

# Peeling the Brownian half-plane

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## Abstract

We establish a new spatial Markov property of the Brownian half-plane. According to this property, if one removes a hull centered at a boundary point, the remaining space equipped with an intrinsic metric is still a Brownian half-plane, which is independent of the part that has been removed. This is an analog of the well-known peeling procedure for random planar maps. We also investigate several distributional properties of hulls centered at a boundary point, and we provide a new construction of the Brownian half-plane giving information about distances from a half-boundary.

## 1 Introduction

This work is concerned with the models of random geometry that arise as scaling limits of large graphs embedded in the sphere. We are especially interested in the Brownian half-plane, which appears as the scaling limit of large planar quadrangulations with a boundary [3], under appropriate conditions on their boundary sizes, or of the infinite random lattice known as the Uniform Infinite Half-Planar Quadrangulation [13]. The Brownian half-plane shares the same local properties as the other well-known models called the Brownian sphere, the Brownian disk or the Brownian plane, but it has the topology of the usual half-plane. In particular, one may define its boundary as the set of all points that have no neighborhood homeomorphic to the open unit disk. One of the main results of the present work is a new spatial Markov property that can be described informally as follows. Let  $r > 0$ , and suppose that one has explored all points of the Brownian half-plane that are at distance smaller than or equal to  $r$  from a distinguished point of the boundary, and also all points that have been disconnected from infinity in this exploration. Then the remaining unexplored region is still a Brownian half-plane, which furthermore is independent of the explored region. Different forms of the spatial Markov property have been obtained in other models (see in particular [24, 25]), but this property takes a nicer form for the Brownian half-plane, where no conditioning is needed to describe the law of the unexplored region. The spatial Markov property of the Brownian half-plane is a kind of continuous version of the peeling process of (finite or infinite) random planar maps, which involves exploring faces of the map one after another, in such a way that the distribution of the unexplored region only depends on the size of its boundary. The peeling process of random planar maps has found a large number of applications (see in particular [2, 6]), and we hope that similar applications can be developed in our continuous setting (see [11] for a first application of the spatial Markov property of the Brownian half-plane).

Let us give a more precise presentation of our main results. We write  $(\mathfrak{H}, D)$  for the random metric space that we call the Brownian half-space. This space comes with a volume measure denoted by  $V$  and a distinguished curve  $(\Lambda(t))_{t \in \mathbb{R}}$  whose range is the boundary  $\partial\mathfrak{H}$ . Therefore, we view  $(\mathfrak{H}, D, V, \Lambda)$  as a curve-decorated measure metric space, in the framework of [13]. The distinguished point on the boundary is the point  $\mathbf{x} = \Lambda(0)$  (this point plays no special role and could be replaced by  $\Lambda(t)$ , for a given  $t \in \mathbb{R}$ , in what follows). For  $r > 0$ , we consider the closed ball of radius  $r$  centered at  $\mathbf{x}$ , which we denote by  $B_r(\mathfrak{H})$ . The hull  $B_r^\bullet(\mathfrak{H})$  is obtained by filling in the “holes” of  $B_r(\mathfrak{H})$ , or, more precisely,  $B_r^\bullet(\mathfrak{H})$  is the complement of the unbounded component of the open set  $\mathfrak{H} \setminus B_r(\mathfrak{H})$ . Write  $\mathfrak{H}_r$  for the closure of  $\mathfrak{H} \setminus B_r^\bullet(\mathfrak{H})$ . According to Theorem 1,  $\mathfrak{H}_r$  equipped with an intrinsic distance  $D_r$ , with the restriction  $V_r$  of the volume measure  $V$ , and with an appropriately defined boundary curve  $\Lambda^r$ , is again a Brownian half-plane. Furthermore,  $\mathfrak{H}_r$  is independent of the hull  $B_r^\bullet(\mathfrak{H})$  also viewed as a random curve-decorated measure metric space (Theorem 2). In the same way as it was done for the peeling of

random planar maps, one can iterate the procedure and remove from  $\mathfrak{H}_r$ , a hull centered at a boundary point  $\mathbf{x}'$  of  $\partial\mathfrak{H}_r$  (which can be chosen as a measurable function of  $B_r^\bullet(\mathfrak{H})$ ) to get another Brownian half-plane, and so on. See Fig. 1 for a schematic illustration.

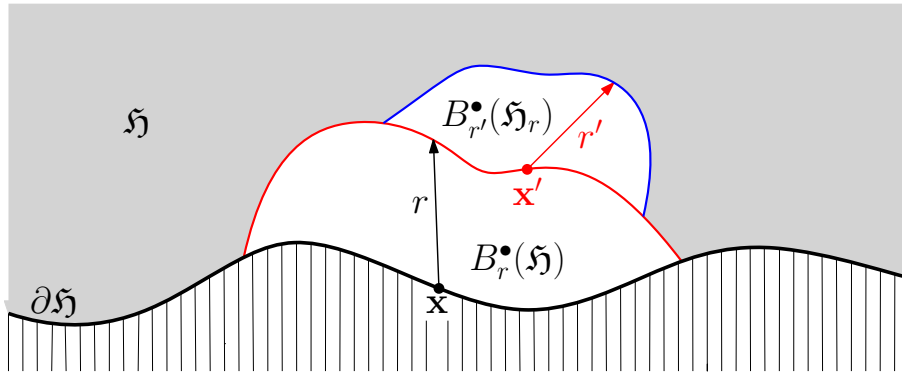


Figure 1: Iterating the peeling procedure. The grey part is still a Brownian half-plane. The hull  $B_{r'}^\bullet(\mathfrak{H}_r)$  is here centered at a point  $\mathbf{x}'$  which can be chosen as a function of  $B_r^\bullet(\mathfrak{H})$ .

Motivated by the spatial Markov property described above, we compute several exact distributions related to the hull  $B_r^\bullet(\mathfrak{H})$ . In particular, the perimeter  $Z_r$  of the hull  $B_r^\bullet(\mathfrak{H})$  (measuring the “size” of the topological boundary of the hull) is exponentially distributed with mean  $2r^2/3$ . More precisely, Proposition 3 gives the joint distribution of  $Z_r$ , of the volume  $V(B_r^\bullet(\mathfrak{H}))$ , and of the sizes of the subsets of  $\partial\mathfrak{H}$  that have been “swallowed” by the hull on both sides of the distinguished point  $\mathbf{x}$ . Furthermore, we also study the perimeter process  $(Z_r)_{r>0}$  and characterize its distribution as that of a self-similar Markov process, which is associated via Lamperti’s representation with a spectrally negative Lévy process whose Laplace exponent has a simple explicit form (Proposition 8).

The proof of our spatial Markov property (Theorems 1 and 2) relies on related results obtained for the Brownian disk in [25], and on a coupling between the Brownian half-plane and the Brownian disk with a large boundary size. As another application of this coupling, we derive a new representation of the Brownian half-plane  $(\mathfrak{H}, D, V, \Lambda)$  in Theorem 11. Like previous constructions, this representation involves random  $\mathbb{R}$ -trees equipped with (nonnegative) Brownian labels. However, in contrast with preceding work, labels now correspond to distances from the “negative half-boundary”  $\partial_1\mathfrak{H} := \{\Lambda(t) : t \leq 0\}$ . As a consequence, the process  $(D(\Lambda(t), \partial_1\mathfrak{H}))_{t \geq 0}$  is distributed as a three-dimensional Bessel process. This should be compared with the identification of the processes  $(D(\Lambda(t), \Lambda(0)))_{t \geq 0}$  and  $(D(\Lambda(-t), \Lambda(0)))_{t \geq 0}$  as two independent five-dimensional Bessel processes, which follows from the Caraceni-Curien construction of the Brownian half-plane [5]. As a side remark, there are several constructions of the Brownian half-plane in terms of labeled trees, where labels may correspond either to distances from a distinguished point of the boundary [5], or to “distances from infinity” [3, 13], or to distances from the boundary [24]. These constructions provide different pieces of information on the Brownian half-plane, but it is somewhat intriguing that there is no simple direct way to prove that they all give rise to the same object (for this, one typically needs to come back to the discrete approximations).

The spatial Markov property of Theorems 1 and 2 suggests several questions. In particular, one may ask about describing a general class of simply connected open subsets  $U$  of  $\mathfrak{H}$  whose intersection with the boundary is an “interval”  $\{\Lambda(t) : \alpha < t < \beta\}$ , such that  $\mathfrak{H} \setminus U$  equipped with an intrinsic metric is again a Brownian half-plane, which furthermore is independent of  $U$  in an appropriate sense (there is an obvious analogy with the strong Markov property of Brownian motion). Iterating the peeling procedure described above gives examples of such subsets  $U$  (see also Section 7). Another appealing but still vague problem would be to characterize the Brownian half-plane by such a general form of the spatial Markov property, together with other properties to be specified.

The paper is organized as follows. Section 2 gives some preliminaries concerning curve-decorated metric spaces and the Brownian snake excursion measures  $\mathbb{N}_x$ , which are our basic tools to construct Brownian trees equipped with Brownian labels. In Section 3, we recall the Caraceni-Curien construction

of the Brownian half-plane, and we prove the spatial Markov property (Theorems 1 and 2). Section 4 is devoted to the calculation of explicit distributions related to the hulls  $B_r^\bullet(\mathfrak{H})$ . In the same direction, two different characterizations of the perimeter process  $(Z_r)_{r>0}$  are discussed in Section 5. Finally, Section 6 presents our new construction of the Brownian half-plane, and Section 7 discusses an analog of Theorem 1 for hulls centered on a segment of the boundary.

## 2 Preliminaries

### 2.1 Curve-decorated spaces

We are interested in (random) metric spaces equipped with additional structures and follow closely [13]. If  $(E, d)$  is a compact metric space, we let  $C_0(\mathbb{R}, E)$  be the space of all continuous functions  $\gamma : \mathbb{R} \rightarrow E$  such that, for every  $\varepsilon > 0$ , there exists  $T > 0$  such that  $d(\gamma(t), \gamma(T)) < \varepsilon$  and  $d(\gamma(-t), \gamma(-T)) < \varepsilon$  for every  $t \geq T$ . By convention, if  $\gamma : [a, b] \rightarrow E$  is only (continuous and) defined on an interval  $[a, b]$ , we view it as an element of  $C_0(\mathbb{R}, E)$  by extending it so that it is constant on  $(-\infty, a]$  and on  $[b, \infty)$ . Following [13], we say that a curve-decorated (compact) measure metric space is a compact metric space  $(E, d)$  equipped with a finite Borel measure  $\mu$  (sometimes called the volume measure) and with a curve  $\gamma \in C_0(\mathbb{R}, E)$  — we then often view  $\gamma(0)$  as the distinguished point of  $E$ . We write  $\mathbb{M}^{\text{GHPU}}$  for the set of all isometry classes of curve-decorated compact measure metric spaces (here  $(E, d, \mu, \gamma)$  and  $(E', d', \mu', \gamma')$  are isometry equivalent if there exists an isometry  $\Phi$  from  $E$  onto  $E'$  such that  $\Phi_*\mu = \mu'$  and  $\gamma' = \Phi \circ \gamma$ ). One can equip  $\mathbb{M}^{\text{GHPU}}$  with the so-called Gromov-Hausdorff-Prokhorov-uniform distance  $d_{\text{GHPU}}$ , which is defined by

$$\begin{aligned} d_{\text{GHPU}}((E_1, d_1, \mu_1, \gamma_1), (E_2, d_2, \mu_2, \gamma_2)) \\ := \inf \left\{ d_{\mathbb{H}}^E(\Phi_1(E_1), \Phi_2(E_2)) \vee d_{\mathbb{P}}^E((\Phi_1)_*\mu_1, (\Phi_2)_*\mu_2) \vee \sup_{t \in \mathbb{R}} d(\Phi_1 \circ \gamma_1(t), \Phi_2 \circ \gamma_2(t)) \right\}, \end{aligned}$$

where the infimum is over all isometric embeddings  $\Phi_1 : E_1 \rightarrow E$  and  $\Phi_2 : E_2 \rightarrow E$  of  $E_1$  and  $E_2$  into the same compact metric space  $(E, d)$ ,  $d_{\mathbb{H}}^E$  is the usual Hausdorff distance between compact subsets of  $E$ , and  $d_{\mathbb{P}}^E$  denotes the Prokhorov metric on the space of all finite Borel measures on  $E$ . By [13, Proposition 1.3],  $d_{\text{GHPU}}$  is a complete separable metric on  $\mathbb{M}^{\text{GHPU}}$ .

We will also be interested in non-compact metric spaces. We again follow closely [13] and restrict ourselves to length spaces for technical reasons. Recall that a metric space is called boundedly compact if all closed balls are compact. We then let  $\mathbb{M}_{\infty}^{\text{GHPU}}$  denote the set of all (isometry classes of) 4-tuples  $\mathfrak{X} = (X, d, \mu, \gamma)$ , where  $(X, d)$  is a boundedly compact length space,  $\mu$  is a Borel measure on  $X$  that assigns finite mass to every compact subset of  $X$ , and  $\gamma : \mathbb{R} \rightarrow X$  is a continuous curve in  $X$ . As previously, we identify  $(X, d, \mu, \gamma)$  and  $(X', d', \mu', \gamma')$  if there is an isometry  $\Phi : X \rightarrow X'$  such that  $\Phi_*\mu = \mu'$  and  $\gamma' = \Phi \circ \gamma$ . We can then define a local version of  $d_{\text{GHPU}}$  as follows. If  $\mathfrak{X} = (X, d, \mu, \gamma) \in \mathbb{M}_{\infty}^{\text{GHPU}}$ , we first need to define the ball of radius  $r > 0$  of  $\mathfrak{X}$  as an element of  $\mathbb{M}^{\text{GHPU}}$ . To this end, for every  $r > 0$ , we define

$$\underline{\tau}_r^\gamma := (-r) \vee \sup\{t < 0 : d(\gamma(0), \gamma(t)) = r\}, \quad \bar{\tau}_r^\gamma := r \wedge \inf\{t > 0 : d(\gamma(0), \gamma(t)) = r\},$$

with the usual conventions  $\sup \emptyset = -\infty$  and  $\inf \emptyset = +\infty$ . We then define  $\mathfrak{B}_r \gamma \in C_0(\mathbb{R}, X)$  by setting  $\mathfrak{B}_r \gamma(t) = \gamma((t \wedge \bar{\tau}_r^\gamma) \vee \underline{\tau}_r^\gamma)$  for every  $t \geq 0$ . Finally we define the “ball”  $\mathfrak{B}_r(\mathfrak{X})$  as the curve-decorated (compact) measure metric space

$$\mathfrak{B}_r(\mathfrak{X}) := (B_r(X), d|_{B_r(X)}, \mu|_{B_r(X)}, \mathfrak{B}_r \gamma),$$

where  $B_r(X)$  denotes the closed ball of radius  $r$  centered at  $\gamma(0)$  in  $X$ . The local Gromov-Hausdorff-Prokhorov-uniform distance on  $\mathbb{M}_{\infty}^{\text{GHPU}}$  is then defined by

$$d_{\text{GHPU}}^{\infty}(\mathfrak{X}, \mathfrak{X}') := \int_0^{\infty} e^{-r} \left( 1 \wedge d_{\text{GHPU}}(\mathfrak{B}_r(\mathfrak{X}), \mathfrak{B}_r(\mathfrak{X}')) \right) dr.$$

According to [13, Proposition 1.7],  $d_{\text{GHPU}}^{\infty}$  is a complete separable metric on  $\mathbb{M}_{\infty}^{\text{GHPU}}$ .

We will also need to deal with pointed measure metric spaces, which simply amounts to considering the special case where the decorating curve  $\gamma$  is constant. Both in the compact and in the non-compact case, we can view the class of pointed spaces as a subclass of  $\mathbb{M}^{\text{GHPU}}$ , resp. of  $\mathbb{M}_{\infty}^{\text{GHPU}}$ , which we equip with the restriction of the distance  $d_{\text{GHPU}}$ , resp. of  $d_{\text{GHPU}}^{\infty}$ . With any curve-decorated space  $(X, d, \mu, \gamma)$  we can associate the pointed space  $(X, d, \mu, \gamma(0))$ , and this mapping is trivially continuous.

## 2.2 Snake trajectories

We now briefly present the formalism of snake trajectories that we will use to define our models of random geometry (we refer to [1] for more details). A (one-dimensional) finite path  $w$  is a continuous mapping  $w : [0, \zeta] \rightarrow \mathbb{R}$ , where  $\zeta = \zeta_{(w)} \geq 0$  is called the lifetime of  $w$ . We let  $\mathfrak{W}$  denote the space of all finite paths, which is equipped with the distance  $d_{\mathfrak{W}}(w, w') := |\zeta_{(w)} - \zeta_{(w')}| + \sup_{t \geq 0} |w(t \wedge \zeta_{(w)}) - w'(t \wedge \zeta_{(w')})|$ . We denote the tip or endpoint of a path  $w \in \mathfrak{W}$  by  $\widehat{w} = w(\zeta_{(w)})$ , and for  $x \in \mathbb{R}$ , we set  $\mathfrak{W}_x := \{w \in \mathfrak{W} : w(0) = x\}$ . The trivial element of  $\mathfrak{W}_x$  with zero lifetime is identified with the point  $x$  of  $\mathbb{R}$ .

*Definition.* Let  $x \in \mathbb{R}$ . A snake trajectory with initial point  $x$  is a continuous mapping  $s \mapsto \omega_s$  from  $\mathbb{R}_+$  into  $\mathfrak{W}_x$  that satisfies the following two properties:

- (i) We have  $\omega_0 = x$  and the number  $\sigma(\omega) := \sup\{s \geq 0 : \omega_s \neq x\}$ , called the duration of the snake trajectory  $\omega$ , is finite (by convention  $\sigma(\omega) = 0$  if  $\omega_s = x$  for every  $s \geq 0$ ).
- (ii) (Snake property) For every  $0 \leq s \leq s'$ , we have  $\omega_s(t) = \omega_{s'}(t)$  for every  $t \in [0, \min_{s \leq r \leq s'} \zeta_{(\omega_r)}]$ .

We write  $\mathcal{S}_x$  for the set of all snake trajectories with initial point  $x$  and  $\mathcal{S} = \bigcup_{x \in \mathbb{R}} \mathcal{S}_x$  for the set of all snake trajectories. If  $\omega \in \mathcal{S}$ , we often write  $W_s(\omega) := \omega_s$  and  $\zeta_s(\omega) := \zeta_{(\omega_s)}$  for every  $s \geq 0$ . We equip  $\mathcal{S}$  with the distance  $d_{\mathcal{S}}(\omega, \omega') := |\sigma(\omega) - \sigma(\omega')| + \sup_{s \geq 0} d_{\mathfrak{W}}(W_s(\omega), W_s(\omega'))$ . We notice that a snake trajectory  $\omega$  is determined by the knowledge of the lifetime function  $s \mapsto \zeta_s(\omega)$  and of the tip function  $s \mapsto \widehat{W}_s(\omega)$ : See [1, Proposition 8].

Let  $\omega \in \mathcal{S}$  be a snake trajectory and  $\sigma = \sigma(\omega)$ . The lifetime function  $s \mapsto \zeta_s(\omega)$  codes a compact  $\mathbb{R}$ -tree, which will be denoted by  $\mathcal{T}_{(\omega)}$  and called the *genealogical tree* of the snake trajectory. This  $\mathbb{R}$ -tree is the quotient space  $\mathcal{T}_{(\omega)} := [0, \sigma] / \sim$  of the interval  $[0, \sigma]$  for the equivalence relation defined by setting  $s \sim s'$  if and only if  $\zeta_s(\omega) = \zeta_{s'}(\omega) = \min_{s \wedge s' \leq r \leq s \vee s'} \zeta_r(\omega)$ . We then equip  $\mathcal{T}_{(\omega)}$  with the distance induced by

$$d_{(\omega)}(s, s') := \zeta_s(\omega) + \zeta_{s'}(\omega) - 2 \min_{s \wedge s' \leq r \leq s \vee s'} \zeta_r(\omega),$$

and we stress that  $d_{(\omega)}(s, s') = 0$  if and only if  $s \sim s'$ . We write  $p_{(\omega)} : [0, \sigma] \rightarrow \mathcal{T}_{(\omega)}$  for the canonical projection, and note that the mapping  $[0, \sigma] \ni t \mapsto p_{(\omega)}(t)$  can be viewed as a cyclic exploration of  $\mathcal{T}_{(\omega)}$ . By convention,  $\mathcal{T}_{(\omega)}$  is rooted at the point  $\rho_{(\omega)} := p_{(\omega)}(0)$ , and the volume measure on  $\mathcal{T}_{(\omega)}$  is defined as the pushforward of Lebesgue measure on  $[0, \sigma]$  under  $p_{(\omega)}$ . If  $u, v \in \mathcal{T}_{(\omega)}$ , we write  $\llbracket u, v \rrbracket$  for the geodesic segment between  $u$  and  $v$  in  $\mathcal{T}_{(\omega)}$ . The segment  $\llbracket \rho_{(\omega)}, u \rrbracket$  is called the ancestral line of  $u$ , and we say that  $u$  is a descendant of  $v$  if  $v \in \llbracket \rho_{(\omega)}, u \rrbracket$ .

