A classification of $\mathbb{C}$-Fuchsian subgroups of Picard modular groups

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Abstract

Given an imaginary quadratic extension K of $\mathbb{Q}$, we give a classification of the maximal nonelementary subgroups of the Picard modular group $\mathrm{PSU}_{1,2}(\mathscr{O}_K)$ preserving a complex geodesic in the complex hyperbolic plane $\mathbb{H}^2_\mathbb{C}$. Complementing work of Holzapfel, Chinburg-Stover and Möller-Toledo, we show that these maximal $\mathbb{C}$-Fuchsian subgroups are arithmetic, arising from a quaternion algebra $\Big(\frac{D\,,\,D_K}{\mathbb{Q}}\Big)$ for some explicit $D \in \mathbb{N}-\{0\}$ and D_K the discriminant of K. We thus prove the existence of infinitely many orbits of K-arithmetic chains in the hypersphere of $\mathbb{P}_2(\mathbb{C})$.
[1]

1 Introduction

Let h be a Hermitian form with signature $(1, 2)$ on $\mathbb{C}^3$. The projective special unitary Lie group PSU_h of h contains exactly two conjugacy classes of connected Lie subgroups locally isomorphic to $\mathrm{SL}_2(\mathbb{R})$. The subgroups in one class, isomorphic to $\mathrm{SU}(1, 1) \simeq \mathrm{SL}_2(\mathbb{R})$, preserve a complex projective line for the projective action of PSU_h on the projective plane $\mathbb{P}_2(\mathbb{C})$, and those of the other class, isomorphic to $\mathrm{SO}_0(2, 1) \simeq \mathrm{PSL}_2(\mathbb{R})$, preserve a totally real subspace. The groups $\mathrm{PSL}_2(\mathbb{R})$ and PSU_h act as the groups of holomorphic isometries, respectively, on the upper halfplane model $\mathbb{H}^2_\mathbb{R}$ of the real hyperbolic space and on the projective model $\mathbb{H}^2_\mathbb{C}$ of the complex hyperbolic plane defined using the form h. If Γ is a discrete subgroup of PSU_h, the intersections of Γ with the connected Lie subgroups locally isomorphic to $\mathrm{SL}_2(\mathbb{R})$ are its *Fuchsian subgroups*, and the Fuchsian subgroups preserving a complex projective line are called $\mathbb{C}$-*Fuchsian subgroups*. We refer to Section 2 for more precise definitions and comments on the terminology.

Let K be an imaginary quadratic number field, with discriminant D_K and ring of integers $\mathscr{O}_K$. We consider the Hermitian form h defined by

$$(z_0, z_1, z_2) \mapsto -z_0\,\overline{z_2} - z_2\,\overline{z_0} + z_1\overline{z_1}\ .$$

The *Picard modular group* $\Gamma_K = \mathrm{PSU}_h(\mathscr{O}_K)$ is a nonuniform arithmetic lattice of PSU_h, see for instance [Hol2, Chap. 5] and subsequent works of Falbel, Parker, Francsics, Lax, Xie-Wang-Jiang, and many others, for information on these groups. In this paper, we classify

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the maximal $\mathbb{C}$-Fuchsian subgroups of Γ_K, and we explicit their arithmetic structures. The results stated in this introduction do not depend on the choice of the Hermitian form h of signature $(2,1)$ defined over K, since the algebraic groups over $\mathbb{Q}$ whose groups of $\mathbb{Q}$ points are $\mathrm{PSU}_h(\mathscr{O}_K)$ depend up to commensurability only on K and not on h, see for instance [Sto, § 3.1].

When $G = \mathrm{PSL}_2(\mathbb{C})$, there is exactly one conjugacy class of connected Lie subgroups of G isomorphic to $\mathrm{PSL}_2(\mathbb{R})$. When Γ is the *Bianchi group* $\mathrm{PSL}_2(\mathscr{O}_K)$, the analogous classification is due to Maclachlan and Reid (see [Mac, MR1] and [MR2, Chap. 9]). They proved that the maximal nonelementary Fuchsian subgroups of $\mathrm{PSL}_2(\mathscr{O}_K)$ are commensurable up to conjugacy in $\mathrm{PSL}_2(\mathbb{C})$ with the stabilisers of the circles $|z|^2 = D$ for $D \in \mathbb{N} - \{0\}$, when $\mathrm{PSL}_2(\mathbb{C})$ acts projectively (by homographies) on the projective line $\mathbb{P}_1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$, and that all these subgroups arise from explicit quaternion algebras over $\mathbb{Q}$. For information on Bianchi groups, see for instance [Fin] and the references of [MR1].

More generally, given a connected semisimple real Lie group G with finite center and without compact factor, there is a nonempty finite set of infinite conjugacy classes of connected Lie subgroups of G locally isomorphic to $\mathrm{SL}_2(\mathbb{R})$, unless G itself is locally isomorphic to $\mathrm{SL}_2(\mathbb{R})$. The structure of the set of these subgroups plays an important role for the classification of the linear representations of G, and for the classification of the groups G themselves, see for instance [Kna, Ser] among others. Given a discrete subgroup Γ of G, it is again interesting to study the *Fuchsian subgroups* of Γ, that is, the intersections of Γ with these Lie subgroups, to classify the maximal ones and to see, when Γ is arithmetic, if its maximal Fuchsian subgroups are also arithmetic (see Proposition 3.1 for a positive answer) with an explicit arithmetic structure. From now on, $G = \mathrm{PSU}_h$.

We first prove (see Proposition 3.2 and just after) that a nonelementary $\mathbb{C}$-Fuchsian subgroup Γ' of Γ_K preserves a unique projective point $[z_0 : z_1 : z_2]$ with z_0, z_1, z_2 relatively prime in $\mathscr{O}_K$. We define the *discriminant* of Γ' as $\Delta_{\Gamma'} = h(z_0, z_1, z_2)$. For any positive integer D, let

$$\Gamma_{K,D} = \mathrm{Stab}_{\Gamma_K}[-D : 0 : 1]\,.$$

In Section 3, we prove the following classification result (see [MR1, Thm. 1] and [MR2, Thm. 9.6.2] in the Bianchi group case).

Theorem 1.1 *Let $D \in \mathbb{N} - \{0\}$. The set of Γ_K-conjugacy classes of maximal nonelementary $\mathbb{C}$-Fuchsian subgroups of Γ_K with discriminant D is finite and nonzero. Every maximal nonelementary $\mathbb{C}$-Fuchsian subgroup of Γ_K with discriminant D is commensurable up to conjugacy in PSU_h with $\Gamma_{K,2D}$.*

In the course of the proof of this result (see Lemma 3.4 and Corollary 4.2), we prove a criterion for when two groups $\Gamma_{K,D}$ for $D \in \mathbb{N} - \{0\}$ are commensurable up to conjugacy in PSU_h. A further application of this condition shows that every maximal nonelementary $\mathbb{C}$-Fuchsian subgroup of Γ_K is commensurable up to conjugacy in PSU_h with $\Gamma_{K,D}$ for a squarefree natural number D.

