACES

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Introduction

In this paper, we prove the following theorem.

THEOREM. *Let k be a number field, and let $X \subset \mathbb{P}_k^n$ $(n \geqslant 3)$ be a cubic hypersurface defined over k. If X contains a set of three conjugate singular points, and if X has rational points over all completions of k, then X has a rational point over k. Moreover, if X is geometrically integral and is not a cone, and if $n \neq 4$, then weak approximation holds for $X_{\mathrm{smooth}} \subset X$; in the case where $n = 4$, the Brauer–Manin obstruction to weak approximation on X_{smooth} is the only one.*

(Concrete equations for such cubic hypersurfaces are given in Proposition 1.6.)

The local assumptions come automatically as soon as $n \geqslant 9$ (Demj'anov, Lewis, Springer, cf. [**4**, § 4]). Thus any cubic hypersurface $X \subset \mathbb{P}_k^n$ with $n \geqslant 9$ with a set of three conjugate singular points has a rational point over the number field k. The same statement is obvious when X contains a k-rational singular point or a pair of conjugate singular points (since X then contains the line through them; note however that some work then remains to be done as far as weak approximation is concerned; see [**3**, 9.8.3]). When $k = \mathbf{Q}$, X is smooth, and $n \geqslant 9$, it is a deep result of Heath-Brown [**7**] that X has a $\mathbf{Q}$-rational point.

The case $n = 3$ of the theorem, part of which goes back to Skolem [**12**] and B. Segre [**11**], is known (Coray [**5**], Coray and Tsfasman [**6**]).

The proof of the above theorem for $n \geqslant 4$ uses techniques similar to those of the joint paper [**3**] of Sansuc, Swinnerton-Dyer, and the first-named author: the fibration method and descent theory. We refer to [**2**] for the account of descent theory. The fibration method is described at some length in the introduction of [**3**]. This method aims at proving the Hasse principle and weak approximation for the (smooth locus of the) total space of a proper (surjective) fibration $f\colon X \to Y$ of k-varieties, once these properties are known for the base space Y and all fibres $f^{-1}(P)$, with $P \in Y(k)$. Here f need not be smooth, but crucial requirements for the method to work are that all geometric fibres of f be integral and that Y be proper over k. The method sometimes also works over a non-proper basis, for example, $Y = \mathbb{A}_k^n$, but one must then appeal to strong approximation on Y.

The basic idea of the present paper is that the situation of the theorem is tailored for the fibration method. Namely, the three conjugate singular points will in general span a plane Π defined over k, and any 3-dimensional linear space H containing Π will cut out on X a cubic surface with three singular points. One then fibres X by means of a pencil, or a linear system (H_λ) of such linear spaces. As mentioned above, one only expects this method to work when all the varieties $H_\lambda \cap X$ are integral, where λ runs through the geometric points of either the

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projective or affine space of parameters. This cannot always be arranged: indeed there are counter-examples to weak approximation in some very special cases when $n=4$ (§ 8). Even in cases where weak approximation actually holds, the fibration method may break down when $n=4$. At this point we have recourse to descent theory, and it is one of the astonishing facts of the present paper that in all dubious cases, the descent varieties always turn out to be essentially cubic hypersurfaces of a type already handled by the fibration method.

We now give a short description of the contents of each section.

In § 1, we give references for the Theorem when $n=3$ and we investigate a number of easy degenerate cases. All the remaining ones are classified into three types. Type (I) for $n=4$ is handled by the fibration method and strong approximation in § 2. The case (I), $n>4$, is then easily obtained by the fibration method together with weak approximation (§ 3). The cases (II), $n=4$, and (III), $n=4$, require the descent method together with the results of § 3 as well as some Brauer group computations which are postponed to § 9. These two cases are handled in § 4 and § 6 respectively. The fibration method and weak approximation, together with Hilbert's irreducibility theorem, easily yield the higher-dimensional cases of (II) and (III) (§ 5 and § 7 respectively). In § 8 we give explicit counter-examples to weak approximation in the case (III), $n=4$. Finally, § 9 computes the value of the Brauer group of a smooth proper model Z of the cubic hypersurface X. For $n=4$, these computations are used in the proofs of § 3 and § 5; for $n>4$, the Brauer group of Z comes entirely from the Brauer group of k. Thus for $n>4$ there is no obstruction of the Manin type to the Hasse principle or weak approximation, which indeed both hold.

Notation. We use $\mathbf{Z}$ to denote the ring of integers, and $\mathbf{Q}$ the rational field.

If k is a field, a k-variety X is an algebraic variety defined over k. If K/k is a field extension, $X_K = X \times_k K$ is the K-variety defined by extending the ground field k to K. By $\bar{k}$ one denotes a separable closure of k, and one writes $\bar{X} = X \times_k \bar{k}$.

A k-variety X is geometrically integral if $\bar{X}$ is reduced and irreducible. The smooth locus X_{smooth} of such a variety is a Zariski-dense open set.

The affine n-space over the field k is denoted by $\mathbb{A}^n_k$, and $\mathbb{P}^n_k$ is the projective n-space over k.

We use $\mathbb{G}_{m,k}$ for the standard multiplicative group, viewed as an algebraic group over k, and for any positive integer n, $\mu_n \subset \mathbb{G}_{m,k}$ is the group of nth roots of unity. If E/k is a finite separable extension, $R_{E/k}\mathbb{G}_m$ is the k-torus obtained from $\mathbb{G}_{m,E}$ by means of Weil descent from E to k.

The cohomology groups are Galois or more generally étale cohomology groups.

For notation and definitions pertaining to descent theory, in particular for torsors under tori, we refer to [**2**].

1. *Reductions*

Let k be a field, with $\operatorname{char}(k)=0$. Let F be a non-zero cubic form in $(n+1)$ variables with coefficients in k and let $X \subset \mathbb{P}^n_k$ be the cubic hypersurface defined by $F=0$.

PROPOSITION 1.1. *Assume that $n \geqslant 3$. If F is not absolutely irreducible, or if X has a conical point, then X has a rational point over k.*

Proof. If F is not absolutely irreducible, either it has a linear factor L defined over k and $L=0$ certainly has non-trivial solutions in k^{n+1}, or it is a product $F=L_1L_2L_3$ of three conjugate linear factors over the algebraic closure $\bar{k}$ of k. Since $n \geqslant 3$, the system $L_1=L_2=L_3=0$ has non-trivial solutions in k^{n+1}. Quite generally, the locus of conical points on a k-variety $X \subset \mathbb{P}_k^n$, if non-empty, contains k-rational points, since it is a linear space globally defined over k.

In the remainder of this section we assume:

(*) X is geometrically integral (equivalently, F is absolutely irreducible) and X is not a cone.

PROPOSITION 1.2. *If X contains a singular k-point, then X is a k-rational variety.*

Proof. Simply parametrize X by means of the lines through the k-point.

PROPOSITION 1.3. *If X contains a k-point, then it is k-unirational.*

Proof. If X is singular in codimension 1, an intersection argument together with Bertini's theorem show that the maximal component of the singular locus consists of one linear space $\mathbb{P}_k^{n-2}$ which is double on X, and X is a k-rational variety. The case where X is regular in codimension 1 is handled in [**3**, Remark 2.3.1].

PROPOSITION 1.4. *If $\bar{X}$ contains a set of at least three conjugate singular points which lie on a line L, then X is a k-rational variety.*

Proof. Consideration of the possible intersections of three lines in a plane shows that given any plane Π through L, the intersection $\Pi \cap X$ is of the shape $2L+L'$, that is L is double on X, and since X is not a cone, Proposition 1.2 applies.

PROPOSITION 1.5. *If X contains a set of three conjugate singular points which do not lie on a line, and if the plane Π which contains them lies entirely in X, then X is a k-rational variety.*

Proof. Let $K=k(\omega)$ be the cubic extension over which the singular points are defined. We may assume that these are given by the equations

$$x_0+\omega_{i+1}x_1+\omega_{i+1}^2x_2=x_0+\omega_{i+2}x_1+\omega_{i+2}^2x_2=x_3=\ldots=x_n=0,$$

where ω_i ($i \bmod 3$) are the various images of ω in $\bar{k}$. A cubic form F defining X then reads:

$$F=\sum_{i \bmod 3}(x_0+\omega_{i+1}x_1+\omega_{i+1}^2x_2)(x_0+\omega_{i+2}x_1+\omega_{i+2}^2x_2)L_i(x_3,\ldots,x_n)$$
$$+\sum_{i \bmod 3}(x_0+\omega_ix_1+\omega_i^2x_2)Q_i(x_3,\ldots,x_n)+C(x_3,\ldots,x_n),$$

where L_i, respectively Q_i, are the various images of a linear form L, respectively a quadratic form Q, with coefficients in $k(\omega)$, and where C is a cubic form with coefficients in k. If one and hence all L_i identically vanish, the plane Π is double on X and we may apply Proposition 1.2. We may thus assume that the forms L_i are not identically zero.

We now consider the graph Γ of the rational map from X to $\mathbb{P}_k^{n-3}$ which sends $(x_0,\ldots,x_n)$ to $(x_3,\ldots,x_n)$. Let F be the function field $k(x_4/x_3,\ldots,x_n/x_3)$. Set $a_i=L_i(x_3,\ldots,x_n)/x_3$, $b_i=Q_i(x_3,\ldots,x_n)/x_3^2$, $c=C(x_3,\ldots,x_n)/x_3^3$.

The generic fibre of the (dominant) projection map $\Gamma \to \mathbb{P}_k^{n-3}$ is isomorphic to a quadric given in projective space $\mathbb{P}_F^3$ with coordinates (X_0, X_1, X_2, X_3) by the vanishing of the form

$$q = \sum_{i \bmod 3} a_i(X_0 + \omega_{i+1}X_1 + \omega_{i+1}^2 X_2)(X_0 + \omega_{i+2}X_1 + \omega_{i+2}^2 X_2)$$

$$+ \sum_{i \bmod 3} b_i(X_0 + \omega_i X_1 + \omega_i^2 X_2)X_3 + cX_3^2.$$

This quadric is geometrically integral since the a_i are non-zero.

Now note that the form q, which is defined over F, is isotropic over the odd-degree extension $F(\omega)$ of F. By a well-known theorem of Springer, it is also isotropic over F. Thus the function field of X, which is equal to the function field of Γ, is purely transcendental over F, and hence also over k.

PROPOSITION 1.6. *If X contains a set of three conjugate singular points which do not lie on a line and such that the plane Π which they span is not contained in X, then X is defined by a cubic form*

$$(1.1)\quad N_{K/k}(x_0 + \omega x_1 + \omega^2 x_2) + \sum_{i=0}^{2} (x_0 + \omega_i x_1 + \omega_i^2 x_2)Q_i(x_3, \dots, x_n) + C(x_3, \dots, x_n) = 0,$$

where $K = k(\omega)$ is the cubic extension over which the singular points are defined, ω_i are the conjugates of ω in $\bar{k}$, each quadratic form Q_i has coefficients in $K_i = k(\omega_i)$ and the forms Q_i are conjugated just as the ω_i are, and where C is a cubic form with coefficients in k. The plane Π is defined by $x_3 = \dots = x_n = 0$.

Proof. We may assume that the plane spanned by the three conjugate singular points is given by $x_3 = \dots = x_n = 0$. Since the plane Π is not contained in X, the trace of X on Π is a cubic curve with three singular points which do not lie on a line. Hence it is a union of three lines, and this accounts for the term $N_{K/k}(x_0 + \omega x_1 + \omega^2 x_2)$. Now a linear change of variables of the type

$$x_0 + \omega_i x_1 + \omega_i^2 x_2 \to x_0 + \omega_i x_1 + \omega_i^2 x_2 + L_i(x_3, \dots, x_n) \quad (i = 0, 1, 2),$$

where L_i is a linear form with coefficients in $k(\omega_i)$, enables one to get rid of the terms which are quadratic in the variables $x_0 + \omega_i x_1 + \omega_i^2 x_2$ $(i = 0,1,2)$.

Equation (1.1) enables us to distinguish several cases, whose study will require different methods.

(I) The greatest common divisor of all forms Q_i and C is of degree at most 1.

(II) The greatest common divisor of all forms Q_i and C is a quadratic form $Q(x_3, \dots, x_n)$ with coefficients in k. In this case, equation (1.1) reads

$$(1.2)\qquad N_{K/k}(x_0 + \omega x_1 + \omega^2 x_2) + Q(x_3, \dots, x_n)L(x_0, \dots, x_n) = 0,$$

where L is a linear form with coefficients in k, such that $L(x_0, x_1, x_2, 0, 0)$ does not identically vanish.

(III) All forms Q_i identically vanish. Equation (1.1) here reads

$$(1.3)\qquad N_{K/k}(x_0 + \omega x_1 + \omega^2 x_2) + C(x_3, \dots, x_n) = 0.$$

THEOREM 1.7. *Let k be a number field, and let $X \subset \mathbb{P}^3_k$ be a cubic surface defined over k. If X is singular, then it satisfies the Hasse principle. If X is geometrically integral and not a cone, and if X contains exactly three singular points, and $X(k) \neq \varnothing$, then X is a k-rational surface. In particular, weak approximation then holds for any smooth model of X.*

Proof. The Hasse principle statement is due to Skolem [**12**]. The hard case of Skolem's result is precisely that of three conjugate singular points. Skolem uses equation (1.1) and simplifies it by allowing quadratic field extensions. The k-rationality statement (which is specific to the case of three singular points) is due to B. Segre [**10**], a modern version being given by Coray [**5**]. Proofs of both results are also given by Coray and Tsfasman ([**6**, Propositions 3.1 and 3.2]). They actually show that if X contains exactly three singular points, then X is k-birational to a Del Pezzo surface Z of degree 6. Such a surface Z is k-rational as soon as $Z(k) \neq \varnothing$, and if k is a number field, it satisfies the Hasse principle (Manin). The degenerate cases (X geometrically reducible, X a cone) are covered by Proposition 1.1 above.

2. *Case* (I), $n = 4$

This section is devoted to the proof of the following theorem.

THEOREM 2.1. *Let $X \subset \mathbb{P}^4_k$ be a geometrically integral cubic hypersurface given by the equation*

$$N_{K/k}(x_0 + \omega x_1 + \omega^2 x_2) + \sum_{i=0}^{2} (x_0 + \omega_i x_1 + \omega_i^2 x_2) Q_i(x_3, x_4) + C(x_3, x_4) = 0, \tag{2.1}$$

where notation is as in Proposition 1.5, and assume that the greatest common divisor of the forms Q_i ($i = 0, 1, 2$) and C is of degree at most 1. Then the Hasse principle and weak approximation hold for any smooth model of X.