By property (ii) in the definition of a snake trajectory, the condition  $p_{(\omega)}(s) = p_{(\omega)}(s')$  implies that  $W_s(\omega) = W_{s'}(\omega)$ . So the mapping  $s \mapsto W_s(\omega)$  can be viewed as defined on the quotient space  $\mathcal{T}_{(\omega)}$ . For  $u \in \mathcal{T}_{(\omega)}$ , we set  $\ell_u(\omega) := \widehat{W}_s(\omega)$  for any  $s \in [0, \sigma]$  such that  $u = p_{(\omega)}(s)$ , and we interpret  $\ell_u(\omega)$  as a “label” assigned to the point  $u$  of  $\mathcal{T}_{(\omega)}$ . We note that the mapping  $u \mapsto \ell_u(\omega)$  is continuous on  $\mathcal{T}_{(\omega)}$ , and we set  $W_*(\omega) := \min\{\ell_u(\omega) : u \in \mathcal{T}_{(\omega)}\}$ . We also observe that, for every  $s \in [0, \sigma]$ , the path  $W_s(\omega)$  records the labels along the ancestral line of  $p_{(\omega)}(s)$ .

We next introduce the important truncation operation on snake trajectories. Let  $x, y \in \mathbb{R}$  with  $y < x$ . For every  $w \in \mathfrak{W}_x$ , set  $\tau_y(w) := \inf\{t \in [0, \zeta_{(w)}] : w(t) = y\} \leq +\infty$ . Then, if  $\omega \in \mathcal{S}_x$ , we set, for every  $s \geq 0$ ,

$$\eta_s(\omega) := \inf \left\{ t \geq 0 : \int_0^t dr \mathbf{1}_{\{\zeta_{(\omega_r)} \leq \tau_y(\omega_r)\}} > s \right\}.$$

Note that the condition  $\zeta_{(\omega_r)} \leq \tau_y(\omega_r)$  holds if and only if  $\tau_y(\omega_r) = \infty$  or  $\tau_y(\omega_r) = \zeta_{(\omega_r)}$ . Then, setting  $\omega'_s = \omega_{\eta_s(\omega)}$  for every  $s \geq 0$  defines an element  $\omega'$  of  $\mathcal{S}_x$ , which will be denoted by  $\text{tr}_y(\omega)$  and called the truncation of  $\omega$  at  $y$  (see [1, Proposition 10]). The effect of the time change  $\eta_s(\omega)$  is to “eliminate” those

paths  $\omega_s$  that hit  $y$  and then survive for a positive amount of time. Informally,  $\mathcal{T}_{(\text{tr}_y(\omega))}$  is obtained from  $\mathcal{T}_{(\omega)}$  by pruning branches at the level where labels first take the value  $y$ .

We finally introduce the excursions of a snake trajectory away from a given level. Let  $\omega \in \mathcal{S}_x$  and  $y < x$ . Let  $(\alpha_j, \beta_j)$ ,  $j \in J$ , be the connected components of the open set  $\{s \in [0, \sigma] : \tau_y(\omega_s) < \zeta_{(\omega_s)}\}$ , and notice that, for every  $j \in J$ , we have  $\omega_{\alpha_j} = \omega_{\beta_j}$ , and  $\zeta_s > \zeta_{(\omega_{\alpha_j})}$  for every  $s \in (\alpha_j, \beta_j)$ . For every  $j \in J$ , we define a snake trajectory  $\omega^j \in \mathcal{S}_y$  by setting

$$\omega_s^j(t) := \omega_{(\alpha_j+s) \wedge \beta_j}(\zeta_{(\omega_{\alpha_j})} + t), \text{ for } 0 \leq t \leq \zeta_{(\omega_s^j)} := \zeta_{(\omega_{(\alpha_j+s) \wedge \beta_j})} - \zeta_{(\omega_{\alpha_j})} \text{ and } s \geq 0.$$

We say that  $\omega^j$ ,  $j \in J$ , are the excursions of  $\omega$  away from  $y$ . We note that, for every  $j \in J$ , the tree  $\mathcal{T}_{(\omega^j)}$  is canonically identified to a subtree of  $\mathcal{T}_{(\omega)}$  consisting of descendants of  $p_{(\omega)}(\alpha_j) = p_{(\omega)}(\beta_j)$ .

### 2.3 The Brownian snake excursion measure on snake trajectories

Let  $x \in \mathbb{R}$ . The Brownian snake excursion measure  $\mathbb{N}_x$  is the  $\sigma$ -finite measure on  $\mathcal{S}_x$  that satisfies the following two properties: Under  $\mathbb{N}_x$ ,

- (i) the distribution of the lifetime function  $(\zeta_s)_{s \geq 0}$  is the Itô measure of positive excursions of linear Brownian motion, normalized so that, for every  $\varepsilon > 0$ ,

$$\mathbb{N}_x \left( \sup_{s \geq 0} \zeta_s > \varepsilon \right) = \frac{1}{2\varepsilon};$$

- (ii) conditionally on  $(\zeta_s)_{s \geq 0}$ , the tip function  $(\widehat{W}_s)_{s \geq 0}$  is a Gaussian process with mean  $x$  and covariance function

$$K(s, s') := \min_{s \wedge s' \leq r \leq s \vee s'} \zeta_r.$$

The measure  $\mathbb{N}_x$  is the excursion measure away from  $x$  for the Markov process in  $\mathfrak{M}_x$  called the Brownian snake. We refer to [16] for a detailed study of the Brownian snake. For every  $y < x$ , we have

$$\mathbb{N}_x(W_* \leq y) = \frac{3}{2(x-y)^2}, \tag{1}$$

where we recall the notation  $W_*(\omega)$  for the minimal label on  $\mathcal{T}_{(\omega)}$ . See e.g. [16, Section VI.1].

**Exit measures.** Let  $x, y \in \mathbb{R}$ , with  $y < x$ . One shows [20, Proposition 34] that the limit

$$L_t^y(\omega) := \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^2} \int_0^t ds \mathbf{1}_{\{\tau_y(W_s(\omega)) = \infty, \widehat{W}_s(\omega) < y + \varepsilon\}} \tag{2}$$

exists uniformly in  $t \geq 0$ ,  $\mathbb{N}_x(d\omega)$  a.e., and defines a continuous nondecreasing function, which is obviously constant on  $[\sigma, \infty)$ . The process  $(L_t^y)_{t \geq 0}$  is called the exit local time at  $y$ , and the exit measure  $\mathcal{Z}_y$  is defined by  $\mathcal{Z}_y := L_\infty^y = L_\sigma^y$ . Informally,  $\mathcal{Z}_y$  measures “how many” paths  $W_s$  hit  $y$  and are stopped at that hitting time. Then,  $\mathbb{N}_x$  a.e., the topological support of the measure  $dL_t^y$  is exactly the set  $\{s \in [0, \sigma] : \tau_y(W_s) = \zeta_s\}$ , and, in particular,  $\mathcal{Z}_y > 0$  if and only if one of the paths  $W_s$  hits  $y$ . The definition of  $\mathcal{Z}_y$  is a special case of the theory of exit measures (see [16, Chapter V] for this general theory).

The exit measure  $\mathcal{Z}_y$  is a function of the truncated snake  $\text{tr}_y(\omega)$ . Indeed, a time change argument shows that the same formula (2) applied to  $\text{tr}_y(\omega)$  instead of  $\omega$  yields a continuous limit  $t \mapsto L_t^y(\text{tr}_y(\omega))$  which is equal to  $L_{\eta_t}^y(\omega)$ , where  $(\eta_s(\omega))_{s \geq 0}$  is the time change used to define  $\text{tr}_y(\omega)$  at the end of Section 2.2. In particular,  $L_\infty^y(\text{tr}_y(\omega)) = L_\infty^y(\omega) = \mathcal{Z}_y(\omega)$ .

**The special Markov property.** Recall the notation introduced in Section 2.2: for  $y < x$  and  $\omega \in \mathcal{S}_x$ , we write  $\omega^j$ ,  $j \in J$ , for the excursions of  $\omega$  away from  $y$ , and  $(\alpha_j, \beta_j)$ ,  $j \in J$ , for the corresponding subintervals of  $[0, \sigma]$ . The special Markov property states that, under  $\mathbb{N}_x$ , conditionally on the truncation  $\text{tr}_y(\omega)$ , the point measure

$$\sum_{j \in J} \delta_{(L_{\alpha_j}^y, \omega^j)}(dt d\omega') \quad (3)$$

is Poisson with intensity  $\mathbf{1}_{[0, \mathcal{Z}_y]}(t) dt \mathbb{N}_y(d\omega')$ . We refer to the Appendix of [19] for a proof. In the following, we will use a minor extension of this result. Suppose that  $x > y > 0$ , then the special Markov property holds in exactly the same form under  $\mathbb{N}_x(\cdot \cap \{W_* > 0\})$ , provided that the intensity measure is replaced by  $\mathbf{1}_{[0, \mathcal{Z}_y]}(t) dt \mathbb{N}_y(d\omega' \cap \{W_*(\omega') > 0\})$ . We leave the easy proof to the reader.

### 3 Spatial Markov property in the Brownian half-plane

#### 3.1 The Caraceni-Curien construction of the Brownian half-plane

In this section, we present the Caraceni-Curien construction of the Brownian half-plane, which we will use to define the boundary curve of hulls. This construction was given in [5], and it was shown in [21] that it is equivalent to the other construction proposed in [13, 3] (still a different equivalent construction appears in [24]).

Consider a process  $\mathcal{R} = (\mathcal{R}_t)_{t \in \mathbb{R}}$  such that  $(\mathcal{R}_t)_{t \geq 0}$  and  $(\mathcal{R}_{-t})_{t \geq 0}$  are two independent five-dimensional Bessel processes started from 0. Conditionally on the process  $\mathcal{R}$ , let  $\mathcal{N}(dt d\omega)$  be a Poisson point measure on  $\mathbb{R} \times \mathcal{S}$  with intensity

$$2 \mathbf{1}_{\{W_*(\omega) > 0\}} dt \mathbb{N}_{\sqrt{3}\mathcal{R}_t}(d\omega).$$

We write

$$\mathcal{N}(dt d\omega) = \sum_{i \in I} \delta_{(t_i, \omega^i)}(dt d\omega).$$

We then let  $\mathfrak{T}$  be the metric space which is obtained from the union

$$\mathbb{R} \cup \left( \bigcup_{i \in I} \mathcal{T}_{(\omega^i)} \right)$$

by identifying the root  $\rho_{(\omega^i)}$  of  $\mathcal{T}_{(\omega^i)}$  with the point  $t_i$  of  $\mathbb{R}$ . The metric  $d_{\mathfrak{T}}$  on  $\mathfrak{T}$  is defined in the obvious manner, so that  $\mathfrak{T}$  equipped with  $d_{\mathfrak{T}}$  is a (non-compact)  $\mathbb{R}$ -tree, the restriction of  $d_{\mathfrak{T}}$  to each tree  $\mathcal{T}_{(\omega^i)}$  is the metric  $d_{(\omega^i)}$ , as defined in Section 2.2, and  $d_{\mathfrak{T}}(u, v) = |v - u|$  if  $u, v \in \mathbb{R}$ . The volume measure on  $\mathfrak{T}$  is the sum of the volume measures on  $\mathcal{T}_{(\omega^i)}$ ,  $i \in I$ .

We can define a clockwise exploration  $(\mathcal{E}_t)_{t \in \mathbb{R}}$  of  $\mathfrak{T}$ , informally by concatenating the functions  $p_{(\omega^i)}$ ,  $i \in I$ , in the order prescribed by the  $t_i$ 's, in such a way that  $\mathcal{E}_0 = 0$ , and  $\{\mathcal{E}_t : t \geq 0\} = \mathbb{R}_+ \cup (\bigcup_{i \in I, t_i > 0} \mathcal{T}_{(\omega^i)})$  (see [25, Section 4.1] for a precise definition in a slightly different setting). The exploration process  $\mathcal{E}$  allows us to define ‘‘intervals’’ on  $\mathfrak{T}$ . For  $u, v \in \mathbb{R}$ , if  $v < u$ , we set  $[u, v]_{\infty} := [u, \infty) \cup (-\infty, v]$ , and if  $u \leq v$ , we let  $[u, v]_{\infty} := [u, v]$  be the usual interval. Then, for any  $a, b \in \mathfrak{T}$ , there is a smallest ‘‘interval’’  $[u, v]_{\infty}$  such that  $\mathcal{E}_u = a$  and  $\mathcal{E}_v = b$ , and we set  $[a, b]_{\infty} := \{\mathcal{E}_t : t \in [u, v]_{\infty}\}$ .

Finally, we define labels on  $\mathfrak{T}$ . If  $a \in \mathbb{R}$ , we take  $\ell_a := \sqrt{3} \mathcal{R}_a$ , and, if  $a \in \mathcal{T}_{(\omega^i)}$ , we let  $\ell_a$  be the label of  $a$  in  $\mathcal{T}_{(\omega^i)}$  (when  $a$  is the root of  $\mathcal{T}_{(\omega^i)}$  the two definitions are consistent). We set, for every  $a, b \in \mathfrak{T}$ ,

$$D^{\circ}(a, b) := \ell_a + \ell_b - 2 \max \left( \min_{c \in [a, b]_{\infty}} \ell_c, \min_{c \in [b, a]_{\infty}} \ell_c \right) \quad (4)$$

and then

$$D(a, b) := \inf_{a_0=a, a_1, \dots, a_{p-1}, a_p=b} \sum_{k=1}^p D^{\circ}(a_{k-1}, a_k) \quad (5)$$

where the infimum is over all choices of the integer  $p \geq 1$  and of the points  $a_1, \dots, a_{p-1}$  in  $\mathfrak{T}$ . Obviously,  $D(a, b) \leq D^\circ(a, b)$ , and one can prove that  $D(a, b) = 0$  if and only if  $D^\circ(a, b) = 0$ .

Then,  $D$  is a pseudo-metric on  $\mathfrak{T}$ , and we set  $\mathfrak{H} := \mathfrak{T}/\{D = 0\}$ , where (here and later) we use the notation  $\mathfrak{T}/\{D = 0\}$  to denote the quotient space of  $\mathfrak{T}$  for the equivalence relation defined by saying that  $a \approx b$  if and only if  $D(a, b) = 0$ . We equip  $\mathfrak{H}$  with the metric induced by  $D$ , which is still denoted by  $D$ . The canonical projection from  $\mathfrak{T}$  onto  $\mathfrak{H}$  is denoted by  $\Pi$ , and the volume measure  $V$  on  $\mathfrak{H}$  is the pushforward of the volume measure on  $\mathfrak{T}$  under  $\Pi$ . We also introduce the ‘‘boundary curve’’  $\Lambda = (\Lambda(t))_{t \in \mathbb{R}}$ , which is simply defined by setting  $\Lambda(t) = \Pi(t)$  for every  $t \in \mathbb{R}$ . We note the easy bound  $D(a, b) \geq |\ell_a - \ell_b|$ , so that the property  $\Pi(a) = \Pi(b)$  implies  $\ell_a = \ell_b$ .

The (random) curve-decorated measure metric space  $(\mathfrak{H}, D, V, \Lambda)$  (or any space having the same distribution) is called the *curve-decorated Brownian half-plane*. It will sometimes be convenient to consider the pointed measure metric space  $(\mathfrak{H}, D, V, \Lambda(0))$ , to which we will refer as the Brownian half-plane. For any fixed  $s \in \mathbb{R}$ , we can replace  $(\Lambda(t))_{t \in \mathbb{R}}$  by  $(\Lambda(s+t))_{t \in \mathbb{R}}$  (resp.  $\Lambda(0)$  by  $\Lambda(s)$ ), without changing the distribution of  $(\mathfrak{H}, D, V, \Lambda)$  (resp. of  $(\mathfrak{H}, D, V, \Lambda(0))$ ). This translation invariance property is not obvious from the construction we have given, but follows from other constructions, especially the one in [3, 13].

We mention that  $(\mathfrak{H}, D)$  is a length space (this can be deduced from the fact that  $D^\circ(a, b)$  is the length of a curve from  $\Pi(a)$  to  $\Pi(b)$  in  $\mathfrak{H}$ , see e.g. Section 4.1 in [24]). The space  $(\mathfrak{H}, D)$  is homeomorphic to the usual closed half-plane, which makes it possible to define its boundary  $\partial\mathfrak{H}$ , and  $\partial\mathfrak{H}$  is exactly the range of  $\Lambda$ . It follows from the results of [22] that the volume measure  $V$  coincides with the Hausdorff measure with gauge function  $h(r) = cr^4 \log \log(1/r)$ , for a suitable constant  $c$ . Moreover,  $\Lambda$  is a standard boundary curve, meaning that the pushforward of Lebesgue measure under  $\Lambda$  is the uniform measure on the boundary, which may be defined by

$$\langle \nu, \varphi \rangle = \lim_{\varepsilon \rightarrow 0} \varepsilon^{-2} \int \varphi(x) \mathbf{1}_{\{D(x, \partial\mathfrak{H}) < \varepsilon\}} V(dx),$$

for any bounded continuous function  $\varphi$  on  $\partial\mathfrak{H}$ . In fact, once  $\Lambda(0)$  is fixed, this property characterizes the boundary curve  $\Lambda$ , up to the replacement of  $(\Lambda(t))_{t \in \mathbb{R}}$  by  $(\Lambda(-t))_{t \in \mathbb{R}}$ . Finally, we note that the curve-decorated Brownian half-plane is scale invariant, in the sense that, for every  $\lambda > 0$ ,  $(\mathfrak{H}, \lambda D, \lambda^4 V, \Lambda(\lambda^{-2} \cdot))$  has the same distribution as  $(\mathfrak{H}, D, V, \Lambda)$ .

To simplify notation, we set  $\mathbf{x} = \Lambda(0) = \Pi(0)$ , which is viewed as the distinguished point of  $\mathfrak{H}$ . It will be important to observe that distances from  $\mathbf{x}$  in  $\mathfrak{H}$  correspond to labels on the tree  $\mathfrak{T}$ : for every  $a \in \mathfrak{T}$ , we have

$$D(\mathbf{x}, \Pi(a)) = \ell_a,$$

as an easy consequence of formulas (4) and (5). As in Section 2.1, we denote the closed ball of radius  $r$  centered at  $\mathbf{x}$  in  $\mathfrak{H}$  by  $B_r(\mathfrak{H})$ . By the previous display, we have

$$B_r(\mathfrak{H}) = \Pi(\{a \in \mathfrak{T} : \ell_a \leq r\}).$$

Let us now turn to hulls. Since  $\mathfrak{H}$  has the topology of the closed half-plane, it follows that for any  $r > 0$ , the set  $\mathfrak{H} \setminus B_r(\mathfrak{H})$  has only one unbounded connected component. The hull of radius  $r$ , denoted by  $B_r^\bullet(\mathfrak{H})$ , is then defined as the complement of this unique unbounded connected component. It is useful to characterize the hull in terms of labels on  $\mathfrak{T}$ . To this end, for every  $a \in \mathfrak{T} \setminus \{0\}$ , let  $m_a$  denote the minimal label on the (unique) geodesic path from  $a$  to infinity in  $\mathfrak{T}$  that does not contain 0. Also set  $m_0 = 0$ . Then we have

$$B_r^\bullet(\mathfrak{H}) = \Pi(\{a \in \mathfrak{T} : m_a \leq r\}). \quad (6)$$

The fact that  $m_a > r$  implies that  $\Pi(a) \notin B_r^\bullet(\mathfrak{H})$  is easy since the image under  $\Pi$  of the geodesic path from  $a$  to  $\infty$  in  $\mathfrak{T}$  gives a path from  $\Pi(a)$  to  $\infty$  in  $\mathfrak{H}$  that does not intersect the ball  $B_r(\mathfrak{H})$ . The converse is a consequence of the so-called cactus bound, which says that any path from  $\Pi(a)$  to  $\infty$  in  $\mathfrak{H}$  has to visit a point whose distance from  $\mathbf{x}$  is (at most)  $m_a$  — see Proposition 3.1 in [17] for a version of this result for the Brownian sphere, whose proof is easily adapted to the present setting.