Recall (see for instance [Gol]) that a *chain* [2] is the intersection of the *Poincaré hypersphere*

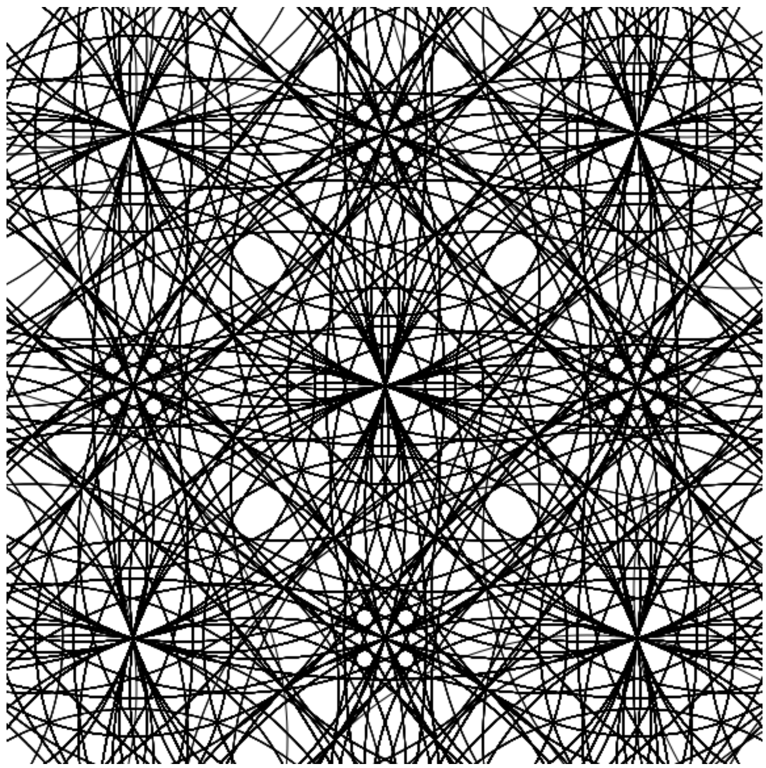
$$\mathscr{HS} = \{[z] \in \mathbb{P}_2(\mathbb{C}) \;:\; h(z) = 0\}$$

with a complex projective line (if nonempty and not a singleton). It is *K-arithmetic* if its stabiliser in Γ_K has a dense orbit in it (see Section 3 for an explanation of the terminology).

[2] a notion attributed to von Staudt in [Car, footnote 3]

Corollary 1.2 *There are infinitely many Γ_K-orbits of K-arithmetic chains in the hypersphere $\mathscr{HS}$.*

The figure below shows part of the image under vertical projection in the Heisenberg group of the orbit under Γ_K of a K-arithmetic chain whose stabiliser has discriminant 10, when $K = \mathbb{Q}[i]$.



We say that a subgroup of PSU_h *arises from a quaternion algebra* A defined over $\mathbb{Q}$ if it is commensurable in PSU_h with $\sigma(A(\mathbb{Z})^1)$ for some $\mathbb{Q}$-algebra morphism $\sigma : A \to \mathscr{M}_3(\mathbb{C})$, where $A(\mathbb{Z})^1$ is the group of norm 1 eleemnts in $A(\mathbb{Z})$. In Section 4, we prove the following result (see [MR2, Thm. 9.6.3] in the Bianchi group case).

Theorem 1.3 *Every nonelementary $\mathbb{C}$-Fuchsian subgroup of Γ_K of discriminant D is conjugate in PSU_h to a subgroup of PSU_h arising from the quaternion algebra $\big(\frac{D,\,D_K}{\mathbb{Q}}\big)$.*

The classification of the quaternion algebras over $\mathbb{Q}$ then allows to classify, up to commensurability and conjugacy in PSU_h, the maximal nonelementary $\mathbb{C}$-Fuchsian subgroups of Γ_K: two such groups, with discriminant D and D' are commensurable up to conjugacy in PSU_h if and only if the quaternion algebras $\big(\frac{D,\,D_K}{\mathbb{Q}}\big)$ and $\big(\frac{D',\,D_K}{\mathbb{Q}}\big)$ are isomorphic (see Corollary 4.2). This holds for instance if and only if the quadratic forms $D_K x^2 + D y^2 - D D_K z^2$ and $D_K x^2 + D' y^2 - D' D_K z^2$ are equivalent over $\mathbb{Q}$ (see for instance [MR2, Coro. 2.3.5]).

The terminology of the following paragraph won't be needed in this paper, and we refer to the quoted references. The existence of a bijection between wide commensurability classes of $\mathbb{C}$-Fuchsian subgroups of Γ_K and isomorphism classes of quaternion algebras over $\mathbb{Q}$ unramified at infinity and ramified at all finite places which do not split in $K/\mathbb{Q}$ is a particular case of a 2011 result of Chinburg-Stover (see Theorem 2.2 in version 3 of

[CS1] and [CS2, Theo. 4.1]), which proves such a result for all arithmetic lattices of simple type (which include the Picard modular groups) in $\mathrm{SU}_{2,1}$. In particular, the existence of this bijection (and our Corollary 4.4) should be attributed to Chinburg-Stover (although they say it was known by experts). Möller-Toledo in [MT] also give a description of the quotients by the maximal $\mathbb{C}$-Fuchsian subgroups of the real hyperbolic planes they preserve, and more generally of all Shimura curves in Shimura surfaces of the first type (which include the complex surfaces $\Gamma_K\backslash\mathbb{H}^2_{\mathbb{C}}$). We believe that our precise correspondence brings interesting effective and geometric information to the picture.

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2 The complex hyperbolic plane

Let h be the nondegenerate Hermitian form

$$h(z) = -z_0\,\overline{z_2} - z_2\,\overline{z_0} + |z_1|^2 = -2\,\mathrm{Re}(z_0\,\overline{z_2}) + |z_1|^2$$

of signature $(1,2)$ on $\mathbb{C}^3$ with coordinates (z_0, z_1, z_2), and let $\langle\cdot,\cdot\rangle$ be the associated Hermitian product. The point $z = (z_0, z_1, z_2) \in \mathbb{C}^3$ and the corresponding element $[z] = [z_0 : z_1 : z_2] \in \mathbb{P}_2(\mathbb{C})$ (using homogeneous coordinates) is *negative, null or positive* according to whether $h(z) < 0$, $h(z) = 0$ or $h(z) > 0$. The *negative/null/positive cone* of h is the subset of negative/null/positive elements of $\mathbb{P}_2(\mathbb{C})$.

The negative cone of h endowed with the distance d defined by

$$\cosh^2 d([z],[w]) = \frac{|\langle z, w\rangle|^2}{h(z)\,h(w)}$$

is the *complex hyperbolic plane* $\mathbb{H}^2_{\mathbb{C}}$. The distance d is the distance of a Riemannian metric with pinched negative sectional curvature $-4 \le K \le -1$. The linear action of the special unitary group of h

$$\mathrm{SU}_h = \{g \in \mathrm{SL}_3(\mathbb{C})\ :\ h\circ g = h\}$$

on $\mathbb{C}^3$ (where $h\circ g : z \mapsto h(gz)$) induces a projective action on $\mathbb{P}_2(\mathbb{C})$ with kernel $\mathbb{U}_3\,\mathrm{Id}$, where $\mathbb{U}_3$ is the group of third roots of unity. This projective action preserves the negative, null and positive cones of h in $\mathbb{P}_2(\mathbb{C})$, and is transitive on each of them. The restriction to $\mathbb{H}^2_{\mathbb{C}}$ of the quotient group $\mathrm{PSU}_h = \mathrm{SU}_h/(\mathbb{U}_3\,\mathrm{Id})$ of SU_h is the holomorphic isometry group of $\mathbb{H}^2_{\mathbb{C}}$. Note that the inclusion $\mathrm{SU}_h \to \mathrm{U}_h$ induces a Lie group isomorphism $\mathrm{PSU}_h \to \mathrm{PU}_h$.

