Boundary Conditions for Petrov-Galerkin Finite Volumes

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Abstract

We consider the Petrov-Galerkin finite volumes based on dual Raviart-Thomas basis functions and the least square method. We propose a numerical scheme for various boundary conditions. First numerical tests indicates good convergence properties.

Keywords: Poisson equation, mixed finite elements.

1 Petrov-Galerkin finite volumes

• The continuous problem

Let Ω be a bidimensional bounded domain in $\mathbb{R}^2$ with a polygonal boundary $\partial\Omega \equiv \Gamma_D \cup \Gamma_N$. We consider the problem for the Laplace operator in Ω with various Dirichlet and Neumann boundary conditions:

$$(1) \qquad -\Delta u = f \ \text{ in } \ \Omega \,, \qquad u = g \ \text{ on } \ \Gamma_D \,, \qquad \frac{\partial u}{\partial n} = \gamma \ \text{ on } \ \Gamma_N \,.$$

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The datum f is supposed to belong to the Hilbert space $L^2(\Omega)$. Let us introduce the momentum $p = \nabla u$ and the Sobolev space $H(\mathrm{div}, \Omega, \Gamma_N) = \{q \in (L^2(\Omega))^2;\ \mathrm{div}\ q \in L^2(\Omega), (q.n) = 0 \text{ on } \Gamma_N\}$, where n is normal direction external to $\partial\Omega$ and $(q.n)$ the tangentiel trace considered in a weak sense [DL72]. Then the problem (1) can be written:

$$
\begin{aligned}
&(2) & -\mathrm{div}\ p &= f \text{ in } \Omega\ , \\
&(3) & p &= \nabla u \text{ in } \Omega\ , \\
&(4) & u &= g \text{ on } \Gamma_D\ , \\
&(5) & \frac{\partial u}{\partial n} &= \gamma \text{ on } \Gamma_N\ .
\end{aligned}
$$

By testing the relation (2) against an arbitrary vector valued function $q \in H(\mathrm{div}, \Omega, \Gamma_N)$ and integrating by parts, it comes:

$$(p, q) + (u, \mathrm{div}\ q) - \int_{\Gamma_D} g(q.n)\, \mathrm{d}s = 0\,.$$

The integration of the relation (3) on the domain Ω after multiplying by a scalar valued function $v \in L^2(\Omega)$ yields: $(\mathrm{div}\ p, v) + (f, v) = 0$. Then the continuous problem (1) takes the following form:

$$
(6) \qquad \left\{
\begin{array}{ll}
u \in L^2(\Omega), \quad p \in H(\mathrm{div}, \Omega)\ , & \\
(p, q) + (u, \mathrm{div}\ q) - \int_{\Gamma_D} g\,(q.n)\, \mathrm{d}s = 0, & \forall q \in H(\mathrm{div}, \Omega, \Gamma_N) \\
(\mathrm{div}\ p, v) + (f, v) = 0, & \forall v \in L^2(\Omega)\ .
\end{array}
\right.
$$

- **Discretization**

In what follows, we denote by $\mathcal{T}$ a mesh that is supposed to be a bidimensional cellular complex in the sense of [Go71] and composed by triangular elements. We define the following components of the mesh $\mathcal{T}$: $\mathcal{T}^0$ the set of vertices (components of the mesh $\mathcal{T}$ of dimension 0), $\mathcal{T}^1$ the set of edges (components of the mesh $\mathcal{T}$ of dimension 1) and $\mathcal{T}^2$ the set of triangles (components of the mesh $\mathcal{T}$ of dimension 2). Let us introduce the three following set of edges:
- The set of **internal** edges $\mathcal{T}_i^1 = \{a \in \mathcal{T}^1, \overline{a} \cap \partial\Omega = \varnothing\}$.
- The set of "**semi boundary**" edges $\mathcal{T}_s^1 = \{a \in \mathcal{T}^1, a \notin \partial\Omega \text{ and } \overline{a} \cap \partial\Omega \neq \varnothing\}$, composed by edges whose one vertex belongs to the boundary and the other one is internal.
- The set of **boundary** edges $\mathcal{T}_b^1 = \{a \in \mathcal{T}^1, \overline{a} \subset \partial\Omega\}$. Let $\mathcal{T}_N^1 = \{a \in \mathcal{T}_b^1, \overline{a} \subset \Gamma_N\}$ the set of boundary edges associated with Neumann condition and $\mathcal{T}_D^1 = \{a \in \mathcal{T}_b^1, \overline{a} \subset \Gamma_D\}$ the set of boundary edges associated with Dirichlet condition; we have $\mathcal{T}_b^1 = \mathcal{T}_D^1 \cup \mathcal{T}_N^1$ (see Figure 1).