It follows from (6) and the preceding observations that the topological boundary of  $B_r^\bullet(\mathfrak{H})$  can also be written as

$$\partial B_r^\bullet(\mathfrak{H}) = \Pi(\{a \in \mathfrak{T} : \ell_a = r \text{ and } \ell_b > r \text{ for every } b \in G_a \setminus \{a\}\}), \quad (7)$$

where we have written  $G_a$  for the geodesic path from  $a$  to infinity in  $\mathfrak{T}$  that does not contain 0. In particular, if

$$\beta_r := \sup\{s \geq 0 : \sqrt{3}\mathcal{R}_{-s} \leq r\}, \quad \gamma_r := \sup\{s \geq 0 : \sqrt{3}\mathcal{R}_s \leq r\},$$

both  $\Pi(-\beta_r)$  and  $\Pi(\gamma_r)$  belong to  $\partial\mathfrak{H} \cap \partial B_r^\bullet(\mathfrak{H})$ . In fact,  $\partial B_r^\bullet(\mathfrak{H})$  is the range of a simple path starting at  $\Pi(-\beta_r)$  and ending at  $\Pi(\gamma_r)$ , which does not intersect  $\partial\mathfrak{H}$  except at its endpoints. To construct this path, we need to consider the exit local times  $(L_s^r(\omega^i))_{s \geq 0}$  for all  $i \in I$  such that  $t_i \notin [-\beta_r, \gamma_r]$ . For every such index  $i$ , we also set  $\alpha_i = \inf\{s \in \mathbb{R} : \mathcal{E}_s \in \mathcal{T}_{(\omega^i)}\}$ . Finally, for every  $s \in \mathbb{R}$ , we set

$$L_s^{\mathfrak{H},r} := \sum_{i \in I: t_i \notin [-\beta_r, \gamma_r]} L_{(s-\alpha_i)^+}^r(\omega^i),$$

which represents the total exit local time at  $r$  accumulated by the exploration process up to time  $s$ . We set

$$Z_r := L_\infty^{\mathfrak{H},r} = \sum_{i \in I: t_i \notin [-\beta_r, \gamma_r]} Z_r(\omega^i).$$

We note that  $Z_r < \infty$ , a.s. Indeed, by [8, Lemma 4.1] (see formula (17) below),  $\mathbb{N}_x(Z_r \mathbf{1}_{\{W_* > 0\}}) = (r/x)^3$  for every  $x > r$ , and thus

$$\mathbb{E}[Z_r \mid \mathcal{R}] = 2 \int_{(-\infty, -\beta_r) \cup (\gamma_r, \infty)} \frac{r^3}{(\sqrt{3}\mathcal{R}_t)^3} dt < \infty.$$

Similarly, the sets  $\{i \in I : t_i < -\beta_r - A, Z_r(\omega^i) > 0\}$  and  $\{i \in I : t_i > \gamma_r + A, Z_r(\omega^i) > 0\}$  are finite, for every  $A > 0$ , as a simple consequence of (1). On the other hand, it also follows from (1) that the sets  $\{i \in I : t_i < -\beta_r, Z_r(\omega^i) > 0\}$  and  $\{i \in I : t_i > \gamma_r, Z_r(\omega^i) > 0\}$  are both infinite.

We write  $Z_r = Z_r' + Z_r''$ , where

$$Z_r' := \sum_{i \in I: t_i \in (-\infty, -\beta_r)} Z_r(\omega^i), \quad Z_r'' := \sum_{i \in I: t_i \in (\gamma_r, \infty)} Z_r(\omega^i). \quad (8)$$

We then define  $\kappa(u)$  for every  $u \in [0, Z_r]$  by setting

$$\begin{aligned} \kappa(Z_r' - u) &:= \inf\{s \in \mathbb{R} : L_s^{\mathfrak{H},r} \geq u\} && \text{if } 0 < u \leq Z_r', \\ \kappa(Z_r' + u) &:= \inf\{s \in \mathbb{R} : L_s^{\mathfrak{H},r} \geq Z_r - u\} && \text{if } 0 \leq u < Z_r'', \end{aligned}$$

and we also set  $\kappa(Z_r) = \kappa(Z_r -)$ . Note that  $\mathcal{E}_{\kappa(0)} = -\beta_r$  and  $\mathcal{E}_{\kappa(Z_r)} = \gamma_r$ .

It follows from the properties of exit local times and formula (7) that the path  $\Lambda^{\bullet,r}$  defined by setting  $\Lambda^{\bullet,r}(u) := \Pi(\mathcal{E}_{\kappa(u)})$  for every  $u \in [0, Z_r]$  is continuous and injective, and its range is exactly  $\partial B_r^\bullet(\mathfrak{H})$ . To verify these properties, first note that the mapping  $u \mapsto \kappa(u)$  is right-continuous on  $[0, Z_r]$ . Then the support property of exit local times implies that, for every  $i \in I$  such that  $t_i \notin [-\beta_r, \gamma_r]$ , the support of the measure  $dL_s^r(\omega^i)$  is exactly the set  $\{s : \tau_r(\omega_s^i) = \zeta_{(\omega_s^i)}\}$ . We can use this to verify that, a.s. for every  $u \in [0, Z_r]$  such that  $\kappa(u-) \neq \kappa(u)$ , we have  $D^\circ(\mathcal{E}_{\kappa(u-)}, \mathcal{E}_{\kappa(u)}) = 0$  and thus  $\Pi(\mathcal{E}_{\kappa(u-)}) = \Pi(\mathcal{E}_{\kappa(u)})$  (we omit a few details here). This gives the continuity of  $\Lambda^{\bullet,r}$ . The injectivity is a consequence of the special Markov property, which implies that  $D^\circ(\mathcal{E}_{\kappa(u)}, \mathcal{E}_{\kappa(v)}) > 0$  if  $0 \leq u < v \leq Z_r$ . Finally, to verify that the range of  $\Lambda^{\bullet,r}$  is  $\partial B_r^\bullet(\mathfrak{H})$ , we use again the support property of exit local times, noting that, for every  $i \in I$  such that  $t_i \notin [-\beta_r, \gamma_r]$ , and every  $s \in [0, \sigma(\omega^i)]$ , we have  $\tau_r(\omega_s^i) = \zeta_{(\omega_s^i)}$  if and only if the point  $a = p_{(\omega^i)}(s)$  satisfies  $\ell_a(\omega^i) = r$  and  $\ell_b(\omega^i) > r$  for any other point  $b$  of the ancestral line of  $a$  in  $\mathcal{T}_{(\omega^i)}$ .

We also observe that  $\{\Lambda^{\bullet,r}(u) : u \in (0, Z_r)\}$  does not intersect  $\partial\mathfrak{H}$  (the special Markov property implies that a point of the form  $\mathcal{E}_{\kappa(u)}$ ,  $u \in (0, Z_r)$ , cannot be in the equivalence class of  $\Lambda(t)$  for any  $t \in \mathbb{R}$ ). We will interpret  $Z_r$  as the perimeter of the hull  $B_r^\bullet(\mathfrak{H})$ .

By Jordan's theorem,  $B_r^\bullet(\mathfrak{H}) \setminus (\partial B_r^\bullet(\mathfrak{H}) \cup \Pi([-\beta_r, \gamma_r]))$  is homeomorphic to the open unit disk, and therefore path-connected. It follows that

$$B_r^\circ(\mathfrak{H}) := B_r^\bullet(\mathfrak{H}) \setminus \partial B_r^\bullet(\mathfrak{H})$$



is also path-connected. We shall be interested in the space

$$\mathfrak{H}_r := \mathfrak{H} \setminus B_r^\circ(\mathfrak{H}).$$

Again by Jordan's theorem,  $\mathfrak{H}_r$  is also homeomorphic to the half-plane, and we can define its boundary curve as follows. For every  $t \in \mathbb{R}$ , we set

$$\Lambda^r(t) := \begin{cases} \Lambda(t - \beta_r) & \text{if } t \leq 0, \\ \Lambda^{\bullet,r}(t) & \text{if } 0 \leq t \leq Z_r, \\ \Lambda(\gamma_r + (t - Z_r)) & \text{if } t \geq Z_r. \end{cases} \quad (9)$$

### 3.2 Peeling the Brownian half-plane

In this section, we present a version of the spatial Markov property for the (curve-decorated) Brownian half-plane. We again fix  $r > 0$ , and consider the space  $\mathfrak{H}_r$  defined in the previous section. We write  $V_r$  for the restriction of the volume measure  $V$  to  $\mathfrak{H}_r$ , and also recall the definition of the boundary curve  $\Lambda^r$ . Finally, we write  $D_r(x, y)$  for the intrinsic metric on the interior  $\mathfrak{H} \setminus B_r^\bullet(\mathfrak{H})$  of  $\mathfrak{H}_r$ : for every  $x, y \in \mathfrak{H} \setminus B_r^\bullet(\mathfrak{H})$ ,  $D_r(x, y)$  is the infimum of lengths of curves connecting  $x$  to  $y$  that stay in  $\mathfrak{H} \setminus B_r^\bullet(\mathfrak{H})$  (lengths of course refer to the distance  $D$  on  $\mathfrak{H}$ ).

**Theorem 1.** *The intrinsic metric  $D_r$  on  $\mathfrak{H} \setminus B_r^\bullet(\mathfrak{H})$  has a continuous extension to  $\mathfrak{H}_r$  which is a metric on  $\mathfrak{H}_r$ , and we keep the notation  $D_r$  for this metric. Then the (random) curve-decorated measure metric space  $(\mathfrak{H}_r, D_r, V_r, \Lambda^r)$  is a curve-decorated Brownian half-plane.*

This theorem is an analog of Theorem 22 in [25], which deals with the Brownian disk, and in fact we will use a coupling between the (curve-decorated) Brownian disk and the curve-decorated Brownian half-plane to reduce the proof to this statement.

*Proof.* By a scaling argument, it is enough to consider the case  $r = 1$ . For every  $S > 0$ , write  $(\mathbb{D}_{(S)}, D_{(S)}, V_{(S)}, \Lambda_{(S)})$  for a curve-decorated free Brownian disk with boundary size  $S$  as defined in [25, Section 4.1] (see also [13]). In view of the coupling with the Brownian half-plane, it will be convenient to make the convention that the decorating curve  $\Lambda_{(S)}$  is indexed by the interval  $[-S/2, S/2]$  instead of  $[0, S]$ : If  $\Lambda_{(S)}^\circ$  is the usual decorating curve indexed by  $[0, S]$ , this simply means that we take  $\Lambda_{(S)}(t) = \Lambda_{(S)}^\circ(t)$  for  $t \in [0, S/2]$  and  $\Lambda_{(S)}(t) = \Lambda_{(S)}^\circ(S + t)$  for  $t \in [-S/2, 0]$ . We will make this convention whenever we consider the curve-decorated free Brownian disk. We note that  $\Lambda_{(S)}(-S/2) = \Lambda_{(S)}(S/2)$  and that  $\Lambda_{(S)}$  is a standard boundary curve, meaning that the pushforward of Lebesgue measure on  $[-S/2, S/2]$  under  $\Lambda_{(S)}$  is the uniform measure on  $\partial\mathbb{D}_{(S)}$  (see [25]). The range of  $\Lambda_{(S)}$  is the boundary  $\partial\mathbb{D}_{(S)}$ , and the distinguished point of  $\mathbb{D}_{(S)}$  is  $\mathbf{x}_{(S)} := \Lambda_{(S)}(0)$ .

For every  $a > 0$  such that  $a < D_{(S)}(\mathbf{x}_{(S)}, \Lambda_{(S)}(S/2))$ , let  $B_a^\bullet(\mathbb{D}_{(S)})$  stand for the hull of radius  $a$  in  $\mathbb{D}_{(S)}$  relative to the point  $\Lambda_{(S)}(S/2) = \Lambda_{(S)}(-S/2)$ . This means that  $\mathbb{D}_{(S)} \setminus B_a^\bullet(\mathbb{D}_{(S)})$  is the connected component containing  $\Lambda_{(S)}(S/2)$  of the complement of the closed ball of radius  $a$  centered at  $\mathbf{x}_{(S)}$ . By convention, if  $a \geq D_{(S)}(\mathbf{x}_{(S)}, \Lambda_{(S)}(S/2))$ , we take  $B_a^\bullet(\mathbb{D}_{(S)}) = \mathbb{D}_{(S)}$ .

Let  $\delta \in (0, 1)$  and  $A > 10$ . By [13, Proposition 4.2] (see also [3, Corollary 3.9] or [21, Lemma 18]) and a scaling argument, we can find  $S_0 > 0$  such that, for every  $S \geq S_0$ , we can couple the curve-decorated measure metric spaces  $(\mathbb{D}_{(S)}, D_{(S)}, V_{(S)}, \Lambda_{(S)})$  and  $(\mathfrak{H}, D, V, \Lambda)$  in such a way that the following holds with probability at least  $1 - \delta$ :  $B_A^\bullet(\mathbb{D}_{(S)}) \neq \mathbb{D}_{(S)}$  and there is a measure-preserving isometry  $\mathfrak{J}$  from  $B_A^\bullet(\mathbb{D}_{(S)})$  onto  $B_A^\bullet(\mathfrak{H})$  such that

$$\mathfrak{J}(\Lambda_{(S)}(t)) = \Lambda(t), \text{ for } \sup\{s \leq 0 : \Lambda_{(S)}(s) \notin B_A^\bullet(\mathbb{D}_{(S)})\} \leq t \leq \inf\{s \geq 0 : \Lambda_{(S)}(s) \notin B_A^\bullet(\mathbb{D}_{(S)})\}. \quad (10)$$

The preceding properties imply that the isometry  $\mathfrak{J}$  maps  $B_1^\bullet(\mathbb{D}_{(S)})$  onto  $B_1^\bullet(\mathfrak{H})$  and  $\partial B_1^\bullet(\mathbb{D}_{(S)})$  onto  $\partial B_1^\bullet(\mathfrak{H})$ .

Note that [13, Proposition 4.2] deals with Brownian disks with a fixed volume, but the result clearly holds also for free Brownian disks. Moreover, [13] considers balls instead of hulls, but it is easy to verify that balls can be replaced by hulls (notice that, both in  $\mathbb{H}$  and in  $\mathbb{D}_{(S)}$ , the probability that the hull of radius  $\varepsilon$  is contained in the ball of radius 1 centered at the distinguished point tends to 1 when  $\varepsilon \rightarrow 0$ , and use again a scaling argument).

The existence of the preceding coupling allows us to transfer properties valid in the Brownian disk to the curve-decorated Brownian half-plane. Let  $\tilde{\mathbb{D}}_{(S)}$  be the closure of  $\mathbb{D}_{(S)} \setminus B_1^\bullet(\mathbb{D}_{(S)})$ , and write  $\tilde{D}_{(S)}$  for the intrinsic metric on  $\mathbb{D}_{(S)} \setminus B_1^\bullet(\mathbb{D}_{(S)})$ . According to [25, Theorem 22],  $\tilde{D}_{(S)}$  has a continuous extension to  $\tilde{\mathbb{D}}_{(S)}$ , which is a metric on  $\tilde{\mathbb{D}}_{(S)}$ . In order to get that the intrinsic metric  $D_1$  on  $\mathfrak{H} \setminus B_1^\bullet(\mathfrak{H})$  has a continuous extension to  $\mathfrak{H}_1$ , we need to verify that  $D_1(x_n, x_m)$  tends to 0 as  $n, m \rightarrow \infty$ , for any sequence  $(x_n)$  in  $\mathfrak{H} \setminus B_1^\bullet(\mathfrak{H})$  that converges to a point of  $\partial B_1^\bullet(\mathfrak{H})$ . However, except on an event of probability at most  $\delta$ , this follows from the preceding coupling and the fact that  $\tilde{D}_{(S)}(y_n, y_m)$  tends to 0 as  $n, m \rightarrow \infty$ , for any sequence  $(y_n)$  in  $\mathbb{D}_{(S)} \setminus B_1^\bullet(\mathbb{D}_{(S)})$  that converges to a point of  $\partial B_1^\bullet(\mathbb{D}_{(S)})$ . Similarly, we get the fact that the extension of  $D_1$  to  $\mathfrak{H}_1$  is a metric on  $\mathfrak{H}_1$  from the corresponding statement in [25, Theorem 22].

From now on, we argue on the event of probability at least  $1 - \delta$  on which one can define the isometry  $\mathfrak{I}$ . Recall the definition of  $\beta_1$  in the previous section, and note that

$$-\beta_1 = \sup\{s \leq 0 : \Lambda(s) \notin B_1^\bullet(\mathfrak{H})\} = \sup\{s \in [-S/2, 0] : \Lambda_{(S)}(s) \notin B_1^\bullet(\mathbb{D}_{(S)})\},$$

where the second equality follows from (10) and the fact that  $\mathfrak{I}$  maps  $B_1^\bullet(\mathbb{D}_{(S)})$  to  $B_1^\bullet(\mathfrak{H})$ . We then define  $\tilde{\mathbf{x}}_{(S)} = \Lambda_{(S)}(-\beta_1)$ , and note that  $\mathfrak{I}(\tilde{\mathbf{x}}_{(S)}) = \mathbf{x}_1 := \Lambda(-\beta_1) = \Lambda^1(0)$ , where  $\Lambda^1$  was defined in (9).

By [25, Theorem 22], we know that  $(\tilde{\mathbb{D}}_{(S)}, \tilde{D}_{(S)})$  equipped with the restriction  $\tilde{V}_{(S)}$  of the volume measure on  $\mathbb{D}_{(S)}$  and the distinguished point  $\tilde{\mathbf{x}}_{(S)}$  is a free Brownian disk with a random boundary size denoted by  $\tilde{Z}_{(S)}$ , which is pointed at a uniform boundary point. In fact, as discussed at the end of [25, Section 4.2], we can also equip  $\tilde{\mathbb{D}}_{(S)}$  with a standard boundary curve  $(\tilde{\Lambda}_{(S)}(t), t \in [-\tilde{Z}_{(S)}/2, \tilde{Z}_{(S)}/2])$ , in such a way that we have in particular  $\tilde{\Lambda}_{(S)}(0) = \tilde{\mathbf{x}}_{(S)}$  and  $\tilde{\Lambda}_{(S)}(t) = \Lambda_{(S)}(-\beta_1 + t)$  for every  $t \in [-(S/2) + \beta_1, 0]$  such that  $t \geq -\tilde{Z}_{(S)}/2$  — note that our convention to index  $\tilde{\Lambda}^{(S)}$  by the interval  $[-\tilde{Z}_{(S)}/2, \tilde{Z}_{(S)}/2]$  makes the definition of  $\tilde{\Lambda}_{(S)}$  look different than in [25]. Then, conditionally on  $\tilde{Z}_{(S)}$ , the 4-tuple  $(\tilde{\mathbb{D}}_{(S)}, \tilde{D}_{(S)}, \tilde{V}_{(S)}, \tilde{\Lambda}_{(S)})$  is a curve-decorated free Brownian disk with boundary size  $\tilde{Z}_{(S)}$ . It is easy to verify that the boundary size  $\tilde{Z}_{(S)}$  tends to  $\infty$  in probability as  $S \rightarrow \infty$  (for instance, because  $\tilde{Z}_{(S)} \geq (S/2) - \beta_1$  on the event that we are considering).

Next set  $a = (A/2) - 1$ , and write  $B_a(\tilde{\mathbb{D}}_{(S)})$  for the closed ball of radius  $a$  centered at  $\tilde{\mathbf{x}}_{(S)}$  in  $\tilde{\mathbb{D}}_{(S)}$ . Note that  $B_a(\tilde{\mathbb{D}}_{(S)}) \subset B_{A/2}(\mathbb{D}_{(S)}) \subset B_{A/2}^\bullet(\mathbb{D}_{(S)})$  (the first inclusion because  $D_{(S)}(\tilde{\mathbf{x}}_{(S)}, \mathbf{x}_{(S)}) = 1$ ). If  $x$  and  $y$  are two points in  $B_a(\tilde{\mathbb{D}}_{(S)}) \setminus \partial \tilde{\mathbb{D}}_{(S)}$  and  $x' = \mathfrak{I}(x)$  and  $y' = \mathfrak{I}(y)$  are the corresponding points of  $\mathfrak{H} \setminus B_1^\bullet(\mathfrak{H})$ , the intrinsic distance  $\tilde{D}_{(S)}(x, y)$  must coincide with  $D_1(x', y')$  — the point is that a curve from  $x'$  to  $y'$  that exits  $B_1^\bullet(\mathfrak{H})$  must have length greater than  $A$ , and thus can be disregarded when computing the intrinsic distance between  $x'$  and  $y'$  (we know that the latter distance is bounded by  $2a \leq A - 2$ , because both  $x'$  and  $y'$  are at distance at most  $a$  from  $\mathbf{x}_1$ ). If  $B_a(\mathfrak{H}_1)$  denotes the closed ball of radius  $a$  centered at  $\mathbf{x}_1$  in  $\mathfrak{H}_1$ , we thus get that  $\mathfrak{I}$  induces an isometry from  $B_a(\tilde{\mathbb{D}}_{(S)})$  onto  $B_a(\mathfrak{H}_1)$ , and this isometry preserves the volume measures. Furthermore, as in Section 2.1, we can also introduce the curve-decorated spaces  $\mathfrak{B}_a(\tilde{\mathbb{D}}_{(S)})$  and  $\mathfrak{B}_a(\mathfrak{H}_1)$  associated with  $B_a(\tilde{\mathbb{D}}_{(S)})$  and  $B_a(\mathfrak{H}_1)$  respectively, so that, in particular, the decorating curve of  $\mathfrak{B}_a(\tilde{\mathbb{D}}_{(S)})$  is

$$\left( \tilde{\Lambda}_{(S)}(t) : -a \vee \sup\{s \leq 0 : \tilde{\Lambda}_{(S)}(s) \notin B_a(\tilde{\mathbb{D}}_{(S)})\} \leq t \leq a \wedge \inf\{s \geq 0 : \tilde{\Lambda}_{(S)}(s) \notin B_a(\tilde{\mathbb{D}}_{(S)})\} \right),$$

and the decorating curve of  $\mathfrak{B}_a(\mathfrak{H}_1)$  is

$$\left( \Lambda^1(t) : -a \vee \sup\{s \leq 0 : \Lambda^1(s) \notin B_a(\mathfrak{H}_1)\} \leq t \leq a \wedge \inf\{s \geq 0 : \Lambda^1(s) \notin B_a(\mathfrak{H}_1)\} \right).$$

One then verifies that, except on a set of small probability when  $S \rightarrow \infty$ , these two curves are defined on the same interval, and the isometry  $\mathfrak{I}$  maps the first one to the second one (we omit a few details here).

