The hull process of the Brownian plane

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Abstract

We study the random metric space called the Brownian plane, which is closely related to the Brownian map and is conjectured to be the universal scaling limit of many discrete random lattices such as the uniform infinite planar triangulation. We obtain a number of explicit distributions for the Brownian plane. In particular, we consider, for every $r > 0$, the hull of radius r, which is obtained by "filling in the holes" in the ball of radius r centered at the root. We introduce a quantity Z_r which is interpreted as the (generalized) length of the boundary of the hull of radius r. We identify the law of the process $(Z_r)_{r>0}$ as the time-reversal of a continuous-state branching process starting from $+\infty$ at time $-\infty$ and conditioned to hit 0 at time 0, and we give an explicit description of the process of hull volumes given the process $(Z_r)_{r>0}$. We obtain an explicit formula for the Laplace transform of the volume of the hull of radius r, and we also determine the conditional distribution of this volume given the length of the boundary. Our proofs involve certain new formulas for super-Brownian motion and the Brownian snake in dimension one, which are of independent interest.

1 Introduction

Much recent work has been devoted to understanding continuous limits of random graphs drawn on the two-dimensonal sphere or in the plane, which are called random planar maps. A fundamental object is the random compact metric space known as the Brownian map, which has been proved to be the universal scaling limit of several important classes of random planar maps conditioned to have a large size (see in particular [1, 3, 6, 23, 30]). The main goal of this work is to study the random (non-compact) metric space called the Brownian plane, which may be viewed as an infinite-volume version of the Brownian map. The Brownian plane was first introduced and studied in [9], where it was shown to be the scaling limit in distribution of the uniform infinite planar quadrangulation (UIPQ) in the local Gromov-Hausdorff sense. The Brownian plane is in fact conjectured to be the universal scaling limit of many discrete random lattices including the uniform infinite planar triangulation (UIPT) introduced by Angel and Schramm [5] and studied then by several authors. It was proved in [9] that the Brownian plane is locally isometric to the Brownian map, in the following sense. Recalling that both the Brownian map and the Brownian plane are equipped with a distinguished point called the root, one can couple these two random metric spaces in such a way that, for every $\delta > 0$, there exists $\varepsilon > 0$ such that the balls of radius ε centered at the root in the two spaces are isometric with probability at least $1 - \delta$. As a consequence, the Brownian plane shares many properties of the Brownian map. On the other hand, the Brownian plane also enjoys the important additional property of invariance under scaling: Multiplying the distance by a constant factor $\lambda > 0$ does not change the distribution of the Brownian plane. This property suggests that the Brownian plane should be more tractable for calculations than the Brownian map, for which very few explicit distributions are known. Our purpose is to obtain such explicit distributions for the Brownian plane, and in particular to give a detailed probabilistic description of the growth of "hulls" centered at the root.

In order to give a more precise presentation of our results, let us introduce some notation. As in [9], we write $(\mathcal{P}_\infty, D_\infty)$ for the Brownian plane, and we let ρ_∞ stand for the distinguished point of $\mathcal{P}_\infty$ called the root. We recall that $\mathcal{P}_\infty$ is equipped with a volume measure, and we write $|A|$ for the volume of a measurable subset of $\mathcal{P}_\infty$. For every $r > 0$, the closed ball of radius r centered at ρ_∞ in $\mathcal{P}_\infty$ is denoted by $B_r(\mathcal{P}_\infty)$. In contrast with the case of Euclidean space, the complement of $B_r(\mathcal{P}_\infty)$ will have infinitely many connected components (see [24] for a detailed discussion of these components in the slightly different setting of the Brownian map) but only one unbounded connected component. We then define the hull of radius r as the complement of the unbounded component of the complement of $B_r(\mathcal{P}_\infty)$, and we denote this hull by $B^\bullet_r(\mathcal{P}_\infty)$. Informally, $B^\bullet_r(\mathcal{P}_\infty)$ is obtained by "filling in the holes" of $B_r(\mathcal{P}_\infty)$ – see Fig. 1 below, and Fig. 3 in Section 5 for a discrete version of the hull.

In what follows, we give a complete description of the law of the process $(|B^\bullet_r(\mathcal{P}_\infty)|)_{r>0}$. To formulate this description, it is convenient to introduce another process $(Z_r)_{r>0}$ which gives for every $r > 0$ the size of the boundary of $B^\bullet_r(\mathcal{P}_\infty)$.

Proposition 1.1. *Let $r > 0$. There exists a positive random variable Z_r such that*

$$\lim_{\varepsilon\to 0} \varepsilon^{-2} |B^\bullet_r(\mathcal{P}_\infty)^c \cap B_{r+\varepsilon}(\mathcal{P}_\infty)| = Z_r$$

in probability.

In view of this proposition, one interprets Z_r as the (generalized) length of the boundary of the hull of radius r (this boundary is expected to be a fractal curve of dimension 2). A key intermediate step in the derivation of our main results is to identify the process $(Z_r)_{r>0}$ as a time-reversed continuous-state branching process. For every $u \geq 0$, set $\psi(u) = \sqrt{8/3}\, u^{3/2}$. The continuous-state branching process with branching mechanism ψ is the Feller Markov process $(X_t)_{t\geq 0}$ with values in $\mathbb{R}_+$, whose semigroup is characterized as follows: for every $x, t \geq 0$ and every $\lambda > 0$,

$$E[e^{-\lambda X_t} \mid X_0 = x] = \exp\Big(-x\Big(\lambda^{-1/2} + \sqrt{2/3}\, t\Big)^{-2}\Big).$$

See subsection 2.1 for a brief discussion of this process. Note that X gets absorbed at 0 in finite time. It is easy to construct a process $(\widetilde{X}_t)_{t\leq 0}$ indexed by the time interval $(-\infty, 0]$ and which is distributed as the process X "started from $+\infty$" at time $-\infty$ and conditioned to hit zero at time 0 (see subsection 2.1 for a more rigorous presentation).

Proposition 1.2. (i) *For every $r > 0$, we have for every $\lambda \geq 0$,*

$$E\Big[\exp(-\lambda Z_r)\Big] = \Big(1 + \frac{2\lambda r^2}{3}\Big)^{-3/2}.$$

Equivalently, Z_r follows a Gamma distribution with parameter $\frac{3}{2}$ and mean r^2.
(ii) *The two processes $(Z_r)_{r>0}$ and $(\widetilde{X}_{-r})_{r>0}$ have the same finite-dimensional marginals.*

We observe that results closely related to Proposition 1.2 have been obtained by Krikun [16, 17] in the discrete setting of the UIPT and the UIPQ.

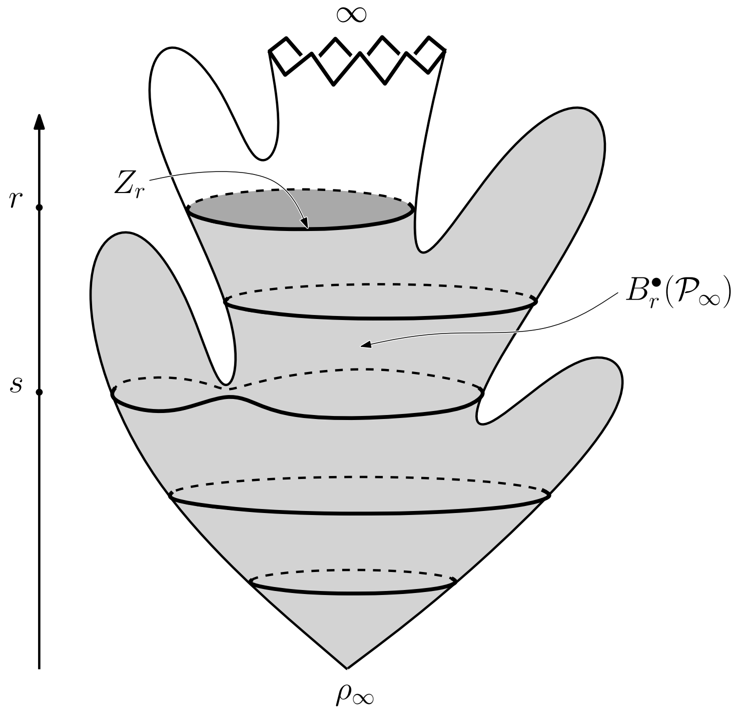

Figure 1: Illustration of the geometric meaning of the processes $(Z_r)_{r\geq 0}$ and $(|B^\bullet_r(\mathcal{P}_\infty)|)_{r\geq 0}$. The Brownian plane is represented as a two-dimensional "cactus" where the height of each point is equal to its distance to the root. The shaded part represents the hull $B^\bullet_r(\mathcal{P}_\infty)$ and Z_r corresponds to the (generalized) length of its boundary. At time s, both processes $Z_.$ and $|B^\bullet_.(\mathcal{P}_\infty)|$ have a jump. Geometrically this corresponds to the creation of a "bubble" above height s.

Part (ii) of the preceding proposition implies that the process $(Z_r)_{r>0}$ has a càdlàg modification, with only negative jumps, and from now on we deal with this modification. We can now state the main results of the present work. For every $r > 0$, we write ΔZ_r for the jump of Z at time r.

Theorem 1.3. *Let $s_1, s_2, \ldots$ be a measurable enumeration of the jumps of Z, and let $\xi_1, \xi_2, \ldots$ be a sequence of i.i.d. real random variables with density*

$$\frac{1}{\sqrt{2\pi x^5}}\, e^{-1/2x}\, \mathbf{1}_{(0,\infty)}(x),$$

which is independent of the process $(Z_r)_{r>0}$. The following identity in distribution of random processes holds:

$$\Big(Z_r, |B^\bullet_r(\mathcal{P}_\infty)|\Big)_{r>0} \overset{\text{(d)}}{=} \Big(Z_r, \sum_{i:s_i\leq r} \xi_i\, (\Delta Z_{s_i})^2\Big)_{r>0}.$$

This theorem identifies the conditional distribution of the process of hull volumes knowing the process of hull boundary lengths, whose distribution is given by the preceding proposition. Informally, each jump time r of Z corresponds to the creation of a new connected component of the complement of the ball $B_r(\mathcal{P}_\infty)$, which is "swallowed" by the hull, leading to a negative jump for the boundary of the hull and a positive jump for its volume. The common distribution of the variables ξ_i should then be interpreted as the law of the volume of a newly created connected component knowing that the "length" of its boundary is equal to 1 (see [4, Proposition 6.4] and especially [10, Proposition 9] for related results about the asymptotic distribution of the volume of a triangulation with a boundary of size tending to infinity). This heuristic discussion is made much more precise in the companion paper [10], where many of the results of the present work are interpreted in terms of asymptotics for the so-called "peeling process" studied by Angel [4] for the UIPT.

The proof of Theorem 1.3 depends on certain explicit calculations of distributions, which are of independent interest.

Theorem 1.4. *Let $r > 0$. For every $\mu > 0$,*

$$E\Big[\exp(-\mu|B_r^\bullet(\mathcal{P}_\infty)|)\Big] = 3^{3/2}\,\cosh((2\mu)^{1/4}r)\,\Big(\cosh^2((2\mu)^{1/4}r)+2\Big)^{-3/2}.$$

Furthermore, for every $\ell > 0$,

$$\begin{aligned}&E\Big[\exp(-\mu|B_r^\bullet(\mathcal{P}_\infty)|)\,\Big|\,Z_r=\ell\Big]\\&\quad= r^3(2\mu)^{3/4}\,\frac{\cosh((2\mu)^{1/4}r)}{\sinh^3((2\mu)^{1/4}r)}\,\exp\Big(-\ell\Big(\sqrt{\frac{\mu}{2}}\Big(3\coth^2((2\mu)^{1/4}r)-2\Big)-\frac{3}{2r^2}\Big)\Big).\end{aligned}$$

In view of the first assertion of the theorem, one may ask whether a similar formula holds for the volume $|B_r(\mathcal{P}_\infty)|$ of the ball of radius r. In principle our methods should also be applicable to this problem, but our calculations did not lead to a tractable expression. One may still compare the expected volumes of the hull and the ball. From the first formula of the theorem, one easily gets that $E[|B_r^\bullet(\mathcal{P}_\infty)|] = r^4/3$. On the other hand, using the method of the proof of [26, Proposition 5], one can verify that $E[|B_r(\mathcal{P}_\infty)|] = 2r^4/21$.

We also note that there is an interesting analogy between the second formula of Theorem 1.4 and classical formulas for Bessel processes (see Corollary 1.8 and Corollary 3.3 in [31, Chapter XI]), which also involve hyperbolic functions – in special cases these formulas can be restated in terms of linear Brownian motion via the Ray-Knight theorems.

The preceding results can also be interpreted in terms of asymptotics for the UIPQ. In the last section of this article, we prove that the process of hull volumes of the UIPQ converges in distribution, modulo a suitable rescaling, to the process $(|B_r^\bullet(\mathcal{P}_\infty)|)_{r>0}$. A similar invariance principle should hold for the UIPT and for more general random lattices such as the ones constructed by Addario-Berry [2] and Stephenson [32].

Our proofs depend on a new representation of the Brownian plane, which is different from the one used in [9]. Roughly speaking, this representation is a continuous analog of the construction of the UIPQ that was given by Chassaing and Durhuus in [7], whereas [9] used a continuous version of the construction in [12]. Similarly as in [9], the representation of the Brownian plane in the present work uses a random infinite real tree $\mathcal{T}_\infty$ whose vertices are assigned real labels. The probabilistic structure of the real tree $\mathcal{T}_\infty$ is more complicated than in [9], but the labels are now nonnegative and correspond to distances from the root in $\mathcal{P}_\infty$ (whereas in [9] labels corresponded in some sense to "distances from infinity"). This is of course similar to the well-known Schaeffer bijection between rooted quadrangulations and well-labeled trees [8]. The fact that labels are distances from the root is important for our purposes, since it allows us to give a simple representation of the hull of radius r: The complement of this hull corresponds to the set of all points a in $\mathcal{T}_\infty$ such that labels stay greater than r along the (tree) geodesic from a to infinity. See formula (16) below. There is a similar interpretation for the boundary of the hull, and a key observation is the fact that the "boundary length" Z_r can be obtained in terms of exit measures from (r, ∞) associated with the "subtrees" branching off the spine of the infinite tree $\mathcal{T}_\infty$ at a level greater than the last occurence of label r on the spine (see formula (18) below).

The construction of the infinite tree $\mathcal{T}_\infty$ and of the labels assigned to its vertices, as well as the subsequent calculations, make a heavy use of the Brownian snake and its properties. In particular the special Markov property of the Brownian snake [18] and its connections with partial differential equations play an important role. Because of the close relation between super-Brownian motion and the Brownian snake, some of the results that follow can be written as statements about super-Brownian motion, which may be of independent interest. In particular, Corollary 4.7, which is essentially equivalent to the second formula of Theorem 1.4, gives the Laplace transform of the total integrated mass of a super-Brownian motion started from $u\delta_a$ (for some $u, a > 0$) knowing that the minimum of the range is equal to 0. Similarly, Corollary 4.9 determines for a super-Brownian motion starting from δ_0 the law of the process whose value at time $r > 0$ is the pair consisting of the exit measure from $(-r, \infty)$ and the mass of those historical paths that do not hit level $-r$.

The paper is organized as follows. Section 2 presents a number of preliminaries. In particular, we recall basic facts about the (one-dimensional) Brownian snake including exit measures and the special Markov property, and its connections with super-Brownian motion. We also state a recent result from [25] giving a decomposition of the Brownian snake knowing its minimal spatial position. The latter result is especially useful in Section 3, where we derive our new representation of the Brownian plane. In order to show that this new construction is equivalent to the one in [9], we use the fact that the distribution of the Brownian plane is characterized by the invariance under scaling and the above-mentioned property stating that the Brownian plane is locally isometric to the Brownian map. Section 4 contains the proof of our main results: Propositions 1.1 and 1.2 are proved in subsection 4.1, Theorem 1.4 is derived in subsection 4.2, and Theorem 1.3 is proved in subsection 4.3. Finally, Section 5 is devoted to our invariance principle relating the hull process of the UIPQ to the process $(|B_r^\bullet(\mathcal{P}_\infty)|)_{r>0}$.

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2 Preliminaries

2.1 A continuous-state branching process

An important role in this work will be played by a particular continuous-state branching process, which was already mentioned in the introduction. We refer to [19, Chapter 2] and references therein for the general theory of continuous-state branching processes, and content ourselves with a brief exposition of the case of interest in this work. We fix a constant $c > 0$. The continuous-state branching process with branching mechanism $\psi(u) = c\,u^{3/2}$ is the Feller Markov process $(X_t)_{t\geq 0}$ with values in $\mathbb{R}_+$, càdlàg paths and no negative jumps, whose semigroup is characterized as follows. If P_x stands for the probability measure under which X starts from $X_0 = x$, then, for every $x, t \geq 0$ and every $\lambda > 0$,

$$E_x[e^{-\lambda X_t}] = e^{-x\,u_t(\lambda)}$$

where the function $u_t(\lambda)$ is determined by the differential equation

$$\frac{\mathrm{d}u_t(\lambda)}{\mathrm{d}t} = -c\,(u_t(\lambda))^{3/2}\,, \quad u_0(\lambda) = \lambda.$$

It follows that $u_t(\lambda) = (\lambda^{-1/2} + \frac{c}{2}\, t)^{-2}$, and thus,

$$E_x[e^{-\lambda X_t}] = \exp\Big(- x\Big(\lambda^{-1/2} + \frac{c}{2}t\Big)^{-2}\Big). \tag{1}$$

By differentiating with respect to λ, we have also

$$E_x[X_t e^{-\lambda X_t}] = x\lambda^{-3/2}\Big(\lambda^{-1/2} + \frac{c}{2}t\Big)^{-3} \exp\Big(- x\Big(\lambda^{-1/2} + \frac{c}{2}t\Big)^{-2}\Big). \tag{2}$$

Let $T := \inf\{t \geq 0 : X_t = 0\}$, and note that $X_t = 0$ for every $t \geq T$, a.s. Since $P_x(T \leq t) = P_x(X_t = 0) = \exp(-\frac{4x}{c^2t^2})$, we readily obtain that the density of T under P_x is (when $x > 0$) the function

$$t \mapsto \phi_t(x) := \frac{8x}{c^2t^3} \exp\Big(- \frac{4x}{c^2t^2}\Big).$$

For future purposes, it will be useful to introduce the process X conditioned on extinction at a fixed time. To this end, we write $q_t(x, \mathrm{d}y)$ for the transition kernels of X. We fix $\rho > 0$ and define the process X "conditioned on extinction at time ρ" as the time-inhomogeneous Markov process indexed by the interval $[0, \rho]$ with values in $(0, \infty)$ (with 0 serving as a cemetery point) whose transition kernel between times s and t is

$$\pi_{s,t}(x, \mathrm{d}y) = \frac{\phi_{\rho-t}(y)}{\phi_{\rho-s}(x)}\, q_{t-s}(x, \mathrm{d}y),$$

if $0 \leq s < t < \rho$ and $x > 0$, and

$$\pi_{s,\rho}(x, \mathrm{d}y) = \delta_0(\mathrm{d}y)$$

if $s \in [0, \rho)$ and $x > 0$. This is just a standard h-transform in a time-inhomogeneous setting, and the interpretation can be justified by the fact that, for every choice of $0 < s_1 < \cdots < s_p < \rho$, the conditional distribution of $(X_{s_1}, \ldots, X_{s_p})$ under $P_x(\cdot \mid \rho \leq T < \rho + \varepsilon)$ converges to $\pi_{0,s_1}(x, \mathrm{d}y_1)\pi_{s_1,s_2}(y_1, \mathrm{d}y_2)\ldots\pi_{s_{p-1},s_p}(y_{p-1}, \mathrm{d}y_p)$ as $\varepsilon \downarrow 0$.