The proof will use the following easy lemmas.

LEMMA 2.2. *Let k be an algebraically closed field, let $n \geqslant 3$ be an integer, let $Q_i(y_3, \ldots, y_n)$ ($i = 0, 1, 2$) be quadratic forms, and let $C(y_3, \ldots, y_n)$ be a cubic form. The form*

$$F = y_0 y_1 y_2 + \sum_{i=0}^{2} y_i Q_i(y_3, \ldots, y_n) + C(y_3, \ldots, y_n)$$

is irreducible unless $Q_0 = Q_1 = 0$ and $C = 0$, or $Q_1 = Q_2 = 0$ and $C = 0$, or $Q_2 = Q_0 = 0$ and $C = 0$. If none of these conditions occur, and if $n = 3$, then the cubic surface $F = 0$ is not a cone.

Proof. If F is not irreducible, some non-zero linear form $L(y_0, \ldots, y_n)$ divides it. Now $L(y_0, y_1, y_2, 0, \ldots, 0)$ divides $y_0 y_1 y_2$, and hence we may assume, for example, that $L = y_0 + M(y_3, \ldots, y_n)$ with M linear. Consideration of the coefficients of $y_1 y_2, y_1, y_2$, in the identity $F(-M(y_3, \ldots, y_n), y_1, \ldots, y_n) = 0$ shows that $Q_1 = Q_2 = 0$ and $M = 0$, and hence $C = 0$. Assume that $n = 3$ and let P be a conical point of X. Then one of the lines in the plane $y_3 = 0$ does not go through P. Now the plane through P and such a line lies entirely in X, whence the form F is reducible.

LEMMA 2.3. *There exists an integer N_0 such that if κ is a finite field, $\#\kappa \geqslant N_0$, and Σ is a geometrically integral cubic surface defined over κ, then Σ has a non-singular κ-rational point.*

Proof. According to Chevalley's theorem, any cubic surface over a finite field κ contains a κ-rational point. The result now follows from a simple discussion of the possible singularities on Σ (if Σ has only isolated singularities, their number is at most 4, see [**6**, Lemma 1.1]; a singular curve on Σ can only consist of a line).

Proof of Theorem 2.1. Given a non-empty Zariski open set W of a smooth k_v-variety W, the set $V(k_v)$ is dense in $W(k_v)$ by the implicit function theorem. Thus it is enough to prove the theorem for some non-empty open set U of the smooth locus of X. We shall take as U the non-empty open set

$$x_4 \prod_{i=0}^{2} \left(\left(\prod_{j \neq i} (x_0 + \omega_j x_1 + \omega_j^2 x_2) \right) + Q_i(x_3, x_4) \right) \neq 0$$

(that U is non-empty follows from Lemma 2.2).

We thus assume that $U(k_v) \neq \varnothing$ for each place v of k, and we give ourselves a finite set S of places of k and points $P_v \in U(k_v)$ for $v \in S$. We want to find a point $P \in U(k)$ which is arbitrarily close simultaneously to each P_v. Given $\lambda \in k$, let H_λ be the hyperplane $x_3 = \lambda x_4$, and let $\Sigma_\lambda = X \cap H_\lambda \subset H_\lambda \simeq \mathbb{P}_k^3$ and $U_\lambda = U \cap H_\lambda$. An immediate computation shows that U_λ is smooth whatever λ. In other terms, H_λ is transversal to X in any (geometric) point of U.

If the forms $Q_i(x_3, x_4)$ $(i = 0, 1, 2)$ and $C(x_3, x_4)$ have a common linear factor, then we may assume that this factor is x_4. Let L be the composite of the extensions $K_i \subset \bar{k}$. The assumptions of Theorem 2.1 (that is, X is of Type (I)) imply that the polynomials $Q_i(t, 1)$ $(i = 0, 1, 2)$ and $C(t, 1)$ in $L[t]$ have no common factor. Thus there exist polynomials $a_i(t)$ $(i = 0, 1, 2)$ and $b(t)$ in $L[t]$ such that

$$\sum_{i=0}^{2} a_i(t) Q_i(t, 1) + b(t) C(t, 1) = 1. \tag{2.2}$$

In order to prove the theorem, we may enlarge the set of places S to any bigger finite set of places (for each new place, we may choose a point $P_v \in U(k_v)$).

We shall enlarge S so that it contains all the real places and so that for any non-archimedean place $v \notin S$:

for any place w of L above v, each ω_i is integral at the place w, and the discriminant $D(1, \omega, \omega^2)$ is a unit at the place v;

for any place w of L above v, all coefficients of $Q_i(x_3, x_4)$, $C(x_3, x_4)$, $a_i(t)$, $b(t)$ are integral at w;

the order of the residue class field at v is bigger than N_0 (for N_0 as in Lemma 2.3).

Let us now fix a place v_0 of k, $v \notin S$, such that $K \otimes_k k_v$ is not a field (that such a place exists is a special case of Tschebotarev's theorem).

For each $v \in S$, let $\lambda_v \in k_v$ be the value of (x_3/x_4) at the point $P_v \in U(k_v)$.

By strong approximation, we may choose $\lambda \in k$ arbitrarily close to each λ_v for

$v \in S$ and integral at each non-archimedean place v for $v \notin S$ and $v \neq v_0$. In particular, the choice of U and the implicit function theorem imply that we may choose λ such that, for each $v \in S$, there exists a (smooth) point $M_v \in U_\lambda(k_v)$ which is as close as we wish to the given point P_v. The equation of $\Sigma_\lambda \subset H_\lambda \simeq \mathbb{P}^3_k$ reads:

$$(2.3)\quad N_{K/k}(x_0+\omega x_1+\omega^2 x_2)+\sum_{i=0}^{2}(x_0+\omega_i x_1+\omega_i^2 x_2)Q_i(\lambda, 1)x_4^2+C(\lambda, 1)x_4^3=0.$$

Since the $Q_i(\lambda, 1)$ $(i=0, 1, 2)$ are conjugate, either all or none of them are zero. It then follows from (2.2) and Lemma 2.2 that the cubic surface Σ_λ, which contains three conjugate singular points in the plane $x_4=0$, is geometrically integral and is not a cone.

We now make a claim: for each place v of k, $\Sigma_{\lambda\,\text{smooth}}(k_v)$ is not empty. This is clear for each place $v \in S$, since $U_\lambda \subset \Sigma_{\lambda\,\text{smooth}}$. Assume that $K \otimes_k k_v$ is not a field. Then one of the singular points in the plane $x_4=0$ is defined over k_v, so that Σ_{λ, k_v} is k_v-birational to $\mathbb{P}^2_{k_v}$, and hence in particular $U_\lambda(k_v) \neq \varnothing$. Assume now that v does not belong to S and that $K \otimes_k k_v$ is a field. Since the coefficients in equation (2.3) all lie in the ring of integers O_v of k_v, we may define the reduction $\Sigma_{\lambda, \kappa_v}$ over the residue class field κ_v of O_v by the same equation (2.3). Let w be a place of L above v. The change of variables

$$y_i = x_0 + \omega_i x_1 + \omega_i^2 x_2$$

has coefficients in O_w, the ring of integers of L_w, and is invertible over O_w (by the choice of S). Thus $\Sigma_{\lambda, \kappa_v} \times \kappa_w$ is given over the residue class field κ_w of O_w by the equation

$$(2.4)\qquad y_0 y_1 y_2 + \sum_{i=0}^{2} Q_i(\lambda, 1) y_i x_4^2 + C(\lambda, 1)x_4^3 = 0.$$

The assumption that $K \otimes_k k_v$ is a field implies that the $Q_i(\lambda, 1)$ $(i=0, 1, 2)$ reduce to elements which are transitively conjugated by $\mathrm{Gal}(\kappa_w/\kappa_v)$, and hence all either vanish or do not vanish. Setting $t=\lambda$ in (2.2) and reducing the equation just obtained shows that the hypotheses of Lemma 2.2 are fulfilled by equation (2.4), so that $\Sigma_{\lambda, \kappa_v}$ is geometrically integral. Thus $\Sigma_{\lambda, \kappa_v}$ contains a smooth κ_v-point according to Lemma 2.3, and hence Σ_λ contains a smooth k_v-point by Hensel's lemma.

Now the cubic surface Σ_λ is geometrically integral, is not a cone, contains three conjugate singular points, and it contains smooth k_v-points for all places v of k. According to Theorem 1.6, its smooth locus contains k-points and satisfies weak approximation (in fact Σ_λ is a k-rational variety); hence we may find a k-point $P \in U_\lambda(k) \subset U(k)$ which is simultaneously as close as we wish to each point $M_v \in U_\lambda(k_v)$, and hence finally to each $P_v \in U(k_v)$.

3. *Case* (I), $n > 4$

THEOREM 3.1. *Let $X \subset \mathbb{P}^n_k$, $n > 4$, be a geometrically integral cubic hypersurface given by the equation*

$$(3.1)\quad F = N_{K/k}(x_0+\omega x_1+\omega^2 x_2) + \sum_{i=0}^{2}(x_0+\omega_i x_1+\omega_i^2 x_2)Q_i(x_3, \ldots, x_n) + C(x_3, \ldots, x_n) = 0,$$

where notation is as in Proposition 1.6, *and assume that the greatest common divisor of the forms Q_i ($i=0, 1, 2$) and C is of degree at most* 1. *Then the Hasse principle and weak approximation hold for any smooth model of X.*

Proof. It is enough to prove the theorem for the smooth locus $U=X_{\text{smooth}}$ of X. We thus assume that $U(k_v)$ is non-empty for each place v of k, and that for each v in a finite set of places S_0 we are given points $P_v \in U(k_v)$.

Let $Z \subset \mathbb{P}^n_k$ be the linear space defined by $x_0=x_1=x_2=0$. By assumption, the closed subvariety of Z given in homogeneous coordinates $(x_3, \ldots, x_n)$ by $Q_0=Q_1=Q_2=0$ and $C=0$ is either a codimension-2 subvariety Y in Z or is the union of such a codimension-2 subvariety and a hyperplane H of Z with multiplicity 1. In the latter case, we may assume this hyperplane to be given by $x_n=0$. Let $V=X \backslash Z \cap X$. Projection induces a k-morphism π: $V \to Z$ which in homogeneous coordinates reads $(x_0, \ldots, x_n) \to (x_3, \ldots, x_n)$, and the fibre $V_M=\pi^{-1}(M)$ of a point $M=(\lambda_3, \ldots, \lambda_n) \in Z$ is simply the complement of $Z(x_0=x_1=x_2=0)$ in the cubic surface

$$(\Sigma_M) \quad N_{K/k}(x_0+\omega x_1+\omega^2 x_2) + \sum_{i=0}^{2} (x_0+\omega_i x_1+\omega_i^2 x_2) Q_i(\lambda_3, \ldots, \lambda_n) t^2 + C(\lambda_3, \ldots, \lambda_n) t^3 = 0.$$

Let $W \subset V$ be the smooth open set ($W \subset U$) defined by the non-vanishing of the first derivative F'_{x_0}. One easily checks that the restriction of π to W is smooth.

If $M \in Z(k)$ does not lie on $Y \cup H$, the cubic surface Σ_M is geometrically integral and is not a cone. (Lemma 2.2 applies here since the $Q_i(\lambda_3, \ldots, \lambda_n)$ ($i=0, 1, 2$) are conjugate and hence are simultaneously zero or non-zero.) Fix such a k-point M_0. It then follows from Lemma 2.3 and from Hensel's lemma that there exists a finite set S_1 of places of k such that for each $v \notin S_1$, there exists a k_v-point in $W \cap \Sigma_{M_0} \subset \Sigma_{M_0\,\text{smooth}}$.

For each place $v \in S_1$, we fix a (smooth) point $P_v \in W(k_v)$. Since $W(k_v)$ is dense in $U(k_v)$ by the implicit function theorem, we may also assume that $P_v \in W(k_v)$ for each $v \in S_0$. Let $S=S_0 \cup S_1$. For each $v \in S$, let $M_v=\pi(P_v)$. Since π: $W \to Z$ is smooth, the implicit function theorem implies that if $N_v \in Z(k_v)$ is close enough to M_v, the set $(W \cap V_{N_v})(k_v)$ is not empty and contains smooth points R_v which are arbitrarily close to P_v.

Let us choose a point $M \in Z(k)$, $M \neq M_0$ such that the line $\Delta=MM_0$ does not meet the subvariety Y (whose codimension in Z is 2) and is transversal to H, and such that for each $v \in S$, the set $V_M(k_v)$ contains a point R_v which is arbitrarily close to P_v (weak approximation in $Z \simeq \mathbb{P}^{n-3}_k$). Now $\pi^{-1}(\Delta)$ is the complement of $Z(x_0=x_1=x_2=0)$ in a cubic hypersurface $X_1 \subset \mathbb{P}^4_k$ given by an equation

$$N_{K/k}(x_0+\omega x_1+\omega^2 x_2) + \sum_{i=0}^{2} (x_0+\omega_i x_1+\omega_i^2 x_2) q_i(x_3, x_4) + c(x_3, x_4) = 0,$$

where the q_i ($i=0, 1, 2$) and c are forms (in new coordinates x_3 and x_4) which satisfy the assumptions of Theorem 2.1, precisely because Δ does not meet the subvariety Y and is transversal to H. Since the projection $W \cap X_1 \to \mathbb{P}^1_k$ given by (x_3, x_4) is smooth, $W \cap X_1$ is a smooth open set of X_1, and this open set contains k_v-points for each place v of k. Theorem 2.1 applies, and we may find a point $P \in (W \cap X_1)(k)$ which is arbitrarily close simultaneously to each point R_v for

$v \in S$. Since P belongs to $W(k)$, it defines a smooth k-point on the original hypersurface and this k-point P may be chosen arbitrarily close to each of the points P_v $(v \in S_0)$.

4. *Case* (II), $n=4$

In this section we shall consider Case (II) for $n=4$. Thus $X \subset \mathbb{P}_k^4$ will be a geometrically integral cubic hypersurface given by an equation

$$(4.1) \qquad N_{K/k}(x_0+\omega x_1+\omega^2 x_2)+Q(x_3, x_4)(L(x_0, x_1, x_2)+M(x_3, x_4))=0,$$

where Q is a non-zero quadratic form with coefficients in k, and L and M are linear forms with coefficients in k, with $L \neq 0$.

