Convergence of an axisymmetric finite element □

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1) Introduction

• Let Ω be a two-dimensional bounded domain. We suppose that its boundary $\partial\Omega$ is decomposed into three components Γ_0 , $\Gamma_{\rm D}$ and $\Gamma_{\rm N}$:

(1.1) $$\partial\Omega \,=\, \overline{\Gamma_0} \cup \overline{\Gamma_{\rm D}} \cup \overline{\Gamma_{\rm N}} \,,\, \Gamma_0 \cap \Gamma_{\rm D} = \emptyset \,,\, \Gamma_0 \cap \Gamma_{\rm N} = \emptyset \,,\, \Gamma_{\rm D} \cap \Gamma_{\rm N} = \emptyset,$$

where Γ_0 is the intersection of $\overline{\Omega}$ with the "axis" $y = 0$:

(1.2) $$\Gamma_0 \,=\, \overline{\Omega} \cap \{(x\,,y) \in {\rm I\!R}^2 \,,\, y = 0\} \,.$$

• Let $f \,:\, \Omega \longrightarrow {\rm I\!R}$ and $g \,:\, \Gamma_{\rm N} \longrightarrow {\rm I\!R}$ be two given functions. We wish to approximate the solution u of the problem

(1.3) $$-\frac{\partial^2 u}{\partial x^2} - \frac{1}{y}\frac{\partial}{\partial y}\left(y\frac{\partial u}{\partial y}\right) + \frac{u}{y^2} \,=\, f \qquad {\rm in}\,\Omega$$

(1.4) $$u \,=\, 0 \qquad {\rm on}\ \Gamma_{\rm D}$$

(1.5) $$\frac{\partial u}{\partial n} \,=\, g \qquad {\rm on}\ \Gamma_{\rm N}$$

where n is the external normal of the boundary $\partial\Omega$.

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• The first question is to formulate the problem (1.3)-(1.5) in order to prove the existence and uniqueness. Our variational formulation follows the approach of Mercier and Raugel [MR82] and is briefly recalled in Section 2. By doing this, it is natural to introduce weighted Sobolev spaces L^2_a, H^1_a and H^2_a associated with axisymmetric problems. The approximation is done with the help of finite elements. We introduce in Section 3 a simplicial conforming mesh $\mathcal{T}$ composed by vertices (set $\mathcal{T}^0$), edges (set $\mathcal{T}^1$) and triangles (set $\mathcal{T}^2$) and we denote by $h_{\mathcal{T}}$ the maximal value of the Lebesgue measure of the edges of the mesh $\mathcal{T}$:

$$h_{\mathcal{T}} = \inf_{a \in \mathcal{T}^1} |a| . \tag{1.6}$$

We propose a new finite element interpolation based on vertices and defining a discrete space $\mathrm{H}^{\surd}_{\mathcal{T}}$ which is "naturally" associated with the Sobolev space H^1_a. The analysis of this new method is not straightforward. Due to the singular weight y, it is necessary to use Clément's interpolate [Cℓ75] and Section 4 summarizes the essential of what has to be known on this subject. In Section 5, we show that if a function u belongs to the space H^2_a, it is possible to define an interpolate $\Pi_{\mathcal{T}} u$ such that the error $\| u - \Pi_{\mathcal{T}} u \|$ measured with the norm in space H^1_a, is of order $h_{\mathcal{T}}$. Then the proof of convergence follows classical arguments with Cea's lemma (see *e.g.* the book [Ci78] of Ciarlet) and is presented in Section 6.

• Some notations:
diam(K) : diameter of the triangle K.
where $|\bullet|$ is the bi-dimensional Lebesgue measure.
classical Sobolev spaces
space $\mathcal{C}^0(\overline{\Omega})$.
semi-norm in $\mathrm{H}^k(\Lambda)$ Sobolev space:

$$|\mathrm{d}^k u|^2 \equiv \sum_{\alpha+\beta=k} \left(\frac{\partial^{\alpha+\beta} u}{\partial x^\alpha \, \partial y^\beta} \right)^2 \tag{1.7}$$

$$|u|^2_{k,\,\Lambda} = \int_\Lambda |\mathrm{d}^k u|^2 \, \mathrm{d}x \, \mathrm{d}y \tag{1.8}$$

2) Weighted Sobolev spaces

• We multiply the equation (1.3) by a test function v null on the portion Γ_{D} of the boundary and we integrate by parts relatively to the measure $y \, \mathrm{d}x \, \mathrm{d}y$. We introduce by this calculus a bilinear form $a(\bullet\,,\bullet)$ and a linear form $< b\,, \bullet >$ according to

$$(2.1)\qquad a(u\,,\,v) \;=\; \int_\Omega y\,\nabla u \bullet \nabla v \;\mathrm{d}x\,\mathrm{d}y \;+\; \int_\Omega \frac{u\,v}{y}\;\mathrm{d}x\,\mathrm{d}y$$

$$(2.2)\qquad <b\,,\,v> = \int_\Omega f\,v\,y\;\mathrm{d}x\,\mathrm{d}y \;+\; \int_{\Gamma_{\mathrm{N}}} g\,v\;y\,\mathrm{d}\gamma\,.$$