If $0 \leq s < t < \rho$ and $x > 0$, the Laplace transform of $\pi_{s,t}(x, \mathrm{d}y)$ is

$$\begin{aligned}\int e^{-\lambda y}\, \pi_{s,t}(x, \mathrm{d}y) &= \frac{1}{\phi_{\rho-s}(x)}\, E_x[\phi_{\rho-t}(X_{t-s})\, e^{-\lambda X_{t-s}}] \\ &= \left(\frac{\rho - s}{\rho - t + (t-s)(1 + \frac{c^2}{4}\lambda(\rho-t)^2)^{1/2}}\right)^3 \\ &\quad \times \exp\left(- \frac{4x}{c^2}\Big(\Big((\frac{c^2\lambda}{4} + (\rho-t)^{-2})^{-1/2} + t - s\Big)^{-2} - (\rho-s)^{-2}\Big)\right),\end{aligned} \tag{3}$$

where the second equality follows from the explicit expression of $\phi_{\rho-s}$ and formula (2).

Finally, let us briefly discuss the process $\widetilde{X}$ which was introduced in Section 1. Simple arguments give the existence of a process $(\widetilde{X}_t)_{t\in(-\infty,0]}$ with càdlàg paths and no negative jumps, which is indexed by the time interval $(-\infty, 0]$ and such that:

- $\widetilde{X}_t > 0$ for every $t < 0$, and $\widetilde{X}_0 = 0$, a.s.;
- $\widetilde{X}_t \longrightarrow +\infty$ as $t \downarrow -\infty$, a.s.;
- for every $x > 0$, if $\widetilde{T}_x := \inf\{t \in (-\infty, 0] : \widetilde{X}_t \leq x\}$, the process $(\widetilde{X}_{(\tilde{T}_x+t)\wedge 0})_{t\geq 0}$ has the same distribution as X started from x.

To get an explicit construction of $\widetilde{X}$, one may concatenate independent copies of the process X started at n and stopped at the hitting time of $n - 1$, for every integer $n \geq 1$. We omit the details.

2.2 Preliminaries about the Brownian snake

We give below a brief presentation of the Brownian snake, referring to the book [19] for more details. We write $\mathcal{W}$ for the set of all finite paths in $\mathbb{R}$. An element of $\mathcal{W}$ is a continuous mapping $\mathrm{w} : [0,\zeta] \longrightarrow \mathbb{R}$, where $\zeta = \zeta_{(\mathrm{w})} \geq 0$ depends on w and is called the lifetime of w. We write $\widehat{\mathrm{w}} = \mathrm{w}(\zeta_{(\mathrm{w})})$ for the endpoint of w. For $x \in \mathbb{R}$, we set $\mathcal{W}_x := \{\mathrm{w} \in \mathcal{W} : \mathrm{w}(0) = x\}$. The trivial path w such that $\mathrm{w}(0) = x$ and $\zeta_{(\mathrm{w})} = 0$ is identified with the point x of $\mathbb{R}$, so that we can view $\mathbb{R}$ as a subset of $\mathcal{W}$. The space $\mathcal{W}$ is equipped with the distance

$$d(\mathrm{w},\mathrm{w}') = |\zeta_{(\mathrm{w})} - \zeta_{(\mathrm{w}')}| + \sup_{t\geq 0} |(\mathrm{w}(t \wedge \zeta_{(\mathrm{w})}) - \mathrm{w}'(t \wedge \zeta_{(\mathrm{w}')})|.$$

The Brownian snake $(W_s)_{s\geq 0}$ is a continuous Markov process with values in $\mathcal{W}$. We will write $\zeta_s = \zeta_{(W_s)}$ for the lifetime process of W_s. The process $(\zeta_s)_{s\geq 0}$ evolves like a reflecting Brownian motion in $\mathbb{R}_+$. Conditionally on $(\zeta_s)_{s\geq 0}$, the evolution of $(W_s)_{s\geq 0}$ can be described informally as follows: When ζ_s decreases, the path W_s is shortened from its tip, and when ζ_s increases the path W_s is extended by adding "little pieces of linear Brownian motion" at its tip. We refer to [19, Chapter IV] for a more rigorous presentation.

It is convenient to assume that the Brownian snake is defined on the canonical space $C(\mathbb{R}_+, \mathcal{W})$ of all continuous functions from $\mathbb{R}_+$ into $\mathcal{W}$, in such a way that, for $\omega = (\omega_s)_{s\geq 0} \in C(\mathbb{R}_+, \mathcal{W})$, we have $W_s(\omega) = \omega_s$. The notation $\mathbb{P}_{\mathrm{w}}$ then stands for the law of the Brownian snake started from w.

For every $x \in \mathbb{R}$, the trivial path x is a regular recurrent point for the Brownian snake, and so we can make sense of the excursion measure $\mathbb{N}_x$ away from x, which is a σ-finite measure on $C(\mathbb{R}_+, \mathcal{W})$. Under $\mathbb{N}_x$, the process $(\zeta_s)_{s\geq 0}$ is distributed according to the Itô measure of positive excursions of linear Brownian motion, which is normalized so that, for every $\varepsilon > 0$,

$$\mathbb{N}_x\Big(\sup_{s\geq 0} \zeta_s > \varepsilon\Big) = \frac{1}{2\varepsilon}.$$

We write $\sigma := \sup\{s \geq 0 : \zeta_s > 0\}$ for the duration of the excursion under $\mathbb{N}_x$. For every $\ell > 0$, we will also use the notation $\mathbb{N}_0^{(\ell)} := \mathbb{N}_0(\cdot \mid \sigma = \ell)$.

We set

$$\mathcal{R} := \{\widehat{W}_s : s \geq 0\}\ ,\ W_* := \inf \mathcal{R} = \inf_{s\geq 0} \widehat{W}_s.$$

We will consider $\mathcal{R}$ and W_* under the excursion measures $\mathbb{N}_x$, and we note that we have also $\mathcal{R} = \{\widehat{W}_s : 0 \leq s \leq \sigma\}$ and $W_* = \min\{\widehat{W}_s : 0 \leq s \leq \sigma\}$, $\mathbb{N}_x$ a.e. Occasionally we also write $\omega_* = W_*(\omega)$ for $\omega \in C(\mathbb{R}_+, \mathcal{W})$.

If $x, y \in \mathbb{R}$ and $y < x$, we have

$$\mathbb{N}_x(y \in \mathcal{R}) = \mathbb{N}_x(W_* \leq y) = \frac{3}{2(x-y)^2} \tag{4}$$

(see e.g. [19, Section VI.1]).

It is known (see e.g. [27, Proposition 2.5]) that $\mathbb{N}_x$ a.e. there is a unique instant $s_{\mathbf{m}} \in [0,\sigma]$ such that $\widehat{W}_{s_{\mathbf{m}}} = W_*$.

Decomposing the Brownian snake at its minimum. We will now recall a key result of [25] that plays an important role in what follows. This result identifies the law of the minimizing path $W_{s_{\mathbf{m}}}$ under $\mathbb{N}_0$, together with the distribution of the "subtrees" that branch off the minimizing path. Let us define these subtrees in a more precise way.

For every $s \geq 0$, we set

$$\hat{\zeta}_s := \zeta_{(s_{\mathbf{m}}+s)\wedge\sigma} \ , \quad \check{\zeta}_s := \zeta_{(s_{\mathbf{m}}-s)\vee 0} \ .$$

We let $(\hat{a}_i, \hat{b}_i)$, $i \in \hat{I}$ be the excursion intervals of $\hat{\zeta}_s$ above its past minimum. Equivalently, the intervals $(\hat{a}_i, \hat{b}_i)$, $i \in \hat{I}$ are the connected components of the set

$$\Big\{ s \geq 0 : \hat{\zeta}_s > \min_{0\leq r\leq s} \hat{\zeta}_r \Big\}.$$

Similarly, we let $(\check{a}_j, \check{b}_j)$, $j \in \check{I}$ be the excursion intervals of $\check{\zeta}_s$ above its past minimum. We may assume that the indexing sets $\hat{I}$ and $\check{I}$ are disjoint. In terms of the tree $\mathcal{T}_\zeta$ coded by the excursion $(\zeta_s)_{0\leq s\leq\sigma}$ under $\mathbb{N}_0$ (see e.g. [20, Section 2]), each interval $(\hat{a}_i, \hat{b}_i)$ or $(\check{a}_j, \check{b}_j)$ corresponds to a subtree of $\mathcal{T}_\zeta$ branching off the ancestral line of the vertex associated with $s_{\mathbf{m}}$. We next consider the spatial displacements corresponding to these subtrees. The properties of the Brownian snake imply that, for every $i \in \hat{I}$, the paths $W_{s_{\mathbf{m}}+s}$, $s \in [\hat{a}_i, \hat{b}_i]$, are the same up to time $\zeta_{s_{\mathbf{m}}+\hat{a}_i} = \zeta_{s_{\mathbf{m}}+\hat{b}_i}$, and similarly for the paths $W_{s_{\mathbf{m}}-s}$, $s \in [\check{a}_j, \check{b}_j]$, for every $j \in \check{I}$. Then, for every $i \in \hat{I}$, we let $W^{[i]} \in C(\mathbb{R}_+, \mathcal{W})$ be defined by

$$W^{[i]}_s(t) = W_{s_{\mathbf{m}}+(\hat{a}_i+s)\wedge\hat{b}_i}(\zeta_{s_{\mathbf{m}}+\hat{a}_i} + t) \ , \quad 0 \leq t \leq \zeta_{s_{\mathbf{m}}+(\hat{a}_i+s)\wedge\hat{b}_i} - \zeta_{s_{\mathbf{m}}+\hat{a}_i}.$$

Similarly, for every $j \in \check{I}$,

$$W^{[j]}_s(t) = W_{s_{\mathbf{m}}-(\check{a}_j+s)\wedge\check{b}_j}(\zeta_{s_{\mathbf{m}}-\check{a}_j} + t) \ , \quad 0 \leq t \leq \zeta_{s_{\mathbf{m}}-(\check{a}_j+s)\wedge\check{b}_j} - \zeta_{s_{\mathbf{m}}-\check{a}_j}.$$

We finally introduce the point measures on $\mathbb{R}_+ \times C(\mathbb{R}_+, \mathcal{W})$ defined by

$$\hat{\mathcal{N}} = \sum_{i\in\hat{I}} \delta_{(\zeta_{s_{\mathbf{m}}+\hat{a}_i}, W^{[i]})} \ , \quad \check{\mathcal{N}} = \sum_{j\in\check{I}} \delta_{(\zeta_{s_{\mathbf{m}}-\check{a}_j}, W^{[j]})}.$$

Theorem 2.1. (i) *Let $a > 0$. Under the excursion measure $\mathbb{N}_0$ and conditionally on $W_* = -a$, the random path $(a + W_{s_{\mathbf{m}}}(\zeta_{s_{\mathbf{m}}} - t))_{0\leq t\leq\zeta_{s_{\mathbf{m}}}}$ is distributed as a nine-dimensional Bessel process started from 0 and stopped at its last passage time at level a.*

(ii) *Under $\mathbb{N}_0$, conditionally on the minimizing path $W_{s_{\mathbf{m}}}$, the point measures $\hat{\mathcal{N}}(\mathrm{d}t, \mathrm{d}\omega)$ and $\check{\mathcal{N}}(\mathrm{d}t, \mathrm{d}\omega)$ are independent and their common conditional distribution is that of a Poisson point measure with intensity*

$$2\,\mathbf{1}_{[0,\zeta_{s_{\mathbf{m}}}]}(t)\,\mathbf{1}_{\{\omega_* > \widehat{W}_{s_{\mathbf{m}}}\}}\,\mathrm{d}t\,\mathbb{N}_{W_{s_{\mathbf{m}}}(t)}(\mathrm{d}\omega).$$

Parts (i) and (ii) of the theorem correspond respectively to Theorem 5 and Theorem 6 of [25]. Note that when applying Theorem 5 of [25], we also use the fact that the time-reversal of a Bessel process of dimension -5 started from a and stopped when hitting 0 is a nine-dimensional Bessel process started from 0 and stopped at its last passage time at level a (see e.g. [31, Exercise XI.1.23]). We refer to [31, Chapter XI] for basic facts about Bessel processes.

Exit measures and the special Markov property. Let D be a nonempty open interval of $\mathbb{R}$, such that $D \neq \mathbb{R}$. We fix $x \in D$ and, for every $\mathrm{w} \in \mathcal{W}_x$, set

$$\tau_D(\mathrm{w}) = \inf\{t \in [0, \zeta_{(\mathrm{w})}] : \mathrm{w}(t) \notin D\},$$

with the usual convention $\inf \varnothing = \infty$. The exit measure $\mathcal{Z}^D$ from D (see [19, Chapter 5]) is a random measure on ∂D, which is defined under $\mathbb{N}_x$ and is supported on the set of all exit points $W_s(\tau_D(W_s))$ for the paths W_s such that $\tau_D(W_s) < \infty$ (note that here ∂D has at most two points, but the following discussion remains valid for the d-dimensional Brownian snake and an arbitrary subdomain D of $\mathbb{R}^d$). Note that $\mathbb{N}_x(\mathcal{Z}^D \neq 0) < \infty$. It is easy to prove, for instance by using Proposition 2.2 below, that

$$\{\mathcal{Z}^D = 0\} = \{\mathcal{R} \subset D\}, \quad \mathbb{N}_x \text{ a.e.} \tag{5}$$

A crucial ingredient of our study is the special Markov property of the Brownian snake [18]. In order to state this property, we first observe that, $\mathbb{N}_x$ a.e., the set

$$\{s \geq 0 : \tau_D(W_s) < \zeta_s\}$$

is open and thus can be written as a union of disjoint open intervals (a_i, b_i), $i \in I$, where I may be empty. From the properties of the Brownian snake, one has, $\mathbb{N}_x$ a.e. for every $i \in I$ and every $s \in [a_i, b_i]$,

$$\tau_D(W_s) = \tau_D(W_{a_i}) = \zeta_{a_i},$$

and more precisely all paths W_s, $s \in [a_i, b_i]$ coincide up to their exit time from D. For every $i \in I$, we then define an element $W^{(i)}$ of $C(\mathbb{R}_+, \mathcal{W})$ by setting, for every $s \geq 0$,

$$W_s^{(i)}(t) := W_{(a_i+s)\wedge b_i}(\zeta_{a_i} + t), \quad \text{for } 0 \leq t \leq \zeta_{(W_s^{(i)})} := \zeta_{(a_i+s)\wedge b_i} - \zeta_{a_i}.$$

Informally, the $W^{(i)}$'s represent the "excursions" of the Brownian snake outside D (the word "outside" is a little misleading here, because although these excursions start from a point of ∂D, they will typically come back inside D).

We also need to introduce a σ-field that contains the information about the paths W_s before they exit D. To this end, we set, for every $s \geq 0$,

$$\eta_s^D := \inf\{r \geq 0 : \int_0^r \mathrm{d}u\, \mathbf{1}_{\{\zeta_u \leq \tau_D(W_u)\}} > s\},$$

and we let $\mathcal{E}^D$ be the σ-field generated by the process $(W_{\eta_s^D})_{s\geq 0}$ and the class of all sets that are $\mathbb{N}_x$-negligible. The random measure $\mathcal{Z}^D$ is measurable with respect to $\mathcal{E}^D$ (see [18, Proposition 2.3]).

We now state the special Markov property [18, Theorem 2.4].

Proposition 2.2. *Under $\mathbb{N}_x$, conditionally on $\mathcal{E}^D$, the point measure*

$$\sum_{i\in I} \delta_{W^{(i)}}$$

is Poisson with intensity

$$\int \mathcal{Z}^D(\mathrm{d}y)\, \mathbb{N}_y.$$

Remarks. (i) Since on the event $\{\mathcal{Z}^D = 0\}$ there are no excursions outside D, the previous proposition is equivalent to the same statement where $\mathbb{N}_x$ is replaced by the probability measure $\mathbb{N}_x(\cdot \mid \mathcal{Z}^D \neq 0)$.
(ii) In what follows we will apply the special Markov property in a conditional form. Suppose that $D = (a, \infty)$ for some $a > 0$ and that $x > a$. Then the preceding statement remains valid if we replace $\mathbb{N}_x$ by $\mathbb{N}_x(\cdot \cap \{\mathcal{R} \subset (0,\infty)\})$, provided we also replace $\int \mathcal{Z}^D(\mathrm{d}y)\, \mathbb{N}_y$ by $\int \mathcal{Z}^D(\mathrm{d}y)\, \mathbb{N}_y(\cdot \cap \{\mathcal{R} \subset (0,\infty)\})$. This follows from the fact that conditioning a Poisson point

measure on having no point on a set of finite intensity is equivalent to removing the points that fall into this set. We omit the details.

For $a < x$, we write $\mathcal{Z}_a := \langle \mathcal{Z}^{(a,\infty)}, 1\rangle$ for the total mass of the exit measure outside (a,∞). We will use the Laplace transform of $\mathcal{Z}_a$ under $\mathbb{N}_x$, which is given by

$$\mathbb{N}_x\Big(1-\exp(-\mu\mathcal{Z}_a)\Big) = \frac{1}{\Big(\mu^{-1/2}+\sqrt{\frac{2}{3}}\,(x-a)\Big)^2}, \tag{6}$$

for every $\mu \geq 0$. This formula is easily derived from the fact that the (nonnegative) function $u(x) = \mathbb{N}_x(1-\exp(-\mu\mathcal{Z}_a))$ defined for $x \in (a,\infty)$ solves the differential equation $u'' = 4u^2$ with boundary conditions $u(a) = \mu$ and $u(\infty) = 0$ (see [19, Chapter V]). On the other hand, an application of the special Markov property shows that, for every $b < a < x$,

$$\mathbb{N}_x(\exp(-\lambda\mathcal{Z}_b) \mid \mathcal{E}^{(a,\infty)}) = \exp\Big(-\mathcal{Z}_a\,\mathbb{N}_a(1-\exp(-\lambda\mathcal{Z}_b))\Big).$$

If we substitute formula (6) in the last display, and compare with (1), we easily get that the process $(\mathcal{Z}_{x-a})_{a>0}$ is Markov under $\mathbb{N}_x$, with the transition kernels of the continuous-state branching process with branching mechanism $\psi(u) = \sqrt{8/3}\,u^{3/2}$. Although $\mathbb{N}_x$ is an infinite measure, the preceding assertion makes sense, simply because we can restrict our attention to the finite measure event $\{\mathcal{Z}_{x-\varepsilon} > 0\}$, for any choice of $\varepsilon > 0$. It follows that $(\mathcal{Z}_{x-a})_{a>0}$ has a càdlàg modification under $\mathbb{N}_x$, which we consider from now on.

