Equidistribution of divergent geodesics in negative curvature

Jouni Parkkonen, Frédéric Paulin, Rafael Sayous

January 7, 2025

Abstract

In the unit tangent bundle of noncompact finite volume negatively curved Riemannian manifolds, we prove the equidistribution towards the measure of maximal entropy for the geodesic flow of the Lebesgue measure along the divergent geodesic flow orbits, as their complexity tends to infinity. We prove the analogous result for geometrically finite tree quotients, where the equidistribution takes place in the quotient space of geodesic lines towards the Bowen-Margulis measure. [1]

1 Introduction

Let M be a finite volume complete connected Riemannian good orbifold with dimension at least 2 and pinched negative curvature at most -1. A (locally) geodesic line ℓ in M (or its unit tangent vector $\dot{\ell}(0)$) is *divergent* if the map $\ell : \mathbb{R} \to M$ is a proper map. In this paper, we define a natural type and a natural complexity of the divergent geodesic lines, and study their counting and equidistribution properties in the unit tangent bundle T^1M of M as their complexity tends to $+\infty$ in any given type.

The study of counting and equidistribution properties of similarly defined divergent flats in finite volume arithmetic nonpositively curved locally symmetric spaces has produced many works, see for instance [TW, Wei, DS1, ShZ, DS2, SoT, DPS]. See also [Pol] for a specific counting problem of divergent geodesics in geometrically finite Kleinian manifolds. Curiously, the problem of the equidistribution of divergent geodesics in general negatively curved manifolds (in particular with variable curvature), does not seem to have been studied so far. One possibility to go around the noncompactness of M, as in [DS1, ShZ, DS2], could be to work in the projective space of locally finite positive Borel measures on T^1M for the quotient topology of the weak-star topology. But as in [DPS], we prefer a more precise result, exhibiting the precise scaling factor, that immediately implies the projective convergence.

Let us introduce some definitions and notations before stating our main result. We denote by $\pi : T^1M \to M$ the footpoint projection, by $(\mathsf{g}^t)_{t\in\mathbb{R}}$ the geodesic flow on T^1M and by h_M the topological entropy of $(\mathsf{g}^t)_{t\in\mathbb{R}}$. We denote by $\operatorname{End}(M)$ the finite set of ends[2] of the locally compact space M. We fix a family $(\mathscr{V}_e)_{e\in\operatorname{End}(M)}$ of closed Margulis neighborhoods

[1] **Keywords:** equidistribution, counting, divergent geodesic, geodesic flow, negative curvature, trees, equilibrium states. **AMS codes:** 37D40, 53C22, 20E08, 37D35, 37A25.

[2] See for instance [BrH, Sect. I.8.27] for the definition.

of the ends of M with pairwise disjoint interiors, see Section 3 for a definition. For every $A \geqslant 0$, we denote by $M^{\leqslant A}$ the closed A-neighborhood in M of $M \smallsetminus \big(\bigcup_{e \in \mathrm{End}(M)} \mathscr{V}_e\big)$.

For every divergent geodesic ℓ in M, there exist two (possibly equal) ends ℓ_- and ℓ_+ of M such that $\lim_{t \to \pm\infty} \ell(t) = \ell_\pm$. The pair (ℓ_-, ℓ_+), which varies in the finite set $\mathrm{End}(M)^2$, is called the *type* of ℓ. Let $t_- = t_-(\ell)$ be the first time at which ℓ exits the interior of $\mathscr{V}_{\ell_-}$, and $t_+ = t_+(\ell)$ be the last time at which ℓ enters the interior of $\mathscr{V}_{\ell_+}$. We define the *complexity* of the divergent geodesic ℓ as

$$\tau(\ell) = t_+ - t_- \geqslant 0\,.$$

As usual in counting problems with symmetry, the *multiplicity* of ℓ is the inverse of the order of its stabilizer in the orbifold M. See Section 3 for details on multiplicities, and note that the multiplicities are 1 when M is a manifold.

We denote by $\mathrm{Div}(M)$ the quotient by the action of $\mathbb{R}$ by translation at the source of the set with multiplicities of divergent geodesics with positive complexity. The set of divergent geodesics with complexity 0 is finite up to the action of $\mathbb{R}$, and can be ignored in our discussion on equidistribution and counting of divergent geodesics. As we want to give counting and equidistribution results of divergent geodesics with prescribed type, for every nonempty subset $\mathscr{T}$ of $\mathrm{End}(M)^2$, we consider the space

$$\mathrm{Div}_{\mathscr{T}}(M) = \{\ell \in \mathrm{Div}(M) : (\ell_-, \ell_+) \in \mathscr{T}\} \tag{1}$$

of divergent geodesics with type contained in $\mathscr{T}$.

We define the *Lebesgue measure* Leb_ℓ of ℓ in T^1M as the pushforward of the Lebesgue measure of $\mathbb{R}$ by the map $t \mapsto \dot{\ell}(t) = \mathsf{g}^t(\dot{\ell}(0))$. The study of equidistribution properties of divergent geodesics is made more complicated by the fact that Leb_ℓ, which is a locally finite measure with support the geodesic flow orbit $\mathsf{g}^{\mathbb{R}}(\dot{\ell}(0))$, is not a finite measure.

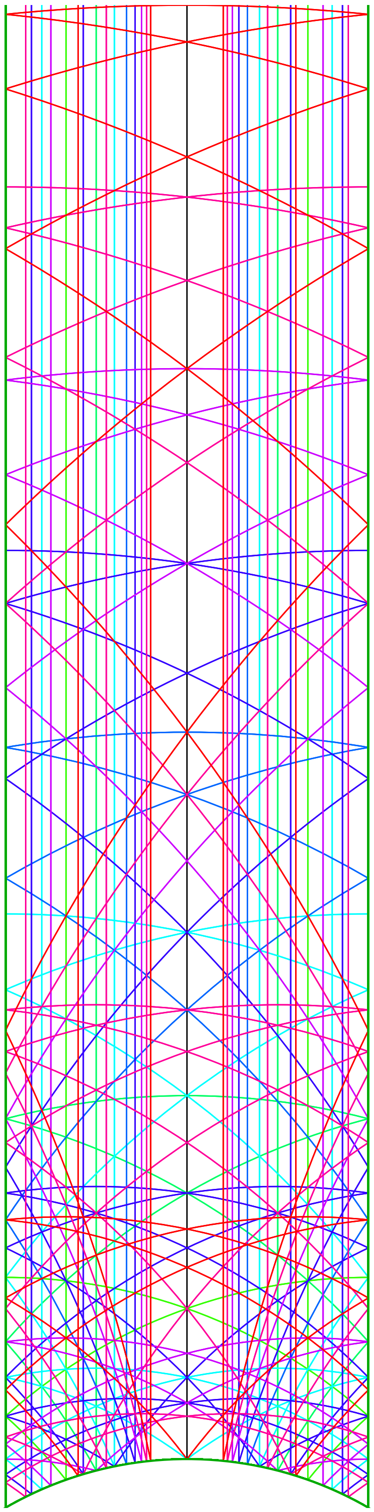
All measures in this paper are locally finite Borel nonnegative measures. We denote by $\|\mu\| \in [0, +\infty]$ the total mass of a measure μ and by $\overset{*}{\rightharpoonup}$ the weak-star convergence of measures on locally compact Hausdorff spaces.

Let m_{BM} be the *Bowen-Margulis measure* on T^1M, which is the Liouville measure when M is locally symmetric. We assume that m_{BM} is finite, which is for instance the case if M is locally symmetric. Then m_{BM} is mixing for the geodesic flow by results of Babillot and Dal'Bo, see for instance [BrPP, §4.2]. Its renormalization to a probability measure $\frac{m_{\mathrm{BM}}}{\|m_{\mathrm{BM}}\|}$ is the unique measure of maximal entropy on M by [OP] (and [DT] that removes the implicit assumption that the sectional curvature of M has bounded derivative in [OP]). The geodesic flow of M has exponential decay of correlation for the Sobolev regularity with respect to m_{BM} for instance if M has constant sectional curvature by [LP], or if M is arithmetic locally symmetric by [KlM1], [KlM2] and [Clo], see for instance [BrPP, §9.1].

For every end $e \in \mathrm{End}(M)$, let $\sigma_e^\pm$ be the (nonzero, finite) *outer/inner skinning measure* on T^1M with support the outer/inner unit normal bundle of $\partial\mathscr{V}_e$.[3] For every nonempty subset $\mathscr{T}$ of $\mathrm{End}(M)^2$, we define a measure on $T^1M \times T^1M$ by

$$\sigma_{\mathscr{T}} = \sum_{(e_-, e_+) \in \mathscr{T}} \sigma^+_{e_-} \otimes \sigma^-_{e_+}\,. \tag{2}$$

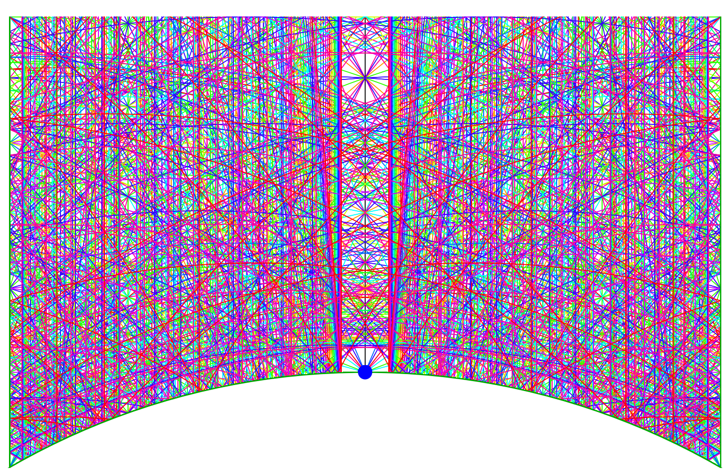
[3]For the definitions, generalising [OhS, §1.2] in constant curvature, see [ParP1] and Section 2.



Example. Let $\mathbb{H}^2_{\mathbb{R}}$ be the upper halfplane model of the real hyperbolic plane with constant curvature -1. The group $G = \mathrm{PSL}_2(\mathbb{R})$ acts isometrically by homographies on the space $\mathbb{H}^2_{\mathbb{R}}$, by the map $(\gamma, z) \mapsto \gamma \cdot z = \frac{az+b}{cz+d}$ for all $z \in \mathbb{H}^2_{\mathbb{R}}$ and $\gamma = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$. Let $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ be the *modular group*, which is a nonuniform arithmetic lattice in $\mathrm{PSL}_2(\mathbb{R})$. Let $M = \Gamma \backslash \mathbb{H}^2_{\mathbb{R}}$ be the *modular curve*, which is a noncompact complete connected real hyperbolic good orbifold, with one end. Its standard Margulis cusp neighbourhood is the Γ-orbit of the horoball H_∞ that consists of the points $z \in \mathbb{H}^2_{\mathbb{R}}$ with $\mathrm{Im}\, z \geqslant 1$.

The figure on the left shows all divergent geodesics of complexity at most $\ln 10$ for the standard Margulis cusp neighbourhood. These divergent geodesics are the images in M of the vertical lines in $\mathbb{H}^2_{\mathbb{R}}$ with points at infinity ∞ and $\frac{p}{q}$, where $p, q \in \mathbb{Z}$, $q > 0$, $|p| \leqslant \frac{q}{2}$, $1 \leqslant q \leqslant 10$. They are represented lifted to the standard fundamental domain of Γ
$$\Big\{ z \in \mathbb{H}^2_{\mathbb{R}} : -\frac{1}{2} < \mathrm{Re}\, z < \frac{1}{2},\ |z| > 1 \Big\}$$
with side identifications given by $z \mapsto z + 1$ and $z \mapsto -\frac{1}{z}$.

The figure below shows the analogous set of divergent geodesics with complexity at most $\ln 30$, in the lower part of the fundamental domain. This illustrates the equidistribution of divergent geodesics stated in the following Theorem 1.1 (1), see also [DS1, Theo. 1.5] in this specific case.



The following is our main result, saying that the divergent geodesics equidistribute towards the Bowen-Margulis measure as their complexity tends to $+\infty$.

Theorem 1.1. *Let M be a finite volume complete connected Riemannian good orbifold with dimension at least 2 and pinched negative curvature at most -1. Let $\mathscr{T}$ be a nonempty* *subset of* $\mathrm{End}(M)^2$*. Assume that the Bowen-Margulis measure on* T^1M *is finite.*

(1) As $T \to +\infty$*, we have*

$$\frac{h_M\,\|m_{\rm BM}\|}{\|\sigma_{\mathscr{T}}\|\;T\;e^{h_M T}}\;\sum_{\ell\in{\rm Div}_{\mathscr{T}}(M)\,:\;\tau(\ell)\leqslant T}{\rm Leb}_\ell\;\overset{*}{\rightharpoonup}\;\frac{m_{\rm BM}}{\|m_{\rm BM}\|}\;.$$

If, furthermore, M *is locally symmetric with exponential decay of correlations, then there exists* $k\in\mathbb{N}$ *such that for every* $\Phi\in C_c^k(T^1M)$ *with support in* $\pi^{-1}(M^{\leqslant A})$*, there is an additive error term in this equidistribution statement when evaluated on* Φ *of the form* $\mathrm{O}_A\big(\frac{\|\Phi\|_{k,\infty}}{T}\big)$*, where* $\|\;\|_{k,\infty}$ *is the* $W^{k,\infty}$*-Sobolev norm.*

(2) As $T \to +\infty$*, we have*

$${\rm Card}\big\{\ell\in{\rm Div}_{\mathscr{T}}(M)\,:\,\tau(\ell)\leqslant T\big\}\;\sim\;\frac{\|\sigma_{\mathscr{T}}\|}{h_M\,\|m_{\rm BM}\|}\;e^{h_M T}\;.$$

If, furthermore, M *is locally symmetric with exponential decay of correlations, then there exists* $\kappa>0$ *and an additive error term in this counting statement of the form* $\mathrm{O}\big(e^{(h_M-\kappa)T}\big)$*.*

As previously mentioned, one of the main difficulties of this paper is that the Lebesgue measure associated with a given divergent geodesic ℓ is an infinite measure. We first reduce the study to the asymptotic distribution of the "compact cores" of the divergent geodesics in Section 3. We then apply the equidistribution results proved in [ParP4] for manifolds in Section 5.

In Section 6, we give a version of Theorem 1.1 for divergent geodesics in geometrically finite quotients of uniform trees. We refer to Section 2, the beginning of Section 6 and [Lub, BaL, Pau2, BrPP] for the definitions and for background.

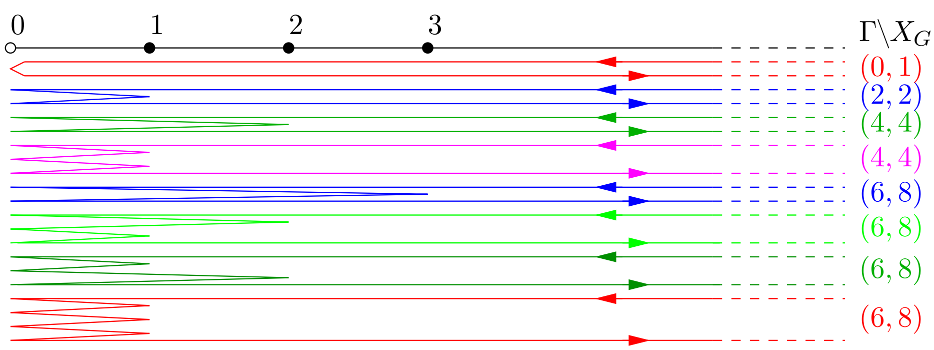
Theorem 1.2. *Let* X *be a uniform simplicial tree without vertices of degree* 1 *or* 2 *and let* Γ *be a finite covolume geometrically finite discrete subgroup of* $\mathrm{Isom}(X)$*. Let* $\mathscr{T}$ *be a nonempty subset of* $\mathrm{End}(\Gamma\backslash X)^2$*. Assume that the length spectrum of* $\Gamma\backslash X$ *is* $\mathbb{Z}$*. As* $N\to+\infty$*, for the weak-star convergence of measures on the locally compact space* $\Gamma\backslash\mathscr{G}X$ *of* Γ*-orbits of geodesic lines in* X*, we have*

$$\frac{(1-e^{-\delta_\Gamma})\,\|m_{\rm BM}\|}{\|\sigma_{\mathscr{T}}\|\;N\;e^{\delta_\Gamma N}}\;\sum_{\ell\in{\rm Div}_{\mathscr{T}}(M)\,:\;\tau(\ell)\leqslant N}{\rm Leb}_\ell\;\overset{*}{\rightharpoonup}\;\frac{m_{\rm BM}}{\|m_{\rm BM}\|}\;.$$

Theorem 1.2 is part of a more general result, Theorem 6.1 proved in Section 6, that covers also the complementary case where the length spectrum is $2\mathbb{Z}$ and gives error terms for the ϵ-locally constant regularity. We develop an analog of the equidistribution results of [ParP4] for tree quotients and horoballs in the proof of Theorem 6.1. In particular, the handling of the error term is much more involved for trees than in the manifold case.