In conclusion, one can couple  $\mathfrak{H}_1$  and the Brownian disk  $\tilde{\mathbb{D}}_{(S)}$  in such a way that the balls  $\mathfrak{B}_a(\tilde{\mathbb{D}}_{(S)})$  and  $\mathfrak{B}_a(\mathfrak{H}_1)$  coincide except on an event of arbitrarily small probability when  $S$  is large. Recalling the coupling of [13, Proposition 4.2] used in the proof, this also means that we can couple  $\mathfrak{H}_1$  with a curve-decorated Brownian half-plane  $\mathfrak{H}'$  in such a way that the balls of radius  $a$  (again viewed

as random curve-decorated measure metric spaces) in both spaces coincide except on an event of arbitrarily small probability. Since this holds for any  $a > 0$ , this suffices to prove that  $(\mathfrak{H}_1, D_1, V_1, \Lambda^1)$  is a curve-decorated Brownian half-plane.  $\square$

We will now show that the space  $(\mathfrak{H}_r, D_r, V_r, \Lambda^r)$  in Theorem 1 is independent of the hull  $B_r^\bullet(\mathfrak{H})$  also viewed as a curve-decorated measure metric space. We first need to introduce the appropriate metric on  $B_r^\bullet(\mathfrak{H})$ . We consider the subset  $\mathcal{K}_r$  of  $\mathfrak{T}$  defined by

$$\mathcal{K}_r = [-\beta_r, \gamma_r] \cup \left( \bigcup_{i \in I: t_i \in [-\beta_r, \gamma_r]} \mathcal{T}_{(\omega^i)} \right) \cup \left( \bigcup_{i \in I: t_i \notin [-\beta_r, \gamma_r]} \{a \in \mathcal{T}_{(\omega^i)} : m_a \leq r\} \right), \quad (11)$$

where we recall that  $m_a$  stands for the minimal label of  $a$  along the geodesic path from  $a$  to  $\infty$  in  $\mathfrak{T}$  that does not contain 0. It follows from (6) that  $B_r^\bullet(\mathfrak{H}) = \Pi(\mathcal{K}_r)$ .

We mention the following simple fact. Let  $a, b \in \mathcal{K}_r$ . Then, in formula (4) defining  $D^\circ(a, b)$ , we may replace the intervals  $[a, b]_\infty$  and  $[b, a]_\infty$  by  $[a, b]_\infty \cap \mathcal{K}_r$  and  $[b, a]_\infty \cap \mathcal{K}_r$  respectively: the point is that, if the interval  $[a, b]_\infty$  contains a point  $c \notin \mathcal{K}_r$ , then, necessarily, it contains another point  $c'$  belonging to  $\mathcal{K}_r$  whose label is  $r$  and is thus smaller than the label of  $c$ . Informally, the definition of  $D^\circ(a, b)$ , when  $a, b \in \mathcal{K}_r$  only depends on the labels on  $\mathcal{K}_r$ , despite the fact that the interval  $[a, b]_\infty$  may not be contained in  $\mathcal{K}_r$ .

For every  $a, b \in \mathcal{K}_r$ , we set

$$D_r^\bullet(a, b) := \inf_{\substack{a_0, a_1, \dots, a_p \in \mathcal{K}_r \\ a_0 = a, a_p = b}} \sum_{i=1}^p D^\circ(a_{i-1}, a_i), \quad (12)$$

where the infimum is over all choices of the integer  $p \geq 1$  and of the finite sequence  $a_0, a_1, \dots, a_p$  in  $\mathcal{K}_r$  such that  $a_0 = a$  and  $a_p = b$ . This is similar to the definition (4) of  $D(a, b)$ , but we restrict the infimum to “intermediate” points  $a_1, \dots, a_{p-1}$  that belong to  $\mathcal{K}_r$ . Clearly, we have  $D(a, b) \leq D_r^\bullet(a, b) \leq D^\circ(a, b)$  for every  $a, b \in \mathcal{K}_r$ . Since the condition  $D(a, b) = 0$  can only hold if  $D^\circ(a, b) = 0$ , we get that, for every  $a, b \in \mathcal{K}_r$ , we have  $D_r^\bullet(a, b) = 0$  if and only if  $D(a, b) = 0$ . Hence  $D_r^\bullet$  induces a metric on  $\Pi(\mathcal{K}_r) = B_r^\bullet(\mathfrak{H})$  and we keep the notation  $D_r^\bullet$  for this metric.

Recall that  $B_r^\circ(\mathfrak{H}) = B_r^\bullet(\mathfrak{H}) \setminus \partial B_r^\bullet(\mathfrak{H})$  is the (topological) interior of  $B_r^\bullet(\mathfrak{H})$ . Since  $B_r^\circ(\mathfrak{H})$  is path-connected, we can define an intrinsic metric on  $B_r^\circ(\mathfrak{H})$ . One can then verify that the restriction of  $D_r^\bullet$  to  $B_r^\circ(\mathfrak{H})$  coincides with the intrinsic distance induced by  $D$  on  $B_r^\circ(\mathfrak{H})$ . We omit the details but refer to the proof of Proposition 6 in [25] for very similar arguments.

We finally define a boundary curve for  $B_r^\bullet(\mathfrak{H})$ . Recall from the previous section the definition of the curve  $\Lambda^{\bullet, r} = (\Lambda^{\bullet, r}(u))_{u \in (0, Z_r)}$  whose range is  $\partial B_r^\bullet(\mathfrak{H})$ . We set, for every  $t \in [0, \gamma_r + \beta_r + Z_r]$ ,

$$\widehat{\Lambda}^{\bullet, r}(t) = \begin{cases} \Lambda(-t) & \text{if } 0 \leq t \leq \beta_r, \\ \Lambda^{\bullet, r}(t - \beta_r) & \text{if } \beta_r \leq t \leq \beta_r + Z_r, \\ \Lambda(\gamma_r + \beta_r + Z_r - t) & \text{if } \beta_r + Z_r \leq t \leq \gamma_r + \beta_r + Z_r. \end{cases}$$

We let  $V_r^\bullet$  denote the restriction of  $V$  to  $B_r^\bullet(\mathfrak{H})$ .

**Theorem 2.** *The random curve-decorated measure metric spaces  $(B_r^\bullet(\mathfrak{H}), D_r^\bullet, V_r^\bullet, \widehat{\Lambda}^{\bullet, r})$  and  $(\mathfrak{H}_r, D_r, V_r, \Lambda^r)$  are independent.*

*Proof.* Again, we may take  $r = 1$ . We will derive Theorem 2 from [25, Theorem 23] by the same coupling argument that we used to prove Theorem 1, and we keep the notation of this proof. In particular, we consider the hull  $B_1^\bullet(\mathbb{D}_{(S)})$  on the event  $\{D_{(S)}(\mathbf{x}_{(S)}, \Lambda_{(S)}(S/2)) > 1\}$ . As it is explained in [25, Section 6.3], we can equip  $B_1^\bullet(\mathbb{D}_{(S)})$  with an (extended) intrinsic metric defined in a way similar to the metric  $D_r^\bullet$  on  $B_r^\bullet(\mathfrak{H})$  and with the restriction of the volume measure on  $\mathbb{D}_{(S)}$ , and we can also define a decorating curve on  $B_1^\bullet(\mathbb{D}_{(S)})$ , which is analogous to  $\widehat{\Lambda}^{\bullet, r}$  (see the discussion before [25, Theorem 23] for more details). We write  $\mathfrak{B}_1^\bullet(\mathbb{D}_{(S)})$  for the resulting curve-decorated measure metric space. Similarly, we write  $\mathfrak{B}_1^\bullet(\mathfrak{H})$  for the curve-decorated measure metric space  $(B_1^\bullet(\mathfrak{H}), D_1^\bullet, V_1^\bullet, \widehat{\Lambda}^{\bullet, r})$ .

Let  $A > 10$  and  $a \in (0, (A/2) - 1)$ . Also fix  $\delta > 0$ . As in the proof of Theorem 1, for every large enough  $S$ , we can couple  $(\mathbb{D}_{(S)}, D_{(S)}, V_{(S)}, \Lambda_{(S)})$  and  $(\mathfrak{H}, D, V, \Lambda)$  in such a way that, except on a set

of probability at most  $\delta$ , there is a measure-preserving isometry  $\mathfrak{J}$  from  $B_A^\bullet(\mathbb{D}_{(S)})$  onto  $B_A^\bullet(\mathfrak{H})$  such that (10) holds. Then it is not hard to verify that  $\mathfrak{J}$  induces an isometry from  $\mathfrak{B}_1^\bullet(\mathbb{D}_{(S)})$  onto  $\mathfrak{B}_1^\bullet(\mathfrak{H})$ , which preserves the volume measures and the decorating curves. As in the proof of Theorem 1, we also know that, except on a set of small probability when  $S$  is large,  $\mathfrak{J}$  induces an isometry from  $\mathfrak{B}_a(\tilde{\mathbb{D}}_{(S)})$  onto  $\mathfrak{B}_a(\mathfrak{H}_1)$ . Summarizing, if  $S$  is large enough, except on event of small probability, we can couple  $(\mathbb{D}_{(S)}, D_{(S)}, V_{(S)}, \Lambda_{(S)})$  and  $(\mathfrak{H}, D, V, \Lambda)$  so that  $(\mathfrak{B}_1^\bullet(\mathbb{D}_{(S)}), \mathfrak{B}_a(\tilde{\mathbb{D}}_{(S)})) = (\mathfrak{B}_1^\bullet(\mathfrak{H}), \mathfrak{B}_a(\mathfrak{H}_1))$ , where the equality is in the sense of isometry between curve-decorated measure metric spaces.

Let  $F$  and  $G$  be two bounded measurable functions defined on the space  $\mathbb{M}^{\text{GHPU}}$  of all curve-decorated compact measure metric spaces. It follows from the preceding considerations that

$$\left| \mathbb{E} \left[ F(\mathfrak{B}_a(\tilde{\mathbb{D}}_{(S)})) G(\mathfrak{B}_1^\bullet(\mathbb{D}_{(S)})) \mathbf{1}_{\{D_{(S)}(\mathbf{x}_{(S)}, \Lambda_{(S)}(S/2)) > 1\}} \right] - \mathbb{E} \left[ F(\mathfrak{B}_a(\mathfrak{H}_1)) G(\mathfrak{B}_1^\bullet(\mathfrak{H})) \right] \right| \xrightarrow{S \rightarrow \infty} 0. \quad (13)$$

On the other hand, Theorems 22 and 23 in [25] imply that

$$\mathbb{E} \left[ F(\tilde{\mathbb{D}}_{(S)}) G(\mathfrak{B}_1^\bullet(\mathbb{D}_{(S)})) \mathbf{1}_{\{D_{(S)}(\mathbf{x}_{(S)}, \Lambda_{(S)}(S/2)) > 1\}} \right] = \mathbb{E} \left[ \Theta_{\tilde{Z}_{(S)}}(F) G(\mathfrak{B}_1^\bullet(\mathbb{D}_{(S)})) \mathbf{1}_{\{D_{(S)}(\mathbf{x}_{(S)}, \Lambda_{(S)}(S/2)) > 1\}} \right]$$

where  $\tilde{\mathbb{D}}_{(S)}$  is also viewed as a curve-decorated measure metric space (as in the proof of Theorem 1) and we write  $\Theta_z$  for the distribution of the (curve-decorated) free Brownian disk with perimeter  $z$ . We can specialize this equality to the case where  $F$  only depends on the ball of radius  $a$ . It follows that

$$\begin{aligned} & \mathbb{E} \left[ F(\mathfrak{B}_a(\tilde{\mathbb{D}}_{(S)})) G(\mathfrak{B}_1^\bullet(\mathbb{D}_{(S)})) \mathbf{1}_{\{D_{(S)}(\mathbf{x}_{(S)}, \Lambda_{(S)}(S/2)) > 1\}} \right] \\ &= \mathbb{E} \left[ \Theta_{\tilde{Z}_{(S)}}(F \circ \mathfrak{B}_a) G(\mathfrak{B}_1^\bullet(\mathbb{D}_{(S)})) \mathbf{1}_{\{D_{(S)}(\mathbf{x}_{(S)}, \Lambda_{(S)}(S/2)) > 1\}} \right]. \end{aligned}$$

The coupling between the curve-decorated Brownian half-plane and the free Brownian disk of perimeter  $z$  ensures that

$$\Theta_z(F \circ \mathfrak{B}_a) \xrightarrow{z \rightarrow \infty} \Theta_\infty(F \circ \mathfrak{B}_a),$$

where  $\Theta_\infty$  is the distribution of the curve-decorated Brownian half-plane. Since  $\tilde{Z}_{(S)}$  tends to  $\infty$  as  $S \rightarrow \infty$ , the last two displays imply that

$$\begin{aligned} & \left| \mathbb{E} \left[ F(\mathfrak{B}_a(\tilde{\mathbb{D}}_{(S)})) G(\mathfrak{B}_1^\bullet(\mathbb{D}_{(S)})) \mathbf{1}_{\{D_{(S)}(\mathbf{x}_{(S)}, \Lambda_{(S)}(S/2)) > 1\}} \right] \right. \\ & \left. - \Theta_\infty(F \circ \mathfrak{B}_a) \mathbb{E} \left[ G(\mathfrak{B}_1^\bullet(\mathbb{D}_{(S)})) \mathbf{1}_{\{D_{(S)}(\mathbf{x}_{(S)}, \Lambda_{(S)}(S/2)) > 1\}} \right] \right| \xrightarrow{S \rightarrow \infty} 0. \end{aligned} \quad (14)$$

Using (13) twice (the second time with  $F = 1$ ), we deduce from (14) that

$$\mathbb{E} \left[ F(\mathfrak{B}_a(\mathfrak{H}_1)) G(\mathfrak{B}_1^\bullet(\mathfrak{H})) \right] = \Theta_\infty(F \circ \mathfrak{B}_a) \mathbb{E} \left[ G(\mathfrak{B}_1^\bullet(\mathfrak{H})) \right].$$

This gives the desired independence property.  $\square$

## 4 Some explicit formulas

In this section, we provide explicit formulas for the joint distribution of the variables  $\beta_r, \gamma_r, Z_r$ , which determine the boundary size of the hull  $B_r^\bullet(\mathfrak{H})$ , and of the volume of the hull  $B_r^\bullet(\mathfrak{H})$  (see Fig. 2). To simplify notation, we write  $\mathcal{V}_r := V(B_r^\bullet(\mathfrak{H}))$  for the latter volume.

**Proposition 3.** *The random variables  $\beta_r, \gamma_r$  and  $Z_r$  are independent. Moreover,  $\beta_r$  and  $\gamma_r$  have the same distribution, whose density is given by*

$$t \mapsto \frac{3^{-3/2} r^3}{\sqrt{2\pi}} t^{-5/2} \exp\left(-\frac{r^2}{6t}\right),$$

and  $Z_r$  is exponentially distributed with mean  $2r^2/3$ . The joint distribution of  $(\beta_r, \gamma_r, Z_r, \mathcal{V}_r)$  is given by the formula

$$\mathbb{E} \left[ \exp\left(-\lambda Z_r - \nu_1 \beta_r - \nu_2 \gamma_r - \mu \mathcal{V}_r\right) \right] = \frac{G_r(\mu, \nu_1) G_r(\mu, \nu_2)}{\frac{2}{3} \lambda r^2 + \sqrt{2\mu} r^2 \left(\coth\left((2\mu)^{\frac{1}{4}} r\right)^2 - \frac{2}{3}\right)}, \quad (15)$$

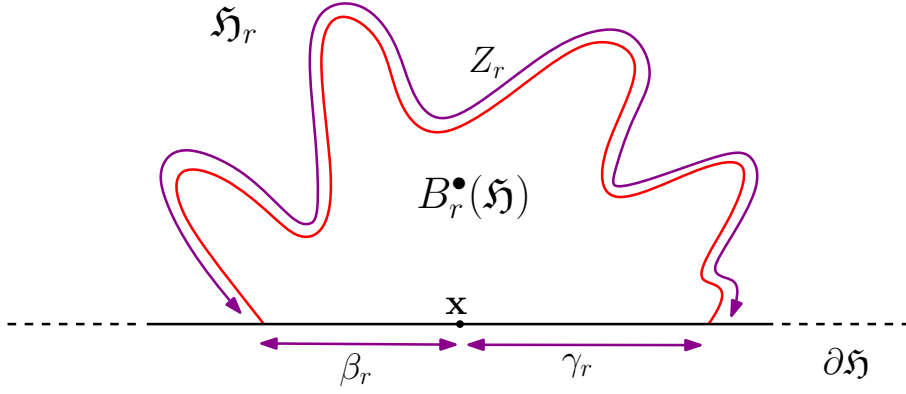


Figure 2: Illustration of the boundary lengths  $\beta_r$ ,  $\gamma_r$  and  $Z_r$ .

where  $G_r$  is the function:

$$G_r(\mu, \nu) := r \exp\left(-r \sqrt{\frac{2}{3}} \sqrt{\sqrt{2\mu} + \nu}\right) \times \left((2\mu)^{\frac{1}{4}} \coth((2\mu)^{\frac{1}{4}} r) + \sqrt{\frac{2}{3}} \sqrt{\sqrt{2\mu} + \nu}\right). \quad (16)$$

*Proof.* Recall the definition of the variables  $Z'_r$  and  $Z''_r$  in (8), such that  $Z'_r + Z''_r = Z_r$ . From the independence of the processes  $(\mathcal{R}_t)_{t \leq 0}$  and  $(\mathcal{R}_t)_{t \geq 0}$  one immediately gets that the pairs  $(\beta_r, Z'_r)$  and  $(\gamma_r, Z''_r)$  are independent and identically distributed. Furthermore, using the well-known fact that  $(\mathcal{R}_{\gamma_r+t})_{t \geq 0}$  is independent of  $\gamma_r$ , one obtains that  $Z''_r$  is independent of  $\gamma_r$ , and similarly  $Z'_r$  is independent of  $\beta_r$ . This discussion shows that  $\beta_r, \gamma_r$  and  $Z_r$  are independent.

By definition,  $\gamma_r$  and  $\beta_r$  are distributed as the last hitting time of  $r/\sqrt{3}$  by a five-dimensional Bessel process started from 0, and their density is well known [12] to be as stated in the proposition. We note that the Laplace transform of  $\gamma_r$  (or  $\beta_r$ ) is given by

$$\mathbb{E}[e^{-\lambda\gamma_r}] = \left(1 + r \sqrt{\frac{2\lambda}{3}}\right) \exp\left(-r \sqrt{\frac{2\lambda}{3}}\right).$$

Let us turn to the distribution of  $Z_r$ . Since, conditionally on  $\mathcal{R}$ , the point measure  $\mathcal{N}(dt d\omega)$  is Poisson with intensity  $2 \mathbf{1}_{\{W_*(\omega) > 0\}} dt \mathbb{N}_{\sqrt{3}\mathcal{R}_t}(d\omega)$ , we have

$$\mathbb{E}[e^{-\lambda Z''_r}] = \mathbb{E}\left[\exp\left(-2 \int_{\gamma_r}^{\infty} dt \mathbb{N}_{\sqrt{3}\mathcal{R}_t}\left((1 - e^{-\lambda Z_r}) \mathbf{1}_{\{W_* > 0\}}\right)\right)\right].$$

From [8, Lemma 4.1], we have, for every  $x > r$ ,

$$\mathbb{N}_x\left((1 - e^{-\lambda Z_r}) \mathbf{1}_{\{W_* > 0\}}\right) = \frac{3}{2} \left(\left(x - r + \left(\frac{2\lambda}{3} + r^{-2}\right)^{-1/2}\right)^{-2} - x^{-2}\right). \quad (17)$$

It follows, that, for  $t > \gamma_r = \sup\{s \geq 0 : \mathcal{R}_s \leq r/\sqrt{3}\}$ ,

$$\mathbb{N}_{\sqrt{3}\mathcal{R}_t}\left((1 - e^{-\lambda Z_r}) \mathbf{1}_{\{W_* > 0\}}\right) = \frac{1}{2} \left((\mathcal{R}_t - b)^{-2} - (\mathcal{R}_t)^{-2}\right), \quad (18)$$

where we have set

$$b = \frac{r}{\sqrt{3}} - (2\lambda + 3r^{-2})^{-1/2} \in \left(0, \frac{r}{\sqrt{3}}\right).$$

**Lemma 4.** For every  $x > 0$ , set  $\mathcal{L}_x := \sup\{t \geq 0 : \mathcal{R}_t \leq x\}$ . Then, for every  $c < x < y$ ,

$$\mathbb{E}\left[\exp\left(-\int_{\mathcal{L}_x}^{\mathcal{L}_y} dt \left((\mathcal{R}_t - c)^{-2} - (\mathcal{R}_t)^{-2}\right)\right)\right] = \frac{y}{x} \times \frac{x - c}{y - c}.$$

The proof is analogous to the proof of Lemma 4.2 in [8] and is deferred to the Appendix. We apply Lemma 4 with  $x = r/\sqrt{3}$ ,  $c = b$  and taking limits as  $y \rightarrow \infty$ . It follows that

$$\mathbb{E}[e^{-\lambda Z''_r}] = \frac{(r/\sqrt{3}) - b}{r/\sqrt{3}} = \left(1 + \frac{2\lambda r^2}{3}\right)^{-1/2}.$$

Since  $Z_r = Z'_r + Z''_r$ , and  $Z'_r$  and  $Z''_r$  are independent and identically distributed, we finally get

$$\mathbb{E}[e^{-\lambda Z_r}] = \left(1 + \frac{2\lambda r^2}{3}\right)^{-1},$$

which gives the desired distribution of  $Z_r$ .