LEMMA 4.1. *The surface X is a cone if and only if*

(i) *$M=0$ and Q is proportional to the square of a linear form, or*

(ii) *$QM \neq 0$ is proportional to the cube of a linear form.*

Proof. If the cubic hypersurface X is a cone, the vertex of this cone is given by the vanishing of all second derivatives of its defining form. The lemma follows by a straightforward computation.

The main problem in this section is weak approximation. Indeed:

LEMMA 4.2. *There is a rational k-point on X. If X is not a cone, then it is even k-unirational, so that k-points are Zariski-dense on X.*

Proof. If M is identically zero, any k-point $(0, 0, 0, x_3, x_4)$ with $Q(x_3, x_4) \neq 0$ is non-singular on X. Assume that M does not vanish identically. If M does not divide Q, the k-point given by

$$x_0=x_1=x_2=M(x_3, x_4)=0$$

is non-singular. If M divides Q, one easily shows that either X is a cone or X is k-birational to an affine space. If X is not a cone, k-unirationality follows from Proposition 1.3.

PROPOSITION 4.3. *If Q is of rank 1 and if X is not a cone, then X is a k-rational variety.*

Proof. The point defined by $x_0=x_1=x_2=Q=0$ is singular on X. The conclusion follows from Proposition 1.2.

THEOREM 4.4. *Let $X \subset \mathbb{P}_k^4$ be the geometrically integral cubic hypersurface given by equation* (4.1). *Assume that $Q(x_3, x_4)$ is of rank 2 and that $M(x_3, x_4)$ is zero or divides Q. Then weak approximation holds for any smooth model of X.*

Proof. According to Lemma 4.1, X is not a cone. If $M \neq 0$ divides Q, the rational point $x_0=x_1=x_2=M=0$ is singular on X, which is k-rational by Proposition 1.2. Assume that $M=0$. An affine model Y of X is given by the system

$$N_{K/k}(x_0+\omega x_1+\omega^2 x_2)+Q(x_3,x_4)=0,$$
$$L(x_0,x_1,x_2)=1.$$

Assume that the coefficient of x_2 in L is non-zero. Then Y is also given by an equation

$$Q(x_3,x_4)+C(x_0,x_1)=0,$$

where C is an irreducible polynomial of the third degree. That weak approximation holds for any smooth model of such a Y is obvious if Q is reducible over k (Y is then k-birational to an affine space). If Q is irreducible, it is a special case of Theorem 9.3 of **[3]** (see also Remark 9.3.2 of **[3]**).

THEOREM 4.5. *Let $X\subset \mathbb{P}^4_k$ be the geometrically integral cubic hypersurface given by equation* (4.1). *Assume that* $LM\neq 0$, rank $Q=2$, *and M does not divide Q.*

(i) *The Brauer–Manin obstruction to weak approximation on a smooth proper model of X is the only one.*

(ii) *If Q does not split into linear factors over the discriminant extension of K/k, then weak approximation holds for any smooth model of X.*

Proof. The proof will be by descent and naturally breaks into a number of steps. The reader who would like to get an overview of the proof is invited to read step (g) first.

(a) *Equations for X.* Let F be the separable quadratic k-algebra $k[t]/Q(t,1)$, and let α be the image of t in F. Let E be the separable k-algebra $K\otimes_k F$. After a suitable k-linear change of coordinates, we may write (4.1) as

$$N_{K/k}(x_0+\omega x_1+\omega^2x_2) + N_{F/k}(x_3-\alpha x_4)\{a\,\mathrm{Tr}_{K/k}(x_0+\omega x_1+\omega^2x_2)+b\,\mathrm{Tr}_{F/k}(x_3-\alpha x_4)\}=0, \tag{4.2}$$

with $a,b\in k^*$.

Let $U\subset X$ be the open set of X defined by $N_{K/k}(x_0+\omega x_1+\omega^2x_2)\neq 0$. By going over to the algebraic closure, one easily checks that either U is smooth or U contains a unique, k-rational, singular point. This last case occurs if and only if $a^3+b^2=0$ (the singular k-point then lies on $x_3=x_4=0$). Since X is not a cone (Lemma 4.1), the second possibility implies that X is a k-rational variety, and the theorem certainly holds. We may assume:

(4.3) The open set U of X defined by $N_{K/k}(x_0+\omega x_1+\omega^2x_2)\neq 0$ is smooth.

Note that this open set U may also be described by means of the *affine* equations

$$\begin{aligned}&N_{K/k}(x_0+\omega x_1+\omega^2x_2)+N_{F/k}(x_3-\alpha x_4)=0,\\ &a\,\mathrm{Tr}_{K/k}(x_0+\omega x_1+\omega^2x_2)+b\,\mathrm{Tr}_{F/k}(x_3-\alpha x_4)=1,\\ &N_{K/k}(x_0+\omega x_1+\omega^2x_2)\neq 0.\end{aligned} \tag{4.4}$$

(b) *The k-torus T.* Let $T=R_{F/k}(R^1_{E/F}\mathbb{G}_m)$ be the k-torus whose associated functor on commutative k-algebras A is:

$$T(A)=\mathrm{Ker}\,N_{E/F}\colon\ (A\otimes_k E)^*\to(A\otimes_k F)^*.$$

There is an exact sequence of k-tori:

$$(4.5)\qquad 1 \longrightarrow T \longrightarrow R_{E/k}\mathbb{G}_m \xrightarrow{N_{E/F}} R_{F/k}\mathbb{G}_m \longrightarrow 1,$$

which makes $R_{E/k}\mathbb{G}_m$ into a torsor over $R_{F/k}\mathbb{G}_m$ under T. For any semi-local k-algebra A, this sequence induces an isomorphism

$$(4.6)\qquad (A \otimes_k F)^*/N_{E/F}(A \otimes_k E)^* \simeq H^1_{\text{ét}}(A, T).$$

(c) *A torsor over U under T.* Given any $c \in F^*$, the function $c^{-1}(x_3 - \alpha x_4)$ belongs to $H^0(U, R_{F/k}\mathbb{G}_m)$, and we may use it to pull back the torsor under T given by (4.5), thus getting a torsor $\mathcal{T}^c$ over U under T. The total space of this torsor is given by the following set of *affine* equations in $\mathbb{A}^{11}$:

$$(4.7)\qquad \begin{aligned} &N_{K/k}(x) + N_{F/k}(x_3 - \alpha x_4) = 0,\\ &a\,\mathrm{Tr}_{K/k}(x) + b\,\mathrm{Tr}_{F/k}(x_3 - \alpha x_4) = 1,\\ &(x_3 - \alpha x_4) = cN_{E/F}(y - \alpha z),\\ &N_{K/k}(x) \neq 0, \end{aligned}$$

where we have introduced variables $(y_0, y_1, y_2, z_0, z_1, z_2)$ and denoted

$$\begin{aligned} x &= x_0 + \omega x_1 + \omega^2 x_2,\\ y &= y_0 + \omega y_1 + \omega^2 y_2,\\ z &= z_0 + \omega z_1 + \omega^2 z_2. \end{aligned}$$

(d) *An extension property.* Let us show:

(4.8) The torsor $\mathcal{T}^c$ over U under T extends to a torsor under T over any smooth compactification Z of U.

According to [**2**, Lemme 2.7.6] or [**3**, Lemma 12.6] and (4.6) above, it is enough to show:

(4.9) For any discrete rank-1 valuation ring A which is a k-algebra and whose fraction field is $k(U)$, the class of $(x_3 - \alpha x_4)$ in $F(U)^*/N_{E/F}E(U)^* \simeq H^1_{\text{ét}}(k(U), T)$ comes from

$$(A \otimes_k F)^*/N_{E/F}(A \otimes_k E)^* \simeq H^1_{\text{ét}}(A, T).$$

By an easy restriction-corestriction argument we may assume that K is a Galois extension of k and that F is a split quadratic extension of k; that is, we may change variables and assume that $Q = x_3x_4$. An affine equation of X now reads:

$$(4.9.1)\qquad N_{K/k}(x_0 + \omega x_1 + \omega^2 x_2) = x_3(L(x_0, x_1, x_2) + cx_3 + d),$$

where L and $cx_3 + d$ are both non-zero with coefficients in k, $cd \neq 0$, and we must prove (4.9) with $F = k$, $E = K$, and x_3 in place of $(x_3 - \alpha x_4)$.

Let A be a discrete valuation ring with $k \subset A$ and with fraction field $k(X)$. Let π be a uniformizing parameter of A. Then $A \otimes_k K$ is a semi-local Dedekind ring with fraction field $K(X)$. If $A \otimes_k K$ is not a discrete valuation ring itself, since $[K : k] = 3$ we conclude that π may be written as the product of a unit in A by the norm of an element of $A \otimes_k K$. Let us now assume that $A \otimes_k K$ is a discrete valuation ring. Then π is a uniformizing parameter for $A \otimes_k K$, and an element $f \in k(X)$ may be written $f = uN_{K/k}(g)$ with u a unit of A and $g \in K(X)$ if and only if its valuation $v_A(f) = v(f)$ is divisible by 3.

Let us show that such is the case for $f = x_3$. In (4.9.1) the linear form $L(x_0, x_1, x_2)$ may be written as a trace $\mathrm{Tr}_{K/k}(\rho(x_0 + \omega x_1 + \omega^2 x_2))$ for a suitable $\rho \in K^*$.

First assume that $v(x_0 + \omega x_1 + \omega^2 x_2) = a \geq 0$. It then follows that $v(L(x_0, x_1, x_2)) \geq a \geq 0$. The valuation of the left-hand side of (4.9.1) is $3a \geq 0$. This implies that $v(x_3) \geq 0$. Assume that $v(x_3) > 0$. Then the valuation of the left-hand side is positive; also the valuation of the left-hand side is $3a$, and hence $a > 0$. We now have $v(L(x_0, x_1, x_2)) \geq a > 0$, $v(x_3) > 0$, whence

$$v(L(x_0, x_1, x_2) + cx_3 + d) = 0;$$

hence finally $v(x_3) = 3a$, so that 3 divides $v(x_3)$.

Now assume that $v(x_0 + \omega x_1 + \omega^2 x_2) = a < 0$. We then have $v(L(x_0, x_1, x_2)) = v(\mathrm{Tr}_{K/k}(\rho(x_0 + \omega x_1 + \omega^2 x_2)) \geq a$. Assume first that $v(x_3) \geq a$. The valuation of the right-hand side of (4.9.1) is then at least $2a$, which contradicts the fact that the valuation of the left-hand side is $3a$. Hence $v(x_3) < a$. We now have $v(L(x_0, x_1, x_2) + cx_3 + d) = v(x_3)$, and (4.9.1) implies that $-3a = 2v(x_3)$, so that 3 divides $v(x_3)$. This completes the proof of (4.9).

(e) *Equations for torsors of a given type.* For Z a smooth compactification of U, there is an exact sequence

$$0 \longrightarrow H^1_{\text{ét}}(k, T) \longrightarrow H^1_{\text{ét}}(Z, T) \xrightarrow{\chi} \mathrm{Hom}_G(\hat{T}, \mathrm{Pic}\, \bar{Z}), \tag{4.10}$$

where the middle term classifies torsors over Z under T, and the right-hand side arrow sends a torsor $\mathcal{T}$ to its *type* $\lambda = \chi(\mathcal{T})$.

According to (4.8), a given torsor $\mathcal{T}^c$ over U as in (4.7) is the restriction of a torsor $\mathcal{T}$ over Z under T. Let λ be the type of such a $\mathcal{T}$.

Comparing the étale cohomology sequences associated to (4.5) over Speck, over Z and over U then shows:

(4.11) The restriction to U of any torsor of type λ may be written as a torsor $\mathcal{T}^c$ as in (4.7) for a suitable $c \in F^*$.

(f) *Factorizing the projection $\mathcal{T}^c \to U$ through a cubic hypersurface of Type* (I). Let us introduce new variables w_0, w_1, w_2, set $w = w_0 + \omega w_1 + \omega^2 w_2$, and consider the k-morphism from $\mathbb{A}^9_k$ with coordinates

$$(w_0, w_1, w_2, y_0, y_1, y_2, z_0, z_1, z_2)$$

to $\mathbb{A}^{11}_k$ with coordinates

$$(x_0, x_1, x_2, x_3, x_4, y_0, y_1, y_2, z_0, z_1, z_2)$$

given by

$$x = w.N_{E/K}(y - \alpha z), \tag{4.12}$$

where x,y and z are as in (4.7). Let

$$d = N_{F/k}(c) \in k^*. \tag{4.13}$$

Then the k-subvariety of $\mathbb{A}^9_k$ defined by

$$\begin{aligned} &N_{K/k}(w) + d = 0, \\ &a\, \mathrm{Tr}_{K/k}(wN_{E/K}(y - \alpha z)) + b\, \mathrm{Tr}_{F/k}(cN_{E/F}(y - \alpha z)) = 1, \\ &N_{E/F}(y - \alpha z) \neq 0, \end{aligned} \tag{4.14}$$

is mapped isomorphically onto $\mathscr{T}^c \subset \mathbb{A}_k^{11}$ (cf. (4.7)), the projection from $\mathscr{T}^c$ as given by (4.14) to U being given by

$$
\begin{aligned}
&x_3 - \alpha x_4 = cN_{E/F}(y - \alpha z),\\
&x_0 + \omega x_1 + \omega^2 x_2 = (w_0 + \omega w_1 + \omega^2 w_2).N_{E/K}(y - \alpha z).
\end{aligned}
\tag{4.15}
$$

Let $W_c \subset \mathbb{P}_k^8$ with homogeneous coordinates $(w_0, w_1, w_2, y_0, y_1, y_2, z_0, z_1, z_2)$ be the cubic hypersurface defined by the equation:

$$N_{K/k}(w) + d\{a \operatorname{Tr}_{K/k}(wN_{E/K}(y - \alpha z)) + b \operatorname{Tr}_{F/k}(cN_{E/F}(y - \alpha z))\} = 0, \tag{4.16}$$

and let $U_c \subset W_c$ be the open set defined by $N_{K/k}(w).N_{E/k}(y - \alpha z) \neq 0$.

