UNRAMIFIED WITT GROUPS OF REAL ANISOTROPIC QUADRICS

J.-L. Colliot-Thélène and R. Sujatha

Abstract. *The unramified Witt group of the real anisotropic conic and the real anisotropic quadric surface are known to be finite groups, cyclic of order 4. We prove that the unramified Witt group of the real anisotropic quadric of dimension d, $3 \leq d \leq 6$ is finite, cyclic of order 8, with the form $< 1 >$ as the generator.*

Let k be a field of characteristic $\neq 2$ and K/k a finitely generated field extension of k. The *unramified Witt group* $W_{nr}(K/k)$ is a subgroup of $W(K)$, the Witt group of quadratic forms over K, and has interesting applications (see [CT-O], [O]). For any poistive integer i, one may also define the unramified subgroup $H^i_{nr}(\mathbb{R}(Q_3), \mathbb{Z}/2)$ of the étale cohomology group $H^i(\mathbb{R}(Q_3), \mathbb{Z}/2)$.

If X/k is a geometrically integral variety over the field k, and $k(X)$ denotes the function field of X, by abuse of terminology, we refer to the group $W_{nr}(k(X)/k)$, as the unramified Witt group of X. In this note, we prove that the unramified Witt groups of the real anisotropic quadrics Q_d of dimension d, $3 \leq d \leq 6$, are finite, cyclic of order 8 with the form $< 1 >$ as the generator. In lower dimensions, a similar result was known for the real anisotropic conic (Theorem 6.2 of [P] with a slight modification for the projective conic) and the anisotropic quadric surface, the groups then being cyclic of order 4. The finiteness of the unramified Witt group for anisotropic quadrics of dimension ≥ 7 is an open question.

Here is the plan of the paper. Section 1 recalls a few basic facts about quadratic forms, the unramified Witt group and unramified cohomology. Section 2 starts with various étale cohomology computations of quadrics (here we use the results of Artin-Verdier-Cox [C] on the étale cohomological dimension of real varieties without a real point). These computations together with Karpenko's computations of the Chow groups of 3-dimensional quadrics [K] enable us to show that the map

$$CH^2(Q_3)/2 \to H^4(Q_3, \mathbb{Z}/2),$$

which is induced by the cycle map into étale cohomology, is an injection. From the properties of the Bloch-Ogus spectral sequence it then follows that the group $H^3_{nr}(\mathbb{R}(Q_3), \mathbb{Z}/2)$ vanishes. Standard properties of unramified cohomology then imply $H^3_{nr}(\mathbb{R}(Q_d), \mathbb{Z}/2) = 0$ for $3 \leq d \leq 6$, and use of a well-known result of Arason [A] together with a result of Merkurjev-Suslin [M-S1] and Rost [R] on Milnor's

1991 *Mathematics Subject Classification.* Primary Classification: 19G12, 19E15, 14F20; Secondary Classification: 14P99, 14C25.

Key words and phrases. Pfister forms, level, unramified cohomology, étale cohomology.

K_3-group lead to $H^3_{nr}(\mathbb{R}(Q_d), \mathbb{Z}/2) = \mathbb{Z}/2$ for $d \geq 7$. Since higher unramified cohomology is rather hard to compute, these results seem to be of independent interest. They also enable us to show that the cycle class map

$$CH^2(Q_d)/2 \to H^4(Q_d, \mathbb{Z}/2)$$

is an injection for all d.

Section 3 applies the vanishing of $H^3_{nr}(\mathbb{R}(Q_3), \mathbb{Z}/2)$ to the computation of the unramified Witt group $W_{nr}(\mathbb{R}(Q_3))$ (which by a standard argument coincides with $W_{nr}(\mathbb{R}(Q_d))$ for any d, $3 \leq d \leq 6$). We use the map $W(\mathbb{R}(Q_3)) \to W(\mathbb{R}(Q_3 \times Q_2))$, whose kernel is known (Arason, Pfister [A]) and investigate the induced map on unramified Witt groups. The image is easily determined because $W_{nr}(\mathbb{R}(Q_3 \times Q_2))$ is isomorphic to $W_{nr}(\mathbb{R}(Q_2))$. Recall that $I^n(F)$ denotes the n-th power of the fundamental ideal of the Witt group of a field F. To control the kernel of the above map, we use the *injection* of $I^3(\mathbb{R}(Q_3))$ into $H^3(\mathbb{R}(Q_3), \mathbb{Z}/2)$. This injection follows from the vanishing of $I^4(\mathbb{R}(Q_3))$ [AEJ] together with the known injectivity of the Milnor map $I^3F/I^4F \to H^3(F, \mathbb{Z}/2)$ for any field F (*char* $F \neq 2$) (Merkurjev-Suslin [M-S1], [R]). The final argument is provided by the vanishing of $H^3_{nr}(\mathbb{R}(Q_3), \mathbb{Z}/2)$, established in section 2.

§1. Preliminaries

All fields considered in this note are of characteristic different from 2 unless otherwise mentioned. Given a field k and a finitely generated field extension K/k, we denote by $\mathcal{V}_K$ the set of all rank one discrete valuations on K, which are trivial on k. Given an element $\nu \in \mathcal{V}_K$, there exists a "second residue homomorphism" $\partial_{\pi_\nu} : W(K) \to W(k(\nu))$ (cf. [Sc, Ch. VI, 2.5]), where π_ν is a parameter for the valuation ν and $k(\nu)$ is the residue field at ν. The homomorphism ∂_{π_ν} depends on the choice of a parameter but the kernel of ∂_{π_ν} depends only on ν and hence may be denoted *ker* ∂_ν. The unramified Witt group of K/k, denoted $W_{nr}(K/k)$, or just $W_{nr}(K)$ when there is no possible confusion of the ground field, is defined as

$$W_{nr}(K/k) = \bigcap_{\nu \in \mathcal{V}_K} ker\ \partial_\nu.$$

Given a k-extension L of K, the natural map $W(K) \to W(L)$ induces a map $W_{nr}(K) \to W_{nr}(L)$ on the corresponding unramified Witt groups [O, Proposition 7.1].

Let X/k be a smooth geometrically integral variety and $k(X)$ denote its function field. We denote the set of codimension one points of X by $X^{(1)}$. Since X is smooth, the elements of $X^{(1)}$ give elements in $\mathcal{V}_{k(X)}$. If $x \in X^{(1)}$ and π_x is a fixed uniformising parameter for the valuation corresponding to x, then we denote the corresponding second residue homomorphism by ∂_x. Given an element $q \in W(k(X))$, it is easily seen that $\partial_x(q) \neq 0$ for finitely many $x \in X^{(1)}$. Clearly there is a complex

$$0 \to W_{nr}(k(X)) \to W(k(X)) \xrightarrow{\oplus_{x \in X^{(1)}} \partial_x} \bigoplus_{x \in X^{(1)}} W(k(x)) \tag{1}$$

where $k(x)$ denotes the residue field at a point $x \in X^{(1)}$.

Recall that a quadratic form ϕ over k is said to be an *n-fold Pfister form* if it is isometric to a form of the type

$$< 1, a_1 > \otimes < 1, a_2 > \otimes \cdots \otimes < 1, a_n >, \ a_i \in k^*.$$

We denote such a form by $\ll a_1, a_2, \cdots a_n \gg$. For a Pfister form, the properties of being isotropic and hyperbolic coincide. The following proposition is well-known ([A] Satz 1.3).

Proposition 1.1. *Let k be a field and ϕ an n-fold Pfister form over k, $n \geq 2$. Let X_ϕ be the projective quadric defined by ϕ and let $k(X_\phi)$ denote the function field of X_ϕ. Then the kernel of the natural map*

$$W(k) \to W(k(X_\phi))$$

is the ideal generated by the class of ϕ. $\square$

Recall that a quadratic form ψ defined over k is said to be a *Pfister neighbour* of an n-fold Pfister form ϕ over k if dimension $\psi \geq 2^{n-1} + 1$ and ψ is isometric to a subform of ϕ. A Pfister neighbour ψ of ϕ has the following property: ψ is isotropic over a field extension $K/k \iff \phi$ is isotropic over K. Further, it is easily seen that the function field $k(X_\rho)$ of the projective quadric defined by a quadratic form ρ over k of rank at least three is a purely transcendental extension of k if and only if the form ρ is isotropic. These properties are used to prove the following

Lemma 1.2. *Let ϕ be an n-fold Pfister form over k and ψ a Pfister neighbour of ϕ. Let X_ϕ (respectively X_ψ) denote the corresponding projective quadric and $k(X_\phi)$ (respectively $k(X_\psi)$) be the function field. Then $W_{nr}(k(X_\phi)) \cong W_{nr}(k(X_\psi))$.*

