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Petrov-Galerkin Finite Volumes

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Petrov-Galerkin Finite Volumes

For an elliptic problem with two space dimensions, we propose to formulate the finite volume method with the help of Petrov-Galerkin mixed finite elementsthat are based on the building of a dual Raviart-Thomas basis.

Pour un problème elliptique bidimensionnel, nous proposons de formuler la méthode des volumes finis avec des éléments finis mixtes de Petrov-Galerkin qui s'appuient sur la construction d'une base duale de Raviart-Thomas.

Introduction

Mixed Finite Elements

Finite Volumes

Finite volumes as mixed Petrov-Galerkin finite elements

Stability analysis

Towards a first Petrov-Galerkin finite volume scheme

Introduction

• Let Ω be a bidimensional bounded convex domain in $\mathbb{R}^2$ with a polygonal boundary $\partial\Omega$. We consider the homogeneous Dirichlet problem for the Laplace operator in the domain Ω :

$$(1) \qquad -\Delta u \,=\, f \qquad \text{in } \Omega\,, \quad u \,=\, 0 \qquad \text{on the boundary } \partial\Omega \text{ of } \Omega.$$

We suppose that the datum f belongs to the space $L^2(\Omega)$. We introduce the momentum p defined by

$$(2) \qquad p \,=\, \nabla u\,.$$

Taking the divergence of both terms arising in equation (2), taking into account the relation (1), we observe that the divergence of momentum p belongs to the space $L^2(\Omega)$. For this reason, we introduce the vectorial Sobolev space

$$H(\mathrm{div},\,\Omega) \,=\, \big\{\, q \in L^2(\Omega) \times L^2(\Omega)\,,\; \mathrm{div}\, q \in L^2(\Omega) \,\big\}.$$

• The variational formulation of the problem (1) with the help of the pair $\xi \,=\, (u,\, p)$ is obtained by testing the definition (2) against a vector valued function q and integrating by parts. With the help of the boundary condition, it comes :

$$(p,\, q) \,+\, (u,\, \mathrm{div}\, q) \,=\, 0, \;\; \forall\, q \in H(\mathrm{div},\, \Omega).$$

Introduction (ii)

Independently, the relations (1), and (2) are integrated on the domain Ω after multiplying by a scalar valued function $v \in L^2(\Omega)$. We obtain :

$$(\operatorname{div} p,\, v) \,+\, (f,\, v) \,=\, 0, \;\forall\, v \in L^2(\Omega).$$

The "mixed" variational formulation is obtained by introducing the product space V defined as

$$V \,=\, L^2(\Omega) \,\times\, H(\operatorname{div},\, \Omega), \quad \|\, (u,\, p)\, \|_V^2 \equiv \|\, u\, \|_0^2 \,+\, \|\, p\, \|_0^2 \,+\, \|\, \operatorname{div} p\, \|_0^2,$$

the following bilinear form $\gamma(\bullet,\, \bullet)$ defined on $V \times V$:

$$(3) \qquad \gamma\big((u,\, p),\, (v,\, q)\big) \,=\, (p,\, q) + (u,\, \operatorname{div} q) + (\operatorname{div} p,\, v)$$

and the linear form $\sigma(\bullet)$ defined on V according to : $< \sigma,\, \zeta >= -(f,\, v),\;\; \zeta = (v,\, q) \in V.$
Then the Dirichlet problem (1) takes the form :

$$(4) \qquad \xi \in V\,,\; \gamma(\xi,\, \zeta) \,=< \sigma,\, \zeta >\,, \qquad \forall\, \zeta \in V\,.$$

Due to classical inf-sup conditions introduced by Babuška in 1971, the problem (4) admits a unique solution $\xi \in V$.