Now (4.16) is exactly as (1.1) in Proposition 1.6, with w_i in place of x_i $(i = 0, 1, 2)$, and the y_i $(i = 0, 1, 2)$ and z_i $(i = 0, 1, 2)$ in place of x_i $(i > 2)$. By going over to $\bar{k}$ and making an obvious change of variables, one easily checks that the forms $da \operatorname{Tr}_{K/k}(wN_{E/K}(y - \alpha z))$ and $db \operatorname{Tr}_{F/k}(cN_{E/F}(y - \alpha z))$ (this last form corresponds to C in (1.1)) are non-zero and have no common factor.

Thus W_c, which is geometrically integral (Lemma 2.2), is a singular cubic hypersurface of Type (I). Using $ab \neq 0$ one can also check that it is not a cone.

Projection $\mathbb{A}_k^9 \backslash 0 \to \mathbb{P}_k^8$ induces a k-morphism from $\mathscr{T}^c$, as given by (4.14), to $U_c \subset W_c$, and one easily checks that it makes $\mathscr{T}^c$ into a μ_3-torsor over U_c. In particular, U_c is smooth since $\mathscr{T}^c$ is smooth itself (it is a torsor under the torus T over the smooth k-variety U). On the other hand, (4.15) expresses the x_i as homogeneous cubic forms in the other variables. Thus the projection from $\mathscr{T}^c$ to $U \subset X \subset \mathbb{P}_k^4$ factorizes through U_c! We have the following diagram of k-morphisms:

$$
\begin{tikzcd}
\mathscr{T}^c \arrow[rd, "q_c"] \arrow[dd] & \\
& U_c \arrow[ld, "r_c"] \\
U &
\end{tikzcd}
\tag{4.17}
$$

(g) *Completion of the proof.* Fix a smooth compactification Z of U. Since $X(k)$ is Zariski-dense in X (Lemmas 4.1 and 4.2), we also have $Z(k) \neq \varnothing$ (and hence certainly $Z(k_v) \neq \varnothing$ for each place v of k). Let S be a finite set of places of k and for each $v \in S$, let $P_v \in U(k_v)$. Applying the extension property (4.8), one sees that the torsor $\mathscr{T}^1$ over U extends to some torsor over Z. Fix such a torsor, and let λ be its type. If there is no Brauer–Manin obstruction to weak approximation on Z, which we now assume, descent theory [2, § 3.7] shows that there exists a torsor $\mathscr{T}$ over Z of type λ such that $\mathscr{T}(k_v) \neq \varnothing$ for each place v of k and that P_v belongs to the projection of $\mathscr{T}(k_v)$ to $Z(k_v)$ for each place $v \in S$. Together with (4.11), this implies:

(4.18) There exists a $c \in F^*$ such that $\mathscr{T}^c(k_v) \neq \varnothing$ for each place v of k and such that each P_v for $v \in S$ lifts to some point $Q_v \in \mathscr{T}^c(k_v)$.

Using diagram (4.17) we now conclude that $U_c(k_v) \neq \varnothing$ for each place v of k and $P_v = r_c(M_v)$ with $M_v = q_c(Q_v) \in U_c(k_v)$ for each $v \in S$. As seen in (f) above, U_c is a smooth open set of the cubic hypersurface $W_c \subset \mathbb{P}_k^8$ and this cubic hypersurface is of Type (I). According to Theorem 3.1, U_c satisfies the Hasse principle and weak approximation, and we may find a k-point $M \in U_c(k)$ which is arbitrarily close simultaneously to each M_v for $v \in S$, thus yielding a k-point $P = r_c(M)$ which is arbitrarily close simultaneously to each P_v for $v \in S$.

This completes the proof of Part (i) of the theorem. Part (ii) follows from the following statement, which guarantees that there is no Brauer–Manin obstruction to weak approximation [**2**, Théorème 3.7.6], and whose proof we defer to § 9:

(4.19) If Q does not split into linear factors over the discriminant extension of K/k, then $H^1(\mathrm{Gal}(\bar{k}/k), \operatorname{Pic} \bar{Z}) = 0$ for any smooth proper model Z of X.

REMARK 4.6. The above proof generally follows the pattern of other proofs in [**3**]. In particular, there is the 'miraculous' change of variables (4.12) in (f) which turns the equations (4.7) of the torsor into the simpler (4.14) (compare with [**3**, (7.12.2), (12.26)]). However, there is one extra argument here: namely (4.14) is not yet simple enough, and we only succeed because of the factorization described in diagram (4.17).

As a matter of fact, the variety U_c, together with the morphism r_c to U, is a torsor over U under a certain torus, and this torsor extends to a torsor over any smooth compactification Z of U. Indeed, the composition of natural inclusions

$$\mu_3 \subset \mathbb{G}_{m,k} \subset R_{E/k}\mathbb{G}_m \quad .$$

induces a homomorphic embedding of μ_3 into the k-torus T. Let S be the k-torus defined by the exact sequence

$$1 \to \mu_3 \to T \to S \to 1.$$

Following equations (4.7), (4.14), (4.16) reveals that the action of $\mu_3 \subset T$ on (4.14) simply reads as

$$(x, y, z) \to (\xi x, \xi y, \xi z) \quad (\xi \in \mu_3).$$

Since quotienting by this action gives precisely the map q_c, it follows that r_c: $U_c \to U$ is the torsor over U under $S = T/\mu_3$ obtained from the torsor $\mathcal{T}^c \to U$ under T by the change of structural group $T \to S$. Now this change of group can be applied to any torsor $\mathcal{T}$ over a smooth compactification Z of U, extending $\mathcal{T}^c$, as given by (4.8), thereby yielding a natural extension of the torsor r_c: $U_c \to U$ over Z.

Thus we could have used torsors of type

$$\hat{S} \longrightarrow \hat{T} \xrightarrow{\lambda} \operatorname{Pic} \bar{Z}$$

as descent varieties.

5. *Case* (II), $n > 4$

In this section, we let $X \subset \mathbb{P}^n_k$, for $n > 4$, be a geometrically integral cubic hypersurface given by an equation

$$(5.1) \quad N_{K/k}(x_0 + \omega x_1 + \omega^2 x_2) + Q(x_3, \dots, x_n)(L(x_0, x_1, x_2) + M(x_3, \dots, x_n)) = 0,$$

where Q is a quadratic form and L and M are linear forms with coefficients in k, with $L \neq 0$.

THEOREM 5.1. *Assume that Q is of maximal rank $(n-2) \geqslant 3$ or that X is not a cone. Then weak approximation holds for any smooth model of X and any such model contains a k-point.*

Proof. First note that X always contains a non-singular k-point. Indeed, it contains the k-linear space H,

$$x_0 = x_1 = x_2 = M(x_3, \ldots, x_n) = 0.$$

Thus if X is not a cone, X contains a non-singular k-point (since it is a k-rational variety otherwise). If Q is of rank at least 3, then there is a k-point on the linear space H with $Q(x_3, \ldots, x_n) \neq 0$, and such a k-point is non-singular on X. As a corollary, we get that X is k-unirational (Proposition 1.3).

If the quadratic form Q is not of maximal rank $(n-2)$, let $(\alpha_3, \ldots, \alpha_n) \in k^{n-2}$ be non-zero and such that all partial derivatives of Q vanish at this point. Then the k-point $(0, 0, 0, \alpha_3, \ldots, \alpha_n)$ is a singular k-point on X. Thus X is k-rational since it is not a cone.

We may now assume that Q is of maximal rank $(n-2) \geq 3$.

Suppose first that $M = 0$. The same argument as in Theorem 4.4 then shows that X contains an affine open set U given by an affine equation:

$$Q(x_3, \ldots, x_n) + C(x_0, x_1) = 0,$$

where C is a polynomial of the third degree and Q is of rank at least 3. Since X is k-unirational, $U_{\text{smooth}}(k)$ is not empty. That U_{smooth} satisfies weak approximation is easy to prove [**3**, Proposition 3.12].

We may now assume that Q is of maximal rank and that M is non-zero.

In this case we shall prove the theorem by reduction to Theorem 4.5. Let S be a finite set of places of k, let $\{M_v\}_{v \in S}$ be a finite set of smooth local points. We may assume that all these points are contained in the open set $x_n \neq 0$. Let $\lambda_v \in k_v$ be the value of the function x_{n-1}/x_n at the point M_v. Changing each M_v slightly if necessary, we may assume that each hyperplane $x_{n-1} = \lambda_v x_n$ is transversal to X at the point M_v. Using weak approximation on k, we may now find $\lambda \in k$ such that

(a) λ is so close to each λ_v for $v \in S$ that there exists a k_v-point P_v very close to M_v and lying on the hyperplane section X_λ given by $x_{n-1} = \lambda x_n$, the point P_v being smooth on X_λ;

(b) $Q(x_3, \ldots, x_{n-1}, \lambda x_{n-1})$ is a quadratic form of maximal rank $(n-3)$ and is irreducible over the discriminant extension of K/k;

(c) $M(x_3, \ldots, x_{n-1}, \lambda x_{n-1})$ does not vanish identically.

Only (b) deserves an explanation. For $n \geq 6$ it is obvious. For $n = 5$, let $P(t)$ be the separable polynomial which is the discriminant of the binary quadratic form $Q(x_3, x_4, tx_4)$, and let $k(\sqrt{d})$ be the discriminant extension attached to K/k. It is a consequence of Hilbert's irreducibility theorem (with the approximation condition) that one may find $\lambda \in k$ satisfying (a) and (b) and such that neither $-P(\lambda)$ nor $-P(\lambda)/d$ are squares, which implies that $-P(\lambda)$ is not a square in $k(\sqrt{d})$.

Now $X_\lambda \subset \mathbb{P}_k^{n-1}$ is given by an equation of Type (II), with L non-zero, M non-zero, and Q irreducible over the discriminant extension of K/k and of maximal rank. An obvious induction argument, together with Theorem 4.5, completes the proof.

For $n \geq 6$, there is a much easier way to prove Theorem 5.1, which we shall now present.

Proposition 5.2. *Let $X \subset \mathbb{P}^n_k$, for $n \geq 4$, be given by* (5.1). *Assume it is not a cone. Then X is k-birational to an affine space or to a k-variety given by an affine equation*

$$\sum_{i=3}^{n-1} a_i x_i^2 + f(x_0, x_1, x_2) = 0,$$

with $a_i \in k^*$ and f a non-zero polynomial.

Proof. First assume that M is identically zero. We may assume that Q is in diagonal form $Q = \sum_{i=3}^{n} a_i x_i^2$. Here all a_i are non-zero, since X is not a cone. The open set $x_n L(x_0, x_1, x_2) \neq 0$ of X is then given by the affine equation

$$\sum_{i=3}^{n-1} a_i x_i^2 + a_n + N_{K/k}(x_0 + \omega x_1 + \omega^2 x_2)/L(x_0, x_1, x_2) = 0.$$

A change of variables $x_i \to x_i L(x_0, x_1, x_2)$ $(i = 3, \dots, n-1)$ and a multiplication by L^2 complete the proof.

Note that we could have set $L(x_0, x_1, x_2) = 1$, as in the proof of the previous theorem. This leads to an affine equation of the shape

$$\sum_{i=3}^{n} a_i x_i^2 + f(x_0, x_1) = 0,$$

so that the conclusion already holds in this case with one more variable in the left-hand quadratic form.

Suppose now that M is not identically zero. We may set $M = x_n$. If $S(x_3, \dots, x_{n-1}) = Q(x_3, \dots, x_{n-1}, 0)$ is a degenerate quadratic form and $(\alpha_3, \dots, \alpha_{n-1}) \in k^{n-3}$ is a non-zero point at which all first derivatives of S vanish, then $(0, 0, 0, \alpha_3, \dots, \alpha_{n-1}, 0)$ is a singular k-point on X, which is then k-birational to an affine space (Lemma 1.2). If S is non-degenerate, setting $x_n = 1$ and performing linear changes of variables, we find that the open set $x_n L(x_0, x_1, x_2) \neq 0$ of X is given by an affine equation

$$\sum_{i=3}^{n-1} a_i x_i^2 + a_n + N_{K/k}(x_0 + \omega x_1 + \omega^2 x_2)/(L(x_0, x_1, x_2) + 1) = 0,$$

where $a_i \in k^*$ $(i = 3, \dots, n-1)$ and $a_n \in k$. A change of variables $x_i \to x_i(L(x_0, x_1, x_2) + 1)$ $(i = 3, \dots, n-1)$ and a multiplication by $(L+1)^2$ complete the proof.

Corollary 5.3. *If X is as above and $n \geq 6$, respectively $n \geq 5$ if $M = 0$, and if Z is a smooth proper model of X, then the Galois module* $\operatorname{Pic} \bar{Z}$ *is stably a permutation module.*

Proof. Apply [**3**, Remark 3.7.1].

Corollary 5.4. *If k is a number field, $n \geq 6$, respectively $n \geq 5$ if $M = 0$, and Z is a smooth model of X, then $Z(k)$ is not empty and weak approximation holds for Z.*

Proof. That $Z(k)$ is not empty follows from the fact that X is k-unirational (beginning of the proof of Theorem 5.1). Now apply [**3**, Proposition 3.12] (whose proof is easy).

6. *Case* (III), $n=4$

In this section we consider a geometrically integral cubic hypersurface $X \subset \mathbb{P}_k^4$ given by the equation

$$N_{K/k}(x_0+\omega x_1+\omega^2 x_2)+C(x_3, x_4)=0, \tag{6.1}$$

where C is a non-zero cubic form with coefficients in k.

LEMMA 6.1. *If C is not separable and if X has smooth k_v-points for all places v of k, then X is a k-rational variety.*

Proof. First assume that $C=-cx_3^3$ with $c \in k^*$. Let T be the 2-dimensional k-torus given by

$$N_{K/k}(x_0+\omega x_1+\omega^2 x_2)=1,$$

and let E be the principal homogeneous space under T given by

$$N_{K/k}(x_0+\omega x_1+\omega^2 x_2)=c.$$

In this case X is a cone, and it is k-birational to the product of $\mathbb{A}_k^1$ by the principal homogeneous space E. Because T is 2-dimensional, $E(k) \neq \varnothing$ implies that E is k-rational [**14**, 4.74] and that E satisfies the Hasse principle [**14**, 6.40].