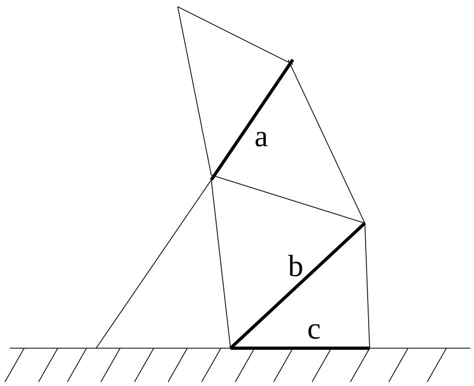

Figure 1. Exemples of edges: $a \in \mathcal{T}_i^1$, $b \in \mathcal{T}_s^1$ and $c \in \mathcal{T}_b^1$.

- **Discrete variational formulation**

We consider two classical finite dimensional spaces $L^2_\tau(\Omega)$ and $H_\tau(\text{div}, \Omega)$. A scalar valued function $v \in L^2_\tau(\Omega)$ is constant in each triangle K of the mesh; a vector valued function $p \in H_\tau(\text{div}, \Omega)$ is a linear combination of Raviart-Thomas [RT77] basis functions φ_a for each edge $a \in \mathcal{T}^1$.

- **Petrov-Galerkin mixed finite element method**

In this section, we use the variational problem (6). As $p_\tau \in H_\tau(\text{div}, \Omega)$, we first treat the Neumann condition (5). We introduce the flux $\gamma_b \equiv \frac{1}{|b|} \int_b \gamma(s)\, \mathrm{d}s$ for each edge $b \in \mathcal{T}^1_N$ and the entire lifting of the Neumann boundary condition: $\overline{\gamma}_\tau = (\sum_{b \in \mathcal{T}^1_N} \gamma_b \varphi_b) \in H_\tau(\text{div}, \Omega)$. Then we have

$$(7)\quad p_\tau = \sum_{a \in \mathcal{T}^1_i \cup \mathcal{T}^1_s \cup \mathcal{T}^1_D} p_a \varphi_a + \sum_{b \in \mathcal{T}^1_N} \gamma_b \varphi_b \equiv \pi_\tau + \overline{\gamma}_\tau \;\in H_\tau(\text{div}, \Omega, \Gamma_N) + H_\tau(\text{div}, \Omega)$$

where $H_\tau(\text{div}, \Omega, \Gamma_N) = H_\tau(\text{div}, \Omega) \cap H(\text{div}, \Omega, \Gamma_N)$. Our method is based on the construction of a discrete functional space $H^*_\tau(\text{div}, \Omega, \Gamma_N)$ generated by vectorial functions φ^*_a, $a \in \mathcal{T}^1$ that are conforming in the space $H(\text{div}, \Omega)$ and represent the dual basis of the family φ_b, $b \in \mathcal{T}^1$ with the L^2 scalar product: $(\varphi_a, \varphi^*_b) = \delta_{ab}$, $\forall\, a, b \in \mathcal{T}^1$. Then the Petrov-Galerkin mixed finite element method consists in replacing the space $H(\text{div}, \Omega, \Gamma_N)$ by the dual space $H^*_\tau(\text{div}, \Omega, \Gamma_N)$ for test functions in the first equation of the discrete formulation (6). This method yields the so-called **Petrov-Galerkin finite volumes scheme**:

$$(8)\quad \begin{cases} u_\tau \in L^2_\tau(\Omega)\,, \quad \pi_\tau \in H_\tau(\text{div}, \Omega, \Gamma_N)\,, \\ (\pi_\tau, q) + (u_\tau, \text{div}\, q) = \int_{\Gamma_D} g(q.n)\, \mathrm{d}s - (\overline{\gamma}_\tau, q), \;\; \forall q \in H^*_\tau(\text{div}, \Omega, \Gamma_N) \\ (\text{div}\, \pi_\tau, v) = -(f, v) - (\text{div} \overline{\gamma}_\tau, v)\,, \qquad\qquad \forall v \in L^2_\tau(\Omega)\,. \end{cases}$$

Now the key point is the construction of the so-called **dual Raviart-Thomas basis functions** φ^*_a, $\forall a \in \mathcal{T}^1$. For an internal edge $a \in \mathcal{T}^1_i$ see [Du02a] and for the case $a \in \mathcal{T}_N^1$ it is usless to construct φ_a^* hence the flux γ_a across Γ_N is done by the Neumann boundary condition. For the cases $a \in \mathcal{T}_s^1$ and $a \in \mathcal{T}_D^1$ we adapt the case the previous ideas.

2 First numerical tests

This section is dedicated to the construction of a dual Raviart-Thomas basis φ_b^* where b is an internal edge of the mesh $\mathcal{T}$. To seek simplicity we will consider here a problem (1) with only an homogeneous Dirichlet boundary condition (*i.e.* $\partial\Omega = \Gamma_D,\ \mathcal{T}_b^1 = \mathcal{T}_D^1,\ \mathcal{T}_N^1 = \varnothing$).