Let us turn to the proof of (15). If  $w$  is a stopped path, we write  $\min(w) := \min\{w(t) : 0 \leq t \leq \zeta(w)\}$ . Then, from formula (11) and the definition of the volume measure, we get that

$$\mathcal{V}_r = \mathcal{V}_r^0 + \mathcal{V}_r^1 + \mathcal{V}_r^2,$$

where

$$\begin{aligned} \mathcal{V}_r^0 &:= \sum_{i \in I: t_i \in (-\infty, -\beta_r) \cup (\gamma_r, \infty)} \int_0^{\sigma(\omega^i)} ds \mathbf{1}_{\{\min(\omega_s^i) \leq r\}}, \\ \mathcal{V}_r^1 &:= \sum_{i \in I: t_i \in [-\beta_r, 0]} \sigma(\omega^i), \\ \mathcal{V}_r^2 &:= \sum_{i \in I: t_i \in [0, \gamma_r]} \sigma(\omega^i). \end{aligned}$$

Using the independence of the processes  $(\mathcal{R}_t)_{t \geq 0}$  and  $(\mathcal{R}_t)_{t \leq 0}$ , the fact that  $(\mathcal{R}_{\gamma_r+t})_{t \geq 0}$  is independent of  $(\mathcal{R}_t)_{0 \leq t \leq \gamma_r}$  (and the analogous property for  $(\mathcal{R}_t)_{t \leq 0}$ ) and properties of Poisson measures, one immediately verifies that the three pairs  $(Z_r, \mathcal{V}_r^0)$ ,  $(\beta_r, \mathcal{V}_r^1)$  and  $(\gamma_r, \mathcal{V}_r^2)$  are independent, and moreover the pairs  $(\beta_r, \mathcal{V}_r^1)$  and  $(\gamma_r, \mathcal{V}_r^2)$  have the same distribution.

Let us start by discussing the pair  $(\gamma_r, \mathcal{V}_r^2)$ . For every  $\mu > 0$ , we have

$$\mathbb{E}\left[\exp(-\mu \mathcal{V}_r^2) \mid (\mathcal{R}_t)_{t \geq 0}\right] = \exp\left(-2 \int_0^{\gamma_r} dt \mathbb{N}_{\sqrt{3}\mathcal{R}_t}\left((1 - e^{-\mu\sigma}) \mathbf{1}_{\{W_* > 0\}}\right)\right).$$

For every  $x > 0$ , set

$$g_\mu(x) = \mathbb{N}_x\left((1 - e^{-\mu\sigma}) \mathbf{1}_{\{W_* > 0\}}\right) = \mathbb{N}_x\left(1 - \mathbf{1}_{\{W_* > 0\}} e^{-\mu\sigma}\right) - \frac{3}{2x^2}.$$

By [10, Lemma 7], we have

$$g_\mu(x) = \sqrt{\frac{\mu}{2}} \left(3 \coth\left((2\mu)^{1/4} x\right)^2 - 2\right) - \frac{3}{2x^2}.$$

**Lemma 5.** *For every  $\mu, \nu > 0$ , we have*

$$\mathbb{E}\left[\exp\left(-\nu\gamma_r - 2 \int_0^{\gamma_r} dt g_\mu(\sqrt{3}\mathcal{R}_t)\right)\right] = G_r(\mu, \nu),$$

where the function  $G_r(\mu, \nu)$  is defined in (16).

We postpone the proof of Lemma 5 until the Appendix and complete the proof of Proposition 3. It readily follows from Lemma 5 that

$$\mathbb{E}[\exp(-\nu\gamma_r - \mu\mathcal{V}_r^2)] = \mathbb{E}\left[\exp\left(-\nu\gamma_r - 2 \int_0^{\gamma_r} dt g_\mu(\sqrt{3}\mathcal{R}_t)\right)\right] = G_r(\mu, \nu). \quad (19)$$

The same formula obviously holds if the pair  $(\gamma_r, \mathcal{V}_r^2)$  is replaced by  $(\beta_r, \mathcal{V}_r^1)$ . It remains to compute  $\mathbb{E}[\exp(-\lambda Z_r - \mu\mathcal{V}_r^0)]$ . An application of the special Markov property gives

$$\mathbb{E}[e^{-\mu\mathcal{V}_r^0} \mid Z_r] = \exp\left(-Z_r \mathbb{N}_r\left((1 - e^{-\mu\sigma}) \mathbf{1}_{\{W_* > 0\}}\right)\right) = \exp(-g_\mu(r) Z_r).$$

Hence, recalling the exponential distribution of  $Z_r$

$$\begin{aligned}\mathbb{E}[\exp(-\lambda Z_r - \mu \mathcal{V}_r^0)] &= \mathbb{E}[\exp(-(\lambda + g_\mu(r))Z_r)] \\ &= \frac{3}{2r^2} \int_0^\infty dt \exp\left(-(\lambda + g_\mu(r))t - \frac{3}{2r^2}t\right) \\ &= \frac{3}{2r^2} \times \frac{1}{\lambda + \sqrt{\frac{\mu}{2}}\left(3 \coth((2\mu)^{1/4}r) - 2\right)}.\end{aligned}\tag{20}$$

Finally, using the independence of the three pairs  $(Z_r, \mathcal{V}_r^0)$ ,  $(\beta_r, \mathcal{V}_r^1)$  and  $(\gamma_r, \mathcal{V}_r^2)$  and formulas (19) and (20), we get

$$\begin{aligned}\mathbb{E}[\exp(-\lambda Z_r - \nu_1 \beta_r - \nu_2 \gamma_r - \mu \mathcal{V}_r)] &= \mathbb{E}[\exp(-\lambda Z_r - \mu \mathcal{V}_r^0)] \times \mathbb{E}[\exp(-\nu_1 \beta_r - \mu \mathcal{V}_r^1)] \times \mathbb{E}[\exp(-\nu_2 \gamma_r - \mu \mathcal{V}_r^2)] \\ &= \frac{3}{2r^2} \times \frac{1}{\lambda + \sqrt{\frac{\mu}{2}}\left(3 \coth((2\mu)^{1/4}r) - 2\right)} \times G_r(\mu, \nu_1) \times G_r(\mu, \nu_2).\end{aligned}$$

This completes the proof of Proposition 3.  $\square$

## 5 The perimeter process

In this section, we study the process  $(Z_t)_{t \geq 0}$ , where, by convention, we take  $Z_0 := 0$ . Our first goal is to compute the finite-dimensional marginals of  $(Z_t)_{t \geq 0}$ . To this end, for every  $t \geq 0$ , we consider the  $\sigma$ -field  $\mathcal{G}_t$  generated by the processes  $(\mathcal{R}_{\gamma_t+r})_{r \geq 0}$  and  $(\mathcal{R}_{-\beta_t-r})_{r \geq 0}$ , the point measure:

$$\sum_{i \in I, t_i \notin [-\beta_t, \gamma_t]} \delta_{(t_i, \omega^i)},$$

and the  $\mathbb{P}$ -negligible sets. Then  $(\mathcal{G}_t)_{t \geq 0}$  is a backward filtration (meaning that  $\mathcal{G}_t \subset \mathcal{G}_s$  if  $s \leq t$ ). We also observe that  $Z_t$  is  $\mathcal{G}_t$ -measurable, since  $Z_t = \sum_{i \in I, t_i \notin [-\beta_t, \gamma_t]} \mathcal{Z}_t(\omega^i)$ .

**Lemma 6.** *Let  $0 \leq s \leq t$ . Then, for every  $\lambda > 0$ , we have:*

$$\mathbb{E}\left[\exp(-\lambda Z_s) \mid \mathcal{G}_t\right] = \left(\frac{t}{s + (t-s)\left(1 + \frac{2\lambda s^2}{3}\right)^{1/2}}\right)^2 \exp\left(-\frac{3Z_t}{2} \left(\frac{1}{(t-s + (\frac{2\lambda}{3} + s^{-2})^{-1/2})^2} - \frac{1}{t^2}\right)\right).$$

In particular, for every fixed  $t > 0$ ,  $(Z_{t-s})_{s \in [0, t]}$  is a time-inhomogeneous Markov process with respect to the filtration  $(\mathcal{G}_{t-s})_{s \in [0, t]}$ . Together with the distribution of  $Z_t$  obtained in Proposition 3, this characterizes the finite-dimensional marginals of  $(Z_t)_{t \geq 0}$ . We mention that there is a striking analogy between Lemma 6 and [8, Proposition 4.3], which was the similar result for the Brownian plane.

*Proof.* Let  $0 < s \leq t$ . We have  $Z_s = Z_s^0 + Z_s^1 + Z_s^2$ , where

$$\begin{aligned}Z_s^0 &:= \sum_{i \in I: t_i \in (-\infty, -\beta_t) \cup (\gamma_t, \infty)} \mathcal{Z}_s(\omega^i), \\ Z_s^1 &:= \sum_{i \in I: t_i \in [-\beta_t, -\beta_s]} \mathcal{Z}_s(\omega^i), \\ Z_s^2 &:= \sum_{i \in I: t_i \in (\gamma_s, \gamma_t]} \mathcal{Z}_s(\omega^i).\end{aligned}$$

The argument is now similar to the proof of Proposition 3. By construction,  $Z_s^1$  and  $Z_s^2$  are independent of  $\mathcal{G}_t$  (hence of  $Z_s^0$ ) and have the same distribution. Furthermore, from (18), we have

$$\begin{aligned}\mathbb{E}\left[\exp(-\lambda Z_s^2)\right] &= \mathbb{E}\left[\exp\left(-2 \int_{\gamma_s}^{\gamma_t} dr \mathbb{N}_{\sqrt{3}\mathcal{R}_r} \left((1 - e^{-\lambda \mathcal{Z}_s}) \mathbf{1}_{\{W_* > 0\}}\right)\right)\right] \\ &= \mathbb{E}\left[\exp\left(-\int_{\gamma_s}^{\gamma_t} dr \left((\mathcal{R}_r - b)^{-2} - (\mathcal{R}_r)^{-2}\right)\right)\right]\end{aligned}$$

where  $b = (s/\sqrt{3}) - (2\lambda + 3s^{-2})^{-1/2}$ . Hence an application of Lemma 4 gives:

$$\mathbb{E}\left[\exp(-\lambda Z_s^2)\right] = \frac{t}{s + (t-s)\left(1 + \frac{2\lambda s^2}{3}\right)^{\frac{1}{2}}}.$$

Finally, by the special Markov property,

$$\begin{aligned}\mathbb{E}\left[\exp(-\lambda Z_s^0) \mid \mathcal{G}_t\right] &= \exp\left(-Z_t \mathbb{N}_t\left((1 - e^{-\lambda Z_s})\mathbf{1}_{\{W_* > 0\}}\right)\right) \\ &= \exp\left(-\frac{3Z_t}{2}\left(\frac{1}{(t-s + (\frac{2\lambda}{3} + s^{-2})^{-\frac{1}{2}})^2} - \frac{1}{t^2}\right)\right),\end{aligned}$$

using (17) in the second equality. This completes the proof of the lemma.  $\square$

Our aim now is to derive two distinct characterizations of the process  $(Z_t)_{t \geq 0}$  from Lemma 6. More precisely, we will prove that:

- viewed backward in time, the process  $(Z_t)_{t \geq 0}$  is an  $h$ -transform of a continuous-state branching process with immigration;
- viewed in the usual forward time direction,  $(Z_t)_{t \geq 0}$  is a self-similar Markov process starting from 0, which can be characterized by the associated Lévy process.

These characterizations are analogous to those provided for the Brownian plane in [8, Proposition 4.4] and [23, Section 11.2]. We start with the first characterization and we will identify the transition kernel whose Laplace transform is given by Lemma 6. We let  $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$  stand for the Skorokhod space of càdlàg functions from  $\mathbb{R}_+$  into  $\mathbb{R}$ . We write  $(Y_t)_{t \geq 0}$  for the canonical process on  $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$ , and  $(\mathcal{F}_t)_{t \geq 0}$  for the canonical filtration. We then define, for every  $x \geq 0$ , the probability measure  $P_x$  as the law of the continuous-state branching process with immigration, with branching mechanism  $\Psi(\lambda) = \sqrt{8/3}\lambda^{3/2}$  and immigration mechanism  $H(\lambda) = \sqrt{8/3}\lambda^{1/2}$ . We refer to [14] for the general theory of continuous-state branching processes with immigration (see also the survey [26]). We have, for every  $s \geq 0$ ,

$$E_x[\exp(-\lambda Y_s)] = \exp\left(-x u_\lambda(s) - \int_0^s H(u_\lambda(r)) dr\right) = \left(1 + s\sqrt{2\lambda/3}\right)^{-2} \exp(-x u_\lambda(s)), \quad (21)$$

where  $u_\lambda(s) := (\lambda^{-1/2} + s\sqrt{2/3})^{-2}$  solves  $\frac{du_\lambda(s)}{ds} = -\Psi(u_\lambda(s))$ , with  $u_0(\lambda) = \lambda$ .

For every  $a > 0$ , we set

$$h_a(x) := a^{-2} \exp\left(-\frac{3}{2a^2}x\right).$$

Then (21) shows that, for every  $s \geq 0$ ,

$$E_x[h_a(Y_s)] = h_{a+s}(x).$$

Let us fix  $t > 0$ . It follows from the last display that the process  $(h_{t-r}(Y_r))_{r \in [0, t]}$  is a martingale under  $P_x$ . Hence, for every  $x > 0$ , we may define a probability measure  $P_x^{(t)}$  on  $\mathbb{D}([0, t], \mathbb{R})$ , the Skorokhod space of càdlàg functions from  $[0, t]$  into  $\mathbb{R}$ , by requiring that, for every  $r \in [0, t]$ ,

$$\frac{dP_x^{(t)}}{dP_x} \Big|_{\mathcal{F}_r} = \frac{h_{t-r}(Y_r)}{h_t(x)}, \quad (22)$$

where we slightly abuse notation by viewing  $P_x$  as a probability measure on  $\mathbb{D}([0, t], \mathbb{R})$  and keeping the same notation  $Y_r$  and  $\mathcal{F}_r$  for the canonical process and the canonical filtration on  $\mathbb{D}([0, t], \mathbb{R})$ . The process

$$\left(\frac{1}{h_{t-r}(Y_r)}\right)_{r \in [0, t]}$$



is a (nonnegative) martingale under  $P_x^{(t)}$  and thus must converge  $P_x^{(t)}$  a.s. as  $r \uparrow t$  to a finite limit. Clearly, this is only possible if  $Y_r$  converges to 0 as  $r \uparrow t$ ,  $P_x^{(t)}$  a.s. It follows from these considerations that we can also view  $P_x^{(t)}$  as a probability measure on  $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$ , which is supported on  $\{Y_s = 0 \text{ for every } s \geq t\}$ .

**Proposition 7.** *Let  $t > 0$  and  $x > 0$ . Conditionally on  $Z_t = x$ , the process  $(Z_{t-r})_{r \leq t}$  has the same finite-dimensional marginals as the process  $(Y_r)_{r \leq t}$  under  $P_x^{(t)}$ .*

*Proof.* Since  $(Z_{t-r})_{r \in [0, t]}$  (under  $\mathbb{P}$ ) and  $(Y_r)_{r \in [0, t]}$  (under  $P_x^{(t)}$ ) are both time-inhomogeneous Markov processes, it is enough to verify that they have the same transition kernels. For  $t > 0$  and  $y \geq 0$ , let  $q_t(y, dz)$  be the distribution of  $Y_t$  under  $P_y$ , and, for every  $0 \leq r < s < t$ ,

$$\pi_{r,s}^{(t)}(y, dz) := \frac{h_{t-s}(z)}{h_{t-r}(y)} q_{s-r}(y, dz),$$

Then the kernels  $\pi_{r,s}^{(t)}(y, dz)$  are the transition kernels of  $(Y_r)_{r \in [0, t]}$  under  $P_x^{(t)}$ . Next, for every  $0 \leq r < s < t$  and  $\lambda \geq 0$ , we have

$$\int \pi_{r,s}^{(t)}(z, dy) \exp(-\lambda y) = \left(\frac{t-r}{t-s}\right)^2 \exp\left(\frac{3}{2(t-r)^2} z\right) E_z\left[\exp\left(-\left(\lambda + \frac{3}{2(t-s)^2}\right) Y_{s-r}\right)\right]$$

and then an application of (21) shows that the right-hand side is equal to

$$\left(\frac{t-r}{t-s + (s-r)\left(1 + \frac{2\lambda(t-s)^2}{3}\right)^{1/2}}\right)^2 \exp\left(-\frac{3z}{2} \left(\frac{1}{(s-r + (\frac{2\lambda}{3} + (t-s)^{-2})^{-1/2})^2} - \frac{1}{(t-r)^2}\right)\right).$$

Comparing with Lemma 6, we get

$$\mathbb{E}\left[\exp(-\lambda Z_{t-s}) \mid \mathcal{G}_{t-r}\right] = \int \pi_{r,s}^{(t)}(Z_{t-r}, dy) \exp(-\lambda y),$$

which completes the proof of the proposition.  $\square$

Proposition 7 entails that the process  $(Z_t)_{t \geq 0}$  possesses a right-continuous modification, which we consider from now on. We turn to the second characterization of  $(Z_t)_{t \geq 0}$ , as a self-similar Markov process. Let  $\psi$  be the function defined for  $q > 0$  by

$$\psi(q) := \sqrt{\frac{8}{3}} q \frac{\Gamma(q+1)}{\Gamma(q+\frac{1}{2})}.$$

Then  $\psi$  is the Laplace exponent of a spectrally negative Lévy process  $(\xi_t)_{t \geq 0}$  (meaning that  $\mathbb{E}[\exp(\lambda \xi_t)] = \exp(t\psi(\lambda))$  for  $\lambda \geq 0$ ). In fact  $\xi$  belongs to the class of hypergeometric Lévy processes discussed in Chapter 4 of [15]. To be precise, the spectrally negative case is a borderline case excluded in Theorem 4.6 of [15], but one can still use Proposition 4.1 and Theorem 4.4 of [15] to get the Lévy-Khintchine representation

$$\psi(q) = 4\sqrt{\frac{2}{3\pi}} q + \frac{1}{\sqrt{6\pi}} \int_{-\infty}^0 (\exp(qy) - 1 + q(1 - \exp(y))) \frac{2 + \exp(y)}{(1 - \exp(y))^{5/2}} \exp(y) dy.$$

For every  $x > 0$ , we then write  $P_x^\uparrow$  for the distribution (on the Skorokhod space  $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$ ) of the self-similar Markov process with index  $1/2$  and initial value  $x$ , which is associated with the Lévy process  $\xi$  via the Lamperti transformation. In other words,  $P_x^\uparrow$  is the law of the process  $(x \exp(\xi_{\alpha(t)}))_{t \geq 0}$ , where, for every  $t \geq 0$ ,  $\alpha(t) := \inf\{s \geq 0 : \sqrt{x} \int_0^s \exp(\frac{1}{2}\xi_r) dr \geq t\}$ . The self-similarity property means that, for every  $\lambda > 0$ , the law of  $(\sqrt{\lambda} Y_{\lambda t})_{t \geq 0}$  under  $P_x^\uparrow$  is  $P_{x\sqrt{\lambda}}^\uparrow$ .

Since  $\xi$  has no positive jumps, we can apply [4, Proposition 1], which shows that  $P_x^\uparrow$  converges weakly as  $x \downarrow 0$  to a limiting law denoted by  $P_0^\uparrow$ , which is characterized by the following two properties holding for every  $t > 0$ :

(i) the law of  $Y_t$  under  $P_0^\uparrow$  is exponential with mean  $2t^2/3$ ;

(ii) under  $P_0^\uparrow$  and conditionally on  $(Y_r)_{0 \leq r \leq t}$ , the process  $(Y_{t+s})_{s \geq 0}$  is distributed according to  $P_{Y_t}^\uparrow$ .