The null cone of h is the *Poincaré hypersphere* $\mathscr{HS}$, which is naturally identified with the boundary at infinity of $\mathbb{H}^2_{\mathbb{C}}$. The *Heisenberg group*

$$\mathrm{Heis}_3 = \{[w_0 : w : 1] \in \mathbb{C}\times\mathbb{C} : 2\,\mathrm{Re}\,w_0 = |w|^2\}$$

acts isometrically on $\mathbb{H}^2_{\mathbb{C}}$ and simply transitively on $\mathscr{H}\mathscr{S}-\{[1:0:0]\}$ by the action induced by the matrix representation

$$[w_0:w:1]\mapsto\begin{pmatrix}1&\overline{w}&w_0\\0&1&w\\0&0&1\end{pmatrix}$$

of Heis_3 in SU_h. The projective transformations induced by these matrices are called *Heisenberg translations.*

If a complex projective line meets $\mathbb{H}^2_{\mathbb{C}}$, its intersection with $\mathbb{H}^2_{\mathbb{C}}$ is a totally geodesic submanifold of $\mathbb{H}^2_{\mathbb{C}}$, called a *complex geodesic*. The intersection of a complex projective line in $\mathbb{P}_2(\mathbb{C})$ with the Poincaré hypersphere is called a *chain*, if nonempty and not reduced to a point. Each complex projective line L in $\mathbb{P}_2(\mathbb{C})$ meeting $\mathbb{H}^2_{\mathbb{C}}$ (or its associated complex geodesic $L\cap\mathbb{H}^2_{\mathbb{C}}$, or its associated chain $L\cap\mathscr{H}\mathscr{S}$) is *polar* to a unique positive point $P_L\in\mathbb{P}_2(\mathbb{C})$, that is, $\langle z,P_L\rangle=0$ for all $z\in L$ (or equivalently $z\in L\cap\mathbb{H}^2_{\mathbb{C}}$ or $z\in L\cap\mathscr{H}\mathscr{S}$). This element P_L is the *polar point* of the projective line L, of the complex geodesic $L\cap\mathbb{H}^2_{\mathbb{C}}$ and of the chain $L\cap\mathscr{H}\mathscr{S}$. Conversely, for each positive point P, there is a unique complex projective line $P^\perp$ polar to P, the *polar line* of P. The intersection of $P^\perp$ with $\mathbb{H}^2_{\mathbb{C}}$ is a complex geodesic.

An easy computation (using for instance Equation (42) in [PP1]) shows that

$$\mathrm{Stab}_{\mathrm{SU}_h}[0:1:0]=\left\{\begin{pmatrix}\zeta a&0&i\zeta b\\0&\zeta^{-2}&0\\-i\zeta c&0&\zeta d\end{pmatrix}\;:\;\begin{array}{c}a,b,c,d\in\mathbb{R},\ \zeta\in\mathbb{C}\\ad-bc=1,\ |\zeta|=1\end{array}\right\}.\tag{1}$$

In particular, $\mathrm{Stab}_{\mathrm{SU}_h}[0:1:0]$ is isomorphic to $(\mathbb{S}^1\times\mathrm{SL}_2(\mathbb{R}))/\{\pm(1,\mathrm{id})\}$. It injects in PSU_h by the canonical projection $\mathrm{SU}_h\to\mathrm{PSU}_h$. Hence $\mathrm{Stab}_{\mathrm{PSU}_h}[0:1:0]$ is also isomorphic to $(\mathbb{S}^1\times\mathrm{SL}_2(\mathbb{R}))/\{\pm(1,\mathrm{id})\}$. More generally, by conjugation, if $P=[z_0:z_1:z_2]$ is a positive point in $\mathbb{P}_2(\mathbb{C})$, then the stabilizer of P in PSU_h is the almost direct product A_PB_P, where A_P is the unique Lie group embedding of $\mathrm{SL}_2(\mathbb{R})$ in PSU_h preserving the complex geodesic polar to P, and B_P is the group of complex reflections with fixed point set the projective line polar to P. The group B_P is isomorphic to $\mathbb{S}^1$ and centralizes A_P. We have $B_P\cap A_P=\{\pm1\}$, and A_P acts on the normal bundle of the complex geodesic polar to P either by parallel translation or by its opposite.

The polar chain of P is

$$C_P=\{[w_0:w_1:w_2]\in\mathbb{P}_2(\mathbb{C})\;:\;h(w_0,w_1,w_2)=\langle(w_0,w_1,w_2),(z_0,z_1,z_2)\rangle=0\}\,,$$

that is $C_P\cap\mathrm{Heis}_3$ is the set of $[w_0:w:1]\in\mathrm{Heis}_3$ satisfying the equation

$$\big(\frac{|w|^2}{2}+i\,\mathrm{Im}\,w_0\big)\,\overline{z_2}-w\,\overline{z_1}+\overline{z_0}=0\,.$$

When $z_2\neq0$, in the coordinates $(w,2\,\mathrm{Im}\,w_0)\in\mathbb{C}\times\mathbb{R}$ of $[w_0:w:1]\in\mathrm{Heis}_3$, this is the equation of an ellipse, whose image under the *vertical projection* $[w_0:w:1]\mapsto w$ is the circle with center $\frac{\overline{z_1}}{\overline{z_2}}$ and radius $\frac{\sqrt{h(z_0,z_1,z_2)}}{|z_2|}$ in $\mathbb{C}$ given by the equation

$$|w|^2-2\,\mathrm{Re}\big(w\,\frac{\overline{z_1}}{\overline{z_2}}\big)+2\,\mathrm{Re}\big(\,\frac{\overline{z_0}}{\overline{z_2}}\,\big)=0\,.$$

If $z_2 = 0$, then $C_P \cap \mathrm{Heis}_3$ is the vertical affine line over $\frac{\overline{z_1}}{\overline{z_2}}$.

We refer to Goldman [Gol, p. 67] and Parker [Par] for the basic properties of $\mathbb{H}^2_{\mathbb{C}}$. These references use different Hermitian forms of signature $(1, 2)$ to define the complex hyperbolic plane, and the curvature is often normalised differently from our definitions. Our choices are consistent with [PP1] and [PP2].

3 Classification of $\mathbb{C}$-Fuchsian subgroups of Γ_K

Before starting to study Fuchsian subgroups of discrete subgroups of PSU_h, let us mention that it is a very general fact that the maximal *nonelementary* (that is, not virtually cyclic) Fuchsian subgroups of arithmetic subgroups of PSU_h are automatically (arithmetic) lattices of the copy of $\mathrm{PSL}_2(\mathbb{R})$ containing them.

Proposition 3.1 *Let G be a semisimple connected real Lie group with finite center and without compact factor, and let Γ be a maximal nonelementary Fuchsian subgroup of an arithmetic subgroup $\widetilde{\Gamma}$ of G. Then Γ is an arithmetic lattice in the copy of the connected Lie group locally isomorphic to* $\mathrm{SL}_2(\mathbb{R})$ *containing it.*

Proof. We refer for instance to [Zim, §3.1] for an elementary introduction to algebraic groups and their Zariski topology. Let $\underline{G}$ be a semisimple connected algebraic group defined over $\mathbb{Q}$, let $\underline{H}$ be a connected algebraic subgroup of $\underline{G}$ defined over $\mathbb{R}$ locally isomorphic to SL_2, and assume that $\Gamma = \underline{H}(\mathbb{R}) \cap G(\mathbb{Z})$ is nonelementary in $\underline{H}(\mathbb{R})$. As a nonelementary subgroup of a group locally isomorphic to SL_2 is Zariski-dense in it, and as the Zariski-closure of a subgroup of $G(\mathbb{Z})$ is defined over $\mathbb{Q}$ (see for instance [Zim, Prop. 3.1.8], we hence have that $\underline{H}$ is defined over $\mathbb{Q}$. Therefore by the Borel-Harish-Chandra theorem [BHC, Thm. 7.8], $\Gamma = \underline{H}(\mathbb{Z})$ is an arithmetic lattice in $\underline{H}(\mathbb{R})$. Since the copies of connected Lie subgroups of G locally isomorphic to $\mathrm{SL}_2(\mathbb{R})$ are algebraic (see for instance [Zim, Prop. 3.1.6]), the result follows. □

One of the main points of the rest of the paper will be to determine explicitly the arithmetic structure of Γ, that is the $\mathbb{Q}$-structure thus constructed on the group locally isomorphic to $\mathrm{SL}_2(\mathbb{R})$ containing it, relating it to the arithmetic structure of $\widetilde{\Gamma}$, that is the given $\mathbb{Q}$-structure on G.