• **Vicinity of an internal edge** $b \in \mathcal{T}_i$

Let denote by $b \equiv (SN)$ an internal edge, by O the middle of SN and by K, L the two triangles that compose the co-boundary (*i.e.* the edge b is included in the boundaries of K and L). The normal n_a is supposed to be oriented from the element K towards the element L and there exist two vertices W and E such that $K = (S, N, W),\ L = (N, S, E)$. Let consider the four edges (N, W), (W, S), (S, E), (E, N) that compose the boundary of the union $K \cup L$ and define four new triangles M, P, Q and R and four new vertices A, B, C and D in the mesh $\mathcal{T}$. So we define the vicinity of the edge $b = (SN)$ as: $\mathcal{V}(b) \equiv (L, K, M, P, Q, R)$, as illustrated on Figure 2.

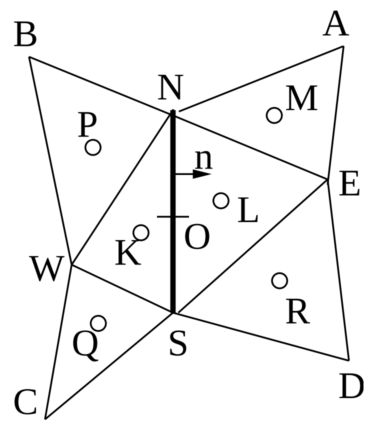

Figure 2. Vicinity $\mathcal{V}(b)$ for an internal edge $b = (SN)$.

• **Hypothesis for the dual Raviart-Thomas basis functions**

We suppose that the Raviart-Thomas dual basis functions satisfy:

-**Hypothesis** (H1): $\varphi_b^* \in H\,(\mathrm{div}, \Omega)\ , (\varphi_a, \varphi_b^*) = \delta_{ab}\ , \forall a, b \in \mathcal{T}^1$.

-**Hypothesis** (H2): For each internal edge $b = (SN)$, the support of the dual Raviart-Thomas basis function φ_b^* is included in the vicinity $\mathcal{V}\,(b) = \mathcal{V}\,(SN)\,$, composed by the triangles K, L, M, P, Q et R.

-**Hypothesis** (H3): The divergence field of φ_b^* is constant in each triangle: div $\varphi_b^* \in L_\tau^2(\Omega)$.

- **Normal gradient for internal edges**

For $b = (SN)$ let φ_b^* be a dual Raviart-Thomas basis function satisfying the hypothesis (H2). Let introduce the following fluxes across the edges SN, EN, NW, WS and SE: $\eta \equiv \int_{SN} \varphi_b^*.n_{SN}\, \mathrm{d}\gamma$, $\alpha \equiv \int_{EN} \varphi_b^*.n_{EN}\, \mathrm{d}\gamma$, $\beta \equiv \int_{NW} \varphi_b^*.n_{NW}\, \mathrm{d}\gamma$, $\gamma \equiv \int_{WS} \varphi_b^*.n_{WS}\, \mathrm{d}\gamma$, $\delta \equiv \int_{SE} \varphi_b^*.n_{SE}\, \mathrm{d}\gamma$. As $\pi_\tau \in H_\tau(\mathrm{div}, \Omega)$, we can write: $\pi_\tau = \sum_{a\in\mathcal{T}^1} p_a \varphi_a$ against the basis generated by the family of functions $\{\varphi_a\}_a$. This leads to:

$$\int_\Omega \pi_\tau \cdot \varphi_b^* \,\mathrm{d}x = \sum_a p_a \int_\Omega \varphi_a \cdot \varphi_b^* \,\mathrm{d}x = p_b = \int_b \pi_\tau \cdot n_b \,\mathrm{d}s = \int_b \nabla u_\tau \cdot n_b \,\mathrm{d}s.$$

Moreover the equality $\pi_\tau = \nabla u_\tau$ can be written weakly with the help of (3) with $q = \varphi_b^*$:

$$\int_\Omega \pi_\tau \cdot \varphi_b^* \,\mathrm{d}x = \int_\Omega \nabla u_\tau \cdot \varphi_b^* \,\mathrm{d}x = - \sum_{K\in\mathcal{V}(b)} u_K \int_{\partial K} \varphi_b^* \cdot n_{\partial K} \,\mathrm{d}s,$$

where ∂K is composed by the three edges of the triangle K (see Figure 2). These two last expressions and with the help of different flux of leads to the following formula for the gradient ∇u_τ accross the edge $b = (SN)$ [Du02a]:

$$(9) \qquad \left\{ \begin{aligned} \int_b \nabla u_\tau \cdot n_b \,\mathrm{d}s = \eta(u_L - u_K) + \alpha(u_M - u_L) + \\ + \beta(u_P - u_K) + \gamma(u_Q - u_K) + \delta(u_R - u_L). \end{aligned} \right.$$

The present finite volume approach is obtained by the representation of the normal interface gradient $\int_b \nabla u_\tau \cdot n_b \,\mathrm{d}s$ as a function of six neigbouring values. If we consider only the two triangles K and L of the co-boundary to define the scheme (*i. e.* $\alpha = \beta = \gamma = \delta = 0$), we found a more simple expression for normal flux (see *e.g.* [BMO96], [EGH2k]). We can found also more elaborate expression in the same spirit, see [No64] and [CVV99].

