Linear and nonlinear elliptic partial differential equations
Final exam

In the sequel, we assume that $\Omega \subset \mathbb{R}^N$ is a bounded open set with $N \geq 2$.

Exercise 1. Let $A = (a_{ij})_{1\leq i,j\leq N}$ be a matrix whose coefficients satisfy $a_{ij} \in L^\infty(\Omega)$ for all $1 \leq i, j \leq N$ as well as the ellipticity condition : there exists $\lambda > 0$ such that

$$A(x)\xi \cdot \xi \geq \lambda|\xi|^2 \quad \text{for a.e. } x \in \Omega \text{ and all } \xi \in \mathbb{R}^N.$$

Let us also consider $h = (h_1, \ldots, h_N)$ with $h_i \in L^2(\Omega)$ for all $1 \leq i \leq N$.

1. Show that there exists a unique $u \in H^1_0(\Omega)$ satisfying

$$(1) \qquad -\operatorname{div}(A\nabla u) = -\operatorname{div} h \quad \text{in } \mathcal{D}'(\Omega).$$

2. Show that (1) is equivalent to

$$(2) \qquad \int_\Omega A\nabla u \cdot \nabla v\, dx = \int_\Omega h \cdot \nabla v\, dx \quad \text{for all } v \in H^1_0(\Omega).$$

3. We next intend to show that, if $h_i \in L^q(\Omega)$ for all $1 \leq i \leq N$ with $q > N$, then $u \in L^\infty(\Omega)$, and

$$\|u\|_{L^\infty(\Omega)} \leq C\|h\|,$$

where $C > 0$ only depends on N, q and Ω.

(a) Let $k \geq 0$. We already know that $(u - k)_+ = \max(u - k, 0) \in H^1(\Omega)$ and $\nabla(u - k)_+ = \mathbf{1}_{\{u\geq k\}}\nabla u$. Show that, actually, $(u - k)_+ \in H^1_0(\Omega)$.

(b) Show that

$$\|\nabla(u - k)_+\|_{L^2(\Omega)} \leq \frac{|\Omega_k|^{\frac{1}{2}-\frac{1}{q}}}{\lambda}\|h\|_{L^q(\Omega)},$$

where $\Omega_k = \{x \in \Omega : u(x) \geq k\}$.

(c) We recall a version of *Sobolev's imbedding* which states that if $N \geq 2$, then there exists a constant $C_* > 0$ such that

$$\|w\|_{L^r(\Omega)} \leq C_*\|\nabla w\|_{L^2(\Omega)} \quad \text{for all } w \in H^1_0(\Omega),$$

where $r = 2N/(N - 2)$ if $N > 2$, while $r \in [2, +\infty[$ is arbitrary if $N = 2$. Show that

$$\|(u - k)_+\|_{L^1(\Omega)} \leq \frac{C_*|\Omega_k|^{\frac{1}{2}-\frac{1}{r}+\frac{1}{p}}}{\lambda}\|h\|_{L^q(\Omega)}.$$

(d) Show that

$$\|(u - k)_+\|_{L^1(\Omega)} = \int_k^\infty |\Omega_s|\, ds.$$

and deduce that

$$(3) \qquad H(k) \leq \frac{C_*\|h\|_{L^q(\Omega)}}{\lambda}[-H'(k)]^{\frac{1}{2}-\frac{1}{r}+\frac{1}{p}},$$

where $H(k) := \int_k^\infty |\Omega_s|\, ds$.

(e) Let $k_0 = \sup\{k \geq 0 : H(k) > 0\}$. Show that

$$k_0 \leq \frac{\beta^\gamma H(0)^{1-\gamma}}{1-\gamma},$$

where $\beta = C_* \|h\|_{L^q(\Omega)}/\lambda$ and $1/\gamma = 1/2 + 1/p - 1/r$.