Mixed Finite Elements

- We introduce a mesh $\mathcal{T}$ that is a bidimensional cellular complex composed in our case by triangular elements K $(K \in \mathcal{E}_\mathcal{T})$, straight edges a $(a \in \mathcal{A}_\mathcal{T})$ and ponctual nodes S $(S \in \mathcal{S}_\mathcal{T})$. We conside also classical finite dimensional spaces $L^2_\mathcal{T}(\Omega)$ and $H_\mathcal{T}(\mathrm{div}, \Omega)$ that approximate respectively the spaces $L^2(\Omega)$ and $H(\mathrm{div}, \Omega)$. A scalar valued function $v \in L^2_\mathcal{T}(\Omega)$ is constant in each triangle K of the mesh :

$$L^2_\mathcal{T}(\Omega) = \left\{ v : \Omega \longrightarrow \mathbb{R}, \forall K \in \mathcal{E}_\mathcal{T}, \exists v_K \in \mathbb{R}, \forall x \in K, v(x) = v_K \right\}.$$

A vector valued function $q \in H_\mathcal{T}(\mathrm{div}, \Omega)$ is a linear combination of Raviart-Thomas (1977) basis functions φ_a of lower degree, defined for each edge $a \in \mathcal{A}_\mathcal{T}$ as follows.

- Let $a \in \mathcal{A}_\mathcal{T}$ be an internal edge of the mesh, denote by S and N the two vertices that compose its boundary (see Figure 1) : $\partial a = \{ S, N \}$ and by K and L the two elements that compose its co-boundary $\partial^c a \equiv \{ K, L \}$ in such a way that the normal direction n is oriented from K towards L and that the pair of vectors $(n, \overrightarrow{\mathrm{SN}})$ is direct, as shown on Figure 1. We denote by W (respectively by E) the third vertex of the triangle K (respectively of the triangle L) : $K = (S, N, W)$, $L = (N, S, E)$.

Mixed Finite Elements (ii)

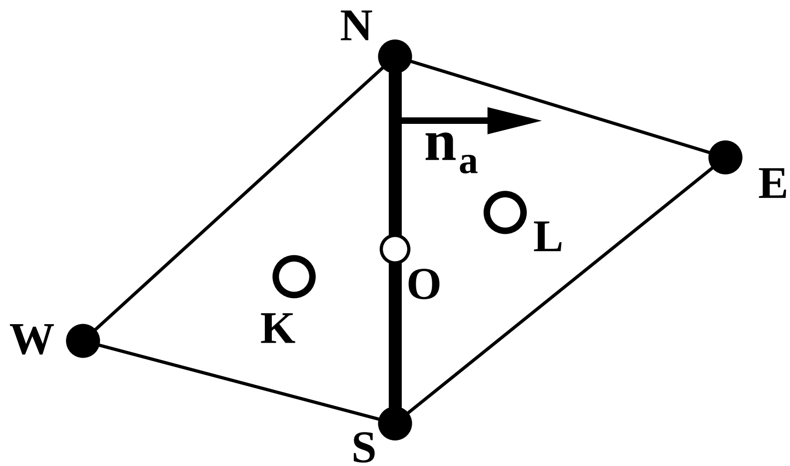

Figure 1. Co-boundary (K, L) of the edge $a = (S, N)$.

The vector valued Raviart-Thomas basis function φ_a is defined by the relations $\varphi_a(x) = \frac{1}{2|K|}\,(x - W)$ when $x \in K$, $\varphi_a(x) = -\frac{1}{2|L|}\,(x - E)$ when $x \in L$ and $\varphi_a(x) = 0$ elsewhere. When the edge a is on the boundary $\partial\Omega$, we suppose that the normal n points towards the exterior of the domain, so the element L is absent. We have in all cases the $H(\text{div}, \Omega)$ conformity : $\varphi_a \in H(\text{div}, \Omega)$ and the degrees of freedom are the fluxes of vector field φ_a for all the edges of the mesh : $\int_b \varphi_a \bullet n \, \mathrm{d}\gamma = \delta_{a,\,b}, \; \forall\, a,\, b \in \mathcal{A}_{\mathcal{T}}$. A vector valued function $q \in H_{\mathcal{T}}(\text{div}, \Omega)$ is a linear combination of the basis functions φ_a : $q = \sum_{a\,\in \mathcal{A}_{\mathcal{T}}} q_a\, \varphi_a \in H_{\mathcal{T}}(\text{div}, \Omega) = < \varphi_b\,,\; b \in \mathcal{A}_{\mathcal{T}} >$.