- **Necessary conditions for normal gradient**

Therefore the explicit representation of the normal gradient of ∇u_τ accross the edge $b = (SN)$ requieres the determination of the coefficients η, α, β, γ and δ as the fluxes of the dual basis function φ_b^*, although we do not have an explicit knowledge of the dual basis function φ_b^*. The orthogonality relations between the Raviart-Thomas basis functions φ_a and the dual basis function

φ_b^* allow to express necessary conditions on these coefficients to determine. So there are three scalar constraints (see [Du02b]):

$$\eta\overrightarrow{KL}+\alpha\overrightarrow{LM}+\beta\overrightarrow{KP}+\gamma\overrightarrow{KQ}+\delta\overrightarrow{LR}=|SN|n_{SN}\ , \tag{10}$$

$$\left\{\begin{array}{l} \alpha\overrightarrow{LM}.\overrightarrow{WA}+\beta\overrightarrow{KP}.\overrightarrow{EB}+\gamma\overrightarrow{KQ}.\overrightarrow{EC}+\delta\overrightarrow{LR}.\overrightarrow{WD}= \\ \qquad = -3|SN|n_{SN}.(\overrightarrow{OL}+\overrightarrow{OK}). \end{array}\right. \tag{11}$$

The constraints (10) express that the relation (9) is **exact** if the filed ∇u_τ is an affine function.

- **Least square finite volumes scheme**

The coefficients $\eta, \alpha, \beta, \gamma$ and δ appear in the flux expression of ∇u_τ across the edge $b = (SN)$. In order to test this method, we must determine these coefficients under the necessary conditions. For this we use a least squares method. We find η by minimizing the functional:

$$I(\eta) \equiv \left(\frac{|SN|}{\eta}n_{SN} - \overrightarrow{KL}\right)^2 + \left(3\frac{|SN|}{\eta}n_{SN}.(\overrightarrow{OL}+\overrightarrow{OK})\right)^2 .$$

To get α, β, γ and δ, we minimize the functional:

$$J(\alpha,\beta,\gamma,\delta) \equiv \left(\frac{\alpha}{\eta}\right)^2 + \left(\frac{\beta}{\eta}\right)^2 + \left(\frac{\gamma}{\eta}\right)^2 + \left(\frac{\delta}{\eta}\right)^2$$

under the conditions (10) and (11) [Bo02].

We have chosen this method to compute α, β, γ and δ, so that we minimize the extra-diagonal term of the global matrix, and maximize the diagonal one in order to increase the chance of stability.

- **First numerical results with fictitious surrounding elements**

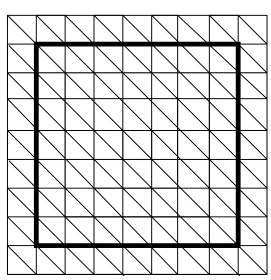

Figure 3. Global mesh composed by the internal mesh $\mathcal{T}$ of the inside domain Ω and by a fictitious mesh outside.

We will use the fictitious mesh method [Le02]. That consists to surround the domain Ω by an extra layer of elemets and we impose the $u_\mathcal{T}(K) = \frac{1}{|K|}\int_K g(x)\,\mathrm{d}x$ for K triangle of this extra layer (see Figure 3). Thus all the edges of the mesh $\mathcal{T}$ can be considered as internal (*i.e.* $\mathcal{T}^1 = \mathcal{T}_i^1$).

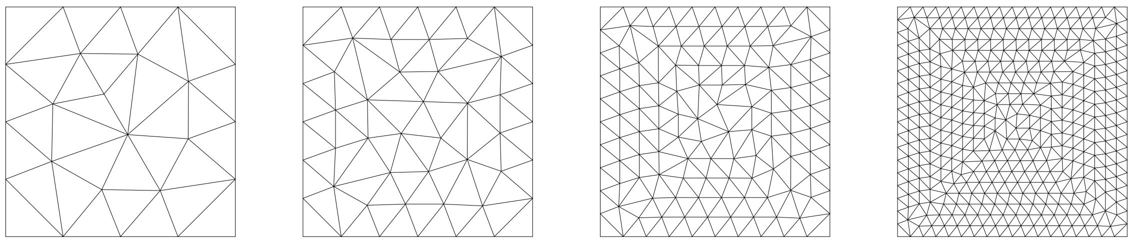

Figure 4. Different mesh refinements.