By [Lub], this theorem in particular applies when X_G is the Bruhat-Tits tree of a rank one simple algebraic group G over a nonarchimedean local field and Γ is an arithmetic lattice in G. For instance, let $K=\mathbb{F}_q((Y^{-1}))$ be the field of formal Laurent series over $\mathbb{F}_q$ with indeterminate Y^{-1} and let $G=\mathrm{PGL}_2(K)$. Then $\Gamma=\mathrm{PGL}_2(\mathbb{F}_q[Y])$ is an arithmetic lattice in G, called the *Nagao lattice* (whose length spectrum is $2\mathbb{Z}$). The quotient of the Bruhat-Tits tree X_G by Γ is then a geodesic ray, called the *modular ray* (see for instance [BrPP, §15.2]) when endowed with its quotient graph of group structure. In this special case, Theorem 6.1 can be deduced from the case $n = 2$ of [DPS, Theo. 1.2], which gives a stronger equidistribution result in $\Gamma\backslash G$ (which factors over $\Gamma\backslash\mathscr{G}X_G$). Note that X_G is a $(q+1)$-regular tree with boundary at infinity the projective line $\mathbb{P}^1(K) = K \cup \{\infty\}$.

Example. When $q = 2$, the divergent geodesics in the modular ray with complexity at most 6 with respect to the maximal precisely invariant family of horoballs in X_G are represented in the following picture (turned horizontally for convenience compared to the analogous picture in $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}^2_{\mathbb{R}}$, all of the divergent geodesics meant to be pinched vertically to $[0, +\infty[$). For each shape of image, the pair (n, m) gives the complexity n and the number m of divergent geodesics in $\Gamma\backslash X_G$ with this complexity and shape (they are no longer determined by the shape of their image in $\Gamma\backslash X_G$). The divergent geodesics in $\Gamma\backslash X_G$ are the images in $\Gamma\backslash X_G$ of the geodesic lines in X_G starting from ∞ and ending at $\frac{P}{Q}$ mod $\mathbb{F}_2[Y]$ with $(P, Q) \in \mathbb{F}_2[Y] \times (\mathbb{F}_2[Y] \smallsetminus \{0\})$. By [Pau1], see also [DPS, §4.3], if $\frac{P}{Q} = \cfrac{1}{a_1 + \cfrac{1}{\cdots + \cfrac{1}{a_k}}}$ is the continued fraction expansion of $\frac{P}{Q}$ with $a_i \in \mathbb{F}_2[Y]$ with degree at least 1, then the complexity of the associated divergent geodesic is $n = 2\sum_{i=1}^{k} \deg a_i$, and $m = (q-1)q^{\frac{n}{2}}$ is the number of choices of the polynomials a_i with a given sequence of degrees $(\deg a_1, \dots, \deg a_k)$ that defines the shape of the image.



Acknowledgements: The authors thank the French-Finnish CNRS IEA PaCap for its support.

2 Background on negative curvature

Let X be either a complete simply connected Riemannian manifold $\widetilde{M}$ with dimension at least 2 and with pinched negative sectional curvature at most -1, or the geometric realisation of a uniform simplicial tree $\mathbb{X}$ without vertices of degree 1 or 2. We denote by $X \cup \partial_\infty X$ the geometric compactification of X. The horoballs in X are closed unless otherwise stated. See for instance [BrH] for background on CAT(-1) geometry and discrete groups, and [Ser, BaL], [BrPP, §2.6] for background on group actions on trees.

Let $x_* \in X$ be a fixed basepoint, with $x_* \in V\mathbb{X}$ when X is a tree. For every $\xi \in \partial_\infty X$, let $\rho_\xi : [0, +\infty[\to X$ be the geodesic ray with origin x_* and point at infinity ξ. The *Busemann cocycle* of X is the map $\beta : X \times X \times \partial_\infty X \to \mathbb{R}$ defined by

$$(x, y, \xi) \mapsto \beta_\xi(x, y) = \lim_{t \to +\infty} d(\rho_\xi(t), x) - d(\rho_\xi(t), y)\,.$$

The *visual distance* d_{x_*} on $\partial_\infty X$ seen from x_* is defined by $d_{x_*}(\xi,\eta) = e^{-\frac{1}{2}(\beta_\xi(x_*,y)+\beta_\eta(x_*,y))}$ where y is the closest point to x_* on the geodesic line $]\xi,\,\xi'[$ between two distinct points at infinity ξ and ξ'. When X is a tree, we have

$$d_{x_*}(\xi,\xi') = e^{-d(x_*,\,]\xi,\,\xi'[)}\,. \tag{3}$$

When X is a manifold, we denote by $\operatorname{Isom}(X)$ the group of isometries of $\widetilde{M}$. When X is a tree, we denote by $\operatorname{Isom}(X)$ the group of graph automorphisms of $\mathbb{X}$ without edge inversion. Let Γ be a nonelementary discrete subgroup of $\operatorname{Isom}(X)$. Let $M = \Gamma\backslash X$ with its quotient orbifold structure when X is a manifold, and $M = \Gamma\backslash X$ with its quotient graph of groups structure when X is a tree. When S is a subset or a point in X, we denote by Γ_S the (global) stabiliser of S in Γ. When X is a tree, we denote by $V\mathbb{X}$ the set of vertices of $\mathbb{X}$, and the *covolume* of Γ is

$$\operatorname{covol}\Gamma = \sum_{[x]\in\Gamma\backslash V\mathbb{X}} \frac{1}{\operatorname{Card}(\Gamma_x)}\,.$$

Let δ_Γ be the critical exponent of Γ. When X is a manifold and the Bowen-Margulis measure $m_{\rm BM}$ is finite, δ_Γ coincides with the topological entropy h_M of the geodesic flow on T^1M (see [OP]):

$$\delta_\Gamma = h_M\ . \tag{4}$$

A continuous mapping $\mathbb{R}\to X$ that is an isometric embedding on a closed interval of $\mathbb{R}$ and constant, with value in $V\mathbb{X}$ when X is a tree, on each complementary component is a *generalized geodesic* in X. We denote by $\breve{\mathscr{G}}X$ the Bartels-Lück metric space of the generalized geodesic lines $\widetilde{\ell}:\mathbb{R}\to X$ in X, considering only the generalized geodesic lines $\widetilde{\ell}$ such that $\widetilde{\ell}(0)$ is a vertex of $\mathbb{X}$ when X is a tree. Its distance is defined, for all $\widetilde{\ell},\widetilde{\ell}'\in\breve{\mathscr{G}}X$ by

$$d(\widetilde{\ell},\widetilde{\ell}') = \int_{-\infty}^{+\infty} d(\widetilde{\ell}(t),\widetilde{\ell}'(t))\; e^{-2\,|t|}\;dt\ . \tag{5}$$

See for instance [BrPP, §2.2] for more information. Geodesic rays in X defined on $\pm\,[0,+\infty[$ are considered as generalized geodesics by being constant on $\mp\,[0,+\infty[$.

We denote the geodesic flow on $\breve{\mathscr{G}}X$ by $(\mathsf{g}^t)_{t\in\mathbb{R}}$ when X is a manifold and by $(\mathsf{g}^t)_{t\in\mathbb{Z}}$ when X is a tree, with $\mathsf{g}^t:\widetilde{\ell}\mapsto(s\mapsto\widetilde{\ell}(s+t))$. Let $\mathscr{G}X$ be the subspace of $\breve{\mathscr{G}}X$ consisting of the geodesic lines of X, which is invariant under the geodesic flow. When X is a manifold, the space $\mathscr{G}X$ is identified with the unit tangent bundle $T^1\widetilde{M}$ by the $\operatorname{Isom}(X)$-equivariant homeomorphism $\widetilde{\ell}\mapsto\dot{\widetilde{\ell}}(0)$. We denote by $\pi:\breve{\mathscr{G}}X\to X$ the footpoint map $\widetilde{\ell}\mapsto\widetilde{\ell}(0)$ and again by $\pi:\Gamma\backslash\breve{\mathscr{G}}X\to\Gamma\backslash X$ its quotient map. They are proper maps. For every $\widetilde{\ell}\in\breve{\mathscr{G}}X$, let $\widetilde{\ell}_\pm=\lim_{t\to\pm\infty}\widetilde{\ell}(t)\in X\cup\partial_\infty X$, which are points at infinity when $\widetilde{\ell}\in\mathscr{G}X$. We denote by $p^-,p^+:\breve{\mathscr{G}}X\to X\cup\partial_\infty X$ the *negative, positive endpoint maps* defined by $\widetilde{\ell}\mapsto\widetilde{\ell}_-,\widetilde{\ell}_+$ respectively.

For every generalized geodesic $\widetilde{w}\in\breve{\mathscr{G}}X$, which is isometric on a maximal interval I, if $w=\Gamma\widetilde{w}\in\Gamma\backslash\breve{\mathscr{G}}X$, we define

- the *length* of w by

$$\lambda(w) = \operatorname{length}(I)\in[0,+\infty]\ , \tag{6}$$

• the *Lebesgue measure* Leb_w of w as the measure on $\Gamma\backslash\breve{\mathscr{G}}X$ which is the pushforward by the map $t \mapsto \mathsf{g}^t w$ of the Lebesgue measure on I when X is a manifold, and of the counting measure on $I \cap \mathbb{Z}$ when X is a tree.

For every $w \in \Gamma\backslash\breve{\mathscr{G}}X$, we have $\|\operatorname{Leb}_w\| = \lambda(w)$. When $\ell \in \Gamma\backslash\mathscr{G}X$, we then have $\|\operatorname{Leb}_\ell\| = +\infty$ and the Lebesgue measure Leb_ℓ is invariant under the action of the geodesic flow on $\Gamma\backslash\mathscr{G}X$, since the Lebesgue measure on $\mathbb{R}$ and the counting measure on $\mathbb{Z}$ are invariant under translations.

We conclude Section 2 by giving details on the construction of the Bowen-Margulis measure $m_{\rm BM}$. We denote by $\partial^2_\infty X$ the complement of the diagonal in $\partial_\infty X \times \partial_\infty X$. *Hopf's parametrisation* with respect to the basepoint x_* is the homeomorphism which identifies $\mathscr{G}X$ with $\partial^2_\infty X \times \mathbb{R}$ when X is a manifold and $\partial^2_\infty X \times \mathbb{Z}$ when X is a tree by the map $\widetilde{\ell} \mapsto (\widetilde{\ell}_-, \widetilde{\ell}_+, s)$, where s is the signed distance to $\widetilde{\ell}(0)$ of the closest point to x_* on the geodesic line $\widetilde{\ell}(\mathbb{R})$. Note that a change of base point only changes the third parameter s by an additive constant. We fix a Patterson-Sullivan density $(\mu_x)_{x\in X}$ when X is a manifold (see [BrPP, §4.1]), and $(\mu_x)_{x\in V\mathbb{X}}$ when X is a tree (see [BrPP, §4.3]), for Γ (with zero potential). Since M has finite covolume, the Patterson-Sullivan measures have full support in $\partial_\infty X$. The *Bowen-Margulis measure* on $\mathscr{G}X$ (associated with this Patterson-Sullivan density) is the measure $\widetilde{m}_{\rm BM}$ on $\mathscr{G}X$ given by the density

$$d\,\widetilde{m}_{\rm BM}(\widetilde{\ell}) = e^{-\delta_\Gamma(\beta_{\widetilde{\ell}_-}(\widetilde{\ell}(0),x_*) + \beta_{\widetilde{\ell}_+}(\widetilde{\ell}(0),x_*))}\,d\mu_{x_*}(\widetilde{\ell}_-)\,d\mu_{x_*}(\widetilde{\ell}_+)\,ds$$

in Hopf's parametrisation of $\mathscr{G}X$ with respect to x_*. When X is a tree, we have

$$d\,\widetilde{m}_{\rm BM}(\widetilde{\ell}) = e^{-2\,\delta_\Gamma\, d(x_*,\,]\widetilde{\ell}_-,\widetilde{\ell}_+[\,)}\,d\mu_{x_*}(\widetilde{\ell}_-)\,d\mu_{x_*}(\widetilde{\ell}_+)\,ds\,. \tag{7}$$

The Bowen-Margulis measure $\widetilde{m}_{\rm BM}$ is independent of x_*, and it is invariant under the actions of the group Γ and of the geodesic flow. Thus, it defines (see [PauPS, §2.6] for the branched cover issues) a measure $m_{\rm BM}$ on $\Gamma\backslash\mathscr{G}X$ which is invariant under the quotient geodesic flow, called the *Bowen-Margulis measure* on $\Gamma\backslash\mathscr{G}X$.

Let D be a nonempty closed convex subset of X, which is the geometric realisation of a subtree of $\mathbb{X}$ when X is a tree. We denote by $\partial^1_\pm D$ the *outer/inner normal bundle of* D, that is the subspace of $\breve{\mathscr{G}}X$ consisting of the positive/negative geodesic rays $\rho : \pm[0,+\infty[\to X$ with $\rho(0) \in \partial D$, $\rho_\pm \in \partial_\infty X \smallsetminus \partial_\infty D$ and $\rho(0)$ the closest point to $\rho(\pm t)$ on D for all $t > 0$, see [BrPP, §2.4] for details. We refer to [ParP1, §3] when X is a manifold and to [BrPP, Chap. 7] in general for more background and for the basic properties of the following measures. The *(outer) skinning measure* on $\partial^1_+ D$ (associated with the above Patterson-Sullivan density) is the measure $\widetilde{\sigma}^+_D$ on $\partial^1_+ D$ defined, using the positive endpoint homeomorphism $p_+ : \rho \mapsto \rho_+$ from $\partial^1_+ D$ to $\partial_\infty X \smallsetminus \partial_\infty D$, by

$$d\,\widetilde{\sigma}^+_D(\rho) = e^{-\delta_\Gamma\,\beta_{\rho_+}(\rho(0),x_*)}\,d\mu_{x_*}(\rho_+)\,.$$

The *(inner) skinning measure* $d\,\widetilde{\sigma}^-_D(\rho) = e^{-\delta_\Gamma\,\beta_{\rho_-}(\rho(0),x_*)}\,d\mu_{x_*}(\rho_-)$ is the similarly defined measure on $\partial^1_- D$. When $D = \{x_*\}$, we immediately have

$$\forall\ \rho \in \partial^1_\pm\{x_*\}, \qquad d\,\widetilde{\sigma}^\pm_{\{x_*\}}(\rho) = d\mu_{\{x_*\}}(\rho_\pm)\,. \tag{8}$$

If the family $(\gamma D)_{\gamma\in\Gamma/\Gamma_D}$ is locally finite in X, we denote by $\sigma^\pm_{\Gamma D}$ the locally finite measure on $\Gamma\backslash\breve{\mathscr{G}}X$ induced by the Γ-invariant locally finite measure $\sum_{\gamma\in\Gamma/\Gamma_D}\gamma_*\,\widetilde{\sigma}^\pm_D$ on $\breve{\mathscr{G}}X$. The support of $\sigma^\pm_{\Gamma D}$ is contained in the image $\partial^1_\pm(\Gamma D)$ of $\partial^1_\pm D$ by the map $\breve{\mathscr{G}}X \to \Gamma\backslash\breve{\mathscr{G}}X$.

3 Generalities on divergent geodesics

We assume from now on that Γ has finite covolume and is furthermore geometrically finite when X is a tree. We recall below all the necessary properties, and we refer to [Pau2] for the definition of geometrical finiteness in the case of trees (implied by the finite covolume assumption when X is a manifold). Note that by [BaL], there are many more finite covolume tree lattices than geometrically finite ones.

The set $\operatorname{End}(M)$ of ends of the locally compact topological space M is finite and discrete, and can be described as follows (see [Bow] for manifolds and [Pau2] for trees). Let $\operatorname{Par}_\Gamma$ be the countable Γ-invariant set of (bounded) parabolic fixed points of elements of Γ, that is the set of points $\xi \in \partial_\infty X$ such that its stabilizer Γ_ξ acts properly and cocompactly on $\partial_\infty X \smallsetminus \{\xi\}$. Let us choose for every $\xi \in \operatorname{Par}_\Gamma$ any geodesic ray $t \mapsto \xi_t$ in X with $\lim_{t\to+\infty} \xi_t = \xi$. Then we have a bijection (independent of the previous choices) from $\Gamma\backslash \operatorname{Par}_\Gamma$ to $\operatorname{End}(M)$ which associates to $\Gamma\xi$ the end of M towards which converges $\Gamma\xi_t$ as $t \to +\infty$. For every end $e \in \operatorname{End}(M)$, we fix a parabolic fixed point $\widehat{e} \in \operatorname{Par}_\Gamma$ such that the above bijection maps $\Gamma\widehat{e}$ to e.