Now assume that $C=cx_3^2x_4$ with $c \in k^*$. In this case X is a k-rational variety, since the open set $x_3 \neq 0$ is given by the affine equation

$$N_{K/k}(x_0+\omega x_1+\omega^2 x_2)+cx_4=0.$$

THEOREM 6.2. *Let $X \subset \mathbb{P}_k^4$ be given by equation* (6.1). *Assume that the form C is separable. Let Z be a smooth proper model of X. If C is irreducible over k, then the Hasse principle and weak approximation hold on Z. If C is reducible over k, then $Z(k)$ is non-empty and the Brauer–Manin obstruction to weak approximation on Z is the only one; if C does not split completely over the discriminant extension of K/k, then weak approximation holds on Z.*

Proof. The proof will be by descent and will follow the same pattern as that of Theorem 4.5. The reader will easily recognize the various steps. We may assume that the coefficient of x_3 in $C(x_3, x_4)$ is non-zero. Let $P(x_3)=-C(x_3, 1)$. This is a separable polynomial of degree 3. Let $U \subset X$ be the smooth open set with affine equation

$$N_{K/k}(x_0+\omega x_1+\omega^2 x_2)=P(x) \neq 0. \tag{6.2}$$

Let L be the separable k-algebra $L=k[x]/P(x)=k[\theta]$, where θ denotes the class of x in L. Note that $P(x)$ then reads

$$P(x)=cN_{L/k}(x-\theta)$$

for some $c \in k^*$. Let M be the separable k-algebra $M=K \otimes_k L$. Let T be the k-torus defined by the exact sequence

$$1 \longrightarrow T \longrightarrow \mathbb{G}_{m,k} \times R_{M/k}\mathbb{G}_m \xrightarrow{(i, N_{M/L})} R_{L/k}\mathbb{G}_m \longrightarrow 1, \tag{6.3}$$

where the right-hand arrow is given on a commutative k-algebra A by

$$A^* \times (A \otimes_k M)^* \to (A \otimes_k L)^*,$$
$$(\lambda, \xi) \to (\lambda . N_{M/L}(\xi)).$$

Given any semi-local commutative k-algebra A, sequence (6.3) induces an isomorphism

$$(A \otimes_k L)^*/A^*.N_{M/L}(A \otimes_k M)^* \simeq H^1_{\text{ét}}(A, T).$$

Note that $(x_3 - \theta)$ defines a k-morphism $U \to R_{L/k}\mathbb{G}_m$, and hence, in particular, an element of $(k(X) \otimes_k L)^*$. We now claim:

(6.4) For any discrete valuation ring $A \subset k(X)$ which contains k and whose field of fractions is $k(X)$, the class of

$$(x_3 - \theta) \in (k(X) \otimes_k L)^*/k(X)^*.N_{M/L}(k(X) \otimes_k M)^* \simeq H^1_{\text{ét}}(k(X), T)$$

belongs to the image of $(A \otimes_k L)^*/A^*.N_{M/L}(A \otimes_k M)^* \simeq H^1_{\text{ét}}(A, T)$.

This is proved by a computation based on (6.2) and is analogous to the proof of [**3**, (12.21)]. Since $H^1_{\text{ét}}(k(X), T)$ is killed by 3, we may allow quadratic extensions and assume that K/k and L/k are cyclic extensions. If $K \otimes_k L$ is not a field, $H^1_{\text{ét}}(k(X), T) = 0$ and the result is clear. We may thus assume that $M = K \otimes_k L$ is a field. Let us write (6.2) as

$$\text{(6.2.1)} \qquad N_{K/k}(x_0 + \omega x_1 + \omega^2 x_2) = c(x - \theta)(x - \theta')(x - \theta'') \neq 0,$$

where θ' and θ'' are the two conjugates of θ in L. Let A be as in (6.4), let p be a uniformizing parameter of A. If p splits in $A \otimes_k K$, then any prime π above p in $A \otimes_k L$ splits in $A \otimes_k M$, and hence belongs to the product $(A \otimes_k M)^*.N_{M/L}(k(X) \otimes_k M)^*$, and hence so does any element of $(k(X) \otimes_k L)^*$. We may therefore assume that p remains prime in $A \otimes_k K$, and hence that any prime π in $A \otimes_k L$ above p remains prime in $A \otimes_k M$.

Let us first consider the case where there is just one prime ideal π above p, that is, p remains prime in $A \otimes_k L$ and hence in $A \otimes_k M$. Let v_π be the associated valuation. If $v_\pi(x - \theta) < 0$, then $v_\pi(x - \theta) = v_\pi(x)$ and $(x - \theta)/x \in (A \otimes_k L)^*$; hence $(x - \theta) \in k(X)^*.(A \otimes_k L)^*$. If $v_\pi(x - \theta) = 0$, then $(x - \theta) \in (A \otimes_k L)^*$. If $v_\pi(x - \theta) > 0$, then $v_\pi(x - \theta') = v_\pi(x - \theta'') = 0$, and from (6.2.1) we conclude that

$$x - \theta \in (A \otimes_k L)^*.N_{M/L}(k(X) \otimes_k M)^*.$$

Let us now consider the case when there are three prime ideals in $A \otimes_k L$ above p, with conjugate generators π_1, π_2, π_3. If $v_{\pi_i}(x - \theta) < 0$ for some i, valuation arguments imply that

$$v_p(x) = v_{\pi_i}(x) = v_{\pi_i}(x - \theta) = v_{\pi_i}(x - \theta') = v_{\pi_i}(x - \theta'').$$

Consideration of the Galois action then reveals that $v_{\pi_i}(x - \theta) = v_{\pi_j}(x - \theta)$ for $j \neq i$. Thus $(x - \theta)/x \in (A \otimes_k L)^*$, and hence $(x - \theta) \in k(X)^*.(A \otimes_k L)^*$. If $v_{\pi_i}(x - \theta) > 0$ for some i, then

$$v_{\pi_i}(x - \theta') = v_{\pi_i}(x - \theta'') = 0.$$

Galois considerations then yield $v_{\pi_j}(x - \theta) = 0$ for $j \neq i$. On the other hand, (6.2.1) now gives

$$v_{\pi_i}(x - \theta) = v_{\pi_i}(N_{K/k}(x_0 + \omega x_1 + \omega^2 x_2)) = 3n$$

for some $n \in \mathbf{Z}$. If we let Π_i be the generator of the prime ideal of $A \otimes_k M$ above π_i, we find that $(x-\theta)/N_{M/L}(\Pi_i)^n$ belongs to $(A \otimes_k L)^*$, whence

$$(x-\theta) \in (A \otimes_k L)^*.N_{M/L}(k(X) \otimes_k M)^*.$$

Finally, if $v_{\pi_i}(x-\theta)=0$ for all i, then certainly $(x-\theta) \in (A \otimes_k L)^*$.

Given $\rho \in L^*$, let $\mathcal{T}^\rho$ be the affine k-variety defined in $\mathbb{A}_k^{14}$ by

$$\begin{aligned} &N_{K/k}(x_0+\omega x_1+\omega^2 x_2)=cN_{L/k}(x-\theta) \neq 0, \\ &x-\theta=\rho.\lambda.N_{M/L}(\xi) \neq 0 \end{aligned} \tag{6.5}$$

(λ is a variable in k and ξ is a variable in M, and hence represents nine variables in k). Note that $\mathcal{T}^\rho$ is a torsor over the smooth k-variety U under the k-torus T. Just as in Theorem 4.5, part (d) of the proof, it follows from (6.4) that this torsor extends to a torsor over any smooth compactification of U.

We refer to § 9 for the proof of the following fact.

(6.6) If C does not split completely as the product of three linear factors over the discriminant extension of K/k, then $H^1(k, \operatorname{Pic} \bar{Z})=0$ for any smooth proper model of X.

If C is reducible and, say, x_3 divides C, then the k-point given by

$$x_0=x_1=x_2=x_3=0$$

is non-singular since C is separable. Hence any smooth proper model Z of X contains a k-point. More is true: since X is not a cone (easy verification), $X(k) \neq \varnothing$ implies that X is k-unirational (Propositions 1.2 and 1.3).

Let us assume that $U(k_v) \neq \varnothing$ for all v (this comes for free if C is reducible). If C splits completely over the discriminant extension of K/k, assume that there is no Brauer–Manin obstruction to weak approximation on smooth proper models of X. Let S be a finite set of places of k, and for each $v \in S$, let P_v be a point of $U(k_v)$. Just as in the proof of Theorem 4.5, it then follows from (6.4), (6.6), and descent theory [**2**, § 3] that there exists $\rho \in L^*$ such that the k-variety $\mathcal{T}^\rho$ has k_v-points for each place $v \in k$ and such that for each place $v \in S$ there exists $M_v \in \mathcal{T}^\rho(k_v)$ which projects down to $P_v \in U(k_v)$. In order to prove the theorem, it will be enough to show that a k-variety such as $\mathcal{T}^\rho$ satisfies the Hasse principle and weak approximation.

The change of variables

$$y_0+\omega y_1+\omega^2 y_2=(x_0+\omega x_1+\omega^2 x_2)/\lambda.N_{M/K}(\xi), \tag{6.7}$$

realizes an isomorphism between $\mathcal{T}^\rho$ and the subvariety of $\mathbb{A}_k^{14}$ (with coordinates $(y_0, y_1, y_2, x, \lambda, \xi)$) given by the following system:

$$\begin{aligned} &(6.8.1)\quad N_{K/k}(y_0+\omega y_1+\omega^2 y_2)=c.N_{L/k}(\rho), \\ &(6.8.2)\quad x-\theta=\rho.\lambda.N_{M/L}(\xi) \neq 0. \end{aligned} \tag{6.8}$$

This k-variety clearly is the product of the k-variety $V_\rho \subset \mathbb{A}_k^3$ given by (6.8.1) with the k-variety $W_\rho \subset \mathbb{A}_k^{11}$ given by (6.8.2). Thus the effect of the change of variables (6.7) has been to *separate variables*.

The variety V_ρ clearly is a principal homogeneous space under a 2-dimensional k-torus. As such, it satisfies the Hasse principle and weak approximation [**14**, 4.74 and 6.40]. It only remains to prove the same statement for W_ρ.

The change of variables

$$u = x.\lambda^{-1}, \quad v = \lambda^{-1},$$

induces a k-isomorphism between W_ρ and the subvariety of $\mathbb{A}_k^{11}$ given by

$$u - v\theta = \rho.N_{M/L}(\xi) \neq 0,$$

which is itself an open set in the closed subvariety of $\mathbb{A}_k^{11}$ given by

$$u - v\theta = \rho.N_{M/L}(\xi). \tag{6.9}$$

Identifying coefficients of 1, θ and θ^2 in this equality, we see that (6.9) is equivalent to the system

$$u = f_\rho(\xi), \quad v = g_\rho(\xi), \quad 0 = h_\rho(\xi), \tag{6.10}$$

where f_ρ, g_ρ, h_ρ are cubic forms in nine variables with coefficients in k. The k-variety defined by this last system is k-isomorphic to the cubic cone in $\mathbb{A}_k^9$ defined by

$$h_\rho(\xi) = 0. \tag{6.11}$$

As will be checked below, h_ρ is an absolutely irreducible form. We now conclude that W_ρ is k-birational to the product of $\mathbb{A}_k^1$ by the geometrically integral cubic hypersurface $X_\rho \subset \mathbb{P}_k^8$ defined by (6.11), and it is enough to prove the Hasse principle and weak approximation for the smooth locus of X_ρ.

Let us introduce variables $u, v, w, u', v', w', u'', v'', w''$, let $\{\theta_i\}_{i=1,2,3}$, respectively $\{\rho_i\}_{i=1,2,3}$, be the image of θ, respectively ρ, in $\bar{k}^3$ under a fixed isomorphism $L \otimes_k \bar{k} \simeq \prod_{i=1}^3 \bar{k}$. Define the following linear forms with coefficients in $\bar{k}$:

$$T_i = u + \theta_i v + \theta_i^2 w, \quad T_i' = u' + \theta_i v' + \theta_i^2 w', \quad T_i'' = u'' + \theta_i v'' + \theta_i^2 w''.$$

From (6.9) we deduce that over $\bar{k}$, $h_\rho(\xi) = h_\rho(u, v, w, u', v', w', u'', v'', w'')$ reads

$$h_\rho = \sum_{i=1}^{3} \alpha_i N_{K/k}(T_i + \omega T_i' + \omega^2 T_i''),$$

where $\alpha_i = (\theta_{i+1} - \theta_{i+2})\rho_i \neq 0$ ($i \bmod 3$) and the forms $N_{K/k}(T_i + \omega T_i' + \omega^2 T_i'')$ have coefficients in $k(\theta_i)$.

Let ω_j ($j = 1, 2, 3$) be the various images of ω in $\bar{k}$. Fix an embedding of K into $\bar{k}$, so that $\omega = \omega_1$.

Define linear forms with coefficients in $\bar{k}$ in the variables $u, v, w, u', v', w', u'', v'', w''$:

$$Y_{i,j} = T_i + \omega_j T_i' + \omega_j^2 T_i'' \quad (j = 1,2,3).$$

The form h_ρ then reads

$$h_\rho = \sum_{i=1}^{3} \alpha_i Y_{i,1} Y_{i,2} Y_{i,3}, \tag{6.12}$$

and hence is irreducible over $\bar{k}$ and one checks that X_ρ is not a cone.

The subvariety $H \subset \mathbb{P}_k^8$ defined by $Y_{i,2} = Y_{i,3} = 0$ ($i = 1, 2, 3$) is a 2-dimensional linear space defined over $K \subset \bar{k}$ and it is contained in the singular locus of X_ρ. Let F be the closed set $Y_{1,1}.Y_{2,1}.Y_{3,1} = 0$. If $P \in H(\bar{k})$ does not belong to F, the tangent cone to X_ρ at P is defined by a quadratic form of rank 6. Let P be a

K-point of H which does not lie on F. The three conjugates of P define a set of conjugate singular points on the cubic hypersurface $X_\rho \subset \mathbb{P}^8_k$.

According to 1.2, 1.4, and 1.5, either X_ρ is a k-rational variety, or X_ρ is defined by an equation of the type

$$(6.13)\quad N_{K/k}(x_0+\omega x_1+\omega^2 x_2) + \sum_{i=0}^{2}(x_0+\omega_i x_1+\omega_i^2 x_2)Q_i(x_3, \ldots, x_8) + C(x_3, \ldots, x_8) = 0.$$

If this equation were of Type (III), the tangent cone to P would be defined by a quadratic form of rank 2, and this has been excluded. Since the tangent cone at each of the conjugates of P is of rank 6, each form Q_i in (6.13) is of rank 4. If (6.13) were of Type (II), each of Q_1, Q_2, Q_3 and C would be divisible by some quadratic form Q of rank 4 defined over k. An obvious change of variables would reduce (6.13) to

$$N_{K/k}(x_0+\omega x_1+\omega^2 x_2) + Q(x_3, \ldots, x_6)L(x_0, \ldots, x_8) = 0,$$

which is the equation of a cone. But X_ρ is not a cone (cf. (6.12)). Hence X_ρ is of Type (I) and according to § 3 the smooth locus of X_ρ satisfies the Hasse principle and weak approximation. This completes the proof of Theorem 6.2.