Mixed Finite Elements (iii)

• The mixed finite element method consist in choosing as discrete linear space the following product :

$$V_{\mathcal{T}} = L^2_{\mathcal{T}}(\Omega) \times H_{\mathcal{T}}(\mathrm{div}, \Omega)$$

and to replace the letter V by $V_{\mathcal{T}}$ inside the variational formulation (4) :

$$\xi_{\mathcal{T}} \in V_{\mathcal{T}}, \quad \gamma(\xi_{\mathcal{T}}, \zeta) = <\sigma, \zeta>, \ \forall \zeta \in V_{\mathcal{T}}.$$

In other terms

$$(5) \qquad \begin{cases} u_{\mathcal{T}} \in L^2_{\mathcal{T}}(\Omega) \,, \quad p_{\mathcal{T}} \in H_{\mathcal{T}}(\mathrm{div}, \Omega) \\ (p_{\mathcal{T}} \,, q) + (u_{\mathcal{T}} \,, \mathrm{div}\, q) \ = \ 0 \,, \qquad \forall\, q \in H_{\mathcal{T}}(\mathrm{div}, \Omega) \\ (\mathrm{div}\, p_{\mathcal{T}} \,, v) + (f \,, v) \ = \ 0 \,, \qquad \forall\, v \in L^2_{\mathcal{T}}(\Omega) \,. \end{cases}$$

The numerical analysis of the relations between the continuous problem (1) and the discrete problem (5) as the mesh $\mathcal{T}$ is more and more refined is classical [Raviart-Thomas, 1977]. The above method is popular in the context of petroleum and nuclear industries but suffers from the fact that the associated linear system is quite difficult to solve from a practical point of view. The introduction of supplementary Lagrange multipliers by Brezzi, Douglas and Marini (1985) allows a simplification of these algebraic aspects, and their interpretation by Croisille in the context of box schemes (2000) gives a good mathematical foundation of a popular numerical method.

• From a theoretical and practical point of view, the resolution of the linear system (5) can be conducted as follows. We introduce the mass-matrix $M_{a,\,b} = (\varphi_a,\, \varphi_b),\; a,\, b \in \mathcal{A}_{\mathcal{T}}$ associated with the Raviart-Thomas vector valued functions. Then the first equation of (5) determines the momentum $p_{\mathcal{T}} = \sum_{a \,\in \mathcal{A}_{\mathcal{T}}} p_{\mathcal{T},a}\, \varphi_a$ as a function of the mean values $u_{\mathcal{T},K}$ for $K \in \mathcal{E}_{\mathcal{T}}$:

$$(6) \qquad p_{\mathcal{T},a} \;=\; - \sum_{b\, \in \mathcal{A}_{\mathcal{T}}} \big(M^{-1}\big)_{a,\,b} \sum_{K\, \in \mathcal{E}_{\mathcal{T}}} u_{\mathcal{T},K} \int_K \mathrm{div}\, \varphi_b \, \mathrm{d}x \;.$$

The representation (6) suffers at our opinion form a major defect : due to the fact that the matrix M^{-1} is full, the discrete gradient $p_{\mathcal{T}}$ is a **global** function of the mean values $u_{\mathcal{T},K}$ and this property contradicts the mathematical foundations of the derivation operator to be **linear** and **local**. An a posteriori correction of this defect has been proposed by Baranger, Maître and Oudin (1996) : with an appropriate numerical integration of the mass matrix M, it is possible to lump it and the discrete gradient in the direction n of the edge a is represented by a formula of the type :

$$(7) \qquad p_{\mathcal{T},a} \;=\; \frac{u_{\mathcal{T},L} - u_{\mathcal{T},K}}{h_a}$$

with the notations of Figure 1.

