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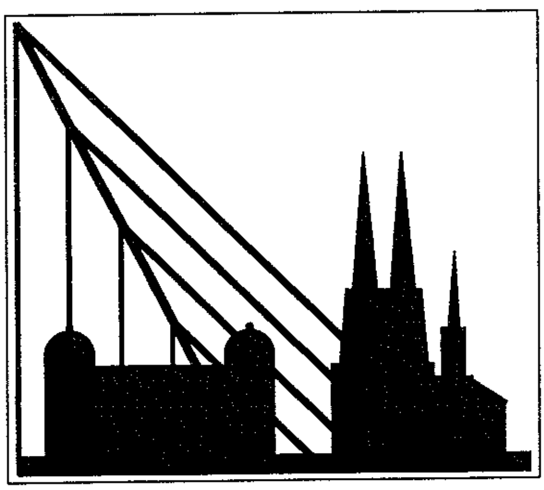

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Nonlinear Interpolation and Total Variation Diminishing Schemes

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1. Introduction.

We consider the following scalar conservation law in one space dimension :

$$(1.1) \quad \frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0 \quad , \quad x \in \mathbb{R} \quad , \quad t > 0 \quad , \quad u(x,t) \in \mathbb{R}$$

where the flux f is supposed to be a convex regular (C^2 class) real function. We focus on the discretization of the Cauchy problem associated with an initial condition u_0 :

$$(1.2) \quad u(x,0) = u_0(x) \quad , \quad x \in \mathbb{R}$$

when the initial datum u_0 satisfies

$$(1.3) \quad u_0 \in L^\infty(\mathbb{R}) \cap BV(\mathbb{R}) \; .$$

2. New Presentation of MUSCL Type Schemes.

We discretize the space variable x on a regular grid x_j with a mesh size $h > 0$. Then (1.1) becomes by a (infinite) system of ordinary differential equations (method of lines) where time t is a continuous variable :

$$(2.1) \quad \frac{du_j}{dt} + \frac{1}{h} \left(f_{j+1/2} - f_{j-1/2} \right) = 0 \quad , \quad j \in \mathbb{Z} \quad , \quad t > 0 \, .$$

The flux $f_{j+1/2}$ is chosen according to the ideas of the MUSCL method introduced by Van Leer [1979] : given a monotone flux function $\phi(\bullet,\bullet)$ (see the definition e.g. in Osher [1985]) and interpolated values $u_{j+1/2}^{\pm}$ on each side of the mid-point $x_{j+1/2}$, we choose :

$$(2.2) \qquad f_{j+1/2}(t) = \phi\left(u^{-}_{j+1/2}(t), u^{+}_{j+1/2}(t)\right) \ , \quad t \geq 0 \quad , \quad j \in \mathbf{Z} \ .$$

The interface values are nonlinear interpolations of the sequence $\{ u_k \}$ at the particular point $x_{j+1/2}$. We restrict ourselves on three point stencils of MUSCL type :

$$(2.3) \qquad u^{-}_{j+1/2}(t) = L(u_{j-1}, u_j, u_{j+1}) \quad ; \quad u^{+}_{j-1/2}(t) = R(u_{j-1}, u_j, u_{j+1}) \ .$$

Under natural hypotheses of homogeneity, translation invariance and left-right symmetry, the functions $L(\bullet,\bullet,\bullet)$ and $R(\bullet,\bullet,\bullet)$ of (2.3) are parameterized by the so-called limiter function $\varphi(\bullet)$ (see Dubois [1990] for details) :

$$(2.4) \qquad L(u, v, w) = v + \frac{1}{2}\varphi\left(\frac{v-u}{w-v}\right)(w-v) \quad ; \quad R(u, v, w) = v - \frac{1}{2}\varphi\left(\frac{w-v}{v-u}\right)(v-u) \ .$$

The hypothesis of monotonicity adds classical constraints on the limiter φ (e.g. Sweby [1984]) and we propose a new notion (introduced in Dubois [1988]) concerning the interpolation functions $L(\bullet,\bullet,\bullet)$ and $R(\bullet,\bullet,\bullet)$ defined in (2.4).