Let K be an imaginary quadratic number field, with D_K its discriminant, $\mathscr{O}_K$ its ring of integers, $\mathtt{tr} : z \mapsto z + \overline{z}$ its trace and $N : z \mapsto |z|^2$ its norm. Recall (see for instance [Sam]) that there exists a squarefree positive integer d such that $K = \mathbb{Q}(i\sqrt{d})$, that $D_K = -d$ and $\mathscr{O}_K = \mathbb{Z}[\frac{1+i\sqrt{d}}{2}]$ if $d \equiv -1 \mod 4$, and that $D_K = -4d$ and $\mathscr{O}_K = \mathbb{Z}[i\sqrt{d}]$ otherwise. Note that $\mathscr{O}_K$ is stable by conjugation, and that $\mathtt{tr}$ and N take integral values on $\mathscr{O}_K$. A *unit* x in $\mathscr{O}_K$ is an invertible element in $\mathscr{O}_K$. Since $N : K^\times \to \mathbb{R}^\times$ is a group morphism, we have $N(x) = 1$ for every unit x in $\mathscr{O}_K$.

The *Picard modular group* of K, that we denote by $\Gamma_K = \mathrm{PSU}_h(\mathscr{O}_K)$, consists of the images in PSU_h of matrices of SU_h with entries in $\mathscr{O}_K$. It is a nonuniform arithmetic lattice by the result of Borel and Harish-Chandra cited above. Note that every nonuniform arithmetic lattice in PSU_h is commensurable to a Picard modular group (see for instance [Sto, § 3.1]).

A discrete subgroup Γ of PSU_h is an *extended $\mathbb{C}$-Fuchsian subgroup* if it satisfies one of the following equivalent conditions

(1) Γ preserves a complex projective line of $\mathbb{P}_2(\mathbb{C})$ meeting $\mathbb{H}^2_{\mathbb{C}}$,
(2) Γ fixes a positive point in $\mathbb{P}_2(\mathbb{C})$,
(3) Γ preserves a chain.

Many references, see for example [FaP1], do not use the word "extended". But as defined in the introduction, in this paper, a $\mathbb{C}$-*Fuchsian subgroup* is a discrete subgroup of PSU_h preserving a complex geodesic in $\mathbb{H}^2_{\mathbb{C}}$ and inducing the parallel transport or its opposite on its unit normal bundle. It is the image of a *Fuchsian group* (that is, a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$) by a Lie group embedding of $\mathrm{SL}_2(\mathbb{R})$ in PSU_h. The extended $\mathbb{C}$-Fuchsian subgroups are then finite extensions of $\mathbb{C}$-Fuchsian subgroups by finite groups of complex reflections fixing the projective line or positive point or chain in the definition above. In particular, up to commensurability, the notions of extended $\mathbb{C}$-Fuchsian subgroups and of $\mathbb{C}$-Fuchsian subgroups coincide. The $\mathbb{C}$-Fuchsian lattices have been studied under a different viewpoint than our geometric one, as fundamental groups of arithmetic curves on ball quotient surfaces or Shimura curves in Shimura surfaces, by many authors, see for instance [Kud, Hol1, Hol2, MT] and their references.

An element of Γ_K is *K-irreducible* if it does not preserve a point or a line defined over K in $\mathbb{P}_2(\mathbb{C})$. An element of $\mathbb{P}_2(\mathbb{C})$ is *rational* if it lies in $\mathbb{P}_2(K)$. Note that the polar line of a positive rational point of $\mathbb{P}_2(\mathbb{C})$ is defined over K. The group $\mathrm{PSU}_h(K)$, image of $\mathrm{SU}_h(K) = \mathrm{SU}_h \cap \mathrm{SL}_3(K)$ in PSU_h, preserves $\mathbb{P}_2(K)$, but in general its projective action on $\mathbb{P}_2(K)$ is transitive neither on the positive, nor on the null, nor on the negative points of $\mathbb{P}_2(K)$.

The Galois group $\mathrm{Gal}(\mathbb{C}|K)$ acts on $\mathbb{P}_2(\mathbb{C})$ by $\sigma[z_0 : z_1 : z_2] = [\sigma z_0 : \sigma z_1 : \sigma z_2]$, and fixes $\mathbb{P}_2(K)$ pointwise. Note that it does not preserve the positive, null, or negative cone of h in $\mathbb{P}_2(\mathbb{C})$. A positive point $z \in \mathbb{P}_2(\mathbb{C})$ is *Hermitian cubic* over K if it is cubic over K (that is, if its orbit under $\mathrm{Gal}(\mathbb{C}|K)$ has exactly three points), and if its other Galois conjugates z', z'' over K are null elements in the polar line of z.

The following result, analog to [MR2, Prop. 9.6.1] in the Bianchi group case, strengthens one direction of [MT, Lem. 1.2].

Proposition 3.2 *A nonelementary extended $\mathbb{C}$-Fuchsian subgroup Γ of Γ_K fixes a unique rational point in $\mathbb{P}_2(\mathbb{C})$. This point is positive and it is the polar point of the unique complex geodesic preserved by Γ.*

Proof. If $\alpha \in \mathrm{PSU}_h$ is loxodromic, let $\alpha_-, \alpha_+ \in \partial_\infty \mathbb{H}^2_{\mathbb{C}}$ be its repelling and attracting fixed points, and let α_0 be its positive fixed point. Since the two projective lines tangent to the hypersphere $\mathscr{HS}$ at α_- and α_+ are invariant under α, their unique intersection point is fixed by Γ, therefore is equal to α_0. In particular, α_0 is polar to the complex projective line through α_-, α_+ (see also [Par, Lemma 6.6] for a more analytic proof).

Let L be the complex projective line preserved by Γ, which meets $\mathbb{H}^2_{\mathbb{C}}$. As Γ is a non elementary discrete group of isomeries of a proper $\mathrm{CAT}(-1)$ space (see for instance [BrH]), there are loxodromic elements $\alpha, \beta \in \Gamma$ such that their sets of fixed points in $\partial_\infty \mathbb{H}^2_{\mathbb{C}} \cap L$ are disjoint. Since L passes through α_-, α_+ as well as through β_-, β_+, and by the uniqueness of the polar point to L, we hence have $\alpha_0 = \beta_0$.

As α and β have infinite order, one of them cannot be K-irreducible. Otherwise, if both were K-irreducible, then by [PP2, Prop. 18], the point $\alpha_0 = \beta_0$ would be Hermitian cubic and its orbit under $\mathrm{Gal}(\mathbb{C}|K)$ would be $\{\alpha_-, \alpha_+, \alpha_0\} = \{\beta_-, \beta_+, \beta_0\}$, a contradiction. Assume then for instance that α preserves a line or a point defined over K. As any projective subspace preserved by α is a combination of α_-, α_+ and α_0, and as α_- and α_+ are not defined over K, it follows that α_0 is rational. $\square$