We finally explain an extension of the special Markov property where we consider excursions outside a random domain. For definiteness, we fix $x = 0$, and for every $a > 0$, we set $\mathcal{E}_a = \mathcal{E}^{(-a,\infty)}$. Let H be a random variable with values in $(0,\infty]$, such that $\mathbb{N}_0(H < \infty) < \infty$, and assume that H is a stopping time of the filtration $(\mathcal{E}_a)_{a>0}$ in the sense that, for every $a > 0$, the event $\{H \leq a\}$ is $\mathcal{E}_a$-measurable. As usual we can define the σ-field $\mathcal{E}_H$ that consists of all events A such that $A \cap \{H \leq a\}$ is $\mathcal{E}_a$-measurable, for every $a > 0$. Since $\mathcal{Z}_{-a}$ is $\mathcal{E}_a$-measurable for every $a > 0$, it follows by standard arguments that the random variable $\mathcal{Z}_{-H}$ is $\mathcal{E}_H$-measurable (at this point it is important that we have taken a càdlàg modification of the process $(\mathcal{Z}_{-a})_{a>0}$).

We may consider the excursions $(W^{H,(i)})_{i\in I}$ of the Brownian snake outside $(-H,\infty)$. These excursions are defined in exactly the same way as in the case where H is deterministic, considering now the connected components of the open set $\{s \geq 0 : W_s(t) < -H \text{ for some } t \in [0,\zeta_s]\}$. We define $\widetilde{W}^{H,(i)}$ by shifting $W^{H,(i)}$ so that it starts from 0.

Proposition 2.3. *Under the probability measure $\mathbb{N}_0(\cdot \mid H < \infty)$, conditionally on the σ-field $\mathcal{E}_H$, the point measure*

$$\sum_{i\in I} \delta_{\widetilde{W}^{H,(i)}}$$

is Poisson with intensity

$$\mathcal{Z}_{-H}\,\mathbb{N}_0.$$

This proposition can be obtained by arguments very similar to the derivation of the strong Markov property of Brownian motion from the simple Markov property: we approximate H with stopping times greater than H that take only countably many values, then use Proposition 2.2 and finally perform a suitable passage to the limit. We leave the details to the reader.

The Brownian snake and super-Brownian motion. The initial motivation for studying the Brownian snake came from its connection with super-Brownian motion, which we briefly recall. Under the excursion measure $\mathbb{N}_x(\mathrm{d}\omega)$, the lifetime process $(\zeta_s(\omega))_{s\geq 0}$ is distributed as a Brownian excursion, and so we can define for every $t \geq 0$ the local time proces $(\ell^t_s(\omega))_{s\geq 0}$ of this excursion at level t. Next let μ be a finite measure on $\mathbb{R}$, and let

$$\mathcal{N}(\mathrm{d}\omega) = \sum_{k\in K} \delta_{\omega_{(k)}}(\mathrm{d}\omega)$$

be a Poisson measure on $C(\mathbb{R}_+, \mathcal{W})$ with intensity $\int \mu(\mathrm{d}x)\, \mathbb{N}_x(\mathrm{d}\omega)$. For every $t > 0$, let $\mathcal{X}_t$ be the random measure on $\mathbb{R}$ defined by setting, for every nonnegative measurable function φ on $\mathbb{R}$,

$$\langle \mathcal{X}_t, \varphi\rangle = \sum_{k\in K} \int_0^{\sigma(\omega_{(k)})} \mathrm{d}\ell^t_s(\omega_{(k)})\, \varphi(\widehat{W}_s(\omega_{(k)})). \tag{7}$$

If we also set $\mathcal{X}_0 = \mu$, the process $(\mathcal{X}_t)_{t\geq 0}$ is then a super-Brownian motion with branching mechanism $\psi_0(u) = 2u^2$ started from μ (see [19, Theorem IV.4]). A nice feature of this construction is the fact that it also gives the associated historical process: Just consider for every $t > 0$ the random measure $\mathbf{X}_t$ defined by setting

$$\langle \mathbf{X}_t, \Phi\rangle = \sum_{k\in K} \int_0^{\sigma(\omega_{(k)})} \mathrm{d}\ell^t_s(\omega_{(k)})\, \Phi(W_s(\omega_{(k)})), \tag{8}$$

for every nonnegative measurable function Φ on $\mathcal{W}$. Some of the forthcoming results are stated in terms of super-Brownian motion and its historical process. Without loss of generality we may and will assume that these processes are obtained by formulas (7) and (8) of the previous construction. This also means that we consider the special branching mechanism $\psi_0(u) = 2u^2$, but of course the case of a general quadratic branching mechanism can then be handled via scaling arguments.

3 The Brownian plane

3.1 The Brownian plane as a random metric space

We start by giving a characterization of the Brownian plane as a random pointed metric space satisfying appropriate properties. We let $\mathbb{K}_{bcl}$ denote the space of all isometry classes of pointed boundedly compact length spaces. The space $\mathbb{K}_{bcl}$ is equipped with the local Gromov-Hausdorff distance d_{LGH} (see [9, Section 2.1]) and is a Polish space, that is, separable and complete for this distance. For $r > 0$ and $F \in \mathbb{K}_{bcl}$, we use the notation $B_r(F)$ for the closed ball of radius r centered at the distinguished point of F. Note that $B_r(F)$ is always viewed as a *pointed* compact metric space.

The Brownian plane $\mathcal{P}_\infty$ is then a random variable taking values in the space $\mathbb{K}_{bcl}$.

Definition 3.1. *Let E_1 and E_2 be two random variables with values in $\mathbb{K}_{bcl}$. We say that E_1 and E_2 are locally isometric if, for every $\delta > 0$, there exists a number $r > 0$ and a coupling of E_1 and E_2 such that the balls $B_r(E_1)$ and $B_r(E_2)$ are isometric with probability at least $1 - \delta$.*

We leave it to the reader to verify that this is an equivalence relation (only transitivity is not obvious). The interest of this definition comes from the next proposition. If E is a (random) metric space and $\lambda > 0$, we use the notation $\lambda \cdot E$ for the same metric space where the distance has been multiplied by λ.

Proposition 3.2. *The distribution of the Brownian plane is characterized in the set of all probability measures on $\mathbb{K}_{bcl}$ by the following two properties:*

(i) *The Brownian plane is locally isometric to the Brownian map.*

(ii) *The Brownian plane is scale invariant, meaning that $\lambda \cdot \mathcal{P}_\infty$ has the same distribution as $\mathcal{P}_\infty$, for every $\lambda > 0$.*

Proof. The fact that property (i) holds is Theorem 1 in [9]. Property (ii) is immediate from the construction in [9], or directly from the convergence (1) in [9, Theorem 1]. So we just have to prove that these two properties characterize the distribution of the Brownian plane. Let E be a random variable with values in $\mathbb{K}_{bcl}$, which is both locally isometric to the Brownian map and scale invariant. Then, E is also locally isometric to the Brownian plane, and, for every $\delta > 0$, we can find $r > 0$ and a coupling of E and $\mathcal{P}_\infty$ such that

$$P[B_r(E) = B_r(\mathcal{P}_\infty)] > 1 - \delta,$$

where the equality is in the sense of isometry between pointed compact metric spaces. Trivially this implies that, for every $a > 0$,

$$P[B_a(\frac{a}{r} \cdot E) = B_a(\frac{a}{r} \cdot \mathcal{P}_\infty)] > 1 - \delta.$$

By scale invariance, $\frac{a}{r} \cdot E$ and $\frac{a}{r} \cdot \mathcal{P}_\infty$ have the same distribution as E and $\mathcal{P}_\infty$ respectively. So we get that for every $\delta > 0$, for every $a > 0$, we can find a coupling of E and $\mathcal{P}_\infty$ such that

$$P[B_a(E) = B_a(\mathcal{P}_\infty)] > 1 - \delta.$$

Recalling the definition of the local Gromov-Hausdorff distance d_{LGH} (see e.g. [9, Section 2.1]) we obtain that, for every $\varepsilon > 0$ and every $\delta > 0$, there exists a coupling of E and $\mathcal{P}_\infty$ such that

$$P[\mathrm{d}_{LGH}(E, \mathcal{P}_\infty) < \varepsilon] > 1 - \delta.$$

Clearly this implies that the Lévy-Prokhorov distance between the distributions of E and $\mathcal{P}_\infty$ is 0 and thus E and $\mathcal{P}_\infty$ have the same distribution. $\square$

3.2 A new construction of the Brownian plane

In this section, we provide a construction of the Brownian plane, which is different from the one in [9]. We then use Proposition 3.2 and Theorem 2.1 to prove the equivalence of the two constructions.

We consider a nine-dimensional Bessel process $R = (R_t)_{t \geq 0}$ starting from 0 and, conditionally on R, two independent Poisson point measures $\mathcal{N}'(\mathrm{d}t, \mathrm{d}\omega)$ and $\mathcal{N}''(\mathrm{d}t, \mathrm{d}\omega)$ on $\mathbb{R}_+ \times C(\mathbb{R}_+, \mathcal{W})$ with the same intensity

$$2\,\mathbf{1}_{\{\mathcal{R}(\omega) \subset (0,\infty)\}}\, \mathrm{d}t\, \mathbb{N}_{R_t}(\mathrm{d}\omega).$$

It will be convenient to write

$$\mathcal{N}' = \sum_{i \in I} \delta_{(t_i, \omega^i)}\ , \quad \mathcal{N}'' = \sum_{i \in J} \delta_{(t_i, \omega^i)},$$

where the indexing sets I and J are disjoint.

We also consider the sum $\mathcal{N} = \mathcal{N}' + \mathcal{N}''$, which conditionally on R is Poisson with intensity

$$4\,\mathbf{1}_{\{\mathcal{R}(\omega)\subset(0,\infty)\}}\,\mathrm{d}t\,\mathbb{N}_{R_t}(\mathrm{d}\omega),$$

and we have

$$\mathcal{N} = \sum_{i\in I\cup J} \delta_{(t_i,\omega^i)}. \tag{9}$$

We start by introducing the infinite random tree that will be crucial in our construction of the Brownian plane. For every $i \in I \cup J$, write $\sigma_i = \sigma(\omega^i)$ and let $(\zeta^i_s)_{s\geq 0}$ be the lifetime process associated with ω^i. Then the function $(\zeta^i_s)_{0\leq s\leq\sigma_i}$ codes a rooted compact real tree, which is denoted by $\mathcal{T}^i$, and we write p_{ζ^i} for the canonical projection from $[0,\sigma_i]$ onto $\mathcal{T}^i$ (see e.g. [20, Section 2] for basic facts about the coding of trees by continuous functions). We construct a random non-compact real tree $\mathcal{T}_\infty$ by grafting to the half-line $[0,\infty)$ (which we call the "spine") the tree $\mathcal{T}^i$ at point t_i, for every $i \in I \cup J$. Formally, the tree $\mathcal{T}_\infty$ is obtained from the disjoint union

$$[0,\infty) \cup \Big(\bigcup_{i\in I\cup J} \mathcal{T}^i\Big)$$

by identifying the point t_i of $[0,\infty)$ with the root ρ_i of $\mathcal{T}^i$, for every $i \in I\cup J$. The metric d_∞ on $\mathcal{T}_\infty$ is determined as follows. The restriction of d_∞ to each tree $\mathcal{T}^i$ is (of course) the metric $d_{\mathcal{T}^i}$ on $\mathcal{T}^i$. If $x \in \mathcal{T}^i$ and $t \in [0,\infty)$, we take $d_\infty(x,t) = d_{\mathcal{T}^i}(x,\rho_i) + |t_i - t|$. If $x \in \mathcal{T}^i$ and $y \in \mathcal{T}^j$, with $i \neq j$, we take $d_\infty(x,y) = d_{\mathcal{T}^i}(x,\rho_i) + |t_i - t_j| + d_{\mathcal{T}^j}(\rho_j, y)$. By convention, $\mathcal{T}_\infty$ is rooted at 0. The infinite tree $\mathcal{T}_\infty$ is equipped with a volume measure $\mathbf{V}$, which puts no mass on the spine and whose restriction to each tree $\mathcal{T}^i$ is the natural volume measure on $\mathcal{T}^i$ defined as the image of Lebesgue measure on $[0,\sigma_i]$ under the projection p_{ζ^i}.

We also define labels on the tree $\mathcal{T}_\infty$. The label Λ_x of a vertex $x \in \mathcal{T}_\infty$ is defined by $\Lambda_x = R_t$ if $x = t$ belongs to the spine $[0,\infty)$, and $\Lambda_x = \widehat{\omega}^i_s$ if $x = p_{\zeta^i}(s)$ belongs to the subtree $\mathcal{T}^i$, for some $i \in I \cup J$. Note that the mapping $x \mapsto \Lambda_x$ is continuous almost surely (for continuity at points of the spine, observe that, for every $K > 0$ and $\varepsilon > 0$, there are a.s. only finitely many values of $i \in I \cup J$ such that $t_i \leq K$ and $\sup\{|\widehat{\omega}^i_s - R_{t_i}| : s \geq 0\} > \varepsilon$). For future use, we also notice that, if $x = p_{\zeta^i}(s)$ belongs to the subtree $\mathcal{T}^i$, the quantities $\omega^i_s(t)$, $0 \leq t \leq \zeta^i_s$ are the labels of the ancestors of x in $\mathcal{T}^i$.

We will use the fact that labels are "transient" in the sense of the following lemma. Recall the notation $\omega_* = W_*(\omega)$.

Lemma 3.3. *We have a.s.*

$$\lim_{r\uparrow\infty}\Big(\inf_{i\in I\cup J, t_i > r} \omega^i_*\Big) = +\infty.$$

Proof. It is enough to verify that, for every $A > 0$, we have

$$\lim_{r\uparrow\infty} P\Big(\inf_{i\in I\cup J,\, t_i\geq r} \omega^i_* < A\Big) = 0.$$

However by construction,

$$\begin{aligned}
&P\Big(\inf_{i\in I\cup J, t_i\geq r} \omega^i_* < A\Big)\\
&\quad = P\Big(\inf_{t\geq r} R_t < A\Big) + E\Big[\mathbf{1}\Big\{\inf_{t\geq r} R_t \geq A\Big\}\Big(1 - \exp\Big(-4\int_r^\infty \mathrm{d}t\,\mathbb{N}_{R_t}(0 < W_* < A)\Big)\Big)\Big]\\
&\quad = P\Big(\inf_{t\geq r} R_t < A\Big) + E\Big[\mathbf{1}\Big\{\inf_{t\geq r} R_t \geq A\Big\}\Big(1 - \exp\Big(-6\int_r^\infty \mathrm{d}t\big(\frac{1}{(R_t - A)^2} - \frac{1}{(R_t)^2}\big)\Big)\Big)\Big],
\end{aligned}$$

using (4). The desired result easily follows from the fact that the integral $\int^\infty \mathrm{d}t\,(R_t)^{-3}$ is convergent. $\square$

Until now, we have not used the fact that $\mathcal{N}$ is decomposed in the form $\mathcal{N} = \mathcal{N}' + \mathcal{N}''$. This decomposition corresponds intuitively to the fact that the trees $\mathcal{T}^i$ are grafted on the left side of the spine $[0,\infty)$ when $i \in I$, and on the right side when $i \in J$. We make this precise by defining an exploration process of the tree. To begin with, we define, for every $u \geq 0$,

$$\tau'_u := \sum_{i\in I} \mathbf{1}_{\{t_i \leq u\}}\,\sigma_i\ , \quad \tau''_u := \sum_{i\in J} \mathbf{1}_{\{t_i \leq u\}}\,\sigma_i\ .$$

Note that both $u \mapsto \tau'_u$ and $u \mapsto \tau''_u$ are nondecreasing and right-continuous. The left limits of these functions are denoted by τ'_{u-} and τ''_{u-} respectively, and $\tau'_{0-} = \tau''_{0-} = 0$ by convention.

Then, for every $s \geq 0$, there is a unique $u \geq 0$, such that $\tau'_{u-} \leq s \leq \tau'_u$, and:

- Either there is a (unique) $i \in I$ such that $u = t_i$, and we set

$$\Theta'_s := p_{\zeta^i}(s - \tau'_{t_i-}).$$

- Or there is no such i and we set $\Theta'_s = u$.

We define similarly $(\Theta''_s)_{s\geq 0}$by replacing $(\tau'_u)_{u\geq 0}$ by $(\tau''_u)_{u\geq 0}$ and I by J. Informally, $(\Theta'_s)_{s\geq 0}$ and $(\Theta''_s)_{s\geq 0}$ correspond to the exploration of respectively the left and the right side of the tree $\mathcal{T}_\infty$. Noting that $\Theta'_0 = \Theta''_0 = 0$, we define $(\Theta_s)_{s\in\mathbb{R}}$ by setting

$$\Theta_s := \begin{cases} \Theta'_s & \text{if } s \geq 0, \\ \Theta''_{-s} & \text{if } s \leq 0. \end{cases}$$

It is straightforward to verify that the mapping $s \mapsto \Theta_s$ is continuous. We also note that the volume measure $\mathbf{V}$ on $\mathcal{T}_\infty$ is the image of Lebesgue measure on $\mathbb{R}$ under the mapping $s \mapsto \Theta_s$.