In consequence of the algebraic expression (2.1) of the bilinear form $a(\bullet,\,\bullet)$, we introduce two notations. If u is some function $\Omega \longrightarrow \mathbb{R}$, we define $u_{\surd}$ and $u^{\surd}$ as two functions $\Omega \longrightarrow \mathbb{R}$ as

$$(2.3)\qquad u_{\surd}(x\,,\,y) \;=\; \frac{1}{\sqrt{y}}\,u(x\,,\,y)\,,\quad (x\,,\,y)\in\Omega$$

$$(2.4)\qquad u^{\surd}(x\,,\,y) \;=\; \sqrt{y}\;u(x\,,\,y)\,,\quad (x\,,\,y)\in\Omega\,.$$

• Following Mercier and Raugel [MR82], we define the three attached Sobolev "axi-spaces"

$$(2.5)\qquad \mathrm{L}^2_a(\Omega) \;=\; \{v : \Omega \longrightarrow \mathbb{R}\,,\; v^{\surd} \in \mathrm{L}^2(\Omega)\}$$

$$(2.6)\qquad \mathrm{H}^1_a(\Omega) \;=\; \{v \in \mathrm{L}^2_a(\Omega)\,,\; v_{\surd} \in \mathrm{L}^2(\Omega)\,,\, (\nabla v)^{\surd} \in (\mathrm{L}^2(\Omega))^2\}$$

$$(2.7)\qquad \mathrm{H}^2_a(\Omega) \;=\; \left\{ \begin{array}{c} v \in \mathrm{H}^1_a(\Omega)\,,\; v_{\surd\surd\surd} \in \mathrm{L}^2(\Omega)\,,\, (\nabla v)_{\surd} \in (\mathrm{L}^2(\Omega))^2\,, \\ (\mathrm{d}^2 v)^{\surd} \in (\mathrm{L}^2(\Omega))^4 \end{array} \right\}.$$

These spaces are Hilbert spaces associated with the following norms and semi-norms defined according to:

$$(2.8)\qquad \|v\|^2_{0,\,a} \;=\; \int_\Omega y\,|v|^2\;\mathrm{d}x\,\mathrm{d}y$$

$$(2.9)\qquad |v|^2_{1,\,a} \;=\; \int_\Omega \Big(\frac{1}{y}\,|v|^2 \;+\; y\,|\nabla v|^2\Big)\;\mathrm{d}x\,\mathrm{d}y$$

$$(2.10)\qquad |v|^2_{2,\,a} \;=\; \int_\Omega \Big(\frac{1}{y^3}\,|v|^2 \;+\; \frac{1}{y}\,|\nabla v|^2 \;+\; y\,|\mathrm{d}^2 v|^2\Big)\;\mathrm{d}x\,\mathrm{d}y$$

$$(2.11)\qquad \|v\|^2_{1,\,a} \;=\; \|v\|^2_{0,\,a} \;+\; |v|^2_{1,\,a}$$

$$(2.12)\qquad \|v\|^2_{2,\,a} \;=\; \|v\|^2_{1,\,a} \;+\; |v|^2_{2,\,a}\,.$$

We do not need here the expression of the associated scalar products.

• Theorem of trace, hypotheses for f and g.

• We observe that the condition

$$(2.13)\qquad u \;=\; 0 \qquad \text{on } \Gamma_0$$

on the axis is completely incorporated inside the choice of the axi-space $\mathrm{H}^1_a(\Omega)$. We introduce the Sobolev space that takes into account the homogeneous Dirichlet boundary condition (1.4):

(2.14) $$V \,=\, \{ v \in \mathrm{H}^1_a(\Omega) \,,\; \gamma v = 0 \text{ on } \Gamma_{\mathrm{D}} \} \,.$$

• Then the problem (1.3)-(1.5) admits the following variational formulation

(2.15) $$\left\{ \begin{array}{l} u \in V \\ a(u \,,\, v) \,= < b \,,\, v > \,,\;\; \forall \, v \in V \,. \end{array} \right.$$

Due to the fact that

(2.16) $$a(v \,,\, v) \,=\, |v|^2_{1,\,a} \,, \quad \forall \, v \in \mathrm{H}^1_a(\Omega) \,,$$

the existence and uniqueness of the solution of problem (2.15) is easy according to the so-called Lax-Milgram-Vishik's lemma and we refer to [MR82] for the study of the ellipticity property.

3) A natural axisymmetric finite element

• Let $\mathcal{T}$ be a conforming mesh of the domain Ω with triangles. Recall our notations: $\mathcal{T}^0$ for the set of vertices, $\mathcal{T}^1$ for edges and $\mathcal{T}^2$ for triangular elements. We first observe that if we consider a function v of the form

(3.1) $$v(x \,,\, y) \,=\, \sqrt{y}\, (a\, x \,+\, b\, y \,+\, c) \,, \qquad (x \,,\, y) \in K \in \mathcal{T}^2 \,,$$

we have

(3.2) $$\sqrt{y}\; \nabla v(x \,,\, y) \,=\, \Big(a\, y \,,\, \frac{1}{2}\, (a\, x \,+\, 3b\, y \,+\, c) \Big) \,.$$

In other terms, if we denote by P_1 the space of polynomials of total degree less or equal to 1, we have:

(3.3) $$v_{\surd} \in P_1 \;\Longrightarrow\; (\nabla v)^{\surd} \in (P_1)^2 \,.$$