In particular, property (i) follows from the formula for the moments  $E_0^\uparrow[(Y_t)^k]$  found in [4, Proposition 1]. At this stage, we note that [4] deals with self-similar Markov process with index 1 (instead of  $1/2$ ), but the results can be applied to  $\sqrt{Y_t}$  under  $P_x^\uparrow$ , which is self-similar with index 1 and such that the Laplace exponent of the associated Lévy process is  $\lambda \mapsto \psi(\lambda/2)$ . The same remark applies to the forthcoming calculations. We finally note that the law of  $(\sqrt{\lambda} Y_{\lambda t})_{t \geq 0}$  under  $P_0^\uparrow$  is  $P_0^\uparrow$ .

**Proposition 8.** *The distribution of  $(Z_t)_{t \geq 0}$  is  $P_0^\uparrow$ .*

*Proof.* We claim that, for every  $t > 0$  and  $\lambda_1, \lambda_2 \geq 0$  such that:

$$\left(1 + t(1 + \lambda_1)^{1/2}\right)^2 \lambda_2 < 1, \quad (23)$$

we have:

$$\mathbb{E}\left[\exp\left(-\frac{3}{2}\lambda_1 Z_1 - \frac{3}{2}\lambda_2 Z_{1+t}\right)\right] = E_0^\uparrow\left[\exp\left(-\frac{3}{2}\lambda_1 Y_1 - \frac{3}{2}\lambda_2 Y_{1+t}\right)\right]. \quad (24)$$

The factors  $3/2$  are present only to help simplifying the expressions in the calculations below. Let us explain why the proposition follows from (24). First, using (24) and a scaling argument, we get that, for every  $0 < s < t$ , the pair  $(Z_s, Z_t)$  is distributed as  $(Y_s, Y_t)$  under  $P_0^\uparrow$ . Then, let  $0 < t_1 < \dots < t_n$  with  $n \geq 3$ . By Lemma 6 and the subsequent comments, we know that  $Z_{t_n}$  is independent of  $(Z_{t_1}, \dots, Z_{t_{n-2}})$  conditionally on  $Z_{t_{n-1}}$ . Moreover, using property (ii) above and the fact that  $(Z_{t_{n-1}}, Z_{t_n})$  is distributed as  $(Y_{t_{n-1}}, Y_{t_n})$  under  $P_0^\uparrow$ , we get that the conditional distribution of  $Z_{t_n}$  knowing  $Z_{t_{n-1}}$  is the law of  $Y_{t_n - t_{n-1}}$  under  $P_{Z_{t_{n-1}}}^\uparrow$ . An induction argument then shows that  $(Z_{t_1}, \dots, Z_{t_n})$  is distributed as  $(Y_{t_1}, \dots, Y_{t_n})$  under  $P_0^\uparrow$ , which was the desired result.

Let us prove (24). Observe that, by Lemma 6 and the known distribution of  $Z_{1+t}$ , we have

$$\mathbb{E}\left[\exp\left(-\frac{3}{2}\lambda_1 Z_1 - \frac{3}{2}\lambda_2 Z_{1+t}\right)\right] = \left(1 + \lambda_1 + \lambda_2 \left(1 + t(1 + \lambda_1)^{1/2}\right)^2\right)^{-1}, \quad (25)$$

for every  $t > 0$ . It remains to compute the right-side hand term of (24). To this end, we use [4, Proposition 1] (applied to the process  $(\sqrt{Y_t})_{t \geq 0}$  under  $\mathbb{P}_x^\uparrow$ ) to get, for every integer  $p \geq 1$ ,

$$E_x^\uparrow[Y_t^p] = x^p + \sum_{k=1}^{2p} \frac{\prod_{\ell=0}^{k-1} \psi(p - \frac{\ell}{2})}{k!} x^{p - \frac{k}{2}} t^k.$$

Next, a direct computation using the definition of  $\psi$  gives

$$E_x^\uparrow[Y_t^p] = \sum_{k=0}^{2p} \left(\frac{2}{3}\right)^{\frac{k}{2}} \binom{2p}{k} \frac{p!}{\Gamma(p + 1 - \frac{k}{2})} x^{p - \frac{k}{2}} t^k.$$

Recall that the law of  $Y_1$  under  $P_0^\uparrow$  is exponential with mean  $2/3$ . By property (ii) again, we have

$$\begin{aligned} \left(\frac{3}{2}\right)^p E_0^\uparrow\left[\exp\left(-\frac{3}{2}\lambda_1 Y_1\right) Y_{1+t}^p\right] &= \left(\frac{3}{2}\right)^p E_0^\uparrow\left[\exp\left(-\frac{3}{2}\lambda_1 Y_1\right) E_{Y_1}^\uparrow\left[Y_t^p\right]\right] \\ &= \frac{3}{2} \sum_{k=0}^{2p} \left(\int_0^\infty dx \exp\left(-\frac{3}{2}(\lambda_1 + 1)x\right) \left(\frac{3x}{2}\right)^{p - \frac{k}{2}}\right) \binom{2p}{k} \frac{p!}{\Gamma(p + 1 - \frac{k}{2})} t^k \\ &= p! \sum_{k=0}^{2p} \binom{2p}{k} \frac{t^k}{(1 + \lambda_1)^{p + 1 - \frac{k}{2}}} \\ &= \frac{p!}{(1 + \lambda_1)} \left((1 + \lambda_1)^{-1/2} + t\right)^{2p}. \end{aligned}$$

It follows that the radius of convergence of the power series

$$z \mapsto \sum_{p=0}^{\infty} \left(\frac{3}{2}\right)^p E_0^\uparrow \left[ \exp\left(-\frac{3}{2}\lambda_1 Y_1\right) Y_{1+t}^p \right] \frac{z^p}{p!}$$

is  $(1+t(1+\lambda_1)^{1/2})^{-2}$ . Hence, for every  $\lambda_1, \lambda_2 \geq 0$  satisfying (23), we get

$$\begin{aligned} E_0^\uparrow \left[ \exp\left(-\frac{3}{2}\lambda_1 Y_1 - \frac{3}{2}\lambda_2 Y_{1+t}\right) \right] &= \frac{1}{(1+\lambda_1)} \sum_{p=0}^{\infty} \left( (1+\lambda_1)^{-1/2} + t \right)^{2p} (-\lambda_2)^p \\ &= \left( 1 + \lambda_1 + \lambda_2 \left( 1 + t(1+\lambda_1)^{1/2} \right)^2 \right)^{-1}, \end{aligned}$$

and, by comparing with (25), we get the desired formula (24).  $\square$

## 6 A new construction of the Brownian half-plane

In this section, we will give a new construction of the Brownian half-plane, which is an analog of a result proved in [25] for the Brownian disk. This construction will be useful in Section 7 when we study the complement of the hull centered on a boundary segment.

### 6.1 A Brownian disk with a random perimeter

The goal of this section is to recall the construction of a Brownian disk with a random perimeter that was given in [25], to which we refer for more details. We consider a normalized Brownian excursion  $(\mathbf{e}_t)_{0 \leq t \leq 1}$ , and, conditionally on  $(\mathbf{e}_t)_{0 \leq t \leq 1}$ , a Poisson point measure  $\mathcal{N} = \sum_{j \in J} \delta_{(t_j, \omega^j)}$  on  $[0, 1] \times \mathcal{S}$  with intensity

$$2 dt \mathbb{N}_{\sqrt{3} \mathbf{e}_t}(d\omega).$$

For every  $j \in J$ , we consider the truncation  $\tilde{\omega}^j := \text{tr}_0(\omega^j)$  of  $\omega^j$  at level 0, and we let  $\mathfrak{T}^*$  be the compact metric space obtained from the disjoint union

$$[0, 1] \cup \left( \bigcup_{j \in J} \mathcal{T}_{(\tilde{\omega}^j)} \right) \quad (26)$$

by identifying the root  $\rho_{(\tilde{\omega}^j)}$  of  $\mathcal{T}_{(\tilde{\omega}^j)}$  with the point  $t_j$  of  $[0, 1]$ , for every  $j \in J$ . The metric  $d_{\mathfrak{T}^*}$  on  $\mathfrak{T}^*$  is defined in the obvious manner, so that the restriction of  $d_{\mathfrak{T}^*}$  to each tree  $\mathcal{T}_{(\tilde{\omega}^j)}$  is the metric  $d_{(\tilde{\omega}^j)}$ , and  $d_{\mathfrak{T}^*}(u, v) = |v - u|$  if  $u, v \in [0, 1]$  (compare with the Caraceni-Curien construction of Section 3.1, and see [25, Section 4] for more details). The volume measure on  $\mathfrak{T}^*$  is the sum of the volume measures on the trees  $\mathcal{T}_{(\tilde{\omega}^j)}$ ,  $j \in J$ . We then assign labels  $(\ell_a^*)_{a \in \mathfrak{T}^*}$  to the points of  $\mathfrak{T}^*$ . If  $a = s \in [0, 1]$ , we take  $\ell_a^* := \sqrt{3} \mathbf{e}_s$ , and if  $a \in \mathcal{T}_{(\tilde{\omega}^j)}$  for some  $j \in J$ , we simply let  $\ell_a^*$  be the label of  $a$  in  $\mathcal{T}_{(\tilde{\omega}^j)}$ . We note that the function  $a \mapsto \ell_a^*$  is continuous on  $\mathfrak{T}^*$ , and that labels  $\ell_a^*$  are nonnegative for every  $a \in \mathfrak{T}^*$  (because we replaced each  $\omega^j$  by its truncation  $\tilde{\omega}^j$ ). We define the boundary of  $\mathfrak{T}^*$  by  $\partial \mathfrak{T}^* := \{a \in \mathfrak{T}^* : \ell_a^* = 0\}$  and note that  $0, 1 \in \partial \mathfrak{T}^*$ .

If  $\Sigma^* := \sum_{j \in J} \sigma(\tilde{\omega}^j)$  is the total mass of the volume measure, we can define a clockwise exploration  $(\mathcal{E}_t^*)_{0 \leq t \leq \Sigma^*}$  of  $\mathfrak{T}^*$  by concatenating the mappings  $p_{(\tilde{\omega}^j)} : [0, \sigma(\tilde{\omega}^j)] \rightarrow \mathcal{T}_{(\tilde{\omega}^j)}$  in the order prescribed by the  $t_j$ 's (again see [25] for more details). Note that  $\mathcal{E}_0^* = 0$  and  $\mathcal{E}_{\Sigma^*}^* = 1$ .

Similarly as in Section 3.1, the clockwise exploration allows us to define ‘‘intervals’’ in  $\mathfrak{T}^*$ . For  $s, t \in [0, \Sigma^*]$ , if  $s > t$ , we set  $[s, t]^* := [s, \Sigma^*] \cup [0, t]$  and if  $s \leq t$ ,  $[s, t]^* := [s, t]$  is the usual interval. Then, for every  $u, v \in \mathfrak{T}^*$ , there is a smallest interval  $[s, t]^*$ , with  $s, t \in [0, \Sigma^*]$ , such that  $\mathcal{E}_s^* = u$  and  $\mathcal{E}_t^* = v$ , and we define

$$[u, v]^* := \{\mathcal{E}_r^* : r \in [s, t]^*\}.$$

We then set, for every  $a, b \in \mathfrak{T}^* \setminus \partial \mathfrak{T}^*$ ,

$$D_\star^\circ(a, b) := \ell_a^* + \ell_b^* - 2 \max \left( \min_{c \in [a, b]^*} \ell_c^*, \min_{c \in [b, a]^*} \ell_c^* \right) \quad (27)$$

if the maximum in the right-hand side is positive, and  $D_\star^\circ(a, b) := \infty$  otherwise. Finally, we set, for every  $a, b \in \mathfrak{T}^\star \setminus \partial\mathfrak{T}^\star$ ,

$$D_\star(a, b) := \inf_{a_0=a, a_1, \dots, a_{p-1}, a_p=b} \sum_{k=1}^p D_\star^\circ(a_{k-1}, a_k) \quad (28)$$

where the infimum is over all choices of the integer  $p \geq 1$  and of the points  $a_1, \dots, a_{p-1}$  in  $\mathfrak{T}^\star \setminus \partial\mathfrak{T}^\star$ . It is not hard to verify that  $D_\star(a, b) < \infty$  (see Proposition 30 (i) in [20] for a very similar argument). The mapping  $(a, b) \mapsto D_\star(a, b)$  is continuous on  $(\mathfrak{T}^\star \setminus \partial\mathfrak{T}^\star) \times (\mathfrak{T}^\star \setminus \partial\mathfrak{T}^\star)$ , and has a unique continuous extension to  $\mathfrak{T}^\star \times \mathfrak{T}^\star$ , which is a pseudo-metric on  $\mathfrak{T}^\star$  [25, Proposition 5]. Moreover, by (27) and (28), we have  $D_\star(a, b) \geq |\ell_a^\star - \ell_b^\star|$ .

We then consider the quotient space  $\mathbb{U} := \mathfrak{T}^\star / \{D_\star = 0\}$ , and the canonical projection  $\Pi_\star : \mathfrak{T}^\star \rightarrow \mathbb{U}$ . The function  $(a, b) \mapsto D_\star(a, b)$  induces a metric on  $\mathbb{U}$ , which we still denote by  $D_\star$ , and the metric space  $(\mathbb{U}, D_\star)$  is equipped with the pushforward of the volume measure on  $\mathfrak{T}^\star$  under  $\Pi_\star$ , which is denoted by  $\mathbf{V}_\star$ . We also write  $\partial_0\mathbb{U} = \Pi_\star([0, 1])$  and  $\partial_1\mathbb{U} = \Pi_\star(\partial\mathfrak{T}^\star)$ . Finally, we note that the equivalence class of 0 (or that of 1) in the quotient space  $\mathbb{U} := \mathfrak{T}^\star / \{D_\star = 0\}$  is a singleton (this follows from [25, Proposition 5], which describes the pairs  $(a, b)$  in  $\mathfrak{T}^\star$  such that  $D^\star(a, b) = 0$ ).

**Theorem 9.** [25, Theorem 16] *The random measure metric space  $(\mathbb{U}, D_\star, \mathbf{V}_\star, \Pi_\star(0))$  is a free Brownian disk with a random boundary size distributed according to the measure  $\frac{3}{2} \mathbf{1}_{\{z>1\}} z^{-5/2} dz$ , which is pointed at a uniform boundary point. Furthermore, the boundary  $\partial\mathbb{U}$  is equal to  $\partial_0\mathbb{U} \cup \partial_1\mathbb{U}$ .*

In contrast with Section 3, we view here the free Brownian disk as a (random) pointed measure metric space: If we condition the boundary size of  $\mathbb{U}$  to be equal to  $S > 0$ , the space  $(\mathbb{U}, D_\star, \mathbf{V}_\star, \Pi_\star(0))$  has the same distribution as the space  $(\mathbb{D}_{(S)}, D_{(S)}, V_{(S)}, \Lambda_{(S)}(0))$ , with the notation introduced at the beginning of the proof of Theorem 1.

We note that labels  $\ell_x^\star$  make sense for  $x \in \mathbb{U}$  (because  $D_\star(a, b) = 0$  implies  $\ell_a^\star = \ell_b^\star$ ) and furthermore we have  $D_\star(x, \partial_1\mathbb{U}) = \ell_x^\star$  for every  $x \in \mathbb{U}$  (see [25, Section 4.2]).

## 6.2 Constructing the Brownian half-plane

Let us start from a three-dimensional Bessel process  $(R_t)_{t \geq 0}$  with  $R_0 = 0$  and then consider a random point measure  $\mathcal{N}_\infty = \sum_{j \in J_\infty} \delta_{(t_j, \omega^j)}$  on  $\mathbb{R}_+ \times \mathcal{S}$ , such that, conditionally on  $(R_t)_{t \geq 0}$ , the measure  $\mathcal{N}_\infty$  is Poisson with intensity:

$$2 dt \mathbb{N}_{\sqrt{3}R_t}(d\omega).$$

For every  $j \in J_\infty$ , we let  $\tilde{\omega}^j$  be the truncation of  $\omega^j$  at level 0.

In a way similar to Sections 3.1 and 6.1, we then introduce the geodesic space  $\mathfrak{T}_\infty$  which is obtained from the disjoint union

$$[0, \infty) \cup \left( \bigcup_{j \in J_\infty} \mathcal{T}_{(\tilde{\omega}^j)} \right)$$

by identifying the root of  $\mathcal{T}_{(\tilde{\omega}^j)}$  with the point  $t_j$  of  $[0, \infty)$ , for every  $j \in J_\infty$ . We interpret  $[0, \infty)$  as the “spine” of  $\mathfrak{T}_\infty$ . We also define the volume measure on  $\mathfrak{T}_\infty$  as the sum of the volume measures on the trees  $\mathcal{T}_{(\tilde{\omega}^j)}$ ,  $j \in J_\infty$ .

We next assign labels to  $\mathfrak{T}_\infty$  by taking  $\ell_u^\infty := \sqrt{3}R_u$ , if  $u \in \mathbb{R}_+$ , and by letting  $\ell_u^\infty$  be the label of  $u$  in  $\mathcal{T}_{(\tilde{\omega}^j)}$  if  $u \in \mathcal{T}_{(\tilde{\omega}^j)}$ . We can also introduce a clockwise exploration  $(\mathcal{E}_t^\infty)_{t \geq 0}$  of  $\mathfrak{T}_\infty$ , defined by concatenating the mappings  $p_{(\tilde{\omega}^j)}$  in the order prescribed by the  $t_j$ 's. As in Sections 3.1 and 6.1, we then define intervals on  $\mathfrak{T}_\infty$ . For  $s, t \in \mathbb{R}_+$ , we set  $[s, t]'$  :=  $[s, \infty) \cup [0, t]$  if  $s > t$  and  $[s, t]'$  :=  $[s, t]$  if  $s \leq t$ . Then, for every  $u, v \in \mathfrak{T}_\infty$ , we set  $[u, v]_\infty^\circ := \{\mathcal{E}_r^\infty : r \in [s, t]'\}$ , where  $s, t \in \mathbb{R}_+$  are such that  $\mathcal{E}_s^\infty = u$  and  $\mathcal{E}_t^\infty = v$  and the interval  $[s, t]'$  is as small as possible.

Let  $\partial\mathfrak{T}_\infty := \{u \in \mathfrak{T}_\infty : \ell_u^\infty = 0\}$ . For every  $a, b \in \mathfrak{T}_\infty \setminus \partial\mathfrak{T}_\infty$ , we set

$$D_\infty^\circ(a, b) := \ell_a^\infty + \ell_b^\infty - 2 \max \left( \min_{c \in [a, b]_\infty^\circ} \ell_c^\infty, \min_{c \in [b, a]_\infty^\circ} \ell_c^\infty \right) \quad (29)$$

if the maximum in the right-hand side is positive, and  $D_\infty^\circ(a, b) := \infty$  otherwise. Finally, in exactly the same way as we defined  $D_\star$  from  $D_\star^\circ$ , we set, for every  $a, b \in \mathfrak{T}_\infty \setminus \partial\mathfrak{T}_\infty$ ,

$$D_\infty(a, b) := \inf_{a_0=a, a_1, \dots, a_{p-1}, a_p=b} \sum_{k=1}^p D_\infty^\circ(a_{k-1}, a_k) \quad (30)$$

where the infimum is over all choices of the integer  $p \geq 1$  and of the points  $a_1, \dots, a_{p-1}$  in  $\mathfrak{T}_\infty \setminus \partial\mathfrak{T}_\infty$ . By arguments similar to the proof of [20, Proposition 30], one verifies that the mapping  $(a, b) \mapsto D_\infty(a, b)$  takes finite values and is continuous on  $(\mathfrak{T}_\infty \setminus \partial\mathfrak{T}_\infty) \times (\mathfrak{T}_\infty \setminus \partial\mathfrak{T}_\infty)$ , and that we have  $D_\infty(a, b) \geq |\ell_a^\infty - \ell_b^\infty|$ .

**Proposition 10.** *The function  $(a, b) \mapsto D_\infty(a, b)$  has a continuous extension to  $\mathfrak{T}_\infty \times \mathfrak{T}_\infty$ , which is a pseudo-metric on  $\mathfrak{T}_\infty$ . Furthermore, the property  $D_\infty(a, b) = 0$  holds if and only if either  $a$  and  $b$  both belong to  $\mathfrak{T}_\infty \setminus \partial\mathfrak{T}_\infty$  and  $D_\infty^\circ(a, b) = 0$ , or  $a$  and  $b$  both belong to  $\partial\mathfrak{T}_\infty$  and we have  $\{a, b\} = \{\mathcal{E}_s^\infty, \mathcal{E}_t^\infty\}$ , for some  $0 \leq s \leq t < \infty$  such that  $\ell_{\mathcal{E}_r^\infty}^\infty > 0$  for every  $r \in (s, t)$ .*

This is an analog of Proposition 5 in [25], which deals with the metric space  $(\mathfrak{T}_\star, D_\star)$  introduced above. The proof is exactly the same and we omit the details.