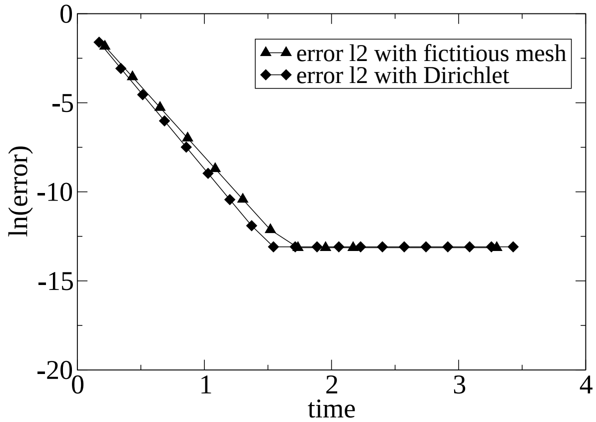

Figure 5. The ℓ^2 errors, for affine exact solution $u(x,y) = 2x + y$, computed with two methods.

We have tested the previous method for a square domain $\Omega =]0,1[^2$ and we have also used a family of unstructured meshes (see Figure 4). At first, we have tested the scheme for the problem (1) with an exact solution given by $u(x,y) = 2x + y$. This problem corresponds to non homogeneous Dirichlet boundary conditions and we used the fictitious mesh methodology. The Figure 5 schows the ℓ^2 relative errors between the exact affine solution $u(x,y) = 2x + y$ and the solution calculated with the Petrov-Galerkin scheme. The curve shows that we can obtain as a relative error the order of the machine precision. That was predicted by the fact that the scheme is **exact** for affine solution u. We have also tested the scheme for a polynomial function $u(x,y) = x(1-x)y(1-y)$ and product of sinus fonctions

$u(x,y) = \sin(\pi x)\sin(\pi y)$, that are exact solutions of the problem (1), with homogeneous Dirichlet boundary conditions.

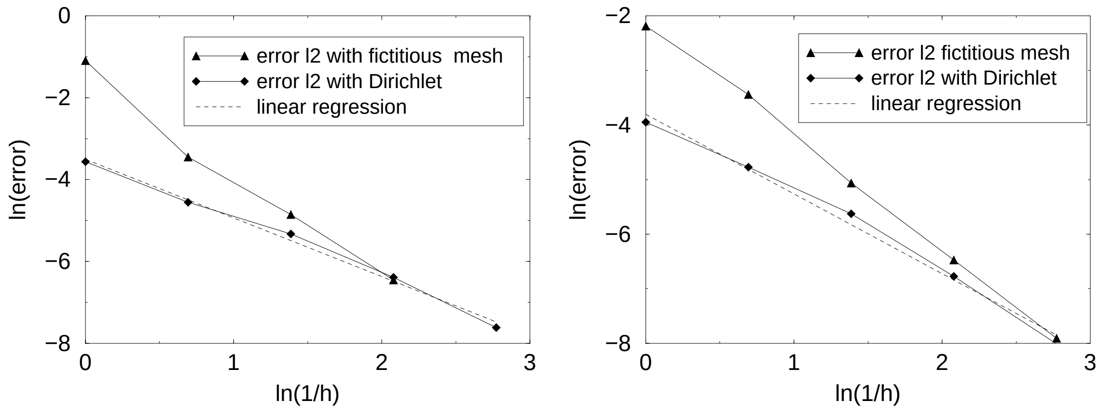

Figure 6. Comparison of the ℓ^2 relative errors *v.s.* $\mathrm{Log}(\frac{1}{h})$, between fictitious mesh method and Dirichlet boundary scheme: left for polynomial fonction $u(x,y) = x(1-x)y(1-y)$, right product of sinus fonction $u(x,y) = \sin(\pi x)\sin(\pi y)$.

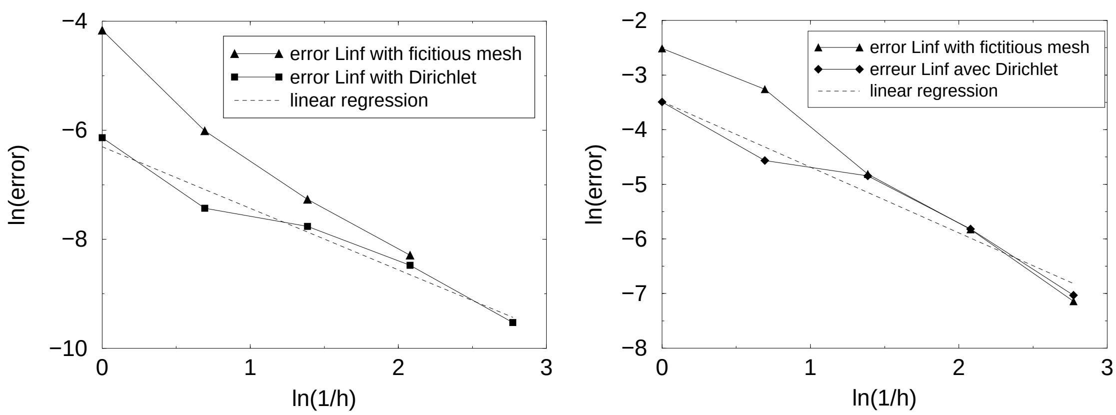

Figure 7. Comparison of the L^∞ relative errors *v.s.* $\mathrm{Log}(\frac{1}{h})$, between fictitious mesh method and Dirichlet boundary scheme for previously described test cases.