• We denote by $P_1^{\surd}$ the linear space

(3.5) $$P_1^{\surd} \,=\, \{ v ,\, v_{\surd} \in P_1 \}.$$

We define the degrees of freedom $< \widetilde{\delta}_S \,,\, v >$ for v sufficiently regular and S vertex of the mesh $\mathcal{T}$ $(S \in \mathcal{T}^0)$ by

(3.6) $$< \widetilde{\delta}_S \,,\, v > = \, v_{\surd}(S), \;\; S \in \mathcal{T}^0 \,.$$

We observe that if the vertex S is not lying on the axis, the number $< \widetilde{\delta}_S \,,\, v >$ is nothing else that the value $v(S)$ divided by $\sqrt{y(S)}$. If S is on the axis, consider this point at the origin to fix the ideas and the representation (3.1) joined with (3.6) claims that $< \widetilde{\delta}_S \,,\, v > \,=\, c$, *id est* is equal to the coefficient of $\sqrt{y}$ that particularizes the approach. We observe that we still have $v(S) = 0$ but a non trivial degree of freedom is still present for such a vertex.

Proposition 1. Unisolvance property of the axi-finite element.
Let $K \in \mathcal{T}^2$ be a triangle of the mesh $\mathcal{T}$, Σ the set of linear forms $< \widetilde{\delta}_S \,, \bullet >$ for S vertex of the triangle K $(S \in \mathcal{T}^0 \cap \partial K)$ and $P_1^{\sqrt{}}$ defined at relation (3.5). Then the triple $(K \,, \Sigma \,, P_1^{\sqrt{}})$ that constituates our axi-finite element is unisolvant.

Proof of Proposition 1.
Given three numbers $\alpha_S \in \mathbb{R}$, there exists a unique function $v \in P_1^{\sqrt{}}$ such that

(3.7) $$< \widetilde{\delta}_S \,, v >= \alpha_S, \; S \in \mathcal{T}^0 \cap \partial K \,.$$

Due to the definition of $\widetilde{\delta}_S$, the relation (3.7) express that $v_{\sqrt{}}(S) = \alpha_S$ and the hypothesis $v \in P_1^{\sqrt{}}$ express that $v_{\sqrt{}} \in P_1$. Then the proof is a consequence of classical arguments for linear finite elements explained *e.g.*in Ciarlet's book. □

Proposition 2. Conformity of the axi-finite element.
The finite element $(K \,, \Sigma \,, P_1^{\sqrt{}})$ is conforming in space $\mathcal{C}^0(\overline{\Omega})$.

Proof of Proposition 2.
The property express that given arbitrary values $\alpha_S \in \mathbb{R}$ for all $S \in \mathcal{T}^0$, the function $v : \Omega \longrightarrow \mathbb{R}$ defined by interpolation in each triangle $K \in \mathcal{T}^2$ by the relation (3.7) is lying in space $\mathcal{C}^0(\overline{\Omega})$. The proof is nothing else that the classical $\mathcal{C}^0-$conformity of the P_1 finite element: $v_{\sqrt{}} \in P_1$ in each triangle and is defined by its values in each vertex. □

- We can introduce our discrete space:

(3.8) $$\mathrm{H}_{\mathcal{T}}^{\sqrt{}} = \{ v \in \mathcal{C}^0(\overline{\Omega}), \, v_{\sqrt{}}|_K \in P_1, \, \forall \, K \in \mathcal{T}^2 \}.$$

We have the property:

Proposition 3. Conformity in the axi-space $\mathrm{H}_a^1(\Omega)$.
The discrete space $\mathrm{H}_{\mathcal{T}}^{\sqrt{}}$ is included in the axi-space $\mathrm{H}_a^1(\Omega)$:

(3.9) $$\mathrm{H}_{\mathcal{T}}^{\sqrt{}} \subset \mathrm{H}_a^1(\Omega) \,.$$

Proof of Proposition 3.
It is a direct consequence of the previous property: $v \in \mathrm{H}_{\mathcal{T}}^{\sqrt{}}$ is continuous then its gradient in the sense of distributions is a classical function. Due to the relation (3.2), this function is clearly in the space $\mathrm{L}^2(\Omega)$. Of course, $v_{\sqrt{}}$ is continuous then the conditions proposed in (2.6) are all valid. □

• The discrete space for the approximation of the variational problem (2.15) is simply

$$V_{\mathcal{T}} \,=\, \mathrm{H}^{\surd}_{\mathcal{T}} \cap V\,. \qquad (3.10)$$

with V introduced in (2.14). The discrete variational formulation takes the form

$$\begin{cases} u_{\mathcal{T}} \in V_{\mathcal{T}} \\ a(u_{\mathcal{T}}\,,\, v) \,=\, < b\,,\, v>\,,\ \forall\, v \in V_{\mathcal{T}}\,. \end{cases} \qquad (3.11)$$

It has a unique solution $u_{\mathcal{T}} \in V_{\mathcal{T}}$ and the question is now to estimate the error $\| u - u_{\mathcal{T}} \|$ measured with the norm in the axi-space $\mathrm{H}^1_a(\Omega)$. For doing this, it is classical to study the interpolation error $\| u - \Pi_{\mathcal{T}} u \|$ when u is sufficiently regular and $\Pi_{\mathcal{T}} u$ is some interpolate of function u.