7. *Case* (III), $n > 4$

THEOREM 7.1. *Let $X \subset \mathbb{P}^n_k$, with $n > 4$, be a geometrically integral cubic hypersurface given by an equation*

$$(7.1)\quad N_{K/k}(x_0+\omega x_1+\omega^2 x_2) + C(x_3, \ldots, x_n) = 0,$$

where C is a cubic form. Assume that C is irreducible over k or that X is not a cone. Then the Hasse principle and weak approximation hold for any smooth model of X.

Proof. Suppose first that the form C is reducible over k. Thus, after a change of coordinates,

$$C(x_3, \ldots, x_n) = x_n Q(x_3, \ldots, x_n),$$

where Q is a quadratic form with coefficients in k. Note that X contains the k-linear space

$$x_0 = x_1 = x_2 = x_n = 0,$$

and hence contains non-singular k-points since it is not a cone (if it contains a singular k-point, it is k-birational to an affine space). Thus if C is reducible, X is k-unirational (Proposition 1.3) and k-rational points are Zariski-dense on it. Let us write

$$Q = \alpha x_n^2 + x_n R(x_3, \ldots, x_{n-1}) + S(x_3, \ldots, x_{n-1}).$$

If the quadratic form S is degenerate and, say, $(a_3, \ldots, a_{n-1}) \in k^{n-3}$ is a non-trivial point at which all first derivatives of S vanish, then the k-point $(0, 0, 0, a_3, \ldots, a_{n-1}, 0)$ is singular on X, which is then k-birational to an affine space since it is not a cone (Lemma 1.2). If the form S is non-degenerate, performing linear changes of variables reduces the equation of X to

$$N_{K/k}(x_0+\omega x_1+\omega^2 x_2) + x_n(S(x_3, \ldots, x_{n-1}) + cx_n^2) = 0,$$

for some $c \in k$. Thus the equation of the affine open set $x_n \neq 0$ of X is given by

$$S(x_3, \ldots, x_{n-1}) = -(c + N_{K/k}(x_0 + \omega x_1 + \omega^2 x_2)).$$

Since the right-hand side is irreducible over k and of total degree 3 and the quadratic form S is of rank $(n-3)$, weak approximation for X_{smooth} is a special case of [**3**, Theorem 9.3] when $n = 5$. For $n \geqslant 6$ it follows from the much easier Proposition 3.12 of [**3**].

We now study the case where the cubic form C is irreducible over k. Let X be such that $X_{\text{smooth}}(k_v) \neq \varnothing$ for all places v of k. We may assume that the coefficient c of x_3^3 is non-zero. Let S be a finite set of places of k and let $\{M_v\}_{v \in S}$ be a finite set of smooth local points. We may suppose that all these points are contained in the smooth open set U of X defined by $x_n . C \neq 0$. Enlarging S, we may also suppose that the finite set S contains all places v of k such that c is not a local norm for the extension K/k. Let $\lambda_v \in k_v$ be the value of the function x_{n-1}/x_n at the point M_v. Slightly changing each M_v if necessary, we may assume that each hyperplane $x_{n-1} = \lambda_v x_n$ is transversal to X at the point M_v. Using weak approximation on k and Hilbert's irreducibility theorem with the approximation condition, we find $\lambda \in k$ such that

(a) λ is so close to each λ_v for $v \in S$ that there exists a k_v-point P_v which is arbitrarily close to M_v and which lies on the hyperplane section X_λ given by $x_{n-1} = \lambda x_n$, the point P_v being smooth on X_λ;

(b) $C(x_3, \ldots, x_{n-1}, \lambda x_{n-1})$ is an irreducible cubic form over k which does not vanish at P_v for $v \in S$.

It is now clear that the smooth open set $C(x_3, \ldots, x_{n-1}, \lambda x_{n-1}) \neq 0$ of the cubic hypersurface $X_\lambda \subset \mathbb{P}_k^{n-1}$ defined by

$$N_{K/k}(x_0 + \omega x_1 + \omega^2 x_2) + C(x_3, \ldots, x_{n-1}, \lambda x_{n-1}) = 0$$

contains k_v-points for all places v (for $v \notin S$, there exist such points with $x_4 = \ldots = x_{n-1} = 0$). Theorem 6.2 and an induction on the dimension of cubic hypersurfaces given by an equation (7.1) with C irreducible over k complete the proof of the theorem.

8. *Counter-examples to weak approximation*

In this section we give counter-examples to weak approximation which reveal that the restriction on C in Theorem 6.2, that is in Case (III), $n = 4$, is necessary. Namely, weak approximation may not hold when the cubic form $C(x_3, x_4)$ splits completely over the discriminant extension associated to the cubic extension K/k.

Similar examples surely exist in Case (II), $n = 4$, that is, when the form Q in Theorem 4.5 splits into linear factors over the discriminant extension of K/k. Exhibiting such examples will be left as an exercise for the reader.

Let $k = \mathbf{Q}(\theta)$, where θ is a primitive cube root of 1, let $F = \mathbf{Q}(\omega)$ and let $K = k(\omega)$, where $\omega = \sqrt[3]{3}$. The cyclic extension K/k is ramified only at the prime $\lambda = 1 - \theta$. Let k_λ, respectively K_λ, respectively F_λ, denote the completions of k, respectively K, respectively F, at the unique prime above 3. Let X be the smooth $\mathbf{Q}$-variety defined in affine space $\mathbb{A}_{\mathbf{Q}}^4$ by the equation

$$N_{F/\mathbf{Q}}(x + \omega y + \omega^2 z) = t^2 - t + 1 = (t + \theta)(t + \theta^2) \neq 0. \tag{8.1}$$

PROPOSITION 8.1. *The variety $X(\mathbf{Q})$ is not empty but is not dense in $X(\mathbf{Q}_3)$ for the 3-adic topology. Similarly, $X(k)$ is not empty and is not dense in $X(k_\lambda)$ for the λ-adic topology.*

Proof. We know that $X(\mathbf{Q}) \subset X(k)$ contains the rational point $(x, y, z, t) = (0, 1, 0, 1)$.

Let $v \neq \lambda$ be a place of k. Let (x, y, z, t) be in $X(k_v)$. Then $t + \theta$ is a local norm at v. Indeed, either v splits in K (this is always so if v is archimedean), in which case the statement is clear, or v is inert. The extension K_v/k_v is then an unramified extension of non-archimedean local fields of degree 3, so that a non-zero element α of k_v is a norm if and only if its valuation $v(\alpha)$ is divisible by 3. Thus the valuation of the left-hand side of (8.1) is divisible by 3. If $v(t+\theta) > 0$, then $v(t+\theta^2) = 0$ and 3 divides $v(t+\theta)$. If $v(t+\theta) < 0$, then $v(t+\theta^2) = v(t+\theta)$, whence 3 divides $2v(t+\theta)$, and hence also $v(t+\theta)$.

We can already conclude that if $M = (x, y, z, t) \in X(k)$, then $t + \theta$, which is a local norm for the cyclic extension K/k at all places v of k except perhaps at $v = \lambda$, must also be a local norm at λ. The function $t + \theta$ is invertible on X and the map from $X(k_\lambda)$ to k_λ^*/NK_λ^* which associates the class of $t + \theta$ to (x, y, z, t) is continuous. Hence for any $(x, y, z, t) \in X(k_\lambda)$ in the closure of $X(k)$, $(t+\theta) \in k_\lambda$ is a norm of K_λ. This certainly implies that for any $(x, y, z, t) \in X(\mathbf{Q}_3)$ in the closure of $X(\mathbf{Q})$, $(t+\theta) \in k_\lambda$ is a norm of K_λ.

If we can produce an element $t \in \mathbf{Q}_3$ such that

(a) $t + \theta$ is not a norm for K_λ/k_λ (which is true if and only if t is not a norm for $F_\lambda/\mathbf{Q}_3$),

(b) $t^2 - t + 1 \neq 0$ is a norm for $F_\lambda/\mathbf{Q}_3$ (which is true if and only if $t^2 - t + 1$ is a norm for K_λ/k_λ),

say $t^2 - t + 1 = N_{F/\mathbf{Q}}(x + \omega y + \omega^2 z) \neq 0$, with $x, y, z \in \mathbf{Q}_3$, then the point $(x, y, z, t) \in X(\mathbf{Q}_3) \subset X(k_\lambda)$ will not belong to the closure of $X(\mathbf{Q})$ in $X(\mathbf{Q}_3)$, nor will it belong to the closure of $X(k)$ in $X(k_\lambda)$.

Now $t = 3$ is the required element of $\mathbf{Q}_3$, as a diligent application of the Artin–Tate formulae (see [**1**, § 4, **F**3]) reveals:

$$[3 + \theta, 3]_3 = 2 \ (3),$$
$$[3^2 - 3 + 1, 3]_3 = 0 \ (3).$$

Here is another example, which was kindly communicated to us by Daniel Coray, and which avoids wild ramification. It was inspired by Swinnerton-Dyer's counter-example to the Hasse principle for cubic surfaces [**13**].

Let $k = \mathbf{Q}$, and let θ be a root of

$$\theta^3 - 7\theta^2 + 14\theta - 7 = 0.$$

Then $K = \mathbf{Q}(\theta)$ is a cyclic extension of $\mathbf{Q}$, unramified away from 7. Let X be the smooth variety defined in $\mathbb{A}^4_{\mathbf{Q}}$ by

$$N_{K/k}(x + \theta y + \theta^2 z) = (t+1)(t+2) \neq 0. \tag{8.2}$$

It contains the $\mathbf{Q}$-point $(0, 2, 0, 6)$. For any place $v \neq 7$ of $\mathbf{Q}$ and any $(x, y, z, t) \in X(k_v)$ one checks, just as above, that $(t + 1)$ is a local norm for the extension K/k at v. In order to see that $X(\mathbf{Q})$ is not dense in $X(\mathbf{Q}_7)$, it is thus enough to find $t \in \mathbf{Q}_7$ such that $(t+1)(t+2) \neq 0$ and $t + 1$ is not a local norm for K/k at $v = 7$ but

$(t+1)(t+2)$ is a local norm. Now $t=1$ is a suitable value. Indeed, $2 \not\equiv \pm 1$ modulo 7 but $6 \equiv -1$ modulo 7 and for the tamely ramified extension $\mathbf{Q}_7(\theta)/\mathbf{Q}_7$, the units congruent to 1 in $\mathbf{Q}_7$ are exactly the norms of units congruent to 1 in $\mathbf{Q}_7(\theta)$.

9. *Computations of Brauer groups*

In this section we compute the Brauer groups of smooth proper models of most geometrically integral non-conical hypersurfaces $X \subset \mathbb{P}^n_k$ $(n \geq 4)$ defined by an (absolutely irreducible) cubic form:

$$(9.1)\quad N_{K/k}(x_0 + \omega x_1 + \omega^2 x_2) + \sum_{i=0}^{2} (x_0 + \omega_i x_1 + \omega_i^2 x_2) Q_i(x_3, \ldots, x_n) + C(x_3, \ldots, x_n) = 0,$$

with $K = k(\omega)$, ω_i, Q_i and C being as in Proposition 1.6. Here the ground field k is an arbitrary field of characteristic zero and $\mathfrak{G}$ is the Galois group $\mathrm{Gal}(\bar{k}/k)$ of an algebraic closure $\bar{k}$ of k. The irreducibility assumption implies that not all forms Q_i and C vanish.

Let Y be the projective space $\mathbb{P}^{n-3}_k$ with coordinates $(y_3, \ldots, y_n)$. Let $M \subset \mathbb{P}^{n-3}_k$ be the closed k-variety which over $\bar{k}$ is the union over $i \bmod 3$ of the closed subvarieties defined by

$$Q_{i+1}(y_3, \ldots, y_n) = Q_{i+2}(y_3, \ldots, y_n) = 0 \quad \text{and} \quad C(y_3, \ldots, y_n) = 0.$$

Let Y_0 be a non-empty Zariski open set of Y whose complement contains M.

Projection $(x_0, \ldots, x_n) \dashrightarrow (x_3, \ldots, x_n)$ induces a rational map $X \dashrightarrow Y$. Let Ξ be the graph of this map. It is the subvariety of $X \times Y$ given by the equations

$$x_i y_j = x_j y_i, \quad \text{where } 3 \leq i < j \leq n.$$

Let F, respectively $\bar{F}$, denote the function field of Y, respectively $\bar{Y}$, and let Ξ_F be the generic fibre of Ξ/Y. Then Ξ_F is isomorphic to the cubic surface in $\mathbb{P}^3_F$ with coordinates (X_0, X_1, X_2, X_3) given by the equation

$$(9.2)\quad N_{K/k}(X_0 + \omega X_1 + \omega^2 X_2) + \sum_{i=0}^{2} b_i (X_0 + \omega_i X_1 + \omega_i^2 X_2) X_3^2 + c X_3^3 = 0,$$

where $b_i = Q_i(y_3, \ldots, y_n)/y_3^2$ $(i = 0, 1, 2)$ and $c = C(y_3, \ldots, y_n)/y_3^3$.

The assumption that X is geometrically integral and Lemma 2.2 imply that Ξ_F is geometrically integral. Let S_i ($i \bmod 3$) be the subvariety of $\Xi_{k(\omega_i)}$ given by

$$(9.3)\quad x_0 + \omega_{i+1} x_1 + \omega_{i+1}^2 x_2 = x_0 + \omega_{i+2} x_1 + \omega_{i+2}^2 x_2 = x_3 = x_4 = \ldots = x_n = 0.$$

Then the generic point of S_i coincides with the rational point P_i on $\Xi \times_F F(\omega_i)$ given by

$$(9.4)\quad X_0 + \omega_{i+1} X_1 + \omega_{i+1}^2 X_2 = X_0 + \omega_{i+2} X_1 + \omega_{i+2}^2 X_2 = X_3 = 0.$$

If $C^2 + 4Q_0Q_1Q_2 = 0$, there is a singular F-point on Ξ_F given by the three equations

$$2b_i(X_0 + \omega_i X_1 + \omega_i^2 X_2) + cX_3 = 0 \quad (i = 1, 2, 3).$$

Thus if $C^2 + 4Q_0Q_1Q_2 = 0$, then the cubic surface Ξ_F is an F-rational variety

(Proposition 1.2 and Lemma 2.2), and hence Z is a k-rational variety and Pic $\bar{Z}$ is a stably permutation module [**2**, Proposition 2.A.1].