This exploration process allows us to define intervals on $\mathcal{T}_\infty$. Let us make the convention that, if $s > t$, the "interval" $[s,t]$ is defined by $[s,t] = [s,\infty) \cup (-\infty, t]$. Then, for every $x, y \in \mathcal{T}_\infty$, there is a smallest interval $[s,t]$, with $s,t \in \mathbb{R}$, such that $\Theta_s = x$ and $\Theta_t = y$, and we define

$$[x,y] := \{\Theta_r : r \in [s,t]\}.$$

Note that $[x,y] \neq [y,x]$ unless $x = y$. We may now turn to our construction of the Brownian plane. We set, for every $x, y \in \mathcal{T}_\infty$,

$$D^\circ_\infty(x,y) = \Lambda_x + \Lambda_y - 2\max\left(\min_{z\in[x,y]} \Lambda_z,\ \min_{z\in[y,x]} \Lambda_z\right), \tag{10}$$

and then

$$D_\infty(x,y) = \inf_{x_0=x,x_1,\ldots,x_p=y} \sum_{i=1}^{p} D^\circ_\infty(x_{i-1}, x_i) \tag{11}$$

where the infimum is over all choices of the integer $p \geq 1$ and of the finite sequence $x_0, x_1, \ldots, x_p$ in $\mathcal{T}_\infty$ such that $x_0 = x$ and $x_p = y$. Note that we have

$$D^\circ_\infty(x,y) \geq D_\infty(x,y) \geq |\Lambda_x - \Lambda_y|, \tag{12}$$

for every $x,y\in\mathcal{T}_\infty$. Furthermore, it is immediate from our definitions that

$$D_\infty(0,x)=D^\circ_\infty(0,x)=\Lambda_x$$

for every $x\in\mathcal{T}_\infty$. As a consequence of the continuity of the mapping $s\mapsto\Lambda_{\Theta_s}$, we have $D^\circ_\infty(x_0,x)\longrightarrow 0$ (hence also $D_\infty(x_0,x)\longrightarrow 0$) as $x\to x_0$, for every $x_0\in\mathcal{T}_\infty$.

It is not hard to verify that D_∞ is a pseudo-distance on $\mathcal{T}_\infty$. We put $x\approx y$ if and only if $D_\infty(x,y)=0$ and we introduce the quotient space $\widetilde{\mathcal{P}}_\infty=\mathcal{T}_\infty/\approx$, which is equipped with the metric induced by D_∞ and with the distinguished point which is the equivalence class of 0. The volume measure on $\widetilde{\mathcal{P}}_\infty$ is the image of the volume measure $\mathbf{V}$ on $\mathcal{T}_\infty$ under the canonical projection.

Theorem 3.4. *The pointed metric space $\widetilde{\mathcal{P}}_\infty$ is locally isometric to the Brownian map and scale invariant. Consequently, $\widetilde{\mathcal{P}}_\infty$ is distributed as the Brownian plane $\mathcal{P}_\infty$.*

Proof. The fact that $\widetilde{\mathcal{P}}_\infty$ is scale invariant is easy from our construction. Hence the difficult part of the proof is to verify that $\widetilde{\mathcal{P}}_\infty$ is locally isometric to the Brownian map. Let us start by briefly recalling the construction of the Brownian map $\mathbf{m}_\infty$. We argue under the conditional excursion measure $\mathbb{N}_0^{(1)}=\mathbb{N}_0(\cdot\mid\sigma=1)$. Under $\mathbb{N}_0^{(1)}$, the lifetime process $(\zeta_s)_{0\le s\le 1}$ is a normalized Brownian excursion, and the tree $\mathcal{T}_\zeta$ coded by $(\zeta_s)_{0\le s\le 1}$ is the so-called CRT. As previously, p_ζ stands for the canonical projection from $[0,1]$ onto $\mathcal{T}_\zeta$. We can define intervals on $\mathcal{T}_\zeta$ in a way analogous to what we did before for $\mathcal{T}_\infty$: If $x,y\in\mathcal{T}_\zeta$, $[x,y]=\{p_\zeta(r):r\in[s,t]\}$, where $[s,t]$ is the smallest interval such that $p_\zeta(s)=x$ and $p_\zeta(t)=y$, using now the convention that the interval $[s,t]$ is defined by $[s,t]=[s,1]\cup[0,t]$ when $s>t$. Then we equip $\mathcal{T}_\zeta$ with Brownian labels by setting $\Gamma_x=\widehat{W}_s$ if $x=p_\zeta(s)$. For every $x,y\in\mathcal{T}_\zeta$, we define $D^\circ(x,y)$, resp. $D(x,y)$, by exactly the same formula as in (10), resp. (11), replacing Λ by Γ. We have again the bound $D(x,y)\ge|\Gamma_x-\Gamma_y|$. We then observe that D is a pseudo-distance on $\mathcal{T}_\zeta$, and the Brownian map $\mathbf{m}_\infty$ is the associated quotient metric space. The distinguished point of $\mathbf{m}_\infty$ is chosen as the (equivalence class of the) vertex $x_{\mathbf{m}}$ of $\mathcal{T}_\zeta$ with minimal label, and we note that $D(x_{\mathbf{m}},x)=\Gamma_x-\Gamma_{x_{\mathbf{m}}}=\Gamma_x-W_*$ for every $x\in\mathcal{T}_\zeta$.

If we replace the normalized Brownian excursion by a Brownian excursion with duration $r>0$, that is, if we argue under $\mathbb{N}_0^{(r)}$, and perform the same construction, simple scaling arguments show that the resulting pointed metric space is distributed as $r^{1/4}\cdot\mathbf{m}_\infty$ and is thus locally isometric to $\mathbf{m}_\infty$ (both are locally isometric to the Brownian plane). Consequently, under the probability measure

$$\mathbb{N}_0(\cdot\mid\sigma>1)=\int_1^\infty\frac{\mathrm{d}r}{2r^{3/2}}\,\mathbb{N}_0^{(r)}(\cdot)$$

the preceding construction also yields a random pointed metric space which is locally isometric to $\mathbf{m}_\infty$. Let us write $\mathbf{M}$ for this random pointed metric space. We will argue that $\mathbf{M}$ is locally isometric to $\widetilde{\mathcal{P}}_\infty$, which will complete the proof. Some of the arguments that follow are similar to those used in [9, Proof of Proposition 4] to verify that the Brownian plane is locally isometric to the Brownian map.

We set for every $b>0$,

$$A_b:=\int_0^\sigma\mathrm{d}s\,\mathbf{1}_{\{\tau_{(-b,\infty)}(W_s)<\infty\}},$$

where we used the notation $\tau_D(\mathrm{w})$ introduced in subsection 2.2. Still with the notation of this subsection, the random variable A_b is $\mathcal{E}_b$-measurable, and it follows that

$$H:=\inf\{b\ge 0:A_b=1\}$$

is a stopping time of the filtration $(\mathcal{E}_a)_{a>0}$. Observe that $\{H < \infty\} = \{\sigma > 1\}$, $\mathbb{N}_0$ a.e. From Proposition 2.3, we get that under the probability measure $\mathbb{N}_0(\cdot \mid \sigma > 1)$, and conditionally on the pair $(H, \mathcal{Z}_{-H})$, the excursions of the Brownian snake outside $(-H, \infty)$ form a Poisson point process with intensity $\mathcal{Z}_{-H}\,\mathbb{N}_{-H}$ (incidentally this also implies that $\mathcal{Z}_{-H} > 0$ a.e. on $\{\sigma > 1\}$). Among the excursions outside $(-H, \infty)$, there is exactly one that attains the minimal value W_*, and conditionally on $H = h$ and $W_* = a$ (with $a < -h$), this excursion is distributed according to $\mathbb{N}_{-h}(\cdot \mid W_* = a)$.

Now compare Theorem 2.1 with the construction of $\widetilde{\mathcal{P}}_\infty$ given above to see that we can find a coupling of the Brownian snake under $\mathbb{N}_0(\cdot \mid \sigma > 1)$ and of the triplet $(R, \mathcal{N}', \mathcal{N}'')$ determining the labeled tree $(\mathcal{T}_\infty, (\Lambda_x)_{x \in \mathcal{T}_\infty})$, in such a way that the following properties hold. There exists a (random) real $\delta > 0$ and an isometry $\mathcal{I}$ from the ball $B_\delta(\mathcal{T}_\zeta)$ (centered at the distinguished vertex $x_{\mathbf{m}} = p_\zeta(s_{\mathbf{m}})$) onto the ball $B_\delta(\mathcal{T}_\infty)$ (centered at 0). This isometry preserves intervals, in the sense that if $x, y \in B_\delta(\mathcal{T}_\zeta)$, $\mathcal{I}([x,y] \cap B_\delta(\mathcal{T}_\zeta)) = [\mathcal{I}(x), \mathcal{I}(y)] \cap B_\delta(\mathcal{T}_\infty)$. Furthermore, the isometry $\mathcal{I}$ preserves labels up to a shift by $-W_*$, meaning that $\Lambda_{\mathcal{I}(x)} = \Gamma_x - W_*$ for every $x \in B_\delta(\mathcal{T}_\zeta)$. Consequently, we have

$$D(x_{\mathbf{m}}, x) = \Gamma_x - W_* = \Lambda_{\mathcal{I}(x)} = D_\infty(0, \mathcal{I}(x))$$

for every $x \in B_\delta(\mathcal{T}_\zeta)$.

Next we can choose $\eta > 0$ small enough so that labels on $\mathcal{T}_\zeta \backslash B_\delta(\mathcal{T}_\zeta)$ are all strictly larger than $W_* + 2\eta$ and labels on $\mathcal{T}_\infty \backslash B_\delta(\mathcal{T}_\infty)$ are all strictly larger than 2η (we use Lemma 3.3 here). In particular, if $x \in \mathcal{T}_\zeta$, the condition $D(x_{\mathbf{m}}, x) \le 2\eta$ implies that $x \in B_\delta(\mathcal{T}_\zeta)$, and, if $x' \in \mathcal{T}_\infty$, the condition $D_\infty(0, x') \le 2\eta$ implies that $x' \in B_\delta(\mathcal{T}_\infty)$. We claim that

$$D(x, y) = D_\infty(\mathcal{I}(x), \mathcal{I}(y)), \tag{13}$$

for every $x, y \in \mathcal{T}_\zeta$ such that $D(x_{\mathbf{m}}, x) \le \eta$ and $D(x_{\mathbf{m}}, y) \le \eta$. To verify this claim, first note that, if $x', y' \in \mathcal{T}_\infty$ are such that $D_\infty(0, x') = \Lambda_{x'} \le 2\eta$ and $D_\infty(0, y') = \Lambda_{y'} \le 2\eta$, we can compute $D^\circ_\infty(x', y')$ using formula (10), and in the right-hand side of this formula we may replace the interval $[x', y']$ by $[x', y'] \cap B_\delta(\mathcal{T}_\infty)$ (because obviously the minimal value of Λ on $[x', y']$ is attained on $[x', y'] \cap B_\delta(\mathcal{T}_\infty)$). A similar replacement may be made in the analogous formula for $D^\circ(x, y)$ when $x, y \in \mathcal{T}_\zeta$ are such that $\Gamma_x \le W_* + 2\eta$ and $\Gamma_y \le W_* + 2\eta$. Using the isometry $\mathcal{I}$, we then obtain that

$$D^\circ(x, y) = D^\circ_\infty(\mathcal{I}(x), \mathcal{I}(y)) \tag{14}$$

for every $x, y \in \mathcal{T}_\zeta$ such that $D(x_{\mathbf{m}}, x) \le 2\eta$ and $D(x_{\mathbf{m}}, y) \le 2\eta$. Then, let $x', y' \in \mathcal{T}_\infty$ be such that $\Lambda_{x'} \le \eta$ and $\Lambda_{y'} \le \eta$. If we use formula (11) to evaluate $D_\infty(x', y')$, we may in the right-hand side of this formula restrict our attention to "intermediate" points x_i whose label Λ_{x_i} is smaller than 2η (indeed if one of the intermediate points has a label strictly greater than 2η, then it follows from (12) that the sum in the right-hand side of (11) is strictly greater than $2\eta \ge D_\infty(x', y')$). A similar observation holds if we use the analog of (11) to compute $D(x, y)$ when $x, y \in \mathcal{T}_\zeta$ are such that $D(x_{\mathbf{m}}, x) \le \eta$ and $D(x_{\mathbf{m}}, y) \le \eta$. Our claim (13) is a consequence of the preceding considerations and (14).

It follows from (13) that $\mathcal{I}$ induces an isometry from the ball $B_\eta(\mathbf{M})$ onto the ball $B_\eta(\widetilde{\mathcal{P}}_\infty)$. This implies that $\mathbf{M}$ is locally isometric to $\widetilde{\mathcal{P}}_\infty$, and the proof is complete. $\square$

In view of Theorem 3.4, we may and will write $\mathcal{P}_\infty$ instead of $\widetilde{\mathcal{P}}_\infty$ for the random metric space that we constructed in the first part of this subsection. We denote the canonical projection from $\mathcal{T}_\infty$ onto $\mathcal{P}_\infty$ by Π. The fact that $D_\infty(x_0, x) \longrightarrow 0$ as $x \to x_0$, for every fixed $x_0 \in \mathcal{T}_\infty$, shows that Π is continuous. The argument of the preceding proof makes it possible to transfer several known properties of the Brownian map to the space $\mathcal{P}_\infty$. First, for every $x, y \in \mathcal{T}_\infty$, we have

$$D_\infty(x, y) = 0 \quad \text{if and only if} \quad D^\circ_\infty(x, y) = 0.$$

Indeed this property will hold for x and y belonging to a sufficiently small ball centered at 0 in $\mathcal{T}_\infty$, by [21, Theorem 3.4] and the coupling argument explained in the preceding proof. The scale invariance of the Brownian plane then completes the argument. Similarly, we have the so-called "cactus bound", for every $x, y \in \mathcal{T}_\infty$ and every continuous path $(\gamma(t))_{0\leq t\leq 1}$ in $\mathcal{P}_\infty$ such that $\gamma(0) = \Pi(x)$ and $\gamma(1) = \Pi(y)$,

$$\min_{0\leq t\leq 1} D_\infty(0, \gamma(t)) \leq \min_{z\in[\![x,y]\!]} \Lambda_z, \tag{15}$$

where $[\![x, y]\!]$ stands for the geodesic segment between x and y in the tree $\mathcal{T}_\infty$. The bound (15) follows from the analogous result for the Brownian map [22, Proposition 3.1] and the coupling argument of the preceding proof.

Since labels correspond to distances from the distinguished point, we have, for every $r > 0$,

$$B_r(\mathcal{P}_\infty) = \Pi\Big(\{x \in \mathcal{T}_\infty : \Lambda_x \leq r\}\Big).$$

Recall the definition of the hull $B^\bullet_r(\mathcal{P}_\infty)$ in Section 1. We claim that

$$B^\bullet_r(\mathcal{P}_\infty) = \mathcal{P}_\infty \setminus \Pi\Big(\{x \in \mathcal{T}_\infty : \Lambda_y > r,\ \forall y \in [\![x, \infty[\![\}\Big), \tag{16}$$

where $[\![x, \infty[\![$ is the geodesic path from x to ∞ in the tree $\mathcal{T}_\infty$. The fact that $B^\bullet_r(\mathcal{P}_\infty)$ is contained in the right-hand side of (16) is easy: If $x \in \mathcal{T}_\infty$ is such that $\Lambda_y > r$ for every $y \in [\![x, \infty[\![$, then $\Pi([\![x, \infty[\![)$ gives a continuous path going from $\Pi(x)$ to ∞ and staying outside the ball $B_r(\mathcal{P}_\infty)$. Conversely, suppose that $x \in \mathcal{T}_\infty$ is such that

$$\min_{y\in[\![x,\infty[\![} \Lambda_y \leq r.$$

Then, if $(\gamma(t))_{t\geq 0}$ is any continuous path going from $\Pi(x)$ to ∞ in $\mathcal{P}_\infty$, the bound (15) leads to

$$\min_{t\geq 0} D(0, \gamma(t)) \leq \min_{y\in[\![x,\infty[\![} \Lambda_y \leq r,$$

and it follows that $\Pi(x) \in B^\bullet_r(\mathcal{P}_\infty)$, proving our claim.

Write $\partial B^\bullet_r(\mathcal{P}_\infty)$ for the topological boundary of $B^\bullet_r(\mathcal{P}_\infty)$. It follows from (16) that

$$\partial B^\bullet_r(\mathcal{P}_\infty) = \Pi\Big(\{x \in \mathcal{T}_\infty : \Lambda_x = r \text{ and } \Lambda_y > r,\ \forall y \in]\!]x, \infty[\![\}\Big), \tag{17}$$

with the obvious notation $]\!]x, \infty[\![$. The latter formula motivates the definition of the (generalized) length of the boundary of $B^\bullet_r(\mathcal{P}_\infty)$. We observe that this boundary contains (the image under Π of) a single point on the spine, corresponding to the last visit of r by the process R,

$$L_r = \sup\{t \geq 0 : R_t = r\}.$$

Any other point $x \in \mathcal{T}_\infty$ such that $\Lambda_x = r$ and $\Lambda_y > r$ for every $y \in]\!]x, \infty[\![$ must be of the form $p_{\zeta^i}(s)$, for some $i \in I \cup J$, with $t_i > L_r$, and some $s \in [0, \sigma_i]$ such that the path ω^i_s hits r exactly at its lifetime. For each fixed i (with $t_i > L_r$), the "quantity" of such values of s is measured by the total mass $\mathcal{Z}_r(\omega^i)$ of the exit measure of ω^i from (r,∞). Here we use the same notation $\mathcal{Z}_r = \langle \mathcal{Z}^{(r,\infty)}, 1\rangle$ as previously.

Following the preceding discussion, we define, for every $r > 0$,

$$Z_r := \int \mathcal{N}(\mathrm{d}t, \mathrm{d}\omega)\, \mathbf{1}_{\{L_r < t\}}\, \mathcal{Z}_r(\omega) = \sum_{i \in I \cup J, t_i > L_r} \mathcal{Z}_r(\omega^i). \tag{18}$$

We observe that the quantities $\mathcal{Z}_r(\omega^i)$ in (18) are well defined since each ω^i is a Brownian snake excursion starting from R_{t_i} and the condition $t_i > L_r$ guarantees that $R_{t_i} > r$. We interpret Z_r as measuring the size of the boundary of the hull $B^\bullet_r(\mathcal{P}_\infty)$.

Note that at the present stage, it is not clear that the random variable Z_r coincides with the one introduced in Proposition 1.1 (which we have not yet proved). At the end of subsection 4.1 below, we will verify that the approximation result of Proposition 1.1 holds with the preceding definition of Z_r.

4 The volume of hulls

4.1 The process of boundary lengths

Our main goal in this subsection is to describe the distribution of the process $(Z_r)_{r>0}$. We fix $a > 0$. By formula (18) and the exponential formula for Poisson measures, we have, for every $\lambda \geq 0$,

$$E\Big[\exp(-\lambda Z_a)\Big] = E\Big[\exp\Big(-4\int_{L_a}^{\infty} \mathrm{d}t\, \mathbb{N}_{R_t}\Big(\mathbf{1}_{\{\mathcal{R}\subset(0,\infty)\}}(1 - e^{-\lambda \mathcal{Z}_a})\Big)\Big)\Big]. \tag{19}$$

The quantity in the right-hand side will be computed via the following two lemmas.