Let Γ be a nonelementary extended $\mathbb{C}$-Fuchsian subgroup of Γ_K. By the previous proposition, Γ fixes a unique rational point P_Γ in $\mathbb{P}_2(\mathbb{C})$, which may be written $P_\Gamma = [z_0 : z_1 : z_2]$ with $z_0, z_1, z_2 \in \mathscr{O}_K$ relatively prime. Such a writing is unique up to the simultaneous multiplication of z_0, z_1, z_2 by a unit in $\mathscr{O}_K$. Since the units in $\mathscr{O}_K$ have norm 1, and since the trace and norm of K take integral values on the integers of K, the number

$$\Delta_\Gamma = h(z_0, z_1, z_2) = N(z_1) - \mathtt{tr}(z_0\,\overline{z_2}) \in \mathbb{Z}$$

is well defined, we call it the *discriminant* of Γ (by analogy with the case of $\mathrm{PSL}_2(\mathscr{O}_K)$, see [MR1, Def 3.2], and as it discriminates Γ up to finite error by Theorem 1.1). As P is positive, we have $\Delta_\Gamma \in \mathbb{N} - \{0\}$. The radius of the vertical projection of the polar chain of P_Γ is hence $\frac{\sqrt{\Delta_\Gamma}}{|z_2|}$. The discriminant of Γ depends only on the conjugacy class of Γ in Γ_K: for every $\gamma \in \Gamma_K$, since by uniqueness we have $P_{\gamma\Gamma\gamma^{-1}} = \gamma P_\Gamma$, we have

$$\Delta_{\gamma\Gamma\gamma^{-1}} = \Delta_\Gamma\ .$$

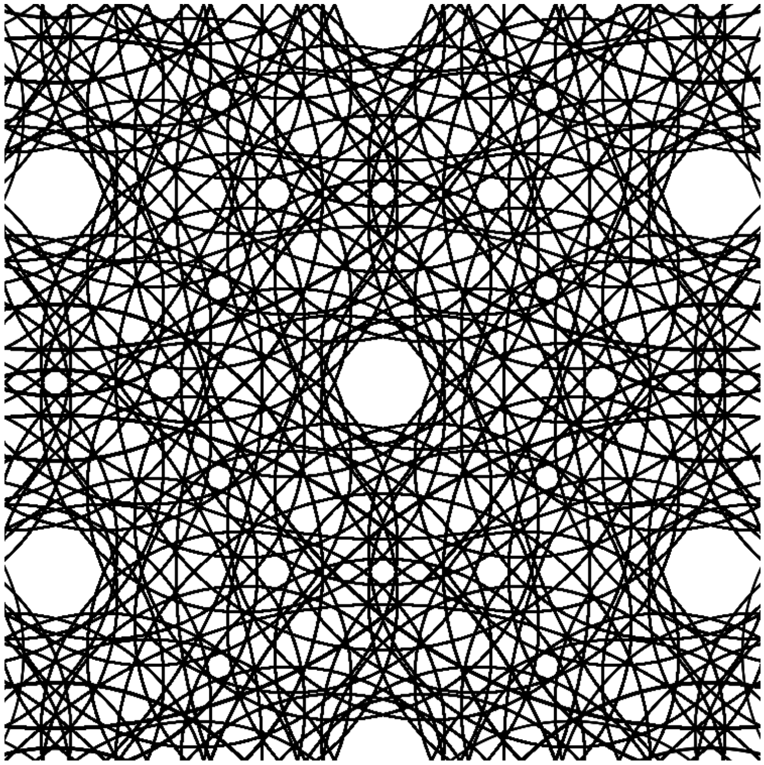
A chain C is *(K-)arithmetic* if its stabiliser in Γ_K has a dense orbit in C. The following result along with Proposition 3.2 justifies this terminology. This result is well known, and it is the other direction of [MT, Lem. 1.2], see also [Hol1, Prop. 1.5,§III.1] and [Kud, §3]. We give a proof, which is a bit different, for the sake of completeness. In Section 4, we give an explicit construction based on quaternion algebras that implies this result.

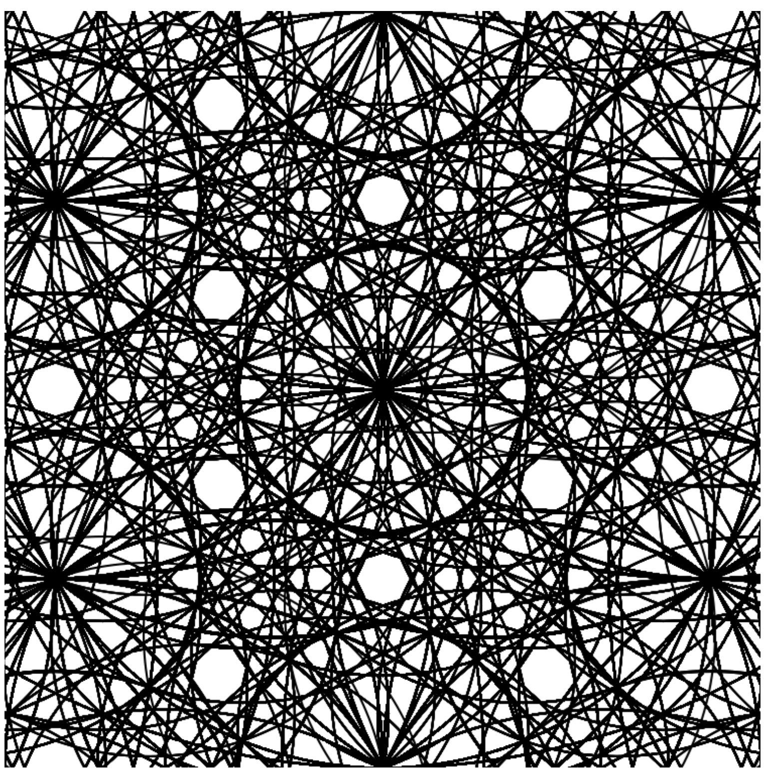
Proposition 3.3 *The stabiliser* $\mathrm{Stab}_{\Gamma_K} P$ *of any positive rational point* $P \in \mathbb{P}_2(K)$ *is a maximal nonelementary extended* $\mathbb{C}$*-Fuchsian subgroup of* Γ_K*, whose invariant chain is arithmetic.*

Proof. Let $\underline{G}$ be the linear algebraic group defined over $\mathbb{Q}$, such that $\underline{G}(\mathbb{Z}) = \mathrm{PSU}_h(\mathscr{O}_K)$ and $\underline{G}(\mathbb{R}) = \mathrm{PSU}_h$. We endow $\mathbb{P}_2(\mathbb{C})$ with the $\mathbb{Q}$-structure $\underline{X}$ whose $\mathbb{Q}$-points are $\mathbb{P}_2(K)$ so that the action of $\underline{G}$ on $\underline{X}$ is defined over $\mathbb{Q}$.

As seen in Section 2, the set of real points of the stabilizer $\mathrm{Stab}_{\underline{G}} P$ is isomorphic to $(\mathbb{S}^1 \times \mathrm{SL}_2(\mathbb{R}))/\{\pm(1, \mathrm{id})\}$ as a real Lie group. The group $\mathrm{Stab}_{\underline{G}} P$ is reductive and it has a (semisimple) Levi subgroup $\underline{H}$ defined over $\mathbb{Q}$, such that $\underline{H}(\mathbb{R})$ is isomorphic to $\mathrm{SL}_2(\mathbb{R})$. By a theorem of Borel-Harish-Chandra [BHC, Theo. 7.8], the group $\underline{H}(\mathbb{Z})$ is an arithmetic lattice in $\underline{H}(\mathbb{R})$, which (preserves the projective line polar to P and) is contained in $\mathrm{Stab}_{\Gamma_K} P$. As $\underline{H}(\mathbb{Z})$ is a lattice in $\underline{H}(\mathbb{R})$, the group $\mathrm{Stab}_{\Gamma_K} P$ is nonelementary and has a dense orbit in the chain $P^\perp \cap \mathscr{H}\mathscr{S}$. $\square$

Recall that in the coordinates $(w, -2\,\mathrm{Im}\ w_0)$ of Heis_3, the chains are ellipses whose images under the vertical projection are Euclidean circles (see also [Gol, §4.3]). The figure in the introduction is the vertical projection of part of the orbit under Γ_K of the chain $[-5 : 0 : 1]^\perp \cap \mathscr{H}\mathscr{S}$ when $K = \mathbb{Q}[i]$, so that Γ_K is the Gauss-Picard modular group, and we use the generating set of Γ_K given by [FFP]. The figure shows the square $|\operatorname{Re} z|, |\operatorname{Im} z| \le 1.5$ in $\mathbb{C}$ with projections of chains whose diameter is at least 1. In the figures below (contrarily to the introduction where $K = \mathbb{Q}(i)$, so that the apparent symmetries are different), $K = \mathbb{Q}[\omega]$, where ω is a primitive third root of unity, so that Γ_K is the Eisenstein-Picard modular group, and we use the generating set of Γ_K [FaP2]. These two pictures illustrate the difference between D odd and D even in the coming proof of Theorem 1.1 (note in particular the difference around the origin).