Proof. Consider the product variety $X_\phi \times_k X_\psi$ and the natural projections given by $\pi_\phi : X_\phi \times_k X_\psi \to X_\phi$ and $\pi_\psi : X_\phi \times_k X_\psi \to X_\psi$. Clearly $k(X_\phi \times_k X_\psi)$, the function field of $X_\phi \times_k X_\psi$, coincides with $k(X_\phi)(X_\psi)$ and $k(X_\psi)(X_\phi)$. Since X_ψ (respectively X_ϕ) has a $k(X_\psi)$-rational point, (respectively $k(X_\phi)$-rational point), namely the generic point, the above remarks imply that X_ϕ (respectively X_ψ) also has a $k(X_\psi)$-rational point (respectively $k(X_\phi)$-rational point). Thus the field $k(X_\phi)(X_\psi)$ is a purely transcendental extension of both $k(X_\phi)$ and $k(X_\psi)$. Since the unramified Witt groups are invariant under purely transcendental extensions (cf. [O, Proposition 7.2]), we have $W_{nr}(k(X_\phi)) \cong W_{nr}(k(X_\phi \times_k X_\psi)) \cong W_{nr}(k(X_\psi))$ and the lemma is proved. $\square$

We now recall some results about cohomology groups. Given a field k and a finitely generated field extension K/k, the "unramified cohomology groups" with coefficients in $\mathbb{Z}/2$, denoted $H^i_{nr}(K/k)$ or just $H^i_{nr}(K)$, $i \geq 1$, if there is no ambiguity of the ground field, are subgroups of the Galois cohomology groups $H^i(G(K_s/K), \mathbb{Z}/2)$ where K_s denotes a separable closure of K (cf. [CT-O]). We denote the Galois cohomology groups $H^i(G(K_s/K), \mathbb{Z}/2)$ by $H^i(K)$. The groups $H^i_{nr}(K)$ are defined as

$$H^i_{nr}(K) = \bigcap_{\nu \in \mathcal{V}_K} ker\ \partial_\nu,$$

where $\mathcal{V}_K$ is the set of rank one valuations on K that are trivial on k, $\nu \in \mathcal{V}_K$ and ∂_ν is the residue homomorphism [CT-O, §1],

$$\partial_\nu : H^i(K) \to H^{i-1}(k(\nu)),$$

with $k(\nu)$ denoting the residue field of the valuation.

The unramified cohomology groups are invariant under purely transcendental extensions [CT-O, Proposition 1.2]. Thus, in the spirit of Lemma 1.2, we have

Lemma 1.3. *Let ϕ be an n-fold Pfister form over k and ψ a Pfister neighbour of ϕ. Let X_ϕ (respectively X_ψ) be the corresponding projective quadric and $k(X_\phi)$ (respectively $k(X_\psi)$) be the function field. Then $H^i_{nr}(k(X_\phi)) \cong H^i_{nr}(k(X_\psi))$.* $\square$

Given an integral variety X/k of dimension d, the sheaves $\mathcal{H}^n$, for $n \geq 0$ are the Zariski sheaves associated to the presheaf $U \longmapsto H^n(U)$, where $H^n(U) = H^n_{ét}(U, \mu_2)$, μ_2 denoting the étale sheaf of square roots of unity. By the results of Bloch-Ogus [B-O], if X is smooth and integral, a flasque resolution of the sheaf $\mathcal{H}^n$ is given by

(2)

$$0 \to \mathcal{H}^n \to i_{\eta *}H^n(k(X)) \to \bigoplus_{x \in X^{(1)}} i_{x*}H^{n-1}(k(x)) \to \cdots \bigoplus_{x\in X^{(d)}} i_{x*}H^{n-d}(k(x)) \to$$

where $k(X)$ is the function field of X, $X^{(i)}$ is the set of points of codimension i in X, η is the generic point and $k(x)$ is the residue field at the point x. Taking the complex of global sections of the above resolution, we see that the homology groups of this complex correspond to the Zariski cohomology groups $H^i(X, \mathcal{H}^n)$ of the sheaf $\mathcal{H}^n$. In fact, if X is also proper, we have $H^n_{nr}(k(X)) = H^0(X, \mathcal{H}^n)$ [CT-O, Remarque 1.1.3]. Thus the groups $H^0(X, \mathcal{H}^i)$ are birational invariants of X (cf. [CT-P, 1.3]).

The "local-to-global" spectral sequence (cf. [B-O, §6 and Remark 6.4])

$$E_2^{p,q} = H^p(X, \mathcal{H}^q) \Longrightarrow H^n(X) = E^n$$

relates the Zariski cohomology groups of the sheaves $\mathcal{H}^n$ to the étale cohomology groups $H^n(X) = H^n_{ét}(X, \mu_2)$.

Finally, we observe that the unramified cohomology groups of a smooth, projective, integral variety X can be related to the unramified Witt group of X whenever the homomorphisms $e_n : I^n(k(X)) \to H^n(k(X))$, (cf. [AEJ]) are well-defined. This is done by considering the induced filtration $(I^m_{nr}(k(X)))_{m\geq 0}$ on $W_{nr}(k(X))$, where, by definition

$$I^m_{nr}(k(X)) = I^m(k(X)) \cap W_{nr}(k(X)),$$

$I(k(X))$ being the fundamental ideal of even dimensional forms and $I^m(k(X))$ its *m-th* power. If the homomorphisms e_n and e_{n-1} are well-defined, we get a commutative diagram [CT-P, Proposition 1.5.1]

$$\begin{array}{ccccccc} 0 \to & I^n_{nr}(k(X)) & \to & I^n(k(X)) & \to & \oplus_{x\in X^{(1)}} I^{n-1}(k(X)) \\ & \downarrow & & \downarrow e_n & & \downarrow e_{n-1} \\ 0 \to & H^0(X, \mathcal{H}^n) & \to & H^n(k(X)) & \to & \oplus_{x\in X^{(1)}} H^{n-1}(k(X)) \end{array} \qquad (3)$$

with the top row a complex, the bottom row exact and notation as before. The maps $e_n : I^nF \to H^n(F)$ are known to be well-defined for all fields F, whenever $n \leq 3$, with kernel precisely equal to $I^{n+1}(F)$. Thus there exist isomorphisms $\overline{e}_n : I^n(F)/I^{n+1}(F) \cong H^n(F)$ for $0 \leq n \leq 3$ (cf. [A], [Me], [M-S1], [R]).

§2. Étale cohomology of quadrics and the cycle map

In this section we compute the étale cohomology groups of certain quadrics. These computations are then used to compute the unramified cohomology group H^3_{nr} of all the real anisotropic quadrics (Proposition 2.7). We also prove the injectivity of the cycle map

$$CH^2(X)/2 \to H^4_{\acute{e}t}(X, \mathbb{Z}/2)$$

for all real anisotropic quadrics (Corollary 2.8). Let k be a field of characterisitic zero. We first record a few results from étale cohomology, given a variety X of dimension d. Cohomology will always be cohomology with coefficients in $\mathbb{Z}/2$. When using arbitrary finite coefficients, the isomorphisms and exact sequences that follow would require Tate twists for the coefficients. However, since $\mathbb{Z}/2$ is *equal* to μ_2, all sheaves $\mu_2^{\otimes i}$, $i \in \mathbb{Z}$ coincide with $\mathbb{Z}/2$.

2.a: Let $Y \subseteq X$ be a closed immersion and $U = X \setminus Y$. Then there exists a long exact localisation sequence [Mi, p 92]

$$.... \to H^n_Y(X) \to H^n(X) \to H^n(U) \to H^{n+1}_Y(X) \to .... \tag{4}$$

where $H^n_Y(X)$ denotes the étale cohomology groups of X with support in Y and coefficients in the sheaf μ_2.

2.b: More generally, let $Z \subseteq Y \subseteq X$ be a sequence of closed immersions. Then there is a long exact sequence

$$.... \to H^i_Z(X) \to H^i_Y(X) \to H^i_{Y\setminus Z}(X \setminus Z) \to H^{i+1}_Z(X) \to .... \tag{5}$$

with notation as in **2.a**.

We refer to [Mi, Ch. VI, §6] for proofs of the purity results cited below, remarking that though it is assumed there that the ground field is algebraically closed, the proofs remain valid over an arbitrary ground field of characteristic zero (see [Ar, SGA 4, XVI 3.8, 3.9, 3.10 and SGA 4, XIX 3.4]).

2.c: (Purity) Let $Y \subseteq X$ be a closed immersion, with both X and Y smooth over k, Y connected and everywhere of codimension c in X. Then there exists an isomorphism

$$H^n_Y(X) \cong H^{n-2c}(Y)$$

with notation as before. We give a brief description of this isomorphism. Let $\mathcal{H}^n_Y(\mathbb{Z}/2)$ denote the Zariski sheaf on Y defined as in [Mi, p.241]. By [SGA 4, XVI, 3.8], there exists an isomorphism of $\mathcal{H}^{2c}_Y(\mathbb{Z}/2)$ with the locally constant sheaf $(\mathbb{Z}/2)_Y$ on Y, i.e. $\mathcal{H}^{2c}_Y(\mathbb{Z}/2) \cong (\mathbb{Z}/2)_Y$. Further, by [SGA 4, XVI, 3.9], we have $\mathcal{H}^i_Y(\mathbb{Z}/2) = 0$ if $i \neq 2c$. Using this information in the spectral sequence [Mi, p.247]

$$H^p_{\acute{e}t}(Y, \mathcal{H}^q_Y(\mathbb{Z}/2)) \Longrightarrow H^n_Y(X)$$

we get an isomorphism

$$H^i(Y) \cong H^{i+2c}_Y(X). \tag{6}$$

This isomorphism gives rise to the *Gysin map*

$$H^i(Y) \cong H_Y^{i+2c}(X) \to H^{i+2c}(X)$$

where the latter map is obtained from the long exact localisation sequence (**2.a**).