There exists a Γ-equivariant family $(H_\xi)_{\xi\in\operatorname{Par}_\Gamma}$ of horoballs H_ξ in X centered at every $\xi \in \operatorname{Par}_\Gamma$ such that

- when X is a tree, for every $\xi \in \operatorname{Par}_\Gamma$, the boundary ∂H_ξ of H_ξ is contained in $V\mathbb{X}$,
- the open horoballs $\overset{\circ}{H}_\xi = H_\xi \smallsetminus \partial H_\xi$, which are the interiors of the horoballs H_ξ, are pairwise disjoint as ξ ranges over $\operatorname{Par}_\Gamma$,
- the quotient $\Gamma\backslash\big(X \smallsetminus \bigcup_{\xi\in\operatorname{Par}_\Gamma} \overset{\circ}{H}_\xi\big)$ is compact.

Note that $\Gamma_{H_\xi} = \Gamma_\xi$ for every $\xi \in \operatorname{Par}_\Gamma$. For every $e \in \operatorname{End}(\Gamma\backslash X)$, the image $\mathscr{V}_e = \Gamma H_{\widehat{e}}$ in $M = \Gamma\backslash X$ of the horoball $H_{\widehat{e}}$ is a neighborhood of the end e, called a *Margulis cusp neighborhood* of e if X is a manifold, and a *cuspidal ray* with point at infinity e if X is a tree, with respect to the family $(H_\xi)_{\xi\in\operatorname{Par}_\Gamma}$.

According to the definition in Section 2, the outer (resp. inner) unit normal bundle $\partial^1_+\mathscr{V}_e = \Gamma\partial^1_+ H_{\widehat{e}}$ (resp. $\partial^1_-\mathscr{V}_e = \Gamma\partial^1_- H_{\widehat{e}}$) of the Margulis cusp neighborhood $\mathscr{V}_e$ is the subset of elements $\Gamma\,\widetilde{\ell}_{|[0,\infty[}$ (resp. $\Gamma\,\widetilde{\ell}_{|]-\infty,0]}$) in $\Gamma\backslash\breve{\mathscr{G}}X$ where $\widetilde{\ell}$ is a geodesic line in $\mathscr{G}X$ with $\widetilde{\ell}_- = \widehat{e}$ (resp. $\widetilde{\ell}_+ = \widehat{e}$) and $\widetilde{\ell}(0) \in \partial H_{\widehat{e}}$. When X is a manifold and $\partial\mathscr{V}_e$ is a submanifold, then the map $\Gamma\,\widetilde{\ell} \mapsto \Gamma\,\dot{\widetilde{\ell}}(0)$ identifies $\partial^1_+\mathscr{V}_e$ and $\partial^1_-\mathscr{V}_e$ with the two connected components of the unit normal bundle of the hypersurface $\partial\mathscr{V}_e$, pointing respectively outwards and inwards from $\mathscr{V}_e$.

A *divergent geodesic* in $\Gamma\backslash\mathscr{G}X$ is an orbit $\ell = \Gamma\widetilde{\ell} \in \Gamma\backslash\mathscr{G}X$ under the action of Γ of a geodesic line $\widetilde{\ell} \in \mathscr{G}X$ in X both of whose points at infinity are in $\operatorname{Par}_\Gamma$. This corresponds to the definition in the introduction whether X is a manifold or not, by the above properties of the family $(H_\xi)_{\xi\in\operatorname{Par}_\Gamma}$, and the fact that a geodesic line that enters in a horoball either converges to its point at infinity or goes through its boundary after a finite time. The *multiplicity* of ℓ (independent of the choice of $\widetilde{\ell}$ and of the action of the geodesic flow on ℓ) is

$$m(\ell) = \frac{1}{\operatorname{Card}\Gamma_{\widetilde{\ell}(\mathbb{R})}}\,. \tag{9}$$

We define

$$\ell_- = \lim_{t=-\infty} \ell(t) \in \operatorname{End}(M) \qquad \text{and} \qquad \ell_+ = \lim_{t=+\infty} \ell(t) \in \operatorname{End}(M),$$

called respectively the *starting end* and *terminating end* of ℓ. The *type* of ℓ is the pair of ends (ℓ_-, ℓ_+) of M. We denote by $\operatorname{Div}(M)$ the set with multiplicities of orbits under the geodesic flow of the divergent geodesics endowed with their multiplicities. For every $\ell \in \operatorname{Div}(M)$, the Lebesgue measure Leb_ℓ on $\Gamma\backslash\mathscr{G}X$ as defined in Section 2 is locally finite, since $\ell : \mathbb{R} \to M$ is a proper map.

Let $\ell \in \Gamma\backslash\mathscr{G}X$ be a divergent geodesic, and choose $\widetilde{\ell} \in \mathscr{G}X$ such that $\ell = \Gamma\widetilde{\ell}$. Let $t_- = t_-(\widetilde{\ell})$ be the time at which $\widetilde{\ell}$ exits $H_{\widetilde{\ell}_-}$, and $t_+ = t_+(\widetilde{\ell})$ be the time at which $\widetilde{\ell}$ enters $H_{\widetilde{\ell}_+}$. The *complexity of ℓ with respect to the family* $(H_\xi)_{\xi\in\operatorname{Par}_\Gamma}$ is

$$\tau(\ell) = t_+ - t_- \,, \tag{10}$$

which is nonnegative since the horoballs in $(H_\xi)_{\xi\in\operatorname{Par}_\Gamma}$ have pairwise disjoint interiors. It is independent of the choice of $\widetilde{\ell}$ and of the action of the geodesic flow on ℓ, hence we will again denote by τ the induced map from $\operatorname{Div}(M)$ to $[0, +\infty[$. This corresponds to the definition in the introduction when X is a manifold.

Note that the Lebesgue measure Leb_ℓ of a divergent geodesic is a locally finite measure on $\Gamma\backslash\mathscr{G}X$, with support the geodesic flow orbit of ℓ, but it is an infinite measure.

The *compact core* of a divergent geodesic ℓ in M is the locally geodesic segment in M denoted by $\alpha_\ell : [0, \tau(\ell)] \to M$ such that $\alpha_\ell(t) = \ell(t_- + t)$ for every $t \in [0, \tau(\ell)]$. We identify it with its extension to a generalized geodesic in $\Gamma\backslash\breve{\mathscr{G}}X$ which is constant on $]-\infty, 0]$ and $[\tau(\ell), +\infty[$. Note that for every $t \in \mathbb{R}$ when X is a manifold and $t \in \mathbb{Z}$ when X is a tree, we have $\alpha_{\mathtt{g}^t\ell} = \alpha_\ell$.

For all $A \in [0, +\infty[$ and $\xi \in \operatorname{Par}_\Gamma$, let $H_\xi[A]$ be the horoball contained in the horoball H_ξ, whose boundary $\partial H_\xi[A]$ is at distance A from the boundary ∂H_ξ of H_ξ. Let $\overset{\circ}{H}_\xi[A]$ be the interior of $H_\xi[A]$. The *A-thick part of M with respect to the family* $(H_\xi)_{\xi\in\operatorname{Par}_\Gamma}$ is the orbispace

$$M^{\leqslant A} = \Gamma\backslash\Big(X \smallsetminus \bigcup_{\xi\in\operatorname{Par}_\Gamma} \overset{\circ}{H}_\xi[A]\Big) \,.$$

As A tends to $+\infty$, the injectivity radius at a point $x \in M \smallsetminus M^{\leqslant A}$ tends to 0 when X is a manifold, and the order of the stabilizer of $x \in M \smallsetminus M^{\leqslant A}$ tends to $+\infty$ when X is a tree. Since Γ is geometrically finite with finite covolume, the A-thick part $M^{\leqslant A}$ is compact of diameter $\operatorname{diam} M^{\leqslant A} \leqslant 2A + \operatorname{diam} M^{\leqslant 0}$. For every compact subset K of M, there exists $A \in [0, +\infty[$ such that K is contained in $M^{\leqslant A}$. Furthermore, any geodesic ray $\rho : [0, +\infty[\to X$ in X from a point $\rho(0)$ in ∂H_ξ to the point at infinity $\lim_{t\to+\infty} \rho(t) = \xi$ meets $\partial H_\xi[A]$ exactly at the point $\rho(A)$.

The following result relates asymptotically the Lebesgue measure of a divergent geodesic to the one of its compact core.

Lemma 3.1. *For every $A \in [0, +\infty[$, for every $f \in C_c(\Gamma\backslash\breve{\mathscr{G}}X)$ with support contained in $\pi^{-1}(M^{\leqslant A})$, for every divergent geodesic $\ell \in \Gamma\backslash\mathscr{G}X$, we have*

$$\big|\ \operatorname{Leb}_\ell(f) - \operatorname{Leb}_{\alpha_\ell}(f)\ \big| \leqslant 2\,A\,\|f\|_\infty \,.$$

Proof. Let A, f, ℓ be as in the statement. Since the above inequality is invariant under the action of the geodesic flow, we may assume that $t_-(\ell) = 0$. By the definitions of the geodesic flow and of the footpoint map, we have $\pi(\mathtt{g}^t\ell) = \ell(t)$, for every $t \in \mathbb{R}$ when X is a manifold and $t \in \mathbb{Z}$ when X is a tree. Hence when $t \leqslant 0$, we have $d(\pi(\mathtt{g}^t\ell), \ell(0)) = |t|$, since for every $\xi \in \mathrm{Par}_\Gamma$, any geodesic ray from the boundary of H_ξ to its point at infinity injects isometrically in $M = \Gamma\backslash X$. Therefore, if $t \leqslant -A$, we have $\pi(\mathsf{g}^t\ell) \notin M^{\leqslant A}$ and in particular $f(\mathsf{g}^t\ell) = 0$ by the assumption on the support of f. Similarly, we have $f(\mathsf{g}^t\ell) = 0$ if $t \geqslant t_+(\ell) + A$.

Assume first that X is a manifold. By the definition of the measures Leb_ℓ and $\mathrm{Leb}_{\alpha_\ell}$, by the definition of the complexity $\tau(\ell) = t_+(\ell)$, we thus have

$$\begin{aligned}\Big|\ \mathrm{Leb}_\ell(f) - \mathrm{Leb}_{\alpha_\ell}(f)\ \Big| &= \Big|\ \int_{-\infty}^{+\infty} f(\mathsf{g}^t\ell)\,dt - \int_0^{\tau(\ell)} f(\mathsf{g}^t\ell)\,dt\ \Big| \\ &= \Big|\ \int_{-A}^{t_+(\ell)+A} f(\mathsf{g}^t\ell)\,dt - \int_0^{t_+(\ell)} f(\mathsf{g}^t\ell)\,dt\ \Big| \\ &= \Big|\ \int_{-A}^{0} f(\mathsf{g}^t\ell)\,dt + \int_{t_+(\ell)}^{t_+(\ell)+A} f(\mathsf{g}^t\ell)\,dt\ \Big| \leqslant 2\,A\,\|f\|_\infty\ .\end{aligned}$$

Though we won't use it when X is a tree, the same proof works, up to replacing A by $\lfloor A \rfloor$, which does not change $\pi^{-1}(M^{\leqslant A})$, and $\int_{t=a}^{b}$ by $\sum_{t=a}^{b}$ for $a, b \in \mathbb{Z} \cup \{\pm\infty\}$. □

4 Divergent geodesics and common perpendiculars

In this section, we give an explicit description of the correspondence between the divergent geodesics in M and the common perpendiculars between Margulis cusp neighborhoods or cuspidal rays in M.

Let $\mathscr{T} \subset \mathrm{End}(M)^2$ be a nonempty set of types of divergent geodesics, and let

$$\mathrm{Div}^+_{\mathscr{T}}(M) = \{\ell \in \mathrm{Div}(M) : (\ell_-, \ell_+) \in \mathscr{T}\ ,\ \tau(\ell) > 0\}$$

Note that the set $\mathrm{Div}^+_{\mathscr{T}}(M)$ differs from the set $\mathrm{Div}_{\mathscr{T}}(M)$ defined in Equation (1) (when X is a manifold, but the definition is valid when X is a tree) only by a finite subset, hence has the same asymptotic distribution property.

Let $e_-, e_+ \in \mathrm{End}(M)$. Since Γ is geometrically finite (by the compactness of $M^{\leqslant 0}$), the cardinality of the set F_{e_-,e_+} of double classes $[\![\gamma]\!] \in \Gamma_{\widehat{e_-}}\backslash\Gamma/\Gamma_{\widehat{e_+}}$, such that $H_{\widehat{e_-}} \cap \gamma H_{\widehat{e_+}}$ is nonempty, is finite. Let $[\![\gamma]\!] \in \Gamma_{\widehat{e_-}}\backslash\Gamma/\Gamma_{\widehat{e_+}} \smallsetminus F_{e_-,e_+}$. In particular, we have $\widehat{e_-} \neq \gamma\,\widehat{e_+}$. The *multiplicity* of $[\![\gamma]\!]$ is defined (independently of the choice of the representative γ of $[\![\gamma]\!]$) by

$$m_{[\![\gamma]\!]} = \frac{1}{\mathrm{Card}(\Gamma_{\widehat{e_-}} \cap \gamma\Gamma_{\widehat{e_+}}\gamma^{-1})}\ . \tag{11}$$

Let $\widetilde{\ell}_{[\![\gamma]\!]}$ be the unique geodesic line in $\mathscr{G}X$ such that $(\widetilde{\ell}_{[\![\gamma]\!]})_- = \widehat{e_-}$, $(\widetilde{\ell}_{[\![\gamma]\!]})_+ = \gamma\,\widehat{e_+}$ and $\widetilde{\ell}_{[\![\gamma]\!]}(0) \in \partial H_{\widehat{e_-}}$. This last condition uniquely defines $\widetilde{\ell}_{[\![\gamma]\!]}$ in its orbit under the geodesic flow. Then $\ell_{[\![\gamma]\!]} = \Gamma\widetilde{\ell}_{[\![\gamma]\!]}$ is a divergent geodesic in M, that does not depend on the choice of the representative γ of $[\![\gamma]\!]$.

We refer to [ParP2] and [BrPP] for background on common perpendiculars between properly immersed locally convex closed subsets of M, especially to [BrPP] when X is a tree. The compact core $\alpha_{\ell_{[\![\gamma]\!]}}$ of $\ell_{[\![\gamma]\!]}$ is a common perpendicular between $\mathscr{V}_{e_-}$ and $\mathscr{V}_{e_+}$, with positive length $\lambda(\alpha_{\ell_{[\![\gamma]\!]}})$. Furthermore, any common perpendicular between two Margulis cusp neighborhoods when X is a manifold, or two cuspidal rays when X is a tree, arises this way: Indeed, there is a unique way to extend a common perpendicular between two disjoint horoballs in X to a geodesic line in X whose points at infinity are the points at infinity of the two horoballs. Let

$$\operatorname{Perp}_{\mathscr{T}}(M) = \bigsqcup_{(e_-,e_+)\in\mathscr{T}} \left(\Gamma_{\widehat{e_-}}\backslash\Gamma/\Gamma_{\widehat{e_+}} \smallsetminus F_{e_-,e_+}\right) .$$

Lemma 4.1. *Assume that X is a manifold. The mapping from $\operatorname{Perp}_{\mathscr{T}}(M)$ to $\operatorname{Div}^+_{\mathscr{T}}(M)$ induced by $[\![\gamma]\!] \mapsto \ell_{[\![\gamma]\!]}$ is a bijection between sets with multiplicities. For every $[\![\gamma]\!] \in \operatorname{Perp}_{\mathscr{T}}(M)$, we have*

$$m(\ell_{[\![\gamma]\!]}) = m_{[\![\gamma]\!]} \qquad \textit{and} \qquad \tau(\ell_{[\![\gamma]\!]}) = \lambda(\alpha_{\ell_{[\![\gamma]\!]}}) . \tag{12}$$

Proof. For every $(e_-, e_+) \in \mathscr{T}$, for every divergent geodesic ℓ in $M = \Gamma\backslash X$ with $\ell_\pm = e_\pm$ and $\tau(\ell) > 0$, let $\widetilde{\ell} \in \mathscr{G}X$ be such that $\ell = \Gamma\widetilde{\ell}$. We may assume that $\widetilde{\ell}_- = \widehat{e_-}$ up to the action of an element of Γ on $\widetilde{\ell}$ unique modulo left multiplication by an element of $\Gamma_{\widehat{e_-}}$, and that $\widetilde{\ell}(0) \in \partial H_{\widehat{e_-}}$ up to the action of a unique element of the geodesic flow on $\widetilde{\ell}$. Then there exists $\gamma_\ell \in \Gamma$ such that $\widetilde{\ell}_+ = \gamma_\ell\, \widehat{e_+}$, and γ_ℓ is unique modulo multiplication on the right by an element of $\Gamma_{\widehat{e_+}}$ and independent of the action of the geodesic flow on $\widetilde{\ell}$. Since $\tau(\ell) > 0$, the horoballs $H_{\widehat{e_-}}$ and $\gamma_\ell H_{\widehat{e_+}}$ are disjoint. Hence the double class $[\![\gamma_\ell]\!] \in \Gamma_{\widehat{e_-}}\backslash\Gamma/\Gamma_{\widehat{e_+}}$ is well defined and does not belong to F_{e_-,e_+}.