We then consider the quotient space  $\mathfrak{H}' = \mathfrak{T}_\infty / \{D_\infty = 0\}$ , and the canonical projection  $\Pi_\infty : \mathfrak{T}_\infty \mapsto \mathfrak{H}'$ . The metric space  $(\mathfrak{H}', D_\infty)$  is equipped with the distinguished point  $\Pi_\infty(0)$  and the pushforward of the volume measure on  $\mathfrak{T}_\infty$  under  $\Pi_\infty$ , which will be denoted by  $\mathbf{V}_\infty$ . We also write  $\partial_0\mathfrak{H}' = \Pi_\infty([0, \infty))$  and  $\partial_1\mathfrak{H}' = \Pi_\infty(\partial\mathfrak{T}_\infty)$ . Then we notice that labels  $\ell_x^\infty$  make sense for  $x \in \mathfrak{H}'$  (again since  $D_\infty(a, b) = 0$  implies  $\ell_a^\infty = \ell_b^\infty$ ).

We finally define a decorating curve  $\Lambda'$  of  $\mathfrak{H}'$ . First, we take  $\Lambda'(t) = \Pi_\infty(t)$  for every  $t \geq 0$ . To define  $\Lambda'(t)$  when  $t \leq 0$ , we set, for every  $s \geq 0$ ,

$$L_s^{\mathfrak{H}', 0} := \sum_{j \in J_\infty} L_{(s-\alpha_j)^+}^0(\tilde{\omega}^j),$$

where  $\alpha_j := \inf\{s \geq 0 : \mathcal{E}_s^\infty \in \mathcal{T}_{(\tilde{\omega}^j)}\}$ , and we recall that  $(L_s^0(\tilde{\omega}^j))_{s \geq 0}$  denotes the exit local time at 0 of the truncated snake trajectory  $\tilde{\omega}^j$  (see the end of Section 2.3). Then, for every  $t \leq 0$ , if

$$\tau_t := \inf\{s \geq 0 : L_s^{\mathfrak{H}', 0} \geq -t\},$$

we define  $\Lambda'(t) := \Pi_\infty(\mathcal{E}_{\tau_t}^\infty)$ . Using arguments similar to those used to study the path  $\Lambda^{\bullet, r}$  at the end of Section 3.1, one verifies that the path  $t \mapsto \Lambda(t)$  is continuous and injective, and we have  $\partial_0\mathfrak{H}' = \{\Lambda'(t) : t \geq 0\}$  and  $\partial_1\mathfrak{H}' = \{\Lambda'(t) : t \leq 0\}$ .

**Theorem 11.** *The random curve-decorated measure metric space  $(\mathfrak{H}', D_\infty, \mathbf{V}_\infty, \Lambda')$  is a curve-decorated Brownian half-plane. Furthermore, we have  $D_\infty(x, \partial_1\mathfrak{H}') = \ell_x^\infty$ , for every  $x \in \mathfrak{H}'$ , and in particular the process  $(D_\infty(\Lambda'(t), \partial_1\mathfrak{H}'))_{t \geq 0}$  is distributed as a three-dimensional Bessel process started from 0.*

Before turning to the proof of Theorem 11, we state a preliminary lemma. For every  $\varepsilon > 0$ , we let  $\mathfrak{T}_\infty^\varepsilon$  be the closed subset of  $\mathfrak{T}_\infty$  consisting of the part  $[0, \varepsilon]$  of the spine and of the subtrees branching off  $[0, \varepsilon]$ .

**Lemma 12.** *For every  $\varepsilon > 0$ , the following properties hold a.s.*

- (i) *Labels vanish both on  $\mathfrak{T}_\infty^\varepsilon \setminus \{0\}$  and on  $\mathfrak{T}_\infty \setminus \mathfrak{T}_\infty^\varepsilon$ .*
- (ii) *We have*

$$\inf_{v \in \mathfrak{T}_\infty \setminus \mathfrak{T}_\infty^\varepsilon} D_\infty(0, v) > 0.$$

*Proof.* (i) To prove that labels vanish on  $\mathfrak{T}_\infty \setminus \mathfrak{T}_\infty^\varepsilon$ , it is enough to verify that there exists  $j \in J_\infty$  with  $t_j > \varepsilon$ , such that  $W_*(\omega^j) \leq 0$ . Recall that, conditionally on  $(R_t)_{t \geq 0}$ , the measure  $\mathcal{N}_\infty$  is Poisson with intensity  $2\mathbf{1}_{\{t \geq 0\}} dt \mathbb{N}_{\sqrt{3}R_t}(d\omega)$ . Consequently, an application of (1) gives

$$\mathbb{P}\left(W_*(\omega^j) > 0 \text{ for every } j \in J_\infty \text{ such that } t_j > \varepsilon\right) = \mathbb{E}\left[\exp\left(-\int_\varepsilon^\infty \frac{dt}{R_t^2}\right)\right].$$

The fact that  $\int_\varepsilon^\infty dt R_t^{-2} = \infty$ , a.s., yields the desired result. The same argument applies to verify that labels vanish on  $\mathfrak{T}_\infty^\varepsilon \setminus \{0\}$ .

(ii) Let  $v_{(\varepsilon)}$  be the last point of  $\mathfrak{T}_\infty^\varepsilon \cap \partial\mathfrak{T}_\infty$  visited by the exploration  $(\mathcal{E}_t^\infty)_{t \geq 0}$  of  $\mathfrak{T}_\infty$  and let  $r_{(\varepsilon)} \in (0, \infty)$  such that  $\mathcal{E}_{r_{(\varepsilon)}}^\infty = v_{(\varepsilon)}$ . We then claim that, for any  $v \in \mathfrak{T}_\infty \setminus \mathfrak{T}_\infty^\varepsilon$ ,

$$D_\infty(0, v) \geq \inf_{u \in \llbracket v_{(\varepsilon)}, \infty \rrbracket} D_\infty(0, u), \quad (31)$$

where  $\llbracket v_{(\varepsilon)}, \infty \rrbracket$  stands for the geodesic line connecting  $v_{(\varepsilon)}$  to  $\infty$  in  $\mathfrak{T}_\infty$ . Let us justify our claim. The continuity of  $v \mapsto D_\infty(0, v)$  allows us to assume that  $v \notin \partial\mathfrak{T}_\infty$ . Then, let  $\delta \in (0, \varepsilon)$ . We observe that, in formula (30) applied to  $D_\infty(\delta, v)$ , we may restrict our attention to points  $a_0, a_1, \dots, a_p$  such that there is (at least) one value of  $j \in \{1, \dots, p-1\}$  such that  $a_j \in \llbracket v_{(\varepsilon)}, \infty \rrbracket$ : if not the case, by letting  $k$  be the first index  $j \in \{1, \dots, p\}$  such that  $a_j \in \{\mathcal{E}_t^\infty : t > r_{(\varepsilon)}\}$ , we would have  $v_{(\varepsilon)} \in [a_{k-1}, a_k]_\infty'$ , and thus  $D_\infty^\circ(a_{k-1}, a_k) = \infty$ . It follows that  $D_\infty(\delta, v)$  is bounded below by the right-hand side of (31), and our claim follows by letting  $\delta \rightarrow 0$ .

The proof then reduces to checking that the right-hand side of (31) is positive. We argue by contradiction. Assume that there is a sequence  $(u_n)_{n \geq 1}$  in  $\llbracket v_{(\varepsilon)}, \infty \rrbracket$  such that  $D_\infty(0, u_n) \rightarrow 0$  as  $n \rightarrow \infty$ . The bound  $D_\infty(a, b) \geq |\ell_a^\infty - \ell_b^\infty|$  ensures that  $\ell_{u_n}^\infty \rightarrow 0$  and this implies that  $u_n \rightarrow v_{(\varepsilon)}$  in  $\mathfrak{T}_\infty$ . It follows that  $D_\infty(0, v_{(\varepsilon)}) = 0$ , which contradicts Proposition 10.  $\square$

*Proof of Theorem 11.* For the sake of simplicity, we will not consider the decorating curve and we will content ourselves with proving that the random pointed measure metric space  $(\mathfrak{H}', D_\infty, \mathbf{V}_\infty, \Pi_\infty(0))$  is a Brownian half-plane, whose boundary is  $\partial_0 \mathfrak{H}' \cup \partial_1 \mathfrak{H}'$ . With a little more work, the arguments that follow can be extended to also include the decorating curve (instead of Theorem 9 above, we need the precise form of [25, Theorem 16] including the decorating curve of  $(\mathbb{U}, D_\star, \mathbf{V}_\star)$ , which is in fact defined in a way very similar to  $\Lambda'$ ).

Recall the construction of the Brownian disk  $(\mathbb{U}, D_\star, \mathbf{V}_\star, \Pi_\star(0))$  in the previous section. The scaling property of the Brownian disk implies that, for every  $\lambda > 0$ , the random pointed measure metric space  $\lambda \cdot \mathbb{U} := (\mathbb{U}, \lambda D_\star, \lambda^4 \mathbf{V}, \Pi_\star(0))$  is a free Brownian disk with a random perimeter distributed according to the measure  $\frac{3}{2} \lambda^{3/2} z^{-5/2} \mathbf{1}_{\{z > \lambda\}} dz$ , which is pointed at a uniform boundary point. We will prove that

$$\lambda \cdot \mathbb{U} \xrightarrow[\lambda \rightarrow \infty]{(d)} \mathfrak{H}' \quad (32)$$

in distribution in the sense of the space  $\mathbb{M}_\infty^{\text{GHPU}}$  (recall that our pointed measure metric spaces are viewed as elements of  $\mathbb{M}_\infty^{\text{GHPU}}$  whose decorating curve is constant). The fact that  $(\mathfrak{H}', D_\infty, \mathbf{V}_\infty, \Pi_\infty(0))$  is a Brownian half-plane will follow since one knows that the Brownian half-plane is the limit in distribution (in the space  $\mathbb{M}_\infty^{\text{GHPU}}$ ) of Brownian disks pointed at a uniform boundary point whose boundary size tends to  $\infty$  — this follows from the coupling argument already used at the beginning of the proof of Theorem 1.

The proof of (32) is based again on a coupling argument. Let  $K > 0$  and  $\delta > 0$ . Our claim (32) will follow if we can prove that, for  $\lambda$  large enough, we can couple  $\mathbb{U}$  and  $\mathfrak{H}'$  in such a way that the balls  $B_K(\lambda \cdot \mathbb{U})$  and  $B_K(\mathfrak{H}')$  are isometric with probability at least  $1 - \delta$  (we require that the isometry preserves the volume measure and the distinguished point). Equivalently, using a scaling argument, it suffices to prove that, for  $\eta > 0$  small enough,  $\mathbb{U}$  and  $\mathfrak{H}'$  can be coupled so that  $B_\eta(\mathbb{U})$  and  $B_\eta(\mathfrak{H}')$  are isometric with probability at least  $1 - \delta$  (again with an isometry preserving the volume measure and the distinguished point).

Recall the point measure  $\mathcal{N} = \sum_{j \in J} \delta_{(t_j, \omega^j)}$  introduced at the beginning of Section 6.1 and used to construct  $\mathfrak{T}_\star$ , and the point measure  $\mathcal{N}_\infty = \sum_{j \in J_\infty} \delta_{(t_j, \omega^j)}$  used to construct  $\mathfrak{T}_\infty$ . For every  $\varepsilon > 0$ , set

$$\mathcal{N}^\varepsilon = \sum_{j \in J, t_j \leq \varepsilon} \delta_{(t_j, \omega^j)}, \quad \mathcal{N}_\infty^\varepsilon = \sum_{j \in J_\infty, t_j \leq \varepsilon} \delta_{(t_j, \omega^j)}.$$

We now fix  $\delta > 0$  and claim that:

1. For  $\varepsilon \in (0, 1)$  small enough,  $(\mathbf{e}, \mathcal{N})$  and  $(R, \mathcal{N}_\infty)$  can be coupled in such a way that the equality  $((\mathbf{e}_t)_{t \leq \varepsilon}, \mathcal{N}^\varepsilon) = ((R_t)_{t \leq \varepsilon}, \mathcal{N}_\infty^\varepsilon)$  holds with probability at least  $1 - \frac{\delta}{2}$ .

2. For  $\varepsilon \in (0, 1)$  small enough, we can choose  $\eta_0 > 0$  so that for every  $0 < \eta \leq \eta_0$ , we have

$$B_\eta(\mathbb{U}) = B_\eta(\mathfrak{H}')$$

on the event where  $((\mathbf{e}_t)_{t \leq \varepsilon}, \mathcal{N}^\varepsilon) = ((R_t)_{t \leq \varepsilon}, \mathcal{N}_\infty^\varepsilon)$ , except possibly on an event of probability at most  $\frac{\delta}{2}$ .

The formula  $B_\eta(\mathbb{U}) = B_\eta(\mathfrak{H}')$  in Property 2 is understood as an equality of pointed measure metric spaces modulo isometries ( $B_\eta(\mathbb{U})$ , resp.  $B_\eta(\mathfrak{H}')$ ), is equipped with the distinguished point and the restriction of the volume measure of  $\mathbb{U}$ , resp. of  $\mathfrak{H}'$ .

As explained above, our claim (32) follows from Properties 1 and 2. Property 1 is a consequence of the following classical fact. For every  $\delta$ , we can find  $\varepsilon \in (0, 1)$  and a coupling between  $\mathbf{e}$  and  $R$  such that the equality  $(\mathbf{e}_t)_{0 \leq t \leq \varepsilon} = (R_t)_{0 \leq t \leq \varepsilon}$  holds with probability at least  $1 - \frac{\delta}{2}$  (see e.g. Proposition 3 in [7] and the proof of Proposition 4 in the same reference for a stronger statement).

It remains to verify Property 2. By Property 1, we can fix  $\varepsilon \in (0, 1)$  small and assume that the event where  $((\mathbf{e}_t)_{t \leq \varepsilon}, \mathcal{N}^\varepsilon) = ((R_t)_{t \leq \varepsilon}, \mathcal{N}_\infty^\varepsilon)$  has probability greater than  $1 - \frac{\delta}{2}$ . We denote the latter event by  $\mathcal{A}_1$ .

By Lemma 12, we can fix  $\eta > 0$  small enough so that the set  $\{v \in \mathfrak{T}_\infty : D_\infty(0, v) \leq 4\eta\}$  is contained in  $\mathfrak{T}_\infty^\varepsilon$ , except possibly on an event of probability at most  $\frac{\delta}{6}$ . Moreover, if the latter property holds, we have also

$$D_\infty(u, v) = \inf_{\substack{u_0=u, u_1, \dots, u_p=v \\ u_1, \dots, u_{p-1} \in \mathfrak{T}_\infty^\varepsilon \setminus \partial \mathfrak{T}_\infty}} \sum_{i=1}^p D_\infty^\circ(u_{i-1}, u_i), \quad (33)$$

for every  $u, v \in \mathfrak{T}_\infty^\varepsilon \setminus \partial \mathfrak{T}_\infty$  such that  $D_\infty(0, u) \leq \eta$  and  $D_\infty(0, v) \leq \eta$ . Let us explain why (33) holds. Suppose that  $u, v \in \mathfrak{T}_\infty^\varepsilon \setminus \partial \mathfrak{T}_\infty$  are such that  $D_\infty(0, u) \leq \eta$ ,  $D_\infty(0, v) \leq \eta$ , so that in particular  $D_\infty(u, v) \leq 2\eta$ . We can find  $u_0 = u, u_1, \dots, u_q = v$  in  $\mathfrak{T}_\infty \setminus \partial \mathfrak{T}_\infty$  such that

$$\sum_{i=1}^q D_\infty^\circ(u_{i-1}, u_i) < D_\infty(u, v) + \eta \leq 3\eta.$$

The triangle inequality then implies that  $D_\infty(u, u_i) < 3\eta$  and  $D_\infty(0, u_i) < 4\eta$ , for every  $i \in \{0, 1, \dots, q\}$ . Since we assumed that  $\{v \in \mathfrak{T}_\infty : D_\infty(0, v) \leq 4\eta\}$  is contained in  $\mathfrak{T}_\infty^\varepsilon$ , it follows that  $u_i \in \mathfrak{T}_\infty^\varepsilon$ , for every  $i \in \{0, 1, \dots, q\}$ . In other words, in formula (30) defining  $D_\infty(u, v)$ , we may restrict the infimum to the case where all  $u_i$ 's belong to  $\mathfrak{T}_\infty^\varepsilon$ . This gives our claim (33). Furthermore, when applying formula (29) to compute the quantities  $D_\infty^\circ(u_{i-1}, u_i)$  in the right-hand side of (33), we can restrict our attention to the case when the interval  $[u_{i-1}, u_i]_\infty'$  (resp.  $[u_i, u_{i-1}]_\infty'$ ) is contained in  $\mathfrak{T}_\infty^\varepsilon$ , since otherwise this interval contains the complement of  $\mathfrak{T}_\infty^\varepsilon$  and then the infimum of labels on  $[u_{i-1}, u_i]_\infty'$  is 0 by Lemma 12 (i).

Let us now discuss  $D_\star(u, v)$  when  $u, v \in B_\eta(\mathbb{U})$ . We write  $\mathfrak{T}^{\star, \varepsilon}$  for the closed subset of  $\mathfrak{T}^\star$  consisting of the part  $[0, \varepsilon]$  of the ‘‘spine’’  $[0, 1]$  and the subtrees branching off  $[0, \varepsilon]$ . Then, we have

$$\inf_{v \in \mathfrak{T}^\star \setminus \mathfrak{T}^{\star, \varepsilon}} D_\star(0, v) > 0, \quad \text{a.s. ,}$$

since we know that the equivalence class of 0 in  $\mathbb{U}$  is a singleton. Hence, for  $\eta > 0$  small, we get that the event where  $\{v \in \mathfrak{T}^\star : D_\star(0, v) \leq 4\eta\} \subset \mathfrak{T}^{\star, \varepsilon}$  has probability at least  $1 - \frac{\delta}{6}$ . On the latter event, the same argument as for (33) then shows that, for every  $u, v \in \mathfrak{T}^{\star, \varepsilon} \setminus \partial \mathfrak{T}^\star$  such that  $D_\star(0, u) \leq \eta$  and  $D_\star(0, v) \leq \eta$ , we have

$$D_\star(u, v) = \inf_{\substack{u_0=u, u_1, \dots, u_p=v \\ u_1, \dots, u_{p-1} \in \mathfrak{T}^{\star, \varepsilon} \setminus \partial \mathfrak{T}^\star}} \sum_{i=1}^p D_\star^\circ(u_{i-1}, u_i), \quad (34)$$

and moreover, when applying formula (27) to compute  $D_\star^\circ(u_{i-1}, u_i)$  we may discard the case when the interval  $[u_{i-1}, u_i]_\star$  (resp.  $[u_i, u_{i-1}]_\star$ ) is not contained in  $\mathfrak{T}^{\star, \varepsilon}$ .

On the event  $\mathcal{A}_1$ , the labeled tree  $\mathfrak{T}^{\star, \varepsilon}$  is identified with  $\mathfrak{T}_\infty^\varepsilon$ . Moreover, fixing  $\eta > 0$  small enough and discarding an event  $\mathcal{A}_2$  of probability at most  $\frac{\delta}{3}$ , we deduce from formulas (33) and (34), that we have  $D_\star(u, v) = D_\infty(u, v)$  whenever  $u, v \in \mathfrak{T}^{\star, \varepsilon} = \mathfrak{T}_\infty^\varepsilon$  are such that  $D_\star(0, u) \vee D_\star(0, v) \leq \eta$  (which is

equivalent to  $D_\infty(0, u) \vee D_\infty(0, v) \leq \eta$  by (33) and (34)). On the event  $\mathcal{A}_1 \setminus \mathcal{A}_2$ , we also know that  $\{v \in \mathfrak{T}_\infty : D_\infty(0, v) \leq 4\eta\} \subset \mathfrak{T}_\infty^\varepsilon$  and  $\{v \in \mathfrak{T}^* : D_\star(0, v) \leq 4\eta\} \subset \mathfrak{T}^{\star, \varepsilon}$ . It then follows from this discussion that, still on the event  $\mathcal{A}_1 \setminus \mathcal{A}_2$ , the identification of  $\mathfrak{T}^{\star, \varepsilon}$  with  $\mathfrak{T}_\infty^\varepsilon$  induces an isometry from  $B_\eta(\mathbb{U})$  onto  $B_\eta(\mathfrak{H}')$ , which clearly preserves the volume measures and the distinguished points. This completes the proof of Property 2 and of the first assertion of the theorem.

Let us finally discuss the boundary  $\partial\mathfrak{H}'$  of  $\mathfrak{H}'$ . Recall that  $\partial\mathfrak{H}'$  is defined as the set of all points of  $\mathfrak{H}'$  that have no neighborhood homeomorphic to the open unit disk. The preceding identification of  $B_\eta(\mathbb{U})$  with  $B_\eta(\mathfrak{H}')$  (except on an event of probability at most  $\delta$ ) also shows that, for every  $\vartheta \in (0, \eta)$ , the set  $\partial\mathfrak{H}' \cap B_\vartheta(\mathfrak{H}')$  is identified with  $\partial\mathbb{U} \cap B_\vartheta(\mathbb{U})$ , which we know to be equal to  $(\partial_0\mathbb{U} \cup \partial_1\mathbb{U}) \cap B_\vartheta(\mathbb{U})$ . It follows that, except possibly on an event of probability at most  $\delta$ , we have  $\partial\mathfrak{H}' \cap B_\vartheta(\mathfrak{H}') = (\partial_0\mathfrak{H}' \cup \partial_1\mathfrak{H}') \cap B_\vartheta(\mathfrak{H}')$ , for every  $\vartheta \in (0, \eta)$ . By scale invariance, the latter equality must hold for every  $\eta > 0$  (except on an event of probability at most  $\delta$ ). Since  $\delta$  was arbitrary, we conclude that  $\partial\mathfrak{H}' = \partial_0\mathfrak{H}' \cup \partial_1\mathfrak{H}'$ .