2.d: Let X, Y as in 2.c. When $n = 2c$, by the isomorphism described above, we get

$$\mathbb{Z}/2 \cong H^0(Y) \cong H_Y^{2c}(X).$$

The image of $1 \in H_Y^{2c}(X)$ under this isomorphism is called the *fundamental class* $s_{Y/X}$ of Y in X.

2.e: Finally, we briefly discuss the cycle map and refer to [Mi, Ch. VI] for details (see also [D, Cycle]). Given a smooth integral variety X/k, let $\mathcal{Z}^n(X)$ denote the free abelian group generated by closed integral subschemes of codimension n in X. The *cycle map* in codimension n is a homomorphism

$$\rho_n : \mathcal{Z}^n(X) \to H^{2n}(X)$$

into the étale cohomology group and is briefly described below. Given a smooth integral closed subvariety Z of codimension n, the element $\rho_n(Z)$ is defined to be the image in $H^{2n}(X)$ of the fundamental class $s_{Z/X}$ in $H_Z^{2n}(X)$ (cf. **2.d**) under the natural map $H_Z^{2n}(X) \to H^{2n}(X)$ (cf. **2.a**). Equivalently, $\rho_n(Z)$ is the image of the non-trivial element of $H^0(Z)$ under the Gysin map $H^0(Z) \to H^{2n}(X)$. Using purity [D], the map so defined can be extended to singular cycles and hence to the whole group $\mathcal{Z}^n(X)$ by linearity.

On the other hand, we have the local-to-global spectral sequence (cf. §1) constructed by Bloch-Ogus [B-O, §3],

$$H^p_{Zar}(X, \mathcal{H}^q) \Longrightarrow H^n_{ét}(X)$$

using the formalism of duality theory and trace map based on [D1, SGA 4, XVIII]. Further, by the results of Bloch-Ogus, we see that $H^p(X, \mathcal{H}^q) = 0$ if $p > q$ [B-O, 6.1 and 6.2] and $H^p(X, \mathcal{H}^p) = CH^p(X)/2$ [B-O, 6.3] (see also [B-O, 7.7], whose proof extends to smooth varieties over arbitrary perfect fields), where $CH^p(X)$ denotes the Chow group of algebraic cycles of codimension p modulo rational equivalence. Therefore the above spectral sequence gives a natural map

$$CH^p(X)/2 \to H^{2p}(X).$$

But this map coincides with the "classical" cycle map ρ_p described above, as is seen using 7.2, 3.7 and 3.9 of [B-O] and working through the definition of ρ_p. In particular, the map ρ_p factors through rational equivalence and we get a homomorphism

$$\rho_p : CH^p(X)/2 \to H^{2p}(X).$$

2.f: If $Y \overset{i}{\hookrightarrow} X$ is a closed embedding of smooth k-varieties, Y connected and everywhere of codimension c, then we have the following commutative diagram:

$$\begin{array}{ccc}
CH^n(Y) & \xrightarrow{i_*} & CH^{n+c}(X) \\
\big\downarrow{\scriptstyle \rho_n} & & \big\downarrow{\scriptstyle \rho_{n+c}} \\
H^{2n}(Y) & \longrightarrow & H^{2n+2c}(X).
\end{array}$$

Here the vertical maps are the cycle maps, the top horizontal map is the natural map induced by the inclusion and the lower horizontal map is the Gysin map

$$H^{2n}(Y) \cong H_Y^{2n+2c}(X) \to H^{2n+2c}(X).$$

That the diagram commutes follows from [Mi, Ch. VI, 9.3] observing that the result remains valid over a ground field of characteristic zero.

For any smooth variety X/k, there is an exact sequence of sheaves in the étale topology as follows,

$$0 \to \mu_2 \to \mathbb{G}_m \xrightarrow{2} \mathbb{G}_m \to 0$$

where $\mathbb{G}_m$ is the sheaf associated to $\mathbb{G}_m(U) = \Gamma(U, \mathcal{O}_U)^*$, the multiplicative group of units of the sections of U, and μ_2 is the étale sheaf of square roots of unity, isomorphic to $\mathbb{Z}/2$. From Kummer theory and Grothendieck's Hilbert theorem 90 [Mi, Ch. III, §4], we get exact sequences

$$0 \to \Gamma(X, \mathcal{O}_X)^*/\Gamma(X, \mathcal{O}_X)^{*2} \to H^1_{\acute{e}t}(X, \mu_2) \to {}_2Pic(X) \to 0 \tag{7}$$

and

$$0 \to Pic(X)/2 \to H^2_{\acute{e}t}(X, \mu_2) \to {}_2Br(X) \to 0 \tag{8}$$

where $\Gamma(X, \mathcal{O}_X)^*$ is the group of units of the global sections over X and ${}_2Pic(X)$ (respectively ${}_2Br(X)$) is the 2-torsion subgroup of the Picard group (respectively étale cohomological Brauer group) of X. We abbreviate the étale cohomology groups $H^n_{\acute{e}t}(U, \mu_2)$ to $H^n(U)$ for a variety U.

For the rest of the discussion, we fix the ground field k to be the field $\mathbb{R}$ of real numbers. The d-dimensional quadric Q_d is defined by $\sum_{i=0}^{d+1} X_i^2 = 0$ in $\mathbb{P}^{d+1}_{\mathbb{R}}$. We denote the corresponding smooth projective complex d-dimensional quadric $Q_d \times_{\mathbb{R}} \mathbb{C}$ by C_d. The following well-known lemma computes the étale cohomology groups of the complex quadrics C_d. By the comparison theorem [Mi, Ch. III, Theorem 3.12], these groups coincide with the usual singular cohomology groups of the underlying complex manifold with coefficients in $\mathbb{Z}/2$.

Lemma 2.1. *Let C_d $(d \geq 2)$ be the d-dimensional complex quadric $Q_d \times_{\mathbb{R}} \mathbb{C}$. Then there exist isomorphisms*

$$H^i(C_d) \cong H^{i-2}(C_{d-2}) \text{ for } i \geq 2,\ d > 2,\ i \neq 2d,\ 2d-1.$$

Further $H^0(C_d) \cong \mathbb{Z}/2,\ H^1(C_d) = 0,\ H^{2d-1}(C_d) = 0$ for all $d \geq 1$ and $H^2(C_2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$.

Proof. Clearly $H^0(C_d) \cong \mathbb{Z}/2$ for all d and hence $H^{2d}(C_d) \cong \mathbb{Z}/2$, by Poincaré duality. The exact sequence (7) gives $H^1(C_d) = 0$, since C_d is a smooth, projective, rational variety and the torsion in the Picard group is well-known [CT-S, Appendix 2.A] to be a birational invariant of smooth projective varieties. By Poincaré duality, we deduce that $H^{2d-1}(C_d) = 0$. The quadric C_2 is isomorphic to $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ and hence by the Künneth formula, $H^2(C_2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$. We now prove the first part of the lemma. By a change of coordinates, we may assume that C_d is defined by the equation $X_0X_1 + \sum_{i=2}^{d+1} X_i^2 = 0$ in $\mathbb{P}^{d+1}_{\mathbb{C}}$. Let $Y = Y_d$ be the hyperplane section of C_d

defined by the hyperplane $X_1 = 0$. The complement $C_d \setminus Y_d$ is then isomorphic to the d-dimensional affine space $\mathbb{A}^d_{\mathbb{C}}$. The point $P = (1 : 0 : 0 \cdots : 0)$ is a singular point on Y_d and $Y_d \setminus P$ is an $\mathbb{A}^1$-bundle over the quadric C_{d-2}. Consider the inclusions $Y_d \subseteq C_d \hookleftarrow \mathbb{A}^d_{\mathbb{C}}$. We have a long exact sequence (cf. **2.a**)

$$\cdots \to H^n_{Y_d}(C_d) \to H^n(C_d) \to H^n(\mathbb{A}^d_{\mathbb{C}}) \to H^{n+1}_{Y_d}(C_d) \to H^{n+1}(C_d) \to \cdots . \tag{9}$$

Since the étale cohomology groups $H^n(\mathbb{A}^d_{\mathbb{C}})$ are trivial for $n > 0$ and $H^0(C_d)$ surjects onto $H^0(\mathbb{A}_d)$, we have $H^0_{Y_d}(C_d) = 0$ and isomorphisms

$$H^n_{Y_d}(C_d) \cong H^n(C_d) \ \textit{for all } n \geq 1. \tag{10}$$

We now consider the chain of inclusions $P \subseteq Y_d \subseteq C_d$ and the corresponding long exact sequence in étale cohomology (cf. **2.b**). We then have a long exact sequence as follows, (see (5))

$$\cdots \to H^n_P(C_d) \to H^n_{Y_d}(C_d) \to H^n_{Y_d\setminus P}(C_d \setminus P) \to H^{n+1}_P(C_d) \to \cdots . \tag{11}$$