Lemma 4.1. *For every $x > a$ and $\lambda \geq 0$,*

$$\mathbb{N}_x\Big(\mathbf{1}_{\{\mathcal{R}\subset(0,\infty)\}}(1 - e^{-\lambda \mathcal{Z}_a})\Big) = \frac{3}{2}\Bigg(\Big(x - a + (\frac{2\lambda}{3} + a^{-2})^{-1/2}\Big)^{-2} - x^{-2}\Bigg).$$

Proof. We have

$$\begin{aligned}
\mathbb{N}_x\Big(\mathbf{1}_{\{\mathcal{R}\subset(0,\infty)\}}(1 - e^{-\lambda \mathcal{Z}_a})\Big) &= \mathbb{N}_x\Big(1 - \mathbf{1}_{\{\mathcal{R}\subset(0,\infty)\}} e^{-\lambda \mathcal{Z}_a}\Big) - \mathbb{N}_x\Big(1 - \mathbf{1}_{\{\mathcal{R}\subset(0,\infty)\}}\Big) \\
&= \mathbb{N}_x\Big(1 - \mathbf{1}_{\{\mathcal{R}\subset(0,\infty)\}} e^{-\lambda \mathcal{Z}_a}\Big) - \frac{3}{2x^2},
\end{aligned}$$

by (4). In order to compute the first term in the right-hand side, we observe that we have $\mathcal{R} \subset (a,\infty) \subset (0,\infty)$ on the event $\{\mathcal{Z}_a = 0\}$, $\mathbb{N}_x$ a.e., by (5). Therefore, we can write

$$\begin{aligned}
\mathbb{N}_x\Big(1 - \mathbf{1}_{\{\mathcal{R}\subset(0,\infty)\}} e^{-\lambda \mathcal{Z}_a}\Big) &= \mathbb{N}_x\Big(\mathbf{1}_{\{\mathcal{Z}_a > 0\}}\Big) - \mathbb{N}_x\Big(\mathbf{1}_{\{\mathcal{Z}_a > 0, \mathcal{R}\subset(0,\infty)\}} e^{-\lambda \mathcal{Z}_a}\Big) \\
&= \mathbb{N}_x\Big(\mathbf{1}_{\{\mathcal{Z}_a > 0\}}\Big) - \mathbb{N}_x\Big(\mathbf{1}_{\{\mathcal{Z}_a > 0\}}\, e^{-\lambda \mathcal{Z}_a}\, \exp\Big(-\frac{3\mathcal{Z}_a}{2a^2}\Big)\Big) \\
&= \mathbb{N}_x\Big(1 - \exp\Big(-(\lambda + \frac{3}{2a^2})\mathcal{Z}_a\Big)\Big).
\end{aligned}$$

In the second equality we used the special Markov property, together with formula (4), to obtain that the conditional probability of the event $\{\mathcal{R} \subset (0,\infty)\}$ given $\mathcal{Z}_a$ is $\exp(-\frac{3\mathcal{Z}_a}{2a^2})$. The formula of the lemma follows from the preceding two displays and (6). ☐

Lemma 4.2. *For every $\alpha \in (0,a)$,*

$$E\Big[\exp\Big(6\int_{L_a}^{\infty}\mathrm{d}t\,\Big(\frac{1}{(R_t)^2}-\frac{1}{(R_t-\alpha)^2}\Big)\Big)\Big]=\Big(\frac{a-\alpha}{a}\Big)^3.$$

Proof. By dominated convergence, we have

$$E\Big[\exp\Big(6\int_{L_a}^{\infty}\mathrm{d}t\,\Big(\frac{1}{(R_t)^2}-\frac{1}{(R_t-\alpha)^2}\Big)\Big)\Big]=\lim_{b\uparrow\infty}\downarrow E\Big[\exp\Big(6\int_{L_a}^{L_b}\mathrm{d}t\,\Big(\frac{1}{(R_t)^2}-\frac{1}{(R_t-\alpha)^2}\Big)\Big)\Big].$$

Let us fix $b>a$. By the time-reversal property of Bessel processes already mentioned after the statement of Theorem 2.1, the process $(\widetilde{R}_t)_{t\geq 0}$ defined by

$$\widetilde{R}_t = R_{(L_b-t)\vee 0}$$

is a Bessel process of dimension -5 started from b. Set $T_a := \inf\{t\geq 0 : \widetilde{R}_t = a\} = L_b - L_a$. Write $(B_t)_{t\geq 0}$ for a one-dimensional Brownian motion which starts from r under the probability measure P_r, and for every $y\in\mathbb{R}$, let $\gamma_y := \inf\{t\geq 0 : B_t = y\}$. Then,

$$\begin{aligned}E\Big[\exp\Big(6\int_{L_a}^{L_b}\mathrm{d}t\,\Big(\frac{1}{(R_t)^2}-\frac{1}{(R_t-\alpha)^2}\Big)\Big)\Big]&=E\Big[\exp\Big(6\int_{0}^{T_a}\mathrm{d}t\,\Big(\frac{1}{(\widetilde{R}_t)^2}-\frac{1}{(\widetilde{R}_t-\alpha)^2}\Big)\Big)\Big]\\&=\Big(\frac{b}{a}\Big)^3 E_b\Big[\exp\Big(-6\int_0^{\gamma_a}\frac{\mathrm{d}t}{(B_t-\alpha)^2}\Big)\Big],\end{aligned}$$

where the last equality is a special case of the classical absolute continuity relations between Bessel processes (see [25, Lemma 1] for this special case). Next observe that

$$E_b\Big[\exp\Big(-6\int_0^{\gamma_a}\frac{\mathrm{d}t}{(B_t-\alpha)^2}\Big)\Big]=E_{b-\alpha}\Big[\exp\Big(-6\int_0^{\gamma_{a-\alpha}}\frac{\mathrm{d}t}{(B_t)^2}\Big)\Big]=\Big(\frac{a-\alpha}{b-\alpha}\Big)^3,$$

where the second equality is well known (and can again be viewed as a consequence of Lemma 1 in [25]). By combining the last two displays, we get

$$E\Big[\exp\Big(6\int_{L_a}^{L_b}\mathrm{d}t\,\Big(\frac{1}{(R_t)^2}-\frac{1}{(R_t-\alpha)^2}\Big)\Big)\Big]=\Big(\frac{b}{a}\Big)^3\times\Big(\frac{a-\alpha}{b-\alpha}\Big)^3,$$

and the desired result follows by letting $b\uparrow\infty$. $\square$

We can now identify the law of Z_a.

Proof of Proposition 1.2 (i). We start from formula (19) and use first Lemma 4.1 and then Lemma 4.2 to obtain, for every $\lambda\geq 0$,

$$\begin{aligned}E\Big[\exp(-\lambda Z_a)\Big]&=E\Big[\exp\Big(6\int_{L_a}^{\infty}\mathrm{d}t\,\Big(\frac{1}{(R_t)^2}-\frac{1}{(R_t-(a-(\frac{2\lambda}{3}+a^{-2})^{-1/2}))^2}\Big)\Big)\Big]\\&=\Big(\frac{a-(a-(\frac{2\lambda}{3}+a^{-2})^{-1/2})}{a}\Big)^3,\end{aligned}$$

which yields the desired result. $\square$

Our next goal is to obtain the law of the whole process $(Z_a)_{a\geq 0}$, where by convention we take $Z_0=0$. To this end it is convenient to introduce a "backward" filtration $(\mathcal{G}_a)_{a\geq 0}$, which we will define after introducing some notation. If $\mathrm{w}\in\mathcal{W}$, we set $\tau_a(\mathrm{w}) := \inf\{t\geq 0 : \mathrm{w}(t)\notin(a,\infty)\}$, with the usual convention $\inf\varnothing = \infty$. Then, let $a \geq 0$ and $x > a$, and let $\omega = (\omega_s)_{s\geq 0} \in C(\mathbb{R}_+, \mathcal{W}_x)$ be such that $\omega_s = x$ for all s large enough. For every $s \geq 0$, we define $\mathrm{tr}_a(\omega)_s \in \mathcal{W}_x$ by the formula

$$\mathrm{tr}_a(\omega)_s = \omega_{\eta_s^{(a)}(\omega)},$$

where, for every $s \geq 0$,

$$\eta_s^{(a)}(\omega) := \inf\{r \geq 0 : \int_0^r \mathrm{d}u\, \mathbf{1}_{\{\zeta(\omega_u)\leq \tau_a(\omega_u)\}} > s\}.$$

From the properties of the Brownian snake, it is easy to verify that $\mathbb{N}_x(\mathrm{d}\omega)$ a.e., $\mathrm{tr}_a(\omega)$ belongs to $C(\mathbb{R}_+, \mathcal{W}_x)$, and the paths $\mathrm{tr}_a(\omega)_s$ do not visit $(-\infty, a)$, and may visit a only at their endpoint (what we have done is removing those paths that hit a and survive for some positive time after hitting a). Note that we are using a particular instance of the time change η_s^D introduced when defining the σ-field $\mathcal{E}^D$ in subsection 2.2 (indeed, the σ-field $\mathcal{E}^{(a,\infty)}$ is generated by the mapping $\omega \mapsto \mathrm{tr}_a(\omega)$ up to negligible sets).

Recall formula (9) for the point measure $\mathcal{N}$. For every $a \geq 0$, we let $\mathcal{G}_a$ be the σ-field generated by the process $(R_{L_a+t})_{t\geq 0}$ and the point measure

$$\mathcal{N}^{(a)} := \sum_{i\in I\cup J, t_i > L_a} \delta_{(t_i, \mathrm{tr}_a(\omega^i))},$$

and by the P-negligible sets. Note that, in the definition of $\mathcal{N}^{(a)}$, we keep only those excursions that start from the "spine" at a time greater than L_a (so that obviously their initial point is greater than a) and we truncate these excursions at level a. Also notice that $\mathcal{N}^{(0)} = \mathcal{N}$.

From our definitions it is clear that $\mathcal{G}_a \supset \mathcal{G}_b$ if $a < b$. Furthermore, it follows from the measurability property of exit measures that Z_a is $\mathcal{G}_a$-measurable, for every $a > 0$ (the point is that $\mathcal{Z}_a(\omega^i)$ is equal a.s. to a measurable function of $\mathrm{tr}_a(\omega^i)$, see [18, Proposition 2.3]). We also notice that, for every $a > 0$, the process $(R_{L_a+t})_{t\geq 0}$ is independent of $(R_t)_{0\leq t\leq L_a}$. This follows from last exit decompositions for diffusion processes, or in a more straightforward way this can be deduced from the time-reversal property already mentioned above.

Proposition 4.3. *Let $0 < a < b$. Then, for every $\lambda \geq 0$,*

$$E[\exp(-\lambda Z_a) \mid \mathcal{G}_b] = \Big(\frac{b}{a + (b-a)(1 + \frac{2\lambda a^2}{3})^{1/2}}\Big)^3 \exp\Big(-\frac{3Z_b}{2}\Big(\frac{1}{(b - a + (\frac{2\lambda}{3} + a^{-2})^{-1/2})^2} - \frac{1}{b^2}\Big)\Big).$$

If $b > 0$ is fixed, the proposition shows that the process $(Z_{b-a})_{0\leq a<b}$ is time-inhomogeneous Markov with respect to the (forward) filtration $(\mathcal{G}_{b-a})_{0\leq a<b}$, and identifies the Laplace transform of the associated transition kernels. Since the law of Z_b is also given by Proposition 1.2 (i), this completely characterizes the law of the process $(Z_a)_{a\geq 0}$. The more explicit description of this law given in Proposition 1.2 (ii) will be derived later.

Proof. Recall that $0 < a < b$ are fixed. We write

$$Z_a = Y_{a,b} + \widetilde{Y}_{a,b},$$

where

$$Y_{a,b} := \sum_{i\in I\cup J, t_i > L_b} \mathcal{Z}_a(\omega^i)\ , \quad \widetilde{Y}_{a,b} := \sum_{i\in I\cup J, L_a < t_i \leq L_b} \mathcal{Z}_a(\omega^i).$$

From the fact that $(R_{L_b+t})_{t\geq 0}$ is independent of $(R_t)_{0\leq t\leq L_b}$ and properties of Poisson measures, it easily follows that $Y_{a,b}$ and $\widetilde{Y}_{a,b}$ are independent, and more precisely $\widetilde{Y}_{a,b}$ is independent of $\sigma(Y_{a,b})\vee\mathcal{G}_b$. This implies that

$$E[\exp(-\lambda Z_a)\mid\mathcal{G}_b]=E[\exp(-\lambda\widetilde{Y}_{a,b})]\,E[\exp(-\lambda Y_{a,b})\mid\mathcal{G}_b]. \tag{20}$$

From the special Markov property (see also the remark following Proposition 2.2), we have

$$\begin{aligned}E[\exp(-\lambda Y_{a,b})\mid\mathcal{G}_b]&=E\Bigg[\prod_{i\in I\cup J,t_i>L_b}\exp(-\lambda\mathcal{Z}_a(\omega^i))\,\Bigg|\,\mathcal{G}_b\Bigg]\\&=\exp\Bigg(-\sum_{i\in I\cup J,t_i>L_b}\mathcal{Z}_b(\omega^i)\,\mathbb{N}_b\Big(\mathbf{1}_{\{\mathcal{R}\subset(0,\infty)\}}(1-e^{-\lambda\mathcal{Z}_a})\Big)\Bigg)\\&=\exp\Big(-Z_b\,\mathbb{N}_b\Big(\mathbf{1}_{\{\mathcal{R}\subset(0,\infty)\}}(1-e^{-\lambda\mathcal{Z}_a})\Big)\Big)\\&=\exp\Bigg(-\frac{3Z_b}{2}\Bigg(\Big(b-a+(\frac{2\lambda}{3}+a^{-2})^{-1/2}\Big)^{-2}-b^{-2}\Bigg)\Bigg),\end{aligned}$$

where the last equality is Lemma 4.1.

Using Proposition 1.2 (i), we have thus,

$$\begin{aligned}E[\exp(-\lambda Y_{a,b})]&=\Bigg(1+b^2\Bigg(\Big(b-a+(\frac{2\lambda}{3}+a^{-2})^{-1/2}\Big)^{-2}-b^{-2}\Bigg)\Bigg)^{-3/2}\\&=\Big(\frac{b}{b-a+(\frac{2\lambda}{3}+a^{-2})^{-1/2}}\Big)^{-3},\end{aligned}$$

and since $Y_{a,b}$ and $\widetilde{Y}_{a,b}$ are independent,

$$\begin{aligned}E[\exp(-\lambda\widetilde{Y}_{a,b})]&=E[\exp(-\lambda Z_a)]\times(E[\exp(-\lambda Y_{a,b})])^{-1}\\&=\Big(1+\frac{2\lambda a^2}{3}\Big)^{-3/2}\Big(\frac{b}{b-a+(\frac{2\lambda}{3}+a^{-2})^{-1/2}}\Big)^{3}\\&=\left(\frac{b}{a+(b-a)\Big(1+\frac{2\lambda a^2}{3}\Big)^{1/2}}\right)^{3}.\end{aligned}$$

The statement of the proposition follows from (20) and the preceding calculations. □

We will now identify the transition kernels whose Laplace transform appears in the previous proposition. To this end, we recall the discussion of subsection 2.1, which we will apply with the particular value $c=\sqrt{8/3}$.

Proposition 4.4. *Let $\rho>0$ and $x>0$. The finite-dimensional marginal distributions of $(Z_{\rho-a})_{0\leq a\leq\rho}$ knowing that $Z_\rho=x$ coincide with those of the continuous-state branching process with branching mechanism $\psi(u)=\sqrt{8/3}\,u^{3/2}$ started from x and conditioned on extinction at time ρ.*

Proof. Recall the notation introduced in subsection 2.1. By comparing the right-hand side of (3) with the formula of Proposition 4.3, we immediately see that, for $0 \leq s < t < \rho$,

$$E[\exp(-\lambda Z_{\rho-t}) \mid \mathcal{G}_{\rho-s}] = \int e^{-\lambda y}\, \pi_{s,t}(Z_{\rho-s}, dy).$$

Arguing inductively, we obtain that, for every $0 < s_1 < \ldots < s_p < \rho$, the conditional distribution of $(Z_{\rho-s_1}, \ldots, Z_{\rho-s_p})$ knowing $\mathcal{G}_\rho$ is $\pi_{0,s_1}(Z_\rho, \mathrm{d}y_1)\pi_{s_1,s_2}(y_1, \mathrm{d}y_2)\ldots\pi_{s_{p-1},s_p}(y_{p-1}, \mathrm{d}y_p)$. The desired result follows. $\square$

We can now complete the proof of Proposition 1.2.

Proof of Proposition 1.2 (ii). We first verify that Z_a and $\widetilde{X}_{-a}$ have the same distribution, for every fixed $a > 0$. Let $\lambda > 0$ and set $f(y) = e^{-\lambda y}$ to simplify notation. By the properties of the process $\widetilde{X}$, we have

$$E[f(\widetilde{X}_{-a})] = \lim_{x\uparrow\infty} E_x[f(X_{T-a})\, \mathbf{1}_{\{a\leq T\}}],$$

where $T = \inf\{t \geq 0 : X_t = 0\}$ as previously. On the other hand, recalling the definition of the functions ϕ_t in subsection 2.1,

$$\begin{aligned}
E_x[f(X_{T-a})\, \mathbf{1}_{\{a\leq T\}}] &= \lim_{n\uparrow\infty} \sum_{k=1}^{\infty} E_x[\mathbf{1}_{\{a+\frac{k-1}{n}<T\leq a+\frac{k}{n}\}}\, f(X_{k/n})] \\
&= \lim_{n\uparrow\infty} \sum_{k=1}^{\infty} E_x\Big[f(X_{k/n})\, P_{X_{k/n}}\Big(a-\frac{1}{n} < T \leq a\Big)\Big] \\
&= \lim_{n\uparrow\infty} \sum_{k=1}^{\infty} E_x\Big[f(X_{k/n}) \int_{a-1/n}^{a} \phi_b(X_{k/n})\, \mathrm{d}b\Big] \\
&= E_x\Big[\int_0^\infty f(X_t)\, \phi_a(X_t)\, \mathrm{d}t\Big],
\end{aligned}$$

where dominated convergence is easily justified by the fact that $E_x[T] < \infty$ and $\phi_b(0) = 0$ for every $b > 0$. Now use the form of ϕ_a together with formula (2) (with $c = \sqrt{8/3}$) to see that the right-hand side of the last display is equal to

$$\begin{aligned}
&\int_0^\infty \mathrm{d}t\, \frac{3x}{a^3}\, (\lambda + \frac{3}{2a^2})^{-3/2} \Big((\lambda + \frac{3}{2a^2})^{-1/2} + \sqrt{\frac{2}{3}}t\Big)^{-3} \exp\Big(-x\Big((\lambda + \frac{3}{2a^2})^{-1/2} + \sqrt{\frac{2}{3}}t\Big)^{-2}\Big) \\
&\quad = (1 + \frac{2\lambda a^2}{3})^{-3/2}\, \Big(1 - \exp(-x(\lambda + \frac{3}{2a^2})^{-1/2})\Big).
\end{aligned}$$

We then let $x \uparrow \infty$ to get that

$$E[\exp(-\lambda \widetilde{X}_{-a})] = (1 + \frac{2\lambda a^2}{3})^{-3/2} = E[\exp(-\lambda Z_a)]$$

by assertion (i) of the proposition.