Convexity hypothesis : The sequence of values

$$(2.5) \qquad u_{j-1} \ , \ u^{+}_{j-1/2} \ , \ u_j \ , \ u^{-}_{j+1/2} \ , \ u_{j+1} \ ,$$

has the same convexity that the given sequence u_{j-1}, u_j, u_{j+1} •

If the limiter φ is defined according to (2.4), then convexity and monotonicity conditions hold if and only if we have :

$$(2.6)(a) \qquad -\infty < r \leq 1 \ : \ \max(0,r) \leq \varphi(r) \leq 1 \ ;$$

$$(2.6)(b) \qquad r \geq 2 \ : \ 1 \leq \varphi(r) \leq \min(r,2) \ ;$$

$$(2.6)(c) \qquad r \geq 1 \ : \ \varphi(r) \leq r\,\varphi\left(\frac{1}{r}\right) .$$

3. Decreasing of the Total Variation.

We have proved in Dubois [1990] under Lipschitz continuity hypotheses for the flux function ϕ that the method of lines (defined by the equalities (2.1)-(2.4)) is mathematically well defined for the short times ($0 \le t \le T$) when the initial datum $\{u_j^0\}$ satisfies (1.3) in a discrete manner. Then we prove the BV stability for short times :

$$(3.1) \qquad TV(t) \equiv \sum_{j \in \mathbb{Z}} \left| u_{j+1}(t) - u_j(t) \right| \le TV(0)$$

under the conditions (2.6) **and** the following one :

$$(3.2) \qquad r \le 0 : \varphi(r) \le (\alpha-2)\, r \quad ; \qquad r > 0 : \varphi(r) \le \alpha \quad ; \qquad 1 \le \alpha \le 2 \text{ fixed}.$$

We remark that the TVD conditions proposed by Sweby [1984] correspond to $\alpha = 2$.
We have also proved the L^∞ stability :

$$(3.3) \qquad \inf_{l \in \mathbb{Z}} u_l^0 \le u_j(t) \le \sup_{l \in \mathbb{Z}} u_l^0 \quad , \qquad j \in \mathbb{Z} \quad , \quad 0 \le t \le T \; .$$

Therefore the method of lines is defined for long times. Problems of that type have been studied previously by Sanders [1983] and Osher [1985]. The difficult point here is that we do **not** assume $u^0 \in L^1$.

4. Order of Accuracy.

The truncation error at point x_j for a mesh size h is defined according to :

$$(4.1) \qquad \rho_j^h(t) \equiv \frac{dv_j^h}{dt} - \frac{1}{h}\left(\phi\left(\bar{v}_{j+1/2}, \overset{+}{v}_{j+1/2} \right) - \phi\left(\bar{v}_{j-1/2}, \overset{+}{v}_{j-1/2} \right) \right)$$

where v_j denotes the value of the solution u of the partial differential equation (1.1) at the point x_j. This truncation error is **second order accurate** in space for regular points ($u_x \ne 0$) if $\varphi(1) = 1$ and φ derivable on each side of the value 1. The latter result remains **true for critical points** ($u_x = 0$) (sonic as well as nonsonic critical value) if we suppose moreover

$$(4.2) \qquad \varphi(-1) + \varphi(3) = 2 \, .$$

The discretization of (2.1)-(2.4) with respect to **time** by the so-called "five point TVD scheme" (see e.g. Sweby [1984]) or the two level second order Runge-Kutta scheme (Heun scheme) can maintain both the TVD property **and** the second order accuracy near extrema under some CFL restriction (the CFL number is smaller than 1/4 for the Heun scheme, see Dubois [1990] for details) if the numerical flux ϕ satisfies the following property :

$$(4.3) \qquad \text{if } df(w) > 0 , \; \frac{\partial \phi}{\partial v}(u=w, v=w) = 0 \quad ; \qquad \text{if } df(w) < 0 , \; \frac{\partial \phi}{\partial u}(u=w, v=w) = 0$$

and if the limiter function φ satisfies the following two extensions of (4.2) :

$$(4.4) \qquad (1-\sigma)\ \varphi\left(\frac{1+\sigma}{-1+\sigma}\right) + (1+\sigma)\ \varphi\left(\frac{3+\sigma}{1+\sigma}\right) \equiv 2 \ , \quad \forall\, \sigma \in [0,\delta] \ , \quad \text{for some } \delta > 0$$

$$(4.5) \qquad (k+1/2)\ \varphi\left(\frac{k-1/2}{k+1/2}\right) + (k-1/2)\ \varphi\left(\frac{k-3/2}{k-1/2}\right) = 1 \ , \qquad \text{for } k = -2, -1, 0, 1, 2 \ .$$

The conditions (4.1), (4.4) and (4.5) are less restrictive than similar relations obtained independently by Wu [1989].