The figures 6 and 7 show, respectively, the ℓ^2 and L^∞ relative errors between the exact solutions and the solution calculated with the Petrov-Galerkin scheme *versus* $\mathrm{Log}(\frac{1}{h})$, where $h = h_\tau \equiv \max_{K\in\mathcal{T}^2}(h_K)$, with h_K is diameter of a triangle K. For the case of polynomial fonction,we have an order of convergence equal to 1.7 for ℓ^2 and 1.5 for L^∞ norms (Figure 6). Finally, for the product of sinus functions, Figure 6 show the order of convergence 1.8 and 1.7, respectively, in the sense of ℓ^2 and L^∞ norms. We conclude here that the scheme with fictitious mesh methodology has good convergence properties. In consequence the internal finite volumes scheme has succeeded for numerical test.

3 Dirichlet boundary condition

Until here, we have considered a scheme for the flux p_a (see (7)), for internal edges, now we propose a scheme for all edges $a \in \mathcal{T}^1$. So, this section is dedicated to the construction of a dual Raviart-Thomas basis φ_b^* when the edge $b = (SN)$ is a semi boundary (*i.e.* $b \in \mathcal{T}_s^1$) or strictly included in the boundary of the mesh $\mathcal{T}$ (*i.e.* $b \in \mathcal{T}_D^1$). We consider the problem (1) with Dirichlet boundary conditions (*i.e.* $\Gamma_N = \varnothing$). We still suppose that the hypotheses (H1), (H2) and (H3) satisfied for the dual basis functions φ_b^*.

• As $b = (SN) \in \mathcal{T}_s^1 \cup \mathcal{T}_D^1$, the vicinity $\mathcal{V}(b)$ of Figure 2 does not exist anymore. In all cases, one triangle that composes the vicinity is at least missing. Moreover, if we denote by $\partial\mathcal{V}(b)$ the boundary edges of the vicinity $\mathcal{V}(b)$ (see Figure 8), the dual function φ_b^* is *a priori* not null any more on the edges $a \in \partial\mathcal{V}(b) \cap \mathcal{T}_D^1$. Thus, to calculate the boundary quantities $\int_{a\in\partial\mathcal{V}(b)\cap\mathcal{T}_D^1} u_\tau \, \varphi_b^* \cdot n_a \, \mathrm{d}s$, we need to adapt the dual fonction φ_b^*.
We set the

-**Hypothesis** (H4)**:** Let s be the curvilinear coordinate equal to zero in the middle of the edge a. We suppose $\int_a \varphi_b^*.n_a \, s \, \mathrm{d}s = 0, \quad a \in \mathcal{T}_D^1 \cap \partial\mathcal{V}(b)$.

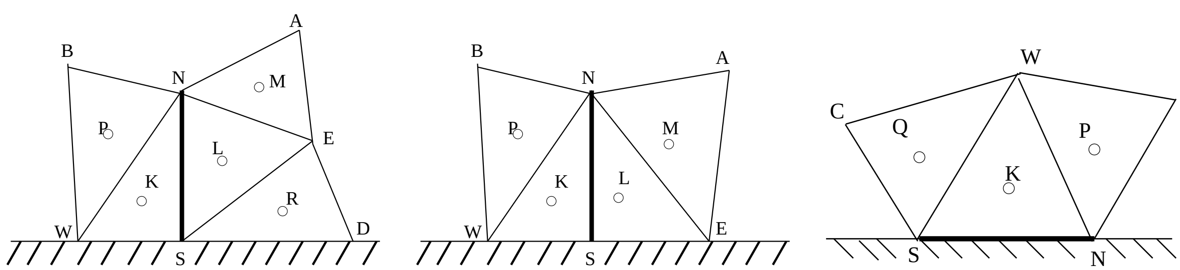

Figure 8. Different cases of the vicinity $\mathcal{V}(b)$ of the edge $b = (SN) \in \mathcal{T}_s^1 \cap \mathcal{T}_D^1$. For example in case on the right ($SN \in \partial\Omega$), $\partial\mathcal{V}(b) = \{SN, BN, BW, CW, CS\}$.

In that way, under the following affine approximation of the solution u on the edge $a \in \partial\mathcal{V}(b) \cap \mathcal{T}_D^1$: $u_\tau(s) = \overline{u}_a + s\nabla u_\tau(0) + \mathrm{O}(s^2)$, we get $\int_a u_\tau \varphi_b^*.n_a ds = \overline{u}_a \int_a \varphi_b^*.n_a ds + \mathrm{O}(|a|^3)$, with $\overline{u}_a \equiv \frac{1}{|a|} \int_a g(s) \, \mathrm{d}s$. This mean value $\overline{u}_a$ is now a natural degree of freedom for the edge $a \in \mathcal{T}_D^1$ associated with the imposed Dirichlet condition.