If $C^2+4Q_0Q_1Q_2\neq 0$, which we now assume, then the points P_i are the only (geometric) singular points on Ξ_F. Let W be the blowing-up of Ξ along the closed subvariety $\bigcup_{i=0}^2 S_i$. Then the generic fibre W_F of W/Y is the blow-up of Ξ_F along its singular locus $\bigcup_{i=0}^2 P_i$. Since Ξ_F is normal and its geometric singular points are rational double points of type either A_1 or A_2, this blow-up is smooth.

Now let V be the Zariski open set of W whose complement is the union of the singular loci of the singular fibres of W/Y and let U be the inverse image of Y_0 under the projection $V\to Y$. The fibres of U/Y_0 are non-empty and are geometrically integral (Lemma 2.2). Let $U_{\bar{F}}$ denote the generic fibre of $\bar{U}/\bar{Y}_0$. We have a natural exact sequence

(9.5) $$\text{Pic } \bar{Y}_0 \to \text{Pic } \bar{U} \to \text{Pic } U_{\bar{F}} \to 0,$$

and it follows, from [**6**, Proposition 3.2], that

(9.6) Pic $U_{\bar{F}}$ is stably a permutation $\mathfrak{G}$-module.

Note that we also have a natural isomorphism

(9.7) $$\bar{k}[Y_0]^* \simeq \bar{k}[U]^*.$$

Indeed, $\bar{k}[U]^*\subset \bar{F}^*$ because U_F/F is proper and geometrically integral. But the map $\bar{U}\to \bar{Y}_0$ is surjective, so that any function in $\bar{F}^*$ whose inverse image on $\bar{U}$ is invertible must belong to $\bar{k}[Y_0]^*$.

LEMMA 9.1. *Let $X\subset \mathbb{P}_k^n$ $(n\geqslant 3)$ be a geometrically integral cubic hypersurface as in* (9.1) *and assume that each Q_i divides C and that each Q_i divides $Q_{i+1}Q_{i+2}$. Then there is a Del Pezzo surface Z of degree* 6 *defined over k such that X is k-birational to $Y\times_k Z=\mathbb{P}_k^{n-3}\times_k Z$.*

Proof. It follows from the assumptions that there are linear K_i-forms $L_i(x_3,\dots,x_n)\neq 0$ which are conjugate under Gal$(\bar{k}/k)$ exactly as the Q_i are, and $\alpha\in k^*$ and $\beta\in k$ such that

(a) $Q_i=\alpha L_{i+1}L_{i+2}$ $(i=0,1,2)$,

(b) $C=\beta\prod_{i=0}^2 L_i$.

Let $a_i=L_i(y_3,\dots,y_n)/y_3\in K_i(Y)$ and consider $\Xi_F\subset\mathbb{P}_F^3$ (cf. (9.2)) in the new coordinates Ξ_0, Ξ_1, Ξ_2, Ξ_3 given by

$$\Xi_0+\omega_i\Xi_1+\omega_i^2\Xi_2=a_i^{-1}(X_0+\omega_iX_1+\omega_i^2X_2),$$
$$\Xi_3=X_3.$$

Then Ξ_F is given by the equation:

$$N_{K/k}(\Xi_0+\omega_i\Xi_1+\omega_i^2\Xi_2)+\alpha\,\text{Tr}_{K/k}(\Xi_0+\omega_i\Xi_1+\omega_i^2\Xi_2)+\beta\Xi_3^3=0.$$

Hence X is k-birational to $Y\times_k W$, where W is the cubic surface over k defined by precisely the same equation as above. Such a cubic surface is k-birational to a Del Pezzo surface of degree 6 [**6**, Theorem 1.3].

PROPOSITION 9.2. *Let $X\subset\mathbb{P}_k^n$ $(n\geqslant 4)$ be a geometrically integral cubic hypersurface defined by a form as in* (9.1). *Assume that X is of Type* (I), *and let Z be a* *smooth proper model of X. Then* $\operatorname{Pic}\bar{Z}$ *is a stably permutation module,* $H^1(\mathfrak{G}, \operatorname{Pic}\bar{Z}) = 0$, *and* $\operatorname{Br} Z/\operatorname{Br} k = 0$.

Proof. If Q_i divides C and each Q_i divides $Q_{i+1}Q_{i+2}$, then the conclusion follows from Lemma 9.1, [**2**, Proposition 2.A.1], and [**6**, Proposition 3.2]. We may thus assume that either Q_i does not divide C or that, for some i, Q_i does not divide $Q_{i+1}Q_{i+2}$. Then there exists a hyperplane $H \subset Y$ defined over k such that $M \backslash M \cap H$ is of codimension at least 2 in $Y \backslash H$. Let Y_0 be the complement of $H_0 \cup M$ in Y. Then $\bar{k}[Y_0]^* = \bar{k}^*$ and $\operatorname{Pic}\bar{Y}_0 = 0$. It now follows from (9.5), (9.6), and (9.7) that $\operatorname{Pic}\bar{Z}$ is stably a permutation module (cf. [**3**, Lemma 3.6] or [**2**, Proposition 2.A.1]). The last two statements are well-known consequences of the first.

PROPOSITION 9.3. *Suppose that X is of Type* (II) *and $n \geq 5$. Then* $H^1(\mathfrak{G}, \operatorname{Pic}\bar{Z}) = 0$ *for any smooth proper model Z of X.*

Proof. When the quadratic form Q (notation of (5.1)) is not of maximal rank, X is a k-rational variety since it is not a cone (cf. the proof of Theorem 5.1); hence $\operatorname{Pic}\bar{Z}$ is a stably permutation module and $H^1(\mathfrak{G}, \operatorname{Pic}\bar{Z}) = 0$. So assume that Q is of maximal rank $(n-2) \geq 3$. Now M is the geometrically integral quadric $Q = 0$. Let Y_0 be the complement of M. We still have $\bar{k}[Y_0]^* = \bar{k}^*$, but now $\operatorname{Pic}\bar{Y}_0 = \mathbf{Z}/2$, and (9.5) and (9.6) only imply that $H^1(\mathfrak{G}, \operatorname{Pic}\bar{U})$ is killed by 2. Since $\bar{k}[Y_0]^* = \bar{k}^*$, this implies that $H^1(\mathfrak{G}, \operatorname{Pic}\bar{Z})$ itself is killed by 2. But the cubic surface U_F becomes rational over the cubic extension $K(Y)$ of $F = k(Y)$ (Proposition 1.2); hence the variety U_K is a K-rational variety. A corestriction argument then shows that $H^1(\mathfrak{G}, \operatorname{Pic}\bar{Z})$ is killed by 3, and this completes the proof of $H^1(\mathfrak{G}, \operatorname{Pic}\bar{Z}) = 0$.

Note that for $n \geq 6$, Corollary 5.3 gives a better result.

PROPOSITION 9.4. *Suppose that X is of Type* (III), *and $n \geq 5$. Then* $H^1(\mathfrak{G}, \operatorname{Pic}\bar{Z}) = 0$ *for any smooth proper model Z of X.*

Proof. The hypersurface X is now given by an equation

$$N_{K/k}(x_0 + \omega x_1 + \omega^2 x_2) + C(x_3, \ldots, x_n) = 0.$$

According to [**2**, Théorème 2.B.1], it is enough to prove the theorem under the assumption $Z(k) \neq \emptyset$. Since Z is smooth and X is proper, this assumption implies that $X(k) \neq \emptyset$; hence X is k-unirational (Proposition 1.3) and hence in turn $X(k)$ is Zariski-dense in X.

Since X_K is a K-rational variety and K/k is of degree 3, standard arguments imply that if $H^1(\mathfrak{g}, \operatorname{Pic} Z_K) = 0$ when K/k is cyclic and $\mathfrak{g} = \operatorname{Gal}(K/k)$, then $H^1(\mathfrak{G}, \operatorname{Pic}\bar{Z}) = 0$ in general.

(a) Let us first assume that the cubic form C is absolutely irreducible. Then M, which is given by the vanishing of C, is absolutely irreducible. Let Y_0 be the complement of M in Y. We have $K[Y_0]^* = K^*$; hence $K[U]^* = K^*$ by the analogue of (9.7) and $\operatorname{Pic} Y_{0,K} = \mathbf{Z}/3$. The analogue of (9.5) at the level of K reads

$$\operatorname{Pic} Y_{0,K} \to \operatorname{Pic} U_K \to \operatorname{Pic} U_{F_K} \to 0,$$

and it follows from [**6**, Proposition 3.2], that $\operatorname{Pic} U_{F_K}$ is stably a permutation $\mathfrak{G}$-module. If the map $\operatorname{Pic} Y_{0,K} \to \operatorname{Pic} U_K$ is zero, then $\operatorname{Pic} U_K$ is stably a permutation module, hence so is $\operatorname{Pic} Z_K$ since $K[U]^* = K^*$, and hence finally $H^1(\mathfrak{g}, \operatorname{Pic} Z_K) = 0$. If the map is non-zero, the above sequence may be completed into an exact sequence

$$0 \to \operatorname{Pic} Y_{0,K} \to \operatorname{Pic} U_K \to \operatorname{Pic} U_{F_K} \to 0.$$

Part of the associated cohomology sequence may be inserted in a diagram:

$$\begin{tikzcd}
(\operatorname{Pic} U_K)^{\mathfrak{g}} \arrow[r] & (\operatorname{Pic} U_{F_K})^{\mathfrak{g}} \arrow[r] & H^1(\mathfrak{g}, \operatorname{Pic} Y_{0,K}) \arrow[r] & H^1(\mathfrak{g}, \operatorname{Pic} U_K) \arrow[r] & 0 \\
\operatorname{Pic} U \arrow[u] \arrow[r] & \operatorname{Pic} U_F \arrow[u] & & &
\end{tikzcd}$$

The assumption $U(k) \neq \varnothing$ ($X(k)$ is Zariski-dense), together with $K[U]^* = K^*$, implies that the map $\operatorname{Pic} U \to (\operatorname{Pic} U_K)^{\mathfrak{g}}$ is an isomorphism (cf. for example [**2**, p. 412]). Thus the cokernel of the map $(\operatorname{Pic} U_K)^{\mathfrak{g}} \to (\operatorname{Pic} U_{F_K})^{\mathfrak{g}}$ is at least as big as the cokernel of the map $\operatorname{Pic} U_F \to (\operatorname{Pic} U_{F_K})^{\mathfrak{g}}$. But U_F is a smooth proper surface which is F-birational to a Severi–Brauer surface over F (see Lemma 9.5 below). Now either this surface has an F-point, in which case its function field is purely transcendental over F, and hence Z is a k-rational variety, so that the conclusion certainly holds, or the cokernel of the map $\operatorname{Pic} U_F \to (\operatorname{Pic} U_{F_K})^{\mathfrak{g}}$ is equal to $\mathbf{Z}/3$. Since $H^1(\mathfrak{g}, \operatorname{Pic} Y_{0,K}) = H^1(\mathbf{Z}/3, \mathbf{Z}/3) = \mathbf{Z}/3$, the above diagram now implies that $H^1(\mathfrak{g}, \operatorname{Pic} U_K) = 0$. Together with $K[U]^* = K^*$ this implies that $H^1(\mathfrak{g}, \operatorname{Pic} Z_K) = 0$.

(b) Now assume that the cubic form C contains a linear factor defined over k. Then X is given by an equation

$$N_{K/k}(x_0 + \omega x_1 + \omega^2 x_2) + L(x_3, \ldots, x_n) Q(x_3, \ldots, x_n) = 0,$$

where L, respectively Q, is a linear, respectively quadratic, form with coefficients in k. Any common zero of x_0, x_1, x_2, L and Q is a singular point on X. Hence X has a singular point defined over a field extension F/k of degree at most two. Now X is F-rational since it is not a cone (Proposition 1.2). On the other hand, $X_{k(\omega)}$ is $k(\omega)$-rational. Standard arguments now imply that $H^1(\mathfrak{G}, \operatorname{Pic} \bar{Z}) = 0$.

(c) We now consider the only case which is left, namely the case when the cubic form C is a constant multiple of the product of three conjugate linear forms. Since X is not a cone, this may only occur when $n = 5$. In this case, after a linear change of variables, an equation for X reads

$$N_{K/k}(x_0 + \omega x_1 + \omega^2 x_2) - c N_{L/k}(x_3 + \tau x_4 + \tau^2 x_5) = 0,$$

where $L = k(\tau)$ is a cubic extension of k and $c \in k^*$. The affine cone over X contains the variety Y_c:

$$N_{K/k}(x_0 + \omega x_1 + \omega^2 x_2) = c N_{L/k}(x_3 + \tau x_4 + \tau^2 x_5) \neq 0,$$

which is a principal homogeneous space under the torus $T_{K,L}$,

$$N_{K/k}(x_0 + \omega x_1 + \omega^2 x_2) = N_{L/k}(x_3 + \tau x_4 + \tau^2 x_5) \neq 0.$$

Since we assumed that $Z(k) \neq \varnothing$, which implied that $X(k)$ is Zariski-dense, Y_c is actually a trivial principal homogeneous space; that is, we may reduce to $c = 1$. Let W

be a smooth compactification of $T_{K,L}$. Now it may be shown that $H^1(\mathfrak{G}, \operatorname{Pic} \bar{W}) = 0$. When L/k is cyclic, this appears in [**8**, Proposition 3.3], which, together with a restriction-corestriction argument, is enough to complete the case under consideration. (As Sansuc (in unpublished work) has shown, the vanishing actually holds for tori of type $T_{K,L}$ in a more general situation, namely with K/k a cyclic field extension of arbitrary degree and L/k any finite field extension.) Now W is k-birational to $Z \times_k \mathbb{P}^1_k$, and therefore $H^1(\mathfrak{G}, \operatorname{Pic} \bar{Z}) = 0$ also.

The previous results are not used in the proofs of this paper, but they give some reason for the Hasse principle and weak approximation to hold: there is no obstruction of the Manin type.

In the proof of Proposition 9.4 and in the proof of Theorem 9.6 below, we use the following lemma, whose proof is hard to locate in the literature.