It is clear by construction that the maps $[\![\gamma]\!] \mapsto \ell_{[\![\gamma]\!]}$ and $\ell \mapsto [\![\gamma_\ell]\!]$ induce maps from $\operatorname{Perp}_{\mathscr{T}}(M)$ to $\operatorname{Div}^+_{\mathscr{T}}(M)$ and from $\operatorname{Div}^+_{\mathscr{T}}(M)$ to $\operatorname{Perp}_{\mathscr{T}}(M)$ that are inverse one of the other. Since X is a manifold (hence nontrivial geodesic segments have a unique extension to a geodesic line), the stabilizer in Γ of the common perpendicular between two disjoint horoballs in X is equal to the stabilizer in Γ of the geodesic line between the points at infinity of the two horoballs. The equalities (12) (the first one being the definition of a bijection between sets with multiplicities) are then immediate by the definitions of the two multiplicities (9) and (11), of the complexity of a divergent geodesic (10) and of the length of a generalized geodesic (6). $\square$

For every $T > 0$, let us consider the counting function (with multiplicities) of the common perpendiculars with length at most T, between two Margulis cusp neighborhoods when X is a manifold, or two cuspidal rays when X is a tree, whose pair of ends belong to $\mathscr{T}$, defined by (using the definition of sums over sets with multiplicities for the second equality below)

$$\mathscr{N}_{\mathscr{T}}(T) = \sum_{\substack{(e_-,e_+)\in\mathscr{T} \\ [\![\gamma]\!]\in\Gamma_{\widehat{e}_-}\backslash\Gamma/\Gamma_{\widehat{e}_+}\smallsetminus F_{e_-,e_+} \\ \lambda(\alpha_{\ell_{[\![\gamma]\!]}})\leqslant T}} m_{[\![\gamma]\!]} = \operatorname{Card}\big\{[\![\gamma]\!] \in \operatorname{Perp}_{\mathscr{T}}(M) : \lambda(\alpha_{\ell_{[\![\gamma]\!]}}) \leqslant T\big\} .$$

For every $(e_-, e_+) \in \mathscr{T}$, we denote respectively by $\sigma^+_{e_-} = \sigma^+_{\mathscr{V}_{e_-}}$ and $\sigma^-_{e_+} = \sigma^-_{\mathscr{V}_{e_+}}$ the outer skinning measure of $\mathscr{V}_{e_-} = \Gamma H_{\widehat{e_-}}$ and inner skinning measure of $\mathscr{V}_{e_+} = \Gamma H_{\widehat{e_+}}$ in $\Gamma\backslash\check{\mathscr{G}}X$, defined at the end of Section 2. They are locally finite measures on $\Gamma\backslash\check{\mathscr{G}}X$ with support contained in $\partial^1_+\mathscr{V}_{e_-}$ and $\partial^1_-\mathscr{V}_{e_+}$ respectively. Since Γ_ξ acts cocompactly on ∂H_ξ for every $\xi \in \operatorname{Par}_\Gamma$ (hence $\partial^1_\pm\mathscr{V}_e$ is compact for every $e \in \operatorname{End}(M)$), and since the limit set of Γ is equal to the whole $\partial_\infty X$, the skinning measures $\sigma^\pm_{e^\mp}$ are finite and nonzero, with support exactly $\partial^1_\pm\mathscr{V}_{e^\mp}$.

Assume till the end of Section 4 that X is a manifold and that the Bowen-Margulis measure is finite and mixing under the geodesic flow, which is the case under the assumptions of Theorem 1.1. By the second claim of [ParP2, Coro. 12] applied with $\Omega^\pm = (\partial^1_\pm H_{\gamma\widehat{e_\mp}})_{\gamma\in\Gamma/\Gamma_{\widehat{e_\mp}}}$ for every $(e_-,e_+)\in\mathscr{T}$, and by a finite summation on $(e_-,e_+)\in\mathscr{T}$, we have as $T\to+\infty$

$$\mathscr{N}_{\mathscr{T}}(T)\sim\sum_{(e_-,e_+)\in\mathscr{T}}\frac{\|\sigma^+_{e_-}\|\,\|\sigma^-_{e_+}\|}{\delta_\Gamma\,\|m_{\rm BM}\|}\,e^{\delta_\Gamma T}\,. \tag{13}$$

Furthermore, if M is locally symmetric with exponential decay of correlation, then by the second claim of [ParP2, Theo. 15 (2)] (whose assumptions are indeed satisfied, since ∂H_ξ is smooth for every $\xi\in{\rm Par}_\Gamma$ and M has finite volume), there exists $\tau>0$ and an additive error term in this counting statement of the form ${\rm O}\left(e^{(\delta_\Gamma-\tau)T}\right)$.

By Equation (2), we have

$$\|\sigma_{\mathscr{T}}\|=\sum_{(e_-,e_+)\in\mathscr{T}}\|\sigma^-_{e_-}\|\,\|\sigma^+_{e_+}\|\,. \tag{14}$$

Claim (2) of Theorem 1.1 follows from Equation (4), from Lemma 4.1, and from Equation (13) with its error term.

5 Equidistribution of divergent geodesics

The aim of this section is to prove Claim (1) of Theorem 1.1. We hence assume that (X,Γ) is as in Section 2 with X a manifold. The main tool is the following theorem proved in [ParP4, Theo. 1] applied with $A^\pm=\mathscr{V}_{e_\pm}$, so that the skinning measures $\sigma^\pm_{e_\mp}=\sigma^\pm_{\mathscr{V}_{e_\mp}}$ are finite and nonzero. Recall from the introduction that if the Bowen-Margulis measure $m_{\rm BM}$ is finite, then it is mixing for the geodesic flow on T^1M, as needed in loc. cit..

Theorem 5.1. *If $m_{\rm BM}$ is finite, then for all $(e_-,e_+)\in\mathscr{T}$, as $T\to+\infty$, we have*

$$\frac{\delta_\Gamma\,\|m_{\rm BM}\|}{T\,e^{\delta_\Gamma T}}\sum_{[\![\gamma]\!]\in{\rm Perp}_{\{(e_-,e_-)\}}(M)\,:\,\lambda(\alpha_{\ell_{[\![\gamma]\!]}})\leqslant T}{\rm Leb}_{\alpha_{\ell_{[\![\gamma]\!]}}}\;\overset{*}{\rightharpoonup}\;\|\sigma^+_{e_-}\|\,\|\sigma^-_{e_+}\|\,\frac{m_{\rm BM}}{\|m_{\rm BM}\|}\,.$$

Furthermore, if M is locally symmetric with exponential decay of correlation, then there exists $k\in\mathbb{N}$ such that for every compact subset K in T^1M and every C^k-smooth function $\Psi:T^1M\to\mathbb{R}$ with support in K, with $\|\;\|_{k,2}$ the $W^{k,2}$-Sobolev norm, we have

$$\frac{\delta_\Gamma\,\|m_{\rm BM}\|}{T\,e^{\delta_\Gamma T}}\sum_{[\![\gamma]\!]\in{\rm Perp}_{\{(e_-,e_-)\}}(M)\,:\,\lambda(\alpha_{\ell_{[\![\gamma]\!]}})\leqslant T}{\rm Leb}_{\alpha_{\ell_{[\![\gamma]\!]}}}(\psi)=\|\sigma^+_{e_-}\|\,\|\sigma^-_{e_+}\|\,\frac{m_{\rm BM}(\psi)}{\|m_{\rm BM}\|}+{\rm O}_K\left(\frac{\|\psi\|_{k,2}}{T}\right).$$

Hence by a finite summation on $(e_-,e_+)\in\mathscr{T}$, by Lemma 4.1 and by Equation (14), for every $f\in C_c(T^1M)$, we have

$$\lim_{T\to+\infty}\frac{\delta_\Gamma\,\|m_{\rm BM}\|}{T\,e^{\delta_\Gamma T}}\sum_{\ell\in{\rm Div}^+_{\mathscr{T}}\,:\,\tau(\ell)\leqslant T}{\rm Leb}_{\alpha_\ell}(f)=\|\sigma_{\mathscr{T}}\|\,\frac{m_{\rm BM}(f)}{\|m_{\rm BM}\|}\,. \tag{15}$$

Furthermore, since $\pi^{-1}(M^{\leqslant A})$ is compact, with k as in Theorem 5.1, there is an error term $\mathrm{O}_A\big(\frac{\|f\|_{k,2}}{T}\big)$ when $f \in C^k(T^1M)$ has support in $\pi^{-1}(M^{\leqslant A})$ and when M is locally symmetric with exponential decay of correlations.

Proof of Theorem 1.1 (1). Let $f \in C_c(T^1M)$ be a test function. Using sums over sets with multiplicities, by Equation (4), by Lemma 3.1, by Equations (13) and (14), and finally by Equation (15), we have

$$\begin{aligned}
&\frac{h_M\,\|m_{\rm BM}\|}{T\,e^{\delta_\Gamma T}}\sum_{\ell\in{\rm Div}_{\mathscr T}:\ \tau(\ell)\leqslant T}{\rm Leb}_\ell(f)\\
&=\frac{\delta_\Gamma\,\|m_{\rm BM}\|}{T\,e^{\delta_\Gamma T}}\sum_{\ell\in{\rm Div}_{\mathscr T}:\ \tau(\ell)\leqslant T}\big({\rm Leb}_{\alpha_\ell}(f)+\mathrm{O}(A\|f\|_\infty)\big)\\
&=\frac{\delta_\Gamma\,\|m_{\rm BM}\|}{T\,e^{\delta_\Gamma T}}\Big(\sum_{\ell\in{\rm Div}_{\mathscr T}:\ \tau(\ell)\leqslant T}{\rm Leb}_{\alpha_\ell}(f)\Big)+\mathrm{O}\Big(\frac{A\|f\|_\infty}{T}\Big) \qquad (16)\\
&\underset{T\to+\infty}{\longrightarrow}\ \|\sigma_{\mathscr T}\|\ \frac{m_{\rm BM}(f)}{\|m_{\rm BM}\|}\,.
\end{aligned}$$

This proves the convergence claim in Theorem 1.1 (1).

Now assume till the end of this proof that M is locally symmetric with exponential decay of correlations. Let $k \in \mathbb{N}$ be given by the error term in Equation (15). Let us fix $f \in C_c^k(T^1M)$ and $A \geqslant 0$ such that the support of f is contained in $\pi^{-1}(M^{\leqslant A})$. Then by Equation (16) and by the error term in Equation (15), we have

$$\begin{aligned}
&\frac{h_M\,\|m_{\rm BM}\|}{T\,e^{h_M T}}\sum_{\ell\in{\rm Div}_{\mathscr T}:\ \tau(\ell)\leqslant T}{\rm Leb}_\ell(f)\\
&=\|\sigma_{\mathscr T}\|\ \frac{m_{\rm BM}(f)}{\|m_{\rm BM}\|}+\mathrm{O}_A\Big(\frac{\|f\|_{k,2}}{T}\Big)+\mathrm{O}\Big(\frac{A\|f\|_\infty}{T}\Big)\\
&=\|\sigma_{\mathscr T}\|\ \frac{m_{\rm BM}(f)}{\|m_{\rm BM}\|}+\mathrm{O}_A\Big(\frac{\|f\|_{k,\infty}}{T}\Big)\,.
\end{aligned}$$

This ends the proof of Theorem 1.1 (1). □

It follows from the arithmeticity results of Margulis and the results of [LP], [KlM1], [KlM2] and [Clo] discussed in the introduction that the only case where the exponential decay of correlations is not known is when X is a complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$ for $n \geqslant 2$ and M is not arithmetic.

6 Equidistribution of divergent geodesics in geometrically finite tree quotients

We assume in this section that X is the geometric realisation of a uniform simplicial tree $\mathbb{X}$ without vertices of degree 1 or 2 and that Γ is a finite covolume geometrically finite nonelementary discrete subgroup of $\mathrm{Isom}(X)$. Recall from Section 2 the notation $M = \Gamma\backslash X$ with its quotient graph of groups structure, and the notation $m_{\rm BM}$ for the Bowen-Margulis measure on $\Gamma\backslash\mathscr{G}X$. Let $\mathscr{T}$ be a nonempty subset of $\mathrm{End}(M)^2$. We refer for instance to [BrPP, §2.6] for background on trees and their (discrete-time) geodesic flow, and to [BrPP, §4.4] for background on the Bowen-Margulis measures of their discrete groups of automorphisms.

The *length spectrum* $\mathbf{L}_\Gamma$ of Γ is the subgroup of $\mathbb{Z}$ generated by the translation lengths in X of the elements of Γ. It is equal to $\mathbb{Z}$ or $2\mathbb{Z}$ under the above assumptions on $(\mathbb{X},\Gamma)$ (see [BrPP, Lem. 4.18], using the fact that since Γ has finite covolume besides being geometrically finite, the minimal nonempty Γ-invariant subtree of $\mathbb{X}$ is equal to $\mathbb{X}$). For instance, when Γ is the Nagao lattice $\mathrm{PGL}_2(\mathbb{F}_q[Y])$ acting on the Bruhat-Tits tree of $(\mathrm{PGL}_2, \mathbb{F}_q((Y^{-1})))$ (see [BaL, §10.2], [BrPP, §15.2]), then $\mathbf{L}_\Gamma = 2\mathbb{Z}$ by [Ser, §II.1.2, Cor.] or [BrPP, page 331]. See [Ser, §II.2.3] for lots of other examples with length spectrum $2\mathbb{Z}$.

We define $\delta'_\Gamma = 1 - e^{-\delta_\Gamma}$, that we will use when $\mathbf{L}_\Gamma = \mathbb{Z}$, and $\delta''_\Gamma = 1 - e^{-2\delta_\Gamma}$, that we will use when $\mathbf{L}_\Gamma = 2\mathbb{Z}$.

We choose a fixed vertex x_* of $\mathbb{X}$ belonging to the horosphere that bounds one of the horoballs of the family $(H_\xi)_{\xi\in\mathrm{Par}_\Gamma}$. Let $V_{\mathrm{even}}\mathbb{X}$ be the set of vertices of $\mathbb{X}$ at even distance from x_* and

$$\mathscr{G}_{\mathrm{even}}X = \{\ell \in \mathscr{G}X : \ell(0) \in V_{\mathrm{even}}\mathbb{X}\} \quad \text{and} \quad \breve{\mathscr{G}}_{\mathrm{even}}\, X = \{\ell \in \breve{\mathscr{G}}X : \ell(0) \in V_{\mathrm{even}}\mathbb{X}\}\,.$$

Note that $\mathscr{G}_{\mathrm{even}}X$ and $\breve{\mathscr{G}}_{\mathrm{even}}\, X$ are clopen subsets of $\mathscr{G}X$ and $\breve{\mathscr{G}}X$, respectively.

When $\mathbf{L}_\Gamma = 2\mathbb{Z}$, then the following properties hold.

• The group Γ preserves $V_{\mathrm{even}}\mathbb{X}$ and $\mathscr{G}_{\mathrm{even}}X$, as well as their complementary subsets in $V\mathbb{X}$ and $\mathscr{G}X$.

• We choose, as we may by the previous point, the Γ-equivariant family $(H_\xi)_{\xi\in\mathrm{Par}_\Gamma}$ so that the distance between two elements of this family is even.

• We denote by $m_{\mathrm{BM,\,even}}$ the restriction of the Bowen-Margulis measure m_{BM} of Γ to $\Gamma\backslash\mathscr{G}_{\mathrm{even}}X$. Since the time-one map g^1 of the geodesic flow exchanges $\mathscr{G}_{\mathrm{even}}X$ and its complementary subset, we have $\|m_{\mathrm{BM,\,even}}\| = \frac{\|m_{\mathrm{BM}}\|}{2}$.

• We define $\mathrm{Div}^+_{\mathscr{T},\,\mathrm{even}}(M)$ as the set with multiplicities (given by Equation (9)) of the orbits under the even-time geodesic flow $(\mathsf{g}^{2t})_{t\in\mathbb{Z}}$ of the divergent geodesics $\underline{\ell} \in \Gamma\backslash\mathscr{G}_{\mathrm{even}}X$ such that $(\underline{\ell}_-, \underline{\ell}_+) \in \mathscr{T}$ and $\tau(\underline{\ell}) > 0$.