The following real function that we call the "Lagrange limiter" defined according to the relations

$$(4.6)(a) \qquad r \le -3 : \varphi(r) = 0 \quad ; \quad -3 \le r \le -1 : \varphi(r) = \frac{r+3}{4} \quad ; \quad -1 \le r \le 0 : \varphi(r) = -\frac{r}{2}$$

$$(4.6)(b) \qquad 0 \le r \le \frac{1}{3} : \varphi(r) = \frac{5r}{2} \quad ; \quad \frac{1}{3} \le r \le 3 : \varphi(r) = \frac{r+3}{4} \quad ; \quad r \ge 3 : \varphi(r) = \frac{3}{2}$$

satisfies both (4.4) and (4.5) and is displayed in Figure 1.

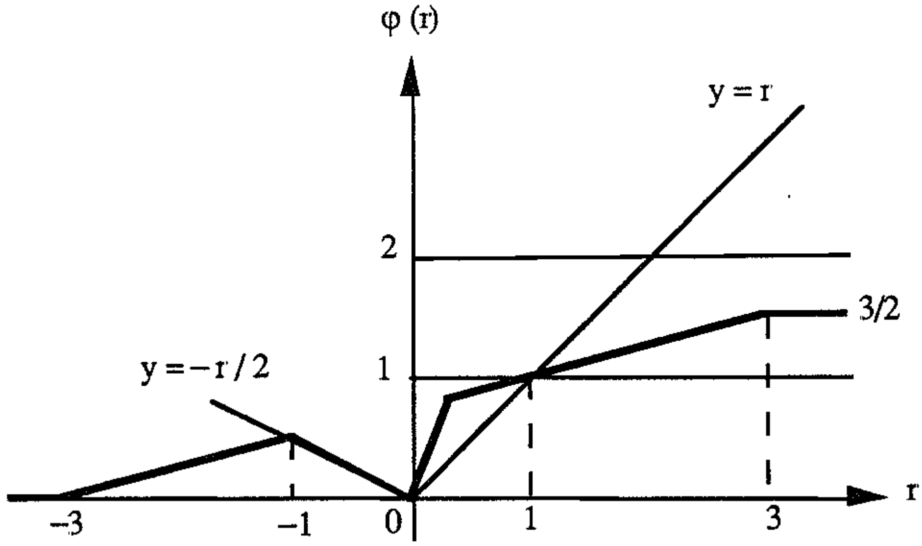


Figure 1. The Lagrange limiter (defined by the relations (4.6)).

5. Numerical Tests with the Advection Equation.

We present approximate solutions of the "Kreiss equation" :

(5.1) $$\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0 , \quad 0 \le x \le 1 , \quad t \ge 0 , \quad a = 1$$

with periodic boundary conditions and an initial regular profile u_0 defined by the relations

(5.2)(a) $$u_0(x) = 1 - \left(x - 1/2\right) + \left(x - 1/2\right)^2 , \quad -\frac{1}{4} \le x \le \frac{3}{4} ,$$

(5.2)(b) $$u_0(0) = \frac{du_0}{dx}(0) = \frac{d^2 u_0}{dx^2} = u_0(1) = \frac{du_0}{dx}(1) = \frac{d^2 u_0}{dx^2}(1) = 0 ,$$

(5.2)(c) $$u_0 \in P_5 \text{ on } \left[0 , \frac{1}{4}\right] \text{ and on } \left[\frac{3}{4} , 1\right] , \quad u_0 \in C^2([0 , 1]) .$$

This profile admits an extremum at x=1/2 and $u_{xxx}(1/2)$ is **not** null. We use meshes with grid spacing h that are powers of 1/2 :

(5.3) $$h = 2^{-k} , \quad k = 3 , 4 , 5 , 6 , 7 , 8 , 9$$

and computation nodes located according to the relations

(5.4) $x_j = (j-1)\,h \quad , \quad j = 1, 2, \ldots, 1/h$.