(f) Deduce that $k_0 \leq C_* |\Omega|^{\frac{1}{2}-\frac{1}{r}+\frac{1}{p}} \|h\|_{L^q(\Omega)}$ and show that

$$u \leq C(N,p,\Omega)\|h\|_{L^q(\Omega)} \quad \text{a.e. in } \Omega.$$

(g) Conclude that

$$\|u\|_{L^\infty(\Omega)} \leq C(N,p,\Omega)\|h\|_{L^q(\Omega)}.$$

Exercise 2. Let $A = (a_{ij})_{1\leq i,j\leq N}$ be a matrix whose coefficients satisfy $a_{ij} \in L^\infty(\Omega)$ for all $1 \leq i, j \leq N$ as well as the ellipticity condition : there exists $\lambda > 0$ such that

$$A(x)\xi \cdot \xi \geq \lambda|\xi|^2 \quad \text{for a.e. } x \in \Omega \text{ and all } \xi \in \mathbb{R}^N.$$

The goal of this exercise is to show the existence of weak solutions to the PDE

$$\begin{cases} -\mathrm{div}(A\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

when the right hand side f only belongs to $L^1(\Omega)$.

1. Assume first that there exists $u \in H_0^1(\Omega)$ such that

$$-\,\mathrm{div}(A\nabla u) = f \text{ in } \mathcal{D}'(\Omega). \tag{4}$$

(a) Let $1 \leq p < N/(N-1)$ and $q = p/(p-1) > N$ its conjugate exponent. For $h = (h_1,\dots,h_N)$ with $h_i \in L^q(\Omega)$ for all $1 \leq i \leq N$, let $v_h \in H_0^1(\Omega)$ be the unique solution of

$$-\mathrm{div}(A^T\nabla v_h) = -\mathrm{div} h \text{ in } \mathcal{D}'(\Omega),$$

where A^T is the transpose of the matrix A. Justify quickly the existence and uniqueness of v_h. Show that

$$\int_\Omega A\nabla u \cdot \nabla v_h \, dx = \int_\Omega f v_h \, dx = \int_\Omega h \cdot \nabla u \, dx.$$

(b) Using the conclusion of exercise 1, deduce that

$$\int_\Omega h \cdot \nabla u \, dx \leq C\|f\|_{L^1(\Omega)}\|h\|_{L^q(\Omega)},$$

where $C > 0$ only depends on Ω, N and p.

(c) Deduce that

$$\|\nabla u\|_{L^p(\Omega)} \leq C\|f\|_{L^1(\Omega)}.$$

2. We now discuss the existence of solutions $u \in W_0^{1,p}(\Omega)$, with $1 \leq p < N/(N-1)$, to (4).

(a) For $f \in L^1(\Omega)$, let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{C}_c^\infty(\Omega)$ such that $f_n \to f$ in $L^1(\Omega)$. For each $n \in \mathbb{N}$, let us consider $u_n \in H_0^1(\Omega)$ such that

$$-\mathrm{div}(A\nabla u_n) = f_n \text{ in } \mathcal{D}'(\Omega).$$

Show that for all $n \in \mathbb{N}$, then $u_n \in W_0^{1,p}(\Omega)$ for any $p < N/(N-1)$, and that if n and $m \in \mathbb{N}$, then

$$\|u_n - u_m\|_{W_0^{1,p}(\Omega)} \leq C\|f_n - f_m\|_{L^1(\Omega)}.$$

(b) Deduce that $u_n \to u^{(p)}$ in $W^{1,p}(\Omega)$, where $u^{(p)} \in W_0^{1,p}(\Omega)$ satisfies (4).

(c) Show that the limit $u^{(p)}$ is independent of p.

Exercise 3. Let $f \in L^2(\Omega)$ and $a : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying

$$\alpha \leq a(x,s) \leq \beta \quad \text{for a.e. } x \in \Omega \text{ and all } s \in \mathbb{R},$$

where $\alpha > 0$ and $\beta > 0$. Using Schauder's fixed point Theorem, show the existence of solutions $u \in H_0^1(\Omega)$ to

$$-\mathrm{div}(a(x,u)\nabla u) = f \text{ in } \mathcal{D}'(\Omega).$$