Using "Purity" (cf. **2.c**), we have isomorphisms

$$H^n_P(C_d) \cong H^{n-2d}(P), \tag{12}$$

since P is a smooth closed point on C_d. Combining these isomorphisms, we get

$$H^n_{Y_d}(C_d) \cong H^n_{Y_d\setminus P}(C_d \setminus P) \ \textit{for all } n \neq 2d,\ 2d-1. \tag{13}$$

Further, since $Y_d \setminus P \subseteq C_d \setminus P$ is smooth of codimension 1, we have

$$H^n_{Y_d\setminus P}(C_d \setminus P) \cong H^{n-2}(Y_d \setminus P). \tag{14}$$

But $Y_d \setminus P$ is an $\mathbb{A}^1$-bundle over C_{d-2} and since the étale cohomology with finite coefficients of an affine bundle over a smooth variety X is isomorphic to the étale cohomology of X, we see that

$$H^{n-2}(Y_d \setminus P) \cong H^{n-2}(C_{d-2}) \ \textit{for all } n \geq 2. \tag{15}$$

The isomorphisms (14), (13) and (10) now complete the proof of the lemma. □

As a corollary, we have the following

Proposition 2.2. *Let C_d be the smooth projective complex quadric of dimension d, $d \geq 1$. Then $H^{2i+1}(C_d) = 0$ for $i \geq 0$ and $H^{2i}(C_d) \cong \mathbb{Z}/2$ for $0 \leq i \leq d$, $i \neq d/2$ if d even. If d is even, then we have $H^d(C_d) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$.* □

Recall that there exists an exact sequence of étale sheaves over a real variety X

$$0 \to \mathbb{Z}/2 \to \pi_*\mathbb{Z}/2 \to \mathbb{Z}/2 \to 0$$

where $\pi : X_{\mathbb{C}} \to X$ is the étale map of degree 2 obtained by base change to $\mathbb{C}$. This induces a long exact sequence

$$\cdots \to H^n(X) \to H^n(X_{\mathbb{C}}) \to H^n(X) \xrightarrow{\cup(-1)} H^{n+1}(X) \to H^{n+1}(X_{\mathbb{C}}) \to \cdots \tag{16}$$

which is a generalisation of the corresponding sequence for the Galois extension $\mathbb{C}/\mathbb{R}$ [A, Corollary 4.6]. The connecting homomorphism is given by the cup-product with the class of (-1) in $H^1(X)$ under the image of the natural map $H^1(\mathit{Spec}\ \mathbb{R}) \to H^1(X)$, noting that $H^1(\mathit{Spec}\ \mathbb{R}) \cong H^1(\mathbb{R}) \cong \mathbb{R}^*/\mathbb{R}^{*2}$. We use this sequence for the quadrics Q_d and the preceding lemma to compute the étale cohomology groups of Q_3 with μ_2 coefficients. We have

Lemma 2.3. *Let Q_3 be the 3-dimensional real anisotropic quadric. Then the étale cohomology groups of Q_3 with $\mathbb{Z}/2$ coefficients are given by*

$$H^i(Q_3) \cong \mathbb{Z}/2 \ for\ i = 0,1,5,6,$$
$$H^i(Q_3) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \ for\ i = 2,3,4,\ and$$
$$H^i(Q_3) \cong 0 \ for\ i \geq 7.$$

Proof. By a result of Artin-Verdier-Cox [C, Theorem 2.1], the étale 2-cohomological dimension of a real variety of dimension d with no $\mathbb{R}$-rational points is $\leq 2d$. Hence $H^i(Q_3) = 0$ for $i \geq 7$. Clearly $H^0(Q_3) \cong \mathbb{Z}/2$. By the Kummer exact sequence (7), we see that $H^1(Q_3) \cong \mathbb{R}^*/\mathbb{R}^{*2} \cong \mathbb{Z}/2$, since ${}_2Pic(Q_3) \subseteq {}_2Pic(C_3) = 0$, and C_3 is rational. Further, by the exact sequence (8), we have $H^2(Q_3) \cong Pic(Q_3)/2 \oplus {}_2Br(Q_3)$. It is well-known that the Picard group $Pic(C_3)$ is isomorphic to $\mathbb{Z}$, the generator being the class of a hyperplane section. For any proper geometrically integral variety $X/\mathbb{R}$, we have an exact sequence (cf. [CT-S, 1.5.0, p.383]

$$(17) \qquad 0 \to Pic(X) \to (Pic(X_{\mathbb{C}}))^G \to Br(\mathbb{R}) \to \mathcal{K} \to H^1(G, Pic(X_{\mathbb{C}}))$$

where $G = Gal(\mathbb{C}/\mathbb{R})$, $X_{\mathbb{C}} = X \times_{\mathbb{R}} \mathbb{C}$, $\mathcal{K}$ denotes the kernel of the natural map $Br(X) \to Br(X_{\mathbb{C}})$ and $(Pic(X_{\mathbb{C}}))^G$ is the subgroup of elements of $Pic(X_{\mathbb{C}})$ fixed under the action of G. Since the generator of $Pic(C_3)$ is defined over $\mathbb{R}$, the inclusion in the above exact sequence gives an isomorphism $Pic(Q_3) \cong Pic(C_3) \cong \mathbb{Z}$. Further, we have $Br(C_3) = 0$, the variety C_3 being rational. Clearly, $H^1(G, Pic(C_3)) = 0$ and hence (17) gives an isomorphism $Br(\mathbb{R}) \cong Br(Q_3) \cong \mathbb{Z}/2$. We remark that this computation is valid for $Q_d,\ d \geq 3$ and hence $Br(Q_d) \cong \mathbb{Z}/2$ for all $d \geq 3$. The exact sequence (16) together with Proposition 2.2 gives an exact sequence

$$(18) \qquad 0 \to H^1(Q_3) \to H^2(Q_3) \to H^2(C_3) \to H^2(Q_3) \to H^3(Q_3) \to 0.$$

By the above computations, we have $H^1(Q_3) \cong \mathbb{Z}/2$ and $H^2(Q_3) \cong (\mathbb{Z}/2)^2$. The previous proposition and an easy computation of dimensions using the exact sequence (18) yields isomorphisms

$$(19) \qquad \mathbb{Z}/2 \oplus \mathbb{Z}/2 \cong H^2(Q_3) \cong H^3(Q_3).$$

We now use a result of Cox [C, Proposition 2.2], which states that if X is a smooth proper real variety of dimension d such that $X(\mathbb{R}) = \emptyset$, then the natural map $H^{2d}(X) \to H^{2d}(X_{\mathbb{C}})$ is zero. The cohomology sequence (16) along with Proposition 2.2 gives exact sequences

$$(20) \qquad 0 \to (\mathbb{Z}/2)^2 \to H^4(Q_3) \to \mathbb{Z}/2 \to H^4(Q_3) \to H^5(Q_3) \to 0$$

and

$$(21) \qquad 0 \to H^5(Q_3) \to H^6(Q_3) \to H^6(C_3) \to H^6(Q_3) \to H^7(Q_3).$$

Since $H^7(Q_3) = 0$, the result of Cox quoted above gives isomorphisms $H^6(Q_3) \cong H^6(C_3) \cong \mathbb{Z}/2$ (Lemma 2.1) and $H^5(Q_3) \cong H^6(Q_3) \cong \mathbb{Z}/2$. This is used in (20) to see that $H^3(Q_3) \cong H^4(Q_3) \cong (\mathbb{Z}/2)^2$. Hence the lemma is proved. $\square$

We need the following result which computes the cohomology group $H^2(Q_2)$ and $Pic(Q_2)$. The results about the Picard group are well-known (cf. [K]), but we include them for the sake of completeness.

Proposition 2.4. *Let Q_2 be the 2-dimensional real anisotropic quadric. Then the Kummer sequence induces an isomorphism*

$$Pic(Q_2)/2 \cong H^2(Q_2)$$

and both groups are isomorphic to $(\mathbb{Z}/2)^2$. The generators of $Pic(Q_2)$ are given by the class of the hyperplane section h and the class ζ of the "double line cycle" defined by $X_0^2 + X_1^2 = 0,\ X_0X_3 + X_1X_2 = 0,\ X_2^2 + X_3^2 = 0$.