Finite Volumes (ii)

• The substitution of the relation (7) inside the second equation of the formulation (5) conducts to a variant of the so-called finite volume method. In an analogous manner, the family of finite volume schemes proposed by Herbin (1995) suppose *a priori* that the discrete gradient in the normal direction admits a representation of the form (7). Nevertheless, the intuition is not correctly satisfied by a scheme such that (7). The finite difference $\frac{u_{\mathcal{T},L} - u_{\mathcal{T},K}}{h_a}$ wish to be a a good approximation of the gradient $p_{\mathcal{T}} = \nabla u_{\mathcal{T}}$ in the direction $\overrightarrow{\mathrm{KL}}$ whereas the coefficient $p_{\mathcal{T},a}$ is an approximation of $\int_{\mathrm{a}} \nabla u_{\mathcal{T}} \bullet n \, \mathrm{d}\tau$ in the **normal** direction (see again the Figure 1). When the mesh $\mathcal{T}$ is composed by general triangles, this approximation is not completely satisfactory and contains a real limitation of these variants of the finite volume method at our opinion.

Finite Volumes (iii)

• In fact, the finite volume method for the approximation of the diffusion operator has been first proposed from empirical considerations. Following *e.g.* Noh (1964) and Patankar (1980), the idea is to represent the normal interface gradient $\int_{\mathrm{a}} \nabla u_{\mathcal{T}} \bullet n \, \mathrm{d}\tau$ as a function of **neighbouring** values. Given an edge a, a vicinity $\mathcal{V}(a)$ is first determined in order to represent the normal gradient $p_{\mathcal{T},a} = \int_{\mathrm{a}} \nabla u_{\mathcal{T}} \bullet n \, \mathrm{d}\tau$ with a formula of the type

$$(8) \qquad \int_{\mathrm{a}} \nabla u_{\mathcal{T}} \bullet n \, \mathrm{d}\tau \;=\; \sum_{K \in \mathcal{V}(a)} g_{a,K} \; u_{\mathcal{T},K} \,.$$

Then the conservation equation $\operatorname{div} p + f = 0$ is integrated inside each cell $K \in \mathcal{E}_{\mathcal{T}}$ is order to determine an equation for the mean values $u_{\mathcal{T},K}$ for all $K \in \mathcal{E}_{\mathcal{T}}$. The difficulties of such approches have been presented by Kershaw (1981) and a variant of such scheme has been first analysed by Coudière, Vila and Villedieu (1999). The key remark that we have done with F. Arnoux (see Du89), also observed by Faille, Gallouët and Herbin (1991) is that the representation (8) must be **exact for linear** functions $u_{\mathcal{T}}$. We took this remark as a starting point for our tridimensional finite volume scheme proposed in 1992. It is also an essential hypothesis for the result proposed by Coudière, Vila and Villedieu.

Finite volumes as mixed Petrov-Galerkin finite elements

• In this contribution, we propose to discretize the variational problem (4) with the Petrov-Galerkin mixed finite element method, first introduced by Thomas and Trujillo (1999). In the way we have proposed in 2000, the idea is to construct a discrete functional space $H^\star_{\mathcal{T}}(\mathrm{div}, \Omega)$ generated by vectorial functions $\varphi^\star_a,\, a \in \mathcal{A}_{\mathcal{T}}$, that are conforming in the space $H(\mathrm{div}, \Omega)$: $\varphi^\star_a \in H(\mathrm{div}, \Omega)$ and to represent exactly the **dual basis** of the family $\{\varphi_b,\, b \in \mathcal{A}_{\mathcal{T}}\}$ with the L^2 scalar product :

$$(\varphi_a \,,\, \varphi^\star_b \,) \;=\; \delta_{a,\,b},\; \forall\, a,\, b \in \mathcal{A}_{\mathcal{T}}.$$