• For every $\ell \in \mathrm{Div}^+_{\mathscr{T},\,\mathrm{even}}(M)$ and $\underline{\ell} \in \Gamma\backslash\mathscr{G}_{\mathrm{even}}X$ a representative of ℓ, we denote by $\mathrm{Leb}_{\ell,\,\mathrm{even}}$ the pushforward measure on $\Gamma\backslash\mathscr{G}X$ by the map $t \mapsto \mathsf{g}^{2t}\underline{\ell}$ of the counting measure on $\mathbb{Z}$. This measure has support contained in $\Gamma\backslash\mathscr{G}_{\mathrm{even}}X$. It does not depend on the choice of the representative $\underline{\ell}$ of ℓ, by the invariance under translation of the counting measure on $\mathbb{Z}$.

We fix β and ε in $]0,1]$. Given a metric space (Y,d), we denote by $C^\beta_c(Y)$ the normed real vector space of (uniformly locally) β-Hölder-continuous functions with compact support on Y, with norm $f \mapsto \|f\|_\beta = \|f\|_\infty + \|f\|'_\beta$, using the convention

$$\|f\|'_\beta = \sup_{0<d(x,y)\leqslant 1} \frac{|f(x)-f(y)|}{d(x,y)^\beta} \tag{17}$$

of for instance [BrPP, §3.1]. We refer for instance to the beginning of Chapter 9 in [BrPP] for the definition of the exponential decay of correlation of m_{BM} or $m_{\mathrm{BM,\,even}}$ for the β-Hölder regularity of the geodesic flow $(\mathsf{g}^t)_{t\in\mathbb{Z}}$ or $(\mathsf{g}^{2t})_{t\in\mathbb{Z}}$.

Since $\Gamma\backslash\mathscr{G}X$ is a totally disconnected metric space, another regularity turns out to be useful, as for instance in [AGP, KPS]. A function $f : \Gamma\backslash\mathscr{G}X \to \mathbb{R}$ is ε-*locally constant* if f is constant on every open ball of radius ε. We denote by $C_c^{\varepsilon\,\mathrm{lc},\,\beta}(\Gamma\backslash\mathscr{G}X)$ the normed real vector space of compactly supported ε-locally constant fonctions f, with norm $\|f\|_{\varepsilon\,\mathrm{lc},\,\beta} = \epsilon^{-\beta}\|f\|_\infty$. By [BrPP, Rem. 3.11], the space $C_c^{\varepsilon\,\mathrm{lc},\,\beta}(\Gamma\backslash\mathscr{G}X)$ is continuously included in $C_c^{\beta}(\Gamma\backslash\mathscr{G}X)$ thanks to the inequality

$$\|\cdot\|_\beta \leqslant 3\,\|\cdot\|_{\varepsilon\,\mathrm{lc},\,\beta}\ . \tag{18}$$

This allows us to state an error term in our theorem, and the proof will juggle between the norm for general β-Hölder-continuous functions and the norm $\|\cdot\|_{\varepsilon\,\mathrm{lc},\,\beta}$.

In what follows, N varies in $\mathbb{N}$.

Theorem 6.1. *Let $X, \Gamma, \mathscr{T}$ be as in the beginning of Section 6, and let $\beta, \varepsilon \in \,]0,1]$.*
(1) Assume that $\mathbf{L}_\Gamma = \mathbb{Z}$. As $N \to +\infty$, for the weak-star convergence of measures on the locally compact space $\Gamma\backslash\mathscr{G}X$, we have

$$\frac{\delta'_\Gamma\ \|m_{\rm BM}\|}{\|\sigma_{\mathscr{T}}\|\ N\ e^{\delta_\Gamma N}}\sum_{\ell\in\mathrm{Div}^+_{\mathscr{T}}(M)\,:\,\tau(\ell)\leqslant N}\mathrm{Leb}_\ell\ \ \overset{*}{\rightharpoonup}\ \ \frac{m_{\rm BM}}{\|m_{\rm BM}\|}\,, \tag{19}$$

with an additive error term of the form

$$\mathrm{O}\Big(\frac{(A+1)\,\|\Phi\|_{\varepsilon\,\mathrm{lc},\,\beta}}{N}\Big)$$

in this equidistribution statement, when evaluated on $\Phi \in C_c^{\varepsilon\,\mathrm{lc},\,\beta}(\Gamma\backslash\mathscr{G}X)$, where $A \geqslant 0$ is such that the support of Φ is contained in $\pi^{-1}(X^{\leqslant A})$.
(2) Assume that $\mathbf{L}_\Gamma = 2\mathbb{Z}$. As $N \to +\infty$, for the weak-star convergence of measures on the locally compact space $\Gamma\backslash\mathscr{G}_{\rm even}X$, we have

$$\frac{\delta''_\Gamma\ \|m_{\rm BM,\,even}\|}{\|\sigma_{\mathscr{T}}\|\ N\ e^{\delta_\Gamma N}}\sum_{\ell\in\mathrm{Div}^+_{\mathscr{T},\,\rm even}(M)\,:\,\tau(\ell)\leqslant 2N}\mathrm{Leb}_{\ell,\,\rm even}\ \ \overset{*}{\rightharpoonup}\ \ \frac{m_{\rm BM,\,even}}{\|m_{\rm BM,\,even}\|}\,,$$

with an additive error term of the form

$$\mathrm{O}\Big(\frac{(A+1)\,\|\Phi\|_{\varepsilon\,\mathrm{lc},\,\beta}}{N}\Big)$$

in this equidistribution statement, when evaluated on $\Phi \in C_c^{\varepsilon\,\mathrm{lc},\,\beta}(\Gamma\backslash\mathscr{G}_{\rm even}X)$, where $A \geqslant 0$ is such that the support of Φ is contained in $\pi^{-1}(X^{\leqslant A})$.

In the coming proof of Theorem 6.1, we will introduce auxiliary points $x_0 \in V\mathbb{X}$, with $x_0 \in V_{\rm even}\mathbb{X}$ when $L_\Gamma = 2\mathbb{Z}$. We will then use the following joint equidistribution result, extracted from [BrPP] as we explain after its statement. Though expressed in the universal cover X, it says that the images in $M = \Gamma\backslash X$ of the initial and terminal generalized geodesics associated with the common perpendiculars between $\mathscr{V}_{e_\pm}$ and Γx_0 jointly equidistribute towards their skinning measures. We first introduce the necessary notation. For every vertex $x \in V\mathbb{X}$ and every parabolic fixed point $\xi \in \mathrm{Par}_\Gamma$ such that $x \notin H_\xi$, we denote by $\widetilde{\alpha}_{\xi,\,x}$ the shortest geodesic segment in $\check{\mathscr{G}}X$ between H_ξ and x, starting at time 0 from H_ξ and arriving at time $d(H_\xi,\,x)$ at x, seen as an element of $\check{\mathscr{G}}X$ being stationary at time $\leqslant 0$ and at time $\geqslant d(H_\xi,\,x)$. Note that $\mathsf{g}^{d(H_\xi,\,x)}\,\widetilde{\alpha}_{\xi,\,x}$ is then the parametrization of the common perpendicular between H_ξ and $\{x\}$ arriving at time 0 at x.

Theorem 6.2. *Let $e_0 \in \operatorname{End}(M)$ and $x_0 \in V\mathbb{X}$.*
(1) Assume that $\mathbf{L}_\Gamma = \mathbb{Z}$. As $N \to +\infty$, for the weak-star convergence of measures on the locally compact space $\breve{\mathscr{G}}X \times \breve{\mathscr{G}}X$, we have

$$\delta'_\Gamma \, e^{-\delta_\Gamma N} \|m_{\rm BM}\| \sum_{\gamma \in \Gamma \,:\, 0 < d(H_{\gamma\,\widehat{e_0}},\, x_0) \leqslant N} \Delta_{\widetilde{\alpha}_{\widehat{e_0},\,\gamma^{-1}x_0}} \otimes \Delta_{\mathsf{g}^{d(H_{\gamma\,\widehat{e_0}},\,x_0)}\widetilde{\alpha}_{\gamma\,\widehat{e_0},\,x_0}} \;\overset{*}{\rightharpoonup}\; \widetilde{\sigma}^+_{H_{\widehat{e_0}}} \otimes \widetilde{\sigma}^-_{\{x_0\}}\,.$$

Furthermore, there exists $\kappa > 0$ such that this convergence has an additive error term, when evaluated on the product function $(w, w') \mapsto (\phi^-(w), \phi^+(w'))$ of any two Hölder-continuous functions $\phi^\pm \in C_c^\beta(\breve{\mathscr{G}}X)$, of the form $\operatorname{O}\big(e^{-\kappa N}\|\phi^-\|_\beta \|\phi^+\|_\beta\big)$.

(2) Assume that $\mathbf{L}_\Gamma = 2\mathbb{Z}$ and that $x_0 \in V_{\rm even}\mathbb{X}$. As $N \to +\infty$, for the weak-star convergence of measures on the locally compact space $\breve{\mathscr{G}}X \times \breve{\mathscr{G}}X$, we have

$$\delta''_\Gamma \, e^{-2\,\delta_\Gamma N} \|m_{\rm BM,\,even}\| \sum_{\gamma \in \Gamma \,:\, 0 < d(H_{\gamma\,\widehat{e_0}},\, x_0) \leqslant 2N} \Delta_{\widetilde{\alpha}_{\widehat{e_0},\,\gamma^{-1}x_0}} \otimes \Delta_{\mathsf{g}^{d(H_{\gamma\,\widehat{e_0}},\,x_0)}\widetilde{\alpha}_{\gamma\,\widehat{e_0},\,x_0}} \;\overset{*}{\rightharpoonup}\; \widetilde{\sigma}^+_{H_{\widehat{e_0}}} \otimes \widetilde{\sigma}^-_{\{x_0\}}\,.$$

Furthermore, there exists $\kappa > 0$ such that this convergence has an additive error term, when evaluated on the product function $(w, w') \mapsto (\phi^-(w), \phi^+(w'))$ of any two Hölder-continuous functions $\phi^\pm \in C_c^\beta(\breve{\mathscr{G}}X)$, of the form $\operatorname{O}\big(e^{-\kappa N}\|\phi^-\|_\beta \|\phi^+\|_\beta\big)$.

Proof. Let us explain more precisely where to find these results in the book [BrPP]. Since $\mathbb{X}$ is a uniform tree and Γ has finite covolume, the Bowen-Margulis measure $m_{\rm BM}$ is finite by [BrPP, Prop. 4.16 (3)]. Since Γ has finite covolume, the minimal nonempty Γ-invariant subtree of $\mathbb{X}$ is $\mathbb{X}$, hence is uniform without vertices of degree 2. We recall the following facts in the two cases of the statement.

(1) When $L_\Gamma = \mathbb{Z}$, the Bowen-Margulis measure $m_{\rm BM}$ is mixing under the geodesic flow $(\mathsf{g}^t)_{t\in\mathbb{Z}}$ by [BrPP, Prop. 4.17] with system of conductances $\widetilde{c} = 0$. The convergence part of Assertion (1) then follows from [BrPP, Theo. 11.9] with system of conductances $\widetilde{c} = 0$, applied with $I^- = \Gamma/\Gamma_{H_{\widehat{e_0}}}$, $I^+ = \Gamma/\Gamma_{x_0}$, $\mathscr{D}^- = (\gamma^- H_{\widehat{e_0}})_{\gamma^- \in I^-}$ and $\mathscr{D}^+ = (\gamma^+\{x_0\})_{\gamma^+ \in \Gamma/\Gamma_{x_0}}$. More precisely, it follows from the discrete-time zero-potential version of Equation (11.1) in [BrPP] whose summation gives [BrPP, Theo. 11.9], applied with i and j the trivial classes in $\Gamma/\Gamma_{H_{\widehat{e_0}}}$ and Γ/Γ_{x_0}, and with a change of variable $\gamma \mapsto \gamma^{-1}$, since $\alpha^-_{i,\,\gamma^{-1}j} = \widetilde{\alpha}_{\widehat{e_0},\,\gamma^{-1}x_0}$ and $\alpha^+_{\gamma i,\,j} = \mathsf{g}^{d(H_{\gamma\,\widehat{e_0}},x_0)}\widetilde{\alpha}_{\gamma\,\widehat{e_0},\,x_0}$ for every $\gamma \in \Gamma$.

The error term part of Assertion (1) follows from [BrPP, Theo. 12.16] with system of conductances $\widetilde{c} = 0$ applied with $\mathbb{D}^-$ the subtree whose geometric realisation is $H_{\widehat{e_0}}$ and $\mathbb{D}^+ = \{x_0\}$, for the following reasons.

- Its hypothesis (1) is satisfied, since $\widehat{e_0}$ is a bounded parabolic fixed point, hence $\Gamma_{H_{\widehat{e_0}}}$ acts cocompactly on $\partial H_{\widehat{e_0}}$, and Γ_{x_0} acts (clearly !) cocompactly on $\{x_0\}$.
- These compactness properties imply that the skinning measures $\sigma^+_{\mathscr{V}_{e_0}}$ and $\sigma^-_{\Gamma\{x_0\}}$ are finite.
- Its hypothesis (2) on the exponential decay of correlations is satisfied by [BrPP, Coro. 9.6 (1)], since Γ is geometrically finite.

(2) When $L_\Gamma = 2\mathbb{Z}$, the restriction $m_{\rm BM,\,even}$ of the Bowen-Margulis measure $m_{\rm BM}$ of Γ to $\Gamma\backslash\mathscr{G}_{\rm even}X$ is mixing under the even-time geodesic flow $(\mathsf{g}^{2t})_{t\in\mathbb{Z}}$ by [BrPP, Prop. 4.17] with system of conductances $\widetilde{c} = 0$. Since the basepoint x_* belongs to the boundary of one of the horoballs of the family $(H_\xi)_{\xi\in{\rm Par}_\Gamma}$, since the distance between two elements of the

family $(H_\xi)_{\xi\in\operatorname{Par}_\Gamma}$ is even, and since the distance of two points of any given horosphere in a simplicial tree is even, for every $\xi \in \operatorname{Par}_\Gamma$, we have $\partial H_\xi \subset V_{\rm even}\mathbb{X}$. In particular, since $x_0 \in V_{\rm even}\mathbb{X}$, for every $\xi \in \operatorname{Par}_\Gamma$ such that $x_0 \notin H_\xi$, the common perpendicular between H_ξ and $\{x_0\}$ has both endpoints in $V_{\rm even}\mathbb{X}$. Furthermore, the skinning measures $\sigma^+_{\mathscr{V}_{e_0}}$ and $\sigma^-_{\Gamma\{x_0\}}$ have support contained in $\Gamma\backslash\breve{\mathscr{G}}_{\rm even}\,X$.

The convergence part of Assertion (2) then follows from [BrPP, Eq. (11.28)] (rather a preliminary version of it without summation, and with a change of variables $\gamma\mapsto\gamma^{-1}$, as above) with system of conductances $\widetilde{c}=0$, applied with $I^-=\Gamma/\Gamma_{H_{\widehat{e_0}}}$, $I^+=\Gamma/\Gamma_{x_0}$, $\mathscr{D}^-=(\gamma^- H_{\widehat{e_0}})_{\gamma^-\in I^-}$ and $\mathscr{D}^+=(\gamma^+\{x_0\})_{\gamma^+\in\Gamma/\Gamma_{x_0}}$, using the definition of δ''_Γ and the fact that $\|m_{\rm BM,\,even}\|=\frac{\|m_{\rm BM}\|}{2}$.

The error term part of Assertion (2) follows from [BrPP, Remark (ii) page 281] with system of conductances $\widetilde{c}=0$ applied with $\mathbb{D}^-$ the subtree whose geometric realisation is $H_{\widehat{e_0}}$ and $\mathbb{D}^+=\{x_0\}$. □

Symmetrically, for all $x\in V\mathbb{X}$ and $\xi\in\operatorname{Par}_\Gamma$ such that $x\notin H_\xi$, we define a generalized geodesic $\widetilde{\alpha}_{x,\,\xi}: t\mapsto\widetilde{\alpha}_{\xi,\,x}(d(H_\xi,\,x)-t)$ with origin $\widetilde{\alpha}_{x,\,\xi}(0)=x$. With the assumptions of Theorem 6.2 (1), as $N\to+\infty$, we similarly have the following weak-star convergence of measures, with the same error term,

$$\delta'_\Gamma\; e^{-\delta_\Gamma N}\|m_{\rm BM}\| \sum_{\gamma\in\Gamma\,:\,0<d(H_{\gamma\widehat{e_0}},\,x_0)\leqslant N} \Delta_{\widetilde{\alpha}_{x_0,\,\gamma\widehat{e_0}}}\otimes\Delta_{\mathtt{g}^{d(H_{\gamma\widehat{e_0}},\,x_0)}\widetilde{\alpha}_{\gamma^{-1}x_0,\,\widehat{e_0}}} \;\overset{*}{\rightharpoonup}\; \widetilde{\sigma}^+_{\{x_0\}}\otimes\widetilde{\sigma}^-_{H_{\widehat{e_0}}}\,. \tag{20}$$

With the assumptions of Theorem 6.2 (2), as $N\to+\infty$, we similarly have, with the same error term,

$$\delta''_\Gamma\; e^{-2\,\delta_\Gamma N}\|m_{\rm BM,\,even}\| \sum_{\gamma\in\Gamma\,:\,0<d(H_{\gamma\widehat{e_0}},\,x_0)\leqslant 2N} \Delta_{\widetilde{\alpha}_{x_0,\,\gamma\widehat{e_0}}}\otimes\Delta_{\mathtt{g}^{d(H_{\gamma\widehat{e_0}},\,x_0)}\widetilde{\alpha}_{\gamma^{-1}x_0,\,\widehat{e_0}}} \;\overset{*}{\rightharpoonup}\; \widetilde{\sigma}^+_{\{x_0\}}\otimes\widetilde{\sigma}^-_{H_{\widehat{e_0}}}\,. \tag{21}$$

Our proof of Theorem 6.1 using Theorem 6.2 is motivated by the proof of Theorem 5.1 in [ParP4, Theo. 1] for good Riemannian orbifolds. We will adapt each step of the main convergence claim therein to the present tree case. Some reduction steps simplify here compared to [ParP4], since we are not considering general convex subsets, but only horoballs. On the other hand, our computation of the error terms in Theorem 6.1 will be much more involved than the one in [ParP4, Theo. 1], leading to a stronger result.