4) Clément's interpolation.

• We recall in this section the essential of what to be known about Clément's interpolation [Cℓ75] in the particular case of affine interpolation with triangles. Let Ω be a bounded bidimensional domain as introduced in Section 1. Let v be a function in space $\mathrm{L}^2(\Omega)$. Let $\mathcal{T}$ be a mesh of the domain Ω and $h_{\mathcal{T}}$ introduced in (1.6). We observe also that $h_{\mathcal{T}}$ is also the maximal diameter of elements in mesh $\mathcal{T}$:

$$h_{\mathcal{T}} \,=\, \sup_{K \in \mathcal{T}^2} \mathrm{diam}(K)\,. \qquad (4.1)$$

Of course, the value $v(S)$ is not defined for a vertex $S \in \mathcal{T}^0$ and the interest of Clément's interpolate is to introduce such an approached value even if v only belongs to the space $\mathrm{L}^2(\Omega)$.

• First, if $S \in \Gamma_{\mathrm{D}}$, we set

$$< \delta^{\mathcal{C}}_S\,,\, v > = \,0\,, \quad S \in \mathcal{T}^0 \cap \Gamma_{\mathrm{D}}\,. \qquad (4.2)$$

If not, for $S \in \mathcal{T}^0$, we introduce the subset Ξ_S of Ω defined by

$$\Xi_S \,=\, \bigcup_{K \in \mathcal{T}^2,\, \partial K \supset S} K \qquad (4.3)$$

and presented on Figure 1. The interpolate value $< \delta^{\mathcal{C}}_S\,,\, v >$ at the vertex S is defined by

$$< \delta^{\mathcal{C}}_S\,,\, v > = \frac{1}{|\Xi_S|} \int_{\Xi_S} v(x)\, \mathrm{d}x\, \mathrm{d}y\,, \quad S \in \mathcal{T}^0\,,\ S \notin \Gamma_{\mathrm{D}}\,. \qquad (4.4)$$

• First we introduce the Clement interpolate $\Pi^{\mathcal{C}} v$ of $v \in \mathrm{L}^2(\Omega)$ with the help of classical P_1 continuous interpolate functions φ_S defined by

(4.5) $$\varphi_S\big|_K \in P_1 \,,\, \forall\, K \in \mathcal{T}^2 \,,\, \varphi_S(S') = \begin{cases} 1 & \text{if } S' = S \\ 0 & \text{if } S' \neq S \,. \end{cases}$$

With Clément [Cℓ75], we set

(4.6) $$\Pi^{\mathcal{C}} v \;=\; \sum_{S \in \mathcal{T}^0} < \delta_S^{\mathcal{C}} \,,\, v > \varphi_S \,.$$

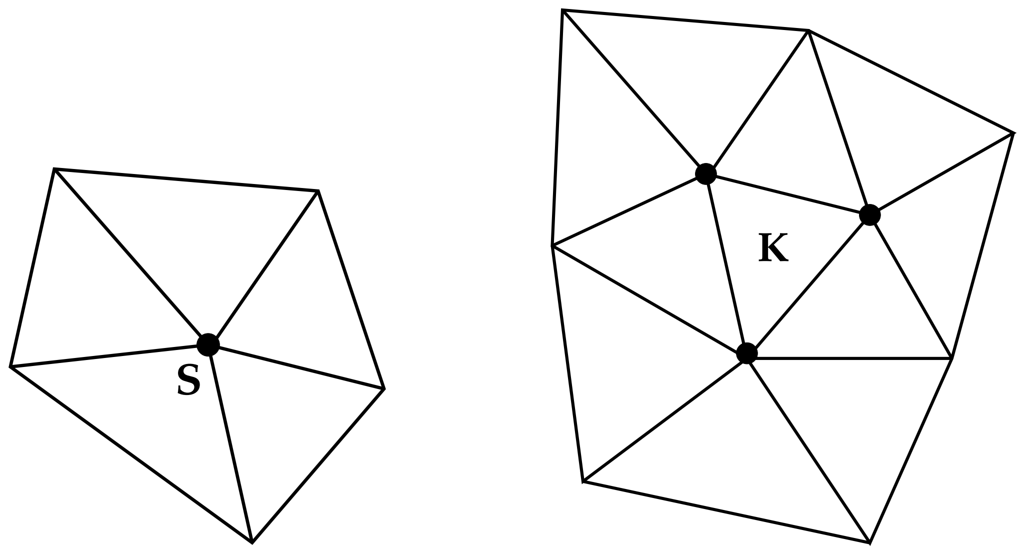

Figure 1. Left: Vicinity Ξ_S of the vertex $S \in \mathcal{T}^0$.
Right: Vicinity Z_K for a given triangle $K \in \mathcal{T}^2$.