LEMMA 9.5. *Let $K = k(\omega)$ be a cubic extension of the field k. Let X be the cubic surface defined by the equation*

$$N_{K/k}(x_0 + \omega x_1 + \omega^2 x_2) + c x_3^3 = 0,$$

where $c \in k^*$. If K/k is a cyclic extension, then X is k-birational to a Severi–Brauer surface defined over k.

Proof. Let $f\colon W \to X$ be the desingularization obtained by blowing up the three double points (of type A_2) P_0, P_1, P_2 on X_K. For $i \in \mathbf{Z}/3$ let $L_i \subset X_K$ be the line through P_{i+1} and P_{i+2}, and let $M_i \subset W_K$ be the strict transform of L_i under f^{-1}. It is shown in [**6**] that one obtains a smooth Del Pezzo surface Y of degree 6 by contracting the exceptional curves of the first kind M_i. The inverse image of each P_i on W_K consists of two smooth projective lines E_i and F_i with $(E_i.E_i) = -2$, $(F_i.F_i) = -2$, $(E_i.F_i) = 1$. Consideration of the tangent cone at P_i, together with the assumption that K/k is a cyclic extension, shows that both lines E_i and F_i are defined over K. By intersection theory one then obtains that the images of the six lines $\{E_i, F_i\}_{i \in \mathbf{Z}/3}$ on Y_K are the six exceptional curves of the first kind on Y_K, which build up a hexagon. Since each of them is defined over K, we may find among them a set of three non-intersecting curves which is globally defined over k and may therefore be blown down over k to produce a Del Pezzo surface of degree 9, that is, a Severi–Brauer surface defined over k.

We now consider cubic *threefolds* of Type (II) and (III). The following result played a crucial rôle in the proof of the Hasse principle and weak approximation (see §§ 4 and 6):

THEOREM 9.6. *Let $X \subset \mathbb{P}^4_k$ be a geometrically integral and non-conical hypersurface defined by a cubic form as in* (9.1). *Let Z be a smooth proper model of X. Then $H^1(\mathfrak{G}, \operatorname{Pic} \bar{Z}) = \operatorname{Br} Z/Br\, k = 0$ if*

(a) *X is of Type* (II), *and the quadratic form $Q(x_3, x_4)$ is irreducible over the discriminant extension of K/k,*

(b) *X is of Type* (III) *and the cubic form $C(x_3, x_4)$ does not split completely over the discriminant extension of K/k.*

Proof. Since X_K is a K-rational variety (Proposition 1.2), a corestriction argument implies that $H^1(\mathfrak{G}, \operatorname{Pic} \bar{Z})$ is killed by 3. Thanks to another corestriction argument, to prove the theorem it is enough to prove that $H^1(\mathrm{Gal}(\bar{k}/k_1), \mathrm{Pic}\,\bar{Z}) = 0$, where k_1 is the discriminant extension of K/k (trivial if K/k is Galois, quadratic otherwise). In other words, it is enough to prove that $H^1(\mathfrak{G}, \mathrm{Pic}\,\bar{Z}) = 0$ when K/k is cyclic, which we now assume. The assumptions in (a) and (b) then read:

(a) in Case (II), Q is irreducible over k,

(b) in Case (III), C does not split completely over k.

Moreover, we may assume that $Z(k) \neq \varnothing$ [**2**, Théorème 2.B.1]. This then implies that $X(k) \neq \varnothing$ (Nishimura's lemma). Since the cubic hypersurface X is not a cone, this implies that X is k-unirational (Proposition 1.3). In particular, $X(k)$ is Zariski-dense in X. We may assume:

(9.8) In Case (III), the cubic form $C(x_3, x_4)$ is separable.

Indeed, if C is not separable, then X is a k-rational variety (use the same proof as in Lemma 6.1).

We shall use the same notation as in the beginning of the section. Here $Y = \mathbb{P}^1_k$, and we let Y_0 be the complement of M. In Case (II), $\bar{Y}_0$ is the complement of two points, and in Case (III) it is the complement of three points. In both cases, $\mathrm{Pic}\,\bar{Y}_0 = 0$. Note that the assumptions imply that $C^2 + 4Q_1Q_2Q_3 \neq 0$, so that the generic fibre $V_F = W_F$ of V/Y is proper, smooth and geometrically integral over $F = k(Y)$. Since the map $\bar{V} \to \bar{Y}$ is surjective, this implies that $\bar{k}[V]^* = \bar{k}^*$. Now, by [**3**, Lemma 3.6] or [**2**, Proposition 2.A.1], there exists an exact sequence of $\mathfrak{G}$-modules:

$$0 \to N_1 \to N_2 \oplus \mathrm{Pic}\,\bar{Z} \to \mathrm{Pic}\,\bar{V} \to 0,$$

where N_1 and N_2 are both permutation $\mathfrak{G}$-modules; hence $H^1(\mathfrak{G}, \mathrm{Pic}\,\bar{Z}) = 0$ as soon as $H^1(\mathfrak{G}, \mathrm{Pic}\,\bar{V}) = 0$.

Note that the restriction map $\mathrm{Pic}\,\bar{V} \to \mathrm{Pic}\,\bar{U}$ is surjective since V is smooth. The exact sequence (9.5) here reduces to an isomorphism between $\mathrm{Pic}\,\bar{U}$ and the stably permutation module $\mathrm{Pic}\,U_{\bar{F}}$. Letting $P_{\bar{V},\bar{U}} = \mathrm{Ker}(\mathrm{Pic}\,\bar{V} \to \mathrm{Pic}\,\bar{U})$, we then get the exact sequence

$$\mathrm{Pic}\,\bar{V}^{\mathfrak{G}} \to \mathrm{Pic}\,\bar{U}^{\mathfrak{G}} \to H^1(\mathfrak{G}, P_{\bar{V},\bar{U}}) \to H^1(\mathfrak{G}, \mathrm{Pic}\,\bar{V}) \to 0. \tag{9.9}$$

From $\bar{k}[V]^* = \bar{k}^*$ and $V(k) \neq \varnothing$ (X is k-unirational) and from the first terms of the Leray spectral sequence for V/k and the étale sheaf $\mathbb{G}_m$, we deduce that $\mathrm{Pic}\,V \simeq \mathrm{Pic}\,\bar{V}^{\mathfrak{G}}$. Together with the isomorphism $\mathrm{Pic}\,\bar{U} \simeq \mathrm{Pic}\,U_{\bar{F}}$, this implies

$$\mathrm{Coker}(\mathrm{Pic}\,\bar{V}^{\mathfrak{G}} \to \mathrm{Pic}\,\bar{U}^{\mathfrak{G}}) \simeq \mathrm{Coker}(\mathrm{Pic}\,U_F \to \mathrm{Pic}\,U_{\bar{F}}^{\mathfrak{G}}). \tag{9.10}$$

Assume that X is of Type (III). From equation (9.2) and the fact that K/k is Galois it follows that U_F is F-birational to a Severi–Brauer surface over F (Lemma 9.5). From (9.10) one then concludes that

$$\mathrm{Coker}(\mathrm{Pic}\,\bar{V}^{\mathfrak{G}} \to \mathrm{Pic}\,\bar{U}^{\mathfrak{G}}) \simeq \mathbf{Z}/3$$

if U_F is not F-rational (which is then equivalent to $U_F(F) = \varnothing$).

Thus by (9.9) the theorem will follow if we can prove

(9.11) $H^1(\mathfrak{G}, P_{\bar{V},\bar{U}}) = 0$ when X is of Type (II), and $H^1(\mathfrak{G}, P_{\bar{V},\bar{U}}) = \mathbf{Z}/3$ when X is of Type (III) but U_F is not an F-rational variety (if it is, Z is k-rational and the theorem follows trivially).

There is an obvious commutative diagram of $\mathfrak{G}$-modules:

$$\begin{array}{ccccccccc} 1 \longrightarrow & \bar{k}[Y]^* & \longrightarrow & \bar{k}[Y_0]^* & \longrightarrow & \mathrm{Div}_{\bar{Y},\bar{Y}_0}\bar{Y} & \xrightarrow{\deg} & \mathbf{Z} & \longrightarrow 0 \\ & \downarrow\simeq & & \downarrow\simeq & & \downarrow & & & \\ 1 \longrightarrow & \bar{k}[V]^* & \longrightarrow & \bar{k}[U]^* & \longrightarrow & \mathrm{Div}_{\bar{V},\bar{U}}\bar{V} & \longrightarrow & P_{\bar{V},\bar{U}} & \longrightarrow 0 \end{array}$$

In this diagram, $\mathrm{Div}_{\bar{Y},\bar{Y}_0}\bar{Y}$, respectively $\mathrm{Div}_{\bar{V},\bar{U}}\bar{V}$, denotes the group of divisors on $\bar{Y}$, respectively $\bar{V}$, with support in $\bar{Y}\setminus\bar{Y}_0$, respectively $\bar{V}\setminus\bar{U}$ (these are permutation modules). From this diagram and Shapiro's lemma we deduce

(9.12) $$H^1(\mathfrak{G}, P_{\bar{V},\bar{U}}) = \mathrm{Ker}\{H^2(\mathfrak{G}, \mathrm{Div}_{\bar{Y},\bar{Y}_0}\bar{Y}) \to H^2(\mathfrak{G}, \mathrm{Div}_{\bar{V},\bar{U}}\bar{V}) \oplus H^2(\mathfrak{G}, \mathbf{Z})\}.$$

Suppose first that X is of Type (II) and let L be the quadratic extension of k defined by the vanishing of Q and let E be the compositum of K and L in $\bar{k}$. Given a field R with $k \subset R \subset \bar{k}$, let $\mathfrak{G}_R = \mathrm{Gal}(\bar{k}/R)$. The $\mathfrak{G}$-module $\mathrm{Div}_{\bar{Y},\bar{Y}_0}\bar{Y}$ is the $\mathfrak{G}$-module $\mathbf{Z}[\mathfrak{G}/\mathfrak{G}_L]$ induced from the trivial $\mathfrak{G}_L$-module $\mathbf{Z}$, and the $\mathfrak{G}$-module $\mathrm{Div}_{\bar{V},\bar{U}}\bar{V}$ is the induced module $\mathbf{Z}[\mathfrak{G}/\mathfrak{G}_E]$. From (9.12) we deduce

(9.13) $$H^1(\mathfrak{G}, P_{\bar{V},\bar{U}}) = \mathrm{Ker}\{H^2(\mathfrak{G}_L, \mathbf{Z}) \xrightarrow{(\mathrm{Res},\mathrm{Cores})} H^2(\mathfrak{G}_E, \mathbf{Z}) \oplus H^2(\mathfrak{G}, \mathbf{Z})\}.$$

In the diagram

(9.14) $$\begin{array}{ccccc} H^2(k, \mathbf{Z}) & \xrightarrow{\mathrm{Res}} & H^2(L, \mathbf{Z}) & \xrightarrow{\mathrm{Cores}} & H^2(k, \mathbf{Z}) \\ \uparrow & & \uparrow & & \\ H^2(K/k, \mathbf{Z}) & \xrightarrow{\simeq} & H^2(E/L, \mathbf{Z}) & & \end{array}$$

the vertical arrows are injections, the composite top map is multiplication by $2 = [L:k]$, the square commutes, and $H^2(K/k, \mathbf{Z})$ is killed by 3. From this and (9.13) one immediately concludes that $H^1(\mathfrak{G}, P_{\bar{V},\bar{U}}) = 0$.

Suppose next that X is of Type (III) and that C is irreducible over K. Let L be the cubic extension of k defined by the vanishing of C, choose an embedding $L \subset \bar{k}$ and let E be the compositum of K and L in $\bar{k}$. Just as above, $\mathrm{Div}_{\bar{Y},\bar{Y}_0}\bar{Y}$ is the induced $\mathfrak{G}$-module $\mathbf{Z}[\mathfrak{G}/\mathfrak{G}_L]$ and $\mathrm{Div}_{\bar{V},\bar{U}}\bar{V}$ is the induced $\mathfrak{G}$-module $\mathbf{Z}[\mathfrak{G}/\mathfrak{G}_E]$. From (9.13) and the same diagram (9.14) as above, it now follows that

$$H^1(\mathfrak{G}, P_{\bar{V},\bar{U}}) \simeq H^2(E/L, \mathbf{Z}) \simeq \mathbf{Z}/3,$$

which proves (9.11) in the case under consideration.

Suppose now that X is of Type (III) and C is irreducible over k but becomes reducible over K. In this case U_F is actually defined over k, as a simple change of variables in equation (9.1) reveals. But since $V(k)$ is Zariski-dense in V, this implies that $U_F(F) \neq \varnothing$ and that U_F is an F-rational variety (and Z a k-rational variety).

Finally, assume that X is of Type (III) and that C is reducible over k, but does not split completely over K. Thus the separable form C is the product of a linear form and a quadratic form Q. Let L be the quadratic extension of k defined by the vanishing of Q. By assumption, this extension is linearly independent from K. Let E be the compositum of K and L in $\bar{k}$. Now

$$\mathrm{Div}_{\bar{Y},\bar{Y}_0}\bar{Y} = \mathbf{Z} \oplus \mathbf{Z}[\mathfrak{G}/\mathfrak{G}_L], \quad \mathrm{Div}_{\bar{V},\bar{U}}\bar{V} = \mathbf{Z}[\mathfrak{G}/\mathfrak{G}_K] \oplus \mathbf{Z}[\mathfrak{G}/\mathfrak{G}_E].$$

Applying (9.12), one finds that

$$H^1(\mathfrak{G}, P_{\bar{V},\bar{U}}) = \operatorname{Ker}\{H^2(K/k, \mathbf{Z}) \oplus H^2(E/L, \mathbf{Z}) \to H^2(k, \mathbf{Z})\},$$

where the arrow is induced by the sum of corestrictions

$$H^2(k, \mathbf{Z}) \oplus H^2(L, \mathbf{Z}) \to H^2(k, \mathbf{Z}).$$

Basically the same argument as that given after (9.14) reveals that the above kernel may be identified with the kernel of the map

$$H^2(K/k, \mathbf{Z}) \oplus H^2(K/k, \mathbf{Z}) \to H^2(K/k, \mathbf{Z}),$$

$$\mathbf{Z}/3 \oplus \mathbf{Z}/3 \to \mathbf{Z}/3,$$

$$(x, y) \to (x + 2y),$$

and hence is isomorphic to $\mathbf{Z}/3$, which completes the proof.

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