The last assertion of the theorem is easy. Let  $x \in \mathfrak{H}'$ . Then, we have  $D_\infty(x, y) \geq \ell_x^\infty$  for every  $y \in \partial_1\mathfrak{H}'$  from the bound  $D_\infty(x, y) \geq |\ell_x^\infty - \ell_y^\infty|$  and the fact that  $\ell_y^\infty = 0$  if  $y \in \partial_1\mathfrak{H}'$ . Conversely, let  $t \geq 0$  be such that  $x = \Pi_\infty(\mathcal{E}_t^\infty)$ , and let  $r = \inf\{s \geq t : \mathcal{E}_s^\infty \in \partial\mathfrak{T}_\infty\}$ . Then,  $y := \Pi_\infty(\mathcal{E}_r^\infty)$  belongs to  $\partial_1\mathfrak{H}'$ , and it is easy to verify that  $D_\infty(x, y) = \ell_x^\infty$ .  $\square$

In the last part of this section, we state and prove a consequence of Theorem 11 that will be useful in the next section when we discuss hulls centered on a boundary segment. For simplicity, we write  $\mathbf{x}' = \Lambda'(0)$  for the distinguished point of  $\mathfrak{H}'$ .

**Proposition 13.** *Let  $\eta > 0$ . Then, almost surely, there exists  $\delta > 0$  such that the following holds. For every  $x \in \mathfrak{H}'$  with  $D_\infty(\mathbf{x}', x) \geq \eta$ , there is a geodesic from  $x$  to  $\partial_1\mathfrak{H}' = \{\Lambda'(t) : t \in (-\infty, 0]\}$  that ends at a point of  $\{\Lambda'(t) : t \in (-\infty, -\delta]\}$ .*

*Proof.* For  $a \in \mathfrak{T}_\infty$ , let  $s$  be the smallest time such that  $\mathcal{E}_s^\infty = a$ , and, for every  $u \in [0, \ell_a^\infty]$ , set

$$\phi_a(u) = \sup\{v \leq s : \ell_{\mathcal{E}_v^\infty}^\infty = u\}$$

so that  $\mathcal{E}_{\phi_a(u)}^\infty$  is the ‘‘last’’ point before  $a$  with label  $u$ . It easily follows from our definitions that  $D_\infty^\circ(\mathcal{E}_{\phi_a(u)}^\infty, \mathcal{E}_{\phi_a(v)}^\infty) = v - u$  for every  $0 \leq u \leq v \leq \ell_a^\infty$ . Note that  $\mathcal{E}_{\phi_a(\ell_a^\infty)}^\infty = a$ , and  $\mathcal{E}_{\phi_a(0)}^\infty \in \partial\mathfrak{T}_\infty$ , so that  $\Pi_\infty(\mathcal{E}_{\phi_a(0)}^\infty) \in \partial_1\mathfrak{H}'$ . Recalling that  $D_\infty(\Pi_\infty(a), \partial_1\mathfrak{H}') = \ell_a^\infty$ , we obtain that the path  $(\Pi_\infty(\mathcal{E}_{\phi_a(\ell_a^\infty - u)}^\infty), 0 \leq u \leq \ell_a^\infty)$  is a geodesic from  $\Pi_\infty(a)$  to  $\partial_1\mathfrak{H}'$ . Let us write  $\Phi_a$  for this geodesic.

Then, almost surely, we can find  $\varepsilon > 0$  such that, for any  $x \in \mathfrak{H}'$  with  $D_\infty(\mathbf{x}', x) \geq \eta$ , we have  $x = \Pi_\infty(a)$  for some  $a \in \mathfrak{T}_\infty \setminus \mathfrak{T}_\infty^\varepsilon$ , with the notation introduced before Lemma 12. But then, the geodesic  $\Phi_a$  hits  $\partial\mathfrak{H}'_1$  at a point of the form  $\Pi_\infty(\mathcal{E}_r^\infty)$  with  $r \geq r_\varepsilon$ , where  $\mathcal{E}_{r_\varepsilon}^\infty$  is the last point of  $\mathfrak{T}_\infty^\varepsilon$  with zero label. By Lemma 12 and the support property of exit local times, we have  $\Pi_\infty(\mathcal{E}_{r_\varepsilon}^\infty) = \Lambda(-\delta)$  for some  $\delta > 0$ , and it follows that the geodesic  $\Phi_a$  hits  $\partial_1\mathfrak{H}'$  at a point of  $\{\Lambda'(t) : t \in (-\infty, -\delta]\}$ .  $\square$

## 7 Hulls centered on a boundary segment

In this section, we give an analog of Theorem 1 for hulls centered on a segment of the boundary. We consider the curve-decorated Brownian half-plane  $(\mathfrak{H}, D, V, \Lambda)$ . Let  $r > 0$  and  $s > 0$ . Let  $\check{B}_r^\circ(\mathfrak{H}, [0, s])$  be the unique unbounded component of the open set

$$\{x \in \mathfrak{H} : D(x, \Lambda([0, s])) > r\}.$$

We write  $\check{B}_r^\bullet(\mathfrak{H}, [0, s])$  for the closure of  $\check{B}_r^\circ(\mathfrak{H}, [0, s])$ , and we set  $B_r^\circ(\mathfrak{H}, [0, s]) = \mathfrak{H} \setminus \check{B}_r^\bullet(\mathfrak{H}, [0, s])$  and  $B_r^\bullet(\mathfrak{H}, [0, s]) = \mathfrak{H} \setminus \check{B}_r^\circ(\mathfrak{H}, [0, s])$ . We also set

$$\mathbf{x}_s := \Lambda(\inf\{t \in \mathbb{R} : \Lambda(t) \notin \check{B}_r^\circ(\mathfrak{H}, [0, s])\}).$$

**Theorem 14.** *The intrinsic metric on  $\check{B}_r^\circ(\mathfrak{H}, [0, s])$  (associated with the metric  $D$  on  $\mathfrak{H}$ ) has a continuous extension to  $\check{B}_r^\bullet(\mathfrak{H}, [0, s])$ , which is a metric on  $\check{B}_r^\bullet(\mathfrak{H}, [0, s])$ . Then the space  $\check{B}_r^\bullet(\mathfrak{H}, [0, s])$  equipped with this extended intrinsic metric, with the restriction of the volume measure  $V$  and with the distinguished point  $\mathbf{x}_s$  is a Brownian half-plane.*



*Remark.* By analogy with Theorem 2, one can also prove that the Brownian half-plane  $\check{B}_r^\bullet(\mathfrak{H}, [0, s])$  of the theorem is independent of the space  $B_r^\bullet(\mathfrak{H}, [0, s])$  equipped with an appropriately defined intrinsic metric, with the restriction of the volume  $V$  and with the distinguished point  $\mathbf{x}_s$ . We will however leave this extension to the reader.

We state a lemma before proving Theorem 14.

**Lemma 15.** *Almost surely, there exists  $\delta \in (0, s)$  such that, for every  $u \in [s - \delta, s]$ ,*

$$\check{B}_r^\bullet(\mathfrak{H}, [0, u]) = \check{B}_r^\bullet(\mathfrak{H}, [0, s]).$$

*Proof.* We claim that, almost surely, for any compact subset  $K$  of  $\mathfrak{H}$  not intersecting  $\Lambda([0, s])$ , there exists  $\delta \in (0, s/2)$  such that

$$D(x, \Lambda([0, s - \delta])) = D(x, \Lambda([0, s])) \quad (35)$$

for every  $x \in K$ . The statement of the lemma easily follows from (35) by taking  $K = \partial B_r^\bullet(\mathfrak{H}, [0, s])$ .

Let us prove (35). We first fix  $\eta > 0$  so that, if  $y \in \mathfrak{H}$  is such that  $D(y, \Lambda([s/2, s])) \leq \eta$ , then  $D(y, \Lambda((-\infty, 0])) > D(y, \Lambda([s/2, s]))$ . Taking  $\eta$  smaller if necessary, we may assume that  $D(y, \Lambda([0, s])) \geq \eta$  for every  $y \in K$ . Thanks to Proposition 13, we can find  $\delta \in (0, s/2)$  such that, for every  $x \in \mathfrak{H}$  with  $D(x, \Lambda(s)) \geq \eta$ , there is a geodesic from  $x$  to  $\Lambda((-\infty, s])$  that ends at a point of  $\Lambda((-\infty, s - \delta])$  (we use the fact that  $(\mathfrak{H}, D, V, \Lambda(s + \cdot))$  has the same distribution as  $(\mathfrak{H}, D, V, \Lambda)$ ). Then, let  $x \in K$ , and consider a geodesic  $\phi$  from  $x$  to  $\Lambda([0, s])$ . If the geodesic  $\phi$  ends at a point of  $\Lambda([0, s - \delta])$ , (35) clearly holds. Otherwise, on the geodesic  $\phi$ , we can find a point  $z$  at distance  $\eta$  from  $\Lambda([s/2, s])$ , and then our choice of  $\eta$  implies that a geodesic from  $z$  to  $\Lambda([0, s])$  must also be a geodesic from  $z$  to  $\Lambda((-\infty, s])$ . By our choice of  $\delta$ , the part of the geodesic  $\phi$  between  $z$  and  $\Lambda([0, s])$  can be replaced (without increasing its length) by a geodesic from  $z$  to  $\Lambda((-\infty, s - \delta])$ , which must end at a point of  $\Lambda([0, s - \delta])$ , again by the choice of  $\eta$ . We conclude that (35) holds.  $\square$

*Proof of Theorem 14.* From Theorems 1 and 2, we know that

$$\mathfrak{H}_1 = \mathfrak{H} \setminus B_1^\circ(\mathfrak{H})$$

equipped with the (extended) intrinsic metric, with the restriction of the volume measure on  $\mathfrak{H}$  and with the boundary curve  $\Lambda^1$  is a curve-decorated Brownian half-plane, which furthermore is independent of the hull  $B_1^\bullet(\mathfrak{H})$  also viewed as a curve-decorated measure metric space (for the appropriate intrinsic metric). In particular, the curve-decorated Brownian half-plane  $\mathfrak{H}_1$  is independent of the perimeter  $Z_1$  of  $B_1^\bullet(\mathfrak{H})$ , which we know to be exponentially distributed with parameter  $3/2$  (Proposition 3). Note that, by definition,  $\Lambda^1([0, Z_1]) = \partial B_1^\bullet(\mathfrak{H})$ .

Now observe that

$$\check{B}_r^\circ(\mathfrak{H}_1, [0, Z_1]) = \mathfrak{H}_1 \setminus B_r^\bullet(\mathfrak{H}_1, [0, Z_1]) = \mathfrak{H} \setminus B_{1+r}^\bullet(\mathfrak{H}),$$

because a curve connecting a point of  $\mathfrak{H}_1$  to infinity stays at distance greater than  $r$  from  $\Lambda^1([0, Z_1]) = \partial B_1^\bullet(\mathfrak{H})$  if and only if it stays at distance greater than  $1 + r$  from  $\mathbf{x}$ . From the last display and Theorems 1 and 2 applied to  $\mathfrak{H}_{1+r}$ , we infer that:

- (i) almost surely, the intrinsic metric on  $\check{B}_r^\circ(\mathfrak{H}_1, [0, Z_1])$  has a continuous extension to its closure  $\check{B}_r^\bullet(\mathfrak{H}_1, [0, Z_1])$ , which is a metric on  $\check{B}_r^\bullet(\mathfrak{H}_1, [0, Z_1])$ ;
- (ii)  $\check{B}_r^\bullet(\mathfrak{H}_1, [0, Z_1])$  equipped with this extended metric (and with the restriction of the volume measure on  $\mathfrak{H}$ , and the distinguished point  $\mathbf{x}_{1+r}$ ) is a Brownian half-plane, which is independent of  $Z_1$ .

In particular, the independence property in (ii) holds because the half-plane  $\mathfrak{H} \setminus B_{1+r}^\bullet(\mathfrak{H})$  is independent of the hull  $B_{1+r}^\bullet(\mathfrak{H})$  and therefore also of  $B_1^\bullet(\mathfrak{H})$ .

We then observe that (i) still holds if  $Z_1$  is replaced by a fixed value  $s > 0$ . Indeed, if this was not true, Lemma 15 would allow us to find  $\delta \in (0, s)$  such that property (i) written with  $Z_1$  replaced by  $u$  would fail for every  $u \in [s - \delta, s]$  with positive probability. Clearly this is a contradiction since  $\mathbb{P}(s - \delta \leq Z_1 \leq s) > 0$ . This gives the first assertion of the theorem.

To get the second assertion, let  $g$  be a bounded continuous function on  $\mathbb{R}_+$ , and let  $F$  be a bounded continuous function on the space  $\mathbb{M}_{\text{GHPU}}^\infty$ . Also write  $\Theta$  for the distribution of the Brownian half-plane (viewed as a pointed measure metric space). It follows from (i) and (ii) that

$$\mathbb{E}[g(Z_1) F(\check{B}_r^\bullet(\mathfrak{H}_1, [0, Z_1]))] = \mathbb{E}[g(Z_1)] \Theta(F) = \frac{3}{2} \Theta(F) \int_0^\infty g(s) e^{-3s/2} ds.$$

On the other hand, we have also, using the independence of  $\mathfrak{H}_1$  and  $Z_1$ ,

$$\mathbb{E}[g(Z_1) F(\check{B}_r^\bullet(\mathfrak{H}_1, [0, Z_1]))] = \frac{3}{2} \int_0^\infty g(s) \mathbb{E}[F(\check{B}_r^\bullet(\mathfrak{H}_1, [0, s]))] e^{-3s/2} ds.$$

It follows that, for Lebesgue almost every  $s > 0$ ,

$$\mathbb{E}[F(\check{B}_r^\bullet(\mathfrak{H}_1, [0, s]))] = \Theta(F),$$

or equivalently, since  $\mathfrak{H}_1$  and  $\mathfrak{H}$  have the same distribution,

$$\mathbb{E}[F(\check{B}_r^\bullet(\mathfrak{H}, [0, s]))] = \Theta(F).$$

To complete the proof, we need to verify that this property in fact holds for *every*  $s > 0$ . To this end, we just notice that the mapping  $u \mapsto \mathbb{E}[F(\check{B}_r^\bullet(\mathfrak{H}, [0, u]))]$  is left-continuous, as an immediate consequence of Lemma 15.  $\square$

## Appendix

In this appendix, we prove Lemmas 4 and 5. Recall that  $(\mathcal{R}_t)_{t \geq 0}$  is a five-dimensional Bessel process started from 0.

*Proof of Lemma 4.* Fix  $0 < x < y$  and  $c$  such that  $0 < c < x$ , and write  $\tilde{\mathcal{R}}_t = \mathcal{R}_{(\mathcal{L}_y - t) \vee 0}$  for every  $t \geq 0$ . Also set  $T_x = \mathcal{L}_y - \mathcal{L}_x = \inf\{t \geq 0 : \tilde{\mathcal{R}}_t = x\}$ . As a consequence of Nagasawa's time-reversal theorem, we know that  $(\tilde{\mathcal{R}}_t)_{t \geq 0}$  is a Bessel process of dimension  $-1$  started at  $y$ , and we have

$$\mathbb{E}\left[\exp\left(-\int_{\mathcal{L}_x}^{\mathcal{L}_y} dt \left((\mathcal{R}_t - c)^{-2} - (\mathcal{R}_t)^{-2}\right)\right)\right] = \mathbb{E}\left[\exp\left(-\int_0^{T_x} dt \left((\tilde{\mathcal{R}}_t - c)^{-2} - (\tilde{\mathcal{R}}_t)^{-2}\right)\right)\right].$$

For every  $0 < u < v$ , let  $B = (B_t)_{t \geq 0}$  stand for a linear Brownian motion that starts at  $v$  under the probability measure  $\mathbb{P}_v$  and let  $T_u^{(B)} = \inf\{t \geq 0 : B_t = u\}$ . Using the absolute continuity properties of Bessel processes with respect to Brownian motion (see e.g. [18, Lemma 1]), we have

$$\begin{aligned} \mathbb{E}\left[\exp\left(-\int_0^{T_x} dt \left((\tilde{\mathcal{R}}_t - c)^{-2} - (\tilde{\mathcal{R}}_t)^{-2}\right)\right)\right] &= \frac{y}{x} \mathbb{E}_y\left[\exp\left(-\int_0^{T_x^{(B)}} dt (B_t - c)^{-2}\right)\right] \\ &= \frac{y}{x} \mathbb{E}_{y-c}\left[\exp\left(-\int_0^{T_{x-c}^{(B)}} dt (B_t)^{-2}\right)\right] \\ &= \frac{y}{x} \times \frac{x-c}{y-c}, \end{aligned}$$

where the last equality is classical and follows from an application of the optional stopping theorem to the martingale

$$\frac{1}{B_{t \wedge T_{x-c}^{(B)}}} \exp\left(-\int_0^{t \wedge T_{x-c}^{(B)}} ds (B_s)^{-2}\right)$$

under  $\mathbb{P}_{y-c}$ .  $\square$

*Proof of Lemma 5.* We use the same notation as in the proof of Lemma 4, taking now  $y = r/\sqrt{3}$ . In particular,  $\tilde{\mathcal{R}}_t = R_{(\mathcal{L}_{y-t})\vee 0}$ , and  $T_x = \inf\{t \geq 0 : \tilde{\mathcal{R}}_t = x\}$  for every  $x \in [0, y]$ . By the same time-reversal argument, we have

$$\begin{aligned} \mathbb{E}\left[\exp\left(-\nu\gamma_r - 2\int_0^{\gamma_r} dt g_\mu(\sqrt{3}\mathcal{R}_t)\right)\right] &= \mathbb{E}\left[\exp\left(-\int_0^{T_0} dt (\nu + 2g_\mu(\sqrt{3}\tilde{\mathcal{R}}_t))\right)\right] \\ &= \lim_{\varepsilon \rightarrow 0} \mathbb{E}\left[\exp\left(-\int_0^{T_\varepsilon} dt (\nu + 2g_\mu(\sqrt{3}\tilde{\mathcal{R}}_t))\right)\right] \\ &= \lim_{\varepsilon \rightarrow 0} \frac{y}{\varepsilon} \mathbb{E}_y\left[\exp\left(-\int_0^{T_\varepsilon^{(B)}} dt (\nu + 2g_\mu(\sqrt{3}B_t) + (B_t)^{-2})\right)\right], \end{aligned}$$

where the last equality relies on the same absolute continuity argument as in the proof of Lemma 4.

Observe that

$$\nu + 2g_\mu(\sqrt{3}B_t) + (B_t)^{-2} = \nu + \sqrt{2\mu}\left(3 \coth\left((2\mu)^{1/4}\sqrt{3}B_t\right)^2 - 2\right).$$

Set  $a := (2\mu)^{1/4}$ , and then, for every  $s > 0$ ,

$$f(s) := \nu + a^2\left(3 \coth(a\sqrt{3}s)^2 - 2\right).$$

Then, if

$$F(s) := \exp\left(-s\sqrt{2(a^2 + \nu)}\right) \left(a \coth(a\sqrt{3}s) + \sqrt{\frac{2}{3}(a^2 + \nu)}\right),$$

a direct calculation shows that

$$F''(s) = 2f(s)F(s).$$

By a simple application of Itô's formula, it follows that, for  $\varepsilon \in (0, y)$ ,

$$F(B_{t \wedge T_\varepsilon^{(B)}}) \exp\left(-\int_0^{t \wedge T_\varepsilon^{(B)}} ds f(B_s)\right)$$

is a (bounded) martingale under  $\mathbb{P}_y$ . The optional stopping theorem then gives

$$\mathbb{E}_y\left[\exp\left(-\int_0^{T_\varepsilon^{(B)}} dt f(B_t)\right)\right] = \frac{F(y)}{F(\varepsilon)},$$

and thus

$$\frac{y}{\varepsilon} \mathbb{E}_y\left[\exp\left(-\int_0^{T_\varepsilon^{(B)}} dt (\nu + 2g_\mu(\sqrt{3}B_t) + (B_t)^{-2})\right)\right] = \frac{y}{\varepsilon} \mathbb{E}_y\left[\exp\left(-\int_0^{T_\varepsilon^{(B)}} dt f(B_t)\right)\right] = \frac{yF(y)}{\varepsilon F(\varepsilon)},$$

which converges when  $\varepsilon \rightarrow 0$  to  $\sqrt{3}yF(y) = rF(r/\sqrt{3}) = G_r(\mu, \nu)$ . This completes the proof.  $\square$

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