• We suppose now that the function v is a bit more regular. The interest of Clément's interpolation is that all the Ciarlet-Raviart [CR72] classical results for Lagrange interpolation in Sobolev spaces can be extented to Clément's. In order to quantify the result, we suppose in the following that the mesh $\mathcal{T}$ belongs to a family $\mathcal{F}$ of meshes such that no infinitesimal angle belongs in the mesh $\mathcal{T}$; in other terms,

(4.7) $$\exists\, C > 0 \,,\, \forall\, \mathcal{T} \in \mathcal{F} \,,\, \forall\, S \in \mathcal{T}^0 \,,\, \sharp\{K \in \mathcal{T}^2,\, K \subset \Xi_S\} \leq C \,.$$

We introduce also the set Z_K for a given triangle $K \in \mathcal{T}^2$ (see again the Figure 1) :

(4.8) $$Z_K \;=\; \{L \in \mathcal{T}^2 \,,\, \overline{K} \cap \overline{L} \neq \emptyset\} \;=\; \bigcup_{S \in \mathcal{T}^0,\, S \subset \partial K} \Xi_S \,.$$

According to the hypothesis (4.7), we have

(4.9) $$\exists\, C > 0 \,,\, \forall\, \mathcal{T} \in \mathcal{F} \,,\, \forall\, K \in \mathcal{T}^2 \,,\, \sharp Z_K \leq C \,.$$

• Consider now a function $v \in \mathrm{H}^1(Z_K)$. Then a main results of Clément's contribution can be stated as

(4.10) $\qquad |v - \Pi^{\mathcal{C}} v|_{0,\,K} \,\leq\, C\, h_{\mathcal{T}}\, |v|_{1,\,Z_K}$

(4.11) $\qquad |v - \Pi^{\mathcal{C}} v|_{1,\,K} \,\leq\, C\, |v|_{1,\,Z_K}$

with a constant $C > 0$ that does not depend on the particular mesh $\mathcal{T}$ chosen in the family $\mathcal{F}$. If the function v is more regular ($v \in \mathrm{H}^2(Z_K)$), we can consolidate the estimate (4.11):

(4.12) $\qquad |v - \Pi^{\mathcal{C}} v|_{1,\,K} \,\leq\, C\, h_{\mathcal{T}}\, |v|_{2,\,Z_K}\,.$

Finally, if v is globally regular, we have

(4.13) $\qquad \| v - \Pi^{\mathcal{C}} v\|_{0,\,\Omega} \,\leq\, C\, h_{\mathcal{T}}\, |v|_{1,\,\Omega}\,.$

5) An interpolation result

• We suppose in this section that a given function u belongs to the space $\mathrm{H}^2_a(\Omega)$ defined in (2.7). It is possible to define the value $u(S)$ for a vertex $S \in \mathcal{T}^0$ due to the Sobolev embedding Theorem (see *e.g.* Brézis [Br83]) that claims that

(5.1) $\qquad \mathrm{H}^2(\Omega) \,\subset\, \mathcal{C}^0(\overline{\Omega})\,.$

The question is now to define or not the number $< \widetilde{\delta}_S \,,\, u >$ introduced in (3.6).

Proposition 4. Lack of regularity.
Let $u \in \mathrm{H}^2_a(\Omega)$ and $u_{\surd}$ introduced in (2.3). Then $u_{\surd}$ belongs to the space $\mathrm{H}^1(\Omega)$ and we have

(5.2) $\qquad \|u_{\surd}\|_{1,\,\Omega} \,\leq\, C\, \|u\|_{2,\,a}$

Proof of Proposition 4.
We set

(5.3) $\qquad v \,\equiv\, u_{\surd}$

and we have the following calculus:

(5.4) $\qquad \nabla v \,=\, -\dfrac{1}{2y\sqrt{y}}\, u\, \nabla y \,+\, \dfrac{1}{\sqrt{y}}\, \nabla u\,.$

Then

$$\int_\Omega |v|^2 \,\mathrm{d}x\,\mathrm{d}y \,\leq\, \int_\Omega \frac{1}{y}\, |u|^2 \,\mathrm{d}x\,\mathrm{d}y \,\leq\, C\, \|u\|^2_{2,\,a}$$

$$\int_\Omega |\nabla v|^2 \,\mathrm{d}x\,\mathrm{d}y \,\leq\, 2 \int_\Omega \Big(\frac{1}{4y^3}\, |u|^2 \,+\, \frac{1}{y}\, |\nabla u|^2 \Big) \,\mathrm{d}x\,\mathrm{d}y \,\leq\, C\, \|u\|^2_{2,\,a}\,.$$

Due to (2.10), the relation (5.2) is established. □

• Il we derive (formally !) the relation (5.4), we get

(5.5) $$\mathrm{d}^2 v \,=\, \frac{3}{4\,y^2 \sqrt{y}}\, u\, \nabla y \bullet \nabla y \,-\, \frac{1}{y \sqrt{y}}\, \nabla u \bullet \nabla y \,+\, \frac{1}{\sqrt{y}}\, \mathrm{d}^2 u$$

and we have not sufficiently powers of y to be sure that we obtain a finite result when we integrate the square of $\mathrm{d}^2 v$. In consequence, the function v is not necessarily continuous. Nevertheless, it is possible to define the Clement interpolate of $u_{\surd}$ relatively to the mesh $\mathcal{T}$ and due to (5.2), this interpolate has good regularity properties. We define our interpolate Πu by conjugation and we set

(5.6) $$\Pi u \,=\, \big(\Pi^{\mathcal{C}} u_{\surd}\big)^{\surd}$$

or equivalently

(5.7) $$\Pi u(x,\, y) \,=\, \sqrt{y}\, \big(\Pi^{\mathcal{C}} v\big)(x,\, y)\,, \qquad (x,\, y) \in K \in \mathcal{T}^2$$

with $v(\bullet)$ introduced in (5.3).