Proof. Let $C_2 \subset \mathbb{P}^3_{\mathbb{C}}$ be the complex quadric $Q_2 \times_{\mathbb{R}} \mathbb{C}$. After a $\mathbb{C}$-linear change of coordinates in $\mathbb{P}^3_{\mathbb{C}}$, we see that the image of the Segre embedding $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \hookrightarrow \mathbb{P}^3_{\mathbb{C}}$ maps isomorphically onto the quadric C_2. Consider the two one-parameter families of lines given by $(\mathbb{P}^1_{\mathbb{C}} \times pt)$ and $(pt \times \mathbb{P}^1_{\mathbb{C}})$. Clearly two lines intersect if and only if they belong to different families. Further, the group $Pic(C_2)$ is generated by the classes of any two intersecting lines. Let e_1 (respectively e_2) be the line defined by $X_0 + iX_1 = 0,\ X_2 - iX_3 = 0$ (resp. $X_0 + iX_1 = 0,\ X_2 + iX_3 = 0$). Then $Pic(C_2) \cong \mathbb{Z}[e_1] \oplus \mathbb{Z}[e_2]$ and under the action of the Galois group $G = Gal(\mathbb{C}/\mathbb{R}) = (1, \sigma)$, the line $\sigma(e_1)$ (resp. $\sigma(e_2)$) is defined by $X_0 - iX_1 = 0,\ X_2 + iX_3 = 0$ (resp. $X_0 - iX_1 = 0,\ X_2 - iX_3 = 0$) and hence $\sigma[e_i] = [e_i],\ i = 1, 2$, where $[e_i]$ denotes the class of e_i in $Pic(C_2)$. Thus the Galois group acts trivially on $Pic(C_2)$ and hence $H^1(G, Pic(C_2)) = 0$. The Brauer group $Br(C_2)$ is trivial, since C_2 is birational to $\mathbb{P}^2_{\mathbb{C}}$. Thus the sequence (17) gives an exact sequence

$$0 \to Pic(Q_2) \to (Pic(C_2))^G \to Br(\mathbb{R}) \to Br(Q_2) \to 0. \tag{22}$$

The natural map $Br(\mathbb{R}) \to Br(\mathbb{R}(Q_2))$ factors as

$$Br(\mathbb{R}) \twoheadrightarrow Br(Q_2) \hookrightarrow Br(\mathbb{R}(Q_2))$$

where $\mathbb{R}(Q_2)$ is the function field of the quadric Q_2, and the second map is an injection since the Brauer group of a smooth variety injects into the Brauer group of its function field [Mi, Ch. IV, Corollary 2.6]. But the quaternion algebra $(-1, -1)/\mathbb{R}$ which is the non-trivial element of $Br(\mathbb{R})$ splits in $Br(\mathbb{R}(Q_2))$, since -1 is a sum of two squares in $\mathbb{R}(Q_2)$. Hence the group $Br(Q_2)$ is trivial and we get an exact sequence

$$\begin{array}{c} 0 \to Pic(Q_2) \to (Pic(C_2))^G \xrightarrow{\eta} \mathbb{Z}/2 \to 0. \\ \quad\quad\quad\quad \shortparallel \\ \quad\quad\quad\quad Pic(C_2) \end{array}$$

Since $[e_1 + e_2]$ is the class of a hyperplane section, it belongs to $Pic(Q_2)$, hence $[e_1]$ and $[e_2]$ map to the same image under η. By the exact sequence above, this element must be 1 and hence an element $[ae_1 + be_2]$ in $Pic(C_2)$ maps to the class of $(a + b)$ under the map η. Thus $Pic(Q_2)$ is identified with the subgroup of $\mathbb{Z}[e_1] \oplus \mathbb{Z}[e_2]$ consisting of elements whose sum of coordinates is even, with the generators being given by the class of the hyperplane $[h] = [e_1] + [e_2]$ and the class $[2e_1]$, which corresponds to the "double line cycle" ζ defined over $\mathbb{R}$ by

$$X_0^2 + X_1^2 = 0,\ X_0X_3 + X_1X_2 = 0,\ X_2^2 + X_3^2 = 0.$$

Therefore we have $Pic(Q_2) \cong \mathbb{Z}[h] \oplus \mathbb{Z}[\zeta]$. We now use this and the vanishing of $Br(Q_2)$ proved above in the Kummer sequence (8) to get isomorphisms

$$(\mathbb{Z}/2)[h] \oplus (\mathbb{Z}/2)[\zeta] \cong Pic(Q_2)/2 \cong H^2(Q_2).$$

Hence the proposition is proved. □

We now prove the injectivity of the cycle map. This is used subsequently to compute the unramified cohomology group $H^0(Q_3, \mathcal{H}^3)$.

Proposition 2.5. *Let Q_d be the real anisotropic quadric of dimension d, $d = 2, 3$. Then the cycle map*

$$\rho_2 : CH^2(Q_d)/2 \to H^4(Q_d)$$

is injective.

Proof. Recall that for a smooth integral variety X, we have the spectral sequence (cf. §2)

$$E_2^{p,q} = H^p_{Zar}(X, \mathcal{H}^q) \Longrightarrow H^n_{ét}(X).$$

The above spectral sequence gives an exact sequence

$$H^3(X) \to H^0(X, \mathcal{H}^3) \to CH^2(X)/2 \to H^4(X). \tag{23}$$

By **2.e**, the map on the extreme right is precisely the cycle map ρ_2. If X is the two dimensional quadric Q_2, then we have $H^0(Q_2, \mathcal{H}^3) \subset H^3(\mathbb{R}(Q_2)) = 0$ by [CT-P, Proposition 1.2.1]. Using this in the exact sequence (23), we see that the cycle map is injective. We now consider the case of the three dimensional quadric Q_3 defined by $\sum_{i=0}^{3} X_i^2 = 0$ in $\mathbb{P}^3_{\mathbb{R}}$. The hyperplane section given by $X_4 = 0$ embeds the two dimensional quadric Q_2 as a smooth closed subvariety of Q_3. Consider the natural map

$$CH^1(Q_2) \xrightarrow{\phi} CH^2(Q_3).$$

By Karpenko's computations of the Chow groups of quadrics, we have an isomorphism $CH^2(Q_3) \cong \mathbb{Z}[h^2] \oplus (\mathbb{Z}/2)[\zeta']$ [K-Me, 1.8], where h is a hyperplane section, $\zeta' = h^2 - \zeta$, and ζ is the "double line cycle" discussed in Proposition 2.4, considered now as an element in $CH^2(Q_3)$. By Proposition 2.4, we have $CH^1(Q_2) \cong \mathbb{Z}[h] \oplus \mathbb{Z}[\zeta]$. Under the map ϕ above, the class $[h]$ maps to $[h^2]$ and $[h - \zeta]$ maps to $[\zeta']$. Therefore the map ϕ is surjective and comparing dimensions, we see that the surjective map gives an isomorphism

$$Pic(Q_2)/2 \cong CH^1(Q_2)/2 \cong CH^2(Q_3)/2 \cong (\mathbb{Z}/2)^2. \tag{24}$$

We now consider the complement $U = Q_3 \setminus Q_2$ which is the three dimensional affine variety defined by $1 + \sum_{i=0}^{3} X_i^2 = 0$ in $\mathbb{A}^4_{\mathbb{R}}$. We have the exact sequence (cf. (4))

$$\cdots \to H^i_{Q_2}(Q_3) \to H^i(Q_3) \to H^i(U) \to H^{i+1}_{Q_2}(Q_3) \to \cdots. \tag{25}$$

Since $U_{\mathbb{C}} = U \times_{\mathbb{R}} \mathbb{C}$ is affine and three dimensional over an algebraically closed field, $H^i(U_{\mathbb{C}}) = 0$ for $i > 3$ [Mi, Ch. VI, Theorem 7.2]. We now use these results in the long exact sequence corresponding to the étale map $U_{\mathbb{C}} \to U$ (cf. (16))

$$\cdots \to H^i(U) \to H^i(U_{\mathbb{C}}) \to H^i(U) \xrightarrow{\cup(-1)} H^{i+1}(U) \to \cdots$$

to get isomorphisms $H^i(U) \cong H^{i+r}(U)$ for all $i \geq 4,\ r \geq 0$. But by the result of Cox, [C, Theorem 2.1], since $U(\mathbb{R}) = \emptyset$, we have $H^i(U) = 0$ for $i > 6$. We use these results in the exact sequence (25). Identifying the group $H^4_{Q_2}(Q_3)$ with $H^2(Q_3)$ (cf. **2.c**), we obtain a diagram

$$\begin{array}{ccccc} Pic\ (Q_2)/2 \cong CH^1(Q_2)/2 & \longrightarrow & CH^2(Q_3)/2 & & \\ \downarrow{\scriptstyle \rho_1} & & \downarrow{\scriptstyle \rho_2} & & \\ H^2(Q_2) \cong H^4_{Q_2}(Q_3) & \longrightarrow & H^4(Q_3) & \longrightarrow & H^4(U) = 0. \end{array} \tag{26}$$