Then $H^\star_{\mathcal{T}}(\mathrm{div}, \Omega) = < \varphi^\star_b \,,\; b \in \mathcal{A}_{\mathcal{T}} >$. The mixed Petrov-Galerkin mixed finite element method consists just in replacing the space $H_{\mathcal{T}}(\mathrm{div}, \Omega)$ by the dual space $H^\star_{\mathcal{T}}(\mathrm{div}, \Omega)$ for **test functions** in the first equation of discrete formulation (5). We obtain by doing this the so-called **Petrov-Galerkin finite volume** scheme :

$$(9) \qquad \begin{cases} u_{\mathcal{T}} \in L^2_{\mathcal{T}}(\Omega) \,, \quad p_{\mathcal{T}} \in H_{\mathcal{T}}(\mathrm{div}, \Omega) \\ (p_{\mathcal{T}} \,,\, q) \,+\, (u_{\mathcal{T}} \,,\, \mathrm{div}\, q) \;=\; 0\,, \qquad \forall\, q \in H^\star_{\mathcal{T}}(\mathrm{div}, \Omega) \\ (\mathrm{div}\, p_{\mathcal{T}} \,,\, v) \,+\, (f \,,\, v) \quad = \; 0\,, \qquad \forall\, v \in L^2_{\mathcal{T}}(\Omega)\,. \end{cases}$$

We introduce a compact form of the previous mixed Petrov-Galerkin formulation with the help of the product space $V^\star_{\mathcal{T}}$ defined by $V^\star_{\mathcal{T}} = L^2_{\mathcal{T}}(\Omega) \times H^\star_{\mathcal{T}}(\mathrm{div}, \Omega)$. Then the discrete variational formulation (9) admits the form :

$$\xi_{\mathcal{T}} \in V_{\mathcal{T}}, \quad \gamma(\xi_{\mathcal{T}},\, \zeta) = < \sigma,\, \zeta >, \forall\, \zeta \in V^\star_{\mathcal{T}}.$$

Stability analysis

• We suppose in the following that the mesh $\mathcal{T}$ is a bidimensional cellular complex composed by triangles as proposed in the previous sections. Following the work of Ciarlet and Raviart (1972), for any element $K \in \mathcal{E}_{\mathcal{T}}$ we denote by h_K the diameter of the triangle K and by ρ_K the diameter of the inscripted ball inside K. We suppose that the mesh $\mathcal{T}$ belongs to a family $\mathcal{U}_\theta$ $(\theta > 0)$ of meshes defined by the condition $\mathcal{T} \in \mathcal{U}_\theta \iff \forall\, K \in \mathcal{E}_{\mathcal{T}}\,,\; \frac{h_K}{\rho_K} \leq \theta$. We suppose also that the dual space $H_{\mathcal{T}}(\mathrm{div},\,\Omega)$ constructed by the previous conditions satisfies the following hypothesis.

Hypothesis 1. Interpolation operator $H_{\mathcal{T}}(\mathrm{div},\,\Omega) \longrightarrow H^\star_{\mathcal{T}}(\mathrm{div},\,\Omega)$. We suppose that the mesh $\mathcal{T}$ belongs to the family $\mathcal{U}_\theta$. Let $H_{\mathcal{T}}(\mathrm{div},\,\Omega) \ni q \longmapsto \Pi\, q \in H_{\mathcal{T}}(\mathrm{div},\,\Omega)$ be the mapping defined by the condition $\Pi \Big(\sum_{a\,\in \mathcal{A}_{\mathcal{T}}} q_a\, \varphi_a \Big) = \sum_{a\,\in \mathcal{A}_{\mathcal{T}}} q_a\, \varphi^\star_a$, $\sum_{a\,\in \mathcal{A}_{\mathcal{T}}} q_a\, \varphi_a \in H^\star_{\mathcal{T}}(\mathrm{div},\,\Omega)$. We suppose that the dual basis $\varphi^\star_a$ is constructed in such a way that there exists strictly positive constants A, B, D, E that only depends on the parameter θ such that we have the following estimations :