• We assume that the mesh $\mathcal{T}$ admits angles that are aware from 0 and π :

(5.8) $$\begin{cases} \exists\,(\alpha\,,\,\beta)\,,\, 0 < \alpha < \dfrac{\pi}{2} < \beta < \pi\,,\, \forall \mathcal{T} \in \mathcal{F}\,,\, \forall K \in \mathcal{T}^2\,, \\ \qquad\qquad\qquad\qquad \forall \theta \text{ angle in } K\,,\, \alpha \leq \theta \leq \beta. \end{cases}$$

We observe that the hypothesis (5.8) clearly implies (4.7). We assume also that the sizes of triangles are quasi-uniform:

(5.9) $$\exists \gamma > 0\,,\, \forall \mathcal{T} \in \mathcal{F}\,,\, \forall a \in \mathcal{T}^1\,,\, \gamma\, h_{\mathcal{T}} \,\leq\, |a| \,\leq\, h_{\mathcal{T}}$$

with $h_{\mathcal{T}}$ introduced at the relation (1.6). We have the following interpolation theorem:

Theorem 1. An interpolation result.
We suppose that the mesh $\mathcal{T}$ belongs to a family $\mathcal{F}$ that satisfy the above hypotheses (5.8) and (5.9). Let $u \in \mathrm{H}^2_a(\Omega)$ and Πu defined by (5.6). Then we have

(5.10) $$\| u - \Pi u \|_{1,\,a} \,\leq\, C\, h_{\mathcal{T}}\, \| u \|_{2,\,a}\,.$$

• In order to prepare the technical points of the proof, we introduce, following [MR82] the sub-domains Ω_+ and Ω_- as

(5.12) $$\Omega_+ \,=\, \{ K \in \mathcal{T}^2\,,\, \mathrm{dist}\,(Z_K\,,\, \Gamma_0) > 0 \}$$

(5.13) $$\Omega_- \,=\, \Omega \setminus \Omega_+\,.$$

Proposition 5. Geometrical lemma.
If the family $\mathcal{F}$ of meshes satisfy the hypotheses (5.8) and (5.9), we have

$$(5.14) \qquad \left\{ \begin{array}{l} \exists\, \delta > 0 \,,\, \forall\, \mathcal{T} \in \mathcal{F} \,,\, \forall\, K \in \mathcal{T}^2 \,, \\ \qquad\qquad (K \,\subset\, \Omega_+) \Longrightarrow (\mathrm{dist}(Z_K \,,\, \Gamma_0) \,\geq\, \delta\; h_{\mathcal{T}}) \,. \end{array} \right.$$

$$(5.15) \qquad \forall\, \mathcal{T} \in \mathcal{F} \,,\, \forall\, K \in \mathcal{T}^2 \,,\, (K \,\subset\, \Omega_-) \Longrightarrow (\forall\, (x ,\, y) \in K \,,\, |y| \leq 2\, h_{\mathcal{T}}) \,.$$

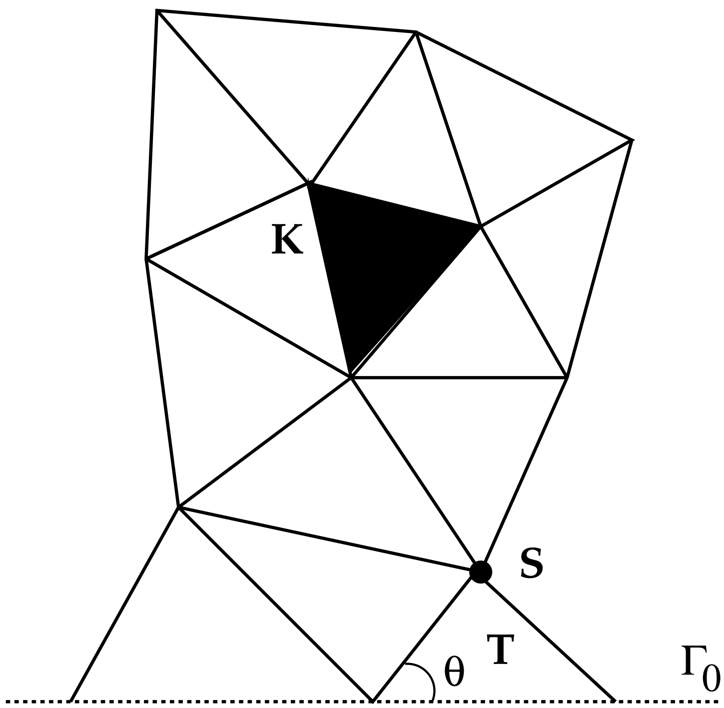

Figure 2. Triangle element K that belongs to the sub-domain Ω_+.

Proof of Proposition 5.