The horizontal map at the top is an isomorphism by (24). We have an isomorphism $Pic(Q_2)/2 \cong H^2(Q_2)$ from the Kummer sequence (8) and Proposition 2.4. Further, by the discussion above, we have $H^4(U) = 0$ and by Lemma 2.3, $H^4(Q_3) \cong (\mathbb{Z}/2)^2$. We claim that the above diagram commutes. To prove this, we first remark that for a smooth variety X/k, we have a commutative diagram

$$\begin{tikzcd} Pic(X)/2 \arrow[r, "\phi"] \arrow[d, "\wr"] & H^2(X) \\ CH^1(X)/2 \arrow[ur, "\rho_1"'] & \end{tikzcd}$$

where ρ_1 is the cycle map and ϕ is the boundary map given by the Kummer sequence (cf. (8)). This is obvious by the very definition of the cycle map (cf. [Mi, Ch. VI, Remark 9.6)]. Further, by **2.f**, the diagram

$$\begin{tikzcd} CH^1(Q_2)/2 \arrow[r] \arrow[d, "\rho_1"] & CH^2(Q_3)/2 \arrow[d, "\rho_2"] \\ H^2(Q_2) \arrow[r] & H^4(Q_3) \end{tikzcd}$$

is commutative. Hence the claim is proved and the left vertical map is an isomorphism in (26). The bottom map is also an isomorphism, being surjective between two groups of the same order. Thus since all maps except the right vertical map are isomorphisms and the square commutes, we see that

$$\rho_2 : CH^2(Q_3)/2 \to H^4(Q_3)$$

is also an isomorphism. Therefore the proposition is proved. $\square$

We now compute the unramified cohomology group $H^0(Q_3, \mathcal{H}^3)$. We have

Theorem 2.6. *Let Q_3 be the three dimensional real anisotropic quadric. Then the unramified cohomology group $H^0(Q_3, \mathcal{H}^3)$ is zero.*

Proof. As observed in the proof of Proposition 2.5 (cf. (23)), the local-to-global spectral sequence gives an exact sequence

$$H^3(Q_3) \to H^0(Q_3, \mathcal{H}^3) \to CH^2(Q_3)/2 \xrightarrow{\rho_2} H^4(Q_3). \tag{27}$$

By Proposition 2.5, the map ρ_2 above is injective. Thus to prove the theorem, it suffices to prove that the homomorphism $H^3(Q_3) \to H^0(Q_3, \mathcal{H}^3)$ is zero. We have a commutative diagram as follows:

$$\begin{tikzcd} H^2(Q_3) \arrow[r, "\cup(-1)"] \arrow[d] & H^3(Q_3) \arrow[d] \\ H^0(Q_3, \mathcal{H}^2) \arrow[r, "\cup(-1)"] & H^0(Q_3, \mathcal{H}^3). \end{tikzcd}$$

The group $H^0(Q_3, \mathcal{H}^2)$, which is by definition (cf. §1) the kernel of the homomorphism $H^2(\mathbb{R}(Q_3)) \xrightarrow{\partial} \oplus_{x \in Q_3^{(1)}} H^1(\mathbb{R}(x))$, is isomorphic to ${}_2Br(Q_3)$, the 2-torsion subgroup of the Brauer group, by the Purity theorem for Brauer groups (cf. [Gr, §§6 and 7]. The proof of Lemma 2.3 (cf. argument following (17)) gives isomorphisms $\mathbb{Z}/2 \cong {}_2Br(Q_3) \cong H^0(Q_3, \mathcal{H}^2)$, the non-trivial element being the image of the quaternion algebra $(-1,-1)/\mathbb{R}$, under the natural map $Br(\mathbb{R}) \to Br(Q_3)$. Again, by the proof of Lemma 2.3, the top horizontal map is an isomorphism. Now, since the element -1 is a sum of four squares in $\mathbb{R}(Q_3)$, the form $\ll 1,1,1 \gg \in I^3(\mathbb{R}(Q_3))$ is isotropic and hence hyperbolic. The maps e_n are known to be well-defined on n-fold Pfister forms (cf. [AEJ]). Since $\ll 1,1,1 \gg$ is isotropic, this implies that $e_3(\ll 1,1,1 \gg) = (-1) \cup (-1) \cup (-1)$ is zero in $H^3(\mathbb{R}(Q_3))$. Thus since $H^0(Q_3, \mathcal{H}^2) \cong \mathbb{Z}/2 < (-1) \cup (-1) >$, the bottom horizontal map in the above commutative square is zero and hence the map $H^3(Q_3) \to H^0(Q_3, \mathcal{H}^3)$ is zero. Therefore the theorem is proved. □

We now record some corollaries to the theorem.

Proposition 2.7. *Let Q_d be the real anisotropic quadric of dimension d and $\mathbb{R}(Q_d)$ be its function field. Then we have $H^3_{nr}(\mathbb{R}(Q_d)) = 0$ for $1 \le d \le 6$ and $H^3_{nr}(\mathbb{R}(Q_d)) \cong \mathbb{Z}/2$ for $d \ge 7$, the non-trivial element being $(-1) \cup (-1) \cup (-1)$.*

Proof. Clearly $H^0(Q_d, \mathcal{H}^3) = H^3_{nr}(\mathbb{R}(Q_d)) = 0$ for $d \le 2$, since the 2-cohomological dimension of $\mathbb{R}(Q_d)$ is d [CT-P, Proposition 1.2.1], and $H^0(Q_d, \mathcal{H}^3) \subset H^3(\mathbb{R}(Q_d))$. The quadrics $Q_d,\ 3 \le d \le 6$ are defined by quadratic forms that are Pfister neighbours of the 3-fold Pfister form $\ll 1,1,1 \gg$ which defines the quadric Q_6. By Lemma 1.3 therefore, the unramified cohomology groups of Q_d for $3 \le d \le 6$ are equal and by the above theorem, $H^0(Q_d, \mathcal{H}^3) = 0$ for $3 \le d \le 6$.

We now assume that $d \ge 7$. For any field F, $char\ F \neq 2$, there exists an exact sequence [A]

$$0 \to \mathbb{Z}/2 < (-1) \cup (-1) \cup (-1) > \to H^3(F) \to H^3(F(Q_6)) \tag{28}$$

where $F(Q_6)$ is the function field of the quadric $\sum_{i=0}^{7} X_i^2 = 0$ in $\mathbb{P}^7_F$. In particular, if $F = \mathbb{R}(Q_d),\ d \ge 7$, restriction of the above exact sequence to the unramified cohomology groups gives an exact sequence

$$0 \to \mathbb{Z}/2 < (-1) \cup (-1) \cup (-1) > \to H^3_{nr}(\mathbb{R}(Q_d)) \to H^3_{nr}(\mathbb{R}(Q_d \times Q_6)). \tag{29}$$

But $\mathbb{R}(Q_d \times Q_6)$ is a purely transcendental extension of $\mathbb{R}(Q_6)$. (In general, $\mathbb{R}(Q_m \times Q_n)$ is a purely transcendental extension of $\mathbb{R}(Q_m)$ when $n \ge m$, since the quadric Q_n has an $\mathbb{R}(Q_m)$-rational point if $n \ge m$). Since the unramified cohomology groups remain invariant under purely transcendental extensions, we see that $H^3_{nr}(\mathbb{R}(Q_6)) \cong H^3_{nr}(\mathbb{R}(Q_d \times Q_6)) = 0$, and the above sequence yields an isomorphism $H^3_{nr}(\mathbb{R}(Q_d)) \cong \mathbb{Z}/2 < (-1) \cup (-1) \cup (-1) >$. We claim that the element $(-1) \cup (-1) \cup (-1)$ is non-zero in $H^3(\mathbb{R}(Q_d))$. To prove this, we use the Merkurjev-Suslin-Rost isomorphism [M-S1] $K_3(F)/2 \cong H^3(F)$, where for a field F , $char\ F \neq 2$, $K_n(F)$ is the $n-th$ Milnor K-group of F and the isomorphism is obtained by mapping the class of the symbol $\{a,b,c\}$ in $K_3(F)/2$ to $(a) \cup (b) \cup (c)$ in $H^3(F)$. Now $(-1) \cup (-1) \cup (-1) = 0$ in $H^3(\mathbb{R}(Q_d))$ implies that $\{-1,-1,-1\}$ belongs to $2K_3(\mathbb{R}(Q_d))$ and this in turn implies that -1 is a reduced norm of the quaternion algebra $(-1,-1)/\mathbb{R}(Q_d)$ [M-S, Theorem 12.1]. But this is impossible as the element -1 is not a sum of four squares in $\mathbb{R}(Q_d)$ if $d \ge 7$ (Pfister, cf. [Sc, Ch. IV, Theorem 4.3]). Hence $H^0(Q_d, \mathcal{H}^3) \cong \mathbb{Z}/2$ and the proposition is proved. □

We also have the following

Corollary 2.8. *Let Q_d be the real anisotropic quadric of dimension d, $d \geq 2$. Then the cycle map*

$$\rho_2 : CH^2(Q_d)/2 \to H^4(Q_d, \mathbb{Z}/2)$$

is injective.

Proof. We use the exact sequence (23). For $2 \leq d \leq 6$, by Proposition 2.7, $H^0(Q_d, \mathcal{H}^3) = 0$ and hence the result holds. For $d \geq 7$, we have $H^0(Q_d, \mathcal{H}^3) \cong \mathbb{Z}/2$, the non-trivial element being $(-1) \cup (-1) \cup (-1)$ which is the image of the non-zero element in $H^3(\mathbb{R})$ under the natural map $H^3(\mathbb{R}) \to H^3(Q_d)$. This implies that the map $H^3(Q_d) \to H^0(Q_d, \mathcal{H}^3)$ in (23) is surjective and hence the result follows. □

§3. Unramified Witt groups of Q_d, $3 \leq d \leq 6$.