The first figure shows part of the orbit of $[-1:0:1]^\perp \cap \mathscr{HS}$ and the second figure shows part of the orbit of $[-2:0:1]^\perp \cap \mathscr{HS}$. They both show the square $|\operatorname{Re} z|, |\operatorname{Im} z| \le 1$ in $\mathbb{C}$ with projections of chains whose diameter is at least 0.5 in the first figure and at least 0.75 in the second.

The first part of Theorem 1.1 in the introduction concerns the classification up to conjugacy in Γ_K of the maximal nonelementary extended $\mathbb{C}$-Fuchsian subgroups of Γ_K. Consider the set $\mathscr{F}_\mathbb{C}$ of maximal nonelementary $\mathbb{C}$-Fuchsian subgroups of Γ_K, on which the group Γ_K acts by conjugation. We will prove that the discriminant map $\Gamma \mapsto \Delta_\Gamma$ on $\mathscr{F}_\mathbb{C}$ induces a finite-to-one map from $\Gamma_K \backslash \mathscr{F}_\mathbb{C}$ onto $\mathbb{N} - \{0\}$. Since every maximal non elementary $\mathbb{C}$-Fuchsian subgroup Γ of Γ_K is contained in a unique maximal non elementary extended $\mathbb{C}$-Fuchsian subgroup $\widehat{\Gamma}$ of Γ_K, and since two maximal non elementary $\mathbb{C}$-Fuchsian subgroups Γ, Γ' of Γ_K are conjugate if $\widehat{\Gamma}, \widehat{\Gamma'}$ are conjugate, this implies the first part of Theorem 1.1.

The second part of Theorem 1.1 concerns the classification up to commensurability and conjugacy in PSU_h. Given a group G and a subgroup H of G, recall that two subgroups Γ, Γ' of H are *commensurable* if $\Gamma \cap \Gamma'$ has finite index in Γ and in Γ', and are *commensurable up to conjugacy in* G (or *commensurable in the wide sense*) if there exists $g \in G$ such that Γ' and $g\Gamma g^{-1}$ are commensurable. Two groups A and B are *abstractly commensurable* if they contain finite index subgroups A' and B' respectivey that are isomorphic.

For any positive integer D, let

$$\Gamma_{K,D} = \mathrm{Stab}_{\Gamma_K}[-D:0:1]\,.$$

The group $\Gamma_{K,D}$ is, by Proposition 3.3, a maximal nonelementary extended $\mathbb{C}$-Fuchsian subgroup, which preserves the projective line $[-D:0:1]^\perp$. Its discriminant is $2D$. We will prove that every element of $\mathscr{F}_\mathbb{C}$ with discriminant D is commensurable up to conjugacy in PSU_h with $\Gamma_{K,2D}$.

Proof of Theorem 1.1. (1) Let $D \in \mathbb{N} - \{0\}$ and let

$$\mathscr{F}_\mathbb{C}(D) = \{\Gamma \in \mathscr{F}_\mathbb{C} \;:\; \Delta_\Gamma = D\}\,.$$

Let

$$P_D = \begin{cases} [-\frac{D}{2}:0:1] \text{ if } D \text{ is even} \\ [0:1:0] \text{ if } D = 1 \\ [-\frac{D-1}{2}:1:1] \text{ if } D > 1 \text{ is odd}\,. \end{cases}$$

By Proposition 3.3, the stabiliser in Γ_K of the positive rational point P_D is a maximal nonelementary extended $\mathbb{C}$-Fuchsian subgroup of Γ_K, with discriminant D. Hence $\mathscr{F}_\mathbb{C}(D)$ is nonempty.

Let $\underline{G}$ be the connected semisimple linear algebraic group defined over $\mathbb{Q}$ such that $\underline{G}(\mathbb{Z}) = \mathrm{SU}_h(\mathscr{O}_K)$ and $\underline{G}(\mathbb{R}) = \mathrm{SU}_h$. Let $\pi : \underline{G} \to \mathrm{GL}(\underline{V})$ be the rational representation such that $\underline{V}(\mathbb{Z}) = (\mathscr{O}_K)^3$, $\underline{V}(\mathbb{R}) = \mathbb{C}^3$ and $\pi_{|\underline{G}(\mathbb{R})}$ is the linear action of SU_h on $\mathbb{C}^3$. Let $\underline{X}_D$ be the closed algebraic submanifold of $\underline{V}$ with equation $h = D$. In particular, $\underline{X}_D$ is defined over $\mathbb{Q}$, and $\underline{X}_D(\mathbb{R})$ is homogeneous under $\underline{G}(\mathbb{R}) = \mathrm{SU}_h$, by Witt's theorem. The map

$$\underline{X}_D(\mathbb{Z}) = \underline{X}_D \cap \underline{V}(\mathbb{Z}) = \{(z_0,z_1,z_2) \in (\mathscr{O}_K)^3 \;:\; h(z_0,z_1,z_2) = D\} \to \mathscr{F}_\mathbb{C}\,,$$

which to (z_0,z_1,z_2) associates the stabiliser of $[z_0 : z_1 : z_2]$ in Γ_K (which is the image of $\underline{G}(\mathbb{Z})$ by the canonical map $\underline{G}(\mathbb{R}) = \mathrm{SU}_h \to \mathrm{PSU}_h$), is well defined by Proposition 3.3 and $\underline{G}(\mathbb{Z})$-equivariant, and its image contains $\mathscr{F}_\mathbb{C}(D)$. Hence the finiteness of $\Gamma_K \backslash \mathscr{F}_\mathbb{C}(D)$ follows from the finiteness of the number of orbits of $\underline{G}(\mathbb{Z})$ on $\underline{X}_D(\mathbb{Z})$, see [BHC, Thm. 6.9].

(2) Let $\Gamma \in \mathscr{F}_{\mathbb{C}}$, and let $D \in \mathbb{N} - \{0\}$ be its discriminant. By Propositions 3.2 and 3.3, and by maximality, there is a unique positive rational point $P = [z_0 : z_1 : z_2]$ with z_0, z_1, z_2 relatively prime in $\mathscr{O}_K$ such that $\Gamma = \operatorname{Stab}_{\Gamma_K} P$ and $D = h(z_0, z_1, z_2)$.

Claim. There exists $\gamma \in \mathrm{PSU}_h(K)$ such that $\gamma P = [-2D : 0 : 1]$.

Assuming this claim for the moment, we conclude the proof of the second part of Theorem 1.1: The groups $\gamma\Gamma\gamma^{-1}$ and $\Gamma_{K,2D}$ are commensurable, since

$$\gamma\big(\operatorname{Stab}_{\Gamma_K} P\big)\gamma^{-1} \cap \Gamma_{K,2D} = \operatorname{Stab}_{\gamma\Gamma_K\gamma^{-1}\cap\Gamma_K} \gamma P = \gamma\big(\operatorname{Stab}_{\Gamma_K\cap\gamma^{-1}\Gamma_K\gamma} P\big)\gamma^{-1}$$

and since $\mathrm{PSU}_h(K)$ is contained in the commensurator of $\Gamma_K = \mathrm{PSU}_h(\mathscr{O}_K)$ in PSU_h by a standard argument of reduction to a common denominator. □

The following result, useful for the proof of the above claim, also gives a natural condition for when two such groups $\Gamma_{K,D}$ for $D \in \mathbb{N} - \{0\}$ are commensurable up to conjugacy in PSU_h. A necessary and sufficient condition when D is even will be given in Corollary 4.2.