- **Normal gradient for non-internal edges**

The expression of the flux of the normal gradient across the edge $b = (SN)$ is slightly modified. We have the following Theorem [Bo02].

Theorem 1 (Normal flux).

$$\int_b \nabla u_\tau \cdot n_b \, \mathrm{d}s = \sum_{a \in \partial\mathcal{V}(b) \cap \mathcal{T}_D^1} \overline{u}_a \int_a \varphi_b^* \cdot n_a \, \mathrm{d}s - \sum_{K \in \mathcal{V}(b)} u_K \int_{\partial K} \varphi_b^* \cdot n \, \mathrm{d}s.$$

Moreover, we estimate the number of necessary conditions on the fluxes η, α, β, γ and δ for boundary edges. We count the number of orthogonality relations that we can write between the dual basis function φ_b^* and the basis functions $\{\varphi_a\}_a$ where the a's are the edges of the vicinity of $\mathcal{V}(b)$. For those non-internal edges, we have the following result [Bo02].

Theorem 2 (Necessary conditions).

$$\eta\overrightarrow{KL} + \alpha\triangle_{NE}\overrightarrow{LM} + \beta\triangle_{NW}\overrightarrow{KP} + \gamma\triangle_{SW}\overrightarrow{KQ} + \delta\triangle_{SE}\overrightarrow{LR} = |SN|\, n_{SN},$$

where $\triangle_a = 1$ if the edge a appears in the set of edges of the vicinity $\mathcal{V}(b)$, 0 elsewhere.

- **Numerical tests**

We have tested the scheme for the same set of exact solutions of the problem (1), but we did not use the method of fictitious mesh (see [Te03]). So, the Dirichlet boundary conditions are now taken into account by the scheme described in this section. Again, we use the least square method to determine the fluxes $\eta, \alpha, \beta, \gamma$ and δ. Figures 5, 6 and 7 show that the method with boundary conditions has similar results of precision and convergence, as the one for the fictitious mesh method. Thus, our scheme is valid for boundary Dirichlet conditions without loss of exacteness property of the scheme for affine solution u.

4 Mixed Dirichlet and Neumann boundary conditions

In this section we consider the problem (1) with both Dirichlet and Neumann boundary conditions, of which the variational formulation is given by (8). For $b \in \mathcal{T}_N^1$, it is useless to construct φ_b^*, hence the flux $p_b = \gamma_b$ is imposed by the Neumann boundary conditions. So, we just have to construct φ_b^* for all

$b \in \{\mathcal{T}_s^1 \cup \mathcal{T}_D^1, \bar{b} \cap \mathcal{T}_N^1 \neq \varnothing\}$. For these edges we have a poor stencil, but there is more information given by the Neumann conditions on the boundary edge. So we suppose for φ_b^*:

- **Hypothesis** (H′1): $(\varphi_a, \varphi_b^*) = \delta_{ab}\ , \forall a \in \mathcal{T}^1 \setminus \mathcal{T}_N^1$.

We remark that the scalar producut (φ_a, φ_b^*) is *a priori* not nul for $a\mathcal{T}_N^1 \cap \partial\mathcal{V}(b)$. Thus the decomposition (7) of $p_\tau \in H_\tau(\mathrm{div}, \Omega, \Gamma_N)$, leads to:

$$(12)\quad \int_\Omega p_\tau \cdot \varphi_b^* \,\mathrm{d}x = \sum_{a\in\mathcal{T}^1\setminus\mathcal{T}_N^1} p_a \int_\Omega \varphi_a \cdot \varphi_b^* \,\mathrm{d}x = p_b + \sum_{a\in\mathcal{T}_N^1\cap\partial\mathcal{V}(b)} \gamma_a \int_\Omega \varphi_a \cdot \varphi_b^* \,\mathrm{d}x.$$

We still suppose hypotheses (H2) and (H3). Hypothesis (H4) is realised if $\partial\mathcal{V}(b) \cap \mathcal{T}_D^1 \neq \varnothing$. Like in the case of the Dirichlet boundary condition, the dual fonction φ_b^* is *a priori* not nul on the edges $a \in \partial\mathcal{V}(b) \cap \mathcal{T}_N^1$. So, we have to calculate the quantities $\int_{a\in\partial\mathcal{V}(b)\cap\mathcal{T}_N^1} u_\tau\ \varphi_b^* \cdot n_a \,\mathrm{d}s$, but we do not have any information about u_τ on the edges $a \in \partial\mathcal{V}(b) \cap \mathcal{T}_N^1$. Thus we suppose

-**Hypothesis** (H5): $\varphi_b^* \cdot n_a = 0,\ \forall a \in \mathcal{T}_N^1$.

Then the quantity $\displaystyle\int_{a\in\partial\mathcal{V}(b)\cap\mathcal{T}_N^1} u_\tau\ \varphi_b^* \cdot n_a \,\mathrm{d}s$ is null, and φ_b^* belongs to the space $H_\tau^*(\mathrm{div}, \Omega, \Gamma_N)$.