Proof of Theorem 6.1. The Bowen-Margulis measure on $\Gamma\backslash\mathscr{G}X$ is finite by for instance [BrPP, Prop. 4.16 (3)]. Since $\operatorname{End}(M)$ is a finite set, we can assume that $\mathscr{T}$ contains only one element (e_-,e_+), then sum the convergence results and the error terms over a general subset $\mathscr{T}$ of the finite set $\operatorname{End}(M)^2$.

(1) Assume that $L_\Gamma=\mathbb{Z}$.

Step 1. Let us prove that if the convergence claim (19) is true when evaluated on $\Phi\in C_c(\Gamma\backslash\mathscr{G}X)$ with support in $\pi^{-1}(M^{\leqslant 0})$, with an error term $\operatorname{O}\big(\frac{\|\Phi\|_{\varepsilon\,{\rm lc},\,\beta}}{N}\big)$ when besides $\Phi\in C_c^{\varepsilon\,{\rm lc},\,\beta}(\Gamma\backslash\mathscr{G}X)$, then Assertion (1) of Theorem 6.1 is true.

Let $A\in\mathbb{N}$, and let $\Phi\in C_c(\Gamma\backslash\mathscr{G}X)$ be such that the support of Φ is contained in the preimage $\pi^{-1}(M^{\leqslant A})$ of the A-thick part of M with respect to the family $(H_\xi)_{\xi\in\operatorname{Par}_\Gamma}$. Let us denote by $\tau_A:\operatorname{Div}(M)\to[0,+\infty[$ the complexity function of the divergent geodesics in M now with respect to the family $(H_\xi[A])_{\xi\in \mathrm{Par}_\Gamma}$. Note that $\tau_A = \tau + 2A$, and that Φ has support in the preimage by π of the 0-thick part of M with respect to the family $(H_\xi[A])_{\xi\in \mathrm{Par}_\Gamma}$.

Let $\mathscr{V}_{e_\pm}[A] = \Gamma H_{\widehat{e_\pm}}[A]$ be the cuspidal ray in M with point at infinity $e_\pm$ with respect to the family $(H_\xi[A])_{\xi\in \mathrm{Par}_\Gamma}$. Let $\sigma^\pm_{e_\mp,\,A} = \sigma^\pm_{\mathscr{V}_{e_\mp}[A]}$ be the outer/inner skinning measure of $\mathscr{V}_{e_\mp}[A]$ in $\Gamma\backslash\check{\mathscr{G}}X$. Let $\sigma_{\mathscr{T},\,A} = \sigma^-_{e_-,\,A}\otimes\sigma^+_{e_+,\,A}$. By [ParP1, Prop. 4 (iii)], since $\mathscr{V}_{e_\pm}$ is the closed A-neighborhood of $\mathscr{V}_{e_\pm}[A]$, we have $\|\sigma^\pm_{e_\mp,\,A}\| = e^{-\delta_\Gamma A}\,\|\sigma^\pm_{e_\mp}\|$. Hence since $\mathscr{T} = \{(e_-,e_+)\}$ and by Equation (2), we have $\|\sigma_{\mathscr{T},\,A}\| = e^{-2\,\delta_\Gamma A}\,\|\sigma_{\mathscr{T}}\|$.

Therefore, under the convergence assumption of Step 1, we have

$$\begin{aligned}
&\frac{\delta'_\Gamma\,\|m_{\rm BM}\|\,e^{-\delta_\Gamma N}}{N\,\|\sigma_{\mathscr{T}}\|}\sum_{\ell\in \mathrm{Div}^+_{\mathscr{T}}(M)\,:\,\tau(\ell)\leqslant N}\mathrm{Leb}_\ell(\Phi)\\
&=\frac{\|\sigma_{\mathscr{T},\,A}\|}{e^{-2\,\delta_\Gamma A}\,\|\sigma_{\mathscr{T}}\|}\,\frac{N+2A}{N}\;\frac{\delta'_\Gamma\,\|m_{\rm BM}\|\,e^{-\delta_\Gamma (N+2A)}}{(N+2A)\,\|\sigma_{\mathscr{T},\,A}\|}\sum_{\ell\in \mathrm{Div}^+_{\mathscr{T}}(M)\,:\,\tau_A(\ell)\leqslant N+2A}\mathrm{Leb}_\ell(\Phi)\\
&\underset{N\to+\infty}{\longrightarrow}\frac{m_{\rm BM}(\Phi)}{\|m_{\rm BM}\|}\,.
\end{aligned}$$

If futhermore $\Phi\in C_c^{\mathrm{lc},\,\beta}(\Gamma\backslash\mathscr{G}X)$, then under the error term assumption of Step 1

$$\begin{aligned}
&\frac{\delta'_\Gamma\,\|m_{\rm BM}\|\,e^{-\delta_\Gamma N}}{N\,\|\sigma_{\mathscr{T}}\|}\sum_{\ell\in \mathrm{Div}^+_{\mathscr{T}}(M)\,:\,\tau(\ell)\leqslant N}\mathrm{Leb}_\ell(\Phi)\\
&=\frac{N+2A}{N}\Big(\frac{m_{\rm BM}(\Phi)}{\|m_{\rm BM}\|}+\mathrm{O}\big(\frac{\|\Phi\|_{\varepsilon\,\mathrm{lc},\,\beta}}{N}\big)\Big)=\frac{m_{\rm BM}(\Phi)}{\|m_{\rm BM}\|}+\mathrm{O}\Big(\frac{(A+1)\,\|\Phi\|_{\varepsilon\,\mathrm{lc},\,\beta}}{N}\Big)\,.
\end{aligned}$$

This concludes the proof of Step 1.

Step 2. Let us now prove the convergence part (19) of Assertion (1) of Theorem 6.1.

Let $N\in\mathbb{N}$. Let $x_0\in V\mathbb{X}\smallsetminus\bigcup_{\xi\in \mathrm{Par}_\Gamma}\overset{\circ}{H}_\xi$, so that its image by the canonical projection $X\to M=\Gamma\backslash X$ belongs to $M^{\leqslant 0}$. We will allow x_0 to vary everywhere in $V\mathbb{X}\smallsetminus\bigcup_{\xi\in \mathrm{Par}_\Gamma}\overset{\circ}{H}_\xi$ at the end of the proof of Step 2.

Using Hopf's parametrisation with respect to x_0 (see Section 2), we define a measure on $\mathscr{G}X$ by

$$\nu_N=\sum_{\gamma^\pm\in\Gamma/\Gamma_{\widehat{e_\pm}}\,:\,0<d(H_{\gamma^-\widehat{e_-}},\,H_{\gamma^+\widehat{e_+}})\leqslant N}\Delta_{\gamma^-\widehat{e_-}}\otimes\Delta_{\gamma^+\widehat{e_+}}\otimes ds\,.$$

This measure is independent of x_0, hence is Γ-invariant, by the invariance of the counting measure ds on $\mathbb{Z}$ under translations. It is locally finite by the local finiteness of the family $(H_\xi)_{\xi\in \mathrm{Par}_\Gamma}$. Since $\mathscr{T}=\{(e_-,e_+)\}$, by the definitions (10) of the complexity τ and (9) of the multiplicities, the induced measure (see for instance [PauPS, §2.6] for ramified cover issues) of ν_N on $\Gamma\backslash\mathscr{G}X$ is equal to $\sum_{\ell\in \mathrm{Div}_{\mathscr{T}}\,:\,\tau(\ell)\leqslant N}\mathrm{Leb}_\ell$ (with the sum convention over sets with multiplicities).

Let $\psi\in C_c(\mathscr{G}X)$ be such that

- ψ is nonnegative,
- ψ has separate variables when considered, via Hopf's parametrisation with respect to x_0, as defined on a subset of the product space $\partial_\infty X\times\partial_\infty X\times\mathbb{Z}$: that is, there exist

two nonnegative continuous functions $\psi^{\pm} : \partial_\infty X \to \mathbb{R}$ and a nonnegative bounded function $\psi^0 : \mathbb{Z} \to \mathbb{R}$ with finite support such that

$$\forall\ (\xi_-, \xi_+, s) \in \partial^2_\infty X \times \mathbb{Z}, \quad \psi(\xi_-, \xi_+, s) = \psi^-(\xi_-)\ \psi^+(\xi_+)\ \psi^0(s)\,, \tag{22}$$

• ψ has support in the compact-open subset $\pi^{-1}(\{x_0\}) = \{\ell \in \mathscr{G}X\ :\ \ell(0) = x_0\}$ of $\mathscr{G}X$. In particular, by the definition of Hopf's parametrisation with respect to x_0, the map ψ^0 is a multiple of the characteristic function $\mathbb{1}_{\{0\}}$ of the singleton $\{0\}$. Since we will, at the end of the proof of Step 2, take linear combinations of such functions ψ, we actually assume that $\psi^0 = \mathbb{1}_{\{0\}}$.

For every $k \in \mathbb{N} \smallsetminus \{0\}$, let

$$\mathscr{A}_k = \big\{(\gamma^-, \gamma^+) \in \Gamma/\Gamma_{\widehat{e_-}} \times \Gamma/\Gamma_{\widehat{e_+}}\ :\ d(H_{\gamma^-\widehat{e_-}}, H_{\gamma^+\widehat{e_+}}) = k\big\}$$

and

$$S_k = \sum_{(\gamma^-,\,\gamma^+)\in\mathscr{A}_k} \psi^-(\gamma^-\widehat{e_-})\ \psi^+(\gamma^+\widehat{e_+})\,. \tag{23}$$

We have

$$\nu_N(\psi) = \sum_{k=1}^{N} S_k\,. \tag{24}$$

Let us subdivide the index set of the sum defining S_k using x_0 as an intermediate point: For all $i, j \in \mathbb{N}$, we define

$$\mathscr{A}^-_i = \{\gamma^- \in \Gamma/\Gamma_{\widehat{e_-}} : d(H_{\gamma^-\widehat{e_-}},\, x_0) = i\}\,,$$
$$\mathscr{A}^+_j = \{\gamma^+ \in \Gamma/\Gamma_{\widehat{e_+}} : d(x_0,\, H_{\gamma^+\widehat{e_+}}) = j\}\,.$$

Thanks to our condition on the support of ψ, if an index (γ^-, γ^+) contributes to the sum defining $\nu_N(\psi)$, then x_0 belongs to the geodesic line $]\gamma^-\widehat{e_-},\, \gamma^+\widehat{e_+}[$ with points at infinity $\gamma^-\widehat{e_-}$ and $\gamma^+\widehat{e_+}$. Since $x_0 \notin \overset{\circ}{H}_{\gamma^-\widehat{e_-}} \cup \overset{\circ}{H}_{\gamma^+\widehat{e_+}}$, the vertex x_0 then belongs to the common perpendicular $[H_{\gamma^-\widehat{e_-}}, H_{\gamma^+\widehat{e_+}}]$ between $H_{\gamma^-\widehat{e_-}}$ and $H_{\gamma^+\widehat{e_+}}$. For such an index (γ^-, γ^+), we then have for every $k \in \mathbb{N} \smallsetminus \{0\}$ the equivalence

$$\big(\exists\ i \in [\![0, k]\!], \quad \gamma^- \in \mathscr{A}^-_i \text{ and } \gamma^+ \in \mathscr{A}^+_{k-i}\big) \iff (\gamma^-, \gamma^+) \in \mathscr{A}_k\,, \tag{25}$$

where $[\![0, k]\!] = [0, k] \cap \mathbb{Z}$. The convergence part of Assertion (1) of Theorem 6.2 applied with $e_0 = e_-$ and an integration over the first factor when projected to $\Gamma\backslash\breve{\mathscr{G}}X$ give, as $N \to \infty$, the following weak-star convergence of measures on $\breve{\mathscr{G}}X$

$$\frac{\delta'_\Gamma\ e^{-\delta_\Gamma N}\ \|m_{\rm BM}\|}{\|\sigma^+_{e_-}\|} \sum_{\gamma^-\in\bigcup_{i=1}^N \mathscr{A}^-_i} \Delta_{\mathsf{g}^{d(H_{\gamma^-\widehat{e_-}},\,x_0)}\widetilde{\alpha}_{\gamma^-\widehat{e_-},\,x_0}} \ \overset{*}{\rightharpoonup}\ \widetilde{\sigma}^-_{\{x_0\}}. \tag{26}$$

Let $\rho_{\gamma^-\widehat{e_-},x_0} :]-\infty, 0] \to X$ be the negative geodesic ray starting from the point at infinity $\gamma^-\widehat{e_-}$ and ending at x_0 at time 0, seen as a generalized geodesic that is constant on $[0, +\infty[$. It belongs to the inner normal bundle $\partial^1_-\{x_0\}$ of the convex subset $\{x_0\}$. It coincides with $\mathsf{g}^{d(H_{\gamma^-\widehat{e_-}},\,x_0)}\widetilde{\alpha}_{\gamma^-\widehat{e_-},\,x_0}$ on $[-d(H_{\gamma^-\widehat{e_-}},\, x_0), +\infty[$. Hence, by Equation (5), we have

$$d(\rho_{\gamma^-\widehat{e_-},x_0},\ \mathtt{g}^{d(H_{\gamma^-\widehat{e_-}},x_0)}\widetilde{\alpha}_{\gamma^-\widehat{e_-},x_0}) = \int_{-\infty}^{-d(H_{\gamma^-\widehat{e_-}},x_0)} 2\big|t + d(H_{\gamma^-\widehat{e_-}},\ x_0)\big|\, e^{2t}\, dt$$

$$= e^{-2\,d(H_{\gamma^-\widehat{e_-}},x_0)} \int_0^{+\infty} 2u\, e^{-2u}\, du = \frac{1}{2}\, e^{-2\,d(H_{\gamma^-\widehat{e_-}},x_0)} \leqslant 1\,. \tag{27}$$

In particular, the distance in $\breve{\mathscr{G}}X$ between $\rho_{\gamma^-\widehat{e_-},x_0}$ and $\mathtt{g}^{d(H_{\gamma^-\widehat{e_-}},x_0)}\widetilde{\alpha}_{\gamma^-\widehat{e_-},x_0}$ tends to 0 uniformly in γ^- as $d(H_{\gamma^-\widehat{e_-}},\ x_0)$ tends to $+\infty$. Therefore Equation (26) gives, as $N \to +\infty$, the following weak-star convergence of measures on $\partial^1_-\{x_0\}$

$$\frac{\delta'_\Gamma\ e^{-\delta_\Gamma N}\ \|m_{\rm BM}\|}{\|\sigma^+_{e_-}\|} \sum_{\gamma^-\in\,\bigcup_{i=1}^N \mathscr{A}_i^-} \Delta_{\rho_{\gamma^-\widehat{e_-},x_0}} \ \overset{*}{\rightharpoonup}\ \ \widetilde{\sigma}^-_{\{x_0\}}\,. \tag{28}$$

The negative endpoint map $\widetilde{\ell} \mapsto \widetilde{\ell}_-$ from $\breve{\mathscr{G}}X$ to $X \cup \partial_\infty X$ restricts to a homeomorphism $p^- : \partial^1_-\{x_0\} \to \partial_\infty X$. Note that by Equation (8) applied with $x_* = x_0$, we have $(p^-)_*\widetilde{\sigma}^-_{\{x_0\}} = \mu_{\{x_0\}}$. Taking the pushforward of Equation (28) by p^- and evaluating it on ψ^-, we hence obtain

$$\lim_{N\to\infty} \frac{\delta'_\Gamma\ e^{-\delta_\Gamma N}\ \|m_{\rm BM}\|}{\|\sigma^+_{e_-}\|} \sum_{\gamma^-\in\,\bigcup_{i=1}^N \mathscr{A}_i^-} \psi^-(\gamma^-\widehat{e_-}) = \mu_{x_0}(\psi^-)\,. \tag{29}$$