Lemma 3.4 *If $D, D' \in \mathbb{N} - \{0\}$ satisfy $D' \in D\, N(\mathscr{O}_K)$, then $\Gamma_{K,D}$ and $\Gamma_{K,D'}$ are commensurable up to conjugacy in $\mathrm{PSU}_h(K)$.*

Proof. Let $D \in \mathbb{N} - \{0\}$ and $N \in N(\mathscr{O}_K) - \{0\}$. As seen above, we only have to prove that there exists $\gamma \in \mathrm{PSU}_h(K)$ such that $\gamma[-D : 0 : 1] = [-DN : 0 : 1]$.

Assume first that $D_K \equiv 0 \mod 4$, so that $\mathscr{O}_K = \mathbb{Z} + \frac{\sqrt{D_K}}{2}\mathbb{Z}$. Since $N \in N(\mathscr{O}_K)$, there exists $x, y \in \mathbb{Z}$ such that $N = x^2 - \frac{D_K}{4}y^2$. It is easy to check using Equation (1) and the fact that $K = \mathbb{Q} + i\sqrt{|D_K|}\mathbb{Q}$ that the matrix

$$\gamma = \begin{pmatrix} x & 0 & -\frac{i}{2}\sqrt{|D_K|}Dy \\ 0 & 1 & 0 \\ -\frac{i}{2}\sqrt{|D_K|}\frac{y}{DN} & 0 & \frac{x}{N} \end{pmatrix}$$

belongs to $\mathrm{SU}_h(K)$. Let γ be its image in $\mathrm{PSU}_h(K)$. It is easy to check that as wanted $\gamma[-D : 0 : 1] = [-DN : 0 : 1]$.

If $D_K \equiv 1 \mod 4$, so that $\mathscr{O}_K = \mathbb{Z} + \frac{1+\sqrt{D_K}}{2}\mathbb{Z}$, the same argument works when γ in the above proof is replaced by the matrix

$$\begin{pmatrix} x + \frac{y}{2} & 0 & -\frac{i}{2}\sqrt{|D_K|}Dy \\ 0 & 1 & 0 \\ -\frac{i}{2}\sqrt{|D_K|}\frac{y}{DN} & 0 & \frac{x+\frac{y}{2}}{N} \end{pmatrix}$$

and the equation $N = x^2 + xy + \frac{1-D_K}{4}y^2$ with $x, y \in \mathbb{Z}$. □

Proof of the claim. As the lattice Γ_K does not preserve the complex geodesic with equation $z_2 = 0$, we may assume that z_2 is nonzero, up to replacing P by an element in its orbit under Γ_K, which does not change the discriminant D of Γ. Let γ_1 be the Heisenberg translation by the element

$$\Big[w_0 = \frac{|z_1|^2}{2\,|z_2|^2} - i\,\mathrm{Im}\,\frac{z_0}{z_2} : w = -\frac{z_1}{z_2} : 1\Big] \in \mathrm{Heis}_3\ ,$$

which belongs to $\mathrm{PSU}_h(K)$. An easy computation shows that

$$\gamma_1[z_0 : z_1 : z_2] = [-D : 0 : 2\,N(z_2)]\ .$$

Let γ_2 be the image in $\mathrm{PSU}_h(K)$ of the diagonal element $\begin{pmatrix} 2N(z_2) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2N(z_2)} \end{pmatrix}$ in $\mathrm{SU}_h(K)$. Then $\gamma_2\gamma_1$ maps P to $[-2\,D\,N(z_2) : 0 : 1]$. By the previous lemma, there exists $\gamma_3 \in \mathrm{PSU}_h(K)$ such that $\gamma_3[-2\,D\,N(z_2) : 0 : 1] = [-2\,D : 0 : 1]$. Hence the claim follows with $\gamma = \gamma_3\gamma_2\gamma_1$. □

4 Quaternion algebras

We refer to [Vig] and [MR2] for generalities on quaternion algebras. Let $a, b \in \mathbb{Z}$ with $a > 0$ and $b < 0$. The quaternion algebra $A = \left(\frac{a,b}{\mathbb{Q}}\right)$ is the 4-dimensional central simple algebra over $\mathbb{Q}$ with standard generators i, j, k satisfying the relations $i^2 = a$, $j^2 = b$ and $ij = -ji = k$. The *(reduced) norm* of an element of A is

$$n(x_0 + x_1 i + x_2 j + x_3 k) = x_0^2 - a x_1^2 - b x_2^2 + ab x_3^2\ .$$

The group of elements in $A(\mathbb{Z}) = \mathbb{Z} + i\mathbb{Z} + j\mathbb{Z} + k\mathbb{Z}$ with norm 1 is denoted by $A(\mathbb{Z})^1$.

Lemma 4.1 *The map $\sigma = \sigma_{a,b} : A \to \mathscr{M}_3(\mathbb{C})$ defined by*

$$(x_0 + x_1 i + x_2 j + x_3 k) \mapsto \begin{pmatrix} x_0 + x_1\sqrt{a} & 0 & (x_2 + x_3\sqrt{a})\sqrt{b} \\ 0 & 1 & 0 \\ (x_2 - x_3\sqrt{a})\sqrt{b} & 0 & x_0 - x_1\sqrt{a} \end{pmatrix}$$

is a morphism of $\mathbb{Q}$-algebras and $\sigma(A(\mathbb{Z})^1)$ is a discrete subgroup of the stabiliser of $[0 : 1 : 0]$ *in* SU_h.

Proof. It is well-known (and easy to check), see for instance [Kat, MR2], that the map $\sigma' : A \to \mathscr{M}_2(\mathbb{R})$ defined by

$$(x_0 + x_1 i + x_2 j + x_3 k) \mapsto \begin{pmatrix} x_0 + x_1\sqrt{a} & (x_2 + x_3\sqrt{a})\sqrt{|b|} \\ -(x_2 - x_3\sqrt{a})\sqrt{|b|} & x_0 - x_1\sqrt{a} \end{pmatrix}$$

is a morphism of $\mathbb{Q}$-algebras and that the image of $A(\mathbb{Z})^1$ is a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$. The map

$$\iota : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & 0 & ib \\ 0 & 1 & 0 \\ -ic & 0 & d \end{pmatrix}$$

is a morphism of $\mathbb{Q}$-algebras, sending $\mathrm{SL}_2(\mathbb{R})$ into the stabiliser of $[0 : 1 : 0]$ in SU_h (see Equation (1)). The claim follows by noting that $\sigma = \iota \circ \sigma'$. □

Proof of Theorem 1.3. By Theorem 1.1, we only have to prove that the maximal $\mathbb{C}$-Fuchsian subgroup F_D of Γ_K stabilising $[-2D : 0 : 1]$ (which has finite index in the extended $\mathbb{C}$-Fuchsian subgroup $\Gamma_{K,2D}$) arises from the quaternion algebra $\left(\frac{D,D_K}{\mathbb{Q}}\right)$. It is easy to check that the element

$$\gamma_0 = -\frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{D} & \sqrt{2D} & \sqrt{D} \\ 1 & 0 & -1 \\ \frac{1}{2\sqrt{D}} & -\frac{1}{\sqrt{2D}} & \frac{1}{2\sqrt{D}} \end{pmatrix}$$

belongs to SU_h and maps $[0:1:0]$ to $[-2D:0:1]$. Hence, using Equation (1), a matrix $M \in \mathrm{SU}_h(\mathscr{O}_K)$ has its image (by the canonical projection $\mathrm{SU}_h \to \mathrm{PSU}_h$) in F_D if and only if there exists $a, d \in \mathbb{R}$ and $b, c \in i\mathbb{R}$ with $ad - bc = 1$ such that $M = \gamma_0 \begin{pmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & d \end{pmatrix} \gamma_0^{-1}$.