- If $\mathrm{dist}(Z_K \,,\, \Gamma_0) > 0$, consider a vertex $S \in \mathcal{T}^0$ such that $\mathrm{dist}(Z_K \,,\, \Gamma_0) = \mathrm{dist}(S \,,\, \Gamma_0)$ (see Figure 2). Then the vertex S belongs to a triangle T that does **not** belong to the family Z_K and $\mathrm{dist}(S \,,\, \Gamma_0) \,\geq\, \sin\theta \bullet \gamma\, h_{\mathcal{T}} \,\geq\, \gamma\, \sin\alpha\, h_{\mathcal{T}}$. This property establishes the relation (5.14) with $\delta = \gamma\, \sin\alpha$.
- If $K \,\subset\, \Omega_-$ and $|y| \geq 2\, h_{\mathcal{T}}$ for a point $(x ,\, y) \in K$, it is clear from the definition of Z_K and is illustrated by the Figure 2 that the distance between Z_K and the axis Γ_0 is strictly positive, then $K \,\subset\, \Omega_+$ and this contradiction establishes the relation (5.15). □

Proof of Theorem 1.
Our proof is constructed in the same spirit that the pioneering work proposed by Mercier and Raugel.

- We first consider the term of order zero in the error $\| u - \Pi u \|_{1,\, a}$ (*c.f.* the relation (2.11)):

$$\begin{aligned}\int_\Omega \frac{1}{y}\,|u-\Pi u|^2\,\mathrm{d}x\,\mathrm{d}y \;&=\; \int_\Omega \frac{1}{y}\,|u-\sqrt{y}\;\Pi^{\mathcal{C}}v|^2\,\mathrm{d}x\,\mathrm{d}y\\ &=\;\int_\Omega |v-\Pi^{\mathcal{C}}v|^2\,\mathrm{d}x\,\mathrm{d}y \;=\; \|v-\Pi^{\mathcal{C}}v\|_{0,\,\Omega}^2\\ &\le\; C\,h_{\mathcal{T}}^2\,|v|_{1,\,\Omega}^2 \qquad\qquad \text{due to (4.13)}\\ &\le\; C\,h_{\mathcal{T}}^2\,\|v\|_{2,\,a}^2 \qquad\qquad \text{according to (5.2)}\end{aligned}$$

$$\int_\Omega \frac{1}{y}\,|u-\Pi u|^2\,\mathrm{d}x\,\mathrm{d}y \;=\; \|v-\Pi^{\mathcal{C}}v\|_{0,\,\Omega}^2 \;\le\; C\,h_{\mathcal{T}}^2\,\|u\|_{2,\,a}^2\,. \tag{5.16}$$

- On the other hand, we have

$$\nabla\Big(\sqrt{y}\;\big(v-\Pi^{\mathcal{C}}v\big)\Big) \;=\; \frac{1}{2\,\sqrt{y}}\,\big(v-\Pi^{\mathcal{C}}v\big)\nabla y \;+\; \sqrt{y}\;\nabla\big(v-\Pi^{\mathcal{C}}v\big)\,.$$

Then

$$\left\{\begin{aligned}&\int_\Omega y\,|\nabla\big(u-\Pi u\big)|^2\,\mathrm{d}x\,\mathrm{d}y \;\le\\ &\qquad\le\;\int_\Omega |v-\Pi^{\mathcal{C}}v|^2\,\mathrm{d}x\,\mathrm{d}y \;+\; 2\,\int_\Omega y^2\,|\nabla\big(v-\Pi^{\mathcal{C}}v\big)\,|^2\,\mathrm{d}x\,\mathrm{d}y\,.\end{aligned}\right. \tag{5.17}$$

The first term in the right hand side of (5.17) is majored with the help of estimation (5.16). We focus now on the second term. We have

$$\left\{\begin{aligned}&\int_\Omega y^2\,|\nabla(v-\Pi^{\mathcal{C}}v)|^2\,\mathrm{d}x\,\mathrm{d}y \;=\\ &\qquad=\;\int_{\Omega_+} y^2\,\nabla(|v-\Pi^{\mathcal{C}}v)|^2\,\mathrm{d}x\,\mathrm{d}y \;+\; \int_{\Omega_-} y^2\,|\nabla(v-\Pi^{\mathcal{C}}v)|^2\,\mathrm{d}x\,\mathrm{d}y\,.\end{aligned}\right.$$

- From the relation (5.16), we have for the internal sub-domain Ω_- :

$$\begin{aligned}\int_{\Omega_-} y^2\,|\nabla(v-\Pi^{\mathcal{C}}v)|^2\,\mathrm{d}x\,\mathrm{d}y \;\le\; 4\,h_{\mathcal{T}}^2\,&\int_{\Omega_-} |\nabla\big(v-\Pi v\big)|^2\,\mathrm{d}x\,\mathrm{d}y\\ &\le\; C\,h_{\mathcal{T}}^2\,|v|_{1,\,\Omega}^2 \qquad\qquad \text{due to (4.11)}\\ &\le\; C\,h_{\mathcal{T}}^2\,\|u\|_{2,\,a}^2 \qquad\qquad \text{thanks to (5.2)}\end{aligned}$$

$$\int_{\Omega_-} y\,|\nabla(v-\Pi^{\mathcal{C}}v)|^2\,\mathrm{d}x\,\mathrm{d}y \;\le\; C\,h_{\mathcal{T}}^2\,\|u\|_{2,\,a}^2\,. \tag{5.18}$$

- We have in the external part of the domain

$$\int_{\Omega_+} y^2\,|\nabla(v-\Pi^{\mathcal{C}}v)|^2\,\mathrm{d}x\,\mathrm{d}y \;=\; \sum_{K\in\mathcal{T}^2,\,K\subset\Omega_+}\int_K y^2\,|\nabla(v-\Pi^{\mathcal{C}}v)|^2\,\mathrm{d}x\,\mathrm{d}y\,.$$