$$
\begin{aligned}
& A\, \| \, q \,\|_0^2 \;\leq\; (\, q \,,\, \Pi\, q\,)\,, && \forall\, q \in H_{\mathcal{T}}(\mathrm{div},\,\Omega) \\
& \| \, \Pi\, q \,\|_0 \;\leq\; B\, \| \, q \,\|_0\,, && \forall\, q \in H_{\mathcal{T}}(\mathrm{div},\,\Omega) \\
& \| \, \mathrm{div}\, \Pi\, q \,\|_0 \;\leq\; D\, \| \, \mathrm{div}\, q \,\|_0\,, && \forall\, q \in H_{\mathcal{T}}(\mathrm{div},\,\Omega) \\
& (\, \mathrm{div}\, q \,,\, \mathrm{div}\, \Pi\, q\,) \;\geq\; E\, \| \, \mathrm{div}\, q \,\|_0^2\,, && \forall\, q \in H_{\mathcal{T}}(\mathrm{div},\,\Omega)\,.
\end{aligned}
$$

Stability analysis (ii)

Proposition 1. Technical lemma about lifting of scalar fields. Let θ be a strictly positive parameter. We suppose that the dual Raviart-Thomas basis satisfies the Hypothesis 1. Then there exists some strictly positive constant F that only depends on the parameter θ such that for any mesh $\mathcal{T}$ that belongs to the family $\mathcal{U}_\theta$ and for any scalar field u constant in each element $K \in \mathcal{E}_\mathcal{T}$ $(u \in L^2_\mathcal{T}(\Omega))$, there exists some vector field $q \in H_\mathcal{T}(\mathrm{div}, \Omega)$ such that $\| \, q \, \|_{H(\mathrm{div}, \Omega)} \leq F \, \| \, u \, \|_0$ and $(\, u \, , \, \mathrm{div} \, q \,) \geq \| \, u \, \|_0^2$.

Proposition 2 Discrete stability. Let θ be a strictly positive parameter. We suppose that the dual Raviart-Thomas basis satisfies the Hypothesis 1. Then we have the following discrete stability for the Petrov-Galerkin mixed formulation (9) :

$\exists \, \beta > 0 \, , \, \forall \mathcal{T} \in \mathcal{U}_\theta, \forall \xi \in V_\mathcal{T}$ such that $\| \, \xi \, \|_V = 1, \; \exists \, \eta \in V^\star_\mathcal{T}, \; \| \, \zeta \, \|_V \leq 1 \, , \gamma(\xi, \zeta) \; \geq \; \beta,$

with $\gamma(\bullet, \bullet)$ defined at the relation (3) and β chosen such that

$$\sqrt{1 - \frac{B + 2D}{A} \beta \, - \, \beta^2} \; \geq \; \Big(1 + F \, \big(1 \, + \, \sqrt{\frac{B + 2A}{A}} \, \big) \Big) \sqrt{\beta}.$$

Stability analysis (iii)

Theorem 1 **Optimal error estimate.** Let Ω be a two-dimensional open convex domain of $\mathbb{R}^2$ with a polygonal boundary, $u \in H^2(\Omega)$ be the solution of the problem (1) considered under variational formulation and $p = \nabla u$ be the associated momentum. Let θ be a strictly positive parameter and $\mathcal{U}_\theta$ a family of meshes $\mathcal{T}$ that satisfy the Hypothesis 1. Let $\xi \equiv (u_{\mathcal{T}}, p_{\mathcal{T}}) \in V_{\mathcal{T}}$ be the solution of the discrete problem (9). Then there exists some constant $C > 0$ that only depends on the parameter θ such that

$$\| u - u_{\mathcal{T}} \|_0 + \| p - p_{\mathcal{T}} \|_{H(\mathrm{div},\,\Omega)} \leq C \, h_{\mathcal{T}} \, \| f \|_0 \, .$$