• **Normal gradient**

The equality $p = \nabla u_\tau$ can be developed as follows:

$$(13)\quad \begin{cases} \displaystyle\int_b \nabla u_\tau \cdot \varphi_b^* \,\mathrm{d}x \ = \sum_{a\in\partial\mathcal{V}(b)\cap\mathcal{T}_D^1} \int_a u_\tau\ \varphi_b^* \cdot n_a \,\mathrm{d}s + \\ \displaystyle\qquad + \sum_{a\in\partial\mathcal{V}(b)\cap\mathcal{T}_N^1} \int_a u_\tau\ \varphi_b^* \cdot n_a \,\mathrm{d}s - \sum_{K\in\mathcal{V}(b)} \int_K u_\tau\ \mathrm{div}\varphi_b^* \,\mathrm{d}x. \end{cases}$$

According to expressions (12), (13) and hypotheses (H4) and (H5), we obtain the following result [Te03]:

Theorem 3 (Normal flux for Dirichlet-Neumann boundary conditions)

$$(14)\quad \begin{cases} \displaystyle\int_b \nabla u_\tau \cdot n_b \,\mathrm{d}s \ = \sum_{a\in\partial\mathcal{V}(b)\cap\mathcal{T}_D^1} \bar{u}_a \int_a \varphi_b^* \cdot n_a \,\mathrm{d}s \\ \displaystyle\qquad - \sum_{a\in\partial\mathcal{V}(b)\cap\mathcal{T}_N^1} \gamma_a \int_\Omega \varphi_a \cdot \varphi_b^* \,\mathrm{d}x - \sum_{K\in\mathcal{V}(b)} u_K \int_{\partial K} \varphi_b^* \cdot n \,\mathrm{d}s. \end{cases}$$

The normal flux is a linear function of the degree of freedom of u_τ in the vicinity of the edge b, as first proposed in [Du92]. For necessary conditions between the differents flux of φ_b^* and $\int_\Omega \varphi_a \cdot \varphi_b^* \, \mathrm{d}x, a \in \partial\mathcal{V}(b) \cap \mathcal{T}_N^1$, we study the following particular case (see [Te03] for the study of all cases):

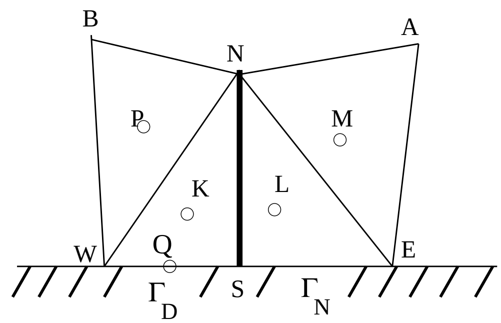

Figure 9. Treatment of a general boundary conditions.

let $b = (SN) \in \mathcal{T}_s^1$, where the vertex N is internal to the domain and the vertex S belongs to $\partial\Omega$. Let the edge $(ES) \in \mathcal{T}_N^1$ and the edge $(WS) \in \mathcal{T}_D^1$ (see Figure 9). In this case two triangles composing the vicinity $\mathcal{V}(b)$ are lost. Then we obtain the following expression for the normal flux:

$$
(15) \qquad \left\{ \begin{aligned} \int_b \nabla u_\tau . n_b \, \mathrm{d}s &= \eta(u_L - u_K) + \alpha(u_M - u_L) + \\ &\qquad + \beta(u_P - u_K) + \gamma(\overline{u}_{WS} - u_K) - \widetilde{\delta}\, \gamma_{ES} \end{aligned} \right.
$$

where $\widetilde{\delta} \equiv \int_\Omega \varphi_{ES} \cdot \varphi_{SN}^* \, \mathrm{d}x$ and $\gamma_{ES} = \frac{1}{ES} \int_{ES} \gamma(s) \, \mathrm{d}s$. Then the four fluxes $\eta, \alpha, \beta, \gamma$ and the unknow coefficient $\widetilde{\delta}$ satisfy the following two scalar constraints [Te03]:

Theorem 4 (Necessary conditions).

$$
\left(\eta \overrightarrow{KL} + \alpha \overrightarrow{LM} + \beta \overrightarrow{KP} + \gamma \overrightarrow{KQ} \right) . \frac{\overrightarrow{SN}}{SN} = - \frac{2|K|}{SN} \widetilde{\delta} \,,
$$

$$
\frac{1}{SN} \left[\left(\eta \overrightarrow{KL} + \alpha \overrightarrow{LM} + \beta \overrightarrow{KP} + \gamma \overrightarrow{KQ} \right) . n_{SN} \right] + \widetilde{\delta} \left(1 - \overrightarrow{WN} . \frac{\overrightarrow{SN}}{SN^2} \right) = 1.
$$

We have proven that these constraints express that the relation (15) is **exact** if the field u_τ is affine.

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