For every $i \in \mathbb{N}$, we define

$$a_i = \sum_{\gamma^-\in\,\mathscr{A}_i^-} \psi^-(\gamma^-\widehat{e_-})\,. \tag{30}$$

Let $\eta > 0$. By Equation (29) and since the sets $\mathscr{A}_i^-$ for $i \in \mathbb{N}$ are pairwise disjoint, there exists $i_0 = i_0(\eta) \in \mathbb{N}\smallsetminus\{0\}$ such that if $s \in \mathbb{N}$ satisfies $s \geqslant i_0$, then

$$\frac{\|\sigma^+_{e_-}\|(\mu_{x_0}(\psi^-) - \eta)}{\delta'_\Gamma\ \|m_{\rm BM}\|}\ e^{\delta_\Gamma s} \leqslant \sum_{i=1}^s a_i \leqslant \frac{\|\sigma^+_{e_-}\|(\mu_{x_0}(\psi^-) + \eta)}{\delta'_\Gamma\ \|m_{\rm BM}\|}\ e^{\delta_\Gamma s}\,. \tag{31}$$

By a similar argument, using Equation (20) with $e_0 = e_+$, taking the difference between the index values $j-1$ and j, integrating on the second factor when projected to $\Gamma\backslash\breve{\mathscr{G}}X$, approximating long geodesic segments with origin x_0 by positive geodesic rays starting from x_0, using the definition $\delta'_\Gamma = 1 - e^{-\delta_\Gamma}$, and pushing forward by the positive endpoint map $p^+ : \partial^1_+\{x_0\} \to \partial_\infty X$, we have

$$\lim_{j\to+\infty} \frac{\|m_{\rm BM}\|\ e^{-\delta_\Gamma j}}{\|\sigma^-_{e_+}\|} \sum_{\gamma^+\in\,\mathscr{A}_j^+} \psi^+(\gamma^+\widehat{e_+}) = \mu_{x_0}(\psi^+)\,. \tag{32}$$

Let us define the smooth functions $f^\pm_\eta : [0, +\infty[\ \to \mathbb{R}$ by

$$\forall\ s \in [0, +\infty[\,,\quad f^\pm_\eta(s) = \frac{\|\sigma^-_{e_+}\|\,(\mu_{x_0}(\psi^+) \pm \eta)}{\|m_{\rm BM}\|}\ e^{\delta_\Gamma(k-s)}\,.$$

Then, as in order to obtain Equation (31), there exists $j_0 = j_0(\eta) \in \mathbb{N} \smallsetminus \{0\}$ such that, for every $j \in \mathbb{N}$ with $j \geqslant j_0$, we have the inequalities

$$f_\eta^-(k-j) \leqslant \sum_{\gamma^+ \in \mathscr{A}_j^+} \psi^+(\gamma^+ \widehat{e_+}) \leqslant f_\eta^+(k-j)\,. \tag{33}$$

We set

$$C_{1,\eta} = \frac{\|\sigma_{e_-}^+\|\,(\mu_{x_0}(\psi^-)+\eta)}{\delta_\Gamma'\,\|m_{\rm BM}\|} \quad \text{and} \quad C_{2,\eta} = \frac{\|\sigma_{e_+}^-\|\,(\mu_{x_0}(\psi^+)+\eta)}{\|m_{\rm BM}\|}\,.$$

Fix $N \in \mathbb{N}$ with $N \geqslant \max\{i_0, j_0\}$. Let $k \in [\![0, N]\!]$. By decomposing the sum S_k defined in Equation (23) using Equation (25) and by using the definition (30), we obtain

$$S_k = \sum_{i=0}^{k} a_i \sum_{\gamma^+ \in \mathscr{A}_{k-i}^+} \psi^+(\gamma^+ \widehat{e_+})\,. \tag{34}$$

We can decompose the above sum $\sum_{i=0}^k$ into $\sum_{i=0}^{k-j_0}$, where we can use the upper bound in Equation (33), and the sum $\sum_{i=k-j_0+1}^k$ which is bounded by $\mathrm{O}_{j_0}(1) \times \mathrm{O}(e^{\delta_\Gamma k}) = \mathrm{O}_{j_0}(e^{\delta_\Gamma k})$ using Equations (29) and (32). Using Abel's summation formula, we obtain

$$\begin{aligned} S_k &\leqslant \sum_{i=0}^{k-j_0} a_i\, f_\eta^+(i) + \mathrm{O}_{j_0}(e^{\delta_\Gamma k}) \\ &= \Big(\sum_{i=0}^{k-j_0} a_i\Big)\, f_\eta^+(k-j_0) - \sum_{i=0}^{k-j_0-1} \Big(\sum_{l=0}^{i} a_l\Big)(f_\eta^+(i+1) - f_\eta^+(i)) + \mathrm{O}_{j_0}(e^{\delta_\Gamma k}) \\ &= \mathrm{O}(e^{\delta_\Gamma k})\,\mathrm{O}_{j_0}(1) - \sum_{i=0}^{k-j_0-1} \big(\sum_{l=0}^{i} a_l\big)(f_\eta^+(i+1) - f_\eta^+(i)) + \mathrm{O}_{j_0}(e^{\delta_\Gamma\, k})\,. \end{aligned} \tag{35}$$

Note that for every $i \in [\![0, k]\!]$, we have

$$\begin{aligned} f_\eta^+(i+1) - f_\eta^+(i) &= e^{\delta_\Gamma(k-i)}\, \frac{\|\sigma_{e_+}^-\|\,(\mu_{x_0}(\psi^+)+\eta)}{\|m_{\rm BM}\|}\,(e^{-\delta_\Gamma} - 1) \\ &= -\,\delta_\Gamma'\, C_{2,\eta}\, e^{\delta_\Gamma(k-i)} = \mathrm{O}(e^{\delta_\Gamma\, k})\,. \end{aligned}$$

Decomposing the sum $\sum_{i=0}^{k-j_0-1}$ in Equation (35) into on the one hand $\sum_{i=0}^{\min\{i_0-1, k-j_0-1\}}$, where we use the above estimate which is uniform in $i \in [\![0, i_0+1]\!]$, and on the other hand $\sum_{i=i_0}^{k-j_0-1}$ (which vanishes if $k < i_0 + j_0 - 1$), where we use the upper bound in Equation (31), we obtain the inequality

$$\begin{aligned} S_k &\leqslant \sum_{i=i_0}^{k-j_0-1} \Big(\sum_{l=0}^{i} a_l\Big)\, \delta_\Gamma'\, C_{2,\eta} e^{\delta_\Gamma(k-i)} + \mathrm{O}_{i_0,j_0}(e^{\delta_\Gamma k}) \\ &\leqslant \delta_\Gamma'\, C_{1,\eta}\, C_{2,\eta}\, k\, e^{\delta_\Gamma k} + \mathrm{O}_{i_0,j_0}(e^{\delta_\Gamma\, k})\,. \end{aligned}$$

By Equation (24) and a computation of the arithmetico-geometric sum (see for instance [ParP3, Lem. 2.1] for a generalization that is well-adapted for the current purpose), for

$N \in \mathbb{N}$ large enough, we hence have

$$\begin{aligned}\nu_N(\psi) &\leqslant \delta'_\Gamma\, C_{1,\eta}\, C_{2,\eta}\, \frac{N\, e^{\delta_\Gamma N}}{1-e^{-\delta_\Gamma}} + \mathrm{O}_{i_0,j_0}(e^{\delta_\Gamma N})\\ &= \frac{\|\sigma^+_{e_-}\|\,\|\sigma^-_{e_+}\|\,(\mu_{x_0}(\psi^-)+\eta)\,(\mu_{x_0}(\psi^+)+\eta)\,N\,e^{\delta_\Gamma N}}{\delta'_\Gamma\,\|m_{\rm BM}\|^2} + \mathrm{O}_{i_0,j_0}(e^{\delta_\Gamma N})\,. \end{aligned} \tag{36}$$

By our assumption on the support of ψ and by the definition (7) of the Bowen-Margulis measure $\widetilde{m}_{\rm BM}$ with $x_* = x_0$, we have

$$\widetilde{m}_{\rm BM}(\psi) = (\mu_{x_0}\otimes\mu_{x_0}\otimes ds)(\psi)\,.$$

Taking in Equation (36) the upper limit as $N\to+\infty$ then letting $\eta\to 0$, we hence obtain

$$\limsup_{N\to+\infty} \frac{\delta'_\Gamma\,\|m_{\rm BM}\|}{N\,e^{\delta_\Gamma N}\,\|\sigma^+_{e_-}\|\,\|\sigma^-_{e_+}\|}\,\nu_N(\psi) \leqslant \frac{1}{\|m_{\rm BM}\|}\,(\mu_{x_0}\otimes\mu_{x_0}\otimes ds)(\psi) = \frac{\widetilde{m}_{\rm BM}(\psi)}{\|m_{\rm BM}\|}\,.$$

A similar argument, this time using the lower bounds in Equation (31) and (33), gives the inequality

$$\liminf_{N\to+\infty} \frac{\delta'_\Gamma\,\|m_{\rm BM}\|}{N\,e^{\delta_\Gamma N}\,\|\sigma^+_{e_-}\|\,\|\sigma^-_{e_+}\|}\,\nu_N(\psi) \geqslant \frac{\widetilde{m}_{\rm BM}(\psi)}{\|m_{\rm BM}\|}\,.$$

Since $\mathscr{T} = \{(e_-,e_+)\}$ and by Equation (2), we have $\|\sigma^+_{e_-}\|\,\|\sigma^-_{e_+}\| = \|\sigma_{\mathscr{T}}\|$. Thus

$$\lim_{N\to+\infty} \frac{\delta'_\Gamma\,\|m_{\rm BM}\|}{N\,e^{\delta_\Gamma N}\,\|\sigma_{\mathscr{T}}\|}\,\nu_N(\psi) = \frac{\widetilde{m}_{\rm BM}(\psi)}{\|m_{\rm BM}\|}\,.$$

Let $p : X \to \Gamma\backslash X$ be the canonical projection. A standard argument of covering the support with sets $\pi^{-1}(\{x_0\})$ for finitely many $x_0 \in V\mathbb{X} \smallsetminus \bigcup_{\xi\in{\rm Par}_\Gamma} \overset{\circ}{H}_\xi$ and of uniform approximation by linear combinations of functions with separate variables gives us the weak-star convergence of measures on $(p\circ\pi)^{-1}(M^{\leqslant 0})$

$$\frac{\delta'_\Gamma\,\|m_{\rm BM}\|\,e^{-\delta_\Gamma N}}{N\,\|\sigma_{\mathscr{T}}\|}\,\nu_N \;\overset{*}{\rightharpoonup}\; \frac{\widetilde{m}_{\rm BM}}{\|m_{\rm BM}\|}\,. \tag{37}$$

Since the measure ν_N on $\mathscr{G}X$ induces the measure $\sum_{\ell\in{\rm Div}_{\mathscr{T}}\,:\,\tau(\ell)\leqslant N} {\rm Leb}_\ell$ on $\Gamma\backslash\mathscr{G}X$ and by the weak-star continuity of taking induced measures (see [PauPS, §2.6]), this concludes the proof of Step 2.

Step 3. Let us now prove the error term part of Assertion (1) of Theorem 6.1.

Let $\beta,\varepsilon \in\,]0,1]$, and let $x_0 \in V\mathbb{X} \smallsetminus \bigcup_{\xi\in{\rm Par}_\Gamma} \overset{\circ}{H}_\xi$. We fix ψ in $C_c^{\varepsilon\,{\rm lc},\,\beta}(\mathscr{G}X)$, where $\mathscr{G}X$ is endowed with the Bartels-Lück distance (5). We assume again that ψ has support contained in $\pi^{-1}(\{x_0\})$ and can be written as in Equation (22), again with $\psi^0 = \mathbb{1}_{\{x_0\}}$. We now prove that we have an error term of the form $\mathrm{O}(\frac{\|\psi\|_{\varepsilon\,{\rm lc},\,\beta}}{N})$ in the weak-star convergence (37) when evaluated on ψ. By the error term part of Step 1, and by a similar lifting and approximation process, this will conclude the proof of Step 3.

Compared to the proof of Step 2, only minor changes are needed. We keep the notation ν_N, $\mathscr{A}_k$, S_k, $\mathscr{A}_i^-$, $\mathscr{A}_j^+$ therein. We may assume that $\psi \neq 0$.

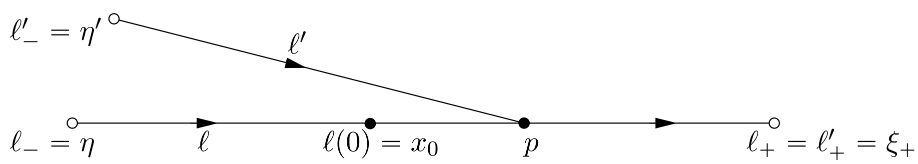
First, let us prove that the function $\psi^- : \partial_\infty X \to \mathbb{R}$ is ε-locally constant for the visual distance d_{x_0} defined in Equation (3) applied with $x_* = x_0$. By symmetry, the same can be done for ψ^+.

Since $\psi \neq 0$, let $(\xi_-, \xi_+) \in \partial^2_\infty X$ be such that $\psi^-(\xi_-)\psi^+(\xi_+) \neq 0$. Let $\eta \in \partial_\infty X$ and $\eta' \in \partial_\infty X$ be at visual distance for d_{x_0} less than ε one of each other. Let us prove that $\psi^-(\eta) = \psi^-(\eta')$, which gives that ψ^- is ε-locally constant for d_{x_0}. We may assume that $\eta \neq \eta'$. Since the functions $\psi^\pm$ are continuous and $\partial_\infty X$ has no isolated point, we may assume that $\eta \neq \xi_+$ and $\eta' \neq \xi_+$.

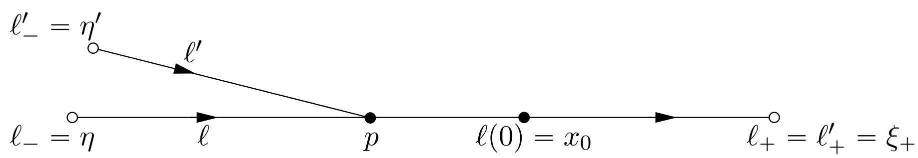
Using Hopf's parametrisation with respect to x_0, we consider the geodesic lines ℓ and ℓ' in $\mathscr{G}X$ with parameters $(\eta, \xi_+, 0)$ and $(\eta', \xi_+, 0)$ respectively. We may assume that one of these two geodesic lines, say ℓ, passes through x_0 (at time 0): Otherwise, our assumption on the support of ψ yields $\psi^-(\eta)\psi^+(\xi_+) = \psi^-(\eta')\psi^+(\xi_+) = 0$, hence $\psi^-(\eta) = \psi^-(\eta')$ since $\psi^+(\xi_+) \neq 0$.

Claim. Let us prove that we also have $\ell'(0) = x_0$.

Proof. Since $\ell_+ = \ell'_+ = \xi_+$, there exists $p \in V\mathbb{X}$ such that the intersection of the images of ℓ and ℓ' is the geodesic ray $[p, \xi_+[$ starting from p with point at infinity ξ_+.



Assume for a contradiction that $p \in [x_0, \xi_+[$, as in the above picture. Then x_0 belongs to the geodesic line $]\eta, \eta'[$. Therefore by Equation (3) applied with $x_* = x_0$, we have $d_{x_0}(\eta, \eta') = e^0 = 1 \geqslant \varepsilon$. This contradicts the fact that $d_{x_0}(\eta, \eta') < \varepsilon$.



Hence p belongs to the open geodesic ray $]\eta, x_0[$, as in the above picture. In particular, x_0 is the closest point to x_0 on the image of ℓ'. By the definition of Hopf's parametrisation with respect to x_0, since the third parameter of ℓ' is 0, we have $\ell'(0) = x_0$, as wanted. □

With p as in the above proof, let $T \in \mathbb{N}$ be such that $p = \ell(-T) = \ell'(-T)$, so that $d(x_0, \,]\eta, \eta'[) = T$. By Equation (5), we have

$$d(\ell, \ell') = \int_{-\infty}^{-T} 2\,|t+T|\, e^{-2\,|t|}\, dt \; = 2\, e^{-2\,T} \int_0^{+\infty} u\, e^{-2\,u}\, du = \frac{1}{2}\, e^{-2\,T}\,.$$

By Equation (3) applied with $x_* = x_0$, we have $d_{x_0}(\eta, \eta') = e^{-T}$. Hence

$$d(\ell, \ell') = \frac{1}{2}\, d_{x_0}(\eta, \eta')^2 < \frac{1}{2}\, \varepsilon^2 < \varepsilon\,.$$

Since ψ is ε-locally constant, this yields $\psi(\ell) = \psi(\ell')$. Dividing by $\psi_+(\xi_+) \neq 0$, we obtain $\psi^-(\eta) = \psi^-(\eta')$. Therefore ψ^- is ε-locally constant, as wanted.