In this section, we prove that $W_{nr}(\mathbb{R}(Q_d)) \cong \mathbb{Z}/8$ for $3 \leq d \leq 6$. We continue with the notation used in the previous sections. Associated to the d-dimensional quadric Q_d is its defining quadratic form of dimension $d+2$, given by $(d+2).< 1 >$, which we denote by ϕ_d. Recall that the *level* of a field F is defined to be the least integer n such that -1 is a sum of n squares in F. A field F is said to be *non-formally real* if its level is finite. The field $\mathbb{R}(Q_n)$ is non-formally real and has level 2^{l_n}, where $2^{l_n} \leq n+1 < 2^{l_n+1}$ (Pfister, cf. [Sc, Ch. IV, Theorem 4.3]). Thus the quadrics Q_n can be grouped as follows:

Level of the function field	*Quadrics*
$2^0 = 1$	Q_0
2^1	Q_1, Q_2
2^2	$Q_3, \cdots, Q_6$
$\vdots$	$\vdots$
2^k	$Q_{2^k-1}, \cdots\cdots, Q_{2^{k+1}-2}$
$\vdots$	$\vdots$

Let B_k denote the bunch of quadrics whose function fields are of level 2^k. The quadratic form $\phi_{2^{k+1}-2}$, attached to the "topmost" quadric $Q_{2^{k+1}-2}$ of B_k is the $(k+1)$-fold Pfister form $\ll 1, 1, \cdots, 1 \gg$ and those defining all the other quadrics in B_k are Pfister neighbours of $\phi_{2^{k+1}-2}$. By Lemma 1.2, we see that the unramified Witt group is an invariant of the bunch B_k. We prove the following well-known lemma, first proved by Arason [A1], and which computes the unramified Witt group of the anisotropic conic. The proof below was outlined to us by Parimala and is valid over any ground field k, *char* $k \neq 2$.

Lemma 3.1. *Let C/k be a smooth projective anisotropic conic and $k(C)$ be its function field. Then any unramified quadratic form ϕ over $k(C)$ comes from a quadratic form over k. In particular, if a quadratic form ϕ over k becomes isotropic over $k(C)$, then the anisotropic part of $\phi \otimes k(C)$ is defined over k.*

Proof. We shall argue by induction on the rank of the unramified quadratic form ϕ, the property being clearly true if the rank is zero. If $\phi = \phi' \perp n\mathbb{H}$ over $k(C)$, where ϕ' is the anisotropic part, $\mathbb{H}$ denotes the hyperbolic form $< 1, -1 >$ (defined over k) and n is positive, then ϕ' is unramified of rank smaller than ϕ, hence the result follows from the induction hypothesis. We may therefore assume that ϕ is anisotropic.

For every closed point $x \in C$, we have $\partial_x(\phi) = 0$, where ∂_x is the second residue homomorphism (cf. §1) corresponding to the point x. This implies (cf. [Sc, Ch. VI, §2]) that given any closed point $x \in C$, there exists a regular quadratic space $\mathcal{E}_x$ over the discrete valuation ring $\mathcal{O}_{C,x}$, whose image under the natural map $\mathcal{O}_{C,x} \hookrightarrow k(C)$ coincides with ϕ. By an easy patching theorem (valid over any integral Dedekind scheme) we get an anisotropic quadratic bundle $\mathcal{E}$ on C such that the restriction of $\mathcal{E}$ at a closed point $x \in C$ is $\mathcal{E}_x$ and $\mathcal{E} \otimes k(C) \cong \phi$.

Given such a bundle $\mathcal{E}$, since $\mathcal{E} \cong \mathcal{E}^*$, the degree of the bundle $\mathcal{E}$ denoted *deg* $\mathcal{E}$, which is the degree of the determinant bundle of $\mathcal{E}$, is zero. Hence by the Riemann-Roch theorem, we have

$$h^0(\mathcal{E}) \geq deg\ \mathcal{E} + (rk\ \mathcal{E})(1 - g(C)) \geq 1,$$

where $h^0(\mathcal{E})$ is the dimension of the k-vector space of global sections of $\mathcal{E}$, rk is the rank and $g(C)$ is the genus of C which is zero. This implies that the bundle $\mathcal{E}$ has a non-trivial global section, say s. The bilinear structure gives a homomorphism

$$b_\phi : \Gamma(C, \mathcal{E}) \times \Gamma(C, \mathcal{E}) \to \Gamma(C, \mathcal{O}_C)$$

where for a sheaf $\mathcal{G}$ on C, $\Gamma(C, \mathcal{G})$ is the k-vector space of global sections of $\mathcal{G}$ and $\mathcal{O}_C$ is the structure sheaf. Since C is smooth, projective, and geometrically integral, the space $\Gamma(C, \mathcal{O}_C)$ is isomorphic to k. Hence the evaluation of the section gives a splitting $\mathcal{E} \cong < a > \perp \mathcal{E}'$ as bilinear spaces, with $0 \neq a = b_\phi(s, s) \in k$ since $\mathcal{E} \otimes k(C)$ is anisotropic. By the induction hypothesis, $\mathcal{E}' \otimes k(C)$ comes from k, and the lemma is proved. $\square$

Remark. Let K/k be a field extension. If given any anisotropic quadratic form q over k, the anisotropic part of $q_K = q \otimes K$ comes from a quadratic form defined over k, the field K is called an *excellent extension* of k. The excellence of the function field of an anisotropic conic over k has been proved by various people. We refer to [VG] and the references cited there for a more extensive treatment of this property.

Corollary 3.2. *(Parimala* [P]*) Let Q_d be the real anisotropic quadric over $\mathbb{R}$, $d = 1, 2$. Then we have $W_{nr}(\mathbb{R}(Q_d)) \cong \mathbb{Z}/4$, with the form $< 1 >$ as the generator.*

Proof. By the remark at the beginning of this section, it suffices to prove the result for the conic Q_1. By the above lemma, the elements of $W_{nr}(\mathbb{R}(Q_1))$ are defined over $\mathbb{R}$, i.e. the natural map $W(\mathbb{R}) \to W_{nr}(\mathbb{R}(Q_1))$ is surjective. But $W(\mathbb{R}) \cong \mathbb{Z}$ with the form $< 1 >$ as the generator. The form $4 < 1 > = \ll 1, 1 \gg$ is isotropic and hence hyperbolic, since -1 is a sum of two squares in $\mathbb{R}(Q_1)$. But $2 < 1 > = < 1, 1 >$ is not zero in $W(\mathbb{R}(Q_1))$ since -1 is not a square in $\mathbb{R}(Q_1)$. Hence $W_{nr}(Q_1) \cong \mathbb{Z}/4$. The result for Q_2 follows from Lemma 1.2. $\square$

We now prove the main theorem.

Theorem 3.3. *Let Q_d be the d-dimensional real anisotropic quadric, $3 \leq d \leq 6$. Then $W_{nr}(\mathbb{R}(Q_d)) \cong \mathbb{Z}/8$, the form $< 1 >$ being the generator.*

Proof. By the remarks at the beginning of this section, it suffices to prove the theorem for the three dimensional anisotropic quadric Q_3, since the quadrics Q_d

with $3 \leq d \leq 6$ are all in B_2. By Proposition 1.1, there is an exact sequence of groups

$$\begin{array}{ll} & 0 \to W(\mathbb{R}(Q_3)) \ll 1,1 \gg \to W(\mathbb{R}(Q_3)) \to W(\mathbb{R}(Q_3 \times Q_2)) \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \shortparallel \\ (30) & \qquad\qquad\qquad\qquad\qquad\qquad\qquad W(\mathbb{R}(Q_3)(Q_2)). \end{array}$$

We remark that the kernel $W(\mathbb{R}(Q_3)) \ll 1,1 \gg$ is a 2-torsion group, since the level of $\mathbb{R}(Q_3)$ is 4 and hence $2. \ll 1,1 \gg = \ll 1,1,1 \gg$ is hyperbolic. Restricting the above exact sequence to the unramified subgroups, we get an exact sequence
(31)

$$0 \to [W(\mathbb{R}(Q_3)) \ll 1,1 \gg] \cap W_{nr}(\mathbb{R}(Q_3)) \to W_{nr}(\mathbb{R}(Q_3)) \to W_{nr}(\mathbb{R}(Q_3 \times Q_2)).$$