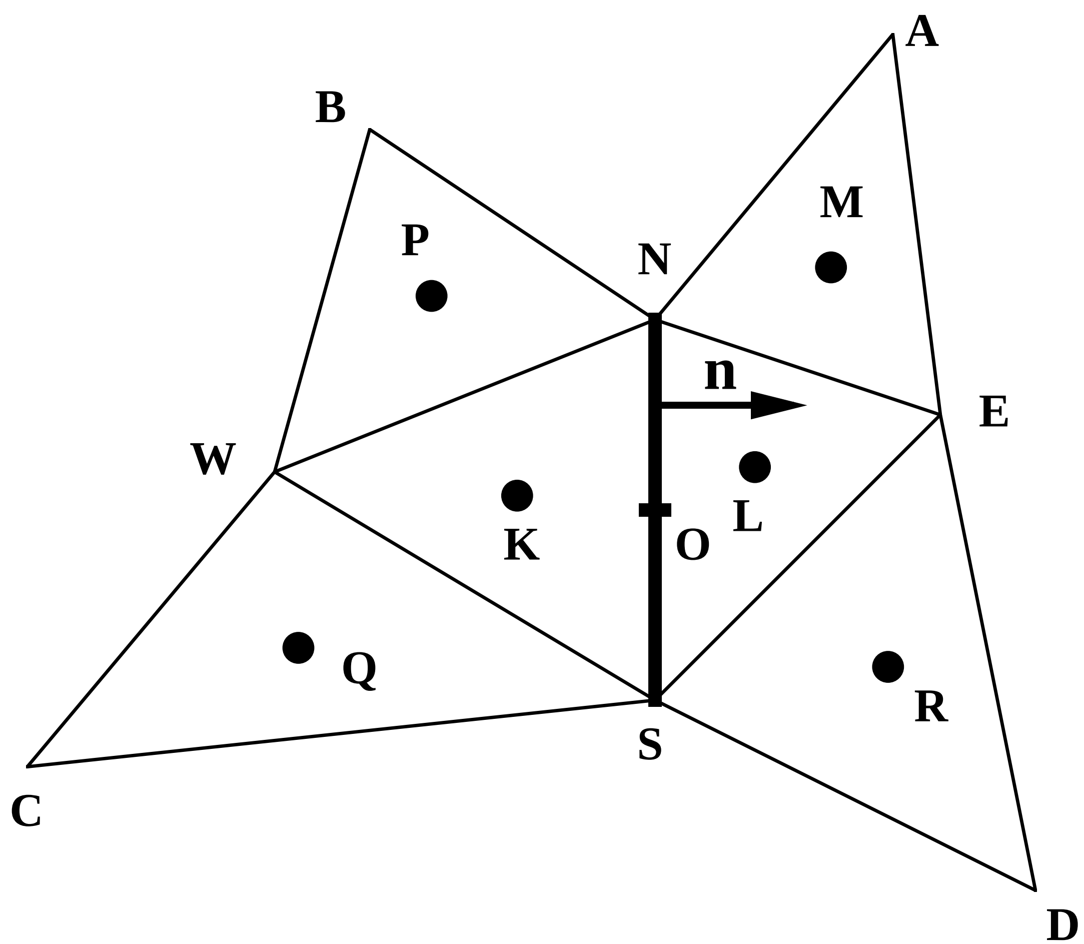

Figure 2. Support $\mathcal{V}(\mathrm{SN})$ of the dual Raviart-Thomas basis function $\varphi^{\star}_{\mathrm{SN}}$.