Knowing that Z_a and $\widetilde{X}_{-a}$ have the same distribution, the proof is completed as follows. We observe that, for every $a > 0$, the law of $(\widetilde{X}_{-a+t})_{0\leq t\leq a}$ conditionally on $\widetilde{X}_{-a} = x$ coincides with the law of X started from x and conditioned on extinction at time a (we leave the easy verification to the reader). By comparing with Proposition 4.4, we get the desired statement. $\square$

As a consequence of Proposition 1.2, the process $(Z_r)_{r>0}$ has a càdlàg modification, and from now on we deal only with this modification. We conclude this subsection by proving Proposition

1.1: We need to verify that our definition of the random variable Z_r matches the approximation given in this proposition.

Proof of Proposition 1.1. If $x \in \mathcal{T}_\infty$ and x is not on the spine, the point $\Pi(x)$ belongs to $B^\bullet_r(\mathcal{P}_\infty)^c \cap B_{r+\varepsilon}(\mathcal{P}_\infty)$ if and only if $\Lambda_x \in (r, r+\varepsilon]$ and $\Lambda_y > r$ for every $y \in [\![x, \infty[\![$. Recalling our notation $\mathbf{V}$ for the volume measure on $\mathcal{T}_\infty$, we can thus write

$$|B^\bullet_r(\mathcal{P}_\infty)^c \cap B_{r+\varepsilon}(\mathcal{P}_\infty)| = \sum_{i \in I \cup J : t_i > L_r} \mathbf{V}(\{x \in \mathcal{T}^i : \Lambda_x \leq r+\varepsilon \text{ and } \Lambda_y > r,\ \forall y \in [\![\rho_i, x]\!]\}).$$

We will first deal with indices i such that $t_i > L_{r+\varepsilon}$, and we set

$$A_\varepsilon := \sum_{i \in I \cup J : t_i > L_{r+\varepsilon}} \mathbf{V}(\{x \in \mathcal{T}^i : \Lambda_x \leq r+\varepsilon \text{ and } \Lambda_y > r,\ \forall y \in [\![\rho_i, x]\!]\})$$

to simplify notation. Recall that if $x \in \mathcal{T}^i$ and $x = p_{\zeta^i}(s)$, we have $\Lambda_x = \widehat{\omega}^i_s$ and $\{\Lambda_y : y \in [\![\rho_i, x]\!]\} = \{\omega^i_s(t) : 0 \leq t \leq \zeta^i_s\}$. An application of the special Markov property shows that the conditional distribution of A_ε knowing $Z_{r+\varepsilon}$ is the law of $U_\varepsilon(Z_{r+\varepsilon})$, where U_ε is a subordinator whose Lévy measure is the "law" of

$$\int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{\widehat{W}_s \leq r+\varepsilon;\ W_s(t) > r,\ \forall t \in [0,\zeta_s]\}},$$

under $\mathbb{N}_{r+\varepsilon}$ (and U_ε is assumed to be independent of $Z_{r+\varepsilon}$). From the first moment formula for the Brownian snake [19, Proposition IV.2], one easily derives that

$$\mathbb{N}_{r+\varepsilon}\Big(\int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{\widehat{W}_s \leq r+\varepsilon;\ W_s(t) > r,\ \forall t \in [0,\zeta_s]\}}\Big) = E_{r+\varepsilon}\Big[\int_0^\infty \mathrm{d}t\, \mathbf{1}_{\{B_t \leq r+\varepsilon\}}\, \mathbf{1}_{\{t < \gamma_r\}}\Big] = \varepsilon^2,$$

where we have used the notation of the proof of Lemma 4.2. On the other hand, scaling arguments show that

$$(U_\varepsilon(t))_{t \geq 0} \overset{\text{(d)}}{=} (\varepsilon^4\, U_1(\frac{t}{\varepsilon^2}))_{r \geq 0},$$

and the law of large numbers implies that $t^{-1}U_1(t)$ converges a.s. to 1 as $t \to \infty$. Since the conditional distribution of $\varepsilon^{-2}A_\varepsilon$ knowing $Z_{r+\varepsilon}$ is the law of $\varepsilon^2 U_1(\frac{Z_{r+\varepsilon}}{\varepsilon^2})$, it follows from the preceding observations that

$$\varepsilon^{-2}A_\varepsilon - Z_{r+\varepsilon} \underset{\varepsilon \to 0}{\longrightarrow} 0$$

in probability. Since $Z_{r+\varepsilon}$ converges to Z_r as $\varepsilon \to 0$, we conclude that

$$\varepsilon^{-2}A_\varepsilon \underset{\varepsilon \to 0}{\longrightarrow} Z_r$$

in probability. To complete the proof, we just have to check that

$$\varepsilon^{-2} \sum_{i \in I \cup J : L_r < t_i \leq L_{r+\varepsilon}} \mathbf{V}(\{x \in \mathcal{T}^i : \Lambda_x \leq r+\varepsilon \text{ and } \Lambda_y > r,\ \forall y \in [\![\rho_i, x]\!]\}) \underset{\varepsilon \to 0}{\longrightarrow} 0$$

in probability. We leave the easy verification to the reader. $\square$

4.2 The law of the volume of the hull

This subsection is devoted to the proof of Theorem 1.4. We fix $a > 0$ and recall our notation $B^\bullet_a(\mathcal{P}_\infty)$ for the hull of radius a in the Brownian plane $\mathcal{P}_\infty$. To simplify notation, we write $B^\bullet_a$ instead of $B^\bullet_a(\mathcal{P}_\infty)$, and we also write $|B^\bullet_a|$ for the volume of this hull. Recall that Z_a is interpreted as a generalized length of the boundary of $B^\bullet_a$.

Thanks to the construction of the Brownian plane explained in subsection 3.2 and to formula (16), we can express the volume $|B^\bullet_a|$ as the sum of two independent contributions, namely, on the one hand, the total volume of those subtrees that branch off the spine below level L_a, and, on the other hand, the contribution of the subtrees that branch off the spine above level L_a (for these, we need to sum, over all indices $i \in I \cup J$ with $t_i > L_a$, the Lebesgue measure of the set of all $s \in [0, \sigma_i]$ such that the path ω^i_s hits level a).

The beginning of this subsection is devoted to calculating the Laplace transform of the first of these two contributions. Thanks to Theorem 2.1, this is also the Laplace transform of σ under the conditional probability measure $\mathbb{N}_a(\cdot \mid W_* = 0)$. This motivates the following calculations.

We recall the notation $\mathcal{Z}_0$ for the (total mass of the) exit measure from $(0,\infty)$, and we also set

$$\mathcal{Y}_0 := \int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{\tau_0(W_s)=\infty\}},$$

where we recall that $\tau_0(\mathrm{w}) = \inf\{t \geq 0 : \mathrm{w}(t) \notin (0,\infty)\}$. We note that $\mathcal{Y}_0 = \sigma$ under the conditional probability measure $\mathbb{N}_a(\cdot \mid W_* = 0)$. Our first goal is to compute, for every $\lambda, \mu > 0$, the function $u_{\lambda,\mu}(x)$ defined for every $x > 0$ by

$$u_{\lambda,\mu}(x) = \mathbb{N}_x(1 - \exp(-\lambda \mathcal{Z}_0 - \mu \mathcal{Y}_0)).$$

Note that $u_{\lambda,0}(x)$ is given by formula (6). On the other hand, the limit of $u_{\lambda,\mu}$ as $\lambda \uparrow \infty$ is

$$u_{\infty,\mu}(x) := \mathbb{N}_x(1 - \mathbf{1}_{\{\mathcal{R}\subset(0,\infty)\}} \exp(-\mu\mathcal{Y}_0)) = \sqrt{\frac{\mu}{2}}\Big(3\coth^2((2\mu)^{1/4}x) - 2\Big) \tag{21}$$

by [14, Lemma 7]. The latter formula is generalized in the next lemma.

Lemma 4.5. *We have, for every $x > 0$:*

- *if $\lambda > \sqrt{\frac{\mu}{2}}$,*

$$u_{\lambda,\mu}(x) = \sqrt{\frac{\mu}{2}}\Bigg(3\Bigg(\coth^2\Bigg((2\mu)^{1/4}x + \coth^{-1}\sqrt{\frac{2}{3} + \frac{1}{3}\sqrt{\frac{2}{\mu}}\lambda}\Bigg)\Bigg) - 2\Bigg);$$

- *if $\lambda < \sqrt{\frac{\mu}{2}}$,*

$$u_{\lambda,\mu}(x) = \sqrt{\frac{\mu}{2}}\Bigg(3\Bigg(\tanh^2\Bigg((2\mu)^{1/4}x + \tanh^{-1}\sqrt{\frac{2}{3} + \frac{1}{3}\sqrt{\frac{2}{\mu}}\lambda}\Bigg)\Bigg) - 2\Bigg).$$

Remark. If $\lambda = \sqrt{\frac{\mu}{2}}$, we have simply

$$u_{\lambda,\mu}(x) = \sqrt{\frac{\mu}{2}}.$$

This can be obtained by a passage to the limit from the previous formulas, but a direct proof is also easy.

Proof. By results due to Dynkin, the function $u_{\lambda,\mu}$ solves the differential equation

$$\begin{cases} \frac{1}{2}u'' = 2u^2 - \mu\ , & \text{on } (0,\infty), \\ u(0) = \lambda. \end{cases} \tag{22}$$

This is indeed a very special case of Theorem 3.1 in [13]. For the reader who is unfamiliar with the general theory of superprocesses, a direct proof can be given along the lines of the proof of Lemma 6 in [14].

It is also easy to verify that

$$\lim_{x\to\infty} u_{\lambda,\mu}(x) = \mathbb{N}_0(1 - e^{-\mu\sigma}) = \sqrt{\frac{\mu}{2}}.$$

The formulas of the lemma then follow by solving equation (22), which requires some tedious but straightforward calculations. □

For future reference, we note that, if $\lambda > \sqrt{\frac{\mu}{2}}$, we have, for every $x > 0$,

$$u_{\lambda,\mu}(x) = u_{\infty,\mu}(x + \theta_\mu(\lambda)), \tag{23}$$

where the function θ_μ, which is defined on $(\sqrt{\frac{\mu}{2}}, \infty)$ by

$$\theta_\mu(\lambda) = (2\mu)^{-1/4} \coth^{-1} \sqrt{\frac{2}{3} + \frac{1}{3}\sqrt{\frac{2}{\mu}}\lambda},$$

is the functional inverse of $u_{\infty,\mu}$. Of course (23) is nothing but the flow property of solutions of (22).

Proposition 4.6. *Let $a > 0$. Then, for every $\mu > 0$,*

$$\mathbb{N}_a\Big(e^{-\mu\mathcal{Y}_0}\,\Big|\, W_* = 0\Big) = -\frac{1}{3}\, a^3\, u'_{\infty,\mu}(a) = a^3 (2\mu)^{3/4}\, \frac{\cosh((2\mu)^{1/4}a)}{\sinh^3((2\mu)^{1/4}a)}.$$

Remark. The conditioning on $\{W_* = 0\}$ may be understood as a limit as $\varepsilon \to 0$ of conditioning on $\{-\varepsilon < W_* \leq 0\}$. Equivalently, we may use Theorem 2.1, which provides an explicit description of the conditional probabilities $\mathbb{N}_0(\cdot \mid W_* = y)$ for every $y < 0$. Recall that $\mathcal{Y}_0 = \sigma$ under $\mathbb{N}_a(\cdot \mid W_* = 0)$.

Proof. We first observe that, for every $\varepsilon > 0$,

$$\mathbb{N}_a(-\varepsilon < W_* \leq 0) = \frac{3}{2a^2} - \frac{3}{2(a+\varepsilon)^2} \underset{\varepsilon\to 0}{\sim} \frac{3\varepsilon}{a^3}, \tag{24}$$

by (4). On the other hand,

$$\begin{aligned} \mathbb{N}_a\Big(e^{-\mu\mathcal{Y}_0}\,\mathbf{1}_{\{-\varepsilon<W_*\leq 0\}}\Big) &= \mathbb{N}_a\Big(e^{-\mu\mathcal{Y}_0}\,\mathbf{1}_{\{\mathcal{Z}_0>0,W_*>-\varepsilon\}}\Big) \\ &= \mathbb{N}_a\Big(e^{-\mu\mathcal{Y}_0}\,\mathbf{1}_{\{\mathcal{Z}_0>0\}}\,\exp(-\mathcal{Z}_0\,\mathbb{N}_0(W_* \leq -\varepsilon))\Big) \\ &= \mathbb{N}_a\Big(\exp\Big(-\mu\mathcal{Y}_0 - \frac{3}{2\varepsilon^2}\mathcal{Z}_0\Big)\,\mathbf{1}_{\{\mathcal{Z}_0>0\}}\Big) \end{aligned}$$

using the special Markov property in the second equality, and then (4). Set $\alpha = \frac{3}{2\varepsilon^2}$ to simplify notation. Then,

$$\begin{aligned}
\mathbb{N}_a\Big(\exp\Big(-\mu\mathcal{Y}_0 - \alpha\mathcal{Z}_0\Big)\mathbf{1}_{\{\mathcal{Z}_0>0\}}\Big) &= \mathbb{N}_a\Big(1 - e^{-\mu\mathcal{Y}_0}\mathbf{1}_{\{\mathcal{Z}_0=0\}}\Big) - \mathbb{N}_a\Big(1 - e^{-\mu\mathcal{Y}_0-\alpha\mathcal{Z}_0}\Big) \\
&= u_{\infty,\mu}(a) - u_{\alpha,\mu}(a) \\
&= u_{\infty,\mu}(a) - u_{\infty,\mu}(a + \theta_\mu(\alpha)) \\
&\underset{\alpha\to\infty}{\sim} -\theta_\mu(\alpha)\, u'_{\infty,\mu}(a),
\end{aligned}$$

using (23) in the last equality. Since

$$\theta_\mu(\alpha) \underset{\alpha\to\infty}{\sim} \sqrt{\frac{3}{2\alpha}},$$

it follows from the preceding discussion that

$$\mathbb{N}_a\Big(e^{-\mu\mathcal{Y}_0}\,\mathbf{1}_{\{-\varepsilon<W_*\leq 0\}}\Big) \underset{\varepsilon\to 0}{\sim} -u'_{\infty,\mu}(a)\,\varepsilon.$$

The result of the proposition follows using also (24). $\square$

We state the next result in terms of super-Brownian motion, although our main motivation comes from our application to the Brownian plane in Theorem 1.4. Recall that, in order to use the connection with the Brownian snake, we always assume that the branching mechanism of super-Brownian motion is $\psi_0(u) = 2u^2$.

Corollary 4.7. *Let $a > 0$ and $r > 0$. Assume that $(\mathcal{X}_t)_{t\geq 0}$ is a super-Brownian motion that starts from $r\delta_a$ under the probability measure $\mathbb{P}_{r\delta_a}$. Set*

$$\Sigma = \int_0^\infty \mathrm{d}t\,\langle \mathcal{X}_t, 1\rangle,$$

and write $\mathcal{R}^{\mathcal{X}}$ for the range of $\mathcal{X}$. Then, for every $\mu > 0$,

$$\begin{aligned}
&\mathbb{E}_{r\delta_a}\Big[e^{-\mu\Sigma}\,\Big|\,\min\mathcal{R}^{\mathcal{X}} = 0\Big] \\
&\quad = a^3(2\mu)^{3/4}\,\frac{\cosh((2\mu)^{1/4}a)}{\sinh^3((2\mu)^{1/4}a)}\,\exp\Big(-r\Big(\sqrt{\frac{\mu}{2}}\Big(3\coth^2((2\mu)^{1/4}a) - 2\Big) - \frac{3}{2a^2}\Big)\Big).
\end{aligned}$$

Proof. We may assume that $(\mathcal{X}_t)_{t\geq 0}$ is constructed from a Poisson point measure $\mathcal{N}$ with intensity $r\mathbb{N}_a$ via formula (7). Then, we immediately verify that

$$\Sigma = \int \mathcal{N}(\mathrm{d}\omega)\,\sigma(\omega)$$

and properties of Poisson measures lead to the formula

$$\mathbb{E}_{r\delta_a}\Big[e^{-\mu\Sigma}\,\Big|\,\min\mathcal{R}^{\mathcal{X}} = 0\Big] = \mathbb{N}_a\Big(e^{-\mu\sigma}\Big|\,\min\mathcal{R} = 0\Big)\,\exp\Big(-r\mathbb{N}_a\Big((1-e^{-\mu\sigma})\mathbf{1}_{\{\min\mathcal{R}>0\}}\Big)\Big).$$

The first term in the right-hand side is given by Proposition 4.6. As for the second term we observe that

$$\mathbb{N}_a\Big((1-e^{-\mu\sigma})\mathbf{1}_{\{\min\mathcal{R}>0\}}\Big) = \mathbb{N}_a\Big(1 - e^{-\mu\sigma}\mathbf{1}_{\{\min\mathcal{R}>0\}}\Big) - \mathbb{N}_a(\min\mathcal{R}\leq 0) = u_{\infty,\mu}(a) - \frac{3}{2a^2},$$

and we use formula (21). This completes the proof. $\square$

Proof of Theorem 1.4. The first formula of the theorem is a straightforward consequence of the second one since we know the distribution of Z_a. More precisely, using Proposition 1.2 (ii), we observe that

$$\begin{aligned} &E\Big[\exp\Big(-Z_a\Big(\sqrt{\frac{\mu}{2}}\Big(3\coth^2((2\mu)^{1/4}a)-2\Big)-\frac{3}{2a^2}\Big)\Big)\Big] \\ &= \Big(1+\frac{2a^2}{3}\Big(\sqrt{\frac{\mu}{2}}\Big(3\coth^2((2\mu)^{1/4}a)-2\Big)-\frac{3}{2a^2}\Big)\Big)^{-3/2} \\ &= 3^{3/2}a^{-3}(2\mu)^{-3/4}\Big(3\coth^2((2\mu)^{1/4}a)-2\Big)^{-3/2}. \end{aligned}$$

If we multiply this quantity by

$$a^3(2\mu)^{3/4}\,\frac{\cosh((2\mu)^{1/4}a)}{\sinh^3((2\mu)^{1/4}a)}$$

we get the desired formula for $E[\exp(-\mu|B^\bullet_a|)]$.