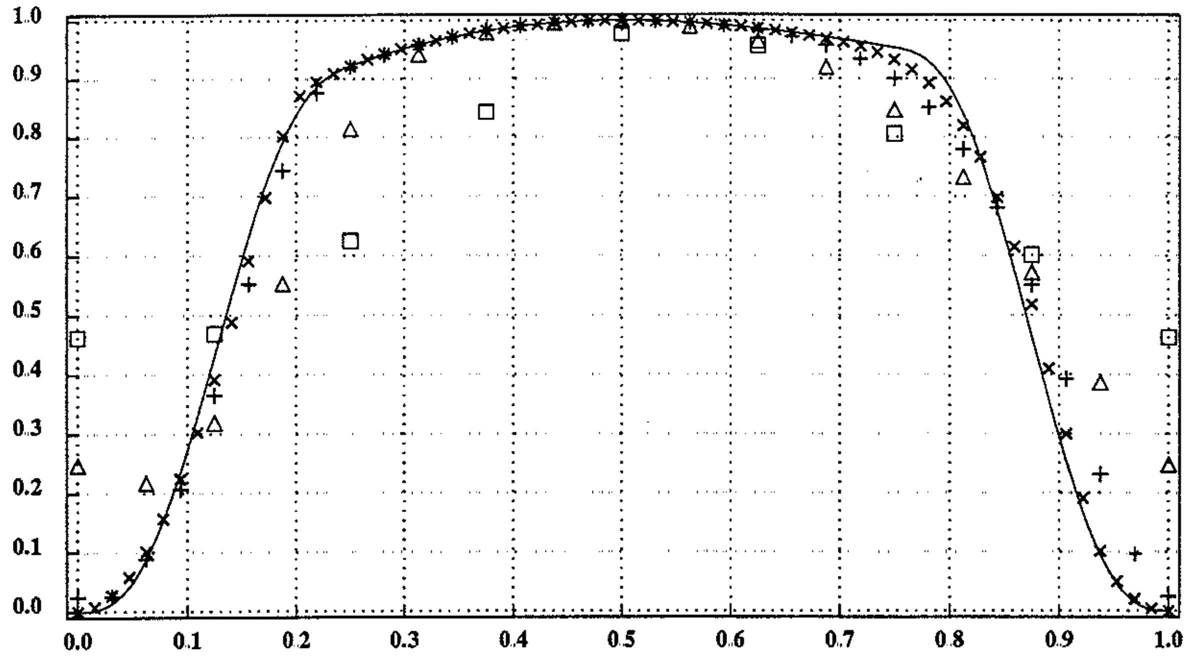
We observe that $x = 1/2$ is **always** a mesh point. We use several schemes associated with classical limiter functions : upwind ($\varphi \equiv 0$), Lax-Wendroff type ($\varphi \equiv 1$), Van Leer 's MUSCL (see Van Leer [1977]), minmod (Harten [1983]), superbee (Roe [1985]), the present Lagrange limiter and the more complicated UNO2 interpolation (Harten-Osher [1986]). Note that with UNO2, the interpolation (2.3) is a five point scheme instead of a 3 point for the other schemes. We use the two level Heun scheme and the following Courant-Friedrichs-Lewy condition :

(5.5) $\Delta t \le \dfrac{h}{2\,a}$.

We compare all the numerical approximations with the exact solution after one period. On Figure 2, we observe that UNO2 and Lagrange limiter give equivalent results. On Figure 3, we compare the L^∞ errors measured in a base 2 logarithm schedule. Despite the fact that the present limiter remains first order in this case at the extremum ($\Delta t \le h / 4$ for satisfying both (4.4) and (4.5) in this case) the L^∞ norm decreases "better that the order 2" which is not the case for the other schemes in this particular example.