Towards a first Petrov-Galerkin finite volume scheme (ii)

Theorem 2 We suppose that the internal edge a links the two vertices S and N (see the Figure where a = SN, O is the middle of SN and n is the associated normal direction), if the support of the dual Raviart-Thomas basis function $\varphi^\star_{\rm SN}$ is the vicinity $\mathcal{V}(a) = \{K,\, L,\, M,\, P,\, Q,\, R\}$ of the edge a composed by six triangles presented on Figure 2 and if the divergence of the dual Raviart-Thomas basis function is equal to a **constant field** in each triangle of $\mathcal{V}(a)$ $({\rm div}\, \varphi^\star_{\rm SN} \in L^2_{\mathcal{T}}(\Omega)),$ then the **five** mean flux values

$$\eta \equiv \int_{\rm SN} \varphi^\star_{\rm SN} \bullet n \,{\rm d}\tau,\;\; \alpha \equiv \int_{\rm EN} \varphi^\star_{\rm SN} \bullet n_{\rm EN} \,{\rm d}\tau\,,\;\; \beta \equiv \int_{\rm WN} \varphi^\star_{\rm SN} \bullet n_{\rm WN} \,{\rm d}\tau\,,$$

$$\gamma \equiv \int_{\rm WS} \varphi^\star_{\rm SN} \bullet n_{\rm WS} \,{\rm d}\tau \;\text{ and }\; \delta \equiv \int_{\rm SE} \varphi^\star_{\rm SN} \bullet n_{\rm SE} \,{\rm d}\tau$$

satisfy the following **three scalar constraints** :

$$\eta\, \overrightarrow{\rm KL} + \alpha\, \overrightarrow{\rm LM} + \beta\, \overrightarrow{\rm PK} + \gamma\, \overrightarrow{\rm QK} + \delta\, \overrightarrow{\rm LR} = \mid \overrightarrow{\rm SN} \mid\, n \tag{10}$$

$$\alpha\, \overrightarrow{\rm LM} \bullet \overrightarrow{\rm WA} + \beta\, \overrightarrow{\rm PK} \bullet \overrightarrow{\rm EB} + \gamma\, \overrightarrow{\rm QK} \bullet \overrightarrow{\rm EC} + \delta\, \overrightarrow{\rm LR} \bullet \overrightarrow{\rm WD} = -3 \mid \overrightarrow{\rm SN} \mid\, n \bullet \big(\overrightarrow{\rm OL} + \overrightarrow{\rm OK}\big). \tag{11}$$

Towards a first Petrov-Galerkin finite volume scheme (iii)

- The finite volume approach is then obtained in the spirit of (8) with a six point scheme for the mean gradient in the normal direction thanks to the first equation of the mixed variational formulation (9) :

$$\int_{\mathrm{SN}} \nabla u_{\mathcal{T}} \bullet n \, \mathrm{d}\tau = \eta(u_{\mathrm{L}} - u_{\mathrm{K}}) + \alpha(u_{\mathrm{M}} - u_{\mathrm{L}}) + \beta(u_{\mathrm{K}} - u_{\mathrm{P}}) + \gamma(u_{\mathrm{K}} - u_{\mathrm{Q}}) + \delta(u_{\mathrm{R}} - u_{\mathrm{L}}). \tag{12}$$

We remark that the constraints (10) express that the relation (12) is **exact** if the field $u_{\mathcal{T}}$ is an affine function. Taking into account the fact that we have five parameters for the definition of the finite volume scheme (relation (12)) and only three constraints (relations (10) and (11)) for these parameters, the stability seems a reasonable goal, even if the problem remains essentially open for general triangular meshes.

- Début d'implémentation en cours en juin 2002 avec le stage de DEA de Sophie Borel au CEA de Saclay, en collaboration avec Christophe Lepottier.