But $\mathbb{R}(Q_3 \times Q_2) = \mathbb{R}(Q_3)(Q_2) = \mathbb{R}(Q_2)(Q_3)$ is a purely transcendental extension of $\mathbb{R}(Q_2)$ and hence $W_{nr}(\mathbb{R}(Q_3 \times Q_2)) \cong W_{nr}(\mathbb{R}(Q_2)) \cong \mathbb{Z}/4 < 1 >$, by Corollary 3.2. We claim that $[W(\mathbb{R}(Q_3)) \ll 1,1 \gg] \cap W_{nr}(\mathbb{R}(Q_3)) \cong \mathbb{Z}/2 \ll 1,1 \gg$. But $I(\mathbb{R}(Q_3)) \subseteq W(\mathbb{R}(Q_3))$ and the quotient $W(\mathbb{R}(Q_3))/I(\mathbb{R}(Q_3)) \cong \mathbb{Z}/2$, with the non-trivial element being the class of the form $< 1 >$. Therefore to prove the claim, it suffices to prove that the group

$$[I(\mathbb{R}(Q_3)) \ll 1,1 \gg] \cap W_{nr}(\mathbb{R}(Q_3)) = 0.$$

But this group is a subgroup of the group $I^3_{nr}(\mathbb{R}(Q_3))$. Further, $I^4(\mathbb{R}(Q_3)) = 0$, this observation being a consequence of the general result (cf. [AEJ]) that if F is a field of transcendence degree m over $\mathbb{R}$, then $I^{m+1}(F)$ is torsion-free. In our situation, since $\mathbb{R}(Q_3)$ is non-formally real, the group $W(\mathbb{R}(Q_3))$ is torsion [Sc, Ch. II, Theorem 7.1]. Also, as mentioned in §1, the maps $\overline{e}_n : I^n(F)/I^{n+1}(F) \to H^n(F)$ are well-defined isomorphisms for $n = 2, 3$. Thus from these remarks and (3) of §1, we see that there is an injection $I^3_{nr}(\mathbb{R}(Q_3)) \hookrightarrow H^3_{nr}(\mathbb{R}(Q_3))$. But the unramified cohomology group $H^3_{nr}(\mathbb{R}(Q_3)) = H^0(Q_3, \mathcal{H}^3) = 0$ by Proposition 2.5. The element $\ll 1,1 \gg$ is not zero in $W(\mathbb{R}(Q_3))$, since -1 is a sum of no less than four squares in $\mathbb{R}(Q_3)$. Thus, the exact sequence (31) reads as

$$\begin{array}{c} 0 \to \mathbb{Z}/2 \ll 1,1 \gg \to W_{nr}(\mathbb{R}(Q_3)) \to \mathbb{Z}/4 < 1 > \\ \qquad\qquad\qquad < 1 > \quad \mapsto \quad < 1 > \end{array}$$

But the form $< 1 > \in W_{nr}(\mathbb{R}(Q_3))$ is an element of order 8 since the level of $\mathbb{R}(Q_3)$ is 4, and hence the above sequence is short exact, non-split and $W_{nr}(\mathbb{R}(Q_3)) \cong \mathbb{Z}/8$ as was to be proved. □

Finally, we note the following Corollary to Theorem 3.3. Given a finitely generated field extension K/k, an element $f \in K^*$ is said to be *unramified* if the form $< f >$ belongs to $W_{nr}(K/k)$. Let Q_d be the real anisotropic quadric of dimension d, $3 \leq d \leq 6$ and $\mathbb{R}(Q_d)$ be its function field. Since the level of $\mathbb{R}(Q_d)$ is 4, the form $< 1,1,1,1,1 >$ is isotropic and hence universal, i.e. it represents every element in $\mathbb{R}(Q_d)^*$. This implies that any element $f \in \mathbb{R}(Q_d)^*$ is a sum of five squares. If further f is unramified, then we have the following

Corollary 3.4. *Let $f \in \mathbb{R}(Q_d)$, $3 \leq d \leq 6$ be unramified. Then f can be represented as a sum of four squares in $\mathbb{R}(Q_d)$. Also, -1 is unramifed and cannot be represented by less than four squares.*

Proof. If f is unramified then the form $\ll 1, 1, -f \gg \in I^3(\mathbb{R}(Q_d))$ is actually an element of $I^3_{nr}(\mathbb{R}(Q_d))$. The homomorphism e_3 maps $I^3_{nr}(\mathbb{R}(Q_d))$ into $H^0(Q_d, \mathcal{H}^d)$ and the latter group is zero by Proposition 2.7. Thus $e_3(\ll 1, 1, -f \gg)$ is zero. But the homomorphism e_3 is injective on Pfister forms [A2, Proposition 2]. Hence the form $\ll 1, 1, -f \gg$ is hyperbolic and f can be represented as a sum of four squares. Further, the element -1, which is clearly unramified, is a sum of no less than four squares in $\mathbb{R}(Q_d)$. Thus the corollary is proved. □

Acknowledgments: The second author would like to thank the University of Regensburg, (especially Professor M. Knebusch and Dr. C. Scheiderer), the University of Paris-Sud, Orsay and the University of Lausanne (especially Professor M. Ojanguren) for hospitality accorded at various stages of completion of this work.

References

[A] Arason, J. Kr., *Kohomologische Invarianten quadratischer Formen*, Journal of Algebra **36** (1975), 448–491.

[A1] Arason, J. Kr., *Appendix to Amenable fields and Pfister extensions*, in Conference on Quadratic Forms (1976), ed. G. Orzech, Queen's papers in pure and applied math. **46** (1977), 319–328.

[A2] Arason, J. Kr., *A proof of Merkurjev's Theorem*, Canadian Math. Soc. Proceedings **4** (1984), 121–129.

[AEJ] Arason, J. Kr., Elman, R., Jacob, B., *Graded Witt ring and Galois cohomology*, Canadian Math. Soc. Proceedings **4** (1984), 17–50.

[Ar] Artin, M., *SGA 4 Tome 3*, Springer Lecture Notes 305, Berlin, Heidelberg, New York: Springer, 1973.

[B-O] Bloch, S., Ogus, A., *Gersten's conjecture and the homology of schemes*, Ann. Sc. Éc. Norm. Supér. **4** *Série* **7** (1974), 181–202.

[CT-O] Colliot-Thélène, J.-L., Ojanguren, M., *Variétés unirationnelles non rationnelles: au-delà de l'exemple d'Artin et Mumford*, Invent. Math. **97** (1989), 141–158.

[CT-P] Colliot-Thélène, J.-L., Parimala, R., *Real components of algebraic varieties and étale cohomology*, Invent. Math. **101** (1990), 81–99.

[CT-S] Colliot-Thélène, J.-L., Sansuc, J.-J., *La descente sur les variétés rationnelles II*, Duke Math. J. **54** (1987), 375–492.

[C] Cox, D.A., *The étale homotopy type of varieties over* $\mathbb{R}$, Proc. Amer. Math. Soc. **76** (1979), 17–22.

[D] Deligne, P., *SGA 4 1/2*, Springer Lecture Notes 569, Berlin, Heidelberg, New York: Springer, 1977.

[D1] Deligne, P., *SGA 4 Tome 3*, Springer Lecture Notes 305, Berlin, Heidelberg, New York: Springer, 1973.

[Gr] Grothendieck, A., *Le groupe de Brauer III: exemples et compléments, in Dix exposés sur la cohomologie des schémas*, North-Holland, Amsterdam, 1968.

[K] Karpenko, N.A., *Algebro-geometric invariants of quadratic forms*, Leningrad Math. Jour. **2** (1991), 119–138.

[K-Me] Karpenko, N.A., Merkurjev, A.S., *Chow groups of projective quadrics*, Leningrad Math. Jour. **2** (1991), 655-671.

[Me] Merkurjev, A. S., *On the norm residue homomorphism of degree two*, Soviet Math. Dokl. **24** (1981), 546–551.

[M-S] Merkurjev, A. S., Suslin, A. A., *K-cohomology of Severi-Brauer varieties and the norm residue homomorphism*, Math. USSR. Izv. **21** (1983), 307–340.

[M-S1] Merkurjev, A. S., Suslin, A. A., *The norm residue homommorphism of degree three*, Izv. Akad. Nauk SSSR Ser. Mat. 54 (1990)= Math. USSR. Izv. **36** (1991), 349-367.

[Mi] Milne, J. S., *Étale Cohomology*, Princeton University Press, 1980.

[O] Ojanguren, M., *The Witt group and a problem of Lüroth*, Ets Editrice Pisa, Pisa, 1991.

[P] Parimala, R., *Witt groups of anisotropic conics, elliptic and hyperelliptic curves*, Jour. Nb. Th. **28** (1988), 69–93.

[R] Rost, M., *Hilbert 90 for* K_3 *for degree-two extensions*, Preprint, Regensburg (1986).

[Sc] Scharlau, W., *Quadratic and Hermitian forms*, Grundlehren der Mathematischen Wissenschaften 270, Berlin, Heidelberg, New York: Springer, 1985.

[VG] Van Geel, J., *Applications of the Riemann Roch theorem for curves to quadratic forms and division algebras*, Recherches de Mathématique n° 7, Institut de Mathématique pure et appliquée, Université Catholique de Louvain, 1991.

Mathématiques, Bâtiment 425, Université de Paris-Sud, 91405-Orsay, France.
E-mail address: colliot@matups.matups.fr

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India.
E-mail address: sujatha@tifrvax.bitnet