Not suprisingly, the second formula of Theorem 1.4 is a consequence of the analogous formula in Corollary 4.7. Let us explain this. Using our representation of the Brownian plane, and formula (16), we can write $|B^\bullet_a|$ as the sum of two independent contributions:

- The contribution of subtrees branching off the spine at a level smaller than L_a. Using Theorem 2.1, we see that this contribution is distributed as σ under the conditional probability measure $\mathbb{N}_a(\cdot \mid W_* = 0)$. We also note that this contribution is independent of the σ-field $\mathcal{G}_a$.
- The contribution of subtrees branching off the spine at a level greater than L_a. This contribution is $\mathcal{G}_a$-measurable. Furthermore, an application of the special Markov property (similar to the one in the proof of Proposition 4.3) shows that its conditional distribution given $Z_a = r$ is the law of
$$\sum_{k\in K}\sigma(\omega_{(k)})$$
where $\sum_{k\in K}\delta_{\omega_{(k)}}$ is a Poisson measure with intensity $r\mathbb{N}_a(\cdot\cap\{W_*>0\})$.

The preceding discussion shows that the conditional distribution of $|B^\bullet_a|$ given $Z_a = r$ coincides with the distribution of Σ under $\mathbb{P}_{r\delta_a}(\cdot \mid \min \mathcal{R}^X = 0)$, with the notation of Corollary 4.7. This completes the proof. □

4.3 The process of hull volumes

Our goal in this subsection is to prove Theorem 1.3. In a way similar to Corollary 4.7, we consider a super-Brownian motion $(\mathcal{X}_t)_{t\geq 0}$, and the probability mesure $\mathbb{P}_{r\delta_0}$ under which this super-Brownian motion starts from $r\delta_0$. We also introduce the associated historical process $(\mathbf{X}_t)_{t\geq 0}$. As previously, we may and will assume that $(\mathcal{X}_t)_{t\geq 0}$ and $(\mathbf{X}_t)_{t\geq 0}$ are constructed from a Poisson measure

$$\mathcal{N} = \sum_{k\in K}\delta_{\omega_{(k)}}$$

with intensity $r\mathbb{N}_0$, via formulas (7) and (8). We then set, for every $a<0$,

$$\mathscr{Z}_a = \sum_{k\in K}\mathcal{Z}_a(\omega_{(k)})$$

and, for every $a \leq 0$,

$$\mathscr{Y}_a = \sum_{k \in K} \mathcal{Y}_a(\omega_{(k)})$$

where

$$\mathcal{Y}_a(\omega) := \int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{\tau_a(W_s(\omega))=\infty\}}.$$

We also set $\mathscr{Z}_0 = r$ by convention.

In the theory of superprocesses [13], $\mathscr{Z}_a$ corresponds to the total mass of the exit measure of the historical process $(\mathbf{X}_t)_{t\geq 0}$ from (a,∞) (for our present purposes, we do not need this interpretation). We also note that, for every $a \leq 0$, we have

$$\mathscr{Y}_a = \int_0^\infty \mathrm{d}t \int \mathbf{X}_t(\mathrm{dw})\, \mathbf{1}_{\{\tau_a(\mathrm{w})=\infty\}},$$

and the right-hand side is the total integrated mass of those historical paths that do not hit a.

As previously, $X = (X_t)_{t\geq 0}$ denotes a continuous-state branching process with branching mechanism $\psi(u) = \sqrt{8/3}\, u^{3/2}$ that starts from r under the probability measure P_r. We will use the "Lévy-Khintchine representation" for ψ: we have

$$\psi(u) = \int \kappa(\mathrm{d}y)\, (e^{-\lambda y} - 1 + \lambda y)$$

where $\kappa(\mathrm{d}y)$ is the measure on $(0,\infty)$ given by

$$\kappa(\mathrm{d}y) = \sqrt{\frac{3}{2\pi}}\, y^{-5/2}\, \mathrm{d}y.$$

Proposition 4.8. *Let $a > 0$. The law under $\mathbb{P}_{r\delta_0}$ of the pair $(\mathscr{Z}_{-a}, \mathscr{Y}_{-a})$ coincides with the law under P_r of the pair*

$$\Big(X_a, \sum_{i: s_i \leq a} \xi_i\, (\Delta X_{s_i})^2\Big) \tag{25}$$

where $s_1, s_2, \ldots$ is a measurable enumeration of the jumps of X, and $\xi_1, \xi_2, \ldots$ is a sequence of i.i.d. real random variables with density

$$\frac{1}{\sqrt{2\pi x^5}}\, e^{-1/2x}\, \mathbf{1}_{(0,\infty)}(x),$$

which is independent of the process $(X_t)_{t\geq 0}$.

Proof. We first observe that, for every $\lambda, \mu > 0$, we have

$$\mathbb{E}_{r\delta_0}[\exp(-\lambda \mathscr{Z}_{-a} - \mu \mathscr{Y}_{-a})] = \exp(-r u_{\lambda,\mu}(a))$$

by the exponential formula for Poisson measures. We will prove that the joint Laplace transform of the pair (25) is given by the same expression.

To this end, we fix $\mu > 0$ and write $\alpha = \sqrt{2\mu}$ to simplify notation. We also set $w_a(\lambda) = u_{\lambda,\mu}(a)$ for every $a \geq 0$. As a consequence of (22) (or directly from Lemma 4.5) we have for every $a, b \geq 0$,

$$w_{a+b} = w_a \circ w_b$$

and $w_0(\lambda) = \lambda$. Furthermore, the derivative of $w_a(\lambda)$ at $a = 0$ is easily computed from the formulas of Lemma 4.5:

$$\frac{\mathrm{d}}{\mathrm{d}a} w_a(\lambda)_{|a=0} = \sqrt{\frac{2}{3}}\, \sqrt{\alpha + \lambda}\, (\alpha - 2\lambda) \tag{26}$$

where we recall that $\alpha = \sqrt{2\mu}$.

Let us consider now the Laplace transform of the pair (25). We first observe that the Laplace transform of the variables ξ_i is given by

$$E[e^{-\beta\xi}] = (1 + \sqrt{2\beta})\, e^{-\sqrt{2\beta}},$$

for every $\beta \geq 0$ (note that $E[\xi e^{-\beta\xi}] = e^{-\sqrt{2\beta}}$ by the well-known formula for the Laplace transform of a positive stable random variable with parameter 1/2). It follows that, for every $\lambda > 0$,

$$E_r\Big[\exp\Big(-\lambda X_a - \mu \sum_{i:\xi_i \leq a} \xi_i (\Delta X_{s_i})^2\Big)\Big] = E_r\Big[\exp(-\lambda X_a) \prod_{0 \leq s \leq a} (1 + \alpha \Delta X_s) e^{-\alpha \Delta X_s}\Big].$$

The additivity property of continuous-state branching processes allows us to write the right-hand side in the form $\exp(-r v_a(\lambda))$, where the function $v_a(\lambda)$ (which of course depends also on α) is such that $v_0(\lambda) = \lambda$. The Markov property of X readily gives the semigroup property

$$v_{a+b} = v_a \circ v_b$$

for every $a, b \geq 0$. To complete the proof of the proposition, it suffices to verify that $w_a = v_a$ for every $a \geq 0$, and to this end it will be enough to prove that

$$\frac{\mathrm{d}}{\mathrm{d}a} w_a(\lambda)_{|a=0} = \frac{\mathrm{d}}{\mathrm{d}a} v_a(\lambda)_{|a=0}. \tag{27}$$

The left-hand side is given by (26). Let us compute the right-hand side. We fix $\lambda > 0$ in what follows.

As we already mentioned, the process X is a Feller process with values in $[0, \infty)$. The exponential function $\varphi_\lambda(x) = e^{-\lambda x}$ belongs to the domain of the generator $\mathcal{L}$ of X, and

$$\mathcal{L}\varphi_\lambda(x) = \psi(\lambda)\, x\, \varphi_\lambda(x),$$

as a straightforward consequence of the formula for the Laplace transform of X_t. Consequently, we have

$$e^{-\lambda X_t} = e^{-\lambda r} + M_t + \psi(\lambda) \int_0^t X_s\, e^{-\lambda X_s}\, \mathrm{d}s,$$

where M is a martingale, which is clearly bounded on every compact interval. For every $t \geq 0$, set

$$V_t := \prod_{0 \leq s \leq t} (1 + \alpha \Delta X_s) e^{-\alpha \Delta X_s},$$

and note that V is a nonnegative nonincreasing process, which is bounded by one. By applying the integration by parts formula, we have

$$V_t e^{-\lambda X_t} = e^{-\lambda r} + \int_0^t V_{s-}\, \mathrm{d}M_s + \psi(\lambda) \int_0^t V_s X_s\, e^{-\lambda X_s} \mathrm{d}s + \int_0^t e^{-\lambda X_s}\, \mathrm{d}V_s. \tag{28}$$

The martingale term $\int_0^t V_{s-}\, dM_s$ has zero expectation. Let us evaluate the expected value of the last term

$$\int_0^t e^{-\lambda X_s}\, \mathrm{d}V_s = \sum_{0 \leq s \leq t} e^{-\lambda X_s} \Delta V_s = \sum_{0 \leq s \leq t} e^{-\lambda X_{s-}} V_{s-} \times e^{-\lambda \Delta X_s} \Big((1 + \alpha \Delta X_s) e^{-\alpha \Delta X_s} - 1\Big).$$

We note that the dual predictable projection of the random measure

$$\sum_{s\geq 0,\Delta X_s>0} \delta_{(s,\Delta X_s)}(\mathrm{d}u,\mathrm{d}x)$$

is the measure

$$X_u\,\mathrm{d}u\,\kappa(\mathrm{d}x)$$

where we recall that $\kappa(dx)$ is the "Lévy measure" associated with X (a simple way to get this is to use the Lamperti transformation to represent X as a time-change of the Lévy process with Lévy measure κ). It follows that

$$E\Big[\int_0^t e^{-\lambda X_s}\,\mathrm{d}V_s\Big] = E\Big[\int_0^t X_s V_s e^{-\lambda X_s}\,\mathrm{d}s\Big] \times \int \kappa(\mathrm{d}x)\,e^{-\lambda x}\Big((1+\alpha x)e^{-\alpha x}-1\Big).$$

By taking expectations in (28), we thus get

$$e^{-rv_t(\lambda)} - e^{-rv_0(\lambda)} = E\Big[\int_0^t X_s V_s e^{-\lambda X_s}\,\mathrm{d}s\Big] \times \Big(\psi(\lambda) + \int \kappa(\mathrm{d}x)\,e^{-\lambda x}\Big((1+\alpha x)e^{-\alpha x}-1\Big)\Big).$$

Note that

$$\frac{1}{t}E\Big[\int_0^t X_s V_s e^{-\lambda X_s}\,\mathrm{d}s\Big] \underset{t\downarrow 0}{\longrightarrow} r\,e^{-\lambda r},$$

and thus it immediately follows from the preceding display that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}a}v_a(\lambda)_{|a=0} &= -\psi(\lambda) - \int \kappa(\mathrm{d}x)\,e^{-\lambda x}\Big((1+\alpha x)e^{-\alpha x}-1\Big)\\ &= -\int \kappa(\mathrm{d}x)\,\Big((1+\alpha x)e^{-(\alpha+\lambda)x}-1+\lambda x\Big).\end{aligned}$$

From the expression of κ, straightforward calculations lead to the formula

$$\int \kappa(\mathrm{d}x)\,\Big((1+\alpha x)e^{-(\alpha+\lambda)x}-1+\lambda x\Big) = -\sqrt{\frac{2}{3}}\,\sqrt{\alpha+\lambda}\,(\alpha-2\lambda)$$

and our claim (27) follows, recalling (26). This completes the proof. □

With the notation introduced in Proposition 4.8, set for every $a\geq 0$,

$$Y_a := \sum_{i:s_i<a} \xi_i\,(\Delta X_{s_i})^2.$$

Corollary 4.9. *The law of the process* $(\mathscr{Z}_{-a},\mathscr{Y}_{-a})_{a\geq 0}$ *under* $\mathbb{P}_{r\delta_0}$ *coincides with the law of* $(X_a,Y_a)_{a\geq 0}$ *under* P_r.

Proof. An application of the special Markov property shows that the process $(\mathscr{Z}_{-a},\mathscr{Y}_{-a})_{a\geq 0}$ is (time-homogeneous) Markov under $\mathbb{P}_{r\delta_0}$, with transition kernel given by

$$\mathbb{E}_{r\delta_0}[g(\mathscr{Z}_{-a-b},\mathscr{Y}_{-a-b}) \mid (\mathscr{Z}_{-a},\mathscr{Y}_{-a})] = \Phi_b(\mathscr{Z}_{-a},\mathscr{Y}_{-a}),$$

where

$$\Phi_b(z,y) = \mathbb{E}_{z\delta_0}[g(\mathscr{Z}_{-b}, y+\mathscr{Y}_{-b})].$$

On the other hand, the Markov property of the continuous-state branching process X also shows that the process $(X_a, Y_a)_{a\geq 0}$ is Markov under P_r, and

$$E_r[g(X_{a+b}, Y_{a+b}) \mid (X_a, Y_a)] = \Psi_b(X_a, Y_a),$$

where

$$\Psi_b(z, y) = E_z[g(X_b, y + Y_b)].$$

By Proposition 4.8, we have $\Phi_b = \Psi_b$, and the desired result follows. $\square$

Remark. We chose to put a strict inequality $s_i < a$ in the definition of Y_a so that the process Y has left-continuous paths, which is also the case for $\mathscr{Y}_{-a}$. On the other hand, both $\mathscr{Z}_{-a}$ and X_a have right-continuous paths.

Proof of Theorem 1.3. Fix $\rho > 0$, and let U follow a Gamma distribution with parameter $\frac{3}{2}$ and mean ρ^2, so that U has the same distribution as Z_ρ, by Proposition 1.2 (i). Suppose that, conditionally given U, $\mathcal{N}$ is a Poisson point measure with intensity $U\,\mathbb{N}_0$ under the probability measure $\mathbb{P}$. We can use formulas (7) and (8) to define a super-Brownian motion $(\mathcal{X}_t)_{t\geq 0}$ started from $U\,\delta_0$ and the associated historical superprocess. We then define $(\mathscr{Z}_a, \mathscr{Y}_a)_{a\leq 0}$ as in the beginning of this subsection. We also write S for the extinction time of $\mathcal{X}$.

The arguments used in the proof of Theorem 1.4, based on our representation of the Brownian plane and formula (16), show that the process $(Z_{\rho-a}, |B^\bullet_\rho| - |B^\bullet_{\rho-a}|)_{0\leq a\leq\rho}$ has the same distribution as $(\mathscr{Z}_{-a}, \mathscr{Y}_{-a})_{0\leq a\leq\rho}$ under $\mathbb{P}(\cdot \mid S = \rho)$. For a precise justification, note that $B^\bullet_\rho \backslash B^\bullet_{\rho-a}$ is the image under Π of those $x \in \mathcal{T}_\infty$ such that $\Lambda_y > \rho - a$ for every $y \in [\![x, \infty[\![$ and there exists $z \in [\![x, \infty[\![$ such that $\Lambda_z \leq \rho$. If x satisfies these properties, either x belongs to one of the subtrees branching off the spine at a level belonging to $]L_{\rho-a}, L_\rho]$, or x belongs to one of the subtrees branching off the spine at a level greater than L_ρ but the label of one of the ancestors of x in this subtree is less than or equal to ρ (and, in both cases, the labels of the ancestors of x in the subtree containing x remain strictly greater than $\rho - a$). The volume of the set of points x corresponding to the second case is handled via the special Markov property for the domain (ρ, ∞), in a way similar to the end of the proof of Theorem 1.4. We obtain that the sum of the two contributions leads to the quantity $\mathscr{Y}_{-a}$ for a super-Brownian motion starting from $Z_\rho\delta_0$ and conditioned on extinction at time ρ.

Write $P_{(U)}$ for a probability measure under which the continuous-state branching process X starts from U (and the process Y is constructed by the formula preceding Corollary 4.9), and let T be the extinction time of X as previously. Recall the process $\widetilde{X}$ from Section 2.1, and also set for every $a \geq 0$,

$$\widetilde{Y}_a = \sum_{i:\tilde{s}_i \geq -a} \xi_i\,(\Delta\widetilde{X}_{\tilde{s}_i})^2,$$

where $\widetilde{s}_1, \widetilde{s}_2, \ldots$ is a measurable enumeration of the jumps of $\widetilde{X}$, and the random variables ξ_i are as in Proposition 4.8 and are supposed to be independent of $\widetilde{X}$.

From Corollary 4.9, we obtain that the law of $(\mathscr{Z}_{-a}, \mathscr{Y}_{-a})_{0\leq a\leq\rho}$ under $\mathbb{P}(\cdot \mid S = \rho)$ coincides with the law of $(X_a, Y_a)_{0\leq a\leq\rho}$ under $P_{(U)}(\cdot \mid T = \rho)$. However, using the final observation of the proof of Proposition 1.2 (ii), the latter law is also the law of $(\widetilde{X}_{-\rho+a}, \widetilde{Y}_\rho - \widetilde{Y}_{\rho-a})_{0\leq a\leq\rho}$.

Summarizing, we have obtained the identity in distribution

$$(Z_{\rho-a}, |B^\bullet_\rho| - |B^\bullet_{\rho-a}|)_{0\leq a\leq\rho} \overset{\text{(d)}}{=} (\widetilde{X}_{-\rho+a}, \widetilde{Y}_\rho - \widetilde{Y}_{\rho-a})_{0\leq a\leq\rho}.$$

This immediately gives

$$(Z_a, |B^\bullet_a|)_{0\leq a\leq\rho} \overset{\text{(d)}}{=} (\widetilde{X}_{-a}, \widetilde{Y}_a)_{0\leq a\leq\rho},$$

from which the statement of Theorem 1.3 follows. □

5 Asymptotics for the UIPQ

We will rely on the Chassaing-Durhuus construction of the UIPQ [7]. The fact that this construction is equivalent to the more usual construction involving local limits of finite quadrangulations can be found in [29]. The Chassaing-Durhuus construction is based on a random infinite labeled discrete ordered tree, which we denote here by $\mathbb{T}$. In a way very analogous to the tree $\mathcal{T}_\infty$ considered above, the tree $\mathbb{T}$ consists of a spine, which is a discrete half-line, and for every vertex of the spine, of two finite subtrees grafted at this vertex respectively to the left and to the right of the spine (if the grafted subtree consists only of the root, this means that we add nothing). The root of $\mathbb{T}$ is the first vertex of the spine. The set of all corners of $\mathbb{T}$ is equipped with a total order induced by the clockwise contour exploration of the tree. Each vertex v of $\mathbb{T}$ is assigned a positive integer label ℓ_v, in such a way that the label of the root is 1 and the labels of two neighboring vertices may differ by at most 1 in absolute value. We will not need the exact distribution of the tree $\mathbb{T}$: See e.g. [26, Section 2.3].

Let us now explain the construction of the UIPQ from the tree $\mathbb{T}$. First the vertex set of Q_∞ is the union of the vertex set $V(\mathbb{T})$ of $\mathbb{T}$ and of an extra vertex denoted by ∂. We then generate the edges of Q_∞ by the following device, which is analogous to the Schaeffer bijection between finite (rooted) quadrangulations and well-labeled trees [8]. All corners of $\mathbb{T}$ with label 1 are linked to ∂ by an edge of Q_∞. Any other corner c is linked by an edge of Q_∞ to the last corner before c (in the clockwise countour order) with strictly smaller label. The resulting collection of edges forms an infinite quadrangulation of the plane, which is the UIPQ Q_∞ (see Fig. 2, and [7] for more details). It easily follows from the construction that the graph distance (in Q_∞) between ∂ and another vertex of Q_∞ is just the label of this vertex in $\mathbb{T}$.