A straightforward computation gives

$$M = \begin{pmatrix} \frac{1}{4}(a+b+c+d+2) & \frac{\sqrt{D}}{2}(a-b+c-d) & \frac{D}{2}(a+b+c+d-2) \\ \frac{1}{4\sqrt{D}}(a+b-c-d) & \frac{1}{2}(a-b-c+d) & \frac{\sqrt{D}}{2}(a+b-c-d) \\ \frac{1}{8D}(a+b+c+d-2) & \frac{1}{4\sqrt{D}}(a-b+c-d) & \frac{1}{4}(a+b+c+d+2) \end{pmatrix}.$$

This matrix has entries in $\mathscr{O}_K$ if and only if

$$\begin{cases} a+b+c+d-2 \in 8D\mathscr{O}_K, \\ a+b-c-d \in 4\sqrt{D}\mathscr{O}_K, \\ a-b+c-d \in 4\sqrt{D}\mathscr{O}_K, \\ a-b-c+d \in 2\mathscr{O}_K\,. \end{cases}$$

Let $u = a+d$, $v = \frac{1}{2\sqrt{D}}(a-d)$, $s' = b+c$ and $t' = \frac{1}{2\sqrt{D}}(b-c)$. Then M has entries in $\mathscr{O}_K$ if and only if

$$\begin{cases} u+s'-2 \in 8D\mathscr{O}_K, \\ v+t' \in 2\mathscr{O}_K, \\ v-t' \in 2\mathscr{O}_K, \\ u-s' \in 2\mathscr{O}_K\,. \end{cases} \tag{2}$$

Let $D'_K = \frac{D_K}{4}$ if $D_K \equiv 0 \mod 4$ and $D'_K = D_K$ otherwise (so that $K = \mathbb{Q}(\sqrt{D'_K})$, see Section 3). Recall that $\mathscr{O}_K \cap \mathbb{R} = \mathbb{Z}$ and $\mathscr{O}_K \cap i\mathbb{R} = \mathbb{Z}\sqrt{D'_K}$. The equations (2) imply in particular that $u, v, s', t' \in \mathscr{O}_K$. Note that $a, d \in \mathbb{R}$ is equivalent to $u, v \in \mathbb{R}$, and $c, b \in i\mathbb{R}$ is equivalent to $s', t' \in i\mathbb{R}$. Hence $u, v \in \mathbb{Z}$ and there exists $s, t \in \mathbb{Z}$ such that $s' = s\sqrt{D'_K}, t' = t\sqrt{D'_K}$. Therefore

$$\gamma_0^{-1}\, F_D\, \gamma_0 = \left\{ \begin{pmatrix} \frac{u}{2} + v\sqrt{D} & 0 & (\frac{s}{2} + t\sqrt{D})\sqrt{D'_K} \\ 0 & 1 & 0 \\ (\frac{s}{2} - t\sqrt{D})\sqrt{D'_K} & 0 & \frac{u}{2} - v\sqrt{D} \end{pmatrix} : \begin{array}{l} u, v, s, t \in \mathbb{Z} \\ v + t\sqrt{D'_K} \in 2\mathscr{O}_K \\ v - t\sqrt{D'_K} \in 2\mathscr{O}_K \\ u - s\sqrt{D'_K} \in 2\mathscr{O}_K \\ u + s\sqrt{D'_K} - 2 \in 8D\mathscr{O}_K \end{array} \right\}.$$

The group $\gamma_0^{-1}\, F_D\, \gamma_0$ is contained in $\sigma_{D,D'_K}(A(\mathbb{Z})^1)$, since the parameters u and s have to be even as a consequence of the defining equations of $\gamma_0^{-1}\, F_D\, \gamma_0$. Furthermore, $\gamma_0^{-1}\, F_D\, \gamma_0$ contains $\sigma_{D,D'_K}(\mathscr{O}^1)$, where $\mathscr{O}$ is the order of A defined by

$$\mathscr{O} = \{x_0 + ix_1 + jx_2 + kx_3 \in A(\mathbb{Z}) \;:\; x_1, x_2, x_3 \equiv 0 \mod 4D\}\,.$$

Indeed, if $x_0+ix_1+jx_2+kx_3\in\mathscr{O}^1$, then with $u=2x_0$, $s=2x_2$, $v=x_1$, $t=x_3$, we have, since $x_0\equiv 1 \mod 4D$ by the condition $n(x_0+ix_1+jx_2+kx_3)=1$,

$$\begin{cases} v\pm t\sqrt{D'_K}\in 2\mathbb{Z}+2\sqrt{D'_K}\,\mathbb{Z}\subset 2\mathscr{O}_K \\ u-s\sqrt{D'_K}\in 2\mathbb{Z}+2\sqrt{D'_K}\,\mathbb{Z}\subset 2\mathscr{O}_K \\ u-2+s\sqrt{D'_K}=2(x_0-1)+2x_2\sqrt{D'_K}\in 8D\mathbb{Z}+8D\sqrt{D'_K}\,\mathbb{Z}\subset 8D\mathscr{O}_K\ . \end{cases}$$

Since $\sigma_{D,D'_K}(\mathscr{O}^1)$ has finite index in $\sigma_{D,D'_K}(A(\mathbb{Z})^1)$ (see for instance [Vig], Coro. 1.5 in Chapt. IV), the groups $\gamma_0^{-1}\,F_D\,\gamma_0$ and $\sigma_{D,D'_K}(A(\mathbb{Z})^1)$ are commensurable.

Since $\big(\frac{D,D'_K}{\mathbb{Q}}\big)=\big(\frac{D,D_K}{\mathbb{Q}}\big)$ as D'_K and D_K differ by a square factor, the result follows. □

Observe that, by Theorem 1.3, a maximal nonelementary $\mathbb{C}$-Fuchsian subgroup of Γ_K of discriminant D is cocompact (in its copy of $\mathrm{SL}_2(\mathbb{R})$) if and only if $\big(\frac{D,D_K}{\mathbb{Q}}\big)$ is a division algebra (see for instance [Kat, Thm. 5.4.1]).

Corollary 4.2 *Let $D,D'\in\mathbb{N}-\{0\}$. The subgroups $\Gamma_{K,2D}$ and $\Gamma_{K,2D'}$ are commensurable up to conjugacy in PSL_h if and only if the quaternion algebras $\big(\frac{D,D_K}{\mathbb{Q}}\big)$ and $\big(\frac{D',D_K}{\mathbb{Q}}\big)$ are isomorphic.*

Proof. We have seen in the previous proof that $\Gamma_{K,2D}$ arises from the quaternion algebra $\big(\frac{D,D_K}{\mathbb{Q}}\big)$. The result hence follows from the fact that two arithmetic Fuschian groups are commensurable up to conjugacy in $\mathrm{SL}_2(\mathbb{R})$ if and only if their associated quaternion algebras are isomorphic (see [Tak]). □

The following corollaries follow from the arguments in [Mac], pages 309 and 310. Corollary 1.2 of the introduction follows from Corollary 4.4 below. We refer for instance to [MR2] for the unexplained terminology below.

Proposition 4.3 *Let A be an indefinite quaternion algebra over $\mathbb{Q}$. There exists an arithmetic $\mathbb{C}$-Fuchsian subgroup of Γ_K whose associated quaternion algebra is A if and only if the primes at which A is ramified are either ramified or inert in K.* □

Corollary 4.4 *(Chinburg-Stover) Every Picard modular group Γ_K contains infinitely many wide commensurability classes in PSU_h of maximal nonelementary $\mathbb{C}$-Fuchsian subgroups.* □

Corollary 4.5 *Any arithmetic Fuchsian group whose associated quaternion algebra is defined over $\mathbb{Q}$ has a finite index subgroup isomorphic to a $\mathbb{C}$-Fuchsian subgroup of some Picard modular group Γ_K.* □

Corollary 4.6 *For all quadratic irrational number fields K and K', there are infinitely many abstract commensurability classes of arithmetic Fuchsian subgroups with representatives in both Picard modular groups Γ_K and $\Gamma_{K'}$.* □

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