As a consequence, since ψ has separate variables, we have

$$\|\psi^-\|_{\varepsilon\,\mathrm{lc},\,\frac{\beta}{2}}\;\|\psi^+\|_{\varepsilon\,\mathrm{lc},\,\frac{\beta}{2}} = \varepsilon^{-\frac{\beta}{2}}\|\psi^-\|_\infty\;\varepsilon^{-\frac{\beta}{2}}\|\psi^+\|_\infty = \varepsilon^{-\beta}\|\psi\|_\infty = \|\psi\|_{\varepsilon\,\mathrm{lc},\,\beta}\,. \tag{38}$$

Let $N\in\mathbb{N}$. By the error term in Theorem 6.2 (1) for the $\frac{\beta}{4}$-Hölder-continuity, there exists $\kappa>0$ such that for every function $\phi^-\in C_c^{\frac{\beta}{4}}(\mathscr{\check G}X)$, Equation (26) becomes

$$\frac{\delta'_\Gamma\; e^{-\delta_\Gamma N}\|m_{\rm BM}\|}{\|\sigma^+_{e_-}\|}\sum_{\gamma^-\in\bigcup_{i=1}^N\mathscr{A}_i^-}\phi^-(\mathtt{g}^{d(H_{\gamma^-\widehat{e_-}},\,x_0)}\widetilde{\alpha}_{\gamma^-\widehat{e_-},\,x_0}) = \widetilde{\sigma}^-_{\{x_0\}}(\phi^-)+\mathrm{O}\left(e^{-\kappa N}\|\phi^-\|_{\frac{\beta}{4}}\right).$$

With $\rho_{\gamma^-\widehat{e_-},\,x_0}$ as defined after Equation (26), since ϕ^- is $\frac{\beta}{4}$-Hölder-continuous and by Equations (17) and (27), for all $i\in\mathbb{N}$ and $\gamma^-\in\mathscr{A}_i^-$, we have

$$\begin{aligned}&\Big|\,\phi^-(\rho_{\gamma^-\widehat{e_-},\,x_0})-\phi^-(\mathtt{g}^{d(H_{\gamma^-\widehat{e_-}},\,x_0)}\widetilde{\alpha}_{\gamma^-\widehat{e_-},\,x_0})\,\Big|\\ \leqslant\;& d\big(\rho_{\gamma^-\widehat{e_-},\,x_0},\;\mathtt{g}^{d(H_{\gamma^-\widehat{e_-}},\,x_0)}\widetilde{\alpha}_{\gamma^-\widehat{e_-},\,x_0}\big)^{\frac{\beta}{4}}\;\|\phi^-\|'_{\frac{\beta}{4}}\\ \leqslant\;& e^{-\frac{\beta}{2}\,d(H_{\gamma^-\widehat{e_-}},\,x_0)}\;\|\phi^-\|'_{\frac{\beta}{4}} = e^{-\frac{\beta}{2}\,i}\;\|\phi^-\|_{\frac{\beta}{4}}\,.\end{aligned}$$

Therefore by a geometric series argument and since $\mathrm{Card}\big(\bigcup_{i=1}^N\mathscr{A}_i^-\big)=\mathrm{O}(e^{\delta_\Gamma N})$, with $\kappa'=\min\{\kappa,\frac{\beta}{2}\}$, for every $\phi^-\in C^{\frac{\beta}{4}}(\partial^1_-\{x_0\})$, Equation (28) becomes

$$\frac{\delta'_\Gamma\; e^{-\delta_\Gamma N}\|m_{\rm BM}\|}{\|\sigma^+_{e_-}\|}\sum_{\gamma^-\in\bigcup_{i=1}^N\mathscr{A}_i^-}\phi^-(\rho_{\gamma^-\widehat{e_-},\,x_0}) = \widetilde{\sigma}^-_{\{x_0\}}(\phi^-)+\mathrm{O}\left(e^{-\kappa' N}\|\phi^-\|_{\frac{\beta}{4}}\right).$$

The negative endpoint map $p^-:\partial^1_-\{x_0\}\to\partial_\infty X$ is $\frac{1}{2}$-Hölder-continuous by a direct adaptation to negative geodesics rays ending at x_0 of the proof of [BrPP, Lem. 3.4 (4)]. Hence if $\phi^-=\psi^-\circ p^-$, then by Equation (17), we have $\|\phi^-\|'_{\frac{\beta}{4}}\leqslant\|\psi^-\|'_{\frac{\beta}{2}}\,(\|p^-\|'_{\frac{1}{2}})^{\frac{\beta}{2}}$ and hence $\|\phi^-\|_{\frac{\beta}{4}}=\mathrm{O}(\|\psi^-\|_{\frac{\beta}{2}})$. Applying the above centered formula to this ϕ^-, Equation (31) thus becomes

$$\forall\; s\in\mathbb{N},\qquad \sum_{i=1}^s a_i = \frac{\|\sigma^+_{e_-}\|\;\mu_{x_0}(\psi^-)}{\delta'_\Gamma\;\|m_{\rm BM}\|}e^{\delta_\Gamma s}+\mathrm{O}(e^{(\delta_\Gamma-\kappa')s}\;\|\psi^-\|_{\frac{\beta}{2}})\,. \tag{39}$$

And similarly, Equation (33) is replaced by

$$\forall\; j\in\mathbb{N},\qquad \sum_{\gamma^+\in\mathscr{A}_j^+}\psi^+(\gamma^+\widehat{e_+}) = \frac{\|\sigma^-_{e_+}\|\;\mu_{x_0}(\psi^+)}{\|m_{\rm BM}\|}\;e^{\delta_\Gamma j}+\mathrm{O}(e^{(\delta_\Gamma-\kappa')j}\;\|\psi^+\|_{\frac{\beta}{2}})\,. \tag{40}$$

Let us define

$$C_- = \frac{\|\sigma^+_{e_-}\|\;\mu_{x_0}(\psi^-)}{\delta'_\Gamma\;\|m_{\rm BM}\|}\quad\text{ and }\quad C_+ = \frac{\|\sigma^-_{e_+}\|\;\mu_{x_0}(\psi^+)}{\|m_{\rm BM}\|}\,.$$

Using Equation (18), we have

$$\mu_{x_0}(\psi^\pm)\leqslant\|\mu_{x_0}\|\;\|\psi^\pm\|_\infty\leqslant\|\mu_{x_0}\|\;\|\psi^\pm\|_{\frac{\beta}{2}} = \mathrm{O}(\|\psi^\pm\|_{\varepsilon\,\mathrm{lc},\,\frac{\beta}{2}})\,.$$

In particular, we have $C_\pm=\mathrm{O}(\|\psi^\pm\|_{\varepsilon\,\mathrm{lc},\,\frac{\beta}{2}})$.

Let $k \in \mathbb{N} \smallsetminus \{0\}$. For every $i \in [\![0, k]\!]$, let $b_i = \sum_{\gamma^+ \in \mathscr{A}^+_{k-i}} \psi^+(\gamma^+ \widehat{e_+})$. Note that we have $b_k = \mathrm{O}(\|\psi^+\|_\infty) = \mathrm{O}(\|\psi^+\|_{\varepsilon\,\mathrm{lc},\,\frac{\beta}{2}})$. By Equation (40), we have

$$b_{i+1} - b_i = \frac{\|\sigma^-_{e_+}\|\;\mu_{x_0}(\psi^+)}{\|m_{\mathrm{BM}}\|}\,(e^{-\delta_\Gamma} - 1)\,e^{\delta_\Gamma(k-i)} + \mathrm{O}(e^{(\delta_\Gamma-\kappa')(k-i)}\;\|\psi^+\|_{\frac{\beta}{2}})\,. \tag{41}$$

Equation (34), Abel's summation formula, Equations (39) and (41), and the convergence of the series $\sum_{l\geqslant 0} e^{-\kappa' l}$ yield

$$\begin{aligned}
S_k &= \sum_{i=0}^{k} a_i\, b_i = \Big(\sum_{i=0}^{k} a_i\Big) b_k - \sum_{i=0}^{k-1}\Big(\sum_{l=0}^{i} a_l\Big)(b_{i+1}-b_i)\\
&= \big(C_-\, e^{\delta_\Gamma\, k} + \mathrm{O}(e^{(\delta_\Gamma-\kappa')k}\;\|\psi^-\|_{\frac{\beta}{2}})\big)\,\mathrm{O}(\|\psi^+\|_\infty)\\
&\quad + \sum_{i=0}^{k-1}\big(C_-\, e^{\delta_\Gamma\, i} + \mathrm{O}(e^{(\delta_\Gamma-\kappa')i}\;\|\psi^-\|_{\frac{\beta}{2}})\big)\big(C_+\,\delta'_k\, e^{\delta_\Gamma(k-i)} + \mathrm{O}(e^{(\delta_\Gamma-\kappa')(k-i)}\;\|\psi^+\|_{\frac{\beta}{2}})\big)\\
&= C_-\,C_+\,\delta'_\Gamma\,k\,e^{\delta_\Gamma k} + \mathrm{O}(e^{\delta_\Gamma k}\|\psi^-\|_{\frac{\beta}{2}}\|\psi^+\|_{\frac{\beta}{2}})\,.
\end{aligned}$$

Hence by Equation (38), we have

$$S_k = C_-\,C_+\,\delta'_\Gamma\,k\,e^{\delta_\Gamma k} + \mathrm{O}(e^{\delta_\Gamma k}\|\psi\|_{\varepsilon\,\mathrm{lc},\,\beta})\,.$$

Thus, by the geometric series estimate

$$\frac{1}{N e^{\delta_\Gamma N}}\sum_{k=0}^{N} k\, e^{\delta_\Gamma k} = \sum_{k=0}^{N}\frac{N-k}{N} e^{-\delta_\Gamma k} = \frac{1}{\delta'_\Gamma} + \mathrm{O}(\frac{1}{N})\,,$$

we have

$$\frac{\delta'_\Gamma\,\|m_{\mathrm{BM}}\|}{\|\sigma^+_{e_-}\|\,\|\sigma^-_{e_+}\|\,N\,e^{\delta_\Gamma\,N}}\,\nu_N(\psi) = \frac{\widetilde{m}_{\mathrm{BM}}(\psi)}{\|m_{\mathrm{BM}}\|} + \mathrm{O}\,\big(\frac{\|\psi\|_{\varepsilon\,\mathrm{lc},\,\beta}}{N}\big)\,.$$

This proves that we have an error term of the form $\mathrm{O}(\frac{\|\psi\|_{\varepsilon\,\mathrm{lc},\,\beta}}{N})$ in the weak-star convergence (37) when evaluated on ψ, hence concluding the proof of Step 3, thereby the one of Assertion (1) of Theorem 6.1.

(2) If we assume that $L_\Gamma = 2\mathbb{Z}$, the proof is completely similar to the one in the case $L_\Gamma = \mathbb{Z}$ using this time Assertion (2) of Theorem 6.2. □

References

[AGP] J. Athreya and A. Ghosh and A. Prasad. *Ultrametric logarithm law, II.* Monat. Math. **167** (2012) 333–356.

[Bab] M. Babillot. *On the mixing property for hyperbolic systems.* Israel J. Math. **129** (2002) 61–76.

[BaL] H. Bass and A. Lubotzky. *Tree lattices.* Prog. in Math. **176**, Birkhäuser, 2001.

[Bow] B. Bowditch. *Geometrical finiteness with variable negative curvature.* Duke Math. J. **77** (1995) 229–274.

[BrH] M. R. Bridson and A. Haefliger. *Metric spaces of non-positive curvature.* Grund. math. Wiss. **319**, Springer Verlag, 1999.

[BrPP] A. Broise-Alamichel, J. Parkkonen, and F. Paulin. *Equidistribution and counting under equilibrium states in negative curvature and trees. Applications to non-Archimedean Diophantine approximation.* With an Appendix by J. Buzzi. Prog. Math. **329**, Birkhäuser, 2019.

[Clo] L. Clozel. *Démonstration de la conjecture τ.* Invent. Math. **151** (2003) 297–328.

[DP] F. Dal'Bo and M. Peigné. *Some negatively curved manifolds with cusps, mixing and counting.* J. reine angew. Math. **497** (1998) 141–169.

[DPS] N.-T. Dang, F. Paulin, and R. Sayous. *Equidistribution of divergent diagonal orbits in positive characteristic.* In preparation.

[DS1] O. David and U. Shapira. *Equidistribution of divergent orbits and continued fraction expansion of rationals.* J. London Math. Soc. **98** (2018) 149–176.

[DS2] O. David and U. Shapira. *Equidistribution of divergent orbits of the diagonal group in the space of lattices.* Erg. Theo. Dyn. Syst. **40** (2020) 1217–1237.

[DT] C. Dilsavor and D. J. Thompson. *Gibbs measures for geodesic flow on CAT(-1) spaces.* Preprint [arXiv:2309.03297].

[KPS] A. Kemarsky and F. Paulin and U. Shapira. *Escape of mass in homogeneous dynamics in positive characteristic.* J. Modern Dyn. 11 (2017) 369–407.

[KlM1] D. Kleinbock and G. Margulis. *Bounded orbits of nonquasiunipotent flows on homogeneous spaces.* Sinai's Moscow Seminar on Dynamical Systems, 141–172, Amer. Math. Soc. Transl. Ser. **171**, Amer. Math. Soc. 1996.

[KlM2] D. Kleinbock and G. Margulis. *Logarithm laws for flows on homogeneous spaces.* Invent. Math. **138** (1999) 451–494.

[LP] J. Li and W. Pan. *Exponential mixing of geodesic flows for geometrically finite hyperbolic manifolds with cusps.* Invent. Math. **231** (2022) 931–1021.

[Lub] A. Lubotzky. *Lattices in rank one Lie groups over local fields.* GAFA **1** (1991) 405–431.

[Mar] G. Margulis. *On some aspects of the theory of Anosov systems.* Mono. Math., Springer Verlag, 2004.

[OhS] H. Oh and N. Shah. *The asymptotic distribution of circles in the orbits of Kleinian groups.* Invent. Math. **187** (2012) 1–35.

[OP] J.-P. Otal and M. Peigné. *Principe variationnel et groupes kleiniens.* Duke Math. J. **125** (2004) 15–44.

[ParP1] J. Parkkonen and F. Paulin. *Skinning measure in negative curvature and equidistribution of equidistant submanifolds.* Erg. Theo. Dyn. Sys. **34** (2014) 1310–1342.

[ParP2] J. Parkkonen and F. Paulin. *Counting common perpendicular arcs in negative curvature.* Erg. Theo. Dyn. Sys. **37** (2017) 900–938.

[ParP3] J. Parkkonen and F. Paulin. *From exponential counting to pair correlations.* Bull. Soc. Math. France **151** (2023) 171–193.

[ParP4] J. Parkkonen and F. Paulin. *Equidistribution of common perpendiculars in negative curvature.* Preprint [arXiv:2410.09216].

[Pau1] F. Paulin. *Groupe modulaire, fractions continues et approximation diophantienne en caractéristique p.* Geom. Dedi. **95** (2002) 65–85.

[Pau2] F. Paulin. *Groupes géométriquement finis d'automorphismes d'arbres et approximation diophantienne dans les arbres*. Manuscripta Math. **113** (2004) 1–23.

[PauPS] F. Paulin, M. Pollicott, and B. Schapira. *Equilibrium states in negative curvature*. Astérisque **373**, Soc. Math. France, 2015.

[Pol] M. Pollicott. *The Schottky-Klein prime function and counting functions for Fenchel double crosses*. Monatsh. Math. **195** (2021) 323–342.

[Ser] J.-P. Serre. *Arbres, amalgames, SL_2*. 3ème éd. corr., Astérisque **46**, Soc. Math. France, 1983.

[ShZ] U. Shapira and C. Zheng. *Limiting distributions of translates of divergent diagonal orbits*. Compos. Math. **155** (2019) 1747–1793.

[SoT] O. Solan and N. Tamam. *On topologically big divergent trajectories*. Duke Math. J. **172** (2023) 3429–3474.

[TW] G. Tomanov and B. Weiss. *Closed orbits for actions of maximal tori on homogeneous spaces*. Duke Math. J. **119** (2003) 367–392.

[Wei] B. Weiss. *Divergent trajectories and $\mathbb{Q}$-rank*. Israel J. Math. **152** (2006) 221–227.

J. P. and R. S., Department of Mathematics and Statistics, P.O. Box 35
40014 University of Jyväskylä, FINLAND.
jouni.t.parkkonen@jyu.fi, sayousr@jyu.fi

F. P. and R. S., Laboratoire de mathématique d'Orsay, UMR 8628 CNRS,
Université Paris-Saclay, 91405 ORSAY Cedex, FRANCE
frederic.paulin@universite-paris-saclay.fr, rafael.sayous@universite-paris-saclay.fr