Let us introduce the left and right contour processes. Starting from the root of $\mathbb{T}$, we list all corners of the left side of $\mathbb{T}$ in clockwise contour order, and, for every $k \geq 0$, we denote the vertex corresponding to the k-th corner in this enumeration by v'_k (in such a way that v'_0 is the root of $\mathbb{T}$). We then write $C^{(L)}_k$ for the generation (distance from the root in $\mathbb{T}$) of v_k, and $V^{(L)}_k = \ell_{v'_k}$. Note that $|C^{(L)}_{k+1} - C^{(L)}_k| = 1$ for every $k \geq 0$. We define similarly $C^{(R)}_k$ and $V^{(R)}_k$ using the exploration in counterclockwise order of the right side of the tree, and the analog of the sequence $(v'_k)_{k\geq 0}$ is denoted by $(v''_k)_{k\geq 0}$. By linear interpolation, we may view all four processes $C^{(L)}, V^{(L)}, C^{(R)}, V^{(R)}$ as indexed by $\mathbb{R}_+$. A key ingredient of the following proof is the convergence [26, Theorem 5],

$$\Big(\frac{1}{k^2}C^{(L)}_{k^4 s}, \sqrt{\frac{3}{2}}\,\frac{1}{k}\,V^{(L)}_{k^4 s}, \frac{1}{k^2}C^{(R)}_{k^4 s}, \sqrt{\frac{3}{2}}\,\frac{1}{k}\,V^{(R)}_{k^4 s}\Big)_{s\geq 0} \xrightarrow[k\to\infty]{\text{(d)}} \Big(h(\Theta'_s), \Lambda_{\Theta'_s}, h(\Theta''_s), \Lambda_{\Theta''_s}\Big)_{s\geq 0}, \tag{29}$$

where we recall that Θ'_s and Θ''_s are the exploration processes of respectively the left and the right side of $\mathcal{T}_\infty$ (see subsection 3.2), and we use the notation $h(\Theta'_s) = d_\infty(0, \Theta'_s)$ for the "height" of Θ'_s in $\mathcal{T}_\infty$. The convergence (29) holds in the sense of weak convergence of the laws on the space $C(\mathbb{R}_+, \mathbb{R}^4)$. We also mention another convergence in distribution concerning labels on the spine. Write u_n for the n-th vertex on the spine of $\mathbb{T}$. Then,

$$\Big(\sqrt{\frac{3}{2}}\,\frac{1}{k}\,\ell_{u_{\lfloor k^2 s\rfloor}}\Big)_{s\geq 0} \xrightarrow[k\to\infty]{\text{(d)}} (R_s)_{s\geq 0} \tag{30}$$

and this convergence in distribution holds jointly with (29). The convergence (30) can be found

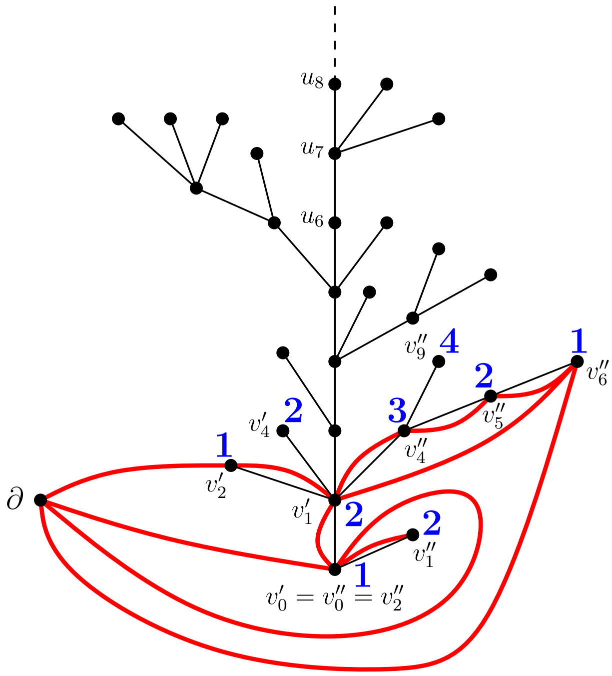

Figure 2: The Chassaing-Durhuus construction. The tree $\mathbb{T}$ is represented in thin lines. A few of the vertices v'_k, v''_k have been indicated together with their labels in bold figures. The edges of Q_∞ incident to 4 particular faces have been drawn in thick lines.

in [26, Proposition 1]. The fact that this convergence holds jointly with (29) is clear from the proof of Theorem 5 in [26].

According to [26, Lemma 3], we have for every $A > 0$,

$$\lim_{K\to\infty} \Big(\sup_{k\geq 1} P\Big(\inf_{t\geq K} \frac{1}{k} V^{(L)}_{k^4 t} < A \Big)\Big) = 0, \tag{31}$$

and by symmetry the analogous statement with $V^{(L)}$ replaced by $V^{(R)}$ also holds. Finally, we note that Lemma 3.3 implies

$$\lim_{s\uparrow\infty} \Lambda_{\Theta'_s} = \lim_{s\uparrow\infty} \Lambda_{\Theta''_s} = +\infty, \quad \text{a.s.} \tag{32}$$

For every integer $k \geq 1$, define the ball $\mathcal{B}_k(Q_\infty)$ as the union of all faces of Q_∞ that are incident to (at least) one vertex at distance smaller than or equal to $k-1$ from ∂. The hull $\mathcal{B}^\bullet_k(Q_\infty)$ is then obtained by adding to $\mathcal{B}_k(Q_\infty)$ the bounded components of the complement of $\mathcal{B}_k(Q_\infty)$ (see Fig. 3). Define the "volume" $|\mathcal{B}^\bullet_k(Q_\infty)|$ as the number of faces contained in $\mathcal{B}^\bullet_k(Q_\infty)$.

Theorem 5.1. *We have*

$$(k^{-4}\,|\mathcal{B}^\bullet_{\lfloor kr\rfloor}(Q_\infty)|)_{r>0} \xrightarrow[k\to\infty]{\text{(d)}} (\frac{1}{2}|B^\bullet_{r\sqrt{3/2}}(\mathcal{P}_\infty)|)_{r>0},$$

in the sense of weak convergence of finite dimensional marginals.

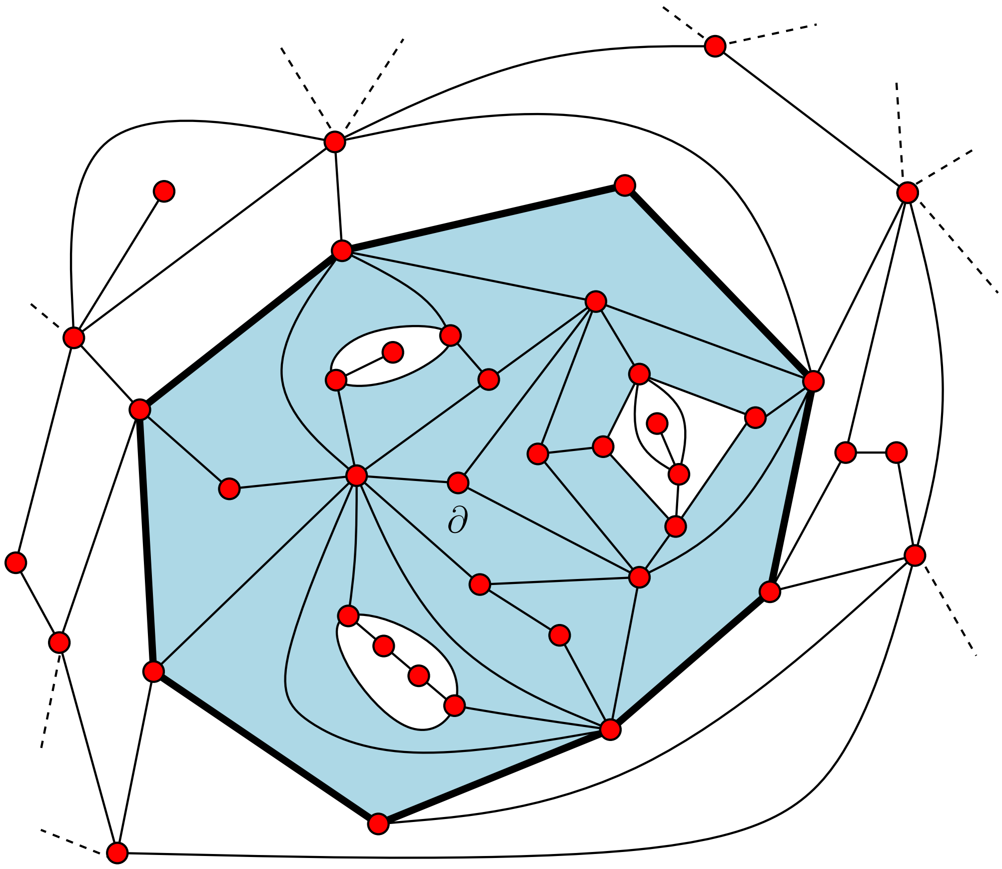

Figure 3: A representation of the UIPQ near the vertex ∂. The shaded part is the ball $\mathcal{B}_2(Q_\infty)$. The hull $\mathcal{B}^\bullet_2(Q_\infty)$, whose boundary is in thick lines on the figure, is obtained by filling in the holes of $\mathcal{B}_2(Q_\infty)$.

Remarks. (i) In the companion paper [10], we use the peeling process to give a different approach to the convergence of the sequence of processes $(k^{-4}\,|\mathcal{B}^\bullet_{\lfloor kr\rfloor}(Q_\infty)|)_{r>0}$. The limit then appears in the form given in Theorem 1.3.
(ii) By scaling, the processes $(\frac{1}{2}|B^\bullet_{r\sqrt{3/2}}(\mathcal{P}_\infty)|)_{r>0}$ and $(|B^\bullet_{(9/8)^{1/4}r}(\mathcal{P}_\infty)|)_{r>0}$ have the same distribution, and we recover the "usual" constant $(9/8)^{1/4}$ (see e.g. [8]). The reason for stating the theorem in the form above is the fact that the convergence then holds jointly with (29) or (30), as the proof will show.

Proof. Instead of dealing with $|\mathcal{B}^\bullet_{\lfloor kr\rfloor}(Q_\infty)|$ we will consider the quantity $\|\mathcal{B}^\bullet_{\lfloor kr\rfloor}(Q_\infty)\|$ defined as the number of vertices that are incident to a face of $\mathcal{B}^\bullet_{\lfloor kr\rfloor}(Q_\infty)$. It is an easy exercise to verify that the desired convergence will follow if we can prove that the statement holds when $|\mathcal{B}^\bullet_{\lfloor kr\rfloor}(Q_\infty)|$ is replaced by $\|\mathcal{B}^\bullet_{\lfloor kr\rfloor}(Q_\infty)\|$ (the underlying idea is the fact that a finite quadrangulation with n faces has $n+2$ vertices, and we also observe that, for every fixed $r>0$, the size of the boundary of $\mathcal{B}^\bullet_{\lfloor kr\rfloor}(Q_\infty)$ is negligible with respect to k^4 – this is clear if we know that the sequence $(k^{-4}\|\mathcal{B}^\bullet_{\lfloor kr\rfloor}(Q_\infty)\|)_{r>0}$ converges to a limit which is continuous in probability).

We will verify that, if $r>0$ is fixed, the sequence $k^{-4}\|\mathcal{B}^\bullet_{\lfloor kr\rfloor}(Q_\infty)\|$ converges in distribution to $\frac{1}{2}|B^\bullet_{r\sqrt{3/2}}(\mathcal{P}_\infty)|$. It will be clear that our method extends to a joint convergence in distribution if we consider a finite number of values of r, yielding the desired statement. To simplify the presentation, we take $r = 1$ in what follows. So our goal is to show that

$$k^{-4}\|\mathcal{B}^\bullet_k(Q_\infty)\| \xrightarrow[k\to\infty]{\rm (d)} \frac{1}{2}|B^\bullet_{\sqrt{3/2}}(\mathcal{P}_\infty)|. \tag{33}$$

If $u \in V(\mathbb{T})$, write $\mathrm{Geo}(u \to \infty)$ for the geodesic path from u to ∞ in $\mathbb{T}$, and set

$$m(u) := \min\{\ell_v : v \in \mathrm{Geo}(u \to \infty)\}.$$

Let $k \geq 1$. We note that:

(i) The condition $m(u) \geq k+3$ ensures that $x \notin \mathcal{B}^\bullet_k(Q_\infty)$. Indeed, from the way edges of Q_∞ are generated, it is easy to construct a path of Q_∞ from u to ∞ that visits only vertices at distance (at least) $m(u) - 1$ from ∂. If $m(u) - 1 \geq k + 2$, none of these vertices can be incident to a face of $\mathcal{B}_k(Q_\infty)$.

(ii) If $m(u) \leq k$ then $x \in \mathcal{B}^\bullet_k(Q_\infty)$. This is an immediate consequence of the discrete "cactus bound" (see [11, Proposition 4.3], in a slightly different setting), which implies that any path of Q_∞ going from u to ∞ visits a vertex at distance less than or equal to $m(u)$ from ∂.

Recall our definition of the "contour sequence" $(v'_k)_{k\geq 0}$ of the left side of the tree. We now extend the definition of v'_k to nonnegative real indices: If $k \geq 1$ and $k-1 < s < k$, we take $v'_s = v'_k$ if $C^{(L)}_k = C^{(L)}_{k-1} + 1$ and $v'_s = v'_{k-1}$ if $C^{(L)}_k = C^{(L)}_{k-1} - 1$. This definition is motivated by the fact that we have $\int_0^\infty ds\, \mathbf{1}\{v'_s = u\} = 2$ for every vertex u in the left side of $\mathbb{T}$ (not on the spine), and the same integral is equal to 1 if u is on the spine and different from the root. We extend similarly the definition of v''_k.

We next observe that, for every fixed $s > 0$,

$$\frac{1}{k}\, m(v'_{k^4 s}) \xrightarrow[k\to\infty]{\rm (d)} \sqrt{\frac{2}{3}}\, \min\{\Lambda_y : y \in [\![\Theta'_s, \infty[\![\}, \tag{34}$$

and this convergence holds jointly with (29). The convergence (34) is essentially a consequence of (29) and (30). Let us only sketch the argument. A first technical ingredient is to replace $m(v'_{k^4 s})$ by a truncated version obtained by replacing $\mathrm{Geo}(u \to \infty)$ in the definition of $m(u)$ by the geodesic from u to the vertex u_{Ak^2}, for some large integer constant A. One then proves, using (29) and (30), that the analog of (34) holds for this truncated version, with a limit equal to $\sqrt{2/3}\, \min\{\Lambda_y : y \in [\![\Theta'_s, A]\!]\}$ (a convenient way is to use a minor variant of the homeomorphism theorem of [28] to see that (29) implies also the convergence of the associated "snakes", which is what we need here). Finally, the fact that the convergence of truncated versions suffices to get (34) is easy using (31) and (32).

If we consider a finite number of values of s, the corresponding convergences (34) hold jointly (and jointly with (29)). Via the method of moments, it easily follows that, for every $A > 0$, and every $a > 0$,

$$\int_0^A \mathrm{d}s\, \mathbf{1}_{\{m(v'_{k^4 s}) \leq a\,k\}} \xrightarrow[k\to\infty]{\rm (d)} \int_0^A \mathrm{d}s\, \mathbf{1}_{\{\min\{\Lambda_y : y \in [\![\Theta'_s, \infty[\![\} \leq \sqrt{3/2}\, a\}}.$$

Thanks to (31) and (32), we can replace A by ∞ and obtain

$$\int_0^\infty \mathrm{d}s\, \mathbf{1}_{\{m(v'_{k^4 s}) \leq a\,k\}} \xrightarrow[k\to\infty]{\rm (d)} \int_0^\infty \mathrm{d}s\, \mathbf{1}_{\{\min\{\Lambda_y : y \in [\![\Theta'_s, \infty[\![\} \leq \sqrt{3/2}\, a\}}.$$

By combining this convergence with the analogous result for the right side of the tree, we get

$$\int_0^\infty \mathrm{d}s\,\mathbf{1}_{\{m(v'_{k^4 s})\leq a\,k\}} + \int_0^\infty \mathrm{d}s\,\mathbf{1}_{\{m(v''_{k^4 s})\leq a\,k\}}$$
$$\xrightarrow[k\to\infty]{\rm (d)} \int_0^\infty \mathrm{d}s\,\mathbf{1}_{\{\min\{\Lambda_y : y\in[\![\Theta'_s,\infty[\![\}\leq\sqrt{3/2}\,a\}} + \int_0^\infty \mathrm{d}s\,\mathbf{1}_{\{\min\{\Lambda_y : y\in[\![\Theta''_s,\infty[\![\}\leq\sqrt{3/2}\,a\}}. \tag{35}$$

By (16), the limit in the previous display is equal to $|B^\bullet_{\sqrt{3/2}\,a}(\mathcal{P}_\infty)|$. On the other hand, previous remarks show that, if $a\,k\geq 1$,

$$\int_0^\infty \mathrm{d}s\,\mathbf{1}_{\{m(v'_{k^4 s})\leq a\,k\}} + \int_0^\infty \mathrm{d}s\,\mathbf{1}_{\{m(v''_{k^4 s})\leq a\,k\}} = 2\,k^{-4}(\#\{u\in V(\mathbb{T}) : m(u)\leq a\,k\}-1).$$

Furthermore, it follows from properties (i) and (ii) stated above that

$$\#\{u\in V(\mathbb{T}) : m(u)\leq k\} \leq \|\mathcal{B}^\bullet_k(Q_\infty)\| \leq \#\{u\in V(\mathbb{T}) : m(u)\leq k+2\}.$$

Our claim (33) now follows from the convergence (35) and the preceding observations, together with the fact that the mapping $r\mapsto |B^\bullet_r(\mathcal{P}_\infty)|$ is continuous in probability. This completes the proof. $\square$

Let us conclude with a comment. It would seem more direct to derive Theorem 5.1 from the fact that the Brownian plane is the Gromov-Hausdorff scaling limit of the UIPQ [9, Theorem 2]. We refrained from doing so because the local Gromov-Hausdorff convergence does not give enough information to handle volumes of balls or hulls. It would have been necessary to establish a type of Gromov-Hausdorff-Prokhorov convergence in our setting, in the spirit of the work of Greven, Pfaffelhuber and Winter [15], who however consider the case of metric spaces equipped with a probability measure. This would require a number of additional technicalities, and for this reason we preferred to rely on the results of [26].

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