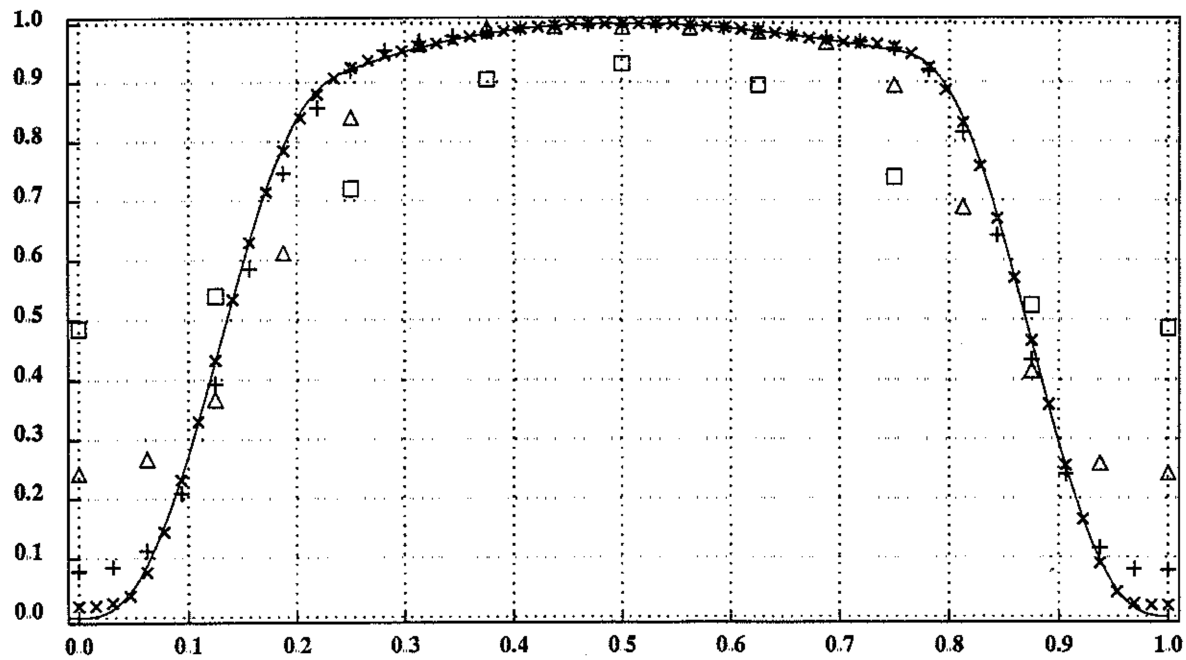
6. Conclusion.

In this paper, we have proposed a concept of convexity preserving property for the approximation of scalar conservation laws with the Van Leer's numerical method. New restrictions have been derived for the construction of the associated limiter functions. We have proved that the method of lines is well posed for u_0 in $L^1 \cap BV$ and new conditions on the limiter have been given to prove the TVD property of the associated solution. We have proved that second order accuracy (in the sense of the truncation error) can be maintained, even at a nonsonic extremum ; this last property and Total Variation Diminishing are compatible with a discretization in time with a two-level Runge-Kutta scheme under new restrictions on the limiter function that have been detailed. An example of such a limiter (the so-called Lagrange limiter) has been proposed and implemented in the case of the advection equation. Theoretical results on orders of convergence have been confirmed numerically.



UNO2 limiter

□ 8 points   △ 16 points   + 32 points
× 64 points   —— exact solution



present Lagrange limiter

Figure 2. Advection of a regular profile Heun scheme cfl = 0.5
Solutions obtained with the UNO2 scheme and the Lagrange limiter