We fix $K\subset\Omega_+$, we introduce $(y_{\min}\,,\,y_{\max})$ according to

$$y_{\min} \;=\; \min_{(x,\,y)\in K} y\,, \qquad y_{\max} \;=\; \max_{(x,\,y)\in K} y\,.$$

Then $y_{\rm min} \leq y \leq y_{\rm max}$, $y_{\rm max} - y_{\rm min} \leq h_{\mathcal{T}}$ and

(5.19) $$y^2 \,\leq\, y_{\rm max}^2 \,\leq\, 2 \left(y_{\rm min}^2 \,+\, h_{\mathcal{T}}^2 \right) , \qquad (x \,,\, y) \in K \,.$$

Now if $(x \,,\, y) \in Z_K$, we have, due to the definition of Z_K and to (5.15):

$y_{\rm min} \,-\, h_{\mathcal{T}} \,\leq\, y \,, \quad y \,\geq\, \delta \, h_{\mathcal{T}}$

and

(5.20) $$\frac{y_{\rm min}^2 + h_{\mathcal{T}}^2}{y^2} \,\leq\, \frac{1}{y^2} \, (2 \, y^2 \,+\, 2 \, h_{\mathcal{T}}^2 \,+\, h_{\mathcal{T}}^2) \,\leq\, 2 \,+\, \frac{3}{\delta^2} \,.$$

• Due to the property (4.12) of Clément's interpolate and to estimate (5.20), we have

$$\begin{aligned} \int_K y^2 \, |\nabla (v - \Pi^{\mathcal{C}} v)|^2 \, {\rm d}x \, {\rm d}y \,&\leq\, 2 \, (y_{\rm min}^2 + h_{\mathcal{T}}^2) \, \int_K |\nabla (v - \Pi^{\mathcal{C}} v)|^2 \, {\rm d}x \, {\rm d}y \\ &\leq\, 2 \, (y_{\rm min}^2 + h_{\mathcal{T}}^2) \, \int_{Z_K} C \, h_{\mathcal{T}}^2 \, |{\rm d}^2 v|^2 \, {\rm d}x \, {\rm d}y \\ &\leq\, C \left(2 \,+\, \frac{3}{\delta^2} \right) h_{\mathcal{T}}^2 \int_{Z_K} y^2 \, |{\rm d}^2 v|^2 \, {\rm d}x \, {\rm d}y \\ &\leq\, C \, h_{\mathcal{T}}^2 \int_{Z_K} \left(\frac{1}{y^3} \, |v|^2 \,+\, \frac{1}{y} \, |\nabla v|^2 \,+\, y \, |{\rm d}^2 v|^2 \right) {\rm d}x \, {\rm d}y \end{aligned}$$

according to (5.5). Then, taking into account (2.10), we have

(5.22) $$\int_{\Omega_+} y^2 \, |\nabla (v - \Pi^{\mathcal{C}} v)|^2 \, {\rm d}x \, {\rm d}y \,\leq\, C \, h_{\mathcal{T}}^2 \, \| u \|_{2,\, a}^2 \,.$$

The inequality (5.11) is a consequence of (5.16), (5.17), (5.18) and (5.22). The theorem is now established. □

6) Convergence of the axi-finite element approximation.

• We suppose now that the data Ω, f and g are chosen in such a way that the solution u of the variational problem (2.13) belongs to the space ${\rm H}_a^2(\Omega)$:

(6.1) $$u \in {\rm H}_a^2(\Omega) \,, \quad u \text{ solution of problem (2.13).}$$

Let $u_{\mathcal{T}} \in {\rm H}_{\mathcal{T}}^{\surd}$ be the solution of the discrete problem (3.11).

Theorem 2. First order approximation

Under the above hypotheses, we have

(6.2) $$\| u - u_{\mathcal{T}} \|_{1,\, a} \,\leq\, C \, h_{\mathcal{T}} \, \| u \|_{2,\, a} \,.$$

Proof of Theorem 2.

It is a classical consequence of the ellipticity of the functional $a(\bullet\,,\,\bullet)$ and of Céa's lemma. We denote by κ the ellipticity constant of the functional. Thus we get

$$\begin{aligned} \kappa\, \| u - u_{\mathcal{T}} \|_{1,\,a}^2 \,&\leq\, a(u - u_{\mathcal{T}}\,,\, u - u_{\mathcal{T}}) \\ &\leq\, C\, a(u - u_{\mathcal{T}}\,,\, u - \Pi u) \qquad \text{take } v = \Pi u - u_{\mathcal{T}} \in \mathrm{H}_{\mathcal{T}}^{\surd} \text{ in (3.11)} \\ &\leq\, C\, \| u - u_{\mathcal{T}} \|_{1,\,a}\, \| u - \Pi u_{\mathcal{T}} \|_{1,\,a}\,. \end{aligned}$$

Then $\| u - u_{\mathcal{T}} \|_{1,\,a} \,\leq\, C\, \| u - \Pi u \|_{1,\,a} \,\leq\, C\, h_{\mathcal{T}}\, \| u \|_{2,\,a}$

due to the theorem 1. □

7) References.

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