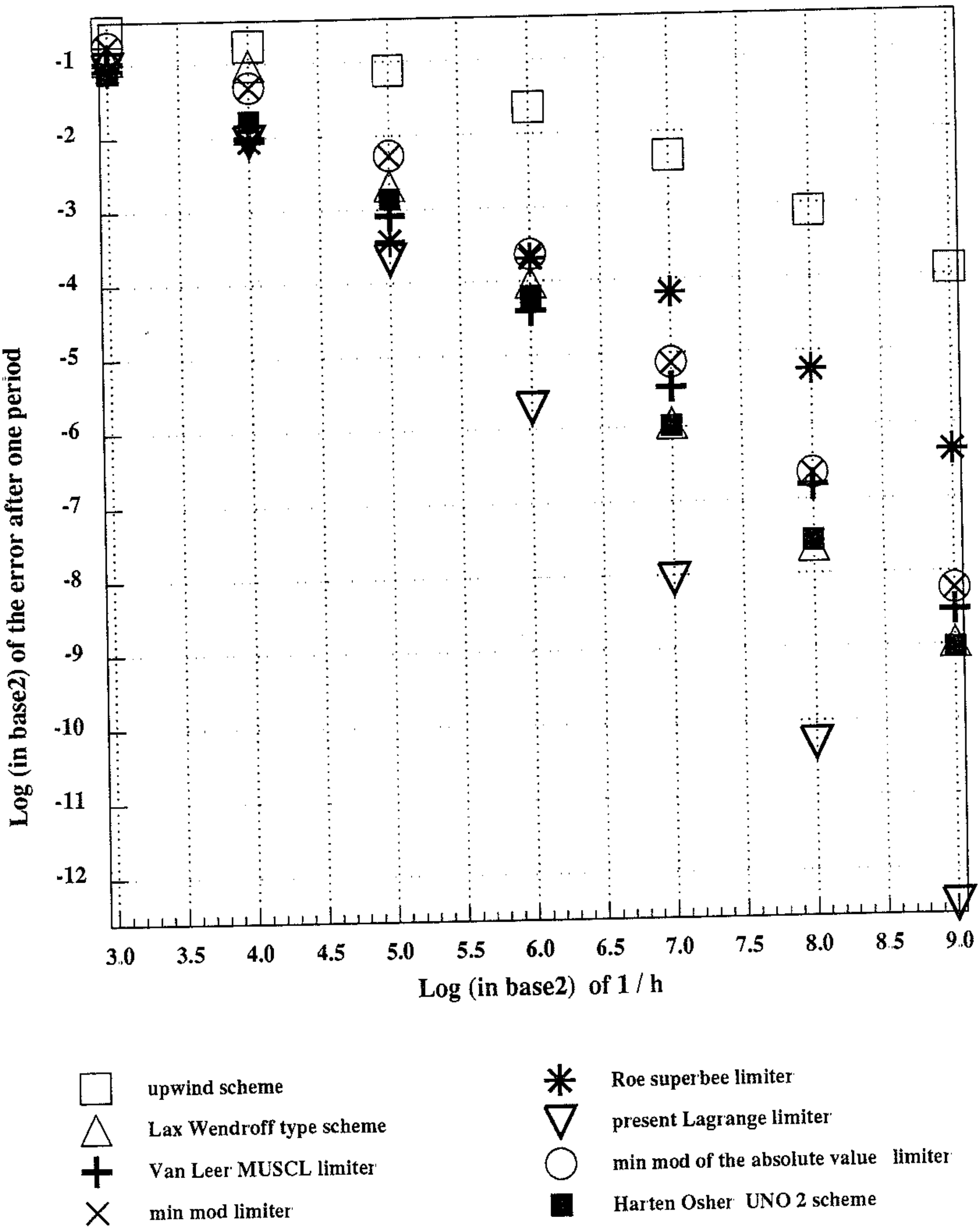


Figure 3. Advection of a regular profile Heun scheme cfl = 0.5
Log base2 of the L infinity norm error

7. References

Dubois F. Interpolation non linéaire et schémas à variation totale décroissante, Internal Report, Aérospatiale Les Mureaux ST/MI n° 40348, november 1988.

Dubois F. Nonlinear Interpolation and Total Variation Diminishing Schemes, Internal Report, Aérospatiale Les Mureaux ST/S n° 46195, june 1990.

Harten A. High Resolution Schemes for Hyperbolic Conservation Laws, J. Comp. Phys., **49**, pp. 357-393, 1983.

Osher S. Convergence of generalized MUSCL schemes , SIAM J. of Numer. Anal., 22, pp. 947-961, 1985.

Roe P. Some Contributions to the Modelling of Discontinuous Flows, in **Lectures in Applied Maths., 22** (Engquist, Osher and Sommerville eds), pp. 161-193, AMS, 1985.

Sanders R. On Convergence of Monotone Finite Difference Schemes With Variable Spatial Differencing, Math. of Comp., **40**, pp. 91-106, 1983.1983.

Sweby P.K. High Resolution Schemes Using Flux Limiters for Hyperbolic Conservation Laws, SIAM J. of Numer. Anal., **21**, pp. 995-1011, 1984.

Van Leer B. Towards the Ultimate Conservative Scheme IV. A New Approach to Numerical Convection, J. Comp. Phys., **23**, pp. 276-299, 1977.

Van Leer B. Towards the Ultimate Conservative Scheme V. A Second Order Sequel to Godunov's Method, J. Comp. Phys., **32**, pp. 101-136, 1979.

Wu H. MmB - A New Class of Accurate High Resolution Schemes for Conservation Laws in Two Dimensions, 8th GAMM Conference on Numerical Methods in Fluid Mechanics, Delft, 